Eastern Economy Edition

Second Edition

Textbook of Differential Calculus

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PHI

Rs. 350.00

TEXTBOOK OF DIFFERENTIAL CALCULUS, 2nd ed.
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ISBN-978-81-203-3853-1

Third Printing (Second Edition) ... ... **August, 2009**

Published by Asoke K. Ghosh, PHI Learning Private Limited, M-97, Connaught Circus, New Delhi-110001 and Printed by Mudrak, 30-A, Patparganj, Delhi-110091.

Contents

Preface

Differential calculus is a powerful mathematical tool that finds its applications in almost every branch of sciences and engineering. The subject also occupies a central position in mathematics from which different lines of development extend in many directions.

The book, now in its second edition, is primarily intended for undergraduate students of mathematics and engineering who have already completed their first course of study in calculus at the senior secondary or intermediate classes.

The text is divided into 11 chapters. Chapters 1 and 2 discuss differentiation and successive differentiation of functions such as trigonometric functions, logarithmic and exponential functions, implicit and explicit functions and their inverse. Leibnitz's theorem, an important theorem on successive differentiation, has also been included in the chapter on successive differentiation.

Taylor's and Maclaurin's (or Stirling's) theorems on expansion of series have been discussed in Chapter 3. In Chapter 4, limit of functions of indeterminate forms is discussed with the help of L'Hospital rule. Chapter 5 deals with the partial differentiation of homogeneous functions. The concept of total differentiation is also discussed in this chapter.

Chapters 6–11 deal with applications of differential calculus such as finding equations of tangents and normal, curvature, asymptotes to a curve, maxima and minima of functions, envelopes and curve tracing. Equations of a tangents, normals, subtangents, subnormals in Cartesian, polar and parametric forms are discussed in Chapter 6. Geometric representation of a curvature, different types of curvatures and radius of curvature at a point and at the origin are discussed in detail in Chapter 7. Chapter 8 presents different methods of finding asymptotes to Cartesian and polar curves. Maxima and minima of functions of two and more than two variables are explained in Chapter 9 with the help of simple geometrical examples. Chapter 10 discusses equation of envelopes. An asymptote to a curve is an important geometrical concept that helps trace a curve; a separate chapter (Chapter 11) is devoted to the concept and methods of tracing curves.

Solved and exercise problems are part of almost every section. All the exercise problems are provided with their answers to build up the confidence of students and encourage them to study the topic in depth. Solved problems and practice exercises have been taken from previous years' examination papers of various universities and competitive examinations. Multiple-choice questions, given at the end of the book, will help students prepare for civil services and other competitive examinations.

The present edition of the book is thoroughly revised as per the latest syllabus of Indian universities to fulfill the need of students. Rolle's theorem, the most important theorem in differential calculus, has also been introduced.

We are very much indebted to our colleagues for their kind cooperation and suggestions while preparing the manuscript. We also sincerely thank PHI Learning especially its editorial and production team, for their painstaking efforts in producing second edition of this book.

We would greatly appreciate receiving suggestions and constructive criticisms from teachers and academics on improving the contents and presentation of the book.

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Introduction

By *differential calculus* we mean the rate of change in one quantity corresponding to the infinitesimal changes in the values of the other. They are so related that the values of one depend on the values of the other. The subject of differential calculus has its origin mainly in the geometrical problems of the determination of the gradient of a curve at a point along the tangent. This subject has a large number of physical concepts such as velocity at an instant, acceleration at an instant, curvature at a point, density at a point, and specific heat at any temperature. Each physical concept appears as a rate of change as against the average rate of change. The fundamental concepts underlies the introduction to the notion of derivatives.

Differential calculus has its origin in the solution of two old problems: one of drawing a tangent line to a curve and the other of calculating the velocity of non-uniform velocity of a particle. These problems were solved in a certain sense by Sir Isaac Newton (English, 1642–1727) and G.W. Leibnitz (German, 1646–1716), and in the process, differential calculus was discovered.

It is applied to geometry, mechanics and other branches of theoretical physics and also to social sciences, such as economics and psychology.

The application of differential equation is essentially based on the notion of measurement. The real number is one of the main functions. In Mechanics, we concerned with the notion of time and, therefore, in the application of Calculus to Mechanics, the first step is to correlate the two notions of time and real numbers. Similar is the case with other notions such as amount of heat, intensity of light, force, etc. Thus it is clear that the knowledge of real numbers is important for the study of the subject. We arrive the set of real numbers from the set of rational numbers.

The set of integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is well known to us. It contains 0, natural numbers and their additive inverse. It is closed for addition, multiplication and subtraction. But the ratio of two integers is not an integer always.

If a and b be two integers and $b \neq 0$, a number expressed in the form of a/b is called a *rational number*. The set of rational numbers contains all integers and fraction. There are numbers $\sqrt{2}$, $\sqrt{3}$, ..., π, e, ..., etc., which cannot be expressed as a/b and they are called *irrational*. The set of real numbers contains both the rational and irrational numbers. The line of real numbers is called *dense*. The number of real numbers between any two different points, how close they may be, is infinite. The greatest number does not exist. However, the symbol ∞ stands for anything greater than which cannot be imagined; and we say that all the real numbers lie between $-\infty$ and ∞.

Some features of real number system R

Constant. A quantity whose value does not change in a problem, associated with a given mathematical operation, is called a *constant* quantity. For example, 7 is a constant, since in any situation the value of 7 will always be taken as 7, it cannot mean any other number. Similarly, 5, 3/5, π, $\sqrt{2}$, e, etc., are constants.

Variable. A quantity which assumes different numerical values in any problem, associated with a given mathematical operation, is called a *variable* quantity. For example, let A be any set and let $x \in A$; the symbol (or letter) x denotes any member of the set A, and is called a *value* of the variable. Generally, constants denoted by the first letters of the alphabet: $a, b, c, \ldots$ and variables are denoted by the last letters of the alphabet: $x, y, z, \ldots$

Independent variable. A quantity which assumes any arbitrary value is called an *independent* variable.

Dependent variable. A quantity whose value depends on the chosen values of another independent quantity is called a *dependent* variable.

Closed and open intervals. Let a, b be two given numbers such that $a < b$. Then the set of numbers x such that $a \leq x \leq b$ is called a *closed* interval denoted by the symbol $[a, b]$. We generally describe the situation as follows:

$$[a, b] = \{x: a \leq x \leq b\}$$

Thus

$$x \in [a, b] \Leftrightarrow a \leq x \leq b.$$

The set of numbers x such that $a < x < b$ is called an *open* interval denoted by the symbol $]a, b[$.

Intervals are called *semi-closed* or *semi-open*, that is $]a, b]$ or $[a, b[$, such that

$$]a, b] = \{x: a < x \leq b\} \quad \text{and} \quad [a, b[= \{x: a \leq x < b\}$$

The number $b - a$ is the length of each of the intervals $[a, b]$, $]a, b[$, $]a, b]$, $[a, b[$.

In differential calculus, the functional relation between two or more variables is studied in much detail with the help of limiting values of indeterminate forms,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for different functions. The technique developed is powerful and is applied for the study of several variables affecting each other simultaneously in different branches of natural and social sciences even in circumstances, when an algebraic method fails.

Important Formulae

1. Important limits and series

(i) $\lim_{x\to a}\dfrac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$, where x is a radian measure

(iii) $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$ or $\lim_{x\to 0}(1+x)^{1/x} = e$

(iv) $\lim_{x\to 0}\dfrac{e^x - 1}{x} = 1$

(v) $\lim_{x\to 0}\dfrac{(1+x)^n - 1}{x} = n$

(vi) $\lim_{n\to\infty} x^n = 0 \quad (-1 < x < 1)$

(vii) *L'Hospital rule:* If $\phi(a) = \psi(a) = 0$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)} = \text{a constant},$$

provided $\psi'(a) \neq 0$.

2. Standard formulae for derivatives

(i) $Dx^n = \dfrac{dx^n}{dx} = nx^{n-1}$

(ii) $\dfrac{da^x}{dx} = a^x \log a$

(iii) $\dfrac{de^{ax}}{dx} = ae^{ax}$

(iv) $\dfrac{d\log x}{dx} = \dfrac{1}{x}$

(v) $\dfrac{d\log_a x}{dx} = \dfrac{1}{x}\log_a e$

(vi) $\dfrac{d\sin x}{dx} = \cos x$

(vii) $\dfrac{d\cos x}{dx} = -\sin x$

(viii) $\dfrac{d\tan x}{dx} = \sec^2 x$

(ix) $\dfrac{d\cot x}{dx} = -\operatorname{cosec}^2 x$ (x) $\dfrac{d\sec x}{dx} = \sec x \tan x$

(xi) $\dfrac{d\operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cot x$ (xii) $\dfrac{d\sin^{-1} x}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

(xiii) $\dfrac{d\cos^{-1} x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$ (xiv) $\dfrac{d\tan^{-1} x}{dx} = \dfrac{1}{1+x^2}$

(xv) $\dfrac{d\cot^{-1} x}{dx} = -\dfrac{1}{1+x^2}$ (xvi) $\dfrac{d\sec^{-1} x}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$

(xvii) $\dfrac{d\operatorname{cosec}^{-1} x}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$ (xviii) $\dfrac{d\sinh x}{dx} = \cosh x$

(xix) $\dfrac{d\cosh x}{dx} = \sinh x$ (xx) $\dfrac{d\tanh x}{dx} = \operatorname{sech}^2 x$

(xxi) $\dfrac{d\coth x}{dx} = -\operatorname{cosech}^2 x$ (xxii) $\dfrac{d\operatorname{sech} x}{dx} = \operatorname{sech} x \tanh x$

(xxiii) $\dfrac{d\operatorname{cosech} x}{dx} = -\operatorname{cosech} x \coth x$

3. Fundamental theorems on differentiation

(i) $\dfrac{d(uv)}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$ (ii) $\dfrac{d(u/v)}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

(iii) Given any function $f(x, y) = 0$,

$$f_x\,dx + f_y\,dy = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

4. Meaning of differential coefficients

(i) *Geometrical*: For any curve $y = f(x)$,

$$\frac{dy}{dx} = f'(x) = \tan\psi,$$

where ψ is inclination of tangent to the curve of (x, y).

(ii) *Rate measure*: For any curve $y = f(x)$,

$$\frac{dy}{dx} = f'(x) = \text{Rate change of } y \text{ with respect to } x.$$

Then

$f'(x) > 0,$ if y and x increase or decrease together

$f'(x) < 0,$ if y decreases when x increases, or vice versa.

5. Successive differentiation (nth derivative, where m and n are positive integers)

(i) $D^n x^n = n!$ (ii) $D^n x^m = \dfrac{m!}{(m-n)!} x^{m-n},\ m > n$

(iii) $D^n(e^{ax}) = a^n e^{ax}$ (iv) $D^n a^x = (\log a)^n a^x$

(v) $D^n \sin(ax + b) = a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$

(vi) $D^n \cos(ax + b) = a^n \cos\left(\dfrac{n\pi}{2} + ax + b\right)$

(vii) $D^n\left(\dfrac{1}{x+a}\right) = \dfrac{(-1)^n n!}{(x+a)^{n+1}}$ (viii) $D^n \log(x + a) = \dfrac{(-1)^{n-1}(n-1)!}{(x+a)^n}$

(ix) $D^n\left(\dfrac{1}{x^2+a^2}\right) = \dfrac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta,\ \theta = \tan^{-1}\left(\dfrac{a}{x}\right)$

(x) Leibnitz's theorem: If u and v are function of x, then

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + u v_n$$

6. Rolle's and Lagrange's mean value theorems

(i) *Rolle's theorem:* If $f(x)$ is continuous in $a \le x \le b$, differentiable in $a < x < b$, and $f(a) = f(b)$, there exists at least one value of c, when $x = c$, such that $f'(c) = 0$, $a < c < b$.

(ii) *Lagrange's mean value theorem*: If $f(x)$ is continuous in $[a, b]$, differentiable in $]a, b[$ then

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad (a < c < b)$$

7. Expansion of functions

(i) *Taylor's infinite series:*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

(ii) *Maclaurin's infinite series:*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

(iii) *Finite series with Lagrange's form of remainder:*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + R_n,$$

where

$$R_n = \frac{h^n}{n!} f^n(x + \theta h), \quad 0 < \theta < 1$$

(iv) $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$

where

$$R_n = \frac{x^n}{n!}f^n(\theta x), \quad 0 < \theta < 1$$

8. Partial differentiation

(i) *Euler's theorem on homogeneous functions*:

(a) If $u(x, y)$ is homogeneous function of degrees n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

(b) If $u(x, y, z)$ be a homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

(ii) *Total and exact differentials:*

(a) If $u = u(x, y)$, then $du = f_x dx + f_y dy$

(b) If $u = u(x, y, z)$, then $du = f_x dx + f_y dy + f_z dz$

(c) If $f(x, y) = 0$, then

$$f_x dx + f_y dy = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}$$

(d) $\phi\, dx + \psi\, dy$ is called exact if

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

9. Tangent and normal

(i) Equation of a tangent at any point (x, y) on the curve:

(a) $y = f(x)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

(b) $f(x, y) = 0$ is

$$(X - x)f_x + (Y - y)f_y = 0$$

(ii) Equation of the normal at (x, y) on $y = f(x)$ or $f(x, y) = 0$ is

$$(X - x)\frac{dx}{dy} + (Y - y) = 0 \quad \text{or} \quad \frac{X - x}{f_x} = \frac{Y - y}{f_y}$$

(iii) *Tangent at the origin:* When a curve passes through the origin, for equation of tangent at (0, 0), equating to zero the terms of the lowest degree the equation of the curve gives the equation of the tangent at the origin.

(iv) *Length of the tangent and normal:*

(a) Cartesian form

$$\text{Tangent} = \frac{y}{y_1}\sqrt{1+y_1^2}, \qquad \text{Normal} = y\sqrt{1+y_1^2}$$

$$\text{Subtangent} = \frac{y}{y_1}, \qquad \text{Subnormal} = yy_1$$

(b) Polar form

$$\text{Subtangent} = r^2\frac{d\theta}{dr}, \qquad \text{Subnormal} = \frac{dr}{d\theta}$$

(v) Arc differential:

$$ds^2 = dx^2 + dy^2$$

$$\frac{dy}{dx} = \tan\psi, \qquad \frac{dy}{ds} = \sin\psi, \qquad \frac{dx}{ds} = \cos\psi$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \qquad \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

(vii) *Polar curves*:

$$\psi = \theta + \phi, \qquad p = r\sin\phi,$$

$$\tan\phi = r\frac{d\theta}{dr}, \qquad \sin\phi = r\frac{d\theta}{ds}, \qquad \cos\phi = \frac{dr}{ds},$$

$$ds^2 = dr^2 + r^2d\theta^2, \qquad \frac{ds}{dr} = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$$

10. Curvature

(i) For $s = f(\psi)$, $y = f(x)$, $x = f(y)$

$$\rho = \frac{ds}{d\psi}, \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \quad \rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

(ii) For $x = f(t)$, $y = F(t)$,

$$\rho = \frac{(f'^2 + F'^2)^{3/2}}{f'F'' - f''F'}$$

(iii) For $r = f(\theta)$, $p = f(r)$, $p = f(\psi)$. Therefore,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}, \quad \rho = r\frac{dr}{dp}, \quad \rho = p + \frac{d^2p}{d\psi^2}$$

(iv) Curvature of the origin (Newton's formulae): When x-axis ($y = 0$) and y-axis ($x = 0$) are tangents

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2}{2y}, \quad \rho = \lim_{\substack{x\to 0\\ y\to 0}}\frac{y^2}{2x}$$

(v) Chord of curvature

Through the pole = $2\rho \sin \phi$
Parallel to x-axis = $2\rho \sin \psi$
Parallel to y-axis = $2\rho \cos \psi$.

11. Asymptotes

(i) If $y = mx + c$ be asymptote to any curve $f(x, y) = 0$. Find m, c. Solving equations $\phi(x, mx + c) = 0$ for infinite roots.

(ii) *Asymptotes parallel to coordinate axes:* For asymptotes parallel to x-axis, equate to zero the coefficient of terms containing the highest power of x in equation $\phi(x, y) = 0$ of the curve, provided it be not constant. Similarly, for asymptotes parallel to y-axis, equate to zero the coefficient of term containing the highest power of y.

(iii) *Method using $\phi_n(m)$, $\phi_{n-1}(m)$:* Write

$$\phi(x, y) = \phi_n(x, y) + \phi_{n-1}(x, y) + \cdots = 0,$$

where $\phi_n(x, y)$ contain all terms of nth degree, and so on.
Put $x = 1$, $y = m$ to obtain $\phi_n(m)$, $\phi_{n-1}(m)$ and differentiate $\phi_n(m)$ with respect to m to find $\phi'_n(m)$. Now $\phi_n(m) = 0$ and $c = -\phi_{n-1}(m)/\phi'_n(m)$ for each m, give different values of m, c for the asymptotes $y = mx + c$.

(iv) If $\phi(x, y) = P_n + F_{n-2} = P_n + F_r$, $0 \le r \le n - 2$, put $P_n = 0$, and each linear non-repeated factor of P_n give one asymptote. In case of repeated factors, if

$$\phi(x, y) = (y - mx)^2 P_{n-2} + F_{n-2} = 0.$$

Then asymptotes are

$$(y - mx) \pm \lim_{\substack{x \to mx \\ x \to \infty}} \sqrt{-\frac{F_{n-2}}{P_{n-2}}} = 0.$$

12. Maxima and Minima

(i) Find $f'(x)$, $f''(x)$, and let $x = c$, satisfy $f'(x) = 0$, then $f(c)$ is a maximum value if $f''(c) < 0$ and $f(c)$ is a minimum value if $f''(c) > 0$.

(ii) Function of two variables $u = f(x, y)$: Solve

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

for stationary point $(x, y) = (a, b)$. Evaluate

$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2} \quad \text{at } (a, b)$$

If $rt - s^2 > 0$, $f(a, b)$ is an extreme value of $f(x, y)$ and it is maximum if $r < 0$, minimum if $r > 0$
If $rt - s^2 < 0$, $f(a, b)$ is not an extreme value
If $rt - s^2 = 0$, doubtful case, investigate further.

Chapter 1

Differential Coefficients

1.1 Introduction

If the value of a variable x changes (increase or decrease) from x_1 to x_2, then the quantity $x_2 - x_1$ is called *increment* in x. In both the cases, we use the term 'increment' to denote the change. A very small increament in x is generally denoted by δx (or Δx). We should remember that just as $\sin\theta$ is not the product of 'sin' and 'θ', similarly δx is not the product of δ and x, rather it is a symbol of infinitesimal increment.

If $y = f(x)$ is a function of x and if there is a change δx in x, there must be a change in the value of y. The corresponding change in y is denoted by δy. Thus we have $y + \delta y = f(x + \delta x)$. Therefore,

$$\delta y = f(x + \delta x) - f(x).$$

If $f(x)$ denotes a finite single-valued function of x defined in a given interval, and $f(x + h)$ denotes the same function of $(x + h)$, h being very small, then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is called the *differential coefficient* or *derivatives* of $f(x)$ with respect to x and is denoted by $f'(x)$. Thus

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

That is, if $y = f(x)$, then

$$\frac{dy}{dx} = f'(x)\,.$$

1.2 Differentiation from the First Principle

The process by which we get the differential coefficient or derivative of a function without application of any standard forms of derivatives or fundamental principles of limits is termed as differentiation from 'first principles' or 'definition' or '*ab initio*'.

Differential coefficient of any function

Let

$$y = f(x). \tag{1.1}$$

Let us also suppose that when the value of x changes from x to $x + \delta x$, the value of y becomes $y + \delta y$. Hence

$$y + \delta y = f(x + \delta x). \tag{1.2}$$

Subtracting Eq. (1.1) from Eq. (1.2), we have

$$\delta y = f(x + \delta x) - f(x).$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Now taking the limit as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Therefore, the differential coefficient of $f(x)$ is

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

The differential coefficient of a function is also called its derivatives.

Left-hand derivative and right-hand derivative

For any function $y = f(x)$, if at $x = x_0$ in its domain, the limit:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

exists, then we say that the function $f(x)$ is differentiable at $x = x_0$ or it possess a derivative. The quantity

$$\lim_{\delta x \to 0-} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

is called the *left-hand* derivative and is denoted by $\mathrm{L}f'(x)$, where the quantity

$$\lim_{\delta x \to 0+} \frac{f(x+\delta x) - f(x)}{\delta x}$$

is called the *right-hand* derivative and is denoted by $\mathrm{R}f'(x)$.

If at $x = x_0$ the left-hand derivative or the right-hand derivative is finite and equal, i.e. $\mathrm{L}f'(x) = \mathrm{R}f'(x)$, then we say that function $f(x)$ is differentiable at $x = x_0$ and differential coefficient of $y = f(x)$ is denoted by $f'(x)$ or by dy/dx.

Theorem 1.1 If $f(x)$ is derivable at $x = a$, then

$$Lf'(x) = Rf'(x) = f'(a) \text{ (say)}.$$

Proof Since $f(x)$ is derivable,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

or

$$f(a + h) - f(a) = h\,[f'(a)] + \varepsilon$$

where $\varepsilon \to 0$ as $h \to 0$. Then

$$\lim_{h \to 0} f(a+h) - f(a) = 0$$

or

$$\lim_{h \to 0} f(a+h) = f(a).$$

Hence $f(x)$ is continuous at $x = a$.

Note: The converse of the above theorem is not necessary true. We also note that every differentiable is continuous.

Example 1.1 Find the differential coefficient of

$$f(x) = \frac{x-1}{2x^2 - 7x + 5}, \qquad x \neq 1$$

$$= -\frac{1}{3}, \qquad x = 1, \text{ at } x = \phi.$$

Solution Here

$$\frac{x-1}{2x^2 - 7x + 5} = \frac{x-1}{(x-1)(2x-5)} = \frac{1}{2x-5}, \qquad \text{when } x \neq 1.$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(1+h)-5} - \frac{1}{-3}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{-3+2h} + \frac{1}{3}}{h}$$

$$= \lim_{h \to 0} \frac{2h}{3(-3+2h)h}$$

$$= \lim_{h \to 0} \frac{2}{3(-3+2h)}$$

$$= -\frac{2}{9}.$$

and

$$\text{LHD} = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2(1-h)-5} - \frac{1}{-3}}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{-3-2h} + \frac{1}{3}}{-h}$$

$$= \lim_{h\to 0} \frac{-2h}{(-h)(3)(-3-2h)}$$

$$= -\frac{2}{9}.$$

Therefore, LHD = RHD. Hence the given function is differentiable at $x = 1$.

Example 1.2 Discuss the function $f(x) = |x|$ and prove that the function is continuous at $x = 0$ but it is not differentiable at $x = 0$.

Solution Since

$$\begin{aligned} f(x) &= x, & x &> 0 \\ &= -x, & x &< 0 \\ &= 0, & x &= 0 \end{aligned}$$

For this $f(0) = 0$ and $f(0 + h) = f(h) = h$. Then

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} h = 0 \cdot$$

Again since $f(0 - h) = f(-h) = h$, we get

$$\lim_{h\to 0} f(0-h) = 0$$

Thus, we see that

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0)$$

Hence $f(x)$ is continuous at $x = 0$.

Now, for differentiability at $x = 0$.

$$\text{R}f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

But $h < 0$, then

$$\text{L}f'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{h} = \lim_{h\to 0} \frac{f(-h) - f(0)}{h} = \lim_{h\to 0} \frac{h}{-h} = -1\,.$$

Thus, we find LHD ≠ RHD at $x = 0$. Hence the given function is not differentiable at $x = 0$.

Example 1.3 Examine the derivability of the function:

$$f(x) = x\sin\frac{1}{x}, \qquad x \neq 0$$
$$= 0, \qquad x = 0$$

at $x = 0$.

Solution We have

$$f(0 + h) = h \sin \frac{1}{h}, \qquad \text{when } h > 0.$$

Then

$$\text{RHL} = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0}\sin\frac{1}{h}$$

So RHL does not exist. Similarly,

$$f(0-h) = (0-h)\sin\frac{1}{0-h} = -h\sin\frac{1}{-h} = h\sin\frac{1}{h}$$

We also get

$$\text{LHL} = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0}\frac{h\sin(1/h) - 0}{-h} = -\lim_{h\to 0}\sin\frac{1}{h}$$

does not exist. Thus the given function is not derivable at $x = 0$.

Example 1.4 If

$$f(x) = x^2\sin\frac{1}{x}, \qquad x \neq 0$$
$$= 0, \qquad x = 0,$$

Then find the differential coefficient of $f(x)$ at $x = 0$.

Solution By definition of differential coefficient of $f(x)$ at $x = 0$,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$
$$= \lim_{h\to 0}\frac{h^2\sin(1/h)}{h}$$
$$= \lim_{h\to 0} h\sin\frac{1}{h}$$
$$= 0, \quad \text{as } \left|\sin\frac{1}{h}\right| \leq 1.$$

Example 1.5 If

$$f(x) = 2 - x, \qquad x \leq 2$$
$$= x - 2, \qquad x > 2$$

then is $f(x)$ differentiable at $x = 2$?

Solution Given $f(2) = 2 - 2 = 0$. For left-hand derivative, $h < 0$. Then $2 + h < 2$. Therefore,

$$f(2 + h) = 2 - (2 + h) = -h.$$

Now

$$f'(2-0) = \lim_{h\to 0-0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1.$$

In case of right-hand derivative, $h > 0$ or $2 + h > 2$. Therefore, $f(2 + h) = 2 + h - 2 = h$. Now

$$f'(2-0) = \lim_{h\to 0+0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

Hence

$$f'(2-0) \neq f'(2+0)$$

Therefore, $f(x)$ is not differentiable at $x = 0$.

Example 1.6 If

$$f(x) = \frac{x}{1+e^{1/x}}, \qquad x \neq 0$$
$$= 0, \qquad x = 0$$

find RHD $f'(0+0)$ and LHD $f'(0-0)$.

Solution We have

$$f'(0-0) = \lim_{h\to 0-0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0-0} \frac{h}{(1+e^{1/h})h} - 0$$
$$= \lim_{h\to 0-0} \frac{1}{1+e^{1/h}}$$
$$= 1, \qquad (\text{as } h < 0)$$

and

$$f'(0+0) = \lim_{h\to 0+0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0+0} \frac{h}{(1+e^{1/h})/h}$$
$$= \lim_{h\to 0+0} \frac{1}{1+e^{1/h}}$$
$$= 0.$$

where $e^{1/h} \to \infty$ as $h \to 0 + 0$. Therefore, $f(0 - 0) \neq f(0 + 0)$. Hence $f(x)$ is not differentiable at $x = 0$. Since LHD and RHD of $f(x)$ at $x = 0$ are finite, $f(x)$ is continuous at $x = 0$.

Example 1.7 Examine the differentiability of $f(x)$ at $x = 1$ and $x = 2$ defined by

$$\begin{aligned} f(x) &= x[x], & 0 \le x < 2 \\ &= (x-1)x, & 2 \le x \le 3 \end{aligned}$$

Solution Derivative of $f(x)$ at $x = 1$ is

$$\begin{aligned} f'(1-0) &= \lim_{h\to 0-0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0-0} \frac{(1+h)[1+h]-[1]}{h} \\ &= \lim_{h\to 0-0} \frac{(1+h)\cdot 0 - 1}{h} \\ &= \lim_{h\to 0-0} \left(-\frac{1}{h}\right) \\ &= \infty \end{aligned}$$

and

$$\begin{aligned} f'(1+0) &= \lim_{h\to 0+0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)[1+h]-[1]}{h} \\ &= \lim_{h\to 0+0} \frac{(1+h)\cdot 1 - 1\cdot 1}{h} \\ &= 1 \end{aligned}$$

Therefore, $h > 0$ and $h \to 0$. Then $1 < 1 + h < 2$. Thus $f'(1-0) \ne f'(1+0)$. Therefore, $f(x)$ is not differentiable at $x = 1$.

Example 1.8 Examine the continuity and differentiability of the function f defined by $f(x) = x \tan^{-1}(1/x)$, $x \ne 0$ and $f(0) = 0$ at the origin.

Solution At the origin $x = 0$, we have

$$\begin{aligned} f'(0+0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h \tan^{-1}(1/h)}{h} \\ &= \lim_{h\to 0} \tan^{-1}\frac{1}{h} \\ &= \tan^{-1}\infty \\ &= \frac{\pi}{2} \end{aligned}$$

and

$$f'(0-0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{h}=\lim_{h\to 0}\frac{f(-h)}{-h}$$

$$=\lim_{h\to 0}\frac{-h\tan^{-1}(-1/h)}{-h}$$

$$=\lim_{h\to 0}\tan^{-1}\left(-\frac{1}{h}\right)$$

$$=\tan^{-1}(-\infty)=-\frac{\pi}{2}.$$

Therefore, $f'(0+0)\neq f'(0-0)$ and $f'(0)$ does not exist. Hence $f(x)$ is not differentiable at $x = 0$.

Example 1.9 The function $f(x) = x|x|$ is differentiable for all values of x.

Solution Here

$$f(x) = x|x| = x^2, \quad \text{if } x \geq 0$$
$$= -x^2, \quad \text{if } x < 0.$$

Since x^2 and $-x^2$ are different functions, $f(x)$ is differentiable except possibly at $x = 0$. Now,

$$f'(0+h)=\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0+}\frac{f(h)}{h}=\lim_{h\to 0+0}\frac{h^2}{h}=0.$$

and

$$f'(0-h)=\lim_{h\to 0-h}\frac{f(0-h)-f(0)}{h}=\lim_{h\to 0-h}\frac{f(h)}{h}=\lim_{h\to 0}\frac{-h^2}{h}=0.$$

Therefore, $f'(0+h)=f'(0-h)$. Hence $f(x)$ is differentiable at $x = 0$. Thus $f(x)$ is differentiable for all values of x.

Example 1.10 The function

$$f(x)=\frac{x}{1+|x|}$$

is differentiable at $x = 0$.

Solution Here

$$f(x)=\frac{x}{1+|x|}, \qquad x\geq 0$$

$$=\frac{x}{1-x}, \qquad x<0$$

Therefore, $x/(1+x)$ is differentiable for $x \geq 0$ and $x/(1-x)$ is also differentiable at $x < 0$.

Now to test the differentiability at $x = 0$.

$$f'(0+)=\lim_{h\to 0+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0+}\frac{h}{(1+h)h}-0=\lim_{x\to 0+}\frac{1}{1+h}=1$$

$$f'(0-)=\lim_{h\to 0-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0-}\frac{h}{(1-h)h}-0=\lim_{h\to 0-}\frac{1}{1-h}=1$$

Since $f'(0+)=f'(0-)$, $f(x)$ is differentiable at $x = 0$. Hence $f(x)$ is differentiable everywhere.

Exercises 1.1

1. If

$$f(x) = \frac{1}{x}\sin x^2, \qquad x \neq 0$$
$$= 0, \qquad x = 0.$$

discuss the continuity and differentiability of $f(x)$ at $x = 0$.

2. Test the continuity and differentiability of the function:

$$f(x) = 1 + x, \qquad x < 2$$
$$= 5 - x, \qquad x > 2.$$

3. Does the differential coefficient of the following function exist at $x = 0$ and $x = 1$, where

$$f(x) = -x, \qquad x < 0$$
$$= x^2, \qquad 0 \le x < 1$$
$$= x^2 - x + 1, \qquad x > 1?$$

4. Discuss the continuity and differentiability of

$$f(x) = x^2, \qquad x < -2$$
$$= 4, \qquad -2 \le x \le 2$$
$$= x^3, \qquad x > 2.$$

5. Examine the continuity and differentiability of $f(x)$ at $x = 0$ if

$$f(x) = \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, \qquad x \neq 0$$
$$= 0, \qquad x = 0.$$

6. Discuss the differentiability of a function f defined as

$$f(x) = 1, \qquad -\infty < x < 0$$
$$= 1 + \sin x, \qquad 0 \le x < \tfrac{\pi}{2}$$
$$= 2 + \left(x - \tfrac{\pi}{2}\right)^2, \qquad \tfrac{\pi}{2} \le x < \infty.$$

7. Discuss the differentiability of a function $f(x)$ defined as

$$f(x) = 2x, \qquad x \ge 1$$
$$= 1 + x^2, \qquad x < 1$$

find $f'(1)$.

8. Given

$$\begin{aligned} f(x) &= x, & 0 \le x < 1 \\ &= 2 - x, & 1 \le x \le 2 \\ &= x - \tfrac{1}{2}x^2, & x \ge 2. \end{aligned}$$

Is $f(x)$ continuous at $x = 1$ and $x = 2$? Does $f'(x)$ exists at these points?

9. Given

$$\begin{aligned} y &= \frac{\sin x}{x}, & x > 0 \\ &= 1 - x\cos x, & x \le 0 \end{aligned}$$

prove that at $x = 0$, the function is continuous but not differentiable.

10. Given $f(x) = \sin|x|$, show that $f'(0)$ exists and $f(x)$ is continuous at $x = 0$.

11. If $f(a) = 2$, $f'(a) = -1$, $g(a) = -1$, $g'(a) = 2$, then what is the value of

$$f(x) = \lim_{x\to 0} \frac{g(x)f(a) - g(a)\,f(x)}{x - a}?$$

Example 1.11 Differentiate e^x from the first principle.

Solution Let

$$f(x) = e^x \tag{1}$$

and

$$f(x + h) = e^{x+h} \tag{2}$$

Then

$$f(x+h) - f(x) = e^{x+h} - e^x = e^x e^h - e^x = e^x(e^h - 1)$$

or

$$\frac{f(x+h) - f(x)}{h} = \frac{e^x(e^h - 1)}{h} = e^x \frac{1}{h}\left[\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right) - 1\right]$$

$$= e^x \frac{1}{h}\left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)$$

$$= e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right).$$

Taking limit, we get

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right)$$

or

$$f'(x) = e^x.$$

Example 1.12 Differentiate a^x from the first principle.

Solution Let

$$f(x) = a^x = e^{x\log a} \quad (1)$$

and

$$f(x+h) = a^{x+h} = e^{(x+h)\log a} \quad (2)$$

Subtracting (1) from (2), we get,

$$f(x+h) - f(x) = e^{(x+h)\log a} - e^{x\log a}$$
$$= e^{x\log a}e^{h\log a} - e^{x\log a}$$
$$= e^{x\log a}\left(e^{h\log a} - 1\right)$$

or

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x\log a}\left(e^{h\log a} - 1\right)}{h}$$
$$= e^{x\log a}\frac{1}{h}\left[\left(1 + h\log a + \frac{h^2}{2!}(\log a)^2 + \cdots\right) - 1\right]$$
$$= e^{x\log a}\frac{1}{h}\left[h\log a + \frac{h^2}{2!}(\log a)^2 + \cdots\right]$$
$$= e^{x\log a}\left[\log a + \frac{h}{2!}(\log a)^2 + \cdots\right]$$

Taking limit both sides, we obtain

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} e^{x\log a}\left[\log a + \frac{h}{2!}(\log a)^2 + \cdots\right] = e^{x\log a}\log a$$

Therefore, $f'(x) = a^x \log_e a$.

Example 1.13 Differentiate $\log_a x$ from the first principle.

Solution Let $f(x) = \log_a x = \log_e x \log_a e$. So,

$$f(x+h) = \log_a(x+h) = \log_e(x+h)\log_a e$$

Then

$$f(x+h) - f(x) = \left(\log_e(x+h) - \log_e x\right)\log_a e$$
$$= \log_e\left(\frac{x+h}{x}\right)\log_a e$$
$$= \log_a e \log_e\left(1 + \frac{h}{x}\right)$$
$$= \log_a e\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots\right)$$

Taking limit, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\log_a e\lim_{h\to 0}\left(\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}-\cdots\right)=(\log_a e)\frac{1}{x}$$

Therefore,

$$f'(x)=\frac{1}{x}\log_a e.$$

Example 1.14 Differentiate $\log_e x$ from the first principle.

Solution Let $f(x) = \log_e x$, and $f(x + h) = \log_e (x + h)$. Then

$$\begin{aligned}f(x+h)-f(x)&=\log_e(x+h)-\log_e x\\&=\log_e\left(\frac{x+h}{x}\right)\\&=\log_e\left(1+\frac{h}{x}\right)\\&=\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\cdots\end{aligned}$$

or

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}&=\lim_{h\to 0}\frac{h\left(\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}-\cdots\right)}{h}\\&=\lim_{h\to 0}\left(\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}-\cdots\right)\\&=\frac{1}{x}.\end{aligned}$$

Therefore,

$$f'(x)=\frac{1}{x}.$$

Example 1.15 Differentiate $\sin^{-1} x$ from the first principle.

Solution Let $f(x) = \sin^{-1}x$ and $f(x + h) = \sin^{-1}(x + h)$. Then

$$\frac{f(x+h)-f(x)}{h}=\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

Put $\sin^{-1}x = \alpha$ or $x = \sin\alpha$, and $\sin^{-1}(x + h) = \alpha + k$ or $x + h = \sin(\alpha + k)$. When $h \to 0$, $k \to 0$, and $h = \sin(\alpha + k) - \sin\alpha$, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

$$=\lim_{k\to 0}\frac{\alpha+k-\alpha}{\sin(\alpha+k)-\sin\alpha}$$

$$=\lim_{k\to 0}\frac{k}{2\cos(\alpha+k/2)}\sin\frac{k}{2}$$

$$=\lim_{k\to 0}\left[\frac{k/2}{\sin(k/2)}\right]\lim_{k\to 0}\frac{1}{\cos(\alpha+k/2)}$$

$$=\frac{1}{\cos\alpha}$$

$$=\frac{1}{\sqrt{1-\sin^2\alpha}}$$

$$=\frac{1}{\sqrt{1-x^2}}.$$

Therefore,

$$f'(x)=\frac{1}{\sqrt{1-x^2}}.$$

Example 1.16 Find the differential coefficient of $\cos^{-1}x$ with respect to x from ab initio.

Solution Let $f(x) = \cos^{-1}x$ and $f(x + h) = \cos^{-1}(x + h)$. Then

$$f(x+h)-f(x)=\cos^{-1}(x+h)-\cos^{-1}x.$$

Substituting $\cos^{-1}x = \alpha$ or $x = \cos\alpha$ and $\cos^{-1}(x + h) = \alpha + k$ or $x + h = \cos(\alpha + k)$, and when $h \to 0$, so $k \to 0$ and $h = \cos(\alpha + k) - \cos\alpha$, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos^{-1}(x+h)-\cos^{-1}x}{h}$$

$$=\lim_{k\to 0}\frac{\alpha-k-\alpha}{\cos(\alpha+k)-\cos\alpha}$$

$$=\lim_{k\to 0}\frac{k}{2\sin\frac{\alpha+k+\alpha}{2}\sin\frac{\alpha-\alpha-k}{2}}$$

$$= \lim_{k\to 0} \frac{k}{2\sin\left(\alpha+\frac{k}{2}\right)\sin\left(-\frac{k}{2}\right)}$$

$$= \lim_{k\to 0} \frac{-k}{2\sin\left(\alpha+\frac{k}{2}\right)\frac{\sin(k/2)}{k/2}}\frac{2}{k}$$

$$= \lim_{k\to 0} -\frac{1}{\sin[\alpha+(k/2)]} \lim_{k\to 0}\left[\frac{\sin(k/2)}{k/2}\right]$$

$$= -\frac{1}{\sin\alpha}$$

$$= -\frac{1}{\sqrt{\cos^2\alpha}}$$

$$= -\frac{1}{\sqrt{1-x^2}}.$$

Therefore,

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Example 1.17 Find from the first principles the differential coefficient of $\tan^{-1}x$ with respect to x.

Solution Let $f(x) = \tan^{-1}x$. Then $f(x+h) = \tan^{-1}(x+h)$. So

$$f(x+h) - f(x) = \tan^{-1}(x+h) - \tan^{-1}x$$

Put $\tan^{-1}x = \alpha$, $x = \tan\alpha$, and $\tan^{-1}(x+h) = \alpha + k$ so $\tan \alpha + k = x + h$. When $h\to 0$, $k\to 0$ and $h = \tan(\alpha+k) - \tan\alpha$. Therefore,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}$$

$$= \lim_{k\to 0}\frac{\alpha+k-\alpha}{\tan(\alpha+k)-\tan\alpha}$$

$$= \lim_{k\to 0}\frac{k}{\frac{\sin(\alpha+k)}{\cos(\alpha+k)}-\frac{\sin\alpha}{\cos\alpha}}$$

$$= \lim_{k\to 0} \frac{k\cos\alpha\cos(\alpha+k)}{\sin(\alpha+k)\cos\alpha - \cos(\alpha+k)\sin\alpha}$$

$$= \lim_{k\to 0} \frac{k\cos\alpha\cos(\alpha+k)}{\sin(\alpha+k-\alpha)}$$

$$= \lim_{h\to 0} \frac{k\cos\alpha\cos(\alpha+k)}{\sin k}$$

$$= \lim_{k\to 0} \frac{k}{\sin k}\cos\alpha \lim_{k\to 0}\cos(\alpha+k)$$

$$= \cos^2\alpha$$

$$= \frac{1}{\sec^2\alpha}$$

$$= \frac{1}{1+\tan^2\alpha}$$

$$= \frac{1}{1+x^2}.$$

Therefore,

$$f'(x) = \frac{1}{1+x^2}.$$

Note: Try yourselves the differential coefficients of $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ and $\cot^{-1}x$, from the first principle.

Example 1.18 Differentiate $\sqrt{\sin x}$ from the first principle.

Solution Let $f(x) = \sqrt{\sin x}$, so $f(x+h) = \sqrt{\sin(x+h)}$. Now,

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h\left[\sqrt{\sin(x+h)}+\sqrt{\sin x}\right]}$$

$$= \lim_{h\to 0}\frac{2\cos(x+h/2)\sin(h/2)}{h\left[\sqrt{\sin(x+h)}+\sqrt{\sin x}\right]}$$

$$= \lim_{h\to 0}\frac{\cos(x+h/2)}{\sqrt{\sin(x+h)}+\sqrt{\sin x}}\lim_{h\to 0}\frac{\sin(h/2)}{h/2}.$$

Therefore,

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}.$$

Example 1.19 Differentiate cos (log x) from the first principle.

Solution Let $f(x) = \cos(\log x)$ and $f(x+h) = \cos[\log(x+h)]$. Therefore,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos[\log(x+h)] - \cos(\log x)}{h}$$

Put $\log x = \alpha$ and $\log(x+h) = \alpha + k$. When $h \to 0$, $k \to 0$, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{k\to 0} \frac{\cos(\alpha + k) - \cos\alpha}{h}$$

$$= \lim_{k\to 0} \frac{-2\sin(\alpha + k/2)\sin(k/2)}{h}$$

$$= -\lim_{k\to 0} \sin\left(\alpha + \frac{k}{2}\right) \lim_{k\to 0} \left[\frac{\sin(k/2)}{k/2}\right]\frac{k}{h}$$

$$= -\sin\alpha \lim_{k\to 0} \frac{k}{h}$$

Now, since

$$k = (\alpha + k) - \alpha$$

$$= \log(x+h) - \log x$$

$$= \log \frac{x+h}{x}$$

$$= \log\left(1 + \frac{h}{x}\right)$$

$$= \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots,$$

We get

$$\lim_{k\to 0} \frac{k}{h} = \lim_{h\to 0} \left(\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \cdots\right) = \frac{1}{x}.$$

Therefore,

$$f'(x) = -\frac{1}{x}\sin\alpha = -\frac{1}{x}\sin(\log x).$$

Example 1.20 Differentiate cosh x from the first principle.

Solution Let $f(x) = \cosh x = (e^x + e^{-x})/2$. Then $f(x+h) = [e^{x+h} + e^{-(x+h)}]/2$.

Now,

$$\frac{d\cosh x}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2}[e^{x+h}+e^{-(x+h)}]-\frac{1}{2}(e^{x}+e^{-x})}{h}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{e^{x+h}-e^{x}}{h} - \frac{1}{2}e^{-x}\lim_{h\to 0}\frac{e^{-h}-1}{-h}$$

$$= \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} \qquad \left(\text{as } \lim_{h\to 0}\frac{e^{h}-1}{h}=1\right)$$

$$= \frac{1}{2}(e^{x}-e^{-x})$$

$$= \sinh x.$$

Therefore, $f'(x) = \sinh x$.

Example 1.21 Find the differential coefficient of $\sinh^{-1}x$ from the first principle.

Solution Let $y = \sinh^{-1}x$, so

$$x = \sinh y = \frac{1}{2}\left(e^{y}-e^{-y}\right) \tag{1}$$

and

$$x+\delta x = \sinh(y+\delta y) = \frac{1}{2}\left[e^{y+\delta y}-e^{-(y+\delta y)}\right] \tag{2}$$

Subtracting (1) from (2), we get

$$\lim_{\delta y\to 0}\frac{\delta x}{\delta y} = \frac{1}{2}\lim_{\delta y\to 0}\frac{\left[e^{y+\delta y}-e^{-(y+\delta y)}\right]-\left(e^{y}-e^{-y}\right)}{\delta y}$$

or

$$\frac{dx}{dy} = \frac{1}{2}\lim_{\delta y\to 0}\frac{e^{y+\delta y}-e^{y}}{\delta y} - \frac{1}{2}\lim_{\delta y\to 0}\frac{e^{-(y+\delta y)}-e^{-y}}{\delta y}$$

$$= \frac{1}{2}e^{y}\lim_{\delta y\to 0}\frac{e^{\delta y}-1}{\delta y} + \frac{1}{2}e^{-y}\lim_{\delta y\to 0}\frac{e^{-\delta y}-1}{-\delta y}$$

$$= \frac{1}{2}(e^{y}+e^{-y}) \qquad \left(\text{as } \lim_{h\to 0}\frac{e^{h}-1}{h}=1\right)$$

$$= \cosh y.$$

Thus

$$\frac{dy}{dx}=\frac{1}{\cosh y}=\frac{1}{\sqrt{\cosh^2 y}}=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

Therefore,

$$\frac{dy}{dx}=\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}}.$$

Note: Similarly the differential coefficients of other inverse hyperbolic functions can be derived from the first principle.

Example 1.22 Differentiate $e^{\cosh 2x}$ with respect to x, from the first principle.

Solution Let $y = e^{\cosh 2x}$ and $y + \delta y = e^{\cosh 2(x+\delta x)}$. Then

$$\delta y = e^{\cosh 2(x+\delta x)} - e^{\cosh 2x}.$$

Therefore,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{e^{\cosh 2(x+\delta x)}-e^{\cosh 2x}}{\delta x}$$

$$=e^{\cosh 2x}\lim_{\delta x\to 0}\frac{e^{\cosh 2(x+\delta x)-\cosh 2x}-1}{\delta x}$$

$$=e^{\cosh 2x}\left[\lim_{\delta x\to 0}\frac{e^{\cosh 2(x+\delta x)-\cosh 2x}-1}{\cosh 2(x+\delta x)-\cosh 2x}\right]\frac{\cosh 2(x+\delta x)-\cosh 2x}{\delta x}$$

$$=e^{\cosh 2x}\left(\lim_{\delta k\to 0}\frac{e^k-1}{k}\right)\lim_{\delta x\to 0}\frac{\cosh 2(x+\delta x)-\cosh 2x}{\delta x}$$

where $\cosh 2x = t$, $\cosh 2(x + \delta x) = t + k$. When $t \to 0$, $\delta x \to 0$, we get

$$\frac{dy}{dx}=2(\sinh 2x)\,e^{\cosh 2x}$$

Example 1.23 Differentiating $\sin x^2$ from the first principle.

Solution Let $y = \sin x^2$, $y + \delta y = \sin (x + \delta x)^2$. Then

$$\delta y = \sin (x+\delta x)^2 - \sin x^2$$

$$=2\cos\left[\frac{(x+\delta x)^2+x^2}{2}\right]\sin\left[\frac{(x+\delta x)^2-x^2}{2}\right]$$

$$=2\cos\left[\frac{2x^2+2x\delta x+(\delta x)^2}{2}\right]\sin\left[\frac{2x\,\delta x+(\delta x)^2}{2}\right]$$

$$=2\cos\left[x^2+x\delta x+\frac{(\delta x)^2}{2}\right]\sin\left[\frac{\delta x\,(2x+\delta x)}{2}\right]$$

or

$$\frac{\delta y}{\delta x} = 2\cos\left[x^2 + x\delta x + \frac{(\delta x)^2}{2}\right]\frac{\sin\left[\delta x(x + \delta x/2)\right]}{\delta x}$$

or

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2\cos\left[x^2 + x\delta x + \frac{(\delta x)^2}{2}\right]\lim_{\delta x \to 0}\frac{\sin\left[\delta x\,(x + \delta x/2)\right]}{\delta x\,(x + \delta x/2)}\left(x + \frac{\delta x}{2}\right)$$

Then

$$\frac{dy}{dx} = 2x\cos x^2.$$

Example 1.24 Differentiate $\sin\sqrt{x}$ from the first principle.

Solution Let $y = f(x) = \sin\sqrt{x}$. When $\sqrt{x} = u$. Then

$$f(x) = \sin u \quad \text{and} \quad f(x + \delta x) = \sin(u + \delta u).$$

Also

$$u + \delta u = \sqrt{x + \delta x}, \quad \text{when } \delta x \to 0, \quad \delta u \to 0$$

Therefore,

$$\delta y = y + \delta y - y = f(x + \delta x) - f(x) = \sin(u + \delta u) - \sin u$$

Then

$$\frac{\delta y}{\delta x} = \frac{\sin(u + \delta u) - \sin u}{\delta x}$$

$$= \frac{\sin(u + \delta u) - \sin u}{\delta u}\frac{\delta u}{\delta x}$$

$$= \cos u \lim_{\delta x \to 0}\frac{\delta u}{\delta x}$$

But we have $\delta u = \sqrt{x + \delta x} - \sqrt{x}$. Then

$$\delta u = \left(\sqrt{x + \delta x} - \sqrt{x}\right)\frac{\sqrt{x + \delta x} + \sqrt{x}}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$= \frac{x + \delta x - x}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$= \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

or

$$\frac{\delta u}{\delta x} = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

When $\delta x \to 0$

$$\frac{du}{dx} = \lim_{\delta x \to 0} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Thus

$$\frac{dy}{dx} = \cos u \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

Example 1.25 Differentiate sec (x/3) from the definition with respect to x.

Solution Let

$$y = \sec\frac{x}{3}, \qquad y + \delta y = \sec\frac{x+\delta x}{3}$$

Then

$$\delta y = \sec\frac{x+\delta x}{3} - \sec\frac{x}{3} = \frac{1}{\cos\frac{x+\delta x}{3}} - \frac{1}{\cos\frac{x}{3}}$$

$$= \frac{\cos\frac{x}{3} - \cos\frac{x+\delta x}{3}}{\cos\left(x + \frac{\delta x}{3}\right)\cos\frac{x}{3}}$$

$$= \frac{2\sin\frac{x+x+\delta x}{6}\sin\frac{x-x-\delta x}{6}}{\cos\frac{x+\delta x}{3}\cos\frac{x}{3}}$$

or

$$\frac{\delta y}{\delta x} = \frac{2\sin\left(\frac{x}{3} + \frac{\delta x}{6}\right)\sin\left(-\frac{\delta x}{6}\right)}{\delta x\cos\left(\frac{x+\delta x}{3}\right)\cos\frac{x}{3}}$$

Taking limit, we get

$$\frac{dy}{dx} = \frac{\lim\limits_{\delta x \to 0} 2\sin\left(\frac{x}{3} + \frac{\delta x}{6}\right)\left[\frac{\sin(\delta x/6)}{\delta x/6}\right]\frac{1}{6}}{\lim\limits_{\delta x \to 0}\cos\left(\frac{x+\delta x}{3}\right)\cos\frac{x}{3}}$$

or

$$\frac{dy}{dx} = \frac{2\sin(x/3)(1)(1/6)}{\cos(x/3)\cos(x/3)} = \frac{1}{3}\tan\frac{x}{3}\sec\frac{x}{3}.$$

Exercises 1.2

1. Obtain the differential coefficients of the following functions from the first principle:

(i) $5x^4$, (ii) $x^{1/2}$, (iii) $x^2 + 5x^3$,

(iv) $\sqrt[3]{x}$, (v) $x^{3/2}$, (vi) $x^{-3/4}$,

(vii) $\sqrt{1/(x+a)}$, (viii) $\sqrt{x^2+1}$, (ix) $(x^2+1)/x$,

(x) $\sqrt{2x-1}$, (xi) $x^8 + x^3$.

2. Find the differential coefficients of the following with respect to x from the definition:

(i) $\sin 4x$, (ii) $\cos 2x$, (iii) $\tan 2x$,

(iv) $\tan kx$, (v) $\tan(x°/3)$, (vi) $\operatorname{cosec} 3x$,

(vii) $\cot 2x$, (viii) $\tan(3x + 1)$, (ix) $\sin(x^2 + 1)$,

(x) $\sqrt{\tan x}$, (xi) $\sin^3 x$, (xii) $\cos x^2$,

(xiii) $\cos^2 x$, (xiv) $\tan(x + a)$, (xv) $\sec(x°/3)$,

(xvi) $\tan^2(ax)$.

3. Differentiate the following functions from ab initio.

(i) $x^2 \cos x$, (ii) $x \sin x$, (iii) $\cos(ax^2 + bx + x)$,

(iv) $x \tan x$, (v) $\sin x \cos x$, (vi) $(\sin x)/x$,

(vii) $\tan(1/x)$.

4. Find the differential coefficients from the first principle of the following functions:

(i) $\tan 3\sqrt{x}$, (ii) $\tan(1 - 3x)$, (iii) $\sec 5x$,

(iv) $\sqrt{\cos x}$, (v) $\sqrt{\sec x}$, (vi) $\sqrt{\cos 3x}$,

(vii) $\sin\sqrt{x}$, (viii) $\cot\sqrt{x}$.

5. Obtain dy/dx of the following functions from the definition:

(i) e^{nx}, (ii) e^{2x+3}, (iii) e^{x^2},

(iv) $e^{\cos x}$, (v) $e^{\tan x}$, (vi) e^x/x,

(vii) $e^{\sqrt{5x}}$, (viii) e^{-3x}.

6. Find the differential coefficients of the following functions from the first principle:

(i) $\log_e(ax + b)$, (ii) $\log \sec x$, (iii) $x \log x$,

(iv) $\log ax$, (v) $\log(\tan x)$, (vi) $\sin(\log x)$,

(vii) $x \log(\sin x)$, (viii) $\log \sin(x/a)$, (ix) $\log \cos x$,

(x) 4^x.

7. Find dy/dx of the following functions from the definition with respect to x:

(i) $\tan^{-1}(x/a)$, (ii) $\sin^{-1}(x/a)$, (iii) $\sec^{-1}x$,
(iv) $\tan^{-1}(2x-3)$, (v) $\sin^{-1}(3x+5)$, (vi) $\tan^{-1}(ax)$,
(vii) $\sin^{-1}(2x)$, (viii) $\sec^{-1}(3x)$, (ix) $\cot^{-1}(mx)$.

1.3 Fundamental Rules for Differentiation

Example 1.26 Find the differential coefficients of

$$y=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}.$$

Solution Since

$$y=\frac{(1-\cos x)+\sin x}{(1+\cos x)+\sin x}$$

$$=\frac{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{2\cos\frac{x}{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$=\tan\frac{x}{2}.$$

Therefore,

$$\frac{dy}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2\cos^2(x/2)}=\frac{1}{1+\cos x}.$$

Example 1.27 Find dy/dx, if

$$y=\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}=\sqrt{\frac{1+\sin x}{1-\sin x}}$$

Solution

Method 1.

Since

$$y=\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}=\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}\,\frac{\sec x+\tan x}{\sec x+\tan x}}$$

$$=\frac{\sec x+\tan x}{\sqrt{\sec^2 x-\tan^2 x}}$$

$$=\sec x+\tan x$$

Then

$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x.$$

Method 2.

We have

$$y = \sqrt{\frac{1+\sin x}{1-\sin x}} = \sqrt{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$$

$$= \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}$$

$$= \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$= \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Then

$$\frac{dy}{dx} = \frac{1}{2}\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Example 1.28 Find dy/dx, if

(i) $y = \dfrac{x \cos x}{x^2+4}$, (ii) $y = \dfrac{x \tan x}{\sec x + \tan x}$.

Solution (i) Differentiating, we get

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}(x\cos x)\right](x^2+4) - \left[\frac{d}{dx}(x^2+4)\right]x\cos x}{(x^2+4)^2}$$

$$= \frac{\left[\frac{dx}{dx}\cos x + x\frac{d}{dx}(\cos x)\right](x^2+4) - (2x)\,x\cos x}{(x^2+4)^2}$$

$$= \frac{[\cos x + x(-\sin x)](x^2+4) - 2x^2\cos x}{(x^2+4)^2}$$

$$= \frac{(4-x^2)\cos x - x(x^2+4)\sin x}{(x^2+4)^2}$$

(ii) Differentiating, we obtain

$$\frac{dy}{dx}=\frac{(\sec x+\tan x)\frac{d}{dx}(x\tan x)-x\tan x\frac{d}{dx}(\sec x+\tan x)}{(\sec x+\tan x)^2}$$

$$=\frac{(\sec x+\tan x)\left[x\frac{d}{dx}(\tan x)+\tan x\frac{dx}{dx}\right]-x\tan x\left[\frac{d}{dx}(\sec x+\tan x)\right]}{(\sec x+\tan x)^2}$$

$$=\frac{(\sec x+\tan x)(x\sec^2 x+\tan x)-x\tan x(\sec x\tan x+\sec^2 x)}{(\sec x+\tan x)^2}$$

$$=\frac{x\sec^3 x+x\tan x\sec^2 x+\sec x\tan x+\tan^2 x-x\sec x\tan^2 x-x\tan x\sec^2 x}{(\sec x+\tan x)^2}$$

$$=\frac{x\sec^3 x-x\sec x\tan^2 x+\sec x\tan x+\tan^2 x}{(\sec x+\tan x)^2}$$

$$=\frac{x\sec x(\sec^2 x-\tan^2 x)+\tan x(\sec x+\tan x)}{(\sec x+\tan x)^2}.$$

Therefore,

$$\frac{dy}{dx}=\frac{x\sec x(\sec x-\tan x)+\tan x}{\sec x+\tan x}$$

Example 1.29 Find *dy*/*dx*, if

$$y=\frac{ax^2+b}{\sin x+\cos x}.$$

Solution On differentiating, we have

$$\frac{dy}{dx}=\frac{\left[\frac{d}{dx}(ax^2+b)\right](\sin x+\cos x)-\left[\frac{d}{dx}(\sin x+\cos x)\right](ax^2+b)}{(\sin x+\cos x)^2}$$

$$=\frac{2ax(\sin x+\cos x)-(ax^2+b)(\cos x-\sin x)}{(\sin x+\cos x)^2}.$$

Example 1.30 Find $f'(\pi/2)$, if

$$f(x)=\frac{1-\cos x}{1+\cos x}$$

Solution Now

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x\cos x+\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

Then

$$f'\left(\frac{\pi}{2}\right)=\frac{2\sin(\pi/2)}{[1+\cos(\pi/2)]^2}=2$$

Exercises 1.3

1. Write down the differential coefficients of the following functions with respect to x:

   (i) $x + 1/x$, (ii) $2\sin x - 5\cos x$, (iii) $-ke^x + 9\cos x$,

   (iv) $1/\sqrt{x^3}$, (v) $\frac{2}{3}x^3+\frac{5}{4}x^2-2x+\frac{1}{5}$, (vi) $\left(\sqrt{x}+1/\sqrt{x}\right)^2$,

   (vii) $x^3\cos x$ (viii) $(x^2+1)(x^3+2x+5)$, (ix) $\frac{5x^4+6x^2-x+1}{x}$,

   (x) $(x^2+1)\sin x$, (xi) $\frac{\sin x}{x}$, (xii) $\frac{e^{\tan x}}{\tan x}$,

   (xiii) $x^4\log x$, (xiv) $\frac{\log x}{x}$, (xv) $\frac{x^3-a^3}{x+a}$,

   (xvi) $\sqrt{x}\left(1+1/\sqrt{x}\right)$, (xvii) $\frac{xe^x-1}{x}$, (xviii) $\frac{a^2-x^2+4x\log x}{x}$.

2. Find dy/dx when each of the following functions is equal y:

   (i) $\frac{x^2}{\sin x}$, (ii) $\frac{\cot x}{x}$, (iii) $\frac{x^n}{\log x}$,

   (iv) $\frac{1}{e^x\tan x}$, (v) $\frac{1+\cos x}{1-\cos x}$, (vi) $\frac{1+\tan x}{1-\tan x}$,

(vii) $\frac{e^x}{1+x^2}$, (viii) $\frac{x}{x^2-1}$, (ix) $\frac{\sin x+\cos x}{\sin x-\cos x}$,

(x) $\frac{x\sin x}{\sin x+\cos x}$, (xi) $\frac{x^2+\sec x}{1+\tan x}$, (xii) $\frac{x^2\sin x}{1+\tan x}$,

(xiii) $\frac{e^x+\sin x}{1+\log x}$, (xiv) $\frac{(x-2)(x^2+2)}{x+2}$, (xv) $\frac{x^2+\sec x}{1+\tan x}$,

(xvi) $\frac{x^2+\operatorname{cosec} x}{1+\cot x}$, (xvii) $\frac{e^x\sec x-\tan x}{1+\tan x}$, (xviii) $\sqrt{\frac{\sec x-1}{\sec x+1}}$,

(xix) $\sqrt{\frac{1-\sin x}{1+\sin x}}$, (xx) $\frac{1}{x^4\sec x}$.

3. Solve:

(i) If $f(x) = \frac{x-4}{2\sqrt{x}}$, then find $f'(x)$ at $x = 4$,

(ii) If $f(x) = \frac{\sin x}{1+x^2} - \sqrt{x}e^x + 5$, find $f'(x)$,

(iii) If $y=\frac{1}{1+x^2}+e^x\sec x$, find dy/dx,

(iv) If $f(x) = \sin x$, find $f'(0)$,

(v) If $f(x) = \frac{x^3}{a^2-x^2}$, find $f'(a/2)$,

4. For what value of x, the differential coefficient of $(x^2 + 1)/(x - 2)$ is zero?

5. For what value of x, the differential coefficient of $x \log x$ is zero?

6. Find the value of differential coefficients of $x^3 - 4x^2 + 3x - 2$, when $x = 2$.

7. If $y = |x|^2 - 4|x| + 2$, find dy/dx at $x = 3$.

8. If $y = |\cos x|$, find dy/dx at $x = 3\pi/4$.

9. If $y = |\cos x - \sin x|$, find $f'(\pi/2)$.

10. Find dy/dx, when

(i) $y = \frac{x^4\sqrt{1+\tan x}}{\cos^2 x}$, (ii) $y=\frac{x^3\sqrt{x^2+4}}{\sqrt{x^2+3}}$.

11. Find the differential coefficient of the following functions:

(i) $\frac{\tan x+\cot x}{\tan x-\cot x}$, (ii) $\frac{\tan x}{x+\sin x}$, (iii) $\frac{x^4+1}{x^2+1}$,

(iv) $\frac{x^2-x+1}{x^2+x+1}$, (v) $\frac{3x+2}{(x+5)(2x-1)+3}$, (vi) $\frac{x^2+\sin x}{x\cos x}$,

(vii) $x^2\sec x+\frac{x^2}{1+\sin x}$.

12. Evaluate:

$$\lim_{h\to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} = \lim_{x\to a} \frac{d}{dx}(x^2 \sin x).$$

13. If $f(x) = (1 - x)/(1 + x)$, find $f'(1)$.

14. If $y = (x - a)(x - b)$, show that at the particular point of the curve where $dy/dx = 0$, x will have the value $(a + b)/2$.

1.4 Differential Coefficient of a Function of Functions

Let y is a differentiable function and u and v differentiable functions of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Let $y = f(u)$ and $u = \phi(x)$, Hence y is a function of x. If $y = f(u)$, $u = \phi(v)$ and $v = \psi(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

1.5 Differential Coefficient of Inverse Functions

Let $y = f(x)$ be a differentiable function in its domain of definition. Let the inverse function of $y = f(x)$ exists and let it be $x = f(y)$. Let δx be a small change in x and δy be the corresponding change in y determined by $y = f(x)$ then corresponding to change in y, change in x determined by $x = \phi(y)$ will be δx. Now,

$$y = f(x), \qquad y + \delta y = f(x + \delta x)$$

Then

$$\delta y = f(x + \delta x) - f(x)$$

Now

$$\delta x \to 0,\ \delta y \to 0.$$

Then

$$\frac{\delta y}{\delta x}\frac{\delta x}{\delta y} = 1 \quad \text{or} \quad \frac{\delta x}{\delta y} = \frac{1}{\delta y/\delta x}$$

or

$$\lim_{\delta x\to 0} \frac{\delta x}{\delta y} = \lim_{\delta x\to 0} \frac{1}{\delta y/\delta x}$$

or

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Example 1.31 If $y = \sin(\cot x)$, then find dy/dx.

Solution We have

$$\frac{dy}{dx} = \frac{d[\sin(\cot x)]}{dx}$$

$$= \frac{d[\sin(\cot x)]}{d(\cot x)} \frac{d(\cot x)}{dx}$$

$$= \cos(\cot x)(-\text{cosec}^2 x)$$

$$= -\text{cosec}^2 x \cos(\cot x).$$

Example 1.32 If $y = (5x^3 + 7x^2 + 11)^{5/2}$, find dy/dx.

Solution We have

$$\frac{dy}{dx} = \frac{d(5x^3 + 7x^2 + 11)^{5/2}}{dx}$$

$$= \frac{d(5x^3 + 7x^2 + 11)^{5/2}}{d(5x^3 + 7x^2 + 11)} \frac{d(5x^3 + 7x^2 + 11)}{dx}$$

$$= \frac{5}{2}(5x^3 + 7x^2 + 11)^{3/2}(15x^2 + 14x)$$

$$= \frac{5}{2}(15x^2 + 14x)(5x^3 + 7x^2 + 11)^{3/2}.$$

Example 1.33 If $y = \sin\sqrt{(x^2 + ax + 1)}$, find dy/dx.

Solution We have

$$\frac{dy}{dx} = \frac{d[\sin\sqrt{x^2 + ax + 1}]}{a\sqrt{x^2 + ax + 1}} \frac{d\sqrt{x^2 + ax + 1}}{d\left(x^2 + ax + 1\right)} \frac{d(x^2 + ax + 1)}{dx}$$

$$= \cos\sqrt{x^2 + ax + 1}\left(\frac{1}{2}\right)\left(x^2 + ax + 1\right)^{-1/2}(2x + a)$$

$$= \frac{(2x + a)\cos\sqrt{x^2 + ax + 1}}{2\sqrt{x^2 + ax + 1}}.$$

Example 1.34 Find dy/dx. If

$$y = \frac{\cot x}{x} + \sqrt{1 - x^2}$$

Solution Let

$$\frac{\cot x}{x} = u \quad \text{and} \quad \sqrt{1 - x^2} = v$$

We also have $y = u + v$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \tag{1}$$

Now $u = \cot x/x$. Then

$$\frac{du}{dx} = \frac{x(-\text{cosec}^2 x) - \cot x}{x^2} = -\frac{x\,\text{cosec}^2 x + \cot x}{x^2}$$

Again $v = (1 - x^2)^{1/2}$. Then

$$\frac{dv}{dx} = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{1 - x^2}}$$

From (1)

$$\frac{dy}{dx} = -\frac{x\,\text{cosec}^2 x + \cot x}{x^2} - \frac{x}{\sqrt{1 - x^2}}$$

Example 1.35 If $y = \sin^2\sqrt{x^2 + 1}$, find dy/dx.

Solution Since $y = \sin^2\sqrt{x^2 + 1}$, we get

$$\frac{dy}{dx} = 2\sin(x^2 + 1)^{1/2}\cos(x^2 + 1)^{1/2}\frac{1}{2}(x^2 + 1)^{-1/2}(2x)$$

$$= \frac{2x\sin(x^2 + 1)^{1/2}\cos(x^2 + 1)^{1/2}}{\sqrt{x^2 + 1}} = \frac{x\sin 2\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}.$$

Example 1.36 Find dy/dx, when $y = \cos\sqrt{\sin\sqrt{x}}$.

Solution On differentiation, we get

$$\frac{dy}{dx} = \frac{d(\cos\sqrt{\sin\sqrt{x}})}{d(\sqrt{\sin\sqrt{x}})}\,\frac{d(\sqrt{\sin\sqrt{x}})}{d(\sin\sqrt{x})}\,\frac{d(\sin\sqrt{x})}{d\sqrt{x}}\,\frac{d\sqrt{x}}{dx}$$

$$= -\sin\sqrt{\sin\sqrt{x}}\left(\frac{1}{2}\right)\left(\sin\sqrt{x}\right)^{-1/2}\cos\sqrt{x}\left(\frac{1}{2}\right)x^{-1/2}$$

$$= -\frac{\cos\sqrt{x}\,\sin\sqrt{\sin\sqrt{x}}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}.$$

Example 1.37 Find dy/dx, when

(i) $y=\dfrac{x\sqrt{4x+3}}{(3x+1)^2}$, (ii) $y=\dfrac{\sqrt{x-a}+\sqrt{x+a}}{\sqrt{x-a}-\sqrt{x+a}}$.

Solution (i) Differentiating the given function with respect to x, we get

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(x\sqrt{4x+3})(3x+1)^2-\frac{d(3x+1)^2}{dx}x\sqrt{4x+3}}{(3x+1)^4}$$

$$=\frac{\left[\sqrt{4x+3}+x\frac{d\sqrt{4x+3}}{d(4x+3)}\frac{d(4x+3)}{dx}\right](3x+1)^2-x\sqrt{4x+3}\,(2)(3x+1)^3}{(3x+1)^4}$$

$$=\frac{\left(\sqrt{4x}+3+\frac{1}{2}x\frac{1}{\sqrt{4x+3}}4\right)(3x+1)^2-6x\,(3x+1)\sqrt{4x+3}}{(3x+1)^4}$$

$$=\frac{[(4x+3)+2x](3x+1)^2-6x(3x+1)(4x+3)}{\sqrt{4x+3}\,(3x+1)^4}$$

$$=\frac{(6x+3)(3x+1)-6x(4x+3)}{(3x+1)^3\sqrt{4x+3}}$$

$$=\frac{3-3x-6x^2}{(3x+1)^3\sqrt{4x+3}}.$$

(ii) Differentiating, we get

$$y=\frac{\sqrt{x-a}+\sqrt{x+a}}{\sqrt{x-a}-\sqrt{x+a}}=\frac{(\sqrt{x-a}+\sqrt{x+a})^2}{x-a-x-a}$$

$$=\frac{x-a+x+a+2\sqrt{x^2-a^2}}{-2a}$$

$$=-\frac{1}{a}(x+\sqrt{x^2-a^2})$$

Then

$$\frac{dy}{dx}=-\frac{1}{a}\left[1+\frac{d\sqrt{x^2-a^2}}{d(x^2-a^2)}\frac{d(x^2-a^2)}{dx}\right]$$

$$=-\frac{1}{a}\left(1+\frac{1}{2\sqrt{x^2-a^2}}2x\right)$$

$$=-\frac{1}{a}\left(1+\frac{x}{\sqrt{x^2-a^2}}\right).$$

Example 1.38 Find dy/dx, when

$$y = \sin\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right).$$

Solution Differentiating, we have

$$\frac{dy}{dx} = \frac{d\sin\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)}{d\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)} \frac{d\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)}{dx}$$

$$= \cos\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \frac{\frac{d(1+\sqrt{x})}{dx}(1-\sqrt{x}) - \frac{d(1-\sqrt{x})}{dx}(1+\sqrt{x})}{(1-\sqrt{x})^2}$$

$$= \cos\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \frac{\frac{1}{2}\frac{1}{\sqrt{x}}(1-\sqrt{x}) - \left(-\frac{1}{2}\frac{1}{\sqrt{x}}\right)(1+\sqrt{x})}{(1-\sqrt{x})^2}$$

$$= \cos\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \frac{(1-\sqrt{x}) + (1+\sqrt{x})}{2\sqrt{x}(1-\sqrt{x})^2}$$

$$= \frac{1}{\sqrt{x}(1-\sqrt{x})^2} \cos\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right).$$

Example 1.39 If $y = x^2 \cos(\log x)$, find dy/dx.

Solution Differentiating, we get

$$\frac{dy}{dx} = \frac{d[x^2 \cos(\log x)]}{dx}$$

$$= x^2 \frac{d[\cos(\log x)]}{dx} + \cos(\log x)\frac{dx^2}{dx}$$

$$= x^2 \frac{d\cos(\log x)}{d(\log x)} \frac{d(\log x)}{dx} + \cos(\log x)(2x)$$

$$= x^2[-\sin(\log x)]\frac{1}{x} + 2x\cos(\log x)$$

$$= -x\sin(\log x) + 2x\cos(\log x).$$

Example 1.40 Find the differential coefficient of

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right).$$

Solution Given

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$$

or

$$\frac{dy}{dx}=2\,\frac{d(\tan^{-1}x)}{dx}=\frac{2}{1+x^2}.$$

Example 1.41 Find dy/dx, when

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right).$$

Solution Since

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

Put $x = \tan\,\theta$, we get

$$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left[\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right]$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

We also have, $y = \theta/2 = (\tan^{-1}x)/2$. Then

$$\frac{dy}{dx}=\frac{1}{2(1+x^2)}.$$

Example 1.42 Find dy/dx, when

$$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

Solution Since

$$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left[\sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}}\right] = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

Then

$$\frac{dy}{dx} = \frac{1}{2}$$

Example 1.43 Find dy/dx if

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right).$$

Solution Since

$$y = \tan^{-1}\frac{\cos x}{1+\sin x} = \tan^{-1}\left[\frac{\cos^2(x/2) - \sin^2(x/2)}{[\sin(x/2) + \cos(x/2)]^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos(x/2) - \sin(x/2)}{\cos(x/2) + \sin(x/2)}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan(x/2)}{1+\tan(x/2)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2}.$$

We get

$$\frac{dy}{dx} = -\frac{1}{2}.$$

Example 1.44 If $y = \log(x + \sqrt{x^2 + a^2})$, find dy/dx.

Solution Given $y = \log\ (x + \sqrt{x^2 + a^2})$, we get

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{1}{2}\left(x^2+a^2\right)^{-1/2}(2x)\right]$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$$

$$=\frac{1}{\sqrt{x^2+a^2}}.$$

Example 1.45 If $y = \log\ (x^2\sqrt{x^2+1})$, find dy/dx.

Solution Given

$$y=\log(x^2\sqrt{x^2+1})=2\log x+\frac{1}{2}\log(x^2+1).$$

We have

$$\frac{dy}{dx}=2\frac{1}{x}+\frac{1}{2}\frac{1}{x^2+1}2x=\frac{2}{x}+\frac{x}{x^2+1}=\frac{3x^2+2}{x(x^2+1)}.$$

Exercises 1.4

1. Find the derivatives of the following functions:

(i) $x^3 \sin 3x$, (ii) $\sin\sqrt{x^2+x+1}$, (iii) $\sqrt{\sin\sqrt{\sin\sqrt{x}}}$,

(iv) $\sin(\sin x)$, (v) $\sin(\tan x)$, (vi) $\sin(\cos ax)$,

(vii) $\sin\sqrt{ax}$, (viii) $\tan^{-1}(\sin x)$, (ix) $\sin(\log x)$,

(x) $\log(\tan x)$, (xi) $\log(\cos x)$, (xii) $\log(\log x)$,

(xiii) $\sqrt{5+2x-4x^2}$, (xiv) $\frac{1}{\sqrt{2x^2+3x-10}}$, (xv) $\sqrt{1+\sin x}$,

(xvi) $\cos\frac{x}{1+\sqrt{x}}$, (xvii) $\sin(\sin\sqrt{x})$, (xviii) $\log(\log x)^2$,

(xix) $\exp(ax^2)$, (xx) $e^{\tan x}$, (xxi) $e^{\sqrt{x}}$,

(xxii) $e^{-x}(\cos(2x+3))$, (xxiii) $\log(ax+b)$, (xxiv) $\log(x+1/x)$,

(xxv) $\log(ax^2+bx+c)$.

2. Find dy/dx, when y be the following functions:

(i) $(\log \sin x)^2$, (ii) $x \log \sec x$, (iii) $e^{ax} \sin bx$,

(iv) $e^{ax} \cos bx$, (v) $e^{ax} \sin (bx + c)$,

(vi) $(e^x + e^{-x})/(e^x - e^{-x})$, (vii) $\sin [\log (1 + x^2)]$,

(viii) $a \log \sec \frac{x}{2a}$, (ix) $\log (\sec x + \tan x)$,

(x) $\log\left(\frac{\pi}{4} + \frac{x}{2}\right)$, (xi) $\log \tan (\pi/4 + x)$ (xii) $\log \left(\frac{1+x}{1-x}\right)$,

(xiii) $\log\left(\frac{1-x^2}{1+x^2}\right)$, (xiv) $\log\left(\frac{ax^2+bx+c}{ax^2-bx+c}\right)$, (xv) $\log\left(\frac{1+\sin x}{1-\sin x}\right)$,

(xvi) $\log\left(\frac{1-\cos x}{1+\cos x}\right)$, (xvii) $\log\left(\frac{ax+b}{ax-b}\right)$, (xviii) $\log\left(\frac{a+b\tan x}{a-b\tan x}\right)$,

(xix) $\log\left[\frac{2+\tan(x/2)}{2-\tan(x/2)}\right]$, (xx) $x \log\left(\frac{x}{ax+b}\right)$, (xxi) $\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$,

(xxii) $\log\left(\frac{1+x\sin x}{1-x\sin x}\right)$.

3. Find dy/dx, when y is equal to

(i) $\cos^3 \sqrt{x}$, (ii) $\sin \sqrt{\cos x}$, (iii) $\sqrt{\cos \sqrt{x}}$,

(iv) $\sqrt{\tan(\tan x)}$, (v) $\sqrt{\sin \sqrt{ax}}$, (vi) $\sin \sqrt{x^2 + ax + 1}$,

(vii) $\sqrt{\cos(1 + x^2)}$, (viii) $1/(1 + \tan^3 x)^2$, (ix) $\sin^{-1} \sqrt{1 - x^2}$,

(x) $\sqrt{\tan(1 + x^2)}$, (xi) $\sqrt{\sin x^2}$, (xii) $\sin^3 \sqrt{ax^2 + bx + c}$,

(xiii) $\sin (\cos \tan \sqrt{x})$, (xiv) $\sin \{\cos [\tan (\cot x)]\}$,

(xv) $\sin \{\cos[\tan (\sec x)]\}$.

4. Find dy/dx when y is equal to

(i) $\sin^m(\alpha x) \cos^n(\beta x)$, (ii) $(2x - 3)^2 \sqrt{4x^2 + 1}$,

(iii) $e^{2x} \sin^{-1}(ax)$, (iv) $\sin^2 (2x + 3) \cos^2(3x + 4)$,

(v) $x^2 \cos (\log x)$, (vi) $e^{\cos \sqrt{x}}$,

(vii) $e^{\sqrt{x}} \sin \sqrt{x}$, (viii) $e^{\sqrt{x}} \log(\cos \sqrt{x})$.

5. Find the differential coefficients of the following functions:

(i) $x \log \sin x$, (ii) $(\log \sin x)^2$, (iii) $\sin [\log (1 + x^2)]$,

(iv) $a \log \sec (x/a)$, (v) $\log \left(\frac{x^2+x+1}{x^2-x+1}\right)$, (vi) $\log \sqrt{\frac{1+\sin x}{1-\cos x}}$.

6. Find dy/dx, when y is equal to

(i) $x\sqrt{x^2+a^2} + a^2 \log\left(x+\sqrt{x^2+a^2}\right)$,

(ii) $\frac{\log x}{\sqrt{x^2-a^2}}$, (iii) $e^{\sin(\log x)}$,

(iv) $x^n \log x + x(\log x)^n$, (v) $\frac{\log x + e^x}{\sin 3x}$,

(vi) $\log\left(\frac{x+\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}\right)$, (vii) $\sec x\sqrt{1+x^2}$,

(viii) $\sqrt{(x^2+3)\sqrt{x^2-1}}$, (ix) $\tan^{-1}(\log x)$,

(x) $\tan^{-1}(x^2e^{-x})$.

7. Find dy/dx when y is equal to:

(i) $\sin\sqrt{x} + \cos^2\sqrt{x}$,

(ii) $\cos(ax^2+bx+c) + \sin^3\sqrt{ax^2+bx+c}$,

(iii) $\cos(5x^2+8) + \frac{1}{\sqrt[3]{4x^2-1}}$, (iv) $\cos\sqrt{1-x^2} + x^4\cot 4x$,

(v) $x\sqrt{1-x^2} + \frac{\tan x}{x}$, (vi) $\log\left(x+\sqrt{x^2+a^2}\right) + \sec^{-1}x/a$,

(vii) $\tan^{-1}\frac{x}{a} + \log\sqrt{\frac{x-a}{x+a}}$.

8. Find dy/dx when each of the following functions is equal to y:

(i) $\sin(\cos^{-1}x)$, (ii) $\cos(\sin^{-1}x)$, (iii) $\tan^{-1}\left(\frac{x+\sqrt{x}}{1-x^{3/2}}\right)$,

(iv) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, (v) $\sin^{-1}(3x-4x^3)$, (vi) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$,

(vii) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, (viii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, (ix) $\tan^{-1}\left(\frac{x}{1-x^2}\right)$,

(x) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$, (xi) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, (xii) $\tan^{-1}\left(\frac{4x}{4-x^2}\right)$,

(xiii) $\tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$, (xiv) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$,

(xv) $\tan^{-1}(\sec x + \tan x)$, (xvi) $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin}-\sqrt{1-\sin x}}\right)$,

(xvii) $\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, (xviii) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$,

(xix) $\sin^{-1}\sqrt{\frac{1-\cos x}{2}}$, (xx) $\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$, (xxi) $\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$,

(xxii) $\tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$, (xxiii) $\cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$, (xxiv) $(1+x)\tan^{-1}\frac{1}{x}$,

(xxv) $\tan^{-1}\left(\frac{a+b\cos x}{b-a\cos x}\right)$, (xxvi) $\cos^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right)$, (xxvii) $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$.

9. Find dy/dx of the following:

(i) $x^{3/2}\tan(5-2x)$, (ii) $\sin^3\sqrt{ax^2+bx+c}$, (iii) $\sqrt{x}\,\sin x+\sin\sqrt{x}$,

(iv) $\sin\sqrt{x}+\cos^2\sqrt{x}$, (v) $(2x^3-1)(x^2+1)^2$, (vi) $(2x^2+3)^{5/3}(x+5)^{-1/3}$,

(vii) $\sin^2(2x+3)\cos^2(3x+4)$,

(viii) $\cos(ax^2+bx+c)+\sin^3\sqrt{ax^2+bx+c}$,

(ix) $\sin\sqrt{1-x^2}+x^2\cos 4x$, (x) $\sin(2ax\sqrt{1-a^2x^2})$,

(xi) $\frac{1}{\sqrt[3]{4x^3-1}}+\cos^2(5x+8)$, (xii) $\frac{5x}{\sqrt[3]{(1-x)^2}}+\cos^2(2x+1)$,

(xiii) $\log\left(\frac{1+x}{1-x}\right)^{1/4}-\frac{1}{2}\tan^{-1}\left(\frac{1}{x}\right)$, (xiv) $\tan^{-1}\left(\frac{x}{a}\right)+\log\sqrt{\frac{x-a}{x+a}}$,

(xv) $2\sin^{-1}\left(\frac{x}{a}\right)+x\sqrt{a^2-x^2}$, (xvi) $\sqrt{1-x^2}+(\sin^{-1}x)x$.

10. Find the differential coefficient of the following functions:

(i) $\tan x$ with respect to x^2, (ii) $\sin x$ with respect to $\cos x$,
(iii) $\sec x$ with respect to $\tan x$, (iv) $\sin x$ with respect to x^3,
(v) $\sin x$ with respect to $\tan x$, (vi) $\tan x$ with respect to $\sin x$,
(vii) $\tan x$ with respect to $\sec x$, (viii) $\tan x$ with respect to $\cot x$,
(ix) $\sin x$ with respect to $\cot 2x$, (x) x^3+x^2+3 with respect to x^3,

(xi) $x/\sqrt{1+x^2}$ with respect to $\tan x$,

(xii) $\sqrt{(1+t)/(1-t)}$ with respect to x where $t=\cos 2t$,

(xiii) $\sqrt{(1-x^2)/(1+x^2)}$ with respect to t, when $x=\tan t$.

(xiv) $\frac{x}{\sin x}$ with respect to $\sin x$, (xv) e^x with respect to $e^x\log x$,

(xvi) $\sin x^2$ with respected to x^2, (xvii) e^{2x} with respect to e^x.

11. Find dy/dx, when y is equal to:

(i) $\cos^2 x^2$, (ii) $1/(\log\cos x)$,

(iii) $\sqrt{\frac{1-\tan x}{1+\tan x}}$, (iv) $\sin\sqrt{\sin x+\cos x}$,

(v) $1/\left(\sqrt{x^2+a^2}+\sqrt{x^2+b^2}\right)$, (vi) $2\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right)+\log\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right)$,

(vii) $\sqrt{x+\sqrt{x+\sqrt{x}}}$,

(viii) $[\log_{\cos x}(\sin x)\log_{\sin x}(\cos x)]^{-1}+\sin^{-1}\left(\frac{2x}{1+x}\right)$, at $x=\pi/4$,

(ix) $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, (x) $\cos^{-1}\left(\frac{2\cos x+3\sin x}{\sqrt{13}}\right)$,

(xi) $\sin^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right)$, (xii) $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$,

(xiii) $\sin^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2}\right)$, (xiv) $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$,

(xv) $\sin^{-1}\left(x\sqrt{1-x^2}+\sqrt{x}\sqrt{1-x^2}\right)$, (xvi) $\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$,

(xvii) $\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$, (xviii) $\tan^{-1}\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$,

(xix) $\tan^{-1}\left(\frac{2+3x}{3-2x}\right)$, (xx) $\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$,

(xxi) $\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-a^{1/3}x^{1/3}}\right)$, (xxii) $\sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$,

(xxiii) $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)+\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$,

(xxiv) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x}\right)$, (xxv) $\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^x$.

12. Differentiate the following functions:

(i) $\sin^{-1}x$ with respect to $\cos^{-1}\left(\sqrt{1-x^2}\right)$,

(ii) $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ with respect to $\tan^{-1}x$,

(iii) $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ with respect to $\sqrt{x}$,

(iv) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$,

(v) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$,

(vi) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$,

(vii) $e^{\sin^{-1}x}$ with respect to $\sin^{-1}x$,

(viii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}x^2$,

(ix) $\sin^3 x$ with respect to $\cos^3 x$.

1.6 Implicit Functions

When independent variable x and dependent variable y occur together in an equation and y cannot be written in terms of only x, then y is said to be an *implicit* function of x. For example,

$$x^3 + y^3 - 3xy = 0 \quad \text{and} \quad xy = \sin(x + y).$$

Let $f(x, y) = 0$, where $f(x, y)$ is an arbitrary function of two variables x and and y and let us suppose that it is not solvable for y. Then differentiate both sides of the given function $f(x, y) = 0$ in x and y with respect to x. It should be remembered that when we differentiate terms containing y, we should multiply it by dy/dx after differentiating and without solving for y to obtain dy/dx.

Example 1.46 Find dy/dx, when $ax^2 + 2hxy + by^2 = 1$.

Solution Differentiating both sides with respect to x, we get

$$a(2x) + 2h\left(x\frac{dy}{dx} + y\right) + b(2y)\frac{dy}{dx} = 0$$

or

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

or

$$2(hx + by)\frac{dy}{dx} + 2(ax + hy) = 0$$

or

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}.$$

Example 1.47 If $x + y = \sin(xy)$, find dy/dx.

Solution Given $x + y = \sin(xy)$, Differentiating the given equation both sides with respect to x, we get

$$1+\frac{dy}{dx}=\cos(xy)\left(y+x\frac{dy}{dx}\right)=y\cos(xy)+x\cos(xy)\frac{dy}{dx}$$

or

$$[1-x\cos(xy)]\frac{dy}{dx}=y\cos(xy)-1$$

or

$$\frac{dy}{dx}=\frac{y\cos(xy)-1}{1-x\cos(xy)}.$$

Example 1.48 If $\sin y = x\sin(a + y)$, show that

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}.$$

Solution From the given relation

$$x=\frac{\sin y}{\sin(a+y)}$$

Differentiating with respect to y, we get

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}=\frac{\sin(a+y-y)}{\sin^2(a+y)}=\frac{\sin a}{\sin^2(a+y)}$$

or

$$\frac{dy}{dx}=\frac{1}{dx/dy}=\frac{1}{(\sin a)/\sin^2(a+y)}=\frac{\sin^2(a+y)}{\sin a}.$$

Example 1.49 If $x^m y^n = (x + y)^{m+n}$, find dy/dx.

Solution Given $x^m y^n = (x + y)^{m+n}$. Differentiating the given equation both sides with respect to x, we get

$$mx^{m-1}y^n+x^m ny^{n-1}\frac{dy}{dx}=(m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$$

or

$$\left[nx^m y^{n-1}-(m+n)(x+y)^{m+n-1}\right]\frac{dy}{dx}=(m+n)(x+y)^{m+n-1}-mx^{m-1}y^n$$

or

$$\frac{dy}{dx}=\frac{(m+n)(x+y)^{m+n-1}-mx^{m-1}y^n}{nx^m y^{n-1}-(m+n)(x+y)^{m+n-1}}$$

$$=\frac{(m+n)(x+y)^{m+n}-mx^{m-1}y^n(x+y)}{nx^m y^{n-1}(x+y)-(m+n)(x+y)^{m+n}}$$

$$=\frac{(m+n)x^m y^n-mx^{m-1}y^n(x+y)}{nx^m y^{n-1}(x+y)-(m+n)x^m y^n}$$

$$=\frac{x^{m-1}y^n(nx-my)}{x^m y^{n-1}(nx-my)}$$

$$=\frac{y}{x}.$$

Example 1.50 If $y = 1/x$, then prove that

$$\frac{dx}{\sqrt{1+x^4}}+\frac{dy}{\sqrt{1+y^4}}=0$$

Solution Given $y = 1/x$, we get

$$\frac{dy}{dx}=-\frac{1}{x^2}.$$

Now

$$\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}}=\frac{\sqrt{1+1/x^4}}{\sqrt{1+x^4}}=\frac{\sqrt{x^4+1}/x^2}{x^2\sqrt{x^4+1}}=\frac{1}{x^2}=-\frac{dy}{dx}$$

or

$$\frac{dx}{\sqrt{1+x^4}}=-\frac{dy}{\sqrt{1+y^4}}$$

or

$$\frac{dx}{\sqrt{1+x^4}}+\frac{dy}{\sqrt{1+y^4}}=0$$

Example 1.51 Find dy/dx, when

$$y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots\infty}}}$$

Solution We have

$$y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$$

Squaring both sides, we get

$$y^2=\sin x+\sqrt{\sin x+\sqrt{\sin x}+\sqrt{\sin x+\cdots}}=\sin x+y$$

Differentiating both sides with respect to x

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx} \quad\text{or}\quad \frac{dy}{dx}=\frac{\cos x}{2y-1}.$$

Example 1.52 If $y=\tan^{-1}(x+y)$, find dy/dx.

Solution Differentiating, we get

$$\frac{dy}{dx}=\frac{1}{1+(x+y)^2}\left(1+\frac{dy}{dx}\right)=\frac{1}{1+(x+y)^2}+\frac{1}{1+(x+y)^2}\frac{dy}{dx}$$

or

$$\frac{dy}{dx}\left[1-\frac{1}{1+(x+y)^2}\right]=\frac{1}{1+(x+y)^2}$$

or

$$\frac{dy}{dx}\left[\frac{1+(x+y)^2-1}{1+(x+y)^2}\right]=\frac{1}{1+(x+y)^2}$$

or

$$\frac{dy}{dx}=\frac{1}{(x+y)^2}.$$

Example 1.53 If $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)$, show that

$$\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$$

Solution Putting $x=\sin\theta$, $y=\sin\phi$, we get

$$\cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$$

or

$$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=2a\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

or

$$\cos\frac{\theta-\phi}{2}=a\sin\frac{\theta-\phi}{2}$$

or

$$\cot\frac{\theta-\phi}{2}=a$$

or

$$\theta-\phi=2\cot^{-1}a$$

or

$$\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a.$$

Differentiating with respect to x,

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

Exercises 1.5

1. Find the differential coefficients of:

(i) $x^4+y^4=a^4$, (ii) $x^3+y^3=3axy$, (iii) $x^2y=(x+2y)^3$,
(iv) $x\sqrt{x}+y\sqrt{x}=1$, (v) $y=\tan(x+y)$, (vi) $x^2y^2=\sin(xy)$,
(vii) $y^2=\tan(x+y^2)$, (viii) $x+y=\tan(xy)$,
(ix) $x\cos y+y\cos x=\tan(x+y)$, (x) $x^2+y^2=a^2$,
(xi) $x^{1/2}+y^{1/2}=a^{1/2}$, (xii) $x^n+y^n=a^n$, (xiii) $x^3y^4=(x+y)^7$,
(xiv) $x^ay^b=(x-y)^{a+b}$, (xv) $xy+x^2y^2=c$,
(xvi) $(x-y)y^n=2\sqrt{x}$, (xvii) $x^2y=(2x+3y)^3$,
(xviii) $x^2y^3=(2x+y^5)$, (xix) $y=\sin(x+y)$, (xx) $y=\sec(x+y)$,
(xxi) $y=\cot(x+y)$, (xxii) $x+y=\sin(x+y)$,
(xxiii) $x-y=\cos(x-y)$, (xxiv) $x-y=\sec(x+y)$,
(xxv) $xy=\sin(x+y)$, (xxvi) $xy=\tan(x+y)$,
(xxvii) $xy=\sec(x+y)$, (xxviii) $x+y=\tan(xy)$,
(xxix) $xy=\sin(xy)$, (xxx) $xy=\tan(xy)$.

2. Find the differential coefficients of the following:

(i) $x\cos y=\sin(x+y)$, (ii) $x\sin y=\cos(x+y)$,
(iii) $x^3y^3=x\cos(xy)$, (iv) $x^2+y^2=\sin(xy)$,
(v) $x^2y^2=\sin(xy)$, (vi) $y=x+y^2\sin^3(x/2)$,
(vii) $x\cos y+y\cos x=\tan(x+y)$, (viii) $x\cos y+y\sin x=\tan(x+y)$.

3. If $y=\sqrt{(1+x^2)}$, then prove that

$$y\frac{dy}{dx}-x=0.$$

4. If $y = x + 1/y$, then prove that

$$(x^2 - y^2 + 3)\frac{dy}{dx} = 1.$$

5. If $y = x + (1/x)$, show that

$$x\frac{dy}{dx} + y - 2x = 0.$$

6. If $y = \sqrt{x} + 1/\sqrt{x}$, then prove that

$$2x\frac{dy}{dx} + y = 2\sqrt{x}.$$

7. If $\cos y = x\cos(b + y)$, then prove that

$$\frac{dy}{dx} = \frac{\cos^2(b+y)}{\sin b}.$$

8. If $x + \sin\sqrt{(xy)} = 0$, then prove that

$$\frac{dy}{dx} = -\frac{\sqrt{y} + y\cos\sqrt{xy}}{x\cos\sqrt{xy}}.$$

9. If

$$y = \sqrt{\cot x + \sqrt{\cot x + \sqrt{\cot x + \cdots}}},$$

prove that

$$\frac{dy}{dx} = \frac{\operatorname{cosec}^2 x}{1 - 2y}.$$

10. If

$$x = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y + \cdots \alpha}}},$$

prove that

$$\frac{dy}{dx} = 2x^2 + y^2 - 3xy.$$

11. If

$$y = \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} + \cdots \infty,$$

prove that

$$\frac{dy}{dx} = \frac{y}{2\sqrt{x}(2y - \sqrt{x})}.$$

12. If $ax^2 + 2hxy + by^2 = 0$, prove that $dy/dx = y/x$.

13. If $y^2 = (x + \sin x)(x - \cot x)$, prove that

$$\frac{dy}{dx} = \frac{x + (x + \sin x - \cos x)(1 + \sin x + \cos x)}{3y^2}.$$

14. If $y = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}.$$

15. If $3y - \sqrt{(9y^2 + 1)} = 5x^2$, prove that

$$\frac{dy}{dx} = \frac{2(5x^2 - 3y)}{3x}.$$

16. If $x + y = \tan^{-1}(xy)$, find dy/dx.

17. If $x \cos y + y \cos x = \tan^{-1}x^2$, find dy/dx.

18. If $e^{xy} = \cos(x^2 + y^2)$, find dy/dx.

19. If $x^3y^3 = \log_e(x + y) \sin e^x$, find dy/dx.

20. If

$$y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots \infty}}},$$

prove that

$$(1 + y^2)\frac{dy}{dx} + y^2 = 0.$$

21. If $e^{x + e^{x + e^{x + \cdots}}}$, prove that

$$\frac{dy}{dx} = \frac{y}{1 - y}.$$

22. Find dy/dx, when:

(i) $\sqrt{x} = y + \sqrt{a}$ at the point $(a, 0)$,

(ii) For the curve $x^2 = y$ at the point $(1, 1)$,

(iii) for $xy + 4 = 0$, at $(2, -2)$.

1.7 Parametric Equations

Consider an equation in two variables x and y:

$$\phi(x, y) = 0 \tag{1.3}$$

If a third variable t can be obtained such that

$$x = f(t), \qquad y = F(t) \tag{1.4}$$

and Eq. (1.3) is satisfied by substituting the values of x and y from Eq. (1.4) then Eq. (1.4) is called the *parametric equations* of (1.3) and t is called the parameter.

Let δt be a small change in t and let δx and δy be corresponding small changes in x and y, respectively. When $\delta t \to 0$, $\delta x \to 0$ and $\delta y \to 0$. Now

$$x + \delta x = f(t + \delta t) \quad \text{and} \quad y + \delta y = F(t + \delta t)$$

Then

$$\delta x = (x + \delta x) - x = f(t + \delta t) - f(t)$$

and

$$\delta y = (y + \delta y) - y = F(t + \delta t) - F(t)$$

Therefore,

$$\frac{\delta y}{\delta x} = \frac{F(t + \delta t) - F(t)}{f(t + \delta t) - f(t)} = \frac{\frac{F(t+\delta t) - F(t)}{\delta t}}{\frac{f(t+\delta t) - f(t)}{\delta t}}$$

or

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta t \to 0} \frac{\frac{F(t+\delta t) - F(t)}{\delta t}}{\frac{f(t+\delta t) - f(t)}{\delta t}}$$

or

$$\frac{dy}{dx} = \lim_{\delta t \to 0} \frac{\frac{F(t+\delta t) - F(t)}{\delta t}}{\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}}$$

$$= \lim_{\delta t \to 0} \frac{\delta y / \delta t}{\delta x / \delta t}$$

$$= \frac{dy/dt}{dx/dt}$$

$$= \frac{dy}{dt} \frac{dt}{dx}$$

Example 1.54 If $x = at^2$ and $y = 2at$, where t is a parameter, find dy/dx.

Solution Differentiating the given functions with respect to t, we get

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}.$$

Example 1.55 If $y = e^t + \cos t$ and $x = \log t + \sin t$, t being parameter, find dy/dx.

Solution We have

$$\frac{dx}{dt} = \frac{1}{t} + \cos t, \qquad \frac{dy}{dt} = e^t - \sin t$$

or

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{e^t-\sin t}{\frac{1}{t}+\cos t}=\frac{t(e^t-\sin t)}{1+t\cos t}$$

Example 1.56 If $x = a\,(t + \sin t)$ and $y = a(1 - \cos t)$, find dy/dx. Also obtain the value of dy/dx, when $t = \pi/2$.

Solution We have

$$\frac{dx}{dt}=a(1+\cos t),\quad \frac{dy}{dt}=a\sin t$$

or

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1+\cos t)}=\frac{\sin t}{1+\cos t}=\frac{2\sin(t/2)\cos(t/2)}{2\cos^2(t/2)}=\frac{\sin(t/2)}{\cos(t/2)}$$

or

$$\frac{dy}{dx}=\tan\frac{t}{2}.$$

Also, when $t = \pi/2$,

$$\frac{dy}{dx}=\tan\frac{\pi}{4}=1.$$

Exercises 1.6

1. Find dy/dx, where θ and t being parameters and when:

   (i) $x = a\sin\theta$, $y = b\cos\theta$,
   (ii) $x = a\cos^2\theta$, and $y = a\sin^2\theta$,
   (iii) $x = a\cos^3\theta$, and $y = a\sin^3\theta$,
   (iv) $x = a\,(1 - \cos\theta)$ and $y = a(\theta + \sin\theta)$,
   (v) $x = a\cos t + b\sin t$ and $y = a\sin t + b\cos t$,
   (vi) $x = t + 1/t$ and $y = t - 1/t$,
   (vii) $x = \sqrt{(1 + t^2)}$ and $y = \sqrt{(1 - t^2)}$,
   (viii) $y = t + \cos t$ and $x = \sin t$,
   (ix) $x = ae^t$ and $y = be^{-t}$,
   (x) $x = e^t + \sin t$ and $y = \log t$.

2. Solve the following:

   (i) If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, θ being a parameter, find dy/dx when $\theta = \pi/2$.

   (ii) If $x = 3at/(1 + t^3)$ and $y = 3at^2/(1 + t^3)$, t being a parameter, find dy/dx when $t = 1/2$.

   (iii) Find dx/dt at $t = \pi/2$, if $x = a\,(1 - \cos t)$ and $y = a(t + \sin t)$, t being a parameter.

3. Find dy/dx, when

(i) $x = a\,[\cos\theta + \log\tan(\theta/2)],\ y = a\sin\theta,$

(ii) $x = a\sin 2\theta(1+\cos 2\theta),\ y = a\cos 2\theta\,(1-\cos 2\theta),$

(iii) $x = a\,(\theta + \sin\theta),\ y = a(1-\cos\theta).$

4. If

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}},$$

find dy/dx at $t = \pi/6$.

5. If $x = \sec\theta - \cos\theta,\ y = \sec^n\theta - \cos^n\theta$, show that

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4).$$

6. If $x = \sin\theta\sqrt{(\cos 2\theta)},\ y = \cos\theta\sqrt{(\sin 2\theta)}$, find dy/dx at $\theta = \pi/4$.

7. If

$$x^2+y^2 = t - \frac{1}{t}, \quad x^4+y^4 = t^2 + \frac{1}{t^2},$$

prove that

$$x^3 y\frac{dy}{dx} = 1.$$

1.8 Logarithmic Differentiation

In the power of a function or if a function is the product of a number of functions, then to get the differential coefficient of such a function, just take logarithm and differentiate next. This process is termed as the *logarithmic differentiation*. For example, if

$$y = [\phi(x)]^{\psi(x)}$$

then

$$\log y = \psi(x)\log\phi(x).$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \psi'(x)\log\phi(x) + \psi(x)\frac{\phi'(x)}{\phi(x)}$$

or

$$\frac{dy}{dx} = y\left[\psi'(x)\log\phi(x) + \psi(x)\frac{\phi'(x)}{\phi(x)}\right]$$

$$= [\phi(x)]^{\psi(x)}\left[\psi'(x)\log\phi(x) + \psi(x)\frac{\phi'(x)}{\phi(x)}\right].$$

Example 1.57 Find dy/dx if (i) $y = x^{\sin x}$, (ii) $y = (\sin x)^{\log x}$.

Solution (i) Since $y = x^{\sin x}$, taking log both sides, we get $\log y = \sin x \log x$. Now differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \log x\right) = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right).$$

(ii) Here $y = (\sin x)^{\log x}$. Taking log both sides, we get $\log y = \log x \log \sin x$. Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin x}\cos x \log x + \frac{1}{x}\log \sin x = \cot x \log x + \frac{1}{x}\log \sin x$$

or

$$\frac{dy}{dx} = y\left(\cot x \log x + \frac{1}{x}\log \sin x\right) = (\sin x)^{\log x}\left(\cot x \log x + \frac{1}{x}\log \sin x\right).$$

Example 1.58 Differentiate $y = x^x + x^{1/x}$.

Solution Now $u = x^x$. Taking log both sides, we get $\log u = x \log x$. Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \log x + 1$$

or

$$\frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \tag{1}$$

Again $v = x^{1/x}$, taking log both sides, we get

$$\log v = \frac{1}{x}\log x,$$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x^2}$$

or

$$\frac{dv}{dx} = v\frac{1 - \log x}{x^2} = x^{1/x}\frac{1 - \log x}{x^2} \tag{2}$$

Adding (1) and (2), we get

$$\frac{du}{dx} + \frac{dv}{dx} = \frac{dy}{dx} = x^x(1 + \log x) + x^{1/x}\frac{1 - \log x}{x^2}.$$

Example 1.59 Differentiate, $y = x^x + (\cot x)^x$.

Solution Now $u = x^x$. Taking log both side, we obtain, $\log u = x \log x$. Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \log x + 1$$

or

$$\frac{du}{dx} = x^x\left(1 + \log x\right) \tag{1}$$

Also let, $v = (\cot x)^x$. So $\log v = x \log \cot x$. Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \log \cot x + x\frac{1}{\cot x}\left(-\operatorname{cosec}^2 x\right)$$

or

$$\frac{dv}{dx} = v\left(\log \cot x - \frac{x \operatorname{cosec}^2 x}{\cot x}\right) = (\cot x)^x (\log \cot x - x \sec x \operatorname{cosec} x) \tag{2}$$

Adding (1) and (2), we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^x(1 + \log x) + (\cot x)^x (\log \cot x - x \sec x \operatorname{cosec} x)$$

Example 1.60 Differentiate $(\sec x)^y = (\tan y)^x$.

Solution Taking log both sides, we get $y \log \sec x = x \log \tan y$. Differentiating with respect to x,

$$\frac{dy}{dx}\log \sec x + y\frac{1}{\sec x}\sec x \tan x = \log \tan y + x\frac{1}{\tan y}\sec^2 y\frac{dy}{dx}$$

$$= \log \tan y + x \sec y \operatorname{cosec} y \frac{dy}{dx}$$

or

$$\frac{dy}{dx}\left(\log \sec x - x \sec y \operatorname{cosec} y\right) = \log \tan y - y \tan x$$

or

$$\frac{dy}{dx} = \frac{\log \tan y - y \tan x}{\log \sec x - x \sec y \operatorname{cosec} y}.$$

Example 1.61 Differentiate:

$$x = y^{y^{y^{\cdot^{\cdot^{\cdot^{\infty}}}}}}$$

Solution Given equation can be written as $x = y^x$. Taking log both sides, we get, $\log x = x \log y$. Differentiating both sides, we get

$$\frac{1}{x} - \log y = x\frac{1}{y}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{y(1 - x\log y)}{x^2}.$$

Example 1.62 Differentiate $x^y y^x = 1$.

Solution Taking log both sides, we get, $y \log x + x \log y = 0$. Differentiating both sides, we also get

$$\frac{dy}{dx}\log x + y\frac{1}{x} + \log y + x\frac{1}{y}\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{y}{x}\frac{x\log y + y}{y\log x + x}.$$

Example 1.63 Differentiate:

$$y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cdots}}}}$$

Solution The given equation can be written as

$$y = \frac{\sin x}{1 + \frac{\cos x}{1+y}} = \frac{\sin x}{\frac{1+y+\cos x}{1+y}} = \frac{(1+y)\sin x}{1 + y + \cos x}$$

or

$$y\ (1 + y + \cos x) = (1 + y)\sin x$$

or

$$y + y^2 + y\cos x = (1 + y)\sin x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - (\sin x)\, y = \frac{dy}{dx}\sin x + (1 + y)\cos x$$

or

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1 + y)\cos x + y\sin x$$

or

$$\frac{dy}{dx} = \frac{(1 + y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}.$$

Example 1.64 Differentiate cos $y = x \cos(\alpha + y)$.

Solution The given equation can be written as

$$x = \frac{\cos y}{\cos(\alpha + y)}.$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{-\sin y \cos(\alpha + y) + \cos y \sin(\alpha + y)}{\cos^2(\alpha + y)} = \frac{\sin(\alpha + y - y)}{\cos^2(\alpha + y)} = \frac{\sin \alpha}{\cos^2(\alpha + y)}$$

or

$$\frac{dy}{dx} = \frac{\cos^2(\alpha + y)}{\sin \alpha}.$$

Example 1.65 If $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, prove that

$$x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}.$$

Solution The given equation can be written as $y = x^y$. Taking log both sides, we get, $\log y = y \log x$. Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\left(\frac{dy}{dx}\right)$$

or

$$\frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

or

$$x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}.$$

Example 1.66 If

$$y = x \log \frac{x}{a + bx},$$

find dy/dx.

Solution We get

$$\frac{y}{x} = \log \frac{x}{a + bx}$$

Differentiating with respect to x, we obtain

$$\frac{x(dy/dx)-y}{x^2}=\frac{a+bx}{x}\frac{a+bx-bx}{(a+bx)^2}=\frac{a}{x(a+bx)}$$

or

$$x\frac{dy}{dx}-y=\frac{ax}{a+bx}$$

or

$$\frac{dy}{dx}=\frac{1}{x}\left(y+\frac{ax}{a+bx}\right).$$

Example 1.67 If $y=(\log x)^{(\log x)^{\cdot^{\cdot^{\infty}}}}$, prove that

$$(x\log x)\frac{dy}{dx}=\frac{y^2}{1-y\log(\log x)}.$$

Solution Since

$$y=(\log x)^{(\log x)^{\cdot^{\cdot^{\infty}}}}=(\log x)^y$$

Taking log both sides, we get, $\log y = y\log(\log x)$. Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}\log(\log x)+\frac{y}{\log x}\frac{1}{x}$$

or

$$\frac{dy}{dx}\left[\frac{1}{y}-\log(\log x)\right]=\frac{y}{x\log x}$$

or

$$x\log x\frac{dy}{dx}=\frac{y^2}{1-y\log(\log x)}.$$

Example 1.68 Find dy/dx, if

$$y=(\sin x)^{\cos x}+(\cos x)^{\sin x}.$$

Solution Since $u=(\sin x)^{\cos x}$, taking log, we get $\log u=\cos x\log\sin x$. Differentiating, we get

$$\frac{1}{u}\frac{du}{dx}=-\sin x\log\sin x+\cos x\frac{1}{\sin x}\cos x$$

or

$$\frac{du}{dx}=u\left(\frac{\cos^2 x}{\sin x}-\sin x\log\sin x\right)=(\sin x)^{\cos x}(\cos x\cot x-\sin x\log\sin x) \quad (1)$$

Again, $v=(\cos x)^{\sin x}$. Taking log both sides, we get $\log v=\sin x\log\cos x$.

Differentiating, we have

$$\frac{1}{v}\frac{dv}{dx} = \cos x \log\cos x + \sin x \frac{1}{\cos x}(-\sin x)$$

or

$$\frac{dv}{dx} = v(\cos x \log\cos x - \sin x \tan x) = (\cos x)^{\sin x}(\cos x \log\cos x - \sin x \tan x) \quad (2)$$

Adding (1) and (2), we get

$$\frac{du}{dx} + \frac{dv}{dx} = \frac{dy}{dx} = (\sin)^{\cos x}(\cos x \cot x - \sin x \log\sin x)$$
$$+ (\cos x)^{\sin x}(\cos x \log\cos x - \sin x \tan x).$$

Example 1.69 Find dy/dx, when $(\tan x)^y + (\cot y)^x = 1$.

Solution Let $u = (\tan x)^y$ and $v = (\cot y)^x$. Then $u + v = 1$, and

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad (1)$$

Now, $u = (\tan x)^y$. Taking log, we get $\log u = y \log \tan x$. Its differentiation gives

$$\frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log\tan y + y\frac{1}{\tan x}\sec^2 x$$

or

$$\frac{du}{dx} = (\tan x)^y\left(\frac{dy}{dx}\log\tan y + y\cot x\sec^2 x\right) \quad (2)$$

Again $v = (\cot y)^x$, $\log v = x \log \cot y$. Differentiating, we have

$$\frac{1}{v}\frac{dv}{dx} = \log\cot y + x\frac{1}{\cot y}\left(-\operatorname{cosec}^2 y\right)\frac{dy}{dx}$$

or

$$\frac{dv}{dx} = (\cot y)^x\left(\log\cot y - x\tan y\operatorname{cosec}^2 y\frac{dy}{dx}\right) \quad (3)$$

Putting (2) and (3) in (1), we find

$$(\tan x)^y\left[\frac{dy}{dx}\log\ (\tan x) + y\cot x\sec^2 x\right] + (\cot y)^x\left(\log\cot y - x\tan y\operatorname{cosec}^2 y\frac{dy}{dx}\right)$$

or

$$\frac{dy}{dx}\left[(\tan x)^y\log(\tan x) - (\cot y)^x x\tan y\operatorname{cosec}^2 y\right]$$
$$+\left[(\tan x)^y\, y\cot x\sec^2 x + (\cot y)^x\log(\cot y)\right] = 0$$

or

$$\frac{dy}{dx} = -\frac{(\tan x)^y\ y\cot x\sec^2 x + (\cot y)^x\ \log(\cos y)}{(\tan x)^y\log(\tan x) - (\cot y)^x x\tan y\operatorname{cosec}^2 y}.$$

Exercises 1.7

1. Find the differential coefficient of the following:

(i) $y = x^x$ (ii) $y = x^y$ (iii) $y = (\sin x)^x$

(iv) $y = (\sin x)^{\cos x}$ (v) $y = (\sin x)^{\tan x}$ (vi) $y = (\sin x)^{\log x}$

(vii) $y = (\cos x)^{\log x}$ (viii) $y = (\cos x)^{\cos x}$

2. Find dy/dx of the following functions:

(i) $\left(1+\frac{1}{x}\right)^x$ (ii) $x + x^{1/x}$ (iii) $(1 + x)^x + x^{1+x}$

(iv) $x^x + x^{\sin x}$ (v) $(\sin x)^x + (\tan x)^x$ (vi) $(\sin x)^x + x^{\sin x}$

(vii) $(\tan x)^{\cot x} + (\cot x)^{\tan x}$ (viii) $x^x + e^{\tan x}$ (ix) $(\sin x)^{\cos^{-1} x}$.

3. Find dy/dx of the following functions:

(i) $x^y = y^x$ (ii) $x^{\tan y} = y^{\tan x}$ (iii) $(\sec x)^y = (\tan y)^x$

(iv) $x^{\sin y} = y^{\sin x}$ (v) $x^y + y^x = c$

(vi) $y = x^{\tan x} + (\tan x)^x$ (vii) $y = (\tan x)^{(\tan x)^{\tan x^{\cdot^{\cdot^{\infty}}}}}$

(viii) $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{(x)})^{\cdot^{\cdot^{\infty}}}}}$ (ix) $x = y^{y^{y^{\cdot^{\cdot^{\infty}}}}}$

(x) $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdot^{\cdot^{\infty}}}}}$.

4. If $e^{xy} = xy$, prove that

$$\frac{dy}{dx} = -\frac{y}{x}.$$

5. If $y = \sqrt{(1-x)(1+x)}$, prove that

$$(1-x^2)\frac{dy}{dx} + xy = 0.$$

6. If $y = x^{1/x}$, show that dy/dx vanishes when $x = e$.

7. Find the differential coefficient of the following functions:

(i) $y = x^{y^x}$ (ii) $y = e^{x^x}$ (iii) $y = x^{x^x}$ (iv) $y = x^{x^y}$.

8. Find dy/dx of the following functions:

(i) $(\cos x)^y = (\sin y)^x$ (ii) $x^y = e^{x-y}$ (iii) $y = x^{e^x}$ (iv) $y = e^{e^x}$.

9. Find dy/dx:

(i) $x^{\sin^{-1} x}$ (ii) $(x \log x)^{\log(\log x)}$ (iii) $(2x + 3)^{3x+5}$ (iv) $x^x\sqrt{x}$.

10. Find dy/dx at $x = 1$, when

$$(\sin y)\sin(\pi x/2) + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan[\log(x+2)] = 0$$

11. Find dy/dx when:

(i) $y=(\cos x)^{(\cos x)^{(\cos x)^{\cdot^{\cdot^{\cdot\infty}}}}}$ (ii) $y=a^{x^{a^{x^{a^{\cdot^{\cdot^{\cdot\infty}}}}}}}$.

12. Find dy/dx, when:

(i) $y=\sqrt{x+\sqrt{x+\sqrt{x+\cdots\infty}}}$ (ii) $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\cdots\infty}}}$

(iii) $y=e^{x+e^{x+e^{x+\cdot^{\cdot^{\cdot\infty}}}}}$ (iv) $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots\infty}}}$

(v) $y=(\tan x)^{(\tan x)^{(\tan x)^{\cdot^{\cdot^{\cdot\infty}}}}}$.

13. If $x^p y^q = (x+y)^{p+q}$, prove that

$$\frac{dy}{dx}=\frac{y}{x}.$$

14. If

$$y=e^{x^{e^x}}+x^{e^{e^x}}+e^{x^{x^e}},$$

find dy/dx.

15. Find the differential coefficient of the following functions:

(i) $\sin^3 x\cos^5 x$ (ii) $\frac{xe^x}{\cos x}$ (iii) $\frac{x}{e^x\sin x}$

(iv) $\frac{x\log x}{e^x\tan x}$ (v) $\sin^m(bx)\cos^n(bx)$ (vi) $x^2+x^{\log x}$

(vii) $x^x+(\log x)^x$ (viii) $10^7+x^{\sin x}$ (ix) $e^x\sin^3 x+(\tan x)^x$

(x) $\frac{10^x(\cot x^2)x^{1/3}}{\sin 2x}$ (xi) $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$ (xii) $\frac{1+x^2}{(1-x^2)^{1/3}}$.

16. If

$$y=a^x+\sqrt{\frac{1+x}{1-x}},$$

find dy/dx at $x=0$.

17. If

$$y=2^{\log_2 x^{2x}}-\left(\tan\frac{\pi x}{4}\right)^{4/(\pi x)}$$

find dy/dx at $x=1$.

18. Find dy/dx, when:

(i) $y = x^{\sin x - \cos x} + \dfrac{x^2 - 1}{x^2 + 1}$ (ii) $y = x^{\cot x} + \dfrac{2x^2 - 3}{x^2 + x + 2}$

(iii) $y = \dfrac{x^2\sqrt{4x+3}}{(3x+1)^2}$ (iv) $y = \dfrac{2(x - \sin x)^{3/2}}{\sqrt{x}}$

(v) $y = \dfrac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+3}}$.

19. Find dy/dx:

(i) $2^x + 2^y = 2^{x+y}$ (ii) $x^y + y^x = a^b$

(iii) $x^y = e^{x-y}$ (iv) $y \log (xy) = x$

(v) $e^{x+y} = xy$ (vi) $x = \exp\left[\tan^{-1}\left(\dfrac{y - x^2}{x^2}\right)\right]$

(vii) $(\sin x)e^x(\log x)^x$.

Chapter 2

Successive Differentiation

2.1 Introduction

It has been shown that the derivative of a function of x is also a functions of x. Thus the derivative of a function may have its derivative without any loss of generality.

If $y = f(x)$,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$$

is called the *first differential coefficient* or *first derivative* of $f(x)$. If the process of differentiation be continued in succession, we obtain, second-, third- and higher -order derivatives, as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\delta x \to 0} \frac{f'(x+\delta x) - f'(x)}{\delta x} = f''(x),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \lim_{\delta x \to 0} \frac{f''(x+\delta x) - f''(x)}{\delta x} = f'''(x)$$

and so on. They are also denoted by

$$y_1 = \frac{dy}{dx} = \mathrm{D}y, \quad y_2 = \frac{d^2y}{dx^2} = \mathrm{D}^2y, \ \ldots, \ y_n = \frac{d^ny}{dx^n} = \mathrm{D}^ny.$$

In successive differentiation, we obtain y_n by method of mathematical induction, for some standard functions, which are used as formulae. We also have Leibnitz's theorem to find y_n for the product of two functions, which will be discussed later in the chapter.

2.2 Successive Differentiation of Some Standard Functions

(a) Let $y = x^m$. Then

$$y_1 = mx^{m-1}$$

$$y_2 = m(m-1)x^{m-2}$$
$$y_3 = m(m-1)(m-2)x^{m-3}$$
$$\vdots$$
$$y_n = m(m-1)(m-2)\cdots(m-n+1)x^{m-n} \quad \text{(for } m > n\text{)}$$

Corollary If $m = n$ be a positive integer and $y = x^n$, then

$$y_n = n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

Also,

$$y_{n+1} = y_{n+2} = \cdots = y_{n+r} = 0,$$

where $n!$ is a constant.

In general,

$$y_{n-r} = n(n-1)(n-2)\cdots(r+1)x^r = \frac{n!}{r!}x^r.$$

(b) Let $y = (ax + b)^m$, where m is any number. Here

$$y_1 = ma(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$
$$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$$

Proceeding in this way, we get

$$y_n = m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}.$$

Corollary If $m = n$, $y_n = n!\,a^n$.

(c) Let $y = e^{ax}$. Here

$$y_1 = ae^{ax},\quad y_2 = a^2e^{ax},\quad y_3 = a^3e^{ax},\quad \ldots$$

Therefore,

$$y_n = a^n e^{ax}.$$

Corollary (i) Let $y = e^x$; then $y_n = e^x$. (ii) Let $y = a^x$ or $y = e^{x\log_e a}$; then $y_n = a^x(\log_e a)^n$.

(d) Let $y = 1/(x + a)$ or $y = (x + a)^{-1}$; then

$$y_1 = (-1)(x+a)^{-2}$$
$$y_2 = (-1)(-2)(x+a)^{-3} = (-1)^2 2!(x+a)^{-3}$$
$$y_3 = (-1)^3 3!(x+a)^{-4}, \text{ etc.}$$

Similarly,

$$y_n = \frac{(-1)^n n!}{(x+a)^{n+1}}.$$

Corollary In general, if

$$y = \frac{1}{(ax+b)^m},$$

where m is an integer greater than n, then

$$y_1 = \frac{-ma}{(ax+b)^{m+1}}$$

$$y_2 = \frac{(-1)m(m+1)a^2}{(ax+b)^{m+2}} = \frac{(-1)^2(m+1)!}{(m-1)!}\frac{a^2}{(ax+b)^{m+2}}$$

Similarly, we get

$$y_n = \frac{(-1)^n(m-n+1)!}{(m-1)!}\frac{a^n}{(ax+b)^{m+n}}$$

(e) Let $y = \log(x+a)$. Here

$$y_1 = \frac{1}{x+a},$$

$$y_2 = \frac{(-1)}{(x+a)^2},$$

$$y_3 = \frac{(-1)(-2)}{(x+a)^3} = \frac{(-1)^2 2!}{(x+a)^3}$$

Therefore,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}$$

Corollary If $y = \log(ax+b)$, then

$$y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

(f) (i) Let $y = \sin(ax+b)$. Then

$$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_2 = a^2\cos\left[\frac{\pi}{2} + (ax+b)\right] = a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right]$$

$$y_3 = a^3\cos\left[\frac{2\pi}{2} + (ax+b)\right] = a^3\sin\left[\frac{3\pi}{2} + (ax+b)\right]$$

Similarly, we get

$$y_n = a^n\sin\left[\frac{n\pi}{2} + (ax+b)\right]$$

(ii) Let $y = \cos(ax + b)$. Then

$$y_1 = -a\sin(ax+b) = a\cos\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_2 = -a^2\sin\left[\frac{\pi}{2} + (ax+b)\right] = a^2\cos\left[\frac{2\pi}{2} + (ax+b)\right]$$

$$y_3 = -a^3\sin\left[\frac{2\pi}{2} + (ax+b)\right] = a^3\cos\left[\frac{3\pi}{2} + (ax+b)\right]$$

Therefore,

$$y_n = a^n\cos\left[\frac{n\pi}{2} + (ax+b)\right]$$

Corollary When $b = 0$, if we obtain for $y = \sin ax$ and $y = \cos ax$

$$y_n = \sin\left(\frac{n\pi}{2} + ax\right) \quad \text{and} \quad y_n = \cos\left(\frac{n\pi}{2} + ax\right).$$

(g) If $y = e^{ax}\cos bx$, then

$$y_n = (a^2 + b^2)^{n/2}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right),$$

and when $y = e^{ax}\sin(bx)$, then

$$y_n = (a^2 + b^2)^{n/2}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

Let

$$u = e^{ax}\cos bx \quad \text{and} \quad v = e^{ax}\sin bx.$$

Then

$$u + iv = e^{ax}(\cos bx + i\sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x}.$$

Therefore,

$$u_n + iv_n = (a+ib)^n e^{(a+ib)x}$$

Putting $a = r\cos\theta$, $b = r\sin\theta$, we get

$$a^2 + b^2 = r^2 \quad \text{and} \quad \tan\theta = \frac{b}{a}.$$

Then

$$(a+ib)^n = r^n(\cos\theta + i\sin\theta)^n = r^n e^{in\theta}.$$

Therefore,

$$\begin{aligned} u_n + iv_n &= r^n e^{in\theta}e^{(a+ib)x} \\ &= r^n e^{ax}e^{i(bx+n\theta)} \\ &= r^n e^{ax}[\cos(bx+n\theta) + i\sin(bx+n\theta)], \end{aligned}$$

where

$$r=\sqrt{a^2+b^2} \quad \text{and} \quad \theta=\tan^{-1}\frac{b}{a}$$

Hence equating real and imaginary parts, we have

$$u_n = \mathrm{D}^n(e^{ax}\cos bx) = (a^2+b^2)^{n/2}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)$$

and

$$v_n = \mathrm{D}^n(e^{ax}\sin bx) = (a^2+b^2)^{n/2}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

2.3 Rational Algebraic Functions

Whenever possible, the nth derivative of any rational algebraic function is generally obtained with the help of partial fraction. However, at times, the final result may involve simplification through complex variables.

Consider the following examples:

(a) If

$$y=\frac{1}{(x+a)(x-a)}=\frac{1}{b-a}\left(\frac{1}{x+a}-\frac{1}{x+b}\right),$$

Then

$$y_n=\frac{1}{b-a}\frac{(-1)^n n!}{1}\left[\frac{1}{(x+a)^{n+1}}-\frac{1}{(x+b)^{n+1}}\right].$$

(b) If

$$y=\frac{1}{x^2+a^2},$$

Then

$$y_n=\frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta,$$

where

$$\theta=\tan^{-1}\left(\frac{a}{x}\right).$$

To prove, let

$$y=\frac{1}{x^2+a^2}=\frac{1}{(x+ia)(x-ia)}=\frac{1}{2ia}\left(\frac{1}{x-ia}-\frac{1}{x+ia}\right)$$

Then

$$y_n=\frac{(-1)^n n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]$$

$$=\frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)}-(x+ia)^{-(n+1)}\right] \qquad (2.1)$$

Putting $x = r\cos\theta$, $a = r\sin\theta$ gives

$$x^2 + a^2 = r^2 \quad \text{and} \quad \tan\theta = \frac{a}{x}. \tag{2.2}$$

Then

$$\begin{aligned}(x - ia)^{-(n+1)} - (x + ia)^{-(n+1)} &= r^{-(n+1)}[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}] \\ &= r^{-(n+1)}[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta \\ &\qquad + i\sin(n+1)\theta] \\ &= r^{-(n+1)}[2i\sin(n+1)\theta]\end{aligned}$$

From Eq. (2.1), we obtain

$$y_n = \frac{(-1)^n n!}{2ia} r^{-(n+1)}[2i\sin(n+1)\theta] = \frac{(-1)^n n!}{a}\left(\frac{\sin\theta}{\theta}\right)^{n+1} \sin(n+1)\theta.$$

Since

$$\frac{1}{r} = r^{-1} = \frac{\sin\theta}{a}, \qquad \text{where} \quad \theta = \tan^{-1}\frac{x}{a},$$

We finally have

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta, \qquad \text{where } \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

Corollary If

$$y = \frac{1}{(x+b)^2 + a^2}$$

Then

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta, \qquad \text{where} \quad \theta = \tan^{-1}\left(\frac{a}{x+b}\right)$$

Corollary If $y = \tan^{-1}x$, then $y_1 = 1/(1+x^2)$. Here putting $a = 1$, $\theta = \tan^{-1}(1/x) = \cot^{-1}x$. Therefore, we get

$$y_n = (-1)^{n-1}(n-1)!\,\sin^n\theta \sin n\theta.$$

Example 2.1 If $y = x^{2n}$, where n is a positive integer, show that

$$y_n = 2^n[1\cdot 3\cdot 5\cdot 7\cdots(2n-1)]\,x^n$$

Solution Given that $y = x^{2n}$, then

$$y_1 = 2nx^{2n-1},$$

$$y_2 = 2n(2n-1)x^{2n-2},$$
$$y_3 = 2n(2n-1)(2n-2)x^{2n-3}, \text{ etc.}$$

Therefore,

$$y_n = 2n(2n-1)(2n-2)\cdots[2n-(n-1)]x^{2n-n}$$
$$= 2n(2n-1)(2n-2)\cdots(n+1)x^n$$
$$= \frac{2n(2n-1)(2n-2)\cdots(n+1)x^n}{n!}n!$$
$$= \frac{(2\cdot4\cdot6\cdot8\cdots2n)[1\cdot3\cdot5\cdot7\cdots(2n-1)]}{n!}x^n$$
$$= \frac{2^n(1\cdot2\cdot3\cdot4\cdots n)[1\cdot3\cdot5\cdot7\cdots(2n-1)]}{n!}x^n$$
$$= \frac{2^n n![1\cdot3\cdot5\cdot7\cdots(2n-1)]}{n!}x^n$$
$$= 2^n[1\cdot3\cdot5\cdot7\cdots(2n-1)]x^n.$$

Example 2.2 Find y_n where

$$y = \frac{1}{6x^2+11x+3}.$$

Solution Here

$$y = \frac{1}{6x^2+11x+3} = \frac{1}{(3x+1)(2x+3)} = \frac{A}{3x+1} + \frac{B}{2x+3},$$

where A and B are arbitrary constants. Now, multiplying both sides by $(3x + 1)(2x + 3)$, we find

$$1 = A(2x+3) + B(3x+1) = (2A+3B)x + (3A+B)$$

Equating constant terms and equal power of x, both sides, we get

$$2A + 3B = 0 \quad \text{and} \quad 3A + B = 1.$$

Solving them, we obtain $A = 3/7, B = -2/7$. Then

$$y = \frac{3}{7}\frac{1}{3x+1} - \frac{2}{7}\frac{1}{2x+3}$$

Differentiating n times, we get

$$y_n = \frac{3}{7}\frac{(-1)^n 3^n n!}{(3x+1)^{n+1}} - \frac{2}{7}\frac{(-1)2^n n!}{(2x+3)^{n+1}} = \frac{(-1)^n n!}{7}\left[\frac{3^{n+1}}{(3x+1)^{n+1}} - \frac{2^{n+1}}{(2x+3)^{n+1}}\right].$$

Example 2.3 If $ax^2 + 2hxy + by^2 = 1$, show that

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}.$$

Solution Here differentiation of $ax^2 + 2hxy + by^2 = 1$ gives

$$2ax + 2h\left(y + x\frac{dy}{dx}\right) + 2hy\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx}(hx + by) + (ax + hy) = 0$$

or

$$-\frac{dy}{dx} = \frac{ax + hy}{hx + by}$$

Differentiating again with respect to x, we get

$$-\frac{d^2y}{dx^2} = \frac{ahy_1(hx + by) - (h + by_1)(ax + hy)}{(hx + by)^2}$$

$$= \frac{\left(a - h\dfrac{ax + by}{hx + by}\right)(hx + by) - \left(h - b\dfrac{ax + by}{hx + by}\right)(ax + hy)}{(hx + by)^2}$$

$$= \frac{[a(hx + by) - h(ax + hy)](hx + by) - [h(hx + by) - b(ax + hy)](ax + hy)}{(hx + by)^2}$$

$$= \frac{y(ab - h^2)(hx + by) - x(h^2 - ab)(ax + hy)}{(hx + by)^3}$$

$$= \frac{(ab - h^2)(ax^2 + 2hxy + by^2)}{(hx + by)^3}$$

$$= -\frac{h^2 - ab}{(hx + by)^3}$$

Hence,

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$$

Example 2.4 If $y = 1/(a - x)$, prove that

$$y_n = \frac{n!}{(a - x)^{n+1}}$$

Solution For given

$$y = \frac{1}{a-x} = (a-x)^{-1}$$

On successive differentiation with respect to x, we get

$$y_1 = (-1)(a-x)^{-2}(-1) = (a-x)^{-2} = \frac{1!}{(a-x)^2}$$

$$y_2 = 1!2(a-x)^{-2} = 2!(a-x)^{-2} = \frac{2!}{(a-x)^3}$$

$$y_3 = 2!3(a-x)^{-4} = 3!(a-x)^{-4} = \frac{3!}{(a-x)^4}$$

$$\vdots$$

$$y_n = \frac{n!}{(a-x)^{n+1}}.$$

Example 2.5 If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, then show that

$$\frac{d^2p}{d\theta^2} + p = \frac{a^2b^2}{p^3}.$$

Solution Differetiating

$$p^2 = a^2\cos^2\theta + b^2\sin^2\theta, \tag{1}$$

with respect to θ, we get

$$2p\frac{dp}{d\theta} = -a^2 \sin 2\theta + b^2 \sin 2\theta = (b^2 - a^2) \sin 2\theta \tag{2}$$

Differentiating again, we obtain

$$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)(\cos^2\theta - \sin^2\theta) = b^2\cos^2\theta + a^2\sin^2\theta - p^2$$

or

$$p\frac{d^2p}{d\theta^2} + p^2 = b^2\cos^2\theta + a^2\sin^2\theta - \left(\frac{dp}{d\theta}\right)^2$$

From (2), we have

$$p\frac{d^2p}{d\theta^2} + p^2 = b^2\cos^2\theta + a^2\sin^2\theta - \frac{(b^2 - a^2)^2 \sin^2\theta\cos^2\theta}{p}$$

$$= p^{-2}[(a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$-(b^2 - a^2)\sin^2\theta\cos^2\theta]$$

$$= p^{-2}a^2b^2(\cos^2\theta + \sin^2\theta)^2$$

$$= p^{-2}a^2b^2$$

or

$$\frac{d^2p}{d\theta^2} + p = \frac{a^2b^2}{p^3}.$$

Example 2.6 If

$$y = x\log\frac{x-1}{x+1},$$

find y_{n-1}.

Solution We have $y = x[\log(x-1) + \log(x+1)]$. Then

$$y_1 = \log(x-1) - \log(x+1) - x\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$= \log(n-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$$

Therefore,

$$y_{n-1} = (-1)^{n-2}\left[\frac{(n-2)!}{(x-1)^{n-1}} - \frac{(n-2)!}{(x+1)^{n-1}} - \frac{(n-1)!}{(x-1)^n} - \frac{(n-1)!}{(x+1)^n}\right]$$

$$= (-1)^{n-2}\left\{\frac{(n-2)!}{(x-1)^n}[(x-1)-(n-1)] - \frac{(n-2)!}{(x+1)^n}[(x+1)+(n-1)]\right\}$$

$$= \frac{(-1)^n}{(-1)^2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

Example 2.7 If

$$y = \frac{1}{x^2+x+1},$$

find y_n.

Solution Given

$$y = \frac{1}{x^2+x+1}$$

$$= \frac{1}{(x+1/2)^2 + 3/4}$$

$$= \frac{1}{\left(x+1/2+i\sqrt{3}/2\right)\left(x+1/2-i\sqrt{3}/2\right)}$$

$$= \frac{2}{i\sqrt{3}}\left[\frac{1}{(2x+1)-i\sqrt{3}} - \frac{1}{(2x+1)+i\sqrt{3}}\right]$$

We get

$$y_n = \frac{2}{i\sqrt{3}}(-1)^n n!\,2^n \left\{\left[(2x+1) - i\sqrt{3}\right]^{-(n+1)} - \left[(2x+1) + i\sqrt{3}\right]^{-(n+1)}\right\}$$

Putting $2x + 1 = 2r\cos\theta$, $\sqrt{3} = 2r\sin\theta$, and applying DeMoivre's theorem, we get

$$4r^2 = (2x+1)^2 + (\sqrt{3})^2 = 4x^2 + 4x + 1 + 3 = 4(x^2 + x + 1)$$

Solving, we get

$$r = (x^2 + x + 1)^{1/2} \quad \text{and} \quad \tan\theta = \frac{\sqrt{3}}{2x+1}.$$

Now,

$$\begin{aligned}
&\left[(2x+1) - \sqrt{3}i\right]^{-(n+1)} - \left[(2x+1) + i\sqrt{3}\right]^{-(n+1)} \\
&= r^{-(n+1)} 2^{-(n+1)} [(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}] \\
&= r^{-(n+1)} 2^{-(n+1)} [\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta] \\
&= r^{-(n+1)} 2^{-(n+1)} 2i\sin(n+1)\theta \\
&= 2^{-n} r^{-(n+1)} i\sin(n+1)\theta.
\end{aligned}$$

Therefore,

$$y_n = \frac{2}{i\sqrt{3}} \frac{(-1)^n n!\,2^n 2^{-n}}{r^{n+1}} i\sin(n+1)\theta.$$

$$= \frac{(-1)^n n!\,2}{r^{n+1}} \sin(n+1)\theta$$

$$= \frac{(-1)^n n!\,2}{(x^2 + x + 1)^{(n+1)/2}} \sin(n+1)\theta, \qquad \text{where } \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)$$

Example 2.8 If

$$y = \frac{ax + b}{a + bx},$$

show that $2y_1y_3 = 3y_2^2$.

Solution Here

$$y = \frac{a}{b} + \left(b - \frac{a^2}{b}\right)\frac{1}{bx + a}$$

Then, we have

$$y_1 = 0 + \left(b - \frac{a^2}{b}\right)\left[\frac{-b}{(bx+a)^2}\right] = \frac{-(b^2 - a^2)}{(bx+a)^2}$$

$$y_2 = \frac{2b(b^2 - a^2)}{(bx+a)^3}$$

$$y_3 = \frac{-6b^2(b^2 - a^2)}{(bx+a)^4}$$

Now

$$2y_1y_3 = 2\frac{-(b^2-a^2)}{(bx+a)^2}\frac{-6b^2(b^2-a^2)}{(bx+a)^4} = 3\left[\frac{2b(b^2-a^2)}{(bx+a)^2}\right]^2 = 3y_2^2$$

Example 2.9 If $\sqrt{(x+y)} + \sqrt{(x-y)} = c$, show that $y_2 = 2/c^2$.

Solution Here $\sqrt{(x+y)} + \sqrt{(x-y)} = c$. Squaring both sides, we get

$$x + y + y - x + 2\sqrt{y^2 - x^2} = c^2$$

or

$$2\sqrt{y^2 - x^2} = c - 2y.$$

Again squaring both sides, we have

$$4(y^2 - x^2) = c^2 - 4cy + 4y^2 \quad \text{or} \quad 4x^2 - 4c^2y + c^4 = 0.$$

Differentiating, we have

$$8x - 4c^2y_1 = 0.$$

Differentiating again, we get

$$2 - c^2y_2 = 0.$$

Hence $y_2 = 2/c^2$.

Example 2.10 Prove that

$$\frac{d^3x}{dy^3} = -\frac{\dfrac{d^3y}{dx^3}\dfrac{dy}{dx} - 3\left(\dfrac{d^2y}{dx^2}\right)^2}{\left(\dfrac{dy}{dx}\right)^5}$$

Solution We have

$$\frac{dx}{dy}=\frac{1}{dy/dx}$$

or

$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{1}{dy/dx}\right)$$

$$=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d^2y}{dx^2}\frac{1}{\frac{dy}{dx}}$$

$$=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

Again,

$$\frac{d^3x}{dy^3}=-\frac{d}{dy}\left[\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}\right]=-\frac{d}{dx}\left[\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}\right]\frac{dx}{dy}$$

$$=-\frac{\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}-3\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^6\frac{dy}{dx}}=-\frac{\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^5}.$$

Example 2.11 If

$$y=\log\left(\frac{x}{a+bx}\right)^x,$$

prove that $x^3y_2=(y-xy_1)^2$.

Solution Here

$$y=\log\left(\frac{x}{a+bx}\right)^x=x\log\frac{x}{a+bx}$$

Then

$$\frac{y}{x} = \log\frac{x}{a+bx} \quad \text{or} \quad e^{y/x} = \frac{x}{a+bx}. \tag{1}$$

Differentiating with respect to x, we get

$$e^{y/x}\frac{y_1x - y}{x^2} = \frac{a}{(a+bx)^2}$$

From (1), we have

$$e^{y/x}(xy_1 - y) = \frac{ax^2}{(a+bx)^2} = ae^{y/x} \quad \text{or} \quad e^{-y/x}(xy_1 - y) = a$$

Differentiating again with respect to x, we obtain

$$e^{-y/x}\frac{-y_1x + y}{x^2}(y_1x - y) + e^{-y/x}[(y_2x + y_1) - y_1] = 0$$

or

$$(y - y_1x)(y_1x - y) + x^2xy_2 = 0$$

or

$$x^3y_2 = (y - xy_1)^2$$

Example 2.12 If

$$x = \cosh\left(\frac{\log y}{m}\right),$$

prove that $(x^2 - 1)y_2 + xy_1 - m^2y = 0$.

Solution We have

$$x = \cosh\left(\frac{\log y}{m}\right) \quad \text{or} \quad \cosh^{-1}x = \frac{1}{m}\log y$$

Differentiating both sides, we get

$$\frac{1}{\sqrt{x^2-1}} = \frac{1}{my}y_1 \quad \text{or} \quad y_1\sqrt{x^2-1} = my.$$

Again differentiating both sides with respect to x, we get

$$y_2\sqrt{x^2-1} + y_1\frac{1}{2}\frac{1}{\sqrt{x^2-1}}2x = my_1$$

or

$$(x^2-1)y_2+xy_1=my_1\sqrt{x^2-1}$$

or

$$(x^2-1)y_2+xy_1=m^2y$$

Hence

$$(x^2-1)y_2+xy_1-m^2y=0.$$

Example 2.13 If

$$y=A\left(x+\sqrt{x^2-1}\right)^n+B\left(x-\sqrt{x^2-1}\right)^n,$$

prove that $(x^2-1)y_2+xy_1-n^2y=0.$

Solution We have

$$y=A\left(x+\sqrt{x^2-1}\right)^n+B\left(x-\sqrt{x^2-1}\right)^n$$

Differentiating both sides with respect to x, we get

$$y_1=nA\left(x+\sqrt{x^2-1}\right)^{n-1}\left(1+\frac{1}{2\sqrt{x^2-1}}2x\right)+nB\left(x-\sqrt{x^2-1}\right)^{n-1}\left(1-\frac{2x}{2\sqrt{x^2-1}}\right)$$

$$=nA\left(x+\sqrt{x^2-1}\right)^{n-1}\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}-nB\left(x-\sqrt{x^2-1}\right)^{n-1}\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

or

$$\sqrt{x^2-1}\,y_1=nA\left(x+\sqrt{x^2-1}\right)^n-nB\left(x-\sqrt{x^2-1}\right)^n.$$

Again differentiating both sides with respect to x, we obtain

$$\sqrt{x^2-1}\,y_2+\frac{1}{2\sqrt{x^2-1}}2xy_1=n^2A\left(x+\sqrt{x^2-1}\right)^{n-1}\left(1+\frac{1}{2\sqrt{x^2-1}}2x\right)$$

$$-n^2B\left(x-\sqrt{x^2-1}\right)^{n-1}\left(1-\frac{1}{2\sqrt{x^2-1}}2x\right).$$

Therefore,

$$(x^2-1)y_2+xy_1=n^2A\left(x+\sqrt{x^2-1}\right)^n+n^2B\left(x-\sqrt{x^2-1}\right)^n,$$

or

$$(x^2-1)y_2+xy_1=n^2\left[A\left(x+\sqrt{x^2-1}\right)^n+B\left(x-\sqrt{x^2-1}\right)^n\right].$$

Hence

$$(x^2-1)y_2+xy_1-n^2y=0.$$

Example 2.14 If $y=e^{a\sin^{-1}x}$, prove that $(1-x^2)y_2-xy_1-a^2y=0$.

Solution We have $y=e^{a\sin^{-1}x}$. Then

$$y_1=e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}} \quad \text{or} \quad y_1\sqrt{1-x^2}=ae^{a\sin^{-1}x}=ay$$

Therefore,

$$y_1^2(1-x^2)=a^2y^2$$

Differentiating, we get

$$2y_1y_2(1-x^2)+(-2x)y_1^2=a^2(2yy_1)$$

or

$$(1-x^2)y_2-xy_1-a^2y=0.$$

Example 2.15 If $y = \sin(m \sin^{-1}x)$, prove that

$$(1-x^2)y_2-xy_1+m^2y=0.$$

Solution We have, $y=\sin(m\sin^{-1}x)$. Then $\sin^{-1}y=m\sin^{-1}x$. Differentiating with respect to x, we get

$$\frac{1}{\sqrt{1-y^2}}y_1=m\frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad y_1\sqrt{1-x^2}=m\sqrt{1-y^2}$$

Squaring both the sides, we have

$$y_1^2(1-x^2)=m^2(1-y^2)$$

Differentiating again with respect to x, we get

$$2y_1y_2(1-x^2)+y_1^2(-2x)=m^2(-2yy_1)$$

or

$$2y_1[y_2(1-x^2)-y_1x]=2y_1(-m^2y)$$

or

$$y_2(1-x^2)-y_1x=-m^2y$$

Therefore,

$$y_2(1-x^2)-xy_1+m^2y=0.$$

Example 2.16 If $y^{1/m}+y^{-1/m}=2x$, prove that $(x^2-1)y_2+xy_1-m^2y=0.$

Solution Dividing $y^{1/m}+y^{-1/m}=2x$ by $y^{-1/m}$, we get

$$y^{2/m}-2xy^{1/m}+1=0.$$

Solving it by quadratic equation, we have

$$y^{1/m}=\frac{1}{2}\left(2x\pm\sqrt{4x^2-4}\right) \quad \text{or} \quad y=\left(x\pm\sqrt{x^2-1}\right)^m \tag{1}$$

Differentiating, we get

$$y_1=m\left(x\pm\sqrt{x^2-1}\right)^{m-1}\left(1\pm\frac{2x}{2\sqrt{x^2-1}}\right)$$

$$=\frac{m\left(x\pm\sqrt{x^2-1}\right)^{m-1}\left(\sqrt{x^2-1}\pm x\right)}{\sqrt{x^2-1}}$$

$$=\frac{\pm m\left(x\pm\sqrt{x^2-1}\right)^{m}}{\sqrt{x^2-1}}$$

From (1),

$$y_1\sqrt{x^2-1}=\pm my \quad \text{or} \quad (x^2-1)y_1^2=m^2y^2.$$

Differentiating again, we get

$$(2x-1)y_1^2+(x^2-1)\,2y_1y_2=m^2(2yy_1)$$

or

$$(x^2-1)y_2+xy_1-m^2y=0\,.$$

Example 2.17 If $y=Ae^{-ax}\cos(bx+c)$, prove that $y_2+2ay_1+(a^2+b^2)y=0.$

Solution We have $y=Ae^{-ax}\cos(bx+c)$

Differentiating with respect to x, we get

$$y_1=-Aae^{-ax}\cos(bx+c)-Ae^{-ax}b\sin(bx+c)$$

$$=-ay-Abe^{-ax}\sin(bx+c).$$

Again, differentiating

$$y_2 = -ay_1 + Aabe^{-ax}\sin(bx+c) - Ab^2e^{-ax}\cos(bx+c)$$

or

$$y_2 + ay_1 = a(-ay - y_1) - b^2 y$$

or

$$y_2 + 2ay_1 + (a^2 + b^2)y = 0.$$

Exercises 2.1

1. Find the nth derivatives of the following functions:

(i) $\dfrac{1}{a-x}$ (ii) $\dfrac{x}{(x-a)(x-b)}$, (iii) $\dfrac{x^2}{x-1}$

(iv) $\dfrac{x^2}{x^2-1}$ (v) $\dfrac{1}{3x^2-11x+6}$ (vi) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$.

2. Find y_n in the following functions:

(i) $(ax+b)^m$ (ii) $e^{ax}\cos bx$ (iii) $e^{3x}\sin 4x$

(iv) $e^{ax}\cos(bx+c)$ (v) $x^2\cos x$ (vi) $x^2\log x$

(vii) x^2e^{ax} (viii) $x^{n-1}\log x$ (ix) $x\sin 4x$

(x) $x^2\sin^2 x$ (xi) $x^3\cos x$ (xii) $x^n\log x$

(xiii) $e^{ax}[a^2x^2 - 2nax + n(n+1)]$.

3. Find d^ny/dx^n of the following:

(i) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (ii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(iii) $y = \tan^{-1}x$ (iv) $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

(v) $y = \sin^{-1}x$.

4. (i) If $y = a\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.
(ii) $y = a\cos(\log x)$, prove that $x^2y^2 + xy_1 + y = 0$.

5. If $y = \cos(m\sin^{-1}x)$, prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$.

6. If $y = \sin(\log y)$, prove that $(1-x^2)\,y_2 - xy_1 - y = 0$.

7. If $y = e^{\cos^{-1}x}$, prove that $(1-x^2)^2 - xy_1 - y = 0$.

8. If $y = e^{a\tan^{-1}x}$, prove that $(1+x^2)y_2 + 2xy_1 - ay_1 = 0$.

9. If $y = [x + \sqrt{(x^2-1)}]^m$, prove that $(x^2-1)y_2 + xy_1 - m^2y = 0$.

10. If $y^{1/m} - y^{-1/m} = 2x$, prove that $(x^2 + 1)y_2 + xy_1 = m^2y$.

11. If

$$x = \cos h\left(\frac{\log y}{m}\right),$$

prove that $(x^2 - 1)y_2 + xy_1 - m^2y = 0$.

12. If

$$x = \sin h\left(\frac{\log y}{m}\right),$$

prove that $(x^2 + 1)y_2 + xy_1 - m^2y = 0$.

13. If $y = e^{ax} \cos bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

14. If $y = (a + bx) \cos mx + (c + dx) \sin mx$, prove that $y_4 + 2m^2y_2 + m^4y = 0$.

15. If $y = (a + bx)e^{-nx}$, prove that

$$\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = 0.$$

16. If $x = \cos \log y$, prove that $(1 - x^2)y_2 - xy_1 - y = 0$.

17. If $$y = \log\frac{x}{a + bx},$$

prove that $x^3y_2 = (y - xy)^2$.

18. If $y = A \sin mx + B \cos mx$, prove that $y_2 = -m^2y$.

19. If $y = Ae^{mx} + Be^{-mx}$, prove that $y_2 = m^2y$.

20. If $p = a^2\cos^2\theta + b^2 \sin^2\theta$, show that

$$p + \frac{d^2p}{d\theta^2} = 2a^2 + 2b^2 - 3p.$$

21. If $y = (\tan^{-1}x)^2$, show that $(x^2 + 1)^2y_2 + 2x\,(x^2 + 1)y_1 = 2$.

22. If

$$y = \sin h\left(\frac{1}{m}\log y\right)$$

show that $(x^2 + 1)\, y_2 + xy_1 = m^2y$.

23. If $y = (a \cos x + b \sin x)e^{-mx}$, show that $y_2 + 2my_1 + (m^2 + 1)y = 0$.

24. If $y = (x^2 - 1)^2$, show that $(x^2 - 1)y_2 + 2xy_1(1 - n) - 2ny = 0$.

25. If $y = x \sin x$, show that,

$$y_n = x\sin\left(x + \frac{n\pi}{2}\right) - n\cos\left(x + \frac{n\pi}{2}\right).$$

26. Find y_2, when

$$y = \cos^{-1}\frac{x}{b}\log\left(\frac{x}{n}\right)^n,$$

27. Find y_n, when

$$y = x\log\frac{x-1}{x+1}.$$

28. Find y_n, when

(i) $y = \tan^{-1}\left(\frac{a+x}{a-x}\right) + \left(\frac{1}{x^4-a^4} + \frac{1}{1+x+x^2+x^3}\right)$

(ii) $y = x\left(a^2+x^2\right)^{-1}$.

29. Find y_n, when

$$y = \tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)$$

30. If $y = e^x \sin^4 x$, find y_n.

31. If $y = \sin^2 x \cos^3 x$, find y_n.

2.4 Leibnitz's Theorem

If $u = \phi(x)$ and $v = \psi(x)$ be a function of x, then the nth derivatives of the product $y = uv$ is denoted as $y_n = (uv)_n$. It is easily obtained from Leibnitz's theorem, which states as follows:

Theorem 2.1 (Leibnitz's Theorem) If u and v are two functions, then the nth derivative of their product $y = uv$ is given by

$$(uv)_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + {}^nC_3 u_{n-3}v_3 + \cdots + {}^nC_{n-1}u_1 v_{n-1} + uv_n,$$

where the suffices with u and v denote the order of differentiations of u and v with respect to x.

Proof Let $y = uv$. Then by actual differentiation, we get

$$y_1 = u_1 v + u v_1$$

$$\begin{aligned} y_2 &= (u_2 v + u_1 v_1) + (u_1 v_1 + u v_2) \\ &= u_2 v + 2u_1 v_1 + u v_2 \\ &= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2 \end{aligned}$$

$$\begin{aligned} y_3 &= (u_3 v + u_2 v_1) + 2(u_2 v_1 + u_1 v_2) + (u_1 v_2 + u v_3) \\ &= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3 \\ &= u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 u v_3. \end{aligned}$$

The theorem also holds good for $n = 2, 3, \ldots$ Let us assume that the theorem is true for $n = m$. Thus

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_{m-1} u_1 v_{m-1} + u v_m.$$

Differentiating both sides, we get

$$\begin{aligned} y_{m+1} &= (u_{m+1} v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1} v_2) + {}^mC_2(u_{m-1} v_2 + u_{m-2} v_3) + \cdots \\ &\quad + {}^mC_{m-1}(u_2 v_{m-1} + u_1 v_m) + (u_1 v_m + u v_{m+1}) \\ &= u_{m+1} v + (1 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2 + \cdots \\ &\quad + ({}^mC_{m-1} + {}^mC_m) u_1 v_m + {}^{m+1}C_{m+1} u v_{m+1}. \end{aligned}$$

But from the binomial coefficients, for all r, we have

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r \qquad \text{and} \qquad {}^mC_m = {}^{m+1}C_{m+1} = 1$$

Therefore,

$$\begin{aligned} y_{m+1} = (uv)_{m+1} &= u_{m+1} v + {}^{m+1}C_1 + u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \cdots \\ &\quad + {}^{m+1}C_m u_1 v_m + u v_{m+1}. \end{aligned}$$

This means that the theorem also holds good for $n = m + 1$. But it is true for $n = 2, 3$ and hence it is true for $n = 4$, and so on. Thus the theorem is true for any positive integral value of n. Then

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_{n-1} u_1 v_{n-1} + u v_n.$$

Note: Using D for d/dx, we have

$$(uv)_n = D^n(uv) = \frac{d^n}{dx^n}(uv).$$

Also,

$${}^nC_1 = n_n, \quad {}^nC_2 = \frac{n(n-1)}{2!}, \text{ etc.}$$

For example, if $y = e^{ax}x^2$, then we can find y_n by putting

$$u = e^{ax} \quad \text{and} \quad v = x^2.$$

Therefore,

$$u_n = a^n e^{ax} \quad \text{and} \quad v_1 = 2x,\ v_2 = 2,\ v_3 = v_4 = \cdots = 0.$$

Hence

$$\begin{aligned} y_n = (uv)_n &= u_n v + n u_{n-1} v_1 + \frac{1}{2} n(n-1)\, u_{n-2} v_2 \\ &= \left[a^n x^2 + n a^{n-1} 2x + \frac{1}{2} n(n-1) a^{n-2}(2)\right] e^{ax} \\ &= [a^2 x^2 + 2anx + n(n-1)] a^{n-2} e^{ax} \end{aligned}$$

Example 2.18 If $y = a \cos(\log x) + b \sin(\log x)$, prove that

$$x^2y_2 + xy_1 + y = 0$$

and

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n$$

Solution We have

$$y = a \cos(\log x) + b \sin(\log x)$$

Differentiating, we get

$$y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

or

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Again differentiating, we have

$$y_1 + xy_2 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

or

$$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)] = -y$$

Therefore,

$$x^2y_2 + xy_1 + y = 0.$$

Now differentiating it n times by Leibnitz's theorem, we get

$$x^2y_{n+2} + {}^nC_1y_{n-1}\,2x + {}^nC_2y_n(2) + y_{n+1}x + {}^nC_1y_n\,(1) + y_n = 0$$

or

$$x^2y_{n+2} + 2nxy_{n-1} + \frac{n(n-1)}{2}2y_n + xy_{n+1} + ny_n + y_n = 0$$

Therefore,

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Example 2.19 If $y = \sin mx + \cos mx$, prove that

$$y_n = m^n\left[1 + (-1)^n \sin(2mx)\right]^{1/2}$$

Solution Differentiating

$$y = \sin mx + \cos mx$$

n times with respect to x, we get

$$y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$$

$$= m^n \left\{\left[\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right]^2\right\}^{1/2}$$

$$= m^n \left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= m^n \left[1 + \sin\,(2mx + n\pi)\right]^{1/2}$$

$$= m^n \left(1 \pm \sin\, 2mx\right)^{1/2}$$

Hence

$$y_n = m^n \left[1 + (-1)^n \sin\, 2mx\right]^{1/2}$$

according as n is even or odd.

Example 2.20 If

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2 - 1)^n\right],$$

prove that $P'_{n+1} = xP'_n + (n+1)P_n$.

Solution Here

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2 - 1)^n\right]$$

putting $n = n + 1$, we get

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}\left[(x^2 - 1)^{n+1}\right]$$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\left\{\frac{d}{dx}\left[(x^2 - 1)^{n+1}\right]\right\}$$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\left[(n+1)(x^2 - 1)^n (2x)\right]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[x(x^2 - 1)^n\right]$$

By Leibnitz's theorem

$$P_{n+1} = \frac{1}{2^n n!}\left[x\frac{d^n}{dx^n}\left(x^2-1\right)^n + {}^nC_1\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]$$

or

$$P_{n+1}(x) = x\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n + \frac{n}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n$$

$$= xP_n + \frac{n}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n$$

Then

$$P'_{n+1}(x) = P_n + xP'_n \; + \frac{n}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = P'_{n+1}(x) = xP'_n + (n+1)P_n \cdot$$

Example 2.21 If

$$u = x^n + \frac{1}{x^n}, \quad v = x + \frac{1}{x},$$

show that

$$(v^2-4)\frac{d^2u}{dv^2} + v\frac{du}{dv} - n^2 v = 0.$$

Solution Since

$$v = x + \frac{1}{x}, \quad \frac{dv}{dx} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

and

$$\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx} = n\left(x^{n-1} - \frac{1}{x^{n+1}}\right)\frac{x^2}{x^2-1} = \frac{nx}{x^2-1}\left(x^n - \frac{1}{x^n}\right)$$

Then

$$\frac{d^2u}{dv^2} = \frac{dx}{dv}\frac{d}{dx}\left[\frac{nx}{x^2-1}\left(x^n - \frac{1}{x^n}\right)\right]$$

$$= \frac{nx^2}{x^2-1}\left[\frac{(x^2-1)-2x^2}{(x^2-1)^2}\left(x^n - \frac{1}{x^n}\right) + \frac{nx}{x^2-1}\left(x^{n-1} + \frac{1}{x^{n-1}}\right)\right]$$

$$= -\frac{(x^2+1)}{x^2-1}x\frac{du}{dv} + \frac{n^2x^2}{(x^2-1)^2}u$$

$$= \frac{x^2}{(x^2-1)^2}\left(-\frac{x^2+1}{x}\frac{du}{dv} + n^2 u\right)$$

or

$$\left(\frac{x^2-1}{x}\right)^2 \frac{d^2u}{dv^2}+\frac{x^2+1}{x}\frac{du}{dv}-n^2u=0.$$

But

$$v^2-4=\left(x+\frac{1}{x}\right)^2-4=\frac{x^2+2+(1/x^2)-4}{x^2}=\frac{(x^2-1)^2}{x^2}$$

or

$$v=x+\frac{1}{x}=\frac{x^2+1}{x}.$$

Therefore,

$$(v^2-4)\frac{d^2u}{dv^2}+v\frac{du}{dv}-n^2u=0.$$

Example 2.22 If $x + y = 1$, prove that

$$\frac{d^n}{dx^n}(x^n y^n)=n!\left(y^n-{}^nC_1 y^{n-1}x+{}^nC_2 y^{n-2}x^2-{}^nC_3 y^{n-3}x^3+\cdots\right)$$

Solution Here $x + y = 1$. Then

$$1+\frac{dy}{dx}=0 \quad \text{or} \quad \frac{dy}{dx}=-1.$$

Differentiating n times by Leibnitz's theorem, we get

$$\frac{d^n}{dx^n}(x^n y^n)=n!y^n+{}^nC_1 n!xy^{n-1}\frac{dy}{dx}+{}^nC_2\frac{n!}{2!}x^2n(n-1)y^{n-2}\left(\frac{dy}{dx}\right)^2+\cdots$$

$$=n!\left[y^n+{}^nC_1y^{n-1}x(-1)+{}^nC_2y^{n-2}(-1)^2x^2+\cdots\right]$$

$$=n!(y^n-{}^nC_1y^{n-2}x+{}^nC_2y^{n-2}x^2-\cdots).$$

Example 2.23 Writing $x^{2n} = x^n x^n$ and using Leibnitz's theorem, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\,2^2\,3^2}+\cdots=\frac{(2n)!}{(n!)^2}.$$

Solution We have

$$D^n(x^n x^n)=D^n(x^{2n})=2n(2n-1)\cdots(n+1)\,x^{2n-n}=\frac{(2n)!}{n!}x^n. \qquad (1)$$

We also have

$$\mathrm{D}^n(x^n x^n) = x^n(\mathrm{D}^n x^n) + {}^n\mathrm{C}_1(\mathrm{D}x^n)(\mathrm{D}^{n-1}x^n) + {}^n\mathrm{C}_2(\mathrm{D}^2x^n)(\mathrm{D}^{n-2}x^n) + \cdots + (\mathrm{D}^n x^n)x^n$$

$$= x^n n! + (n)nx^{n-1}n!x + \frac{n(n-1)}{1\cdot 2}n(n-1)x^{n-2}\frac{n!}{2}x^2 + \cdots + x^n n!$$

$$= n!x^n\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \cdots\right] \tag{2}$$

Equating right-hand sides of (1) and (2), and dividing by $(n!x^n)$ both the sides, we get

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \cdots = \frac{2n!}{(n!)^2}$$

Example 2.24 If $y = (x^2 - 1)^n$, then show that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

Solution Since $y = (x^2 - 1)^n$, $\log y = n \log (x^2 - 1)$. Differentiating it, we get

$$\frac{y_1}{y} = \frac{2nx}{x^2-1} \qquad \text{or} \qquad (x^2-1)y_1 - 2nxy = 0$$

Applying Leibnitz's theorem and differentiating $(n + 1)$ times, we have

$$(x^2-1)y_{n+2} - {}^{n+1}\mathrm{C}_1(2x)y_{n+1} + {}^{n+1}\mathrm{C}_2 2y_n - 2nxy_{n+1} - 2n\,{}^{n+1}\mathrm{C}_1 y_n = 0$$

or

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

Example 2.25 If $y = \sin (m \sin^{-1}x)$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

Solution Here $y = \sin (m \sin^{-1}x)$. Then

$$y_1 = \cos (m \sin^{-1}x)\ \frac{m}{\sqrt{1-x^2}}$$

or

$$(1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$
$$= m^2[1-\sin^2(m\sin^{-1}x)]$$
$$= m^2(1-y^2)$$

Differentiating it again n times by Leibnitz's theorem, we have

$$D^n[(1-x^2)y_2]-D^n(xy_1)+D^n(m^2y)=0$$

or

$$\left[(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{1}{2}n(n-1)(-2)y_n\right]-[xy_{n+1}+(n)(1)y_n]+m^2y_n=0$$

or

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+[m^2-n-n(n-1)]y_n=0$$

or

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0.$$

Example 2.26 If

$$x=\cosh\left(\frac{1}{m}\log y\right),$$

then prove that

$$(x^2-1)y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)y_n=0,$$

and deduce

$$\lim_{x\to 0}\frac{y_{n+2}}{y_n}=n^2-m^2.$$

Solution Here

$$x=\cosh\left(\frac{1}{m}\log y\right) \tag{1}$$

Differentiating with respect to x, we have

$$1=\sinh\left(\frac{1}{m}\log y\right)\frac{1}{m}\frac{1}{y}y_1$$

or

$$my=y_1\sinh\left(\frac{1}{m}\log y\right)$$

or

$$m^2y^2=y_1^2\sinh^2\left(\frac{1}{m}\log y\right)=y_1^2\left[\cosh^2\left(\frac{1}{m}\log y\right)-1\right]$$

From (1), we obtain

$$m^2y^2=(x^2-1)y_1^2 \qquad \text{or} \qquad (x^2-1)y_1^2=m^2y^2$$

Differentiating again, we get

$$(x^2-1)(2y_1y_2)+2xy_1^2=2m^2yy_1$$

or

$$(x^2-1)y_2+xy_1=m^2y$$

Now, differentiating n times by Leibnitz's theorem, we have

$$(x^2-1)y_{n+2}+{}^nC_1y_{n+1}(2x)+{}^nC_2y_n(2)+xy_{n+1}+{}^nC_1y_1-m^2y_n=0$$

or

$$(x^2-1)y_{n+2}+2nxy_{n+1}+\frac{n(n-1)}{2}2y_n+xy_{n+1}+ny_n-m^2y_n=0$$

or

$$(x^2-1)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-m^2y_n=0$$

or

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-n+n-m^2\right)y_n=0$$

Therefore,

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

Now taking the limit as $x\to 0$, we get

$$y_{n+2}=(n^2-m^2)y_n.$$

Hence

$$\lim_{x\to 0}\frac{y_{n+2}}{y_n}=n^2-m^2.$$

Example 2.27 If

$$u_n=\frac{d^n}{dx^n}(x^n\log x),$$

show that $u_n=nu_{n-1}+(n-1)!$ and hence deduce that

$$u_n=n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

Solution We have

$$u_n=\frac{d^n}{dx^n}(x^n\log x)$$
$$=\frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}\left(x^n\log x\right)\right] \quad (1)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^n \frac{1}{x} + nx^{n-1} \log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[n\left(x^{n-1} \log x\right) + x^{n-1}\right]$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} \log x\right) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$= nu_{n-1} + (n-1)! \tag{2}$$

For the next part, we replace n by $(n-1)$ in (2), we get $(n-2)!$.

$$u_{n-1} = (n-1)u_{n-2} - (n-2)!$$

Putting this in (2) we get,

$$u_n = n[(n-1)u_{n-2} + (n-2)!] + (n-1)!$$

$$= n(n-1)u_{n-2} + n(n-2)! + (n-1)!$$

$$= \frac{n!}{(n-2)!}u_{n-2} + \frac{n!}{n-1} + \frac{n!}{n} \tag{3}$$

Again replacing n by $(n-2)$ in (2), we get

$$u_{n-2} = (n-2)u_{n-3} + (n-3)!$$

Putting this in (3), we have

$$u_n = n(n-1)\left[(n-2)u_{n-3} + \frac{(n-3)!}{n-3}\right] + n(n-2)! + (n-1)!$$

$$= n(n-1)(n-2)u_{n-3} + n(n-1)(n-3)! + n(n-2)! + (n-1)!$$

$$= \frac{n!}{(n-3)!}u_{n-3} + \frac{n!}{(n-2)} + \frac{n!}{n-1} + \frac{n!}{n}$$

Similarly, we get

$$u_n = \frac{n!}{1!}u_1 + \frac{n!}{2} + \frac{n!}{3} + \cdots + \frac{n!}{n}. \tag{4}$$

Putting $n = 1$ in (1), we find

$$u_1 = \frac{d}{dx}(x \log x) = \log x + x\frac{1}{x} = \log x + 1.$$

Putting this value in (4), we get

$$u_n = n!(\log x + 1) + \frac{n!}{2} + \frac{n!}{3} + \cdots + \frac{n!}{n}$$
$$= n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

Example 2.28 If $y = x^2e^x$, show that

$$\frac{d^ny}{dx^n} = \frac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y.$$

Solution Here

$$y = x^2e^x. \tag{1}$$

Differentiating, we get

$$y_1 = 2xe^x + x^2e^x$$
$$y_2 = 2e^x + 4xe^x + x^2e^x$$

Differentiating (1) n times by Leibnitz's theorem, we have

$$y_n = e^x x^2 + ne^x(2x) + \frac{n(n-1)}{2}e^x(2)$$
$$= x^2e^x + 2nxe^x + n(n-1)e^x \tag{2}$$

Now, we get,

$$\frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$$
$$= \frac{1}{2}n(n-1)(2+4x+x^2)e^x - n(n-2)(2x+x^2)e^x + \frac{1}{2}(n-1)(n-2)x^2e^x,$$
$$= x^2e^x\left[\frac{1}{2}(n-1)(n+n-2) - n(n-2)\right] + 2xe^x\left(n^2 - n - n^2 + 2n\right) + n(n-1)e^x$$
$$= x^2e^x(n^2 - 2n + 1 - n^2 + 2n) + 2nxe^x + n(n-1)e^x$$
$$= x^2e^x + 2nxe^x + n(n-1)e^x$$
$$= y_n$$

Example 2.29 If $y = (\sin^{-1}x)^2$, prove that

(a) $(1 - x^2)y_2 - xy_1 = 2$

(b) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$

(c) $\lim_{x\to 0}(y_{n+2}/y_n) = n^2$.

Solution Here $y = (\sin^{-1}x)^2$. Then

$$y_1 = (2\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

or

$$y_1\sqrt{1-x^2} = 2\sin^{-1}x$$

or

$$y_1^2(1-x^2) = 4\left(\sin^{-1}x\right)^2 = 4y$$

(a) Differentiating, we get

$$(1 - x^2)(2y_1y_2) + y_1^2(-2x) = 4y_1,$$

or

$$(1 - x^2)y_2 - xy_1 = 2 \tag{1}$$

(b) Again, according to Leibnitz's theorem, differentiating n times, we get

$$(1-x^2)y_{n+2} + 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - [n(n-1)+n]y_n = 0$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \tag{2}$$

(c) Dividing both sides by y_n, we obtain

$$(1-x^2)\frac{y_{n+2}}{y_n} - (2n+1)x\frac{y_{n+1}}{y_n} - n^2 = 0$$

Now, $(\sin^{-1}x)^2$ is an even function of x. So, if n be even, y_n contains a constant term, while y_{n+1} has a multiple of x as the lowest-degree term, when expanded in ascending powers if x. Hence

$$\lim_{x\to 0}(2n+1)x\frac{y_{n+1}}{y_n} = 0$$

Therefore,

$$\lim_{x\to 0}\frac{y_{n+2}}{y_n} = n^2.$$

Note: If n is odd, y_n contains a multiple of x as the lowest-degree term, and so does y_{n+2}. While y_{n+1} contains a constant term. Hence the limit of $(2n + 1)xy_{n+1}$ exists and is $\neq 0$. Hence the limit of $(1 - x^2)y_{n+1}/y_n$ exists. Therefore, the limit of y_{n+1}/y_n exists and is $\neq n^2$ in this case.

Example 2.30 The first second, third and fourth differential coefficients of y with respect to x are denoted by t, a, b, c respectively, and those of x with respect to y by θ, α, β, γ. Show that

$$\frac{3ac-5b^2}{t^4} = \frac{3\alpha\gamma-5\beta^2}{\theta^4}.$$

Solution Let

$$\theta = \frac{dx}{dy} = \frac{1}{t}$$

Then

$$\alpha = \frac{d\theta}{dy} = \frac{dx}{dy}\frac{1}{t} = \frac{1}{t}\left(-\frac{a}{t^2}\right) = -\frac{a}{t^3}$$

$$\beta = \frac{d\alpha}{dy} = \frac{dx}{dy}\frac{d}{dx}\left(-\frac{a}{t^3}\right) = \frac{1}{t}\left(-\frac{bt^3 - 3a^2t^2}{t^6}\right) = \frac{3a^2 - bt}{t^5}$$

$$\gamma = \frac{d\beta}{dy} = \frac{dx}{dy}\frac{d}{dx}\left(\frac{3a^2 - bt}{t^5}\right)$$

$$= \frac{(6ab - ct - ab)t^5 - 5(3a^2 - bt)t^4a}{t^{11}}$$

$$= \frac{(5ab - ct)t - 5a\,(3a^2 - bt)}{t^7}$$

$$= -\frac{ct^2 + 15a^3 - 10abt}{t^7}$$

Therefore,

$$\frac{3\alpha\gamma - 5\beta^2}{\theta^4} = \frac{3a(ct^2 - 10abt + 15a^3) - 5(3a^2 - bt)^2}{\theta^4 t^{10}}$$

$$= \frac{(3ac - 5b^2)t^2}{\theta^4 t^{10}}$$

$$= \frac{3ac - 5b^2}{t^4}.$$

Example 2.31 Show that

$$\frac{1}{n!}\frac{d^n}{dx^n}\left[x^n(\log x)^n\right] = 1 + S_1 \log x + \frac{S_2}{2!}(\log x)^2 + \cdots + \frac{s_n}{n!}(\log x)^n,$$

where $S_r \equiv$ the sum of the products of the first n natural numbers, taken r at a time.

Solution Let $y = x^n(\log x)^n$. Put $\log x = z$, so $x = e^z$. Therefore, $y = e^{nz}z^n$. Since $x = e^z$,

$$\frac{dx}{dx} = e^z.$$

Now

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}e^{-z}=e^{-z}\frac{d}{dz}(y)=e^{-z}\frac{dy}{dz}$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right)=\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$=\left[e^{-z}\frac{d^2y}{dz^2}+\frac{dy}{dz}(-1)e^{-z}\right]e^{-z}=e^{-2z}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$=e^{-2z}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)=e^{-2z}(\mathrm{D}^2-\mathrm{D})y,\quad \left(\text{where } \mathrm{D}=\frac{d}{dz}\right).$$

Proceeding in similar manner, we get

$$\frac{d^ny}{dx^n}=e^{-nz}\left[\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\cdots(\mathrm{D}-n+1)\right]y$$

Now

$$\frac{1}{n!}\frac{d^n}{dx^n}\left[x^n(\log x)^n\right]=\frac{1}{n!}\frac{d^ny}{dx^n}$$

$$=\frac{1}{n!}e^{-nz}[\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\cdots(\mathrm{D}-n+1)e^{nz}z^n]$$

$$=\frac{1}{n!}e^{-nz}e^{nz}[(\mathrm{D}+n)(\mathrm{D}+n-1)\cdots(\mathrm{D}+1)]z^n$$

$$=\frac{1}{n!}[(\mathrm{D}+1)(\mathrm{D}+2)\cdots(\mathrm{D}+n-1)(\mathrm{D}+n)]z^n$$

$$=\frac{1}{n!}[\mathrm{D}^n+(1+2+3+\cdots+n)\mathrm{D}^{n-1}+(1\cdot 2+2\cdot 3+3\cdot 4)\mathrm{D}^{n-2}$$

$$+\cdots+(1\cdot 2\cdot 3\cdots n)]z^n$$

$$=\frac{1}{n!}[\mathrm{D}^n+S_1\mathrm{D}^{n-1}+S_2\mathrm{D}^{n-2}+\cdots+S_n]z^n$$

$$=\frac{1}{n!}\left(n!+S_1\frac{n!}{1!}z+S_2\frac{n!}{2!}z^2+\cdots+S_n\frac{n!}{n!}z^n\right)$$

$$=1+S_1\log x+\frac{S_2}{2!}(\log x)^2+\frac{S_3}{3!}(\log x)^3+\cdots+\frac{S_n}{n!}(\log x)^n.$$

Example 2.32 If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2-1)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)y_n = 0.$$

Solution Differentiating $y^{1/m} + y^{-1/m} = 2x$, we get

$$\frac{1}{m}y^{1/m-1}y_1 + \left(-\frac{1}{m}\right)y^{-1/m-1}\,y_1 = 2$$

or

$$\frac{1}{m}\frac{y^{1/m}}{y}y_1 - \frac{1}{m}\frac{y^{-1/m}}{y}y_1 = 2$$

or

$$\frac{1}{m}\frac{y_1}{y_2}\left(y^{1/m} - y^{-1/m}\right) = 2$$

Squaring, we get

$$\frac{y_1^2}{m^2y^2}\left(y^{1/m} - y^{-1/m}\right)^2 = 4$$

or

$$y_1^2\left[(y^{1/m} + y^{-1/m})^2 - 4y^{1/m}y^{-1/m}\right] = 4m^2y^2$$

or

$$y_1^2(4x^2-4) = 4m^2y^2$$

or

$$(x^2-1)y_1^2 = m^2y^2$$

Differentiating it again, we obtain

$$(x^2-1)(2y_1y_2) + y_1^2(2x) = m^2(2yy_1)$$

or

$$(x^2-1)y_2 + xy_1 = m^2y$$

or

$$(x^2-1)y_2 + xy_1 - m^2y = 0$$

Differentiating n times using Leibnitz's theorem, we get

$$(x^2-1)y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + (xy_{n+1} + {}^nC_1y_n) - m^2y_n = 0$$

or

$$(x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}(2y_n) + xy_{n+1} + ny_n - m^2y_n = 0$$

or

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$$

or

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Exercises 2.2

1. Find y_n for the following functions, where y be equal to:

   (i) xe^x (ii) $x^2 \log x$ (iii) $x^2 \sin^2 x$

   (iv) $x^2 \cos x$ (v) $x^3 e^{ax}$.

2. (i) If $y = \tan^{-1}x$, then prove that $(1 + x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$
   (ii) If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \ldots$, show that $(n + 1)\, a_{n+1} + (n - 1)\, a_{n-1} = ma_n$.

3. If $y = \sin^{-1}x$, then prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.

4. If $y = \log [x + \sqrt{(1 + x^2)}]^2$, prove that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0$.

5. If $y = e^{x^2}$, prove that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$.

6. If $y = x \sin x$, prove that

$$y_n = x \sin\left(x + \frac{n\pi}{2}\right) - n \cos\left(x + \frac{n\pi}{2}\right).$$

   Also deduce that $y_2 - y_1 = 4(y/x)$.

7. If $y = e^{-x} \cos x$, prove that $y_4 + 4y = 0$.

8. If

$$\cos^{-1}\frac{y}{b} = \log\left(\frac{x}{n}\right)^n,$$

   prove that $x^2y_{n+2} + (2n + 1)xy_{n+1} + 2n^2y_n = 0$.

9. If $y = x^{n-1} \log x$, prove that $y_n = (n - 1)!/x$.

10. If $y = x^3 \log x$, find y_n.

11. If $y = x^n/(1 + x)$, find y_n.

12. If $y = (\log x)/x^{n+1}$, prove that

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\left(\log x - \sum_{r=1}^{n}\frac{1}{r}\right).$$

13. If $y = ax^{n+1} + bx^{-n}$, prove that $x^2y_2 = n(n + 1)y$.

14. If $y = Ae^{-ax} \cos (bx + c)$, find y_2.

15. If $y = \log [x + \sqrt{(1 + x^2)}]$, then $(y_{n+2})_0 = -n^2(y_n)_0$.

16. If $y = e^{a\sin^{-1}x}$, then $(y_{n+2})_0 = (n^2 + a^2)(y_n)_0$.

17. If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + \ldots$, show that $(n+1)(n+2)\, a_{n+2} = n^2 a_n$.

18. If $x = \tan (\log y)$, then show that

$$(1 + x^2)y_{n+1} + (2nx - 1)y_n + n(n - 1)y_{n-1} = 0.$$

19. If $y = e^{x^2/2} \cos x$, then show that

$$(y_{2n}+2)_0 - 4^n (y_{2n})_0 + 2n(2n-1)(y_{2n}-2)_0 = 0.$$

20. If $y = e^{m\cos^{-1}x}$, then show that

$$(1-x^2)y_{n+1} - (2n+1)xy_n - (n^2 - m^2)y_n = 0.$$

21. If $y = e^{a\sin^{-1}x}$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0.$$

22. If $y = Ae^{ax} + Be^{bx}$, show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0.$$

23. If

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right),$$

show that

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$$

24. If $y = \cos(m \cos^{-1}x)$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0.$$

25. If $y = e^{\tan^{-1}x}$, prove that

$$(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0.$$

26. If $y = \cosh(\sin^{-1}x)$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0.$$

27. Find y_n, where

$$y = \frac{x+1}{(x+2)(x+3)},$$

28. Find y_n, where

$$y = \frac{x^2}{(x-1)^2(x+2)}$$

29. If $y = x^3/(x^2 - 1)$, find y_n.

30. Find y_n, when

$$y = \frac{x^4}{(x-1)(x-2)},$$

31. Find y_n, when

$$y = \frac{1}{(x-1)^3(x-2)},$$

32. Find y_n, where

$$y = \frac{1}{(x+1)(x^2+1)},$$

33. Find y_n, where

$$y = \tan^{-1}\left(\frac{1+x}{1+x}\right),$$

34. If $y = x^4 \log x$, prove that

$$y_n = (-1)^{n-1}\frac{(n-5)!}{x^{n-4}}24, \qquad \text{when } n \geq 5.$$

35. If $y = x^2 e^{2x} \sin x$, find y_n.

36. If

$$y = (a + bx) \cos mx + (c + dx) \sin mx,$$

prove that $y_4 + 2m^2 y_2 + m^4 y = 0$.

37. If $y = (a \cos x + b \sin x)e^{-mx}$, prove that $y_2 + 2my_1 + (m^2 + 1)y = 0$.

Chapter 3

Expansions

3.1 Introduction

Explicit functions can be expanded in ascending integral powers of the independent variables in the following ways:

1. By using Taylor's or Maclaurin's theorem,
2. By using either algebra or trigonometry,
3. By using differential equation,
4. By differentiating the known series.

Expansion of functions in finite or infinite terms has often been found very useful to solve many problems in mathematics. We have a number of such convergent series like $(a + x)^n$, e^x, $\log(1 + x)$, $\sin x$, $\cos x$, $\tan^{-1}x$, etc., in ascending powers of x. All such expansions have certain definite forms and values, called their *sums*. In calculus, we try to expand any function $f(x + h)$ in general, in terms of derivatives. In this context, the main problem is to investigate if $f(x + h)$ can be expanded in ascending powers of h or x. This is done with the help of Rolle's theorem followed by Lagrange's mean value theorem. These theorems provide theoretical background for the expansion of the following two series developed by Taylor and Maclaurin, respectively:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

and

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

However, Taylor's theorem occupies a fundamental position in any scheme of expansion. Hence, we begin with Taylor's series.

3.2 Rolle's Theorem

If the function $f(x)$ be defined as

(i) $f(x)$ is continuous at every point of the closed interval

$$a \leq x \leq b,$$

(ii) $f'(x)$ exists at every point of the open interval

$$a < x < b,$$

(iii) $f(a) = f(b)$

Then $\exists$ at least one value c of x at which

$$f'(c) = 0, \quad \text{where } a < c < b.$$

Proof As the value of the function at $x = a$, b are equal, the following cases may arise:

Case I. $f(x)$ is constant throughout $a \leq x \leq b$.

Case II. $f(x)$ is not constant throughout $a \leq x \leq b$.

Case I. $f(x)$ is constant in $a \leq x \leq b$, by definition of derivative, $f'(x) = 0$ at all points in the interval $a \leq x \leq b$ and so the theorem is proved (Fig. 3.1).

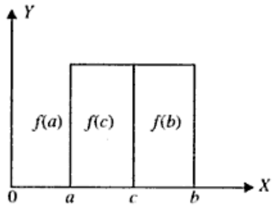


Fig. 3.1 Rolle's theorem (Case I).

Case II. If $f(x)$ is not constant in $a \leq x \leq b$ (Fig. 3.2).

But $f(x)$ is continuous in the closed interval $a \leq x \leq b$, so by property of continuous function we know that $f(x)$ is bounded in $[a, b]$ and attains its bounds in the interval. As $f(a) = f(b)$, the least upper-bound M or the greatest lower bound m will be attained at a point c other than a and b. For otherwise $M = m$, which implies $f(x)$ is constant. Here, $f(x) \leq M = m \leq f(x)\ \forall\ a \leq x \leq b$.

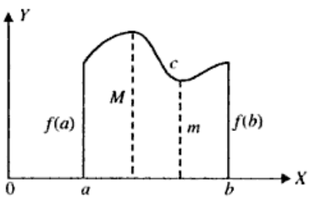


Fig. 3.2 Rolle's theorem (Case II).

Let $f(c) = m$, where $a < c < b$. As $m \le f(x)\ \forall\ a < x < b$, $f(c) \le f(x)\ \forall\ a < x < b$. Therefore,

$$f(c) \le f(c \pm h) \text{ where } a < c \pm h < b \text{ is positive}$$

or

$$f(c \pm h) - f(c) \ge 0$$

or

$$\frac{f(c+h) - f(c)}{h} \ge 0 \quad \text{and} \quad \frac{f(c-h) - f(c)}{-h} \le 0 \tag{3.1}$$

But $f'(x)$ exists at $x = c$. Then

$$\underset{h \to 0}{lt} \frac{f(c+h) - f(c)}{h} = \underset{h \to 0}{lt} \frac{f(c-h) - f(c)}{-h} = f'(c)\cdot$$

By Eq. (3.1), when $h \to 0$, we get

$$\underset{h \to 0}{lt} \frac{f(c+h) - f(c)}{h} \ge 0 \quad \text{and} \quad \underset{h \to 0}{lt} \frac{f(c-h) - f(c)}{-h} \le 0$$

Therefore,

$$f'(c) \ge 0 \quad \text{and} \quad f'(c) \le 0.$$

Hence

$$f'(c) = 0 \text{ where } a < c < b.$$

By similar argument if $M = f(c)$ we can prove $f'(c) = 0$ where $a < c < b$.

Corollary It is evident that Rolle's theorem is also applicable in the case when $f(a) = 0 = f(b)$, i.e. when $x = a, b$ satisfies $f(x) = 0$. Hence we have the following theorem in the theory of equations:

"A real root of the equation $f'(x) = 0$ lies between every adjacent two of the real roots of the equation $f(x) = 0$." This result is important in the theory equations.

The generalization of Rolle's theorem is usually known as "the first mean value theorem", or "the law of the mean" (refer to Fig. 3.3).

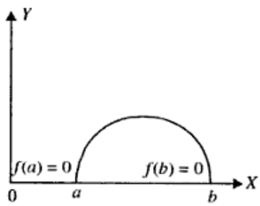


Fig. 3.3 Rolle's theorem (corollary).

3.3 Lagrange's Mean Value Theorem

If the function $f(x)$ be defined as

(i) $f(x)$ is continuous at every point of the closed interval

$$a \le x \le b,$$

(ii) $f'(x)$ exists at every point of the open interval $a < x < b$, then there is a value c of x for which

$$\frac{f(b)-f(a)}{b-c} = f'(c), \text{ where } a < c < b.$$

Proof Let us consider the function $\phi(x)$, where

$$\phi(x) = f(b) - f(x) - \frac{b-x}{b-a}\{f(b)-f(a)\} \tag{3.2}$$

Given that $f(x)$ is continuous in $a \le x \le b$ and $(b - x)$ is also continuous. But we know that the algebraic sum of continuous function is also continuous. Hence $f(x)$ must be continuous in $a \le x \le b$. Differentiating Eq. (3.2) w.r. to x, we get

$$\phi'(x) = -f'(x) + \frac{f(b)-f(a)}{b-a} \tag{3.3}$$

Also, given that $f'(x)$ exists in $a < x < b$. Therefore, $\phi'(x)$ must exists in $a < x < b$. Putting $x = a$ in Eq. (3.2), we get

$$\begin{aligned} \phi(a) &= f(b) - f(a) - \frac{b-a}{b-a}\{f(b) - f(a)\} \\ &= f(b) - f(a) - \{f(b) - f(a)\} \\ &= 0 \end{aligned}$$

Again, putting $x = b$ in Eq. (3.2), we get

$$\phi(b) = f(b) - f(b) - \frac{b-b}{b-a}[f(b) - f(a)] = 0 - 0 = 0$$

Therefore,

$$\phi(a) = \phi(b).$$

Hence the function $\phi(x)$ under consideration satisfies all the conditions of Rolle's theorem. Therefore,

$\phi(x)$ is continuous in $[a, b]$
$\phi'(x)$ exists in $]a, b[$,
$\phi(a) = \phi(b)$.

Hence, by Rolle's theorem, $\exists$ a value $x = c$, where $a < c < b$, at which

$$\phi'(c) = 0 \tag{3.4}$$

Putting $x = c$ in Eq. (3.3), we get

$$\phi'(c) = -f'(c) + \frac{f(b)-f(a)}{b-a}$$

From Eq. (3.4)

$$0 = -f'(c) + \frac{f(b)-f(a)}{b-a},$$

therefore,

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

3.4 Cauchy's Mean Value Theorem

If two function $f(x)$ and $\phi(x)$ be defined as

(i) $f(x)$ and $\phi(x)$ are continuous at every point in a closed interval $[a, b]$.
(ii) $f'(x)$ and $\phi'(x)$ exists at every point in the open interval $]a, b[$.
(iii) $\phi'(x) \neq 0 \ \forall \ x \in \ ]a, b[$, then $\exists$ at least one value 'c' of $x \in \ ]a, b[$, such that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(c)}.$$

Proof Let us consider a function $\psi(x)$ such that

$$\psi(x) = f(x) + A\ \phi(x) \tag{3.5}$$

where A is a constant to be determined such that

$$\psi(b) = \psi(a) \tag{3.6}$$

Putting $x = a$ in Eq. (3.5) we get

$$\psi(a) = f(a) + A\ \phi(a)$$

Again, putting $x = b$ in Eq. (3.5) we get

$$\psi(b) = f(b) + A\ \phi(b)$$

Subtituting these values of $\psi(a) < \psi(b)$ in Eq. (3.6), we get

$$f(b) + A\ \phi(b) = f(a) + A\ \phi(a)$$

or

$$f(b) - f(a) = -A[\phi(b) - \phi(a)]$$

Then

$$-A = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} \tag{3.7}$$

Since $\phi'(x) \neq 0$ anywhere in $]a, b[$, therefore $\phi(b) \neq \phi(a)$, so that A is always finite and determinate.

It is given that $f(x)$ and $\phi(x)$ are continuous in the closed interval $[a, b]$.

We know that the algebraic sum of continuous functions is continuous. Therefore $f(x) + A\ \phi(x)$, i.e. $\psi(x)$ must be continuous in $[a, b]$.

Differentiating Eq. (3.5) w.r. to x we get

$$\psi'(x) = f'(x) + A\ \phi'(x) \tag{3.8}$$

Since $f'(x)$ and $\phi'(x)$ exists at every point in the open interval $]a, b[$. Therefore, $\psi'(x)$ must exists in $]a, b[$. Hence $\psi(x)$ satisfies all the conditions of Rolle's theorem. By Rolle's theorem $\exists$ a value $x = c$ where $a < c < b$ at which

$$\psi'(c) = 0 \tag{3.9}$$

Putting $x = c$ in Eq. (3.8), we get

$$\psi'(c) = f'(c) + A\ \phi'(c) \tag{3.10}$$

From Eqs. (3.9) and (3.10), we get

$$f'(c) + A\ \phi'(c) = 0$$

or

$$-A = \frac{f'(c)}{\phi'(c)} \tag{3.11}$$

Therefore, from Eqs. (3.7) and (3.11), we get

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}$$

Hence, the theorem is proved.

Another form of Cauchy's Mean Value Theorem

On substituting ath for b and $a + \theta h$ for c, where θ is a number between 0 and 1, we obtain another form of Cauchy's mean value theorem:

$$\frac{f(a+h) - f(a)}{\phi(a+h) - \phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}, \text{ where } 0 < \theta < 1.$$

3.5 Taylor's Theorem

Under certain circumstances, the 'theorem' states that the function $f(x + h)$ can be expanded in powers of h, i.e.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Proof Let us suppose that the expansion of $f(x + h)$ is possible in ascending powers of h and the series is convergent, so that

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \cdots \tag{3.12}$$

where a_0, a_1, a_2, a_3, ... are functions of x alone, not containing h, and are to be determined. Now, differentiating Eq. (3.12) with respect to h, we get

$$f'(x + h) = a_1 + 2a_2 h + 3a_3 h^2 + 4a_4 h^3 + \cdots \tag{3.13}$$

$$f''(x + h) = 2a_2 + 6a_3 h + 12a_4 h^2 + 20a_5 b^3 + \cdots \tag{3.14}$$

$$f'''(x + h) = 6a_3 + 24a_4 h + \cdots \tag{3.15}$$

$$\vdots$$

Putting $h = 0$ in Eqs. (3.12)–(3.15), we have

$$f(x) = a_0$$

$$f'(x) = a_1$$

$$f''(x) = 2a_2, \quad a_2 = f''(x)/2$$

$$f'''(x) = 6a_3, \quad a_3 = f'''(x)/6$$

Now substituting these values of a_0, a_1, a_2, a_3, ... in Eq. (3.12), we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \tag{3.16}$$

This is called *Taylor's series without remainder*, or *Taylor's infinite series*. Many useful series are deduced from it.

Corollary Put $x = a$ and $h = x$, in Eq. (3.16), we have

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots$$

Corollary Again by putting $a = 0$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

It is called *Stirling's* or *Maclaurin's series*, which is discussed now.

3.6 Maclaurin's (or Stirling's) Theorem

Under certain circumtances, the theorem states that if the function $f(x)$ can be expanded in a convergent series of the integral powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Proof Let us suppose that the expansion of $f(x)$, in ascending powers of x as an infinite series, as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \tag{3.17}$$

where $a_0, a_1, a_2, a_3, \ldots$ are constants and are to be determined. Now, differentiating Eq. (3.17) with respect to x, we get,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots \tag{3.18}$$

Differentiating again, we have

$$f''(x) = 2a_2 + 6a_3x + \cdots \tag{3.19}$$

$$f'''(x) = 6a_3 + \text{higher powers of } x. \tag{3.20}$$

Putting $x = 0$, we get

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2 \quad \text{or} \quad a_2 = \frac{f''(0)}{2!}, \cdots$$

Now substituting these values in Eq. (3.17), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

It is called *Maclaurin's* series.

Corollary Let $y = f(x)$, $y_1 = f'(x)$, $y_2 = f''(x)$, ... Putting $x = 0$, we write

$$f(0) = (y)_0, \quad f'(0) = (y_1)_0, \quad f''(0) = (y_2)_0, \ \ldots$$

Therefore,

$$y = f(x) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$$

This is the another form of Maclaurin's series.

Corollary Putting $x = 0$ and $h = x$ in Taylor's series, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

This is known as Maclaurin's series that is frequently used in problems concerning expansion.

Corollary Failure of Taylor's series and Maclaurin's series.

(i) If $f(x)$, or any one of its derivative, is infinite,
(ii) If the series of the expansion is not convergent.

Consider the following examples:

(a) Let $f(x) = e^{1/x}$. When $x \neq 0$ and $f(0) = 0$, then

$$f'(x) = -\frac{1}{x^2} e^{1/x}$$

Therefore, $f'(0) = \infty$. This implies the differential coefficient of any order for the function $e^{1/x}$ is not finite. Therefore, the expansion of $e^{1/x}$, in ascending powers of x, is not possible.

(b) Let $f(x) = \sqrt{x}$. Here $f'(0), f''(0), f'''(0), \ldots$ are all infinite. Hence the expansion of $\sqrt{x}$ in ascending powers of x is not possible.

(c) Let $f(x) = \log x$. Here $f'(0), f''(0), f'''(0), \ldots$ are all infinite. Hence the expansion of $\log x$ in ascending powers of x is not possible.

Note: The second condition of Taylor's series is that the RHS should be convergent. As we noted the remaider series after the nth term by R_n, which is

$$\frac{x^n}{n!} f^n(\theta x);$$

it should be necessary that $R_n \to 0$, as $n \to \infty$.

Example 3.1 Find the value of c in Rolle's theorem, where $0 < c < 2$. and $f(x) = x(x - 2)$.

Solution Here

$$f(x) = x(x - 2) \tag{1}$$

Putting $x = 0$ in (1), we get

$$f(0) = 0(0 - 2) = 0$$

Again putting $x = 2$ in (1), we get

$$f(2) = 2(2 - 2) = 2 \times 0 = 0$$

or

$$f(0) = f(2)$$

Obviously $f(x)$ is continuous in [0, 2] and $f'(x)$ exists in]0, 2[and $f(0) = f(2)$. Hence, by Rolle's Theorem,

$$f'(c) = 0 \tag{2}$$

Differentiating w.r. to x, we get

$$f'(x) = 1(x-2) + x(1) = 2x - 2$$

Putting $x = c$, we get

$$f'(c) = 2c - 2 \tag{3}$$

From (2) and (3), we get

$$f'(c) = 2c - 2 = 0$$

or

$$2c - 2 = 0$$

or

$$2c = 2$$

or

$$c = 1.$$

Example 3.2 Find the value of c in mean value theorem, viz

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

where $f(x) = \sqrt{x}$; $a = 9$, and $b = 16$.

Solution We have

$$f(x) = \sqrt{x} \tag{1}$$

Putting $x = a$,

$$f(a) = \sqrt{a}$$

For $a = 9$,

$$f(9) = \sqrt{9} = 3.$$

Putting $x = b$ in (1), we have

$$f(b) = \sqrt{b} = \sqrt{16} = 4. \qquad \because b = 16, \text{ given}$$

Differentiating (1) with respect to x, we get

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Putting $x = c$, we get

$$f'(c) = \frac{1}{2\sqrt{c}}$$

By mean value theorem, we have

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

or

$$\frac{4-3}{16-9} = \frac{1}{2\sqrt{c}}$$

or

$$\frac{1}{7} = \frac{1}{2\sqrt{c}}$$

or

$$\sqrt{c}=\frac{7}{2}$$

Hence,

$$c=\frac{49}{4}.$$

Example 3.3 If $a = 1$, $h = 1$ and $f(x) = x^2$ in Lagrange's mean value theorem

$$f(a + h) = f(a) + hf'(a + \theta h),\ 0 < \theta < 1,$$

find θ.

Solution

Here $f(x) = x^2$. Therefore,

$$f(a + h) = f(1 + 1) = f(2) = 2^2 = 4$$

Again $f'(x) = 2x$. Therefore,

$$f'(a + \theta h) = f'(1 + \theta\cdot 1) = f'(1 + \theta) = 2(1 + \theta)$$

By Lagrange's mean value theorem, we have

$$f(a + h) = f(a) + hf'(a + \theta h)$$

or

$$4 = 1 + 1\cdot 2(1 + \theta)$$

or

$$3 = 2 + 2\theta$$

or

$$2\theta = 1$$

Hence,

$$\theta=\frac{1}{2}.$$

Example 3.4 Show that $1 + x\log_e\left(x+\sqrt{x^2+1}\right)\ge\sqrt{1+x^2}\ \forall\ x\ge 0.$

Solution Let us consider

$$f(x) = 1 + x\log_e\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2} \tag{1}$$

Evidently $f(x)$ is continuous and differentiable $\forall\ x \ge 0$. Therefore, by Lagrange's theorem, we have

$$\frac{f(x)-f(0)}{x-0}=f'(\theta x) \tag{2}$$

where θ lies between 0 and 1. From (1),

$$f'(x)=1\cdot\log_e\left(x+\sqrt{x^2+1}\right)+\frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)-\frac{2x}{2\sqrt{x^2+1}}$$

$$=\log_e\left(x+\sqrt{x^2+1}\right)$$

From (2), we have

$$\frac{\left[1+x\log_e\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2}\right]-[0]}{x-0}=\log_e\left(\theta x+\sqrt{\theta^2x^2+1}\right)$$

or

$$1+x\log_e\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2}=x\log_e\left(\theta x+\sqrt{\theta^2x^2+1}\right)\geq 0,\quad \forall\, x\geq 0$$

or

$$1+x\log_e\left(x+\sqrt{x^2+1}\right)\geq\sqrt{1+x^2}$$

Hence the desired result.

Example 3.5 In the equation

$$f(x+h)=f(x)+hf'(x)+\cdots+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(x+\theta h)$$

if $f^n(x)$ is continuous, then prove that

$$\lim_{h\to 0}\theta=\frac{1}{n}.$$

Solution Given that the equation, we have

$$f(x+h)=f(x)+hf'(x)+\cdots+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(x+\theta h) \tag{1}$$

Since $f^n(x)$ is continuous, by Taylor's theorem

$$f(x+h)=f(x)+hf'(x)+\cdots+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(x)+\frac{h^n}{\underline{|n}}f^n(x+\theta_1 h) \tag{2}$$

where, $0<\theta_1<1$. From (1) and (2), we find that

$$\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(x)+\frac{h^n}{\underline{|n}}f^n(x+\theta_1 h)=\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(x+\theta h)$$

or

$$f^{n-1}(x)+\frac{h}{n}f^n(x+\theta_1 h)=f^{n-1}(x+\theta h)$$

or

$$\frac{h}{n}f^n(x+\theta_1 h)=f^{n-1}(x+\theta h)-f^{n-1}(x)$$

or

$$\frac{1}{n}f^n(x+\theta_1 h)=\theta\cdot\frac{f^{n-1}(x+\theta h)-f^{n-1}(x)}{x+\theta h-x} \tag{3}$$

By Lagrange's mean value theorem, we have

$$\frac{f^{n-1}(x+\theta h)-f^{n-1}(x)}{(x+\theta h)-(x)}=f^n(x+\theta\theta_2 h),\quad \text{where } 0<\theta_2<1.$$

Therefore, (3) becomes

$$\frac{1}{n} f^n(x+\theta_1 h) = \theta \cdot f^n(x+\theta\theta_2 h).$$

Let $h \to 0$

$$\frac{1}{n} f^n(x) = f^n(x). \lim_{h\to 0} \theta$$

Hence,

$$\lim_{h\to 0} \theta = \frac{1}{n}.$$

Example 3.6 Use Taylor's series to expand the following functions:

(i) $\cos x$ (ii) $\dfrac{\log(1+x)}{1+x}$

(iii) $\log \sin (x + h)$ (iv) $\tan^{-1}(x + h)$ (v) e^{x+h}

in ascending powers of x for three terms.

Solution Taylor's series for the expansion of $f(a + x)$ in ascending powers of x is

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \cdots$$

(i) For expansion of $\cos x$, let

$$f(x) = \sin x \quad \text{and} \quad \alpha = \frac{\pi}{2}$$

in Taylor's series. So

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

and

$$f(\alpha + x) = f\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + x\right) = \cos x$$

Also, since

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{IV}(x) = \sin x, \cdot$$

Therefore,

$$f'\left(\frac{\pi}{2}\right) = 0, \quad f''\left(\frac{\pi}{2}\right) = -1, \quad f'''\left(\frac{\pi}{2}\right) = 0, \quad f^{IV}\left(\frac{\pi}{2}\right) = 1, \cdots$$

Substituting these values is Taylor's series in Eq. (3.12), we get

$$f(a+x) = \cos x = f\left(\frac{\pi}{2}\right) + xf'\left(\frac{\pi}{2}\right) + \frac{x^2}{2!} f''\left(\frac{\pi}{2}\right)$$

$$+ \frac{x^3}{3!} f'''\left(\frac{\pi}{2}\right) + \frac{x^4}{4!} + f^{IV}\left(\frac{\pi}{2}\right) + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(ii) For expansion of

$$\frac{\log(1+x)}{1+x},$$

let

$$f(x)=\frac{\log x}{x} \quad \text{and} \quad f(1+x)=\frac{\log(1+x)}{1+x}=f(a+x).$$

Here $a = 1$, and we have to find $f(a), f'(a), f''(a), \ldots$ to apply in Taylor's series. We also have

$$f'(x)=\frac{1}{x^2}-\frac{\log x}{x^2}=\frac{1}{x^2}(1-\log x)$$

$$f''(x)=\frac{-2}{x^3}(1-\log x)+\frac{1}{x^2}\left(-\frac{1}{x}\right)=-\frac{1}{x^3}(3-2\log x)$$

$$f'''(x)=\frac{3}{x^4}(3-2\log x)-\frac{1}{x^3}\left(-\frac{2}{x}\right)=\frac{1}{x^4}(1-6\log x).$$

Putting $x = 1$, we get,

$$f(1)=0,\quad f'(1)=1,\quad f''(1)=-3,\quad f'''(1)=1,\cdots$$

and so on. Substituting these values in Taylor's series, we obtain

$$f(1+x)=\frac{\log(1+x)}{1+x}=f(1)+xf'(1)+\frac{x^2}{2!}f''(1)+\frac{x^3}{3!}f'''(1)+\cdots$$

$$=x-\frac{3}{2}x^2+\frac{11}{6}x^3-\cdots$$

$$=x-\left(1+\frac{1}{2}\right)x^2+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3-\cdots$$

(iii) For expansion of $\log \sin(x+h)$, let $f(x) = \log \sin x$. Then

$$f'(x)=\cot x,$$

$$f''(x)=-\operatorname{cosec}^2 x,$$

$$f'''(x)=2\operatorname{cosec}^2 x\cot x.$$

According to Taylor's theorem

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots$$

Therefore,

$$\log\sin(x+h)=\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\operatorname{cosec} x\cot x+\cdots$$

(iv) For expansion of $\tan^{-1}(x + h)$, let $f(x) = \tan^{-1} x$ and

$$f^n(x) = (-1)^{n-1}\,(n-1)!\sin^n\alpha\sin n\alpha,$$

where $x = \cot\alpha$.

Now putting $n = 1, 2, 3, \dots$ successively, we get

$$f'(x) = \sin\alpha\sin\alpha$$

$$f''(x) = -\sin^2 a\sin 2a$$

$$f'''(x) = 2!\sin^3\alpha\sin 3\alpha$$

Hence, according to Taylor's theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

or

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\sin\alpha - \frac{h^2}{2!}\sin^2\alpha\sin 2\alpha$$

$$+\frac{h^3}{3!}2!\sin^3\alpha\sin 3\alpha + \cdots$$

$$= \tan^{-1}x + h\sin\alpha\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2}$$

$$+ (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - \cdots$$

(v) For expansion of e^{x+h}, consider

$$f(x+h) = e^{x+h} \quad \text{and} \quad f(x) = e^x.$$

Therefore,

$$f'(x) = e^x = f''(x) = f'''(x) = \cdots$$

We know that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$= e^x + he^x + \frac{h^2}{2!}e^x + \cdots$$

$$= e^x\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)$$

Example 3.7 Expand e^x in ascending powers of $(x - 1)$.

Solution We have

$$e^x = e^{(x-1)+1} = f(y+1), \qquad \text{where } y = x - 1.$$

By Taylor's theorem

$$f(y+1) = f(1) + yf'(1)\frac{y^2}{2!}f''(1) + \frac{y^3}{3!}f'''(1) + \cdots$$

Here

$$f(y+1) = e^{(x-1)+1} = e^{y+1}$$

Therefore,

$$f'(y+1) = f''(y+1) = f'''(y+1) = e^{y+1}$$

Putting $y = 0$, then

$$f(1) = e, \quad f'(1) = e, \quad f''(1) = e, \quad f'''(1) = e, \cdots$$

Hence

$$f(y+1) = e + ey + e\frac{y^2}{2!} + e\frac{y^3}{3!} + \cdots$$

Therefore,

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

Example 3.8 Use Maclaurin's series to expand the following:

(i) $\sin x$, (ii) $\log(1 + x)$, (iii) a^x, (iv) $\log(1 + \sin x)$, (v) $e^{ax}\cos bx$.

Solution We know that Maclaurin's series for the expansion of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

(i) For the expansion of $\sin x$, $f(x) = \sin x$, then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{\text{IV}}(x) = \sin x$$

$$f^{\text{V}}(x) = \cos x$$

Putting $x = 0$, we have

$$f(0) = f''(0) = f^{\text{IV}}(0) = \cdots = 0$$

and

$$f'(0) = 1, \quad f'''(0) = -1, \quad f^{\text{V}}(0) = 1, \cdots$$

Therefore,

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(ii) For the expansion of $\log(1 + x)$, let $f(x) = \log(1 + x)$. Then

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{(-1)^2 2!}{(1+x)^2}, \cdots$$

Putting $x = 0$, we get

$$f(0) = \log 1 = 0, \qquad f'(0) = 1, \qquad f''(0) = -1, \qquad f'''(0) = (-1)\, 2! \cdots$$

and in general

$$f^n(0) = (-1)^{n-1}(n-1)!$$

Therefore,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

(iii) For the expansion of a^x, let $f(x) = a^x$. Then

$$e^{x \log a} = e^{kx}, \qquad \text{where } k = \log a.$$

Therefore,

$$f'(x) = ke^{kx}, \quad f''(x) = k^2 e^{kx}, \quad f'''(x) = k^3 e^{kx}, \cdots$$

Hence

$$a^x = 1 + kx + k^2 \frac{x^2}{2!} + k^3 \frac{x^3}{3!} + \cdots$$

$$= 1 + (\log a)\, x + (\log a)^2 \frac{x^2}{2!} + (\log a)^3 \frac{x^3}{3!} + \cdots$$

(iv) For the expansion of $\log(1 + \sin x)$, let

$$y = f(x) = \log(1 + \sin x).$$

Then

$$y_1 = \frac{\cos x}{1 + \sin x}$$

$$= \frac{\sin(\pi/2 - x)}{1 + \cos(\pi/2 - x)}$$

$$= \frac{2 \sin(\pi/4 - x/2) \cos(\pi/4 - x/2)}{2 \cos^2(\pi/4 - x/2)}$$

When $x = 0$, we have $(y)_0 = \log 1 = 0$. Now, differentiating the given function with respect to x, we get

$$y_1 = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \tan\frac{\pi}{4}$$

at $x = 0$ $(y_1)_0 = 1$. Differentiating again, we obtain

$$y_2 = -\frac{1}{2}\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) = -\frac{1}{2}\left[1 + \tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] = -\frac{1}{2}\left(1 + y_1^2\right)$$

and

$$y_3 = -\frac{1}{2}(2y_1 y_2) = -y_1 y_2$$

$$y_4 = -(y_2 y_2 + y_1 y_2) = -(y_2^2 + y_1 y_2)$$

$$y_5 = -(2y_2 y_3 + y_2 y_3 + y_1 y_4) = -(3y_2 y_3 + y_1 y_4)$$

Now at $x = 0$,

$$(y_2)_0 = -\frac{1}{2}(1+1) = -1$$
$$(y_3)_0 = (-1)(-1) = 1,$$
$$(y_4)_0 = -(1^2+1) = -2$$
$$(y_5)_0 = -(-3-2) = 5$$
$$\vdots$$

Substituting these values in Maclaurin's series, we find

$$f(x) = f(0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$$

Hence

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \cdots$$

(v) For the expansion of $e^{ax} \cos bx$, let $y = f(x) = e^{ax} \cos bx$.

Here $f(0) = 1$, when $x = 0$. Hence by successive differentiation,

$$y_n = f^n(x) = D^n(e^{ax} \cos bx) = r^n e^{ax} \cos(bx + n\phi)$$

where

$$r = \sqrt{a^2+b^2} \quad \text{and} \quad \tan\phi = \frac{b}{a}.$$

When $x = 0$, we get

$$(y_n)_0 = f^n(0) = r^n \cos n\phi = (a^2+b^2)^{n/2} \cos n\phi. \tag{1}$$

Now

$$\tan\phi = \frac{b}{a} \quad \text{or} \quad \cos\phi = \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{r}.$$

Therefore, $r \cos \phi = a$. Putting $n = 1, 2, 3, \ldots$ in (1), we obtain

$$(y_1)_0 = r\cos\phi = a$$
$$(y_2)_0 = r^2 \cos 2\phi = r^2(2\cos^2\phi - 1) = 2a^2 - (a^2+b^2) = a^2 - b^2$$
$$\begin{aligned}(y_3)_0 &= r^3 \cos 3\phi = r^3(4\cos^3\phi - 3\cos\phi) \\ &= r\cos\phi\,(4r^2\cos^2\phi - 3r^2) \\ &= a[4a^2 - 3(a^2+b^2)] \\ &= a(a^2 - 3b^2)\end{aligned}$$

Apply Maclaurin's series:

$$f(x) = f(0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$$

We get

$$e^{ax}\cos bx = 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + a(a^2 - 3b^2)\frac{x^3}{3!} + \cdots$$

Example 3.9 Prove that

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^2}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots$$

Solution Let

$$y = \sin^{-1}x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \tag{1}$$

$$y_1 = \frac{1}{\sqrt{1-x^2}} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots \tag{2}$$

For $|x| < 1$, applying binomial theorem, we get

$$y_1 = (1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots \tag{3}$$

Comparing (2) and (3), we have

$$a_2 = a_4 = a_6 = \cdots = a_{2n} = 0$$

and

$$a_1 = 1, \qquad 3a_3 = \frac{1}{2}, \qquad 5a_5 = \frac{1\cdot 3}{2\cdot 4}, \cdots$$

Substituting in (1), we get

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots$$

Example 3.10 If $e^x \sin x = \Sigma a_nx^n$, prove that

$$a_n - \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} - \frac{a_{n-3}}{3!} + \cdots = \frac{\sin(n\pi/2)}{n!}.$$

Solution Since $e^x \sin x = \Sigma a_nx^n$, we get

$$\sin x = \left(\sum a_nx^n\right)e^{-x} = \left(a_0 + a_1x + a_2x^2 + \cdots\right)e^{-x} \tag{1}$$

Now

$$f(x) = \sin x, \qquad f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$$

Therefore, by Maclaurin's series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots,$$

we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \cdots$$

Then from (1),

$$\sin x = (a_0 + a_1 x + \cdots + a_n x)\left(1 - x + \frac{x^2}{2!} - \cdots\right)$$

Equating coefficients of x_n, we get

$$a_n - \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} - \cdots \ = \sin\frac{n\pi/2}{n!}.$$

Example 3.11 Expand sec *x*.

Solution Let

$$\sec x = \frac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (1)$$

Therefore,

$$\begin{aligned} 1 &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\cos x \\ &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \end{aligned}$$

Equating the constant term and the coefficients of x, x^2, x^3, x^4, we get

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = 0, \quad a_4 = \frac{5}{24}$$

Substituting these values in (1), we get

$$\sec x = \frac{1}{\cos x} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$$

Example 3.12 Expand

$$\frac{e^x}{e^x + 1}$$

as far as the term in x^3.

Solution Let

$$\frac{e^x}{e^x + 1} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (1)$$

or

$$e^x = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(e^x + 1)$$

or

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\left(2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

Equating the constant term and the coefficients of x, x^2, x^3, we have

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{4}, \quad a_2 = 0, \quad a_3 = -\frac{1}{48}$$

Substituting these values in (1), we get

$$\frac{e^x}{e^x + 1} = \frac{1}{2} + \frac{1}{4}x - \frac{1}{48}x^3 + \cdots$$

Example 3.13 Expand $e^{\sin x}$ as far as the term involving x^4.

Solution Let

$$f(x) = e^{\sin x}$$
$$f'(x) = e^{\sin x} \cos x$$
$$f''(x) = e^{\sin x} \cos^2 x - e^{\sin x} \sin x$$
$$f'''(x) = e^{\sin x} \cos^2 x - \frac{3}{2} e^{\sin x} \sin 2x - e^{\sin x} \cos x$$
$$f^{IV}(x) = e^{\sin x} \cos^4 x - e^{\sin x}\, 3\cos^2 x \sin x - \frac{3}{2} e^{\sin x} \cos x \sin 2x$$
$$- 3e^{\sin x} \cos 2x - e^{\sin x} \cos^2 x + e^{\sin x} \sin x$$

When $x = 0$, we obtain

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 0, \quad f^{\mathrm{IV}}(0) = -3, \cdots$$

Applying Maclaurin's theorem:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{\mathrm{IV}}(0) + \cdots$$

We get

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

Example 3.14 Apply Maclaurin's theorem to expand $e^{\sin^{-1} x}$ as far as the term involving x^4.

Solution Let $y = e^{\sin^{-1} x}$. Differentiating, we get

$$y_1 = e^{\sin^{-1} x} \frac{1}{\sqrt{1 - x^2}} \quad \text{or} \quad (1 - x^2)\, y_1^2 = y^2$$

Differentiating again, we have

$$2y_1 y_2 (1 - x^2) - 2xy_1^2 = 2yy_1 \quad \text{or} \quad (1 - x^2) y_2 - xy_1 = y.$$

By Leibnitz's theorem,

$$y_{n+2}(1-x^2) - {}^nC_1 2xy_{n+1} - {}^nC_2 2y_n - xy_{n+1} - {}^nC_1 y_n = y_n$$

or

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - 2\frac{n(n-1)}{2}y_n - xy_{n+1} - ny_n = y_n$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0.$$

Putting $x = 0$,

$$y_{n+2_0} = (n^2+1)(y_n)_0$$

Now,

$$(y)_0 = e^0 = 1$$
$$(y_1)_0 = e^0 = 1$$
$$(y_2)_0 = (y_0)_0 = (y)_0 = 1$$
$$(y_3)_0 = (1^2+1)(y_1)_0 = 2$$
$$(y_4)_0 = (2^2+1)(y_2)_0 = 5.$$
$$\vdots$$

By Maclaurin's series, we get

$$e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \frac{2}{3!}x^3 + \frac{5}{4!}x^4 + \cdots$$

Example 3.15 If $e^{e^x} = \Sigma a_n x^n$, prove that

$$a_{n+1} = \frac{1}{n+1}\left(a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \cdots + \frac{a_0}{n!}\right)$$

Solution We have

$$e^{e^x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + a_{n+1}x^{n+1} + \cdots$$

Differentiating both sides with respect to x, we get

$$e^{e^x}e^x = a_1 + 2a_2x + \cdots + (n+1)\,a_{n+1}x^n + \cdots$$

or

$$(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n + \cdots)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= a_1 + 2a_2x + \cdots + (n+1)a_{n+1}x^n + \cdots$$

Equating the coefficient of x^n both the sides, we get

$$a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \frac{a_{n-3}}{3!} + \cdots + \frac{a_n}{n!} = (n+1)a_{n+1}$$

Hence

$$a_{n+1} = \frac{1}{n+1}\left(a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \cdots + \frac{a_0}{n!}\right).$$

Example 3.16 Prove that

$$f(mx) = f(x) + (m-1)xf'(x) + \frac{(m-1)^2}{2!}x^2 f''(x) + \cdots$$

Solution

$$f(mx) = f[x + (m-1)x] = f(x+h), \quad \text{where } h = (m-1)x$$
$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$
$$= f(x) + (m-1)xf'(x) + \frac{(m-1)^2}{2!}x^2 f''(x) + \cdots$$

Example 3.17 Expand $\sinh^{-1}x$.

Solution Let $y = \sinh^{-1}x$, then

$$y_1 = \frac{1}{\sqrt{1+x^2}} \quad \text{or} \quad \left(1+x^2\right)y^2 = 1$$

When $n = 0$, we get $f(0) = 0$, $f'(0) = 1$.

Differentiating (1) again, we get

$$(1+x^2)(2yy_1) + y_1^2(2x) = 0 \quad \text{or} \quad (1+x^2)y_2 + xy_1 = 0.$$

For $n = 0$, $(y_2)_0 = 0$. Differentiating n times, according to Leibnitz's theorem,

$$\left[(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2}(2y_n)\right] + \left[xy_{n+1} + n(1)y_n\right] = 0$$

or

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$

Putting $x = 0$, so $(y_{n+2})_0 = -n^2(y_n)_0$. Now putting successively $n = 1$, $n = 3$, $n = 2$, $n = 4$.

$$(y_3)_0 = -(1)^2(1) = -1$$
$$(y_5)_0 = -3^2(y_3)_0 = 3^2$$
$$(y_4)_0 = -2^2(y_2)_0 = 0$$
$$(y_6)_0 = 0$$
$$\vdots$$

Hence by Maclaurin's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$= 0 + x(1) + 0 + \frac{x^3}{3!}(-1) + 0 + \frac{x^5}{5!}3^2 + \cdots$$

$$= x - \frac{x^3}{3!} + 3^2\frac{x^5}{5!} - \cdots$$

Hence

$$\sinh^{-1}x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \cdots$$

Example 3.18 Expand log cos x by Maclaurin's theorem.

Solution Let $y = \log\cos x$. When $x = 0, f(0) = \log 1 = 0$. Differentiating successively, we get

$$y_1 = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$y_2 = -\sec^2 x = -(1 + \tan^2 x) = -(1 + y_1^2)$$

$$y_3 = -2y_1y_2$$

$$y_4 = -2(y_1y_3 + y_2^2)$$

$$y_5 = -2[(y_1y_4 + y_3y_2) + 2y_2y_3] = -2y_1y_4 - 6y_2y_3$$

$$y_6 = -2(y_1y_5 + y_4y_2) - 6(y_2y_4 + y_3^2)$$

$$\vdots$$

Putting $x = 0$, we obtain $(y_1)_0 = 0$, $(y_2)_0 = -1$, $(y_3)_0 = 0$, $(y_4)_0 = -2$, $(y_5)_0 = 0$, $(y_6)_0 = -16$, $\cdots$ Therefore,

$$f(x) = 0 + 0 + \frac{x^2}{2!}(-1) + \frac{x^4}{4!}(-2) + \frac{x^6}{6!}(1 - 16) + \cdots$$

$$\log\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots$$

$$= \frac{-x^2}{2!} - 2\frac{x^4}{4!} - 16\frac{x^6}{6!} - \cdots$$

Exercises 3.1

1. If $f(x) = (x - 1)(x - 2)(x - 3)$; $x \in [0, 4]$, find c.

2. Find 'c' so that $f'(c) = [f(b) - f(a)]/(b - a)$ in the following cases:

   (i) $f(x) = x^2 - 3x - 1$; $a = -11/7$, $b = 13/7$
   (ii) $f(x) = \sqrt{(x^2 - 4)}$; $a = 2$, $b = 3$
   (iii) $f(x) = e^x$; $a = 0$, $b = 1$
   (iv) $f(x) = \log x$; $a = 1/2$, $b = 2$.

3. In the mean value theorem, viz.,

$$f(a + h) = f(a) + hf'(a + \theta h)$$

   (i) If $a = 2$, $h = 1$ and $f(x) = \dfrac{1}{x}$, then find the value of θ.
   (ii) If $f(x) = \sin x$, find the limiting value of θ, when $h \to 0$.
   (iii) If $f(x) = e^x$, express the value of θ in terms of a and h.

4. In the equation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^{n-1}}{\underline{|n-1}} f^{n-1}(x) + \frac{h^n}{\underline{|n}} f^n(x + \theta_n h),$$

prove that the limiting value of θ as h is indefinitely diminished and other conditions being usual is $1/(n + 1)$.

5. Show that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a + \theta h),$$

where $0 < \theta < 1$ and prove that

$$\lim_{h \to 0} \theta = \frac{1}{3}.$$

6. Prove the following results by Taylor's theorem:

   (i) $\sin(x + h) = \sin x + h \cos x - \dfrac{h^3}{2!} \sin x - \ldots$

   (ii) $\sin^{-1}(x + h) = \sin^{-1} x + \dfrac{h}{\sqrt{1 - x^2}} + \dfrac{x}{(1 - x^2)^{3/2}} \dfrac{h^2}{2!} + \cdots$

   (iii) $\tan\left(\dfrac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \dfrac{8}{3}x^3 + \dfrac{10}{3}x^4 + \cdots$

   (iv) $\sec^{-1}(x + h) = \sec^{-1} x + \dfrac{h}{x\sqrt{x^2 - 1}} - \dfrac{2x^2 - 1}{x^2(x^2 - 1)^{3/2}} \dfrac{h^2}{2!} + \cdots$

   (v) $\log[\log(1 + e^x)] = \log 2 + \dfrac{x}{2} + \dfrac{1}{4}\dfrac{x^2}{2!} - \dfrac{1}{8}\dfrac{x^4}{4!} + \cdots$

7. Prove that

$$e^{x\cos x} = 1 + x + \frac{x^2}{2!} - 2\frac{x^3}{3!} + \cdots$$

8. Prove that

$$\log(1 + \sin^2 x) = x^2 - \frac{5}{6}x^4 + \cdots$$

9. Prove that

$$e^x \log(1 + x) = x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + \cdots$$

10. Prove that

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

11. Prove that

$$\frac{e^x}{\cos x} = 1 + x + 2\frac{x^2}{2!} + 4\frac{x^3}{3!} + \cdots$$

12. Prove the following:

(i) $\log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots$

(ii) $\frac{\log \sin x}{x} = -\frac{x^2}{6} - \frac{x^4}{180} - \cdots$

(iii) $\log\left(\frac{\tan x}{x}\right) = \frac{x^2}{3} + \frac{7}{90}x^4 + \cdots$

(iv) $\log\left(\frac{\tan^{-1} x}{x}\right) = -\frac{x^2}{3} + \frac{13}{90}x^4 - \cdots$

(v) $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} - \cdots$

(vi) $\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$

(vii) $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^6}{5} + \cdots$

13. Prove that

$$\frac{1}{2}[\log(1 + x)]^2 = \frac{x^2}{2} - \left(1 + \frac{1}{2}\right)\frac{x^3}{3} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{x^4}{4} + \cdots$$

14. Prove that

$$e^x \sin x = x + \frac{2x^2}{2!} + \frac{2x^3}{3!} - \frac{4x^5}{5!} - \cdots$$

15. Prove that

$$e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots$$

16. Prove that

$$\cos x \cosh x = 1 - \frac{2^2 x^4}{4!} + \frac{2^4 x^8}{8!} - \frac{2^6 x^{12}}{12!} + \cdots$$

17. Prove that

$$\sin x \cosh x = x + \frac{2x^3}{3!} - \frac{4x^5}{5!} + \cdots$$

18. Prove that

$$e^{ax} \cosh x = 1 + ax + (a^2+1)\frac{x^2}{2!} + (a^3+3a)\frac{x^3}{3!} + \cdots$$

19. Prove that

$$e^{ax} \sin bx = bx + abx^2 - \frac{3a^2b - b^3}{3!}x^3 + \cdots + \frac{(a^2+b^2)^{n/2}}{n!} x^n \sin\left(n \tan^{-1}\frac{b}{a}\right) + \cdots$$

20. Prove that

(i) $e^{\sec x} = e\left(1 + \frac{x^2}{2!} + \frac{8x^4}{4!} + \cdots\right)$

(ii) $e^{x\sin x} = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^5 + \cdots$

(iii) $e^{x\sec x} = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$

(iv) $\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots$

21. Expand $e^{ax} \cos bx$ by Maclaurin's theorem and deduce that

$$e^{x\cos\alpha} \cos(x \sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \cdots$$

22. Prove taht

(i) $x \cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots$

(ii) $x \operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$

23. Expand $x \coth x$.

24. Expand $\log [x + \sqrt{(1 + x^2)}]$ by Maclaurin's theorem in ascending powers of x and find the general term.

25. Expand $\frac{\log(1+x)}{1+x}$ in powers of x as far as x^3.

26. Expand $\log [1 - \log (1 - x)]$ in powers of x by Maclaurin's theorem up to the terms of x^3 and deduce the expansion of $\log [1 + \log (1 + x)]$.

27. Expand $\sin x$ in powers of $(x - \pi/2)$.

28. Prove that

$$a^x = 1 + x \log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \cdots + \frac{x^n}{n!}\left(\log_e a^n\right)(\theta x).$$

29. Show that

$$\sin \alpha x = \alpha x - \frac{\alpha^3 x^3}{3!} + \frac{\alpha^5 x^5}{5!} - \cdots \frac{a^n x^n}{n!} \sin \frac{n\pi}{2}$$

and that the remainder after r terms may be expressed as

$$\frac{\alpha^r x^r}{r!} \sin\left(\alpha\theta x + \frac{r\pi}{2}\right).$$

30. Show that the remainder after r terms of $e^{ax} \cos bx$ have been taken is

$$\frac{(a^2 + b^2)^{r/2}}{r!} x^r e^{a\theta x} \cos\left(b\theta x + r \tan^{-1} \frac{b}{a}\right).$$

31. Prove that

$$e^{a + bx + cx^2 + dx^3 + \cdots} = e^a\left(1 + bx + \frac{b^2 + 2c}{2!} + \frac{b^3 + 6bc + 6d}{3!} + \cdots\right)$$

32. If $u = f(x)$, show that

$$f\left(\frac{x}{2}\right) = u - \frac{x}{2}\frac{du}{dr} + \frac{1}{2!}\left(\frac{x}{2}\right)^2 \frac{d^2u}{dx^2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 \frac{d^3u}{dx^3} + \cdots$$

33. If $e^x = \log (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots)$, prove that

$$(n+1)a_{n+1} = a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \frac{a_{n-3}}{3!} + \cdots + \frac{a_0}{n!}.$$

34. If A_0, A_1, etc., to the successive coefficients in the expansion of $y = e^{\cos mx + \sin mx}$, prove that

$$A_{n+1} = \frac{m}{n+1}\left[A_n + \sum_1^n \frac{m^r}{r!} A_{n-r}\left(\cos\frac{\pi r}{2} - \sin\frac{\pi r}{2}\right)\right].$$

35. Given that

$$\sin\log(1+x)=\frac{A_1}{1!}x+\frac{A_2}{2!}x^2+\frac{A_3}{3!}x^3+\cdots$$

and

$$\cos\log(1+x)=1+\frac{B_1}{1!}x+\frac{B_2}{2!}x^2+\frac{B_3}{3!}x^3+\cdots$$

calculate the first-five coefficients of each expansion.

36. From the expansion of

$$\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right),$$

deduce

$$\tan^{-1}x=\frac{x}{1+x^2}\left[1+\frac{2}{3}\frac{x^2}{1+x^2}+\frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2+\cdots\right].$$

Also, establish the series:

(i) $\dfrac{\pi}{2}=1+\dfrac{1}{3}+\dfrac{1}{3}\dfrac{2}{5}+\dfrac{1}{3}\dfrac{2}{5}\dfrac{3}{7}+\cdots$

(ii) $\dfrac{2\pi}{3\sqrt{3}}=1+\dfrac{1}{3}\left(\dfrac{1}{2}\right)+\dfrac{1}{3}\dfrac{2}{5}\left(\dfrac{1}{2}\right)^2+\dfrac{1}{3}\dfrac{2}{5}\dfrac{3}{7}\left(\dfrac{1}{2}\right)^3+\cdots$

Chapter 4

Indeterminate Forms

4.1 Introduction

The function $f(x) = f(a)$, when $x = a$, is called the *value* of any function $f(x)$, provided it is finite and unique. However, at times, $f(a)$ may assume any of the following forms. These forms are called *indeterminate* forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0 \times \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

In such cases, $x = a$ is called a *limiting point* of the functions $f(x)$. Although its value $f(a)$ does not exist there, we may evaluate its limiting value $\lim_{x \to a} f(x)$ in certain cases.

Let us suppose a rational function

$$f(x) = \frac{\psi(x)}{\phi(x)},$$

where $\lim_{x \to a} \psi(x)$ and $\lim_{x \to a} \phi(x)$ are both zeros. Then $\lim_{x \to a} f(x)$ cannot be equal to

$$\frac{\lim\limits_{x \to a} \psi(x)}{\lim\limits_{x \to a} \phi(x)}$$

as it becomes of the form $\frac{0}{0}$, which is meaningless and undefined. This form $\frac{0}{0}$ may be taken fundamental because all other indeterminate forms may be reduced to a problem corresponding to the form $\frac{0}{0}$. This form can be evaluated by L'Hospital theorem, which states as

$$\lim_{x \to a} \frac{\psi(x)}{\phi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} = \lim_{x \to a} \frac{\psi''(x)}{\phi''(x)} = \cdots$$

In other words, this means that the numerator and the denominator should be differentiated repeatedly unless the form $\frac{0}{0}$ vanishes and the limit obtained is finite zero, constant or infinite.

4.2 L'Hospital Rule

Form $\frac{0}{0}$

Let $\phi(x)$ and $\psi(x)$ be functions of x capable of being expanded by Taylor's theorem, and if $\psi(a) = 0 = \phi(a)$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)},$$

provided the latter limit exists.

Let $x = a + h$. Then $x \to a$ as $h \to 0$. Now according to Taylor's theorem, we have

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \cdots$$

and

$$\psi(a+h) = \psi(a) + h\psi'(a) + \frac{h^2}{2!}\psi''(a) + \cdots$$

Therefore,

$$\frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi(a) + h\phi'(a) + (h^2/2!)\phi''(a) + \cdots}{\psi(a) + h\psi'(a) + (h^2/2!)\,\psi''(a) + \cdots}$$

$$= \frac{h\phi'(a) + (h^2/2!)\phi''(a) + \cdots}{h\psi'(a) + (h^2/2!)\psi''(a) + \cdots} \qquad [\text{as } \phi(a) = 0 = \psi(a)]$$

$$= \frac{\phi'(a) + (h/2!)\,\phi''(a) + \cdots}{\psi'(a) + (h/2!)\,\psi''(a) + \cdots}$$

Now, taking $\lim_{h\to 0}$, when $x \to a$.

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{h\to 0}\frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi'(a)}{\psi'(a)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

Again, if

$$\frac{\phi'(a)}{\psi'(a)} = \frac{0}{0}$$

Then proceeding as before, we get

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \frac{\phi''(a)}{\psi''(a)} = \lim_{x\to a}\frac{\phi''(x)}{\psi''(x)}$$

Generally if

$$\phi'(a) = 0 = \phi''(a) = \cdots = \phi^{n-1}(a)$$

and

$$\psi'(a) = 0 = \psi''(a) = \cdots = \psi^{n-1}(a),$$

Then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \frac{\phi^n(a)}{\psi^n(a)} = \lim_{x \to a} \frac{\phi^n(x)}{\psi^n(x)}.$$

In other words, under this rule, we shall go on differentiating successively the numerator $\phi(x)$ and the denominator $\psi(x)$ separately. Each time we shall put $x = a$ in the differential coefficient. This process will continue till the form $\frac{0}{0}$ stops to occur.

Example 4.1 Evaluate:

$$\lim_{x \to (0)} \frac{e^x - e^{\sin x}}{x - \sin x}$$

Solution It is of the form of $\frac{0}{0}$ as $x \to 0$. Given

$$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos x}{1 - \cos x} \qquad \left(\text{form } \frac{0}{0}\right)$$

Differentiating again numerator and denominator separately, we get

$$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} \sin x}{\sin x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} 2 \cos x \sin x + e^{\sin x} \cos x \sin x + e^{\sin x} \cos x}{\cos x}$$

$$= \frac{1 - 1 + 0 + 0 + 1}{1}$$

$$= 1$$

Example 4.2 Evaluate:

$$\lim_{x \to \pi/4} \frac{1 - \tan x}{1 - \sqrt{2} \sin x}.$$

Solution It is of the form of $\frac{0}{0}$ as $x \to \pi/4$. Thus differentiating numerator and denominator, we get

$$\lim_{x \to \pi/4} \frac{-\sec^2 x}{-\sqrt{2} \cos x} = \frac{\sec^2(\pi/4)}{\sqrt{2} \cos(\pi/4)} = \frac{(\sqrt{2})^2}{\sqrt{2}(1/\sqrt{2})} = \frac{2}{1} = 2$$

Example 4.3 Evaluate:

$$\lim_{x\to 0}\frac{\log(1+kx^2)}{1-\cos x}.$$

Solution It is of the form of $\frac{0}{0}$, as $x \to 0$. Differentiating numerator and denominator of the given function, we get

$$\lim_{x\to 0}\frac{\log(1+kx^2)}{1-\cos x}=\lim_{x\to 0}\frac{2kx/(1+kx^2)}{\sin x}$$

$$=\lim_{x\to 0}\frac{2kx}{(1+kx^2)\sin x}$$

$$=\lim_{x\to 0}\frac{2k}{2k\,x\sin x+(1+kx^2)\cos x}$$

$$=\frac{2k}{0+(1+0)1}$$

$$=2k.$$

Example 4.4 Prove that

$$\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\frac{3}{2}.$$

Solution It is of the form of $\frac{0}{0}$, as $x \to 0$. Now

$$\frac{xe^x-\log(1+x)}{x^2}=\frac{x\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}{x^2}$$

$$=\frac{\left(x^2+\frac{x^2}{2}\right)+x^3\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots}{x^2}$$

$$=\frac{3}{2}+\frac{1}{6}x+\text{higher powers of }x$$

Therefore,

$$\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\lim_{x\to 0}\left(\frac{3}{2}+\frac{1}{6}x+\text{higher powers of }x\right)=\frac{3}{2}$$

Example 4.5 Find the value of a and b, so that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$$

Solution Now, the expression

$$x\left[1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\right]-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)$$

$$=\frac{(1+a-b)x+\left(b/6-a/2\right)x^3+\cdots}{x^3}$$

or

$$\lim_{x\to 0}\left[\frac{(1+a-b)x+\left(b/6-a/2\right)x^3+\cdots}{x^3}\right]=1$$

Equating both sides, we get

$$1+a-b=0 \quad \text{and} \quad \frac{b}{6}-\frac{a}{2}=1$$

or

$$a-b=-1 \quad \text{and} \quad b-3a=6$$

Solving them, we get $a = -5/2$, $b = -3/2$.

Example 4.6 Evaluate:

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}.$$

Solution It is of the form of $\frac{0}{0}$, as $x \to 0$. Now, we have

$$(1+x)^{1/x}=\exp\left[\frac{1}{x}\log\left(1+x\right)\right]$$

$$=\exp\left[\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)\right]$$

$$=\exp\left(1-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)$$

$$=e\exp\left(-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^3}{3} - \cdots\right)^2\right.$$

$$\left. + \frac{1}{3!}\left(-\frac{x}{2} + \frac{x^3}{3} - \cdots\right)^3 + \cdots\right]$$

$$= e\left\{1 + \left(-\frac{x}{2} + \frac{x^2}{3}\right) + \frac{1}{2}\left[\left(\frac{x^2}{4} - 2\times\frac{1}{6}x^2\right)\right.\right.$$

$$\left.\left. + x^4\left(\frac{1}{9} + 2\times\frac{1}{2}\times\frac{1}{4}\right) + \frac{1}{6}\left(-\frac{x^2}{8} + 3\frac{x^2}{4}\frac{x^2}{3} + \cdots\right) + \cdots\right]\right\}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right) + \frac{1}{2}\left(\frac{x^2}{4} - \frac{x^3}{3} + \frac{13}{36}x^4 - \cdots\right)\right.$$

$$\left. + \frac{1}{6}\left(-\frac{x^3}{8} + \frac{x^4}{4} + \cdots\right) + \cdots\right]$$

$$= e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \cdots\right)$$

Therefore,

$$\frac{(1+x)^{1/x} - e}{x} = \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \cdots\right) - e}{x}$$

$$= \frac{-\frac{e}{2}x + \frac{11}{24}ex^2 - \cdots}{x}$$

$$= -\frac{e}{2} + \frac{11}{24}ex - \cdots$$

Hence

$$\lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = -\frac{e}{2}.$$

Example 4.7 Evaluate:

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}.$$

Solution The given limit

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x} = \lim_{x\to 0}\frac{\frac{d}{dx}\left[e^x - e^{-x} - 2\log(1+x)\right]}{\frac{d}{dx}(x\sin x)}$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} - 2/(1+x)}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x} + 2\frac{1}{(1+x)^2}}{\cos x + \cos x - x\sin x}$$

$$= 1$$

Example 4.8 Evaluate:

$$\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}.$$

Solution It is of the form of $\frac{0}{0}$, as $x \to 0$. Now

$$\lim_{x\to 0}\frac{\cos x - x\sin x - 1/(1+x)}{2x} = \lim_{x\to 0}\frac{-\sin x - \sin x - x\cos x + 1/(1+x)^2}{2} = \frac{1}{2}.$$

Example 4.9 Evaluate:

$$\lim_{\theta\to 0}\frac{e^{\theta} + e^{-\theta} + 2\cos\theta - 4}{\theta^4}.$$

Solution If is of the form of $\frac{0}{0}$ as $\theta \to 0$. Then

$$\text{The given limit} = \lim_{\theta\to 0}\frac{e^{\theta} + e^{-\theta} + 2\cos\theta - 4}{\theta^4}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to 0}\frac{e^{\theta} - e^{-\theta} - 2\sin\theta}{4\theta^3}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to 0}\frac{e^{\theta} + e^{-\theta} - 2\cos\theta}{12\theta^2}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to 0}\frac{e^{\theta} - e^{-\theta} + 2\sin\theta}{24\theta}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta \to 0} \frac{e^{\theta} - e^{-\theta} + 2\cos\theta}{24}$$

$$= \frac{4}{24}$$

$$= \frac{1}{6}.$$

Form $\frac{\infty}{\infty}$

Let $\phi(a) = \infty$, $\psi(a) = \infty$, so that $\frac{\phi(x)}{\psi(x)}$ takes the form $\frac{\infty}{\infty}$, when x approaches indefinitely near a value a. In order to bring it to the form $\frac{0}{0}$, we write it as follows:

$$\frac{\phi(x)}{\psi(x)} = \frac{1/\psi(x)}{1/\phi(x)}$$

Here

$$\frac{1}{\psi(a)} = \frac{1}{\infty} = 0 \qquad \text{and} \qquad \frac{1}{\phi(a)} = \frac{1}{\infty} = 0,$$

we may consider this as taking the form $\frac{0}{0}$, and therefore we may apply the previous rule.

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{1/\psi(x)}{1/\phi(x)} = \lim_{x \to a} \frac{\psi'(x)/[\psi(x)]^2}{\phi'(x)/[\phi(x)]^2} = \lim_{x \to a} \left[\frac{\phi(x)}{\psi(x)}\right]^2 \frac{\psi'(x)}{\phi'(x)}$$

Therefore,

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \left[\lim_{x \to 0} \frac{\phi(x)}{\psi(x)}\right]^2 \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}$$

Hence unless

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)}$$

be zero or infinite, we have

$$1 = \left[\lim_{x \to a} \frac{\phi(x)}{\psi(x)}\right]\left[\lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}\right],$$

or

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Example 4.10 Find

$$\lim_{\theta \to \pi/2} \frac{\log(\theta - \pi/2)}{\tan\theta}$$

Solution Given

$$\lim_{\theta\to\pi/2}\frac{\log(\theta-\pi/2)}{\tan\theta}=\lim_{\theta\to\pi/2}\frac{1/(\theta-\pi/2)}{\sec^2\theta}, \qquad \left[\frac{\infty}{\infty}\right]$$

$$=\lim_{\theta\to\pi/2}\frac{\cos^2\theta}{\theta-\pi/2}, \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{\theta\to\pi/2}\left(-\frac{2\cos\theta\sin\theta}{1}\right)$$

$$=0.$$

Example 4.11 Evaluate:

$$\lim_{x\to\infty}\frac{x^n}{e^x}.$$

Solution This is of the form of $\frac{\infty}{\infty}$. Now, we have

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}$$

$$=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$\vdots$$

$$=\lim_{x\to\infty}\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{e^x}$$

$$=\lim_{x\to\infty}\frac{n!}{e^x}$$

$$=0.$$

Example 4.12 Evaluate: $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$.

Solution It is of the form of $\frac{\infty}{\infty}$.

$$\text{Given limit}=\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$$

$$=\lim_{x\to a}\frac{1/(x-a)}{e^x/(e^x-e^a)}$$

$$=\lim_{x\to a}\frac{e^x-e^a}{x-a}\frac{1}{e^x} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to a}\frac{e^x}{(x-a)e^x+e^x}$$

$$=\frac{e^a}{e^a}$$

$$=1.$$

Example 4.13 Evaluate: $\lim_{x\to 0} \log(\tan x)^{\tan 2x}$.

Solution Since we know $\log_a m \log_e a = \log_e m$, the given expression can be written as

$$\lim_{x\to 0} \frac{\log_e(\tan 2x)}{\log_e(\tan x)}, \qquad \left[\frac{\infty}{\infty}\right]$$

Therefore,

$$\lim_{x\to 0} \frac{\log_e \tan(2x)}{\log_e \tan x} = \lim_{x\to 0} \frac{(1/\tan 2x)\times \sec^2(2x)(2)}{(1/\tan x)\sec^2 x}$$

$$= \lim_{x\to 0} \frac{\dfrac{\cos 2x}{\sin 2x}\dfrac{1}{\cos^2(2x)}2}{\dfrac{\cos x}{\sin x}\dfrac{1}{\cos^2 x}}$$

$$= \lim_{x\to 0} \frac{2}{\sin 2x \cos 2x} \sin x \cos x$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\sin 2x \cos 2x}$$

$$= \lim_{x\to 0} \frac{1}{\cos 2x}$$

$$= 1.$$

Example 4.14 Evaluate:

$$\lim_{x\to 0} \frac{\log x^2}{\log \cot^2 x}.$$

Solution It is of the form of $\frac{\infty}{\infty}$. Therefore,

$$\lim_{x\to 0} \frac{\log x^2}{\log \cot^2 x} = \lim_{x\to 0} \frac{1/x}{\cot x(-\operatorname{cosec}^2 x)},$$

$$= -\lim_{x\to 0} \frac{\sin x \cos x}{x}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}(-1)\frac{\cos^2 x - \sin^2 x}{1}$$

$$= -1.$$

Form ∞ − ∞

Suppose $\phi(a) = \infty$ and $\psi(a) = \infty$, so that $\phi(x) - \psi(x)$ takes the form $\infty - \infty$. When L'Hospital rule is applied to this expression approaches a value a.

Let us suppose

$$u = \phi(x) - \psi(x) = \psi(x)\left[\frac{\phi(x)}{\psi(x)} - 1\right].$$

Here, if

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = 1,$$

the limit of u becomes

$$\psi(a) \times (\text{a quantity which is not zero})$$

Therefore, the function tends to infinity. But if

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = 1,$$

then the function takes the form of $\infty \times 0$.

Example 4.15 Evaluate:

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right].$$

Solution We write

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right] = \lim_{x\to 2} \frac{\log(x-1)-(x-2)}{(x-2)\log(x-1)}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 2} \frac{1/(x-1)-1}{\log(x-1)+[1/(x-1)](x-2)}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 2} \frac{-\dfrac{1}{(x-1)^2}}{\dfrac{1}{x-1} + \dfrac{(x-1)-(x-2)}{(x-1)^2}}$$

$$= -\lim_{x\to 2}\left[\frac{1}{(x-1)^2}\,\frac{(x-1)^2}{x-1+1}\right]$$

$$= \frac{1}{2}.$$

Example 4.16 Evaluate:

$$\lim_{x\to\pi/2}\left(x\tan x - \frac{\pi}{2}\sec x\right)$$

Solution We have

$$\text{The given limit} = \lim_{x\to\pi/2}\left(x\tan x - \frac{\pi}{2}\sec x\right) \qquad [\infty - \infty]$$

$$= \lim_{x\to\pi/2}\left(\frac{x\sin x}{\cos x} - \frac{\pi}{2\cos x}\right)$$

$$= \lim_{x\to\pi/2}\frac{2x\sin x - \pi}{2\cos x}$$

$$= \lim_{x\to\pi/2}\frac{2x\cos x + 2\sin x}{-2\sin x}$$

$$= \frac{2}{-2}$$

$$= -1.$$

Example 4.17 Find the limit: $\frac{1}{x} - \cot x$, when $x \to 0$.

Solution It is of the form of $\infty - \infty$.

$$\text{The given limit} = \lim_{x\to 0}\left(\frac{1}{x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x}, \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x}$$

$$= 0.$$

Example 4.18 Evaluate:

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right).$$

Solution It is of the form of $\infty - \infty$. Then

$$\text{The given limit} = \lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\sin 2x - 2x}{2x\sin^2 x + x^2\sin 2x}$$

$$= \lim_{x\to 0}\frac{2\cos 2x - 2}{2\sin^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}$$

$$= \lim_{x \to 0} \frac{\cos 2x - 1}{\sin^2 x + 2x \sin 2x + x^2 \cos 2x}$$

$$= \lim_{x \to 0} \frac{-2 \sin 2x}{\sin 2x + 2 \sin 2x + 4x \cos 2x + 2x \cos 2x - 2x^2 \sin 2x}$$

$$= \lim_{x \to 0} \frac{-2 \sin 2x}{3 \sin 2x + 6x \cos 2x - 2x^2 \sin 2x}$$

$$= \lim_{x \to 0} \frac{-4 \cos 2x}{6 \cos 2x + 6 \cos 2x - 12x \sin 2x - 4x \sin 2x - 4x^2 \cos 2x}$$

$$= -\frac{1}{3}.$$

Example 4.19 Evaluate:

$$\lim_{x \to 0} \left(\frac{1}{2x^2} - \frac{1}{x \tan 2x} \right).$$

Solution It is of the form of $\infty - \infty$ as $x \to 0$.

$$\text{The given limit} = \lim_{x \to 0} \left(\frac{1}{2x^2} - \frac{1}{x \tan 2x} \right)$$

$$= \lim_{x \to 0} \left(\frac{1}{2x^2} - \frac{\cos x}{2x \sin x} \right)$$

$$= \lim_{x \to 0} \frac{\sin x - x \cos x}{2x^2 \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x - \cos x + x \sin x}{4x \sin x + 2x^2 \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x + x \cos x}{4 \sin x + 4x \cos x + 4x \cos x - 2x^2 \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x + \cos x - x \sin x}{4 \cos x + 8 \cos x - 8x \sin x - 4x \sin x - 2x^2 \cos x}$$

$$= \frac{1}{6}.$$

Forms 0^0, ∞^0 and 1^∞

Let $y = u^v$, where u and v being functions of x. Then $\log y = v \log_e u$. Now

$$\log_e l = 0, \qquad \log_e \infty = \infty, \qquad \log_e 0 = -\infty,$$

and therefore when the expression u^v takes one of the forms 0° ∞°, 1^∞, where $\log y$ takes the indeterminate form, $0 \times \infty$. The need is therefore to take the logarithm and proceed to reciprocal method.

Example 4.20 Find $\lim_{x\to 0} x^x$.

Solution It is of the form of 0^0. Taking logarithm both sides, we get

$$\begin{aligned}\lim_{x\to 0} \log x^x &= \lim_{x\to 0} x\log x \\ &= \lim_{x\to 0} \frac{\log x}{1/x} \\ &= \lim_{x\to 0} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0} (-x) = 0.\end{aligned}$$

Therefore,

$$\lim_{x\to 0} x^x = e^0 = 1.$$

Example 4.21 Find

$$\lim_{x\to \pi/2} (\sin x)^{\tan x}.$$

Solution It is of the form of 1^∞. Now, we get

$$\begin{aligned}\lim_{x\to \pi/2} (\sin x)^{\tan x} &= \lim_{x\to \pi/2} e^{\tan x \log \sin x} \\ &= \lim_{x\to \pi/2} \tan x \log\sin x \\ &= \lim_{x\to \pi/2} \frac{\log\sin x}{\cot x} \\ &= \lim_{x\to \pi/2} \frac{(1/\sin x)\cos x}{-\operatorname{cosec}^2 x} \\ &= \lim_{x\to \pi/2} \frac{\cot x}{\operatorname{cosec}^2 x} \\ &= \lim_{x\to \pi/2} (-\sin x \cos x) \\ &= 0.\end{aligned}$$

Therefore, the required limit = e^0 = 1.

Example 4.22 Determine, $\lim_{x\to 0} (\cos x)^{1/x^2}$.

Solution It is of the form of 1^∞. We have

$$y = (\cos x)^{1/x^2}.$$

Therefore,

$$\log y = \frac{1}{x^2} \log \cos x .$$

Taking limit both sides, as $x \to 0$, we get

$$\begin{aligned}\lim_{x\to 0} \log y &= \lim_{x\to 0} \frac{\log \cos x}{x^2}\\ &= \lim_{x\to 0} \frac{(1/\cos x)\sin x(-1)}{2x}\\ &= \lim_{x\to 0} -\frac{\tan x}{2x}\\ &= -\frac{1}{2}.\end{aligned}$$

or

$$\lim_{x\to 0} y = e^{-1/2}$$

Hence

$$\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}.$$

Example 4.23 Evaluate:

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/2}.$$

Solution It is of the form of 1^∞. Taking log both sides, we get

$$\log y = \frac{1}{2}\left(\log \frac{\tan x}{x}\right) = \frac{1}{2} \log\left[1 + \left(\frac{x^2}{3} + \frac{2}{15}x^4\right)\right]$$

Since we have,

$$\begin{aligned}\log\left(\frac{\tan x}{x}\right) &= \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)^2 + \cdots\\ &= \frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{2}\frac{1}{9}\right) + \cdots\\ &= \frac{x^2}{3} + \frac{7}{90}x^4 + \text{higher powers of } x,\end{aligned}$$

We have

$$\log y = \frac{x^2}{3} + \frac{7}{90}x^4 + \cdots$$

Therefore,

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left(\frac{x^2}{3}+\frac{7}{90}x^4+\cdots\right)=0$$

or

$$\lim_{x\to 0} y = e^0 = 1$$

Hence

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/2}=1.$$

Example 4.24 Evaluate:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}.$$

Solution Let

$$y=\left(\frac{\sin x}{x}\right)^{1/x^2},$$

Taking logarithm both sides

$$\log y = \frac{1}{x^2}\log\frac{\sin x}{x}. \qquad (1)$$

But

$$\sin x = x - \frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

or

$$\frac{\sin x}{x} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots$$

or

$$\log\frac{\sin x}{x} = \log\left[1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\right)\right]$$

$$=\log\left[1-\left(\frac{x^2}{6}-\frac{x^4}{120}+\cdots\right)\right]$$

$$=-\left(\frac{x^2}{6}-\frac{x^4}{120}+\cdots\right)-\frac{1}{2}\left(\frac{x^2}{6}-\frac{x^4}{120}+\cdots\right)^2+\frac{1}{3}\left(\frac{x^2}{6}-\frac{x^4}{120}+\cdots\right)^3+\cdots$$

$$=-\frac{x^2}{6}-\frac{x^4}{180}+\cdots$$

Therefore, from (1), we have

$$\log y = \frac{1}{x^2}\left(-\frac{x^2}{6} - \frac{x^4}{180} + \cdots\right) = -\frac{1}{6} - \frac{x^2}{180} + \text{higher power of } x.$$

Therefore,

$$\lim_{x\to 0} \log y = -\frac{1}{6} \qquad \text{or} \qquad \lim_{x\to 0} y = e^{-1/6}$$

Hence

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}.$$

Example 4.25 Evaluate:

$$\lim_{x\to 0} (\cos x)^{\cot^2 x}.$$

Solution It is of the form of 1^∞. Let

$$y = (\cos x)^{\cot^2 x},$$

or

$$\log y = \cot^2 x \log \cos x = \frac{\log \cos x}{\tan^2 x}, \qquad \left[\frac{0}{0}\right]$$

Taking limit both sides, we get

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{(1/\cos x)(-\sin x)}{2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} -\frac{1}{2\sec^2 x}$$

$$= -\frac{1}{2}.$$

Therefore,

$$\lim_{x\to 0} y = e^{-1/2}$$

Example 4.26 Evaluate:

$$\lim_{x\to a}\left(2 - \frac{x}{a}\right)^{\tan[\pi x/(2a)]}$$

Solution It is of the form of 1^∞. Let

$$y = \left(2 - \frac{x}{a}\right)^{\tan[\pi x/(2a)]}$$

or

$$\log y = \tan\frac{\pi x}{2a}\log\left(2-\frac{x}{a}\right) = \frac{\log\left(2-\frac{x}{a}\right)}{\cot\frac{\pi x}{2a}}, \qquad \left[\frac{0}{0}\right]$$

Therefore,

$$\lim_{x\to a}\log y = \lim_{x\to a}\frac{\frac{1}{2-x/a}}{\left(-\text{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)}$$

$$= \lim_{x\to a}\frac{-1/a}{\left(-\text{cosec}^2\frac{\pi}{2}\right)\left(\frac{\pi}{2a}\right)}$$

$$= \frac{2}{\pi}$$

or

$$\lim_{x\to a} y = e^{2/\pi}$$

Thus

$$\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan[\pi x/(2a)]} = e^{2/\pi}.$$

Example 4.27 Evaluate $(\sin x)^{\tan x}$ when $x \to 0$ and $x \to r$.

Solution Let $y = (\sin x)^{\tan x}$. Taking logarithms both sides, we get

$$\log y = \tan x \log\sin x = \frac{\log\sin x}{\cot x}$$

or

$$\log y = \frac{\log\sin x}{\cot x}, \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \frac{(1/\sin x)\cos x}{-\text{cosec}^2 x}$$

$$= -\frac{\cos x}{\sin x}\sin^2 x$$

$$= -\sin x\cos x$$

$$= 0 \qquad \text{(when } x \to 0,\ x \to r\text{)}$$

Therefore,

$$\lim_{x \to 0,r} y = e^0 = 1.$$

Thus

$$\lim_{x \to 0,r} (\sin x)^{\tan x} = 1.$$

Example 4.28 Find

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x.$$

Solution Let

$$y = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x,$$

$$\log y = \lim_{x \to \infty} x \log\left(1 + \frac{a}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\log(1 + a/x)}{1/x}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{1 + a/x}(-a/x)}{-1/x^2}$$

$$= \lim_{x \to \infty} \frac{a}{1 + a/x}$$

$$= a$$

or

$$y = e^a.$$

Thus

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

Example 4.29 Evaluate:

$$\lim_{x \to 0} (\cot x)^{1/\log x}.$$

Solution It is of the form of ∞^0. Let

$$y = \lim_{x \to 0} (\cot x)^{1/\log x}$$

Taking logarithms both sides, we get

$$\log y = \lim_{x\to 0} \frac{1}{\log x} \log\cot x, \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{(1/\cot x)(-\text{cosec}^2 x)}{x}$$

$$= \lim_{x\to 0} \frac{-x}{\sin x \cos x}$$

$$= \lim_{x\to 0} \left(-\frac{2x}{\sin 2x}\right)$$

$$= -1.$$

or

$$y = e^{-1}.$$

Therefore,

$$\lim_{x\to 0} (\cot x)^{1/\log x} = e^{-1}.$$

Example 4.30 Evaluate:

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2},$$

Solution It is of the form of 1^∞. Let

$$y = \left(\frac{\tan x}{x}\right)^{1/x^2},$$

Taking log, we get

$$\log y = \frac{\log\tan x - \log x}{x^2}, \qquad \left[\frac{0}{0}\right]$$

$$= \frac{\dfrac{\sec^2 x}{\tan x} - \dfrac{1}{x}}{2x}$$

$$= \frac{x\sec^2 x - \tan x}{2x^2 \tan x}$$

$$= \frac{\sec^2 x + 2x\sec^2 x \tan x - \sec^2 x}{4x\tan x + 2x^2 \sec^2 x}$$

$$= \frac{\sec^2 x \tan x}{2\tan x + x\sec^2 x}$$

or

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\sin^2 x \dfrac{\tan x}{x}}{2\dfrac{\tan x}{x} + \sec^2 x} = \frac{1}{3}$$

or

$$\lim_{x\to 0} y = e^{1/3}$$

Thus

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}.$$

Exercises 4.1

1. Determine the limits of the following:

(i) $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ (ii) $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$ (iii) $\lim_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$

(iv) $\lim_{x\to 0} \dfrac{\sin x \log(1+\sin x)}{\log(1+\sin x)}$ (v) $\lim_{x\to 0} \dfrac{x^n - a^n}{x - a}$ (vi) $\lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{\tan^3 x}$

(vii) $\lim_{x\to 0} \dfrac{x - \sin x \cos x}{x^3}$ (viii) $\lim_{x\to 1} \dfrac{1 + \log x - x}{1 - 2x + x^2}$ (ix) $\lim_{x\to 0} \dfrac{\log(1-x^2)}{\log\cos x}$

(x) $\lim_{x\to \frac{1}{2}} \dfrac{\cos^2(\pi x)}{e^{2x} - 2ex}$ (xi) $\lim_{x\to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$ (xii) $\lim_{x\to 0} \dfrac{xe^x - \log(x+1)}{\cosh x - \cos x}$

(xiii) $\lim_{x\to 0} \dfrac{\sin hx - x}{\sin x - x\cos x}$ (xiv) $\lim_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

(xv) $\lim_{x\to 0} \dfrac{x - \log(1+x)}{x^2}$.

2. If the limit of

$$\frac{\sin 2x + a\sin x}{x^3}$$

is finite, as $x \to 0$, find the value of a and the limit.

3. Find the limit of

$$\frac{1 + \sin x - \cos x + \log(1-x)}{x\tan^2 x}, \quad \text{when } x \to 0.$$

4. Evaluate:

(i) $\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}$ (ii) $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

(iii) $\lim_{x \to 0} \frac{\sin x - \log(e^x \cos x)}{x \sin x}$ (iv) $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$

(v) $\lim_{x \to 0} \frac{x - \log(1+x)}{1 - \cos x}$ (vi) $\lim_{x \to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$

(vii) $\lim_{x \to 0} \log(1-x)\cot\left(\frac{\pi}{2}x\right)$ (viii) $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$.

5. Evaluate the following limits:

(i) $\lim_{x \to \pi/2} \frac{\tan x}{\tan 3x}$ (ii) $\lim_{x \to 1} \log(1-x)\cot\frac{\pi x}{2}$ (iii) $\lim_{x \to 0} \frac{\log \tan x}{\log x}$

(iv) $\lim_{x \to 0} \frac{\log \sin x}{\cot x}$ (v) $\lim_{x \to 0} \frac{\log \tan 2x}{\log \tan x}$ (vi) $\lim_{x \to \pi/2} \frac{\tan 5x}{\tan x}$

6. Determine the following limits:

(i) $\lim_{x \to 0} \sin x \log x^2$ (ii) $\lim_{x \to 0} x \log \tan x$ (iii) $\lim_{x \to 0} x \log \sin^2 x$

(iv) $\lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right)$ (v) $\lim_{x \to 0} x^2 \log x^2$ (vi) $\lim_{x \to 0} x \log x$

(vii) $\lim_{x \to 1} \operatorname{cosec}(n\pi) \log x$ (viii) $\lim_{x \to a} (a-x)\tan\frac{\pi x}{2a}$.

7. Determine the following limits:

(i) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$ (ii) $\lim_{x \to 1} \left(\frac{\pi \cot \pi x}{x} + \frac{3x^2 - 1}{x^2 - x}\right)$

(iii) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^x - 1}\right)$ (iv) $\lim_{x \to 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2}\right)$

(v) $\lim_{x \to 1} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1}\right)$ (vi) $\lim_{x \to 0} \left(\frac{a}{x} - \cot\frac{x}{a}\right)$

(vii) $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$ (viii) $\lim_{x \to 0} \left(\frac{1}{2x^2} - \frac{1}{2x\tan x}\right)$.

8. Determine the following limits:

(i) $\lim_{x\to 0} (\cot^2 x)^{\sin x}$ (ii) $\lim_{x\to a} (x-a)^{x-a}$

(iii) $\lim_{x\to 1} (1-x^2)^{1/\log(1-x)}$ (iv) $\lim_{x\to \pi/4} (\tan x)^{\tan^2 x}$ (v) $\lim_{x\to \pi/4} \cos x^{\cos x}$

(vi) $\lim_{x\to 0} (\cot x)^{\sin 2x}$ (vii) $\lim_{x\to 0} x^{(1-x)^{-1}}$ (viii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$

(ix) $\lim_{x\to 1} \frac{1}{x^{x-1}}$ (x) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$ (xi) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$

(xii) $\lim_{x\to \infty}\left(1+\frac{1}{x^2}\right)^{x}$ (xiii) $\lim_{x\to 0}\left(\frac{2x+1}{x+1}\right)^{x^{-1}}$.

9. Determine the following limits:

(i) $\lim_{x\to \infty} 2^x \tan\frac{a}{2^x}$ (ii) $\lim_{x\to \infty} \frac{\log x}{x^2}$ (iii) $\lim_{x\to \infty} \frac{\log x}{x}$

(iv) $\lim_{x\to \infty} \frac{\log x}{x^{1/2}}$ (v) $\lim_{x\to \infty} \frac{\log x}{x^3}$ (vi) $\lim_{x\to \infty}\left(\frac{x^x}{x!}\right)^{1/x}$

(vii) $\lim_{x\to \infty} a^x \sin\frac{b}{a^x}$.

10. Determine the limit:

$$\sqrt{a^2-x^2}\cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right), \quad \text{when } x \to a.$$

11. Evaluate:

$$\lim_{x\to 0}\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}.$$

12. Evaluate:

(i) $\lim_{x\to 0}\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}$ (ii) $\lim_{x\to 0}\frac{e^x - e^{-x} + 2\log(1+x)}{x^2 \sin x}$.

13. Find the limit of:

$$\frac{x\sin(\sin x) - \sin^2 x}{x^6}, \quad \text{when } x \to 0.$$

14. Determine:

$$\lim_{x\to \pi/2}\left(\frac{1+\cos x}{1-\cos x}\right)^{1/\cos x}$$

15. Evaluate:

(i) $\lim_{x\to 0}(1+\sin x)^{\cot x}$ (ii) $\lim_{a\to b}\dfrac{a^x\sin bx-b^x\sin ax}{\tan bx-\tan ax}$

(iii) $\lim_{x\to 0}\dfrac{\tan x\tan^{-1}x-x^2}{x^6}$ (iv) $\lim_{x\to 0}\dfrac{\log_{\sin^2 x}\cos x}{\log_{\sin^2(x/2)}\cos(x/2)}$

(v) $\lim_{x\to 0}\dfrac{\log(1+x\sin x)}{\cos x-1}$.

16. Evaluate:

$$\lim_{x\to 1/2}\frac{(x-4x^4)^{1/2}-(x/4)^{1/3}}{(1-8x^3)^{1/4}}.$$

17. Determine the following limits:

(i) $\lim_{x\to\pi/2}(2x\tan x-\pi\sec x)$ (ii) $\lim_{x\to 1}\dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$

(iii) $\lim_{x\to 0}\left(\dfrac{x-1}{2x^2}+\dfrac{e^{-x}}{2x\sin x}\right)$ (iv) $\lim_{x\to 0}\dfrac{1+x\cos x-\cosh x-\log(1+x)}{\tan x-x}$

(v) $\lim_{x\to 0}\dfrac{\log(1+x)\log(1-x)-\log(1-x^2)}{x^4}$ (vi) $\lim_{x\to 0}(\cos x)^{\cot x}$

(vii) $\lim_{x\to 0}\dfrac{(1+x)e^{-x}-(1-x)e^x}{x(e^x-e^{-x})-2x^2e^{-x}}$ (viii) $\lim_{x\to 0}\dfrac{3x\log\left(\dfrac{\sin x}{x}\right)^2+x^3}{(x-\sin x)(1-\cos x)}$.

18. Obtain the limits of the following:

(i) $\lim_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$ (ii) $\lim_{x\to 0}\dfrac{x^2\sin(1/x)}{\tan x}$ (iii) $\lim_{x\to\pi/2}(\sin x)^{\tan^2 x}$

(iv) $\lim_{x\to 0}(\cos ax)^{b/x^2}$ (v) $\lim_{x\to 0}2\left(\dfrac{\cosh x-1}{x^3}\right)^{1/x^2}$

(vi) $\lim_{x\to 0}\left(\sin^2\dfrac{\pi}{2-ax}\right)^{\sec^2[\pi/(2-bx)]}$ (vii) $\lim_{x\to\pi/2}\dfrac{a^{\sin x}-a}{\log\sin x}$

(viii) $\lim_{x\to 1}\dfrac{x^x-x}{x-1-\log x}$ (ix) $\lim_{x\to\pi/2}(\sec x)^{\cot x}$ (x) $\lim_{x\to 1}\dfrac{1-4\sin^2(\pi x/6)}{1-x^2}$

(xi) $\lim_{a\to b}\dfrac{a^b-b^a}{a^a-b^b}$.

19. Find the limit:

$$\frac{\cot x}{\pi - x}, \quad \text{when } x \to \frac{\pi}{2}.$$

20. Evaluate:

$$\lim_{x \to 0} \frac{(1+x)^{1/x} - e + ex/2}{x^2}$$

21. Find:

$$\lim_{x \to a} (a - x) \tan\left(\frac{\pi x}{2a}\right).$$

22. Find:

(i) $\lim_{x \to 1} \frac{1}{x^{1-x}}$ (ii) $\lim_{x \to 0} (\cot x)^{\sin 2x}$ (iii) $\lim_{x \to 0} \left(\frac{2x+1}{x+1}\right)^{1/x}$

(iv) $\lim_{x \to 1} (1 - x^2)^{1/\log(1-x)}$ (v) $\lim_{x \to \pi/2} \left(\frac{x-1}{2x^2} + \frac{e^{-x}}{2x \sinh x}\right)$

(vi) $\lim_{x \to 1} \frac{1 - x + \log x}{1 - \sqrt{2x - x^2}}$ (vii) $\lim_{x \to \infty} x(a^{1/x} - 1)$ (viii) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$

(ix) $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$ (x) $\lim_{x \to 1} \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2}$ (xi) $\lim_{x \to 0} \frac{\log_{\sec(x/2)} \cos x}{\log_{\sec x} \cos(x/2)}$

(xii) $\lim_{x \to \pi/2} (1 - \sin x)^{\tan x}$ (xiii) $\lim_{x \to 0} x^2 \log x^2$ (xiv) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$

(xv) $\lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x$ (xvi) $\lim_{x \to 0} \frac{\log x}{\cot x}$

(xvii) $\lim_{x \to 0} \frac{\sin x \sin^{-1} x - x^2}{x^4}$.

23. Find a if the limit of

$$\frac{\tan^2(ax) + 8 \tan ax}{\sin 4x} = 1, \quad \text{when } x \to 0.$$

24. Find a if

$$\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$$

is finite.

25. If

$$\lim_{x \to 0} \frac{(1 + ax \sin x) - b \cos x}{x^4}$$

be finite, find the value of a and b and the limit.

Chapter 5

Partial Differentiation

5.1 Introduction

If several variables are functionally related, measuring the rate of change of one variable with respect to another keeping others constant, is often called *partial differentiation*. In partial differentiation, the function which is partially differentiated, is always considered as the function of only one variable—the one with respect to which the partial differentiation is to be carried out. Whereas, all the other variables are treated as constants.

If $u = f(x, y)$ be a function of x and y, the differential coefficient of u with respect to x, treating y as constant, is called *partial derivatives* of u with respect to x. It is denoted as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), \text{ or } f_x$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+\partial y) - f(x, y)}{h} = f_y$$

where $\partial/\partial x$, $\partial/\partial y$ are symbolic operators. Now let $u = f(x, y, z)$ be any function of three variables x, y, z, then

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{f((x+h), y, z) - f(x, y, z)}{h} = f_x$$

this means u has been differentiated with respect to x only, where the remaining variables y and z have been treated as constants. Similarly,

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

It means u has been differentiated with respect to y only, where the remaining variables x and z have been treated as constants, and

$$\frac{\partial u}{\partial z} = \lim_{\partial t \to 0} \frac{f(x, y, z + \partial t) - f(x, y, z)}{\partial t}$$

This means u has been differentiated with respect to z only, where the remaining variables x and y have been treated as constants.

In this manner, if u is a function of more than three variables, then the definition of partial derivatives for the function u can be likewise extended. Generally, if $u = f(x_1, x_2, ..., x_n)$ is a function of n variables, $x_1, x_2, x_3, ..., x_n$, then $\partial u/\partial x$ means that u has been differentiated with respect to x_1 only while the remaining variables $x_2, x_3, ..., x_n$ have been treated as constants. Similarly,

$$\frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, ..., \frac{\partial u}{\partial x_n}$$

can be defined.

Partial derivative, $\partial u/\partial x$, $\partial u/\partial y$ are functions of x, y. Each may possess partial derivatives with respect to these two independent variables. These are known as the *second-order* partial derivatives of $u = f(x, y)$. We denote these second-order partial derivatives as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = u_{xx} \text{ or } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = u_{yy} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = u_{xy} \text{ or } f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = u_{yx} \text{ or } f_{yx}$$

In general, $f_{xy} = f_{yx}$, when $f(x, y)$ is discontinuous of (x, y), partial derivatives $f_{xy} \neq f_{yx}$. In this text, we only concerned about order of partial differentiation, which has no effect on the values of derivatives, and in general

$$f_{xy} = f_{yx}, \qquad f_{xyz} = f_{yzx} = f_{zxy} = ..., \text{ etc.}$$

For example, consider the function:

$$f(x, y) = ax^3 + 2hxy^2 + by^3$$

Here

$$f_x = 3ax^2 + 2hy^2$$

$$f_y = 4hxy + 3by^2$$

$$f_{yx} = 4hy$$

$$f_{xy} = 4hy$$

Therefore, $f_{xy} = f_{yx}$. There are, however, cases in which $f_{yx} \neq f_{xy}$.

Consider another example,

$$f(x, y) = \frac{x^3 + y^3}{x - y}, \qquad x \neq 0; y \neq 0$$

$$f(0, 0) = 0$$

It can be easily verified that at the origin, $f_{xy} \neq f_{yx}$.

Example 5.1 When $u = x^3 + y^3 - 3axy$, find

$$\frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2}.$$

Solution We have $u = x^3 + y^3 - 3axy$. Then

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay, \qquad \frac{\partial u}{\partial y} = 3y^2 - 3ax,$$

and

$$\frac{\partial^2 u}{\partial x^2} = 6x, \qquad \frac{\partial^2 u}{\partial y^2} = 6y.$$

Example 5.2 If

$$u = \log \frac{x^2 + y^2}{xy},$$

prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Solution Here

$$u = \log \frac{x^2 + y^2}{xy} = \log(x^2 + y^2) - \log(xy) \tag{1}$$

Then

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{y}{xy} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

and

$$\frac{\partial^2 x}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2} - \frac{1}{x}\right) = 2x \frac{2y}{(x^2 + y^2)^2} = -\frac{4xy}{(x^2 + y^2)^2} \tag{2}$$

Again, from (1), we have

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{x}{xy} = \frac{2y}{x^2 + y^2} - \frac{1}{y}$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2} - \frac{1}{y}\right) = 2y\left[-\frac{2x}{(x^2 + y^2)^2}\right] = -\frac{4xy}{(x^2 + y^2)^2} \tag{3}$$

Therefore, from (2) and (3),

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Example 5.3 If

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right),$$

prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Solution We have

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Then

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2 + y^2)}{x^2 + y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

Similarly,

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{2y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

Example 5.4 If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

Solution We have $u = e^{xyz}$. Therefore,

$$\frac{\partial u}{\partial z} = e^{xyz} yx$$

and

$$\frac{\partial^2 u}{\partial y \partial z} = x(e^{xyz} + xyze^{xyz}) = xe^{xyz} + x^2 yze^{xyz}.$$

Again

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}[(xyz + 1) + (x^2y^2z^2 + 2xyz)] = e^{xyz}(1 + 3xyz + x^2y^2z^2).$$

Example 5.5 If

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right),$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Solution We have

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right).$$

Then

$$\frac{\partial u}{\partial x} = \frac{y}{\sqrt{y^2 - x^2}}\frac{1}{y} - \frac{x^2}{x^2 + y^2}\frac{y}{x^2} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

or

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \tag{1}$$

Again

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - (x/y)^2}}x\left(-\frac{1}{y^2}\right) + \frac{1}{1 + (y/x)^2}\frac{1}{x}$$

$$= -\frac{y}{\sqrt{y^2 - x^2}}\frac{x}{y^2} + \frac{x^2}{x^2 + y^2}\frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

or

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \tag{2}$$

Now, adding Eqs. (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} - \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} = 0.$$

Example 5.6 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$,

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=-\frac{9}{(x+y+z)^2}$.

Solution Here $u = \log (x^3 + y^3 + z^3 - 3xyz)$. Therefore,

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz},$$

$$\frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3z^2-xy}{x^3+y^3+z^3-3xyz}.$$

(i) Adding them, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-3xyz)}$$

$$=\frac{3}{x+y+z}$$

(ii) For the second part

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\frac{3}{x+y+z}$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$=3\frac{-3}{(x+y+z)^2}$$

$$=-\frac{9}{(x+y+z)^2}.$$

Example 5.7 If

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1,$$

prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Solution Differentiating partially with respect to x, we get

$$\left[\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}\right]=0$$

or

$$\frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=0$$

or

$$\frac{\partial u}{\partial x}=\frac{2x}{(a^2+u)\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]} \tag{1}$$

or

$$\left(\frac{\partial u}{\partial x}\right)^2=\frac{4x^2}{(a^2+u)^2\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2}$$

$$=\frac{4x^2}{(a^2+u)^2\left[\sum\frac{x^2}{(a^2+u)^2}\right]^2}$$

Similarly,

$$\left(\frac{\partial u}{\partial y}\right)^2=\frac{4y^2}{(b^2+u)^2}\frac{1}{\left[\sum\frac{x^2}{(a^2+u)^2}\right]^2},$$

and

$$\left(\frac{\partial u}{\partial z}\right)^2=\frac{4z^2}{(c^2+u)^2}\frac{1}{\left[\sum\frac{x^2}{(a^2+u)^2}\right]^2}$$

Adding them, we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4\left[\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right]}{\left[\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right]^2}$$

$$=\frac{4}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}} \qquad (2)$$

Again, from (1)

$$x\frac{\partial u}{\partial x}=\frac{2x^2/(a^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

$$y\frac{\partial u}{\partial y}=\frac{2y^2/(b^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

$$z\frac{\partial u}{\partial z}=\frac{2z^2/(c^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Adding them, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2\sum\left[x^2/(a^2+u)\right]}{\sum\left[x^2/(a^2+u)^2\right]}=\frac{2}{\sum\left[x^2/(a^2+u)^2\right]}. \qquad (3)$$

Since from the given equation

$$\sum\frac{x^2}{(a^2+u)^2}=1,$$

From (2) and (3), we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Example 5.8 If $u = \log(\tan x + \tan y + \tan z)$, show that

$$(\sin 2x)\frac{\partial u}{\partial x}+(\sin 2y)\frac{\partial u}{\partial y}+(\sin 2z)\frac{\partial u}{\partial z}=2.$$

Solution Here $u = \log(\tan x + \tan y + \tan z)$. Then

$$\frac{\partial u}{\partial x}=\frac{\sec^2 x}{\tan x+\tan y+\tan z}$$

or

$$(\sin 2y)\frac{\partial u}{\partial y} = \frac{2\sin x\cos x}{\tan x + \tan y + \tan z}\,\frac{1}{\cos^2 x} = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

Similarly,

$$(\sin 2y)\frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$$

and

$$(\sin 2z)\frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

Adding them, we obtain

$$(\sin 2x)\frac{\partial u}{\partial x} + (\sin 2y)\frac{\partial u}{\partial y} + (\sin 2z)\frac{\partial u}{\partial z} = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2.$$

Example 5.9 Prove that $\nabla^2 r = 0$, where

$$\frac{1}{r^2} = x^2 + y^2 + z^2.$$

Or if

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

∇^2 stands for the Laplace operator:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Solution We have

$$\frac{1}{r^2} = x^2 + y^2 + z^2 \tag{1}$$

Differentiating partially with respect to x, we get,

$$-\frac{2}{r^3}\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{1}{r^3}\frac{\partial r}{\partial x} = -x \tag{2}$$

Differentiating again with respect to x, we get

$$-\frac{3}{r^4}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial xz} = -1,$$

or

$$-\frac{3}{r^4}(-xr^3)^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial x^2}=-1,$$

or

$$-3x^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial x^2}=-1. \tag{3}$$

Similarly,

$$-3y^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial y^2}=-1, \tag{4}$$

and

$$-3z^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial z^2}=-1 \tag{5}$$

Adding (3)–(5), we get

$$-3r^2(x^2+y^2+z^2)+\frac{1}{r^3}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial z^2}\right)=-3$$

or

$$-3r^2\frac{1}{r^2}+\frac{1}{r^3}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)r=-3$$

or

$$\frac{1}{r^3}\nabla^2 r=0.$$

Hence $\nabla^2 r=0$.

Example 5.10 If $u=f(r)$, where $r^2=x^2+y^2$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

Solution Given $r^2=x^2+y^2$, we get

$$\frac{\partial r}{\partial x}=\frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y}=\frac{y}{r}.$$

Now from (1) as $u=f(r)$,

$$\frac{\partial u}{\partial x}=f'(r)\frac{\partial r}{\partial x}=\frac{x}{r}f'(r) \tag{1}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \frac{\partial}{\partial x}\frac{\left[xf'(r) - xf'(r)\frac{\partial r}{\partial x}\right]}{r^2}$$

$$= \frac{r\left[xf''(r)\frac{\partial r}{\partial x} + f'(r)\right] - xf'(r)\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r[xf''(r)(x/r) + f'(r) - xf'(r)(x/r)}{r^2}$$

$$= \frac{1}{r^2}\left[x^2 f''(r) + rf'(r) - \frac{x^2}{r}f'(r)\right].$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} + \frac{1}{r^2}\left[y^2 f''(r) + rf'(r) - \frac{y^2}{r}f'(r)\right].$$

Adding them, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[(x^2 + y^2)f''(r) + 2rf'(r) - \frac{x^2 + y^2}{r}f'(r)\right].$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r^2}f'(r).$$

Example 5.11 If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

Solution Given $x^x y^y z^z = c$, taking log both sides, we get

$$x \log x + y \log y + z \log z = \log c$$

or

$$z \log z = \log c - x \log x - y \log y$$

Differentiating partially with respect to x, we get

$$(1 + \log z)\frac{\partial z}{\partial x} = -(1 + \log x)$$

or

$$\frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z}$$

Now,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1+\log y}{1+\log z}\right)$$

$$= -(1+\log y)\left[-\frac{1}{(1+\log z)^2}\right]\frac{1}{z}\frac{\partial z}{\partial x}$$

$$= \frac{1+\log y}{(1+\log z)^2}\frac{1}{z}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$= -\frac{1}{z}\frac{(1+\log x)(1+\log y)}{(1+\log z)^3}$$

Hence at the point $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x}\frac{(1+\log x)(1+\log x)}{(1+\log x)^3}$$

$$= -\frac{1}{x}\frac{1}{1+\log x}$$

$$= -\frac{1}{x}\frac{1}{\log_e e + \log_e x}$$

$$= -\frac{1}{x \log (ex)}$$

$$= -[x \log (ex)]^{-1}.$$

Example 5.12 If

$$u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}.$$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Solution Since

$$u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y},$$

we have

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2} + \frac{1}{y}.$$

Similarly,

$$\frac{\partial u}{\partial y}=\frac{1}{z}-\frac{x}{y^2} \quad \text{and} \quad \frac{\partial u}{\partial z}=-\frac{y}{z^2}+\frac{1}{x}.$$

Now, we get

$$\begin{aligned} x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} &= x\left(-\frac{z}{x^2}+\frac{1}{y}\right)+y\left(\frac{1}{z}-\frac{x}{y^2}\right)+z\left(-\frac{y}{z^2}+\frac{1}{x}\right) \\ &= -\frac{z}{x}+\frac{x}{y}+\frac{y}{z}-\frac{x}{y}-\frac{y}{z}+\frac{z}{x} \\ &= 0. \end{aligned}$$

Example 5.13 If $u=\tan^{-1}(y/x)$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Solution Here

$$u=\tan^{-1}\left(\frac{y}{x}\right) \quad \text{or} \quad \tan u=\frac{y}{x} \tag{1}$$

Differentiating, we get

$$\sec^2 u\frac{\partial u}{\partial x}=-\frac{y}{x^2}$$

or

$$(1+\tan^2 u)\ \frac{\partial u}{\partial x}=-\frac{y}{x^2}$$

or

$$\left(1+\frac{y^2}{x^2}\right)\frac{\partial u}{\partial x}=-\frac{y}{x^2}$$

or

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2+y^2}$$

Differentiating partially again with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2}=+\frac{2xy}{(x^2+y^2)^2}.$$

Again from (1),

$$\sec^2 u\,\frac{\partial u}{\partial y}=\frac{1}{x}$$

or

$$\frac{x^2+y^2}{x^2}\frac{\partial u}{\partial y}=\frac{1}{x}$$

or

$$\frac{\partial u}{\partial y}=\frac{x}{x^2+y^2}$$

or

$$\frac{\partial^2 u}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2}$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2xy}{(x^2+y^2)^2}-\frac{2xy}{(x^2+y^2)^2}=0.$$

Thus

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\cdot$$

Example 5.14 If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, then find the value of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Solution Here $u = 2(ax + by)^2 - (x^2 + y^2)$. Then

$$\frac{\partial u}{\partial x}=4a(ax+by)-2x$$

and

$$\frac{\partial^2 u}{\partial x^2}=4a^2-2$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2}=4b^2-2$$

Adding them, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=4a^2-2+4b^2-2=4(a^2+b^2)-4=(4)(1)-4=0, \text{ as } a^2+b^2=1.$$

Example 5.15 If $u = \sin(\sqrt{x}+\sqrt{y})$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$$

Solution Here since $u = \sin(\sqrt{x} + \sqrt{y})$, we have

$$\frac{\partial u}{\partial x} = \cos\left(\sqrt{x} + \sqrt{y}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

and

$$\frac{\partial u}{\partial y} = \cos\left(\sqrt{x} + \sqrt{y}\right)\left(\frac{1}{2}y^{-1/2}\right)$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cos\left(\sqrt{x} + \sqrt{y}\right)\left(\frac{1}{2}\right)\left(x^{1/2} + y^{1/2}\right)$$

$$= \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\cos\left(\sqrt{x} + \sqrt{y}\right)$$

Exercises 5.1

1. Find $\partial u/\partial x$ and $\partial u/\partial y$, when

   (i) $u = \sin(x/y)$ (ii) $y = \tan^{-1}(x/y)$ (iii) $u = \log(x^2y + xy^2)$

   (iv) $u = \log(x^2 + y^2)$ (v) $u = x^y$.

2. When $u = \cos y + y \cos x$, verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

3. If

$$u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right),$$

   show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{3/2}}.$$

4. If $x^3 + y^3 - x^3y^2z = 0$, find $\partial z/\partial x$ and $\partial z/\partial y$ when $x = y = 1$.

5. If $u = x^3y^2 + x^2y^3$, show that $u_{xy} = u_{yx}$.

6. If $u = \log\sqrt{x^2 + y^2}$, prove that $u_{xx} + u_{yy} = 0$.

7. If $u = \sqrt{x^2 + y^2}$, prove that $u_{xx} + u_{yy} = u^{-1}$.

8. If $u = \sin^{-1}x$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

9. If $u = f\left(\frac{y}{x}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

10. If

$$u = \sin\left(\sin\frac{y}{x}\right),$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

11. Find the second-order partial derivative of: (i) e^{x^y}, (ii) $(x^2 + y^2)^{3/2}$.

12. If

$$f(x, y) = y^{-1/2} \exp\left[-\frac{(x-a)^2}{4y}\right],$$

show that $f_{xy} = f_{yx}$.

13. If $u = f(x + ay) + f(x - ay)$, show that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

14. If $u = r^m$, where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}.$$

15. If $u = \log \frac{x^4 + y^4}{x+y}$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

16. If $u = xf(x + y) + yF(x + y)$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

17. If

$$u = \sin\frac{y}{x},$$

then prove that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

18. If f and g are two differentiable functions and are connected by $y = f(x + at) + g(x - at)$, where a is a constant, show that

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

19. If

$$u = 2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right),$$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \cot\frac{u}{2} = 0.$$

20. If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

21. If $u = x^2 + y^2 + z^2$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$$

22. If $u = f(x^2 + y^2)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

23. If $u = x^2y + y^2z + xz^2$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2.$$

24. If $u = (x^2 + y^2)/(x + y)$, prove that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right).$$

25. If $u = a\sin(x/y) + b\cos(y/b)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

26. If

$$f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix},$$

prove that

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0.$$

27. If $u = \tan(y + ax) + (y - ax)^{3/2}$, prove that

$$\frac{\partial^2 u}{\partial x^2}=a^2\frac{\partial^2 u}{\partial y^2}.$$

28. If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$, and $a^2 + b^2 + c^2 = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

29. If $V = z\tan^{-1}(y/x)$, prove that $V_{xx} + V_{yy} + V_{zz} = 0$.

30. If $u = \log(x^2 + y^2 + z^2)$, prove that

$$x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial x\partial z}=z\frac{\partial^2 u}{\partial x\partial y}.$$

31. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\text{(i)}\ \frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0 \qquad \text{(ii)}\ \frac{1}{r}\frac{\partial x}{\partial \theta}=r\frac{\partial \theta}{\partial x}.$$

32. If

$$u=\frac{\sin(ct-x)}{x},$$

prove that

$$e^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{2}{x}\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial t^2}$$

33. If

$$u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}},$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$

5.2 Degree of Homogeneous Functions

If the sum of indices of different variables contained in each term of an algebraic expression be n, it is called a *homogeneous function* of degree n, which may be any number, positive or negative, including zero.

Let $u = f(x, y)$ be a function of x and y. If the sum of the powers of x and y in each term of $f(x, y)$ be equal, then $f(x, y)$ is called a homogeneous function. In particular, if the sum of the powers of x and y in each term of $f(x, y)$ be 3, then $f(x, y)$ is called a homogeneous function of order 3; if the sum of the powers of x and y in each term of $f(x, y)$ be 4, then $f(x, y)$ is called the homogeneous function of order 4. In general, if the sum of the powers of x and y of each term of $f(x, y)$ be n, then $f(x, y)$ is called a homogeneous function of order n. For example, $x^3 + y^3 + 3x^2y$ is a homogeneous function of degree 3; $x^4 + 4x^2y^2 + y^4$ is a homogeneous function of 4th order; etc.

In general, if $f(x, y)$ be a homogeneous function of nth order, then its form will be

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n. \qquad (5.1)$$

The sum of powers of x and y in each term of Eq. (5.1), or in general term $a_rx^{n-r}y^r$ is $(n - r) + r = n$. Therefore $f(x, y)$, as given (1), is a homogeneous function of nth order in x and y.

The above homogeneous function, as in Eq. (5.1), may also be written in the form:

$$f(x,y) = x^n\left[a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + a_3\left(\frac{y}{x}\right)^3 + \cdots + a_n\left(\frac{y}{x}\right)^n\right] = x^n\phi\left(\frac{y}{x}\right),$$

where ϕ is a function of y/x. Hence if $f(x, y)$ be a homogeneous function of nth order, we can write $f(x, y)$ as

$$f(x, y) = x^n\phi\left(\frac{y}{x}\right)$$

Alternatively, if function $f(x, y)$ is said to be homogeneous of degree n in x and y, $f(tx, ty) = t^nf(x, y)$, where t is a parameter.

The function $f(x, y)$ can also be written as

$$x^nf\left(\frac{y}{x}\right) \quad \text{or} \quad y^n\phi\left(\frac{y}{x}\right),$$

if $f(x, y)$ is homogeneous Similarly, $f(x, y, z)$ is a homogeneous function of nth order then

$$f(tx, ty, tz) = t^3f(x, y, z), \text{ etc.}$$

Rational homogeneous function

If P and Q be homogeneous functions of degree n and r, respectively, then $f = P/Q$ is a homogeneous function of degree $(n - 1)$. Following are some examples of rational homogeneous functions:

(i) $$f(x,y) = \frac{x^4 + y^4}{x + y} = \frac{x^4[1 + (y/x)^4]}{x(1 + y/x)} = x^3\phi\left(\frac{y}{x}\right)$$

Therefore, $f(x, y)$ is a homogeneous of order $n = (4 - 1) = 3$.

$$\text{(ii)}\quad f(x,y) = \frac{x^3 + y^3}{x^2 y + xy^2} = \frac{x^3\left[1 + (y/x)^3\right]}{x^3\left[(y/x) + (y/x)^2\right]} = x^0 \phi\left(\frac{y}{x}\right)$$

Therefore, $f(x, y)$ is a homogeneous function of degree $(3 - 3) = 0$.

(iii)
$$\begin{aligned} f(x, y) &= \frac{\sqrt{y} + \sqrt{x}}{y + x} \\ &= \frac{\sqrt{x}\left(1 + \sqrt{y/x}\right)}{x(1 + y/x)} \\ &= \frac{1}{\sqrt{x}} \frac{\left[1 + (y/x)^{1/2}\right]}{1 + y/x} \\ &= \frac{x^{-1/2}\left[1 + (y/x)^{1/2}\right]}{1 + y/x} \\ &= x^{-1/2} \phi \frac{y}{x} \end{aligned}$$

So, $f(x, y)$ is a homogeneous function of degree $(1/2 - 1) = -1/2$.

$$\text{(iv)}\quad f(x, y, z) = \frac{x^3 + y^3 + z^3}{x + y + z}$$

Here, we put, $x = xt$, $y = yt$ and $z = zt$. Thus

$$f(tx, ty, tz) = \frac{t^3(x^3 + y^3 + z^3)}{t(x + y + z)} = t^2 \int (x, y, z)$$

Therefore, $f(x, y, z)$ is a homogeneous function of degree $n = (3 - 1) = 2$.

We also note that

$$\sin\frac{x}{y}, \cos\frac{x}{y}, \tan\frac{x}{y}, \log\left(1 + \frac{x}{y}\right), e^{x/y}, \cdots$$

are homogeneous functions of degree $n = 0$, since each of them can be expressed as

$$x^n \phi\left(\frac{y}{x}\right), \quad \text{with } n = 0.$$

Their expansions like

$$\sin\frac{x}{y} = \frac{x}{y} - \frac{1}{3!}\left(\frac{x}{y}\right)^3 + \frac{1}{5!}\left(\frac{x}{y}\right)^5 - \cdots$$

also indicate that they are homogeneous of degree zero.

Similarly,

$$\sin^{-1}\left(\frac{x}{y}\right),\quad \cos^{-1}\left(\frac{x}{y}\right),\quad \tan^{-1}\frac{x}{y},\quad \sin\left(\frac{x}{y}\right)^k,\quad \sin^{-1}\left(\frac{x}{y}\right)^k,\ldots$$

are also homogeneous functions of degree zero. But

$$u=\sin\left(\frac{x^4+y^4}{x}\right),\qquad v=\log\left(\frac{x^4+y^4}{x+z}\right)$$

are not homogeneous, since numerators and denominators of

$$\frac{x^4+y^4}{x} \quad \text{and} \quad \frac{x^4+y^4}{x+z}$$

are not homogeneous equations of the same degree. However,

$$\sin^{-1}u=\frac{x^4+y^4}{x} \quad \text{and} \quad e^v=\frac{x^4+y^4}{x+z}$$

are homogeneous functions of degree $n = 4 - 1 = 3$ each.

The correct identification of homogeneous functions and evaluation of their degrees are very essential for the application of Euler's theorem discussed now.

Theorem 5.1 (Euler's theorem) If $u = f(x, y)$ be a homogeneous function of x and y of order n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$$

for all x, y belong to the domain of the function.

Proof We have,

$$u = x^n\phi\frac{y}{x},$$

as u is a homogeneous function. Therefore,

$$\begin{aligned}\frac{\partial u}{\partial x}&=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\\ &=nx^{n-1}\phi\left(\frac{y}{x}\right)-x^{n-1}y\phi'\left(\frac{y}{x}\right)\end{aligned} \tag{5.2}$$

and

$$\frac{\partial u}{\partial y}=x^n\phi'\left(\frac{y}{x}\right)\frac{1}{x} \tag{5.3}$$

Multiplying Eqs. (5.2) and (5.3) by x and y, respectively and adding, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)-x^ny\phi'\left(\frac{y}{x}\right)+x^ny\phi'\left(\frac{y}{x}\right)=nx^n\phi\left(\frac{y}{x}\right)=nu.$$

Hence the theorem.

Alternative proof Let

$$u = Ax^{\alpha}y^{\beta} + Bx^{\alpha_1}y^{\beta_1} + \cdots = \sum Ax^{\alpha}y^{\beta} \tag{5.4}$$

where

$$\alpha + \beta = \alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \cdots = \alpha_n + \beta_n = n\,.$$

Therefore,

$$\frac{\partial u}{\partial x} = A\alpha x^{\alpha-1}y^{\beta} + Bx^{\alpha_1-1}y^{\beta_1} + \cdots$$

and

$$\frac{\partial u}{\partial y} = A\beta x^{\alpha}y^{\beta-1} + B\beta_1 x^{\alpha_1}y^{\beta_1-1} + \cdots$$

Then

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= A\alpha x^{\alpha}y^{\beta} + B\alpha_1 x^{\alpha_1}y^{\beta_1} + \cdots + (A\beta x^{\alpha}y^{\beta} + B\beta_1\, x^{\alpha_1}y^{\beta_1} + \cdots) \\ &= (\alpha + \beta)\, Ax^{\alpha}y^{\beta} + (\alpha_1 + \beta_1)Bx^{\alpha_1}y^{\beta_1} + \cdots \\ &= n(Ax^{\alpha}y^{\beta} + Bx^{\alpha_1}y^{\beta_1} + \cdots) \end{aligned}$$

Since

$$\alpha + \beta = \alpha_1 + \beta_1 = \cdots = n.$$

Thus from Eq. (5.4)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\cdot$$

Hence the theorem.

In general, Euler's theorem can be extended as a homogeneous function of any number of variables. Thus if u be a homogeneous function of three variables x, y, z of degree n, then

$$u = Ax^{\alpha}y^{\beta}z^{\gamma} + Bx^{\alpha_1}y^{\beta_1}z^{\gamma_1} + \cdots = \sum Ax^{\alpha}y^{\beta}z^{\gamma}$$

where

$$\alpha + \beta + \gamma = \alpha_1 + \beta_1 + \gamma_1 = \cdots = n.$$

Therefore,

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= x\sum A\alpha x^{\alpha-1}y^{\beta}z^{\gamma} + y\left(\sum A\beta_x^{\alpha}y^{\beta-1}z^{\gamma}\right) + z\sum A\gamma x^{\alpha}y^{\beta}z^{\gamma-1} \\ &= (\alpha + \beta + \gamma)\sum Ax^{\alpha}y^{\beta}z^{\gamma} \\ &= nu \end{aligned}$$

Thus

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu\,.$$

Hence the theorem.

Theorem 5.2 If $u = f(x, y)$ be a homogeneous function of degree n, then

(a) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=(n-1)\dfrac{\partial u}{\partial x}$

(b) $x\dfrac{\partial^2 x}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}=(n-1)\dfrac{\partial u}{\partial x}$

(c) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u$

where

$$\frac{\partial^2 u}{\partial x\partial y}\quad\text{and}\quad\frac{\partial^2 u}{\partial y\partial x}$$

are equal.

Proof Since $u = f(x, y)$ is a homogeneous function of nth order, from Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \tag{5.5}$$

Differentiating Eq. (5.5) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

Therefore,

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}=(n-1)\frac{\partial u}{\partial x} \tag{5.6}$$

Again differentiating Eq. (5.5) partially with respect to y, we obtain

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

Therefore,

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}=(n-1)\frac{\partial u}{\partial x} \tag{5.7}$$

Now multiplying Eq. (5.6) by x and Eq. (5.7) by y and then adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = (n-1)nu = n(n-1)u$$

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

5.3 Total Differential

If $y = f(x)$, then $dy/dx = f'(x)$ and $dy = f'(x)\,dx$, is called *differential of* y, when y is a function of x only. Now consider $u = f(x, y)$, which is a function of two variables, x and y only. In this case, u varies partly with x and y is constant, and partly with y when x is constant. Hence the total variation in u can be logically measured in terms of partial derivatives f_x and f_y and total differential of u is denoted as

$$du = f_x\,dx + f_y\,dy.$$

Theorem 5.3 (Theorem on total differential) Let $u = f(x, y)$, where u is a function of two variables x and y.

Let x changes to $x + h$ and y changes $y + k$, so u changes to $u + \partial u$. Then

$$u + \partial u = f(x + h, y + k)$$

or

$$\partial u = f(x + h, y + k) - f(x, y)$$

$$= \frac{f(x+h, y+k) - f(x, y+k)}{h} h + \frac{f(x, y+k) - f(x, y)}{k} k \qquad (5.8)$$

Proceeding to the limit, when $h \to 0$

$$\frac{f(x+h, y+k) - f(x, y+k)}{h}$$

becomes

$$\frac{\partial}{\partial x} f(x, y+k)$$

and when $k \to 0$,

$$\frac{\partial f(x, y)}{\partial x}$$

becomes

$$\frac{\partial u}{\partial x}$$

Also,

$$\frac{f(x, y+k) - f(x, y)}{k}$$

becomes

$$\frac{\partial f(x, y)}{\partial y} \text{ or } \frac{\partial u}{\partial y}.$$

Thus the values of the ratio $\delta u : h : k$ may be expressed as $du : dx : dy$.

Hence Eq. (5.8) becomes

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Hence the proof.

In general, if $u = f(x_1, x_2, x_3, \ldots, x_n)$, we have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \cdots + \frac{\partial u}{\partial x_n} dx_n .$$

5.3.1 Total Differential Coefficient

If $u = f(x_1, x_2)$, where x_1 and x_2 are known functions of a single variable, x, we have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 .$$

Also, since

$$du = \frac{du}{dx} dx, \quad dx_1 = \frac{dx_1}{dx} dx, \quad dx_2 = \frac{dx_2}{dx} dx,$$

we obtain

$$\frac{du}{dx} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dx} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dx}.$$

In general, if $u = f(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are known functions of x, we obtain

$$\frac{du}{dx} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dx} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dx} + \cdots + \frac{\partial u}{\partial x_n} \frac{dx_n}{dx}.$$

Further if $x_1, x_2, \ldots, x_n$ be each known function of several variables $x, y, z, \ldots$, we shall have the same way

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial x} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial x}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial y} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial y}.$$

Corollary If $u = f(x, y)$, where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \frac{dy}{dx}, \quad \text{since } \frac{dx}{dx} = 1.$$

5.3.2 Differentiation of an Implicit Function

If $f(x, y) = 0$, then $f(x + h, y + k) = 0$. Here

$$\frac{f(x+h, y+k) - f(x, y+k)}{h} + \frac{f(x, y+k) - f(x, y)}{k}\frac{k}{h} = 0$$

where h and k are indefinitely diminished. Therefore,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

This is very useful formula for the determination of dy/dx in cases where the relation between x and y is an implicit one, of which the solution of y in terms of x is inconvenient or impossible.

5.4 Exact Differential

Converse of total differential leads to what is called exact or perfect differential in calculus. If ϕ and ψ be two functions involving two variables x and y or constant, then the expression

$$\phi\, dx + \psi\, dy \tag{5.9}$$

is called *exact* or *perfect* differential, if a function $u = f(x, y)$ exists such that

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \phi\, dx + \psi\, dy. \tag{5.10}$$

Necessary condition. Comparing coefficients of dx and dy both sides of Eq. (5.10), we find

$$\frac{\partial f}{\partial x} = \phi \quad \text{and} \quad \frac{\partial f}{\partial y} = \psi$$

or

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial \psi}{\partial y}$$

or

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \psi}{\partial x}.$$

But, in general

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

is the necessary condition if Eq. (5.9).

Sufficient condition. Any expression $\phi\, dx + \psi\, dy$ can be an exact differential only when we can find a function $u(x, y)$, such that for all values of dx, dy,

$$\phi\, dx + \psi\, dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy. \tag{5.11}$$

In this case, we have to establish Eq. (5.11). Given

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \tag{5.12}$$

Let us suppose $dy = 0$, which implies y is constant.
Let $V = \int \phi dx$, so that y being constant,

$$\frac{\partial V}{\partial x} = \phi. \tag{5.13}$$

From Eq. (5.12)

$$\frac{\partial \phi}{\partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial \psi}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial x}\left(\psi - \frac{\partial V}{\partial y}\right) = 0,$$

which is possible only when

$$\psi - \frac{\partial v}{\partial y}$$

is constant or a function of y only. Hence let

$$\psi - \frac{\partial v}{\partial y} = f'(y) \quad \text{or} \quad \psi = \frac{\partial v}{\partial y} + f'(y). \tag{5.14}$$

Next let $u = v + f(y)$ be a function of x, y. Therefore, from Eqs. (5.13) and (5.14),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \phi, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + f'(y) = \psi$$

Therefore,

$$\phi dx + \psi dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy,$$

where u is a function of x and y. Hence it is an exact differential.

Example 5.16 Verify Euler's theorem for the function: $u = x^3 + x^2y + y^3$.

Solution Here $u = x^3 + x^2y + y^3$. This is a homogeneous function of degree 3, i.e. $n = 3$. Now,

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy \quad \text{and} \quad \frac{\partial u}{\partial y} = 2x + 3y^2$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(3x^2 + 2xy) + y(2x + 3y^2) = 3(x^3 + x^2y + y^3) = 3u = nu$$

Example 5.17 If $u = x^2/y^2$, verify Euler's theorem.

Solution Since $u = x^2/y^2$, the function is homogeneous of degree $(2 - 2) = 0$ and hence $n = 0$. Also,

$$\frac{\partial u}{\partial x} = \frac{2x}{y^2}, \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 y^{-2}) = \frac{-2x^2}{y^3}.$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(\frac{2x}{y}\right) + y\left(-\frac{2x^2}{y^3}\right) = \frac{2x^2}{y^2} - \frac{2x^2}{y^2} = 0 = nu \qquad \text{(as } n = 0).$$

Example 5.18 Verify Euler's theorem for the expression $x^n \sin(y/x)$.

Solution Let

$$u = x^n \sin\frac{y}{x}.$$

Then

$$\frac{\partial u}{\partial x} = nx^{n-1}\sin\frac{y}{x} + x^n \cos\frac{y}{x}\, y\left(-\frac{1}{x^2}\right)$$

or

$$x\frac{\partial u}{\partial x} = -x^{n-1} y\cos\frac{y}{x} + n\,x^n \sin\frac{y}{x}.$$

Again,

$$\frac{\partial u}{\partial y} = x^n\left(\cos\frac{y}{x}\right)\frac{1}{x} = x^{n-1}\cos\frac{y}{x}$$

or

$$y\frac{\partial u}{\partial y} = x^{n-1} y\cos\frac{y}{x}.$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \sin\frac{y}{x} = nu.$$

Example 5.19 Verify Euler's theorem for the function:

$$u = \frac{x(x^3 - y^3)}{x^3 + y^3}$$

Solution Here

$$u = \frac{x(x^3 - y^3)}{x^3 + y^3} = \frac{x^4(1 - y^3/x^3)}{x^3(1 + y^3/x^3)} = x\phi\left(\frac{y}{x}\right).$$

It is a homogeneous function of degree one, i.e. $n = (4 - 3) = 1$. Now, taking logarithm both sides, we get

$$\log u = \log x + \log (x^3 - y^3) - \log (x^3 + y^3)$$

Differentiating partially with respect to x, taking y constant, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3 - y^3} - \frac{3x^2}{x^3 + y^3}$$

or

$$\frac{1}{u}x\frac{\partial u}{\partial x} = 1 + \frac{3x^3}{x^3 - y^3} - \frac{3x^3}{x^3 + y^3} \tag{1}$$

Similarly,

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{-3y^2}{x^3 - y^3} - \frac{3y^2}{x^3 + y^3}$$

or

$$\frac{1}{u}y\frac{\partial u}{\partial y} = \frac{-3y^3}{x^3 - y^3} - \frac{3y^3}{x^3 + y^3} \tag{2}$$

Adding (1) and (2), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 1 + \frac{3(x^3 - y^3)}{x^3 - y^3} - \frac{3(x^3 + y^3)}{x^3 + y^3} = 1 + 3 - 3 = 1$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{(where } n = 1\text{)}$$

Example 5.20 If

$$u = \log\left(\frac{x^2 + y^2}{x + y}\right),$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$$

Solution Given

$$u = \log\left(\frac{x^2 + y^2}{x + y}\right)$$

or

$$e^u = \frac{x^2 + y^2}{x + y} = \frac{x^2(1 + y^2/x^2)}{x(1 + y/x)} = x\phi\left(\frac{y}{x}\right).$$

Therefore, e^u is a homogeneous function of degree one, i.e. $n = 2 - 1 = 1$. Let $e^u = v$. Therefore, according to Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = (1)(v) \tag{1}$$

But $V = e^u$, then

$$\frac{\partial v}{\partial x} = e^u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = e^u\frac{\partial u}{\partial y}$$

Putting in (1)

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

Thus

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{e^u}{e^u} = 1.$$

Example 5.21 If

$$u = \cos^{-1}\frac{x-y}{x+y},$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Solution Here

$$u = \cos^{-1}\left(\frac{x-y}{x+y}\right) \quad \text{or} \quad \cos u = \frac{x-y}{x+y} = \frac{x(1-y/x)}{x(1+y/x)} = x^0\,\phi\left(\frac{y}{x}\right).$$

Therefore, cos u is a homogeneous function of zero degree. That is, $n = (1 - 1) = 0$. Let $v = \cos u$. According to Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0. \tag{1}$$

But $v = \cos u$, then

$$\frac{\partial v}{\partial x} = -\sin u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\sin u\frac{\partial u}{\partial y}.$$

Therefore,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = (-\sin u)\,x\frac{\partial u}{\partial x} + (-\sin u)y\frac{\partial u}{\partial y} = -\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x}\right).$$

From (1), we get

$$-\sin x\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0\,.$$

Thus

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Example 5.22 If

$$u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right),$$

show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Solution Here

$$u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

Then

$$\sin u=\frac{x^2+y^2}{x+y}=\frac{x^2(1+y^2/x^2)}{x(1+y/x)}=\frac{x(1+y^2/x^2)}{1+y/x}=x\phi\left(\frac{y}{x}\right).$$

Therefore, $\sin u$ is a homogeneous function of degree one, i.e. $n = 2 - 1 = 1$.

Let $v = \sin u$. Then by Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\sin u. \tag{1}$$

But $v = \sin u$. Therefore,

$$\frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}$$

or

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

From (1),

$$\sin u = \cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

Then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}=\tan u\,.$$

Example 5.23 If

$$u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right),$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$$

Solution Here

$$u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \quad \text{or} \quad \sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

or

$$\sin u=\frac{x(1+y/x)}{\sqrt{x}\left(1+\sqrt{y/x}\right)}=x^{1/2}\,\phi\left(\frac{y}{x}\right)$$

Hence sin u is a homogeneous function of degree 1/2, i.e. $n = 1 - (1/2) = 1/2$.

Let $v = \sin u$, then by Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v.$$

Now,

$$\frac{\partial v}{\partial x}=\cos u\,\frac{\partial u}{\partial x},\qquad \frac{\partial v}{\partial y}=\cos u\,\frac{\partial u}{\partial y}$$

Thus

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

From (1),

$$\frac{1}{2}v=\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or

$$\frac{1}{2}\frac{\sin u}{\cos u}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}.$$

Thus

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$$

Example 5.24 If

$$u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right),$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$

Solution Here

$$u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$$

or

$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3(1 + y^3/y^3)}{x(1 - y/x)} = x^2\phi\left(\frac{y}{x}\right)$$

Therefore, tan u is a homogeneous function of degree 2, i.e. $n = 3 - 1 = 2$.

Let $v = \tan u$. By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v. \tag{1}$$

Now,

$$\frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}, \qquad \frac{\partial v}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

Therefore,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

From (1),

$$2v = \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right),$$

or

$$\frac{2\tan u}{\sec^2 u} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y},$$

or

$$\frac{\sin u}{\cos u}\cos^2 u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

or

$$2\sin u\cos u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

Thus

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

Example 5.25 If

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right),$$

Show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Solution Let

$$\theta=\sin^{-1}\frac{x}{y} \quad \text{or} \quad \sin\theta=\frac{x}{y}$$

Then

$$\tan\theta=\frac{x}{\sqrt{y^2-x^2}} \quad \text{or} \quad \theta=\tan^{-1}\left(\frac{x}{\sqrt{y^2-x^2}}\right)$$

Then,

$$u=\tan^{-1}\left(\frac{x}{\sqrt{y^2-x^2}}\right)+\tan^{-1}\left(\frac{y}{x}\right)$$

$$=\tan^{-1}\frac{x\sqrt{y^2-x^2}+y/x}{\left(x/\sqrt{y^2-x^2}\right)(y/x)}$$

$$=\tan^{-1}\frac{x+y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2}-xy}$$

or

$$\tan u=\frac{x+y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2}-xy}=\frac{x^2\left[1+(y/x)\sqrt{(y/x)^2-1}\right]}{x^2\left[\sqrt{(y/x)^2-1}-(y/x)\right]}=x^0\phi\frac{y}{x}$$

Therefore, $\tan u$ is a homogeneous function of degree 0, i.e. $n = 2 - 2 = 0$.

Let $v = \tan u$. By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0. \tag{1}$$

Then

$$\frac{\partial v}{\partial x}=\sec^2 u\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}.$$

Therefore,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

From (1), we get

$$0=\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \text{or} \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Example 5.26 If

$$u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}},$$

show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

Solution Here

$$\operatorname{cosec} u=\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

or

$$\sin u=\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}=v \quad \text{(say)}$$

Here v is homogeneous function of degree $n = -1/12$, as

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right)=-\frac{1}{12}$$

Then

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-\frac{1}{12}v$$

or

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=-\frac{1}{12}\sin u. \tag{1}$$

Also

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=n(n-1)v$$

$$=-\frac{1}{12}\left(-\frac{1}{12}-1\right)\sin u$$

$$=\frac{13}{144}\sin u \tag{2}$$

Putting $v = \sin u$, we get

$$\frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}=-\sin u\left(\frac{\partial u}{\partial x}\right)^2+\cos u\frac{\partial^2 u}{\partial x^2}.$$

Also,

$$\frac{\partial^2 v}{\partial x \partial y} = -\sin u \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \cos u \frac{\partial^2 u}{\partial x \partial y}$$

Again,

$$\frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = -\sin u \left(\frac{\partial u}{\partial y}\right)^2 + \cos u \frac{\partial^2 u}{\partial x^2}$$

Putting these values in (1) and (2), we get

$$\cos u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = -\frac{1}{12}\sin u$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \cdot \qquad (3)$$

From (2),

$$\cos u \left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) - \sin u \left[x^2 \left(\frac{\partial u}{\partial x}\right)^2 + 2xy \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + y^2 \left(\frac{\partial u}{\partial y}\right)^2 \right]$$

$$= \frac{13}{144}\sin u$$

or

$$\cos u \left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) - \sin u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)^2 = \frac{13}{144}\sin u$$

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12} \right).$$

Example 5.27 If u be a homogeneous function of the nth degree in x, y, z and if $u = f(x, y, z)$, where X, Y, Z are 1st derivatives of u with respect to x, y, z respectively, prove that

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \frac{n}{n-1}u.$$

Solution Since u is a homogeneous function of nth degree in three independent variables x, y, z, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \qquad (1)$$

Also,

$$u = f(X, Y, Z),$$

where

$$X = \frac{\partial u}{\partial x}, \quad Y = \frac{\partial u}{\partial y}, \quad Z = \frac{\partial u}{\partial z}.$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial x} \tag{2}$$

Again, we have

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial y} \tag{3}$$

Also,

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial z} \tag{4}$$

Multiplying (2), (3) and (4) by x, y and z, respectively, and adding, we get

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= \frac{\partial f}{\partial X}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right] \\ &+ \frac{\partial f}{\partial Y}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right] \\ &+ \frac{\partial f}{\partial Z}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) + z\frac{\partial x}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right], \end{aligned}$$

or

$$\begin{aligned} nu &= \frac{\partial f}{\partial X}\left(x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z}\right) \\ &+ \frac{\partial f}{\partial Y}\left(x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial y \partial z}\right) \\ &+ \frac{\partial f}{\partial Z}\left(x\frac{\partial^2 u}{\partial z \partial x} + y\frac{\partial^2 u}{\partial z \partial y} + z\frac{\partial^2 u}{\partial z^2}\right) \end{aligned} \tag{5}$$

Differentiating (1) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = n\frac{\partial u}{\partial x}$$

or

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = (n-1)\frac{\partial u}{\partial x} = (n-1)\,.$$

Similarly, differentiating (1) partially with respect to y

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+z\frac{\partial^2 u}{\partial y\partial z}=(n-1)Y$$

and

$$x\frac{\partial^2 u}{\partial z\partial x}+y\frac{\partial^2 u}{\partial z\partial y}+z\frac{\partial^2 u}{\partial z^2}=(n-1)Z\cdot$$

Substituting in (5), we get

$$nu=(n-1)\left(X\frac{\partial f}{\partial X}+Y\frac{\partial f}{\partial Y}+Z\frac{\partial f}{\partial Z}\right).$$

Hence

$$X\frac{\partial f}{\partial X}+Y\frac{\partial f}{\partial Y}+Z\frac{\partial f}{\partial Z}=\frac{n}{n-1}u.$$

Example 5.28 If $u = F(x - y, y - z, z - x)$ and $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ all exist, prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

Solution We have

$$u=F(x-y,\ y-z,\ z-x) \tag{1}$$

Let $X = x - y$, $Y = y - z$, $Z = z - x$. Therefore,

$$\frac{\partial X}{\partial x}=1,\quad \frac{\partial X}{\partial y}=-1,\quad \frac{\partial X}{\partial z}=0,$$
$$\frac{\partial Y}{\partial x}=0,\quad \frac{\partial Y}{\partial y}=1,\quad \frac{\partial Y}{\partial z}=-1,$$
$$\frac{\partial Z}{\partial x}=-1,\quad \frac{\partial Z}{\partial y}=0,\quad \frac{\partial Z}{\partial z}=1.$$

Therefore from (1), $u = F(X, Y, Z)$. Then

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial F}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial F}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial F}{\partial Z}\frac{\partial Z}{\partial x}\\&=\frac{\partial F}{\partial X}(1)+\frac{\partial F}{\partial y}(0)+\frac{\partial F}{\partial z}(-1)\\&=\frac{\partial F}{\partial X}-\frac{\partial F}{\partial Z}\end{aligned} \tag{2}$$

Again,

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial F}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial y}\\ &= \frac{\partial F}{\partial X}(-1) + \frac{\partial F}{\partial Y}(1) + \frac{\partial F}{\partial Z}(0)\\ &= \frac{\partial F}{\partial Y} - \frac{\partial F}{\partial X} \qquad (3)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial F}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial z}\\ &= \frac{\partial F}{\partial X}(0) + \frac{\partial F}{\partial Y}(-1) + \frac{\partial F}{\partial z}(1)\\ &= \frac{\partial F}{\partial Z} - \frac{\partial F}{\partial Y} \qquad (4)\end{aligned}$$

Adding (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Example 5.29 If $F(x, y, z) = 0$, find $\partial z/\partial x$, $\partial z/\partial y$.

Solution We have

$$F(x, y, z) = u \qquad (1)$$

Then

$$\begin{aligned}du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\\ &= (0)\,dx + (0)\,dy + (0)\,dz\\ &= 0 \qquad [\text{as } u = 0] \qquad (2)\end{aligned}$$

Now, $\partial z/\partial x$ be considered where y is constant, then from (2), $dy = 0$. Therefore, putting $du = 0$ and $dy = 0$ in (2), we find the value of dz/dx, we get

$$\left(\frac{dz}{dx}\right)_y = \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y}, \qquad \text{provided } \frac{\partial F}{\partial y} \neq 0$$

and

$$\left(\frac{dz}{dy}\right)_x = \frac{\partial z}{\partial y} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \qquad \text{provided } \frac{\partial F}{\partial z} \neq 0.$$

Example 5.30 If A, B, C be the angles of the triangle such that $\sin^2 A + \sin^2 B + \sin^2 C$ = constant, prove that

$$\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}.$$

Solution We know that in a ΔABC, $A + B + C = \pi$. Differentiating with respect to B, we get

$$\frac{dA}{dB} + 1 + \frac{dC}{dB} = 0$$

or

$$\frac{dC}{dB} = -\left(1 + \frac{dA}{dB}\right) \qquad (1)$$

By question,

$$\sin^2 A + \sin^2 B + \sin^2 C = \text{constant}$$

Differentiating with respect to B, we get

$$(\sin 2A)\frac{dA}{dB} + \sin 2B + (\sin 2C)\frac{dC}{dB} = 0$$

Using (1), we get

$$(\sin 2A)\frac{dA}{dB} + \sin 2B - (\sin 2C)\left(1 + \frac{dA}{dB}\right) = 0$$

or

$$\frac{dA}{dB}(\sin 2A - \sin 2C) = \sin 2C - \sin 2B$$

or

$$\frac{dA}{dB}(2)\cos\frac{2(A+C)}{2}\sin\frac{2(A-C)}{2} = 2\frac{\cos 2(C+B)}{2}\sin\frac{C-B}{2}$$

or

$$\frac{dA}{dB}\cos(A+C)\sin(A-C) = \cos(C+B)\sin(C-B)$$

or

$$\frac{dA}{dB}\cos(\pi - B)(\sin A\cos C - \cos A\sin C) = \cos(\pi - A)(\sin C\cos B - \cos C\sin B)$$

or

$$-\frac{dA}{dB}\cos B(\sin A\cos C - \cos A\sin C) = -\cos A(\sin C\cos B - \cos C\sin A)$$

Dividing both sides by $\cos A \cos B \cos C$, we get

$$\frac{dA}{dB}(\tan A - \tan C) = \tan C - \tan B \quad \text{or} \quad \frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}.$$

Exercises 5.2

1. If $u = e^{-mx}\sin(nt - mx)$, prove that

$$\frac{\partial u}{\partial f} = \frac{n}{2m^2}\frac{\partial^2 u}{\partial x^2}.$$

2. If H be a homogeneous function of degree n in x and y and $u = (x^2 + y^2)^{-n/2}$, show that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) = 0.$$

3. Verify Euler's theorem for the following functions:

(i) $u = \tan^{-1}(y/x)$ (ii) $u = \sin^{-1}(y/x)$

(iii) $u = x^2\cos^{-1}(y/x)$ (iv) $u = \log(x^2 + y^2)$.

4. If

$$u = \cos^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right),$$

then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

5. If

$$u = \sin^{-1}\left(\frac{x^3 + y^3}{x - y}\right),$$

then prove that

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\tan u.$$

6. If $u = (x^2 + y^2)/(x + y)$, prove that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right).$$

7. If $u = a\sin(x/y) + b\cos(x/y)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

8. If $u = 3x^2yz + 5xy^2z + 4z^4$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u.$$

9. If $u = x^2y + y^2z + z^2x$, show that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=(x+y+z)^2.$$

10. If $u=t^{-3/2}e^{-r^2/(4kt)}$, prove that

$$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right).$$

11. If $u\ (Ax^n + Bx^{-n}) \cos n(y - \alpha)$, where A, B, n, α are constants, then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{x^2}\frac{\partial^2 u}{\partial y^2}=0.$$

12. If

$$u=\sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right),$$

show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

13. If

$$u=\sin\left(\sqrt{x}+\sqrt{y}\right),$$

show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$$

14. If

$$u=\tan^{-1}\left(\frac{x^2+y^2}{x-y}\right),$$

show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u.$$

15. If

$$u=x^3\log\frac{y}{x},$$

verify Euler's theorem.

16. If $P\,dx + Q\,dy + R\,dz$ can be made a perfect differential of some function of x, y, z by multiplying each term by a common factor, prove that

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

17. If z and u be the function of x and y defined by

$$[z-\phi(u)]^2 = x^2(y^2-u^2), \qquad [z-\phi(u)]^2\phi'(u) = ux^2,$$

prove that

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = xy.$$

18. If

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right),$$

show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(1-4\sin^2 u\right)\sin\,2u.$$

19. If $u = \log\,(x^2 + y^2 + z^2)$, prove that

$$x\frac{\partial^2 u}{\partial y\partial z} = y\frac{\partial^2 u}{\partial z\partial x} = z\frac{\partial^2 u}{\partial x\partial y}.$$

20. If $u\, f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

21. If $z = (x + y) + (x + y)\,\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y\partial x}\right)=y\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x\partial y}\right).$$

22. If $u = 3xy - y^3 + (y^2 - 2x)^{3/2}$, verify that

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x} \quad\text{and}\quad \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2.$$

23. Verify Euler's theorem for

$$u=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}.$$

24. If

$$u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{y}{x}\right); \quad xy\neq 0,$$

prove that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}.$$

25. If $u=f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then write polar form of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

26. If $z=f(x, y)$, $x = u - v$, $y = uv$, prove that

$$(u+v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}-v\frac{\partial x}{\partial v}.$$

27. Transform the following equations:

(i) $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+y=0$, by putting $x = \tan z$.

(ii) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$, given that $x = \cos\theta$.

28. If $z = f(u, v)$ and $u = x^2 - 2xy - y^2$, $v = y$, show that

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=0$$

is equivalent to $\partial z/\partial v = 0$, $x\neq y$.

29. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2-2x)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0.$$

30. If

$$u=\frac{x^{1/4}+3y^{1/4}}{x^{1/3}+y^{1/3}},$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=\frac{1}{12}u$$

31. If

$$u = f(x, y) = (x^2 + y^2)^{1/3},$$

prove that

(i) $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{2}{3}f$ (ii) $x^2\frac{\partial f}{\partial x^2}+y^2\frac{\partial f}{\partial y^2}+2xy\frac{\partial f}{\partial x\partial y}=-\frac{2}{9}f.$

32. If Δ be the area of any triangle, prove that

(i) $\frac{d\Delta}{R}$ = cos A da + cos B cb + cos C dc, where R is the circum-radius.

(ii) $\frac{d\Delta}{\Delta}=\frac{da}{a}+\frac{db}{b}$, if $C=\frac{\pi}{2}$.

33. If a Δ ABC inscribed in a fixed circle be slightly varied in such a way as to have its vertices always on the circle, prove that

$$\frac{\partial a}{\cos A}+\frac{\partial b}{\cos B}+\frac{\partial c}{\cos C}=0$$

34. If the measurement of the side c of any ΔABC depends upon a, b and c, prove that

$$dc = \cos B\,\partial a + \cos A\,\partial b + a\sin B\,\partial c.$$

35. If Δ be the area of any triangle ABC, prove that

$$\frac{d\Delta}{\Delta}=\frac{db}{b}+\frac{dc}{c}+\frac{dA}{\tan A}.$$

36. Verify if the following expressions are exact differential:

(i) $2xy\,dx + x^2dy$

(ii) $x^2y^2\,dx + x^2\,ydy$

(iii) $\frac{y\,dx}{x^2+y^2}-\frac{x\,dy}{x^2+y^2}$

(iv) $\frac{-y}{x^2}dx+\frac{1}{x}dy.$

37. Find the total differential of: (i) $ax^2 + 2hxy + by^2 = 0$ (ii) $x^y + y^x = a^b$.

Chapter 6

Tangents and Normals

6.1 Tangent

The tangent at a point P to a given curve is defined as the limiting position of the secant PQ, (if such limit exists), as the point Q tends to P along the curve.

In this chapter the applications of derivatives to plane geometry will be discussed.

6.1.1 Equation of a Tangent in Cartesian Form

Let $y = f(x)$ be the equation of a curve and let P(x, y) be any point on the curve and let Q$(x + \delta x, y + \delta y)$ be any other point on the curve very near to P (Fig. 6.1).

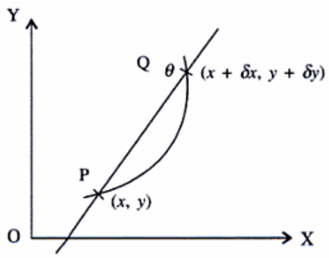


Fig. 6.1 A straight line through two points.

Let X, Y denote the current coordinates. Then the equation of the secant PQ is

$$Y - y = \frac{y + \delta y - y}{x + \delta x - x}(X - x) = \frac{\delta y}{\delta x}(X - x).$$

This line will be tangent at P, when Q → P and $\delta x \to 0$. Therefore, the equation of the tangent at P(x, y) is

$$Y - y = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}(X - x) = \frac{dy}{dx}(X - x)$$

Thus, the tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x).$$

dy/dx can also be written as the gradient $m = \tan \psi = f'(x)$. Then the equation of tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Second form. When the curve is $f(x, y) = 0$, in this case

$$\frac{dy}{dx} = -\frac{\delta f/\delta x}{\delta f/\delta y} = -\frac{f_x}{f_y}.$$

Thus, the equation of the tangent to the curve at (x, y) is

$$Y - y = -\frac{f_x}{f_y}(X - x)$$

or

$$(X - x)f_x + (Y - y)f_y = 0.$$

Note: For the sake of convenience, the current coordinates in the equation of the tangent and normal are denoted by (X, Y) and those of any particular point be (x, y). The current coordinates in the equation of the curve are however denoted by (x, y). So try to convert the current coordinates (x, y) in the form of (X, Y), wherever necessary. If the coordinates of a particular point be other than (x, y) then (x, y) may be used as current coordinates.

6.1.2 Geometrical Interpretation of *dy/dx*

The equation of the tangent to the curve $y = f(x)$ at $P(x, y)$

$$Y - y = \frac{dy}{dx}(X - x)$$

can be written as

$$Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right),$$

which is of the form $y = mx + c$, the standard equation of a straight line (Fig. 6.2).

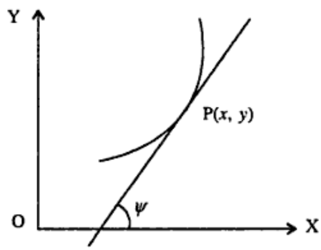


Fig. 6.2 Geometrical representation of *dy/dx*.

Hence we conclude that

$$\frac{dy}{dx} = m$$

of the tangent at P(*x*, *y*). If ψ be the angle which the tangent to the curve at P makes with the positive direction of *x*-axis, then

$$\tan\psi = m = \frac{dy}{dx}.$$

Hence *dy*/*dx* is trigonometrical tangent of the angle, with a geometrical tangent to the curve at (*x*, *y*) makes with a positive direction of *x*-axis.

Corollary tan ψ = *dy*/*dx* is called the *gradient* of the curve at the point (*x*, *y*) and ψ is called the *inclination* of the curve.

Corollary If the tangent at (*x*, *y*) is parallel to *x*-axis then $\psi = 0$. Therefore from

$$\tan\psi = \frac{dy}{dx},$$

we have

$$\tan 0° = \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = 0.$$

Corollary If the tangent at (*x*, *y*) is perpendicular to *x*-axis, i.e. parallel to *y*-axis then $\psi = \pi/2$. Therefore

$$\tan\psi = \tan\frac{\pi}{2} = \frac{dy}{dx}.$$

Thus,

$$\infty = \frac{dy}{dx} = \frac{1}{dx/dy} \quad \text{or} \quad \frac{dx}{dy} = 0.$$

Example 6.1 Find the equation of the tangent to the circle

$$x^2 + y^2 + 4x + 2y - 5 = 0$$

at (1, –2).

Solution Let

$$f(x, y) = x^2 + y^2 + 4x + 2y - 5$$

or

$$\frac{dy}{dx} = f'(x, y) = 2x + 2y\frac{dy}{dx} + 4 + 2\frac{dy}{dx} = 0$$

or

$$f'(x,y) = \frac{x+2}{y+1} \quad \text{or} \quad f'(1,-2) = -\frac{1+2}{-2+1} = 3$$

is the gradient of the tangent at (1, –2). Therefore, the equation of the tangent

$$y - y_1 = m(x - x_1)$$

at (1, –2) is

$$3x - y - 5 = 0.$$

6.1.3 Equation of Tangent in Symmetric Form

Let the equation of the curve be $f(x, y) = 0$; and $f(x, y)$ be an algebraic function of x and y of degree n. For homogeneous; we take a suitable power of z, where $z = 1$. Thus the function is altered by

$$f(x,y,z)=0, \qquad \text{where } z=1. \tag{6.1}$$

By Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z)$$

or

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=(n)(0)$$

or

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0.$$

or

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-z\frac{\partial f}{\partial z} \tag{6.2}$$

But the equation of tangent to curve $f(x, y) = 0$ is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0$$

From Eq. (6.2),

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-z\frac{\partial f}{\partial z}$$

or

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0$$

or

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0 \tag{6.3}$$

where z, the coefficient of df/dz, for the sake of symmetry, has been replaced by Z.

Corollary Remember that both Z and z are to be put equal to one after the differentiation has been performed.

Equation (6.3) is the simplest form of equation of the tangent and it will be very convenient in practice.

Example 6.2 Find the equation of tangent at (x, y) of the curve:

$$x^3 + ax^2 + by + cy^2 = 0.$$

Solution Here $x^3 + ax^2 + by + cy^2 = 0$. Let

$$f(x, y) = x^3 + ax^2 + by + cy^2 = 0$$

For homogeneity, we write

$$x^3 + ax^2z + byz^2 + cy^2z = 0, \qquad \text{where } z = 1.$$

Now, we have

$$f(x, y, z) = x^3 + ax^2z + byz^2 + cy^2z = 0$$

Then

$$\frac{\partial f}{\partial x} = 3x^2 + 2axz, \qquad \frac{\partial f}{\partial y} = bz^2 + 2cyz, \qquad \frac{\partial f}{\partial z} = ax^2 + 2byz + cy^2.$$

Here the equation of the tangent to the curve at (x, y) is

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0$$

or

$$X(3x^2 + 2axz) + Y(bz^2 + 2cyz) + Z(ax^2 + 2byz + cy^2) = 0$$

or

$$X(3x^2 + 2ax) + Y(b + 2cy) + ax^2 + 2by + cy^2 = 0,$$

as both Z and $z = 1$.

6.1.4 Equation of a Tangent in Parametric Form

If the form of the curve be in parametric, then $x = \phi(t)$, $y = \psi(t)$. Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\phi'(t)}.$$

Hence the equation of the tangent to the curve at the point (t) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y - \psi(t) = \frac{\psi'(t)}{\phi'(t)}[X - \phi(t)]$$

6.1.5 Tangent at the Origin

Let the equation be $y = f(x)$. Now the equation of a tangent at any point (x, y) to the curve is

$$Y - y = \frac{dy}{dx}(X - x).$$

If the curve passes through the origin (0, 0), the equation of the tangent at (0, 0) to the curve is

$$Y - 0 = \left(\frac{dy}{dx}\right)_{x=0,y=0}(X - 0) \quad \text{or} \quad Y = \left(\frac{dy}{dx}\right)_{(0,0)} X$$

Corollary If the curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent at the origin is obtained by equating the lowest degree terms to zero in the equation. For example, let

$$f(x, y) = 3x^3 + 4x^2y^2 - 7y^2 + 2x - y = 0. \tag{6.4}$$

Here the lowest-degree term is $(2x - y)$. Hence the equation of the tangent at the origin is $2x - y = 0$.

Verification: Differentiating Eq. (6.4) with respect to x, we get

$$9x^2 + 8xy^2 + 8x^2y\frac{dy}{dx} - 14y\frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$

At (0, 0),

$$\frac{dy}{dx} = 2.$$

Hence the equation of the tangent at the origin is

$$y = \left(\frac{dy}{dx}\right)_{(0,0)} x \quad \text{or} \quad 2x - y = 0.$$

Example 6.3 Let $f(x, y) = x^3 + y^3 - 3xy = 0$. Find the equation(s) of tangent.

Solution Here the lowest-degree term is $-3xy = 0$, i.e. $xy = 0$. Therefore, equations of the tangents at the origin are separately $x = 0$, $y = 0$.

Example 6.4 Find the equation of the tangent to the parabola $y^2 = 4ax$ at the origin.

Solution Since $y^2 = 4ax$, we get

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dx}{dy} = \frac{y}{2a}.$$

At the origin (0, 0)

$$\left(\frac{dx}{dy}\right)_{(0,0)} = \left(\frac{y}{2a}\right)_{(0,0)} = 0.$$

Therefore,

$$x - 0 = \left(\frac{dx}{dy}\right)_{(0,0)} (y - 0)$$

Thus $x = 0$ is the equation of the tangent.

6.2 Angle of Intersection of Two Curves

The angle between two curves is the angle between the tangents at the two curves at their points of intersection.

Let the two curves be $f(x, y) = 0$ and $F(x, y) = 0$ intersecting at the point (x, y). Let PT and PN be the two tangents to the curves respectively, such that $\angle$TPN is the required angle. Let the tangents PT and PN make angles ψ and ψ_1 respectively with the positive direction of x-axis. From Fig. 6.3.

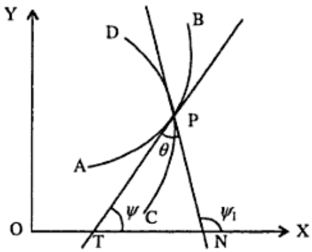


Fig. 6.3 Angle of intersection of two curves.

$$\tan\psi = \frac{dy}{dx} = -\frac{f_x}{f_y}$$

and

$$\tan\psi_1 = \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Let θ be the angle between the tangents. Then

$$\theta = \psi_1 - \psi$$

or

$$\tan\theta = \tan(\psi_1 - \psi)$$

$$= \frac{\tan\psi_1 - \tan\psi}{1 + \tan\psi_1 \tan\psi}$$

$$= \frac{-(F_x/F_y) + (f_x/f_y)}{1 + (F_x/F_y)(f_x/f_y)}$$

$$= \frac{f_x F_y - F_x f_y}{f_y F_y + F_x f_x}$$

If the curves cut orthogonally at P(*x*, *y*), then $\theta = \pi/2$. Therefore,

$$\tan\frac{\pi}{2} = \frac{f_x F_y - F_x f_y}{f_y F_y + F_x f_x}$$

or

$$\infty = \frac{f_x F_y - F_x f_y}{f_y F_y + F_x f_x}$$

or

$$f_x F_x + f_y F_y = 0.$$

Therefore, the condition for orthogonality is

$$f_x F_x + f_y F_y = 0 \quad \text{or} \quad \frac{\partial f}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial F}{\partial y} = 0$$

Corollary If the curves touch at P(*x*, *y*), then $\theta = 0$. Therefore,

$$\tan 0° = \frac{f_x F_y - F_x f_y}{f_x F_x + f_y F_y}$$

Then

$$f_x F_y - F_x f_y = 0 \quad \text{or} \quad f_x F_y = f_y F_x.$$

Thus the condition for parallelism is

$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}\frac{\partial F}{\partial x}.$$

6.3 Normal at a Point of a Curve

A *normal* at any point of a curve is a straight line through that point and perpendicular to the tangent at that point.

Let the equation of any straight line through the point P(*x*, *y*) be

$$Y - y = m(X - x) \tag{6.5}$$

Also, the equation to the tangent to curve $f(x, y) = 0$ at the point P(*x*, *y*) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y = X\frac{dy}{dx} + \left(y - x\frac{dy}{dx}\right) \tag{6.6}$$

Since the tangent and the normal are mutually parpendicular (Fig. 6.4), product of their m's = −1. That is,

$$m\frac{dy}{dx}=-1 \quad \text{or} \quad m=-\frac{dx}{dy}.$$

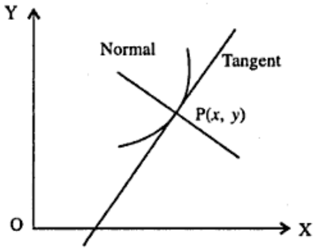


Fig. 6.4 Normal at a point of a curve.

Substituting this value of m in Eq. (6.5), the required equation to the normal to the curve $y = f(x)$ at (x, y) becomes

$$Y-y=-\frac{dx}{dy}(X-x)$$

or

$$(Y-y)\frac{dy}{dx}+(X-x)=0\,. \tag{6.7}$$

Similarly, if the curve is $f(x, y) = 0$, the equation of the normal at (x, y) is

$$\frac{X-x}{f_x}=\frac{Y-y}{f_y} \tag{6.8}$$

where

$$\frac{dy}{dx}=-\frac{f_x}{f_y}.$$

Example 6.5 Find the equation of the normal at the point (x, y) to the curve

$$\frac{y}{a}=\log\sec\frac{x}{a}.$$

Solution Here the given curve is

$$\frac{y}{a}=\log\sec\frac{x}{a}$$

Differentiating with respect to x, we get

$$\frac{1}{a}\frac{dy}{dx} = \frac{1}{\sec(x/a)} \sec\frac{x}{a}\tan\frac{x}{a}\frac{1}{a} \quad \text{or} \quad \frac{dy}{dx} = \tan\frac{x}{a}.$$

Hence the equation of the normal at (x, y) of the curve is

$$(X - x) + (Y - y)\frac{dy}{dx} = 0 \quad \text{or} \quad (X - x) + (Y - y)\tan\frac{x}{a} = 0.$$

6.4 Cartesian Subtangent, Subnormal and Other Geometrical Results

Let P be a point (x, y) on the curve $y = f(x)$ (Fig. 6.5). Draw PM perpendicular to OX, then PM = y.

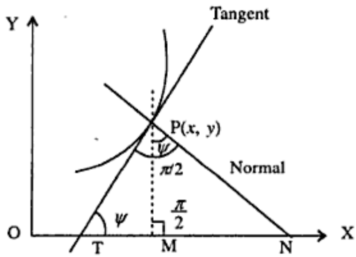


Fig. 6.5 Cartesian subtangent and subnormal.

Let the tangent and the normal at P meet OX at T and N, respectively. Then the length TM is called the *subtangent* at P, the length MN is called the *subnormal* at P, the length PT is called the *length of the tangent* and PN is called the *length of the normal.*

Also let $\angle PTN = \psi$, $\angle NPT = \pi/2$. Then $\angle PNT = \pi/2 - \psi$, and $\angle MPN = \psi$, where $\tan\psi = dy/dx$. In the right-angled triangle PMT,

$$\tan\psi = \frac{PM}{MT} = \frac{y}{MT}$$

or

$$MT = \frac{y}{\tan\psi} = \frac{y}{dy/dx} = \frac{y}{y_1}.$$

Also, in the right-angled triangle PMN,

$$\tan\psi = \frac{MN}{PM} = \frac{MN}{y},$$

Therefore,

$$MN = y \tan \psi = yy_1.$$

Again, in the right-angled triangle PMT,

$$\sin \psi = \frac{PM}{PT} = \frac{y}{PT}$$

Then

$$PT = y \operatorname{cosec} \psi = y\sqrt{1 + \cot^2 \psi} = y\sqrt{1 + \frac{1}{y_1^2}} = \frac{y}{y_1}\sqrt{1 + y_1^2}.$$

Therefore,

$$\text{Length of the tangent } PT = \frac{y}{y_1}\sqrt{1 + y_1^2}.$$

In the right-angled triangle PMN,

$$\sec \psi = \frac{PN}{PM} = \frac{PN}{y}$$

Then

$$PN = y \sec \psi = y\sqrt{1 + \tan^2 \psi} = y\sqrt{1 + y_1^2}.$$

Therefore,

$$\text{Length of the normal } PN = y\sqrt{1 + y_1^2}.$$

Corollary We know that the equation of the tangent to the curve $y = f(x)$ at P(x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X\frac{dy}{dx} - Y + \left(y - x\frac{dy}{dx}\right) = 0.$$

From O draw OD perpendicular to the tangent, PT, as shown in Fig. 6.6.

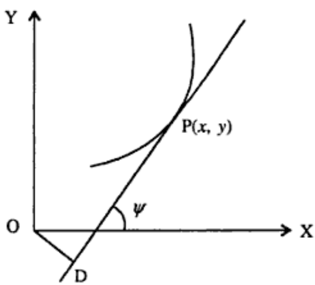


Fig. 6.6 Length of perpendicular from origin to tangent.

The length of the perpendicular D from the origin to this tangent is

$$\frac{(0)\frac{dy}{dx} - 0 + \left(y - x\frac{dy}{dx}\right)}{\sqrt{\left(\frac{dy}{dx}\right)^2 + (-1)^2}} = \frac{y - xy_1}{\sqrt{1 + y_1^2}}$$

Corollary We know that the equation of the normal to the curve $y = f(x)$ is

$$(Y - y)\frac{dy}{dx} + (X - x) = 0$$

or

$$\left(y\frac{dy}{dx} + x\right) - \left(X + Y\frac{dy}{dx}\right) = 0.$$

From O draw OE perpendicular to the normal PN, as shown in Fig. 6.7.

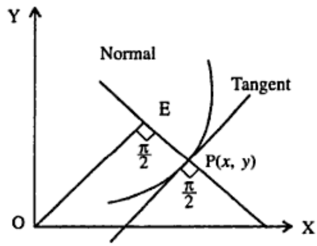


Fig. 6.7 Length of perpendicular from origin to normal.

Then the length of perpendicular from the origin to this normal OE is

$$\frac{\left(y\frac{dy}{dx} + x\right) - \left[0 + (0)\frac{dy}{dx}\right]}{\sqrt{(-1)^2 + \left(-\frac{dy}{dx}\right)^2}} = \frac{x + yy_1}{\sqrt{1 + y_1^2}}.$$

Example 6.6 Find the subtangent and the subnormal at (x, y) on the curve $y^3 = a^2x$.

Solution Here, the given equation of the curve is $y^3 = a^2x$. Differentiating with respect to x, we get,

$$3y^2\frac{dy}{dx} = a^2,$$

or

$$\frac{dy}{dx}=\frac{a^2}{3y^2}=\frac{a^2y}{3y^3}=\frac{a^2y}{3a^2x}=\frac{y}{3x}.$$

Therefore,

$$\text{Subtangent} = \frac{y}{dy/dx}=\frac{y(3x)}{y}=3x$$

and

$$\text{Subnormal} = y\frac{dy}{dx}=y\frac{y}{3x}=\frac{y^2}{3x}.$$

Corollary The product of the Cartesian subtangent and subnormal at any point on the curve $y = f(x)$ is equal to the square of the ordinate of that point. Therefore,

$$(\text{Subtangent}) \times (\text{Subnormal}) = \frac{y}{y_1}\times yy_1 = y^2 = \text{Square of the ordinate.}$$

6.5 Derivatives of Arc Length in Cartesian Form

Let P(x, y) be any point on the curve and Q($x + \delta x$, $y + \delta y$) be any other point on the same curve very near to P (Fig. 6.8). Let s be the length of the arc AP and $s + \delta s$ that of the arc AQ measured from a fixed point A on the curve such that arc PQ = $(s + \delta s) - s = \delta s$. Now PM and QN are perpendiculars to OX and PR perpendicular to QN.

Here, $OM = x$ and $ON = x + \delta x$, such that $MN = PR = ON - OM = \delta x$. Again, $PM = y$ and $QN = y + \delta y$, such that $QR = QN - RN = QN - PM = \delta y$. Now from the right-angled triangle PRQ

$$(\text{Chord } PQ)^2 = PR^2 + QR^2 = (\delta x)^2 + (\delta y)^2.$$

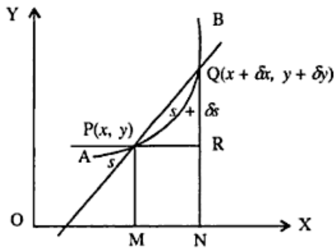


Fig. 6.8 Arc length.

Dividing both sides by $(\delta x)^2$, we get

$$\left(\frac{\text{Chord } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

or

$$\left(\frac{\text{Chord } PQ}{\delta s}\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2.$$

When Q → P, $\delta x \to 0$ and

$$\lim_{Q \to P} \frac{\text{Chord } PQ}{\text{Arc } PQ} = 1.$$

We have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{6.9}$$

Multiplying both sides by dx/dy, we get

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dy} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Note: (i) If the tangent at P(x, y) to the curve makes an angle ψ with the x-axis, then

$$\tan\psi = \frac{dy}{dx}.$$

Therefore,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2\psi = \sec^2\psi$$

or

$$\cos\psi = \frac{dx}{ds} \tag{6.10}$$

Also,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \cot^2\psi} = \operatorname{cosec}\psi = \frac{1}{\sin\psi}$$

or

$$\sin \psi = \frac{dy}{ds} \tag{6.11}$$

(ii) If the tangent to the curve at P(x, y) be parallel to x-axis then $\psi = 0$ or $\tan\ \psi = 0$. That is,

$$\frac{dy}{dx} = 0.$$

(iii) If the tangent to the curve at P(x, y) be perpendicular to x-axis, i.e. parallel to y-axis, then $\psi = 90°$. Therefore,

$$\tan 90° = \tan \psi = \frac{dy}{dx}$$

or

$$\frac{1}{0} = \frac{1}{dx/dy}$$

or

$$\frac{dx}{dy} = 0.$$

(iv) Length of the curve be in the form: $x = f(t)$, $y = \phi(t)$. Then

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ \frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Example 6.7 If $y^2 = 4ax$, prove that

$$\frac{ds}{dx} = \sqrt{\frac{a+x}{x}}.$$

Solution Here $y^2 = 4ax$. Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 4a \qquad \text{or} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{4a^2}{y^2} = \frac{4a^2}{4ax} = \frac{a}{x}$$

Therefore,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{a}{x}} = \sqrt{\frac{a+x}{x}}.$$

Example 6.8 Show that the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

touches the straight line

$$\frac{x}{a}+\frac{y}{b}=2$$

at the point (a, b), whatever the value of n.

Solution Equations of the tangent to the curve at (a, b) is

$$(x-a)f_x+(y-b)f_y=0 \tag{1}$$

Here

$$f_x=\frac{nx^{n-1}}{a^n} \quad \text{and} \quad f_y=\frac{ny^{n-1}}{b^n}.$$

At (a, b),

$$f_x=\frac{n}{a} \quad \text{and} \quad f_y=\frac{n}{b}$$

Substituting these values of f_x and f_y, we get the required result as

$$(x-a)\frac{n}{a}+(y-b)\frac{n}{b}=0\cdot$$

Thus

$$\frac{x}{a}+\frac{y}{b}=2.$$

Example 6.9 If $lx + my = 1$ touches the curve $(ax)^n + (by)^n = 1$, show that

$$\left(\frac{l}{a}\right)^{n/(n-1)}+\left(\frac{m}{b}\right)^{n/(n-1)}=1.$$

Solution Let us suppose that the given line touches the curve at (x, y) (Fig. 6.9). The equation of the tangent to the curve at (x, y), is

$$(X-x)f_x+(Y-y)f_y=0. \tag{1}$$

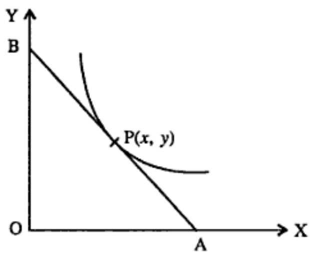


Fig. 6.9 Solution to Example 6.9.

But the equation of the curve is

$$(ax)^n + (by)^n = 1$$

where

$$f_x = na^n x^{n-1} \quad \text{and} \quad f_y = nb^n y^{n-1}.$$

Then

$$(X - x)na^n x^{n-1} + (Y - y)nb^n y^{n-1} = 0$$

or

$$a^n x^{n-1} X - a^n x^n + Yb^n y^{n-1} - b^n y^n = 0$$

or

$$a^n x^{n-1} X + Yb^n y^{n-1} \left[(ax)^n + (by)^n\right] = 0$$

or

$$a^n x^{n-1} X + b^n y^{n-1} Y = 1$$

or

$$X \frac{a^n x^n}{x} + Y \frac{b^n y^n}{y} = 1.$$

Here the equation

$$lx + my = 1 \tag{2}$$

be the tangent.

Equating (1) and (2), we get

$$\frac{l}{a^n x^{n-1}} = \frac{m}{b^n y^{n-1}} = 1$$

or

$$x = \left(\frac{l}{a^n}\right)^{1/(n-1)} \quad \text{and} \quad y = \left(\frac{m}{b^n}\right)^{1/(n-1)}$$

Putting x and y in the given equation of the curve, we obtain

$$a^n \left(\frac{l}{a^n}\right)^{n/(n-1)} + b^n \left(\frac{m}{b^n}\right)^{n/(n-1)} = 1$$

or

$$\frac{a^n}{(a^n)^{n/(n-1)}} l^{n/(n-1)} + \frac{b^n}{(b^n)^{n/(n-1)}} m^{n/(n-1)} = 1$$

or

$$\left(\frac{l}{a}\right)^{n/(n-1)} + \left(\frac{m}{b}\right)^{n/(n-1)} = 1.$$

Example 6.10 Show that the portion of the tangent at any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between the axes is of constant length.

Solution Let the tangent to the curve at P meets the coordinates axes at A and B such that AB is the portion of the tangent intercepted between the axes (Fig. 6.10).

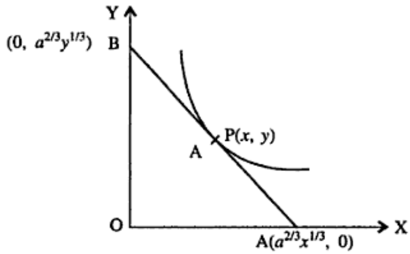


Fig. 6.10 Solution to Example 6.10.

Now the equation of the tangent to the curve at P(*x*, *y*) is

$$(X - x)f_x + (Y - y)f_y = 0$$

or

$$(X - x)\frac{2}{3}x^{-1/3} + (Y - y)\frac{2}{3}y^{-1/3} = 0$$

or

$$(X - x)x^{-1/3} + (Y - y)y^{-1/3} = 0$$

or

$$Xx^{-1/3} + Yy^{-1/3} = x^{2/3} + y^{2/3} = a^{2/3}.$$

Since the tangent passes through A, i.e. *y*-coordinate is 0, we get,

$$Xx^{-1/3} = a^{2/3} \text{ or } X = \frac{a^{2/3}}{x^{-1/3}} = a^{2/3}x^{1/3}.$$

Hence the coordinates of A are $(a^{2/3}x^{1/3}, 0)$.

Similarly, we have $X = 0$, at B on *y*-axis. Therefore,

$$Yy^{-1/3} = a^{2/3} \text{ or } Y = a^{2/3}y^{1/3}.$$

Hence the coordinates of B are $(0, a^{2/3}y^{1/3})$.

Now

$$\begin{aligned} AB^2 &= (a^{2/3}x^{1/3})^2 + (-a^{2/3}y^{1/3})^2 \\ &= a^{4/3}x^{2/3} + a^{4/3}y^{2/3} \\ &= a^{4/3}(x^{2/3} + y^{2/3}) \\ &= a^{4/3}a^{2/3} \\ &= a^2. \end{aligned}$$

Thus $AB = a =$ constant.

Example 6.11 Show that in the curve $x^{m+n} = a^{m-n}y^{2n}$, the mth power of the subtangent varies as the nth power of the subnormal.

Solution We have

$$\text{Subtangent} = \frac{y}{y_1} \quad \text{and} \quad \text{subnormal} = yy_1.$$

According to the question

$$(\text{Subtangent})^m \propto (\text{Subnormal})^n$$

or

$$\frac{(\text{Subtangent})^m}{(\text{Subnormal})^n} = \text{constant}$$

or

$$\frac{(y/y_1)^m}{(yy_1)^n} = \text{constant}$$

or

$$\frac{y^{m-n}}{y_1^{m+n}} = \text{constant.} \tag{1}$$

Now the equation to the curve is $x^{m+n} = a^{m-n}y^{2n}$. Taking logarithm both the sides $(m+n)\log x = (m-n)\log a + 2n\log y$. Differentiating with respect to x, we get

$$(m+n)\frac{1}{x} = 2n\frac{1}{y}y_1 \quad \text{or} \quad y_1 = \frac{y}{x}\frac{m+n}{2n}.$$

Now,

$$\text{LHS of (1)} = \frac{y^{m-n}}{\left(\frac{y}{x}\frac{m+n}{2n}\right)^{m+n}}$$

$$= \frac{y^{m-n}x^{m+n}}{y^{m+n}\left(\frac{m+n}{2n}\right)^{m+n}}$$

$$= \frac{x^{m+n}}{y^{2n}\left(\frac{m+n}{2n}\right)^{m+n}}$$

$$= \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}}$$

$$= \text{constant}$$

Thus LHS = RHS.

Example 6.12 In the catenary

$$y = c\cosh\frac{x}{c}$$

show that the length of the perpendicular from the foot of the ordinate on the tangent is of constant length.

Solution Let PT be the tangent to the curve at P and MN be perpendicular from M, the foot of the ordinate on the tangent (Fig. 6.11). We have to prove that MN = constant.

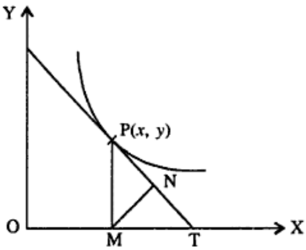


Fig. 6.11 Solution to Example 6.12.

Now the equation to the tangent at P(x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$X\frac{dy}{dx} - Y + \left(y - x\frac{dy}{dx}\right) = 0 \qquad (1)$$

Therefore,

MN = Length of the perpendicular from M(x, 0) to the tangent.

$$= \frac{x(dy/dx) + y - x(dy/dx)}{\sqrt{1 + (dy/dx)^2}}$$

$$= \frac{y}{\sqrt{1 + (dy/dx)^2}} \qquad (2)$$

Here the curve is

$$y = c\cosh\frac{x}{c}$$

Then

$$\frac{dy}{dx} = c\left(\sinh \frac{x}{c}\right)\frac{1}{c} = \sinh \frac{x}{c}$$

Putting this value in (2), we get

$$MN = \frac{y}{\sqrt{1+\sinh^2(x/c)}} = \frac{c\cosh(x/c)}{\cosh(x/c)} = c = \text{constant.}$$

Example 6.13 Prove that the sum of the intercepts of the tangent to the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

upon the coordinate axes is constant.

Solution Let the tangent to the curve at P meets the coordinate axes at A and B, respectively. Thus OA and OB are the intercepts of the tangent upon the axes. We have to prove that OA + OB = constant. Equation of the tangent at P is

$$(X-x)f_x + (Y-y)f_y = 0$$

or

$$(X-x)\frac{1}{2}x^{-1/2} + (Y-y)\frac{1}{2}y^{-1/2} = 0$$

as

$$f_x = \frac{1}{2}x^{-1/2}, \quad f_y = \frac{1}{2}y^{-1/2}$$

Then, we have

$$Xx^{-1/2} + Yy^{-1/2} = \sqrt{x} + \sqrt{y} = \sqrt{a}.$$

Now, the point A is the intersection of the tangent and the axis of x. That is,

$$Xx^{-1/2} + Yy^{-1/2} = \sqrt{a} \text{ and } Y = 0$$

Therefore,

$$Xx^{-1/2} = \sqrt{a} \quad \text{or} \quad X = \sqrt{a}\sqrt{x}.$$

Hence coordinates of A are $(\surd a \surd x, 0)$.

Similarly, point B is the intersection of the tangent and the y-axis. Then

$$Xx^{-1/2} + Yy^{-1/2} = \sqrt{a} \quad \text{or} \quad X = 0.$$

Therefore,

$$Yy^{-1/2} = \sqrt{a} \quad \text{or} \quad Y = \sqrt{a}\sqrt{y}.$$

Hence coordinates of B are $(0, \sqrt{a}\sqrt{y})$. Now

$$OA + OB = \sqrt{a}\sqrt{x} + \sqrt{a}\sqrt{y} = \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a}\sqrt{a} = a \quad \text{(a constant).}$$

Example 6.14 In the curve

$$y = a\cosh\frac{x}{a},$$

prove that the length of the portion of the normal intercepted between the curve and the x-axis, varies as y^2.

Solution Here

$$y = a\cosh\frac{x}{a},$$

Then

$$\frac{dy}{dx} = a\left(\sinh\frac{x}{a}\right)\frac{1}{a} = \sinh\frac{x}{a}.$$

Now

$$\begin{aligned} \text{Length of the normal} &= y\sqrt{1 + y_1^2} \\ &= y\sqrt{1 + \sinh^2\frac{x}{a}} \\ &= y\cosh\frac{x}{a} \\ &= y\frac{y}{a} \\ &\propto y^2. \end{aligned}$$

Example 6.15 Find the condition that the conics $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ shall cut orthogonally.

Solution Let the equations of conics are

$$f(x, y) = ax^2 + by^2 - 1 = 0 \tag{1}$$

and

$$F(x, y) \equiv a_1x^2 + b_1y^2 - 1 = 0 \tag{2}$$

Since the two curves are orthogonal,

$$f_xF_x + f_yF_y = 0.$$

or

$$(2ax)(2a_1x) + (2by)(2b_1y) = 0$$

or

$$aa_1x^2 + bb_1y^2 = 0 \tag{3}$$

Now eliminating x and y among (1), (2) and (3), we have

$$\begin{vmatrix} aa_1 & bb_1 & 0 \\ a & b & 1 \\ a_1 & b_1 & 1 \end{vmatrix} = 0$$

or

$$aa_1(b - b_1) - bb_1(a_1 - a) = 0$$

or

$$aba_1 - aa_1b_1 - a_1bb_1 + abb_1 = 0.$$

Dividing by aa_1bb_1, we get

$$\frac{1}{a} - \frac{1}{a_1} = \frac{1}{b} - \frac{1}{b_1}.$$

This is the required condition.

Example 6.16 Show that the part of the tangent to $xy = c^2$ included between the coordinate axes is bisected at the point of tangency.

Solution The equation of the curve is

$$xy = c^2 \tag{1}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{y}{x}.$$

The equation of the tangent to (1) at (x, y), is

$$Y - y = -\frac{y}{x}(X - x) \quad \text{or} \quad \frac{X}{2x} + \frac{Y}{2y} = 1.$$

This tangent makes intercepts $2x$ and $2y$ on the axes of x and y, that is, the tangent meets the axes of x and y at points $(2x, 0)$ and $(0, 2y)$, whose middle point is (x, y), the point of contact of the tangent.

Example 6.17 Prove that

$$\frac{x}{a} + \frac{y}{b} = 1$$

touches the curve $y = be^{-x/a}$ at the point, where the curve crosses the axis of y.

Solution Here the equation of y-axis is $x = 0$. The point at which the curve crosses the axis of y is $(0, b)$. Now the equation of the tangent at $(0, b)$ to the curve is

$$Y - y = (X - x)\frac{dy}{dx},$$

where $y = b$, $x = 0$. At $y = b$, $x = 0$,

$$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a} = \frac{-b}{a}.$$

The equation of the tangent becomes

$$Y - b = (X - 0)\left(-\frac{b}{a}\right)$$

or

$$Y - b = -\frac{b}{a}X.$$

or

$$\frac{X}{a} + \frac{Y}{b} = 1.$$

If we take (x, y) instead of (X, Y) for the current coordinates on the tangent, the equation becomes

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Example 6.18 Find the equation of the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ which makes an angle ψ with the x-axis.

Solution Let $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta.$$

or

$$\frac{dx}{dy} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

or

$$\theta = \frac{\pi}{2} - \psi\left(as - \frac{dx}{dy} = \tan\psi\right)$$

The equation of the normal at $P(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = \cot\theta\,(x - a\cos^3\theta)$$

or

$$y\sin\theta - a\sin^4\theta = x\cos\theta - a\cos^4\theta$$

or

$$x\cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta) = a\cos 2\theta$$

or

$$x\cos\left(\frac{\pi}{2}-\psi\right)-y\sin\left(\frac{\pi}{2}-\psi\right)=a\cos 2(\pi/2\ -\psi)$$

or

$$-x\sin\psi+y\cos\psi=a\cos 2\psi.$$

Example 6.19 Prove that the tangent to any point of the curve

$$x = a(t + \sin t\cot t), \qquad y = a(1 + \sin t)^2$$

makes an angle with x-axis.

Solution Here $x = a(t + \sin t\cos t)$. Then

$$\frac{dx}{dt}=a(1+\cos 2t) \quad \text{and} \quad y=a(1+\sin t)^2$$

and

$$\frac{dy}{dt}=2a\cos t(1+\sin t)=a\,(2\cos t+\sin 2t)$$

Therefore,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a(2\cos t+\sin 2t)}{a(1+\cos 2t)}$$

$$=\frac{1+\sin t}{\cos t}=\frac{\cos(t/2)+\sin(t/2)}{\cos(t/2)-\sin(t/2)}$$

$$=\frac{1+\tan(t/2)}{1-\tan(t/2)}$$

$$=\tan\left(\frac{\pi}{4}+\frac{t}{2}\right)$$

Then

$$\tan\psi=\tan\frac{1}{4}(\pi+t) \quad \text{or} \quad \psi=\frac{\pi}{4}+\frac{t}{2}=\frac{1}{4}(\pi+2t).$$

Example 6.20 Prove that the normal at any point of the curve

$$x = a(\cos\theta + \theta\sin\theta), \qquad y = a(\sin\theta - \theta\cos\theta)$$

is at a constant distance from the origin.

Solution Here

$$\frac{dx}{d\theta}=-a\sin\theta+a\sin\theta+a\theta\cos\theta=a\theta\cos\theta$$

$$\frac{dy}{d\theta}=-a\cos\theta-a\cos\theta+a\sin\theta=a\theta\sin\theta$$

Then

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

Now

$$\text{Slope of the normal} = \frac{-dx}{dy}=-\frac{\cos\theta}{\sin\theta}$$

The equation of the normal at θ is

$$y-(a\sin\theta-a\theta\cos\theta)=-\frac{\cos\theta}{\sin\theta}(x-a\cos\theta-a\theta\sin\theta)$$

or

$$y\sin\theta - a\sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta.$$

Therefore, the equation of the normal is $x\cos\theta + y\sin\theta - a = 0$.

Example 6.21 Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ cut orthogonally.

Solution Let $P(x_1, y_1)$ be the point of intersection of the given curves. Differentiating $x^3 - 3xy^2 + 2 = 0$ with respect to x, we get

$$3x^2-3y^2-6xy\frac{dy}{dx}=0 \quad \text{or} \quad \frac{dy}{dx}=\frac{x^2-y^2}{2xy}$$

Then

$$m_1=\left(\frac{dy}{dx}\right)_{x_1,y_1}=\frac{x_1^2-y_1^2}{2x_1y_1}$$

Also, differentiating $3x^2y - y^3 - 2 = 0$ with respect to x, we obtain

$$6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0 \quad \text{or} \quad \frac{dy}{dx}=\frac{-2xy}{x^2-y^2}$$

Then

$$m_2=\left(\frac{dy}{dx}\right)_{x_1,y_1}=-\frac{2x_1y_1}{x_1^2-y_1^2}.$$

Thus

$$m_1m_2=-1$$

which shows the given curves cut orthogonally.

Example 6.22 Prove the following:

(a) The subtangent on $x^my^n = a^{m+n}$ varies as the abscissa.
(b) The subtangent on $y = ae^{-a^2x^2}$ varies inversely as abscissa.
(c) The sum of the tangent and subtangent on $y = a\log(x^2 - a^2)$ varies as the product of the coordinates.

or

$$\frac{X}{3x^2\left(\frac{1}{3x}+\frac{1}{2a}\right)}+\frac{Y}{2ay\left(\frac{1}{3x}+\frac{1}{2a}\right)}=1.$$

If it cuts off equal intercepts from the coordinate axes, we have

$$3x^2\left(\frac{1}{3x}+\frac{1}{2a}\right)=2ay\left(\frac{1}{3x}+\frac{1}{2a}\right).$$

Then

$$3x^2=2ay \quad \text{and} \quad x=-\frac{2a}{3}.$$

Since the normal passess through the origin, the equation becomes

$$\frac{X}{3x^2}+\frac{Y}{2ay}=0.$$

The intercept being equal to zero on $3x^2 = 2ay$. But

$$ay^2=x^3 \quad \text{or} \quad x^3=a\left(\frac{3x^2}{2a}\right)^2$$

Then $4a^2x^3 = 9ax^4$ gives either $x = 0$, or $x = 4a/9$. But, when $x = 0$, $y = 0$, the normal passes through the origin (0, 0). Hence the required abscissa is $4a/9$.

Example 6.25 Show that the normal to the parabola $y^2 = 4ax$ touches the curve $27ay^2 = 4(x - 2a)^3$.

Solution Given the equation of the parabola is

$$y^2 = 4ax \tag{1}$$

Differentiating, we get

$$\frac{dy}{dx}=\frac{4a}{2y} \quad \text{or} \quad \frac{dx}{dy}=\frac{y}{2a}.$$

At any point $(at^2, 2at)$ on the parabola (1), we have

$$\frac{dx}{dy}=\frac{y}{2a}=\frac{2at}{2a}=t.$$

The equation of the normal to (1), at $(at^2, 2at)$ is

$$Y-y+(X-x)\frac{dx}{dy}=0 \qquad \text{or} \qquad y+tx=2at+at^3 \tag{2}$$

Any point on the curve

$$27ay^2 = 4(x - 2a)^3 \tag{3}$$

is $x = 2a + 3am^2$, $y = 2am^2$. Therefore,

$$\frac{dx}{dm}=6am, \quad \frac{dy}{dm}=6am^2.$$

Example 6.30 For the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

show that

$$\frac{ds}{d\phi}=a\sqrt{1-e^2\sin^2\phi}$$

where $x = a \sin\phi$.

Solution We have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

and

$$x = a \sin\phi \tag{2}$$

Then

$$\frac{a^2\sin^2\phi}{a^2}+\frac{y^2}{b^2}=1 \quad \text{or} \quad y = b \cos\phi. \tag{3}$$

Differentiating (2) and (3), we get

$$dx = a \cos\phi \, d\phi \quad \text{and} \quad dy = -b \sin\phi \, d\phi.$$

Therefore,

$$(ds)^2 = (dy)^2 + (dx)^2 = b^2 \sin^2\phi \, (d\phi)^2 + a^2\cos^2\phi \, (d\phi)^2$$

or

$$\left(\frac{ds}{d\phi}\right)^2 = b^2 \sin^2\phi + a^2\cos^2\phi.$$

We know that the equation of the ellipse, $b^2 = a^2(1 - e^2)$. Then

$$\left(\frac{ds}{d\phi}\right)^2 = a^2(1-e^2)\sin^2\phi + a^2\cos^2\phi$$
$$= a^2(\sin^2\phi - e^2\sin^2\phi + \cos^2\phi)$$
$$= a^2(1-e^2\sin^2\phi)$$

or

$$\frac{ds}{d\phi}=a\sqrt{1-e^2\sin^2\phi}$$

Exercises 6.1

1. Find the tangent and the normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it cuts the axis.

2. Find the equations of the tangents and normals at the point (x, y) on each of the following curves:

   (i) $x^{1/3}+y^{1/3}=a^{2/3}$ (ii) $x^3+y^3-3axy=0$ (iii) $x^m/a^m+y^m/b^m=1$.

3. Find the equation to the tangent and normal at the point $\theta=\pi/2$ to the curve $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

4. Find the equation of the normal to the curve $xy=(a+x)^2$ which makes equal intercepts on the coordinate axes.

5. If the line $x\cos\alpha+y\sin\alpha=p$ touches the curve

   $$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1,$$

   prove that

   $$(a\cos\alpha)^{m/(m-1)}+(b\sin\alpha)^{m/(m-1)}=p^{m/(m-1)}.$$

6. Find the condition that the line $x\cos\alpha+y\sin\alpha=p$ may touch the curve $x^m y^n=a^{m+n}$.

7. For the curve $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, prove that

   $$\psi=\frac{1}{2}\theta \text{ and } \frac{ds}{d\theta}=2a\cos\frac{\theta}{2}.$$

8. Show that the normal to the ellipse

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

   touches the curve $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

9. If the tangent at (x_1, y_1), to the curve $x^3+y^3=a^3$ meets the curve again in (x_2, y_2), prove that

   $$\frac{x_2}{x_1}+\frac{y_2}{y_1}=-1.$$

10. Prove that the tangent to the curve $x^3+y^3-3axy=0$ is parallel to the x-axis at the point, where it meets the parabola $x^2=ay$.

11. Prove that the tangent to the curve $ax^2+2hxy+by^2=1$ is perpendicular to the x-axis at points, where the line $hx+by=0$ intersects the curve.

35. Prove that in an ellipse the subnormal varies as the abscissa.

36. Find the length of tangent, normal, subtangent and subnormal at the point θ of the curve $x = a \cos^3\theta$, $y = a \sin^3\theta$.

37. Prove that the subnormal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies inversely as the cube of its abscissa.

38. Find the condition that the tangent at any two points P, Q on the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angle.

39. Find the portion of the normal intercepted between x-axis and the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

40. Find the condition that the normal at point P(x, y) to the curve meets the x-axis at

$$x = \frac{3a}{2}(\sinh\theta\cosh\theta + \theta), \quad y = a\cosh^2\theta.$$

41. Prove that for the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

length of normal varies as the perpendicular from the origin on tangent.

42. Show that the curves $y^2 = 2x$ and $2xy = k$ cut at right angle, if $k^2 = 8$.

43. Prove that the portion of the tangent to the curve:

$$\frac{x + \sqrt{a^2 - y^2}}{a} = \log\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right)$$

intercepted between the point of contact and the x-axis is constant.

44. Find the equation of the tangent to the curve $y^2 = 4x + 5$ which is parallel to the line $y = 2x + 1$.

45. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, the tangents meet the axis at A and B. Find the locus of Q(OA, OB).

46. Find the tangent and the normal on

$$x = \frac{2at^2}{1+t^2}, \quad y = \frac{2at^3}{1+t^2}, \quad \text{when } t = \frac{1}{2}.$$

47. Find the tangent and the normal when

$$x = 2a\cos\theta - a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta \quad \text{at} \quad \theta = \frac{\pi}{2}$$

48. Find the tangent and the normal on

$$(x^2 + y^2)x - a(x^2 - y^2) = 0 \quad \text{at} \quad x = \frac{-3a}{5}.$$

Let P(r, θ) be any point on the curve $r = f(\theta)$ and Q($r + \delta r$, $\theta + \delta\theta$) be any other point on the same curve very near to P (Fig. 6.14). Let PQ be the secant through PQ and PT, a tangent at P. Draw PN perpendicular to OQ.

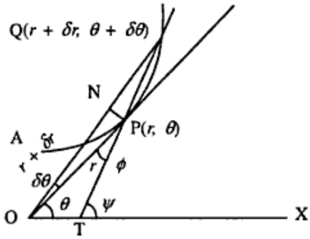


Fig. 6.14 Angle between radius vector and tangent.

Now $\angle POT = \theta$ and $\angle QOT = \theta + \delta\theta$ such that $\angle QOP = \delta\theta$. Again $PN = r \sin \delta\theta$, $ON = r \cos \delta\theta$. Let $\angle OPT = \phi$. From right-angled triangle ΔPQN,

$$\tan PQN = \frac{PN}{NQ}$$

$$= \frac{PN}{OQ - ON}$$

$$= \frac{r \sin \delta\theta}{(r + \delta r) - r \cos \delta\theta}$$

$$= \frac{r \sin \delta\theta}{r(1 - \cos \delta\theta) + \delta r}$$

$$= \frac{r \sin \delta\theta}{2r \sin^2(\delta\theta/2) + \delta r}$$

$$= \frac{r \dfrac{\sin \delta\theta}{\delta\theta}}{2r\left[\dfrac{\sin(\delta\theta/2)}{\delta\theta/2}\right]^2 \dfrac{\delta\theta}{4} + \dfrac{\delta r}{\delta\theta}}$$

Now in the limit when Q → P, $\delta\theta \to 0$ and the secant PQ becomes the tangent at P and hence $\angle PQN = \angle OPT = \phi$. Therefore,

$$\tan \phi = \lim_{\delta\theta \to 0} \frac{r \dfrac{\sin \delta\theta}{\delta\theta}}{2r\left[\dfrac{\sin(\delta\theta/2)}{d\theta/2}\right]^2 \dfrac{\delta\theta}{4} + \dfrac{\delta r}{\delta\theta}} = \frac{(r)(1)}{2r(1)^2(0) + dr/d\theta} = r\frac{\delta\theta}{dr}.$$

Again since

$$\text{cosec}^2\phi = 1 + \cot^2\phi = 1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad \text{or} \qquad ds^2 = r^2 d\theta^2 + dr^2.$$

Perpendicular from pole to tangent

Let $ON = p$ be the length of the perpendicular from the pole O to the tangent PT to the curve $r = f(\theta)$ at any point P (Fig. 6.15). Then from right-angled triangle ONP,

$$\sin\phi = \frac{ON}{OP} = \frac{p}{r} \quad \text{or} \quad p = r\sin\phi \tag{6.12}$$

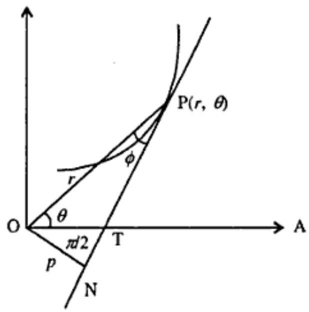


Fig. 6.15 Perpendicular from pole to tangent.

Now

$$\begin{aligned}
\frac{1}{p^2} &= \frac{1}{r^2\sin^2\phi} \\
&= \frac{1}{r^2}\text{cosec}^2\phi \\
&= \frac{1}{r^2}(1 + \cot^2\phi) \\
&= \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right] \left[\text{as } \tan\phi = r\frac{d\theta}{dr}\right] \\
&= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] \\
&= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2
\end{aligned} \tag{6.13}$$

The symbol u is generally used to denote $1/r$, that is,

$$u = \frac{1}{r}, \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

or

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Hence from Eq. (6.13)

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

Angle of intersection of two curves (in polar form)

Let the two curves $r = f(\theta)$, and $r = \phi(\theta)$ intersect at P and let PT_1, PT_2 be the tangents at P to the curves (Fig. 6.16). Let

$$\angle OPT_1 = \phi_1 \text{ and } \angle OPT_2 = \phi_2.$$

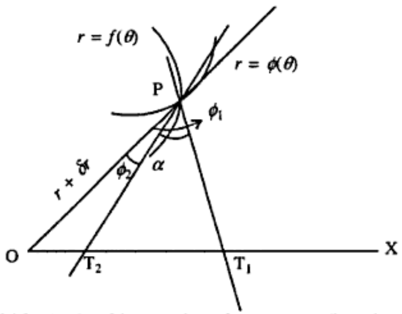


Fig. 6.16 Angle of intersection of two curves (in polar form).

Also, let α ($=\phi_1 - \phi_2$) be the angle between the curves such that $\angle T_1PT_2 = \alpha$. Therefore,

$$\tan\alpha = \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}$$

But

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)}$$

and

$$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{\phi(\theta)}{\phi'(\theta)}$$

Therefore,

$$\tan\alpha = \frac{\dfrac{f(\theta)}{f'(\theta)} - \dfrac{\phi(\theta)}{\phi'(\theta)}}{1 + \dfrac{f(\theta)}{f'(\theta)}\dfrac{\phi(\theta)}{\phi'(\theta)}} = \frac{f(\theta)\phi'(\theta) - \phi(\theta)f'(\theta)}{f'(\theta)\phi'(\theta) + f(\theta)\phi(\theta)}.$$

Corollary If $f\phi + f'\phi' = 0$, therefore $\alpha = \pi/2$ and the two carves instersect orthogonally, i.e. at right angles.

6.7 Polar Subtangent and Polar Subnormal

Let P(r, θ) be any point on the curve $r = f(\theta)$. Refer to Fig. 6.17. Draw TON perpendicular to the radius vector OP. Let tangent and normal to the curve at P intersect TON at T, N, respectively. Then OT and ON are called *polar subtangent* and *polar subnormal* to the curve at P.

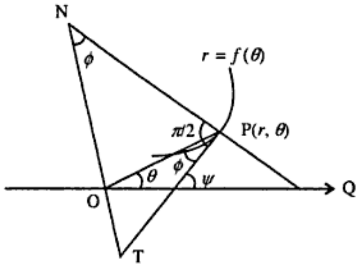


Fig. 6.17 Polar subtangent and polar subnormal.

Since $\angle TPN = \angle PON = \dfrac{\pi}{2}$ and $\angle OPT = \angle ONP = \phi$,

from triangles OPT and OPN, we get

$$OT = r\tan\phi = r\frac{rd\theta}{dr} \quad \text{and} \quad ON = r\cot\phi = r\frac{dr}{r\,d\theta}.$$

Therefore,

$$\text{Polar subtangent, } OT = r^2\frac{d\theta}{dr} \text{ and Polar subnormal, } ON = \frac{dr}{d\theta}.$$

Corollary Since

$$OT \cdot ON = r^2\frac{d\theta}{dr}\frac{dr}{d\theta} = r^2$$

Let p be the length of the perpendicular from the origin to the tangent (6.14), then

$$p = \frac{xf_x + yf_x}{\sqrt{f_x^2 + f_y^2}} \tag{6.15}$$

Also

$$r^2 = x^2 + y^2 \tag{6.16}$$

and

$$f(x, y) = 0. \tag{6.17}$$

Eliminating x and y from Eqs. (6.15)–(6.17), the required pedal equation is obtained.

Pedal equation deduced from polar equation

Let the equation be $f(r, \theta) = 0$. Let the pole be taken at the point with regard to which it is required to find the pedal equation of the curve.

If p be the length of the perpendicular draw from the pole to the tangent at (r, θ) to the curve, then

$$p = r \sin \phi \tag{6.18}$$

$$\tan \phi = r \frac{d\theta}{dr} \tag{6.19}$$

$$f(r, \theta) = 0 \tag{6.20}$$

If we eliminate θ and ϕ from Eqs. (6.18)–(6.20), we get the required pedal equation, i.e. relation between p and r.

Example 6.31 Find the pedal equation of the curve $r = ae^{\theta \cot \alpha}$.

Solution Here the given curve is

$$r = ae^{\theta \cot \alpha} \tag{1}$$

Differentiating with respect to θ, we get

$$\frac{dr}{d\theta} = ae^{\theta \cot \alpha} \cot \alpha = \frac{r}{\tan \alpha}$$

We know that

$$\tan \phi = r \frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r \tan \alpha}{r} = \tan \alpha.$$

Therefore, $\phi = \alpha$. We also know that $p = r \sin \phi$. Hence $p = r \sin \alpha$.

This is the required pedal equation.

Example 6.32 Show that for the curve $r = a^{\theta}$, the polar subtangent and the polar subnormal have constant ratio.

Solution Here the given equation is $r = a^\theta$ or $r = e^{\theta \log a}$. Then

$$\frac{dr}{d\theta} = e^{\theta \log a} \log a = r \log a$$

Therefore,

$$\frac{\text{Polar subtangent}}{\text{Polar subnormal}} = \frac{r^2(d\theta/dr)}{dr/d\theta}$$
$$= \frac{r^2}{(dr/d\theta)^2}$$
$$= \frac{r^2}{r^2(\log a)^2}$$
$$= \frac{1}{(\log a)^2}$$
$$= \text{Constant.}$$

Example 6.33 Prove that the polar subtangent for the cardiode $r = a(1 - \cos\theta)$ is

$$\frac{2a \sin^2(\theta/2)}{\cos(\theta/2)}.$$

Solution Here

$$\text{Polar subtangent} = r^2 \frac{d\theta}{dr}$$

Then

$$r^2 = a^2\left(2\sin^2\frac{\theta}{2}\right)^2 = 4a^2 \sin^4\frac{\theta}{2}$$

Therefore

$$\text{Polar subtangent } = \frac{4a^2 \sin^4(\theta/2)}{2a \sin(\theta/2)\cos(\theta/2)} = \frac{2a \sin^2(\theta/2)}{\cos \theta/2}.$$

Example 6.34 Find the angle of intersection of the curves:

$$r = a(1 + \cos\theta), \qquad r = b(1 - \cos\theta).$$

Solution The first curve is $r = a(1 + \cos\ \theta)$. Then

$$\frac{dr}{d\theta} = -a \sin\theta$$

Then

$$p^2 = r^2 \sin^2\phi$$
$$= r^2 \sin^2 \frac{\theta}{2} \qquad \left[\text{since } \phi = \frac{\theta}{2} \text{ from (1)}\right]$$
$$= \frac{r^3}{2a}.$$

Example 6.37 For the parabola $2a/r = 1 - \cos\theta$, prove the following results:

(i) $\phi = \pi - \frac{\theta}{2}$, (ii) $p = \frac{a}{\sin(\theta/2)}$, (iii) $p^2 = ar$,

(iv) polar subtangent = $2a$ cosec θ.

Solution We know that

$$p = r\sin\phi, \quad \tan\phi = r\frac{d\theta}{dr}.$$

Now,

$$\frac{2a}{r} = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

Differentiating, we get

$$2a\left(-\frac{1}{r^2}\right)\frac{dr}{d\theta} = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Therefore,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a}{\sin^2(\theta/2)}\,\frac{-a}{r^2\sin(\theta/2)\cos(\theta/2)}$$
$$= \frac{a}{\sin^2(\theta/2)}\,\frac{\sin^4(\theta/2)}{a^2}\,\frac{-a}{\sin(\theta/2)\cos(\theta/2)}$$
$$= -\tan\frac{\theta}{2}$$
$$= \tan\left(\pi - \frac{\theta}{2}\right)$$

Then

$$\phi = \pi - \frac{\theta}{2} \tag{1}$$

From (1), we get

$$p = r\sin\phi, = r\sin\left(\pi - \frac{\theta}{2}\right) = \frac{a}{\sin^2\theta}\sin\left(\pi - \frac{\theta}{2}\right) = \frac{a}{\sin(\theta/2)}$$

Example 6.39 Find the pedal equation of $r^m = a^m \cos m\theta$.

Solution Taking log both sides, we get, $m \log r = m \log a + \log \cos m\theta$. Differentiating both sides with respect to θ, we get

$$\frac{m}{r}\frac{dr}{d\theta} = -\frac{\sin m\theta}{\cos m\theta} m$$

or

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan m\theta$$

or

$$\cot\phi = -\tan m\theta = \cot\left(\frac{\pi}{2} + m\theta\right)$$

Then

$$\phi = \frac{\phi}{2} + m\theta.$$

Now

$$p = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta = r\frac{r^m}{a^m}$$

Hence the required pedal equation is

$$p = \frac{r^{m+1}}{a^m}.$$

Example 6.40 Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with respect to its centre is

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}.$$

Solution The equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

From the coordinate geometry, the equation of the tangent at (x, y) is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1$$

Now the length of the perpendicular on the tangent from the centre (0, 0) is

$$p = \frac{1}{\sqrt{(x/a^2)^2 + (y/b^2)^2}} \quad \text{or} \quad \frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} \tag{2}$$

Also,

$$r^2 = x^2 + y^2 \tag{3}$$

Eliminating x^2 and y^2 from (1), (2) and (3), we get

$$\begin{vmatrix} 1/a^2 & 1/b^2 & 1 \\ 1/a^4 & 1/b^4 & 1/p^2 \\ 1 & 1 & r^2 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} a^2 & b^2 & p^2 \\ a^4 & b^4 & r^2p^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solving, we get

$$\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$$

Therefore,

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2},$$

which is the required pedal equation.

Example 6.41 Find the pedal equation of the parabola $y^2 = 4ax$ with regard to its focus.

Solution Here the equation of the parabola $by^2 = 4ax$. The focus of the parabola is $(a, 0)$. So, transfer the origin $(0, 0)$ to the focus $(a, 0)$. Therefore, the equation to the parabola becomes

$$(y + a)^2 = 4a(x + a) \quad \text{or} \quad y^2 = 4ax + 4a^2 \tag{1}$$

By the coordinate geometry, the equation of the tangent to the parabola (1) at the point (x, y) is

$$Yy = 2a(X + x) + 4a^2$$

or

$$2aX - yY + 4a^2 + 2ax = 0.$$

Now, from (1), the length of the perpendicular (p) from origin upon the tangent is

$$p = \frac{4a^2 + 2ax}{\sqrt{4a^2 + y}} = \frac{4a^2 + 2ax}{\sqrt{4a^2 + 4ax + 4a^2}}$$

or

$$p = \frac{2a(2a + x)}{\sqrt{4a(2a + x)}} = \sqrt{a}\sqrt{2a + x}$$

or

$$p^2 = a(2a + x)^2 \tag{2}$$

7. Show that the curves $r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$ intersect at an angle which is independent of a and b.

8. Show that the pedal equation of the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.

9. Show that the pedal equation of $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$ is $r = (\sqrt{2})p$

10. Obtain the pedal equation of $x^2 - y^2 = a^2$.

11. Show that:

    (i) Pedal equation of $r^2 \cos 2\theta = a^2$ is $pr = a^2$.
    (ii) Pedal equation of $r^2 = \cos 2\theta$ is $r^3 = a^2 p$.
    (iii) Pedal equation of $r = a(1 + \cos\theta)$ is $r^3 = 2ap^2$.

12. In the equiangular spiral $r = ae^{\theta\cot\alpha}$, prove that $dr/ds = \cos\alpha$.

13. For the curve $r = a(1 + \cos\theta)$, prove that $p = 2a\cos^3(\theta/2)$.

14. Obtain the pedal equation of the circle $x^2 + y^2 = 2ax$.

15. Show that the pedal equation of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 + 3p^2 = a^2$.

16. Find the pedal equation of the curve $l/r = 1 + e\cos\theta$.

17. Prove that the pedal equation of the curve $2a = r(1 + \cos\theta)$ is $p^2 = ar$.

18. Prove that the pedal equation of the curve $r^n = a^n \sin n\theta$ is $p = r^{n+1}/a^n$.

19. Prove that the pedal equation of the curve $r \sin n\theta = a$ is

$$p = \frac{ar}{\sqrt{a^2 + n^2(r^2 - a^2)}}.$$

20. For the curve

$$r\cos\frac{\sqrt{a^2 - b^2}}{a}\theta = \sqrt{a^2 - b^2},$$

show that

$$p = \frac{ar}{\sqrt{b^2 + r^2}}.$$

21. Prove that $r^2 = a^2 - 3p^2$ is the pedal equation of the curve: $x = a\cos^3 t$, $y = a\sin^3 t$.

22. Prove that the pedal equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is $pr = a^2$.

23. If ϕ be the angle, which the tangent to a curve makes with the radius vector drawn from the origin, prove that

$$\tan\phi = \frac{x(dy/dx) - y}{x + y(dy/dx)}.$$

24. Show that:

(i) Pedal equation of $r\left(1+\sin\frac{\theta}{2}\right)^2 = a$ is $ar^3 = 4p^4$.

(ii) Pedal equation of $r^m = a^m \sin m\theta + b^m \cos m\theta$ is $r^{m+1} = p\sqrt{a^{2m}+b^{2m}}$.

(iii) Pedal equation of $r = a$ sech $n\theta$ is $\frac{1}{p^2} = \frac{A}{r^2} + B$, where A and B are constants.

25. For any curve, prove that:

(i) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$, (ii) $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2-p^2}}$.

26. For the involute of a circle

$$\theta = \frac{r^2-a^2}{a} - \cos^{-1}\left(\frac{a}{r}\right),$$

prove that $\cos\phi = a/r$.

Chapter 7

Curvature

7.1 Introduction

A *curve* is that which changes its direction at each point and as such the tangent goes on changing as the point moves on the curve. The rate of change in the direction of tangent is different for different curves. The special feature of any curve is called *bending* or *curvature*. In mathematics, when we use the word 'curvature' it also means 'bend'.

Let us take two curves AB and CD (Fig. 7.1). It is obvious that the curve AB has a larger bend of the point P, whereas the curve CD has the least bend at

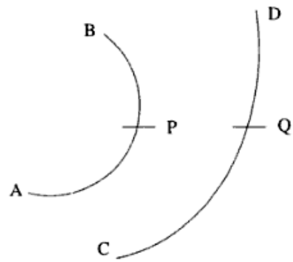


Fig. 7.1 Two curves with different bendings or curvatures.

the point Q. It means the rate of change of the tangent at P is large compared to that at the point Q. Thus bend is determined by the rate of change in direction of the tangent. When we say the curvature of any curve it means a point on the curve. How to measure the curvature of a curve at a given point. For that the measurement of curvature at any point on a curve taking into account its length and the direction of the tangent. The basic formula depends upon the intrinsic equation of a curve and all the other formulae are derived from it.

Let P be any point on the curve and Q, R be the other points very near to P (Fig. 7.2). Thus we can draw one and only circle through these three points P, Q, R. Now, we tend the points Q and R to coincide at P. We have a limiting

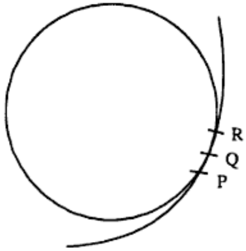


Fig. 7.2 Curvature at a point of a curve.

position of the circle PQR. This circle is called the *circle of curvature*. When Q, R → P_1, the limiting position of the circle PQR is called the *point of curvature*. The centre of this circle is called *centre of curvature* and its radius is called *radius of curvature*. The chord drawn inside the circle of curvature through P is called the *chord of curvature*.

7.2 Average Curvature

Let P be any point on the curve and Q be any other point on the same curve which is very near to P (Fig. 7.3). Let A be the fixed point on the curve such that the arc $AP = s$ and the arc $AQ = s + \delta s$ then the curve $PQ = \delta s$. Let PT and QN

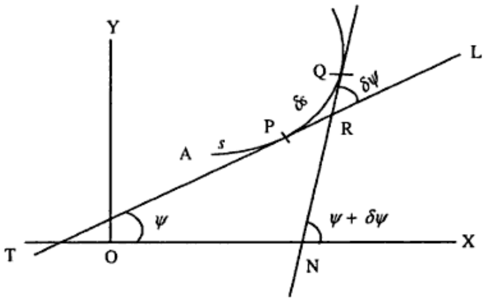


Fig. 7.3 Average curvature.

be the tangents to the curve P and Q respectively such that $\angle PTN = \psi$ and $\angle QNX = \psi + \delta\psi$. Then $\angle TRN = \angle QRL = \delta\psi$. Hence $\delta\psi$ is the change in the inclination of the tangent line as the point of contact of the tangent line describes the arc PQ. Thus the angle $\angle RTQ = \delta\psi$ is called the *angle of contingence* of the arc PQ. Hence the angle of contingence of any arc is the difference of the angles which the tangents at its extremities make with any given fixed straight line, generally x-axis. That is, it measures the change in the direction of the tangent when the point moves on the curve through an arc length δs. It is clear that the whole bending of the curvature which the curve undergoes between P and Q is greater or less according as the angle of contingence RTQ is greater or less.

The fraction,

$$\frac{\text{Angle of contingency}}{\text{Arc length}}$$

is called *average curvature* or *average bending* of the arc. Thus

$$\text{Average curvature of the arc } PQ = \frac{\delta\psi}{\delta s}$$

The limiting value of the average curvature, when Q → P along the curve, $\delta s \to 0$, i.e. $\lim_{\delta s \to 0} \to 0\ \delta\psi/\delta s$ is called the curvature at P and denoted by K. The *curvature* K at a point P, whose distance from a fixed point on the curve is s and the inclination of the tangent at P makes an angle ψ with the positive direction of the x-axis, is

$$K = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

Hence the curvature at a point is the arc-rate of the turning of the tangent at the point. The sign of the curvature K is positive if ψ increases as s increases and it is negative if ψ decreases as s decreases. Though the curvature can be positive or negative it indicates the absolute value of K, i.e. curvature

$$K = \left|\frac{d\psi}{ds}\right|$$

7.3 Intrinsic Equation of a Curve

Suppose any curve AP is given on a plane. Let its tangent at A be parallel to OX, or the initial line (Fig. 7.4). Draw a tangent at P to meet OX at T making $\angle XTP = \psi$. Let $AP = s$, the arc length of AP of the curve. (s, ψ) be the position

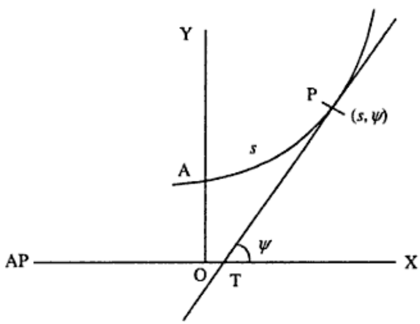


Fig. 7.4 Intrinsic equation of a curve.

of P on the curve, and ψ/s is called the average rate of curvature of the curve along AP. As s and ψ vary, the coordinate (s, ψ) is the position of different points on the curve uniquely. The coordinates of o is (0, 0) and any functional relation $s = f(\psi)$ is called *intrinsic equation* of the curve.

If $y = f(x)$ be the equation of any curve, its intrinsic equation is obtained by eliminating x between

$$\frac{dy}{dx} = \tan\psi = f'(x) \quad \text{and} \quad \frac{ds}{dx} = \sqrt{1 + f'(x)^2}.$$

A list of intrinsic equations of some useful curves is given below:

(i) Circle: $s = a\psi$,
(ii) Catenary: $s = c\tan\psi$ and $s = \log(\sec\psi + \tan\psi)$,
(iii) Cycloid: $s = 4a\sin\psi$,
(iv) Tractrix: $s = c\log\sec x$,
(v) Cardiod: $s = 4a(1 + \cos\psi/2) = 8a\sin^2\psi/6$,
(vi) Equiangular spiral: $s = a(e^{m\psi} - 1)$.

7.4 Geometrical Representation of Curvature

Theorem 7.1 If the normals at two consecutive points P and Q on a curve intersect at N, then the radius of curvature ρ of the curve at P is given by

$$\rho = \lim_{Q \to P} PN.$$

Proof Let the tangents at two consecutive points $P(s, \psi)$ and $Q(s + \delta s, \psi + \delta\psi)$ intersect at L. Then $\angle QLR = \delta\psi$ and arc $PQ = \delta s$ (Fig. 7.5). Since normals at P and Q intersect at N,

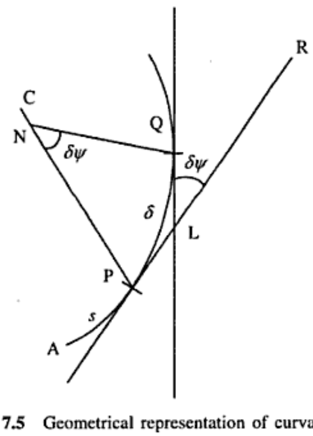


Fig. 7.5 Geometrical representation of curvature.

$$\angle NQL = \angle NPL = \frac{\pi}{2} \quad \text{and} \quad \angle PNQ = \delta\psi.$$

Also,

$$\angle PQN = \frac{\pi}{2} - \angle PQL.$$

Hence from ΔPNQ,

$$\frac{PQ}{\sin\delta\psi} = \frac{PN}{\sin\angle PQN} = \frac{PN}{\sin(\pi/2 - \angle PQL)} = \frac{PN}{\cos\angle PQL}$$

Therefore,

$$PN = \frac{PQ\cos\angle PQL}{\sin\delta\psi}.$$

Now, when Q → P, $\delta\psi \to 0$, $\angle PQL \to 0$ and the chord $PQ \to \delta s$. Hence Q → P implies $\delta\psi \to 0$. Therefore,

$$\lim_{Q\to P} PN = \lim_{\delta\psi\to 0} \frac{PQ\cos\angle PQL}{\sin\delta\psi}$$

$$= \lim_{\delta\psi\to 0} \frac{\delta s}{\delta\psi}\frac{\delta\psi}{\sin\delta\psi}\cos\angle PQL$$

$$= \frac{ds}{d\psi}(1)\cos 0$$

$$= \rho.$$

As Q → P, N shifted towards the centre of curvature C on the normal PN till it coincides with C, and the limit $PN \to PC = \rho$.

7.5 Circle of Curvature

The radius of any circle is its radius of curvatures, too. Let C be the centre and $CA = CP = a$ is the radius of any circle (Fig. 7.6).

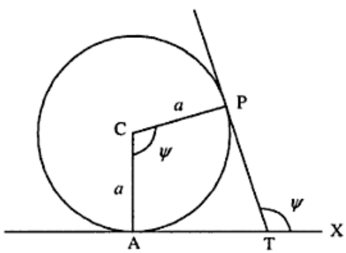


Fig. 7.6 Circle of curvature.

Note: If at any point, y becomes infinite, the above formula is not applicable and in that case, we consider the equation of the curve as $x = F(y)$.

(b) Let the curve be $x = F(y)$. If the tangent to the curve at the point (x, y) makes an angle ψ with the axis of x, then

$$\frac{dx}{dy} = \cot \psi.$$

Differentiating with respect to y, we get

$$\begin{aligned}\frac{d^2x}{dy^2} &= -\operatorname{cosec}^2\psi \frac{d\psi}{dy} \\ &= -\operatorname{cosec}^2\psi \frac{d\psi}{ds}\frac{ds}{dy} \\ &= -\operatorname{cosec}^2\psi \frac{1}{\rho} \operatorname{cosec}\psi \qquad \left(\text{as } \frac{dy}{ds} = \sin\psi\right).\end{aligned}$$

Therefore,

$$x_2 = -\frac{\operatorname{cosec}^3\psi}{\rho} \quad \text{or} \quad \rho = -\frac{\operatorname{cosec}^3\psi}{x_2}$$

But

$$\operatorname{cosec}\psi = \sqrt{1 + \cot^2\psi} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + x_1^2}$$

Then

$$\rho = -\frac{(1 + x_1^2)^{3/2}}{x_2}, \qquad \text{where } x_2 \neq 0.$$

Here suffixes denote differentiation with respect to y.

Example 7.1 Find the radius of curvature for the catenary $y = c \cosh (x/c)$.

Solution Differentiating with respect to x, we get

$$\frac{dy}{dx} = e\left(\sinh\frac{x}{c}\right)\frac{1}{c} = \sinh\frac{x}{c}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \left(\cosh\frac{x}{c}\right)\frac{1}{c}$$

Then

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{c\left(\cosh^2\frac{x}{c}\right)^{3/2}}{\cosh\frac{x}{c}} = \frac{c\cosh^3\frac{x}{c}}{\cosh\frac{x}{c}}$$

$$= c\cosh^2\frac{x}{c} = \frac{y^2}{c}$$

Example 7.2 Find the radius of curvature at the point (x, y) of the parabola $x^2 = 4ay$.

Solution The equation of the parabola $x^2 = 4ay$ can be rewritten as $x = 2\sqrt{(ay)}$. Differentiating it with respect to y, we get

$$\frac{dx}{dy} = 2\sqrt{a}\frac{1}{2\sqrt{y}} \quad \text{or} \quad \frac{dx}{dy} = \frac{\sqrt{a}}{\sqrt{y}}$$

Again differentiating, we get

$$\frac{d^2x}{dy^2} = -\frac{1}{2}\frac{\sqrt{a}}{y^{3/2}}.$$

We have

$$\rho = \frac{\left[1+(dx/dy)^2\right]^{3/2}}{d^2x/dy^2} = \frac{[1+(a/y)]^{3/2}}{(-1/2)(\sqrt{a}/y^{(3/2)})} = \frac{2(y+a)^{3/2}}{\sqrt{a}},$$

taking its absolute value.

Corollary Since

$$\sin\psi = \frac{dy}{ds}, \quad \cos\psi = \frac{dx}{ds}$$

Differentiating with respect to s, we get

$$\cos\psi\frac{d\psi}{ds} = \frac{d^2y}{ds^2} \quad \text{and} \quad -\sin\psi\frac{d\psi}{ds} = \frac{d^2x}{ds^2}$$

Therefore,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = -\frac{d^2x}{ds^2}\frac{ds}{dy} = \frac{d^2y}{ds^2}\frac{ds}{dx}$$

Also, squaring and adding, we get

$$\left(\frac{d\psi}{ds}\right)^2\left(\cos^2\psi+\sin^2\psi\right)=\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2x}{ds^2}\right)^2$$

or

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2.$$

The above two formulae are useful when equation of any curve be available as functions of s, x, y.

Implicit form

Let the equation of the curve be $f(x, y) = 0$. We have

$$\frac{dy}{dx}=-\frac{f_x}{f_y} \quad \text{or} \quad f_x+f_y\frac{dy}{dx}=0 \tag{7.1}$$

Differentiating with respect to x, we get

$$\frac{df_x}{dx}+\frac{df_y}{dx}\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0$$

or

$$\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_x}{\partial y}\frac{dy}{dx}\right)+\left(\frac{\partial f_y}{\partial x}+\frac{\partial f_y}{\partial y}\frac{dy}{dx}\right)\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0$$

Assuming that $f_{xy} = f_{yx}$ and using (7.1), we get

$$f_{xx}+2f_{xy}\left(-\frac{f_x}{f_y}\right)+f_{yy}\left(-\frac{f_x}{fy}\right)^2+f_y\frac{d^2y}{dx^2}=0$$

or

$$\frac{d^2y}{dx^2}=-\frac{f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2}{f_y^3} \tag{7.2}$$

We know that

$$\rho=\frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}=\frac{\left[1+(-f_x/f_y)^2\right]^{3/2}}{-(f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2)/f_y^3}$$

Then

$$\rho=\frac{(f_x^2+f_y^2)^{3/2}}{f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2},$$

taking the absolute value and $f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2\neq 0$.

Parametric form

Let the equation of the curve be $x = f(t)$, $y = \phi(t)$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left[\frac{\phi'(t)}{f'(t)}\right]\frac{1}{dx/dt}$$

$$= \frac{f'(t)\phi''(t) - \phi'(t)f''(t)}{[f'(t)]^2}\frac{1}{f'(t)}$$

If we put $f'(t) = x', \phi'(t) = y', \ldots$ then

$$\frac{dy}{dx} = \frac{y'}{x'} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^2}$$

But the formula,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left[1 + (y'/x')^2\right]^{3/2}}{(x'y'' - y'x'')/x'^3} = \frac{(x'^2 + y'^2)^{3/2}}{x'^2}\frac{x'^2}{x'y'' - y'x''}$$

Then

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''},$$

where the denominator $\neq 0$.

Example 7.3 Find the radius of curvature for the circle: $x = r\cos\theta$, $y = r\sin\theta$.

Solution Here $x = r\cos\theta$, and $y = r\sin\theta$. Differentiating, we get

$$x' = -r\sin\theta, \qquad y' = r\cos\theta$$

$$x'' = -r\cos\theta, \qquad y'' = -r\sin\theta$$

We also have

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}}{r^2(\sin^2\theta + \cos^2\theta)} = \frac{r^3(\sin^2\theta + \cos^2\theta)}{r^2} = r$$

Polar form

Let the equation of a curve be in polar form, $r = f(\theta)$. We know that

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\frac{d\theta}{d\psi} \tag{7.3}$$

Differentiating with respect to p, we get

$$2p + 2\frac{dp}{d\psi}\frac{d^2p}{d\psi^2}\frac{d\psi}{dp} = 2r\frac{dr}{dp},$$

or

$$p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp} = \rho,$$

or

$$\rho = p + \frac{d^2p}{d\psi^2}.$$

Corollary From Eq. (7.4) it can be easily seen that the projection of the radius vector on the tangent = $r\cos\phi = dp/d\psi$. Hence, geometrically, $dp/d\psi$ represents the projection of the radius vector on the tangent.

Theorem 7.2 For any curve, prove that the formula:

$$\rho = \frac{r}{\sin\phi(1 + d\phi/d\theta)} \quad \text{or} \quad \sin\phi\left(1 + \frac{d\phi}{d\theta}\right) = \frac{r}{\rho}.$$

Proof We have

$$\begin{aligned}\sin\phi\left(1 + \frac{d\phi}{d\theta}\right) &= \sin\phi + \sin\phi\frac{d\phi}{d\theta}\\ &= r\frac{d\theta}{ds} + r\frac{d\theta}{ds}\frac{d\phi}{d\theta}\\ &= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)\\ &= r\frac{d}{ds}(\theta + \phi)\\ &= r\frac{d\psi}{ds}\\ &= r\frac{1}{\rho}\end{aligned}$$

Example 7.4 Prove that the radius of curvature of hypocycloids, $p = A\sin B\psi$, varies as p.

Solution We have $p = A\sin B\psi$. Differentiating with respect to ψ, we get

$$\frac{dp}{d\psi} = AB\cos B\psi$$

Again differentiating, we get

$$\frac{d^2p}{d\psi^2} = -AB^2\sin B\psi = B^2p.$$

Now,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Putting the values of y_1 and y_2 from (1) and (2), we have $\rho = 3(axy)^{1/3}$.

Example 7.7 Show that the radius of curvature at a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

Solution Proceeding as in Example 7.6, we find its radius of curvature at any point (x, y) as $\rho = 3(axy)^{1/3}$. Here putting $x = a\cos^3\theta$ and $y = a\sin^3\theta$, we get

$$\rho = [a(a\cos^3\theta)a(\sin^3\theta)]^{1/3} = 3a\sin\theta\cos\theta.$$

Example 7.8 Find the radius of curvature of the curve: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = 0$.

Solution Since $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, we get on differentiation

$$\frac{dx}{d\theta} = a(1+\cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a\sin\theta.$$

Now,

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\frac{d\theta}{d\psi} \tag{1}$$

where

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\cos^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)}$$

$$= 2a\cos\frac{\theta}{2}$$

with

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

Therefore,

$$\tan\psi = \tan\frac{\theta}{2} \quad \text{or} \quad \psi = \frac{\theta}{2}.$$

Differentiating with respect to θ, we get

$$\frac{d\psi}{d\theta} = \frac{1}{2}.$$

Putting the values of $ds/d\theta$ and $d\psi/d\theta$ in (1), we have

$$\rho = \left(2a\cos\frac{\theta}{2}\right)2 = 4a\cos\frac{\theta}{2}.$$

At $\theta = 0$, $\rho = 4a\cos 0 = 4a$.

Alternative method: We have $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Then

$$\frac{dx}{d\theta} = a(1+\cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta$$

Since,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{+a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\sec^2\frac{\theta}{2}\right)\frac{d\theta}{dx} = \frac{1}{2}\left(\sec^2\frac{\theta}{2}\right)\frac{1}{a(1+\cos\theta)} = \frac{1}{4a}\frac{1}{\cos^4(\theta/2)}$$

We get

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1+\tan^2(\theta/2)]^{3/2}}{1/(4a)\cos^4(\theta/2)} = 4a\cos\frac{\theta}{2}$$

Note: Radius of curvature of all the parametric curves with the same constant, can be found in a similar manner.

Example 7.9 For the curve $r^n = a^n \cos n\theta$, show that the radius of curvature varies inversely as the $(n-1)$th power of the radius vector.

Solution Here $r^n = a^n\cos n\theta$, taking log both sides, we get $n\log r = n\log a + \log\cos n\theta$. Differentiating with respect to θ, we have

$$n\frac{1}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{r_1}{r} = -\tan n\theta \tag{1}$$

Therefore,

$$1 + \left(\frac{r_1}{r}\right)^2 = 1 + \tan^2(n\theta) \quad \text{or} \quad r^2 + r_1^2 = r^2\sec^2(n\theta) \tag{2}$$

Differentiating again, we get

$$\frac{r_2 r - r_1^2}{r^2} = -n\sec^2(n\theta)$$

or

$$rr_2 - r_1^2 = -n^2r^2\sec^2(n\theta). \tag{3}$$

From (2) and (3),

$$r^2 + 2r_1^2 - rr_2 = (1+n)r^2\sec^2(n\theta) \tag{4}$$

or

$$\frac{dx}{ds} = \sqrt{\frac{x}{x+a}}.$$

Now,

$$\frac{d\rho}{ds} = \frac{d\rho}{dx}\frac{dx}{ds} = \frac{3}{\sqrt{a}}\sqrt{x+a}\sqrt{\frac{x}{x+a}} = \frac{3}{\sqrt{a}}\sqrt{x}$$

$$\frac{d^2\rho}{ds^2} = \frac{d}{ds}\left(\frac{3}{\sqrt{a}}\sqrt{x}\right) = \frac{d}{dx}\left(\frac{3}{\sqrt{a}}\sqrt{x}\right)\frac{dx}{ds} = \frac{3}{\sqrt{a}}\frac{1}{2\sqrt{x}}\sqrt{\frac{x}{x+a}} = \frac{3}{2\sqrt{a(x+a)}}$$

Then

$$3\rho\frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9 = \frac{9}{a}(x+a) - \frac{9x}{a} - 9 = \frac{9}{a}(x+a-x-a) = 0$$

Example 7.11 In the curve $p = r^{n+1}/a^n$, show that the radius of curvature varies inversely as the $(n-1)$th power of the radius vector.

Solution We have

$$p = \frac{r^{n+1}}{a^n}. \tag{1}$$

Then

$$\frac{dp}{dr} = (n+1)\frac{r^n}{a^n}$$

But

$$\rho = r\frac{dr}{dp} = r\frac{a^n}{(n+1)r^n} = \frac{a^n}{n+1}\frac{1}{r^{n-1}}$$

Therefore,

$$\rho \propto \frac{1}{r^{n-1}}.$$

Example 7.12 For the curve $y^2 = c^2 + s^2$, prove that $s = c\tan\psi$ and $\rho = y^2/c$.

Solution We have $y^2 = c^2 + s^2$. Differentiating, we get

$$2y\frac{dy}{dx} = 2s\frac{ds}{dx}$$

Then

$$y\tan\psi = s\sec\psi \quad \text{or} \quad \sin\psi = \frac{s}{y}. \tag{2}$$

Now,

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{\sin\psi}{\sqrt{1-\sin^2\psi}} = \frac{s/y}{\sqrt{1-s^2/y^2}} = \frac{s}{\sqrt{y^2-s^2}} = \frac{s}{c}$$

or

$$s = c \tan \psi \tag{3}$$

Differentiating to ψ, we get

$$\frac{ds}{d\psi} = c\sec^2\psi = c(1+\tan^2\psi) = c\left(1+\frac{s^2}{c^2}\right) = \frac{c(c^2+s^2)}{c^2} = \frac{y^2}{c}$$

Therefore,

$$\rho = \frac{ds}{d\psi} = \frac{y^2}{c}.$$

Example 7.13 If ρ and ρ' be the radii of curvature of a curve and of its pedal at corresponding points, show that $\rho'(2r^2 - p\rho) = r^3$.

Solution Here, for the pedal curve

$$r_1 = p \tag{1}$$

and

$$p_1 r = p^2 \quad \text{or} \quad p_1 = \frac{p^2}{r} \tag{2}$$

We know that

$$\rho = r\frac{dr}{dp} \tag{3}$$

$$\rho' = r_1\frac{dr_1}{dp_1} \tag{4}$$

From (1)

$$dr_1 = dp \tag{5}$$

From (2)

$$dp_1 = 2p\frac{1}{r}dp - \frac{1}{r^2}p^2dr \tag{6}$$

Therefore, using (1), (5) and (6) in (4)

$$\rho' = p\frac{dp}{(2p/r)dp - (p^2/r^2)dr} = \frac{dp}{2(dp/r) - (p/r^3)\,\rho\, dp} \quad \text{[Using (3)]}$$

Therefore,

$$\rho' = \frac{r^3}{2r^2 - p\rho} \quad \text{or} \quad \rho'(2r^2 - p\rho) = r^3.$$

Example 7.14 Find the radius of curvature at any point ($a \cos t$, $b \sin t$) of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution Let the coordinates of any point on the ellipse be $x = a \cos t$, $y = b \sin t$. On differentiation, we get

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = b\cos t,$$

Therefore,

$$\sin\psi = \frac{s}{4a} \quad \text{or} \quad s = 4a\sin\psi.$$

Then

$$\frac{ds}{d\psi} = 4a\cos\psi$$

$$= 4a\sqrt{1-\sin^2\psi}$$

$$= 4a\sqrt{1-\frac{s^2}{16a^2}}$$

$$= 4a\sqrt{1-\frac{8ay}{16a^2}}$$

$$= 4a\left(1-\frac{y}{2a}\right)^{1/2}.$$

Example 7.16 Find the radius of curvature of the curve $y = a \log \sec (x/a)$.

Solution Differentiating, we get

$$\frac{dy}{dx} = \frac{\sec(x/a)\tan(x/a)}{\sec(x/a)} = \tan\frac{x}{a}, \qquad \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\frac{x}{a}.$$

Then

$$\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1+\tan^2(x/2)]^{3/2}}{(1/a)\sec^2(x/a)} = a\sec\frac{x}{a}.$$

Example 7.17 If $x = 6t^2 - 3t^4$, $y = 8t^3$ be two curves, find the radius of curvature.

Solution Here $x = 6t^2 - 3t^4$, $y = 8t^3$. Then

$$\frac{dx}{dt} = 12t - 12t^3, \qquad \frac{dy}{dt} = 24t^2.$$

Now,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{24t^2}{12t(1-t^2)} = \frac{2t}{1-t^2}$$

and

$$\frac{d^2y}{dx^2} = \frac{2(1-t^2)-2t+2t}{(1-t^2)^2}\frac{dt}{dx} = \frac{2(1+3t^2)}{(1-t^2)^2}\frac{1}{12t(1-t^2)} = \frac{1+3t^2}{6t(1-t^2)^3}.$$

Therefore,

$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left[1+\dfrac{4t^2}{(1-t^2)^2}\right]^{3/2}}{\dfrac{1+3t^2}{6t(1-t^2)^3}}.$$

We also have

$$\rho_1 = a\left(1+t^2\right)^{3/2}, \qquad \rho_2 = a\left(1+\frac{1}{t^2}\right)^{3/2}.$$

Then

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = a^{-2/3}(1+t^2)(1+t^2)^{-1} = a^{-2/3}.$$

Example 7.20 If ρ_1, ρ_2 be the radii of the curvature at the extremities of any chord of $r = a(1 + \cos\theta)$, which passes through the pole, then

$$\text{(i) } \rho_1^2 + \rho_2^2 = \frac{16a^2}{9}, \qquad \text{(ii) } \frac{\rho^2}{r} = \text{constant.}$$

Solution Here $r = a(1 + \cos\theta)$. Then

$$\frac{dr}{d\theta} = -a\sin\theta, \qquad \frac{d^2r}{d\theta^2} = -a\cos\theta$$

Therefore,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2\cos^2\theta + a^2\cos\theta}$$

$$= \frac{(2a^2+2a^2\cos\theta)^{3/2}}{a^2(3+3\cos\theta)}$$

$$= \frac{2^{3/2}a}{3}(1+\cos\theta)^{1/2}$$

$$= \frac{4a}{3}\cos\frac{\theta}{2}.$$

The vectorial angles of two extremities of the chord will be θ and $(\pi + \theta)$. Then

$$\rho_1 = \frac{4a}{3}\cos\frac{\theta}{2}, \qquad \rho_2 = \frac{4a}{3}\cos\left(\frac{\pi+\theta}{2}\right) = \frac{4a}{3}\sin\frac{\theta}{2}.$$

(i) We have

$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{16a^2}{9}.$$

(ii) We also have

$$\rho^2 = \frac{16a^2}{9}\frac{\cos^2\theta}{2} = \frac{8r}{9} \quad \text{or} \quad \frac{\rho^2}{r} = \frac{8}{9} \text{ (constant)}$$

Example 7.21 If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that

$$\rho_1^{2/3} + \rho_2^{2/3} = (a^2+b^2)(ab)^{-2/3}.$$

Solution Since $x = a\cos t$, $y = b\sin t$ is the point on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$,

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = b\cos t.$$

and

$$\frac{dy}{dx} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2 t\,\frac{dx}{dt} = -\frac{b}{a}\operatorname{cosec}^3 t$$

Therefore,

$$\rho = \frac{[1 + (b^2/a^2)\cos^2 t]^{3/2}}{(-b/a)\operatorname{cosec}^3 t} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

Since P and Q be the extremities of two conjugate diameters, the parameters of P and Q are t and $(t + \pi/2)$. Also,

$$\rho_1^{2/3} = \frac{a^2\sin^2 t + b^2\cos^2 t}{(ab)^{2/3}} \quad \text{and} \quad \rho_2^{2/3}\,\frac{a^2\cos^2 t + b^2\sin^2 t}{(ab)^{2/3}}$$

Therefore,

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2 + b^2}{(ab)^{2/3}} = (a^2 + b^2)(ab)^{-2/3}.$$

Exercises 7.1

1. Find the radius of curvature at any point (s, ψ) of the following curves:
   (i) $s = a\psi$, (ii) $s = 4a\sin\psi$, (iii) $s = c\tan\psi$,
   (iv) $s = c\log\sec\psi$ (v) $s = a\log\tan(\pi/4 + \psi/2)$.

2. Find the radius of curvature at any point of the curve $s = a(e^{\psi} - 1)$.

3. Find the radius of curvature at (s, ψ) of the curve of parabola $s = a\tan\psi\sec\psi + a\log(\sec\psi + \tan\psi)$.

4. Find the radius of curvature at the point (x, y) of the following curves:
   (i) $y^2 = 4ax$, (ii) $xy = c^2$,
   (iii) $ay^2 = x^3$, (iv) $y = ae^{x/a}$,
   (v) $x^2/a^2 + y^2/b^2 = 1$, (vi) $x^2/a^2 - y^2/b^2 = 1$,
   (vii) $y = 4\sin x - \sin 2x$, at $x = \pi/2$, (viii) $x^2y = a^2(x - y)$, at $x = a$,
   (ix) $y = e^{-x^2}$, at $(0, 1)$, (x) $x = a\cos t/t$, $y = a\sin t/t$,
   (xi) $y = x^3 - 2x^2 + 7x$ at the origin,
   (xii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $x = y$.
   (xiii) $\sqrt{x/a} - \sqrt{y/b} = 1$, at the point where it touches the coordinate axes,
   (xiv) $x^3 + y^3 = 2$ at $(1, 1)$,
   (xv) $x^2y = a(x^2 + y^2)$ at $(-2a, 2a)$.

5. Find the radius of curvatures $xy^2 = 16(x + 4)$ at the point (–4, 0).

6. Find the radius of curvature at $(3a/2, 3a/2)$ on the curve $x^3 + y^3 = 3axy$.

7. In the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that the radius of curvature at the end of the major axis is equal to the semi-latus return of the ellipse.

8. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

9. Show that the radius of curvature at a point of the curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \operatorname{cost} t)$ is at.

10. Show that the radius of curvature at a point of the curve $x = a \sin 2\theta (1 + \cos 2\theta)$, $y = a \cos 2\theta (1 - \cos 2\theta)$ is $4a \cos 3\theta$ at $\theta = 0$, $\theta = 4a$.

11. Find the radius of curvature for the curve $y = \sqrt{(s^2 + c^2)}$.

12. Prove that the radius of curvature of the catenary $y = a \cosh(x/a)$ at any point is equal in length to the portion of the normal intercepted between the curve and the axis of x.

13. Show that for the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, the radius of curvature at any point is twice the portion of the normal intercepted between the curve and the axis of x.

14. Prove the following for any curve:

(i) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$ (ii) $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2 - p^2}}$ (iii) $\sin^2 \phi \dfrac{d\phi}{d\theta} + r\dfrac{d^2 r}{ds^2} = 0$

15. Show that when the angle between the tangent to a curve and the radius vector of the point of contact has a maximum or minimum value, $\rho = r^2/p$.

16. If $x = c \log [s + \sqrt{(s^2 + c^2)}]$, prove that $c\rho = c^2 + s^2$.

17. For the curve $r^2 \cos 2\theta = a^2$, prove that $\rho = -r^3/a^2$.

18. For the cardioide $r = a(1 + \cos \theta)$, prove that $3\rho = 2\sqrt{(2ar)}$, i.e. ρ varies as $\sqrt{r}$.

19. For the curve $y = ax/(a + x)$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2.$$

20. Find ρ at any point (r, θ) on the following curves:

(i) $r = a \sin\theta + b \cos\theta$ (ii) $r = a\theta$ (iii) $r = a \sec^2(\theta/2)$

(iv) $r = a \cos\theta$ (v) $r^2 = a^2 \cos 2\theta$ (vi) $r^n = a^n \sin n\theta$

(vii) $r = a(1 - \cos \theta)$ (viii) $r = 2a \cos\theta - a$, at $\theta = 0$

(ix) $r = a \sin\theta$.

21. Find ρ at any pint (p, r) on the following curves:

(i) $r^2 = 2ap$ (ii) $r^3 = a^2p$ (iii) $p = r \sin \alpha$

(iv) $p^2 = ar$ (v) $r^2 + 3p^2 = a^2$ (vi) $a^2b^2/p + r^2 = a^2 + b^2$.

22. If $r = a \sec 2\theta$, prove that $r^4 + 3p^3\rho = 0$.

23. Prove that $\rho = r \operatorname{cosec} \alpha$ for the curve $r = ae^{\theta \cot \alpha}$.

24. Find the radius of curvature at any point on the curves:

(i) $p = a(1 + \sin\psi)$ (ii) $p^2 = a^2\cos 2\psi$.

25. Prove that the radius of curvature of the curve $ax^2 + by^2 = 1$ varies as the cube of the length of the normal.

26. Prove that $p^3\rho = a^4$ if $p^2 + a^2\cos 2\psi = 0$ in usual notations.

27. Find the radius of curvature of the curve

$$y^2 = x^2\frac{a+x}{a-x} \quad \text{at } (-a, 0).$$

28. Show that the radius of curvature at any point on the curve $r = ae^{\theta}\cot\theta$ subtends a right angle at the pole.

29. For any curve, prove that

$$\frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right), \quad \text{where } \tan\phi = r\frac{d\theta}{dr}.$$

30. For any curve, prove that

(i) $\rho = \left[\left(\dfrac{dx}{d\psi}\right)^2 + \left(\dfrac{dy}{d\psi}\right)^2\right]^{1/2}$ (ii) $\dfrac{1}{\rho} = -\dfrac{d^2x/ds^2}{dy/ds} = \dfrac{d^2y/ds^2}{dx/ds}$

(iii) $\dfrac{1}{\rho} = \dfrac{(1/r) - (1/r)(dr/d\theta)^2 - (d^2r/d\theta^2)}{\left[1-(dr/ds)^2\right]^{1/2}}$ (iv) $\rho = \dfrac{r(d\theta/ds)}{r(d\theta/ds)^2 - (d^2r/ds^2)}$

31. Prove that the least value of the radius of curvature of the curve

$$x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$$

is $9a/10$ at the point $x = 0$.

32. Find the point on the parabola $y^2 = 4x$ if the radius of curvature at that point is $\sqrt{(125/27)}$.

33. Find the radius if curvature at the point having the same abscissa of the curves $xy = a^2$, $x^3 = 3a^2y$.

34. Show that at the points, where the curves $r = a\theta$ and $r\theta = a$ intersect, their curvatures are in the ratio 3:1.

35. Prove that for the curve

$$s = a \log \cot\left(\frac{\pi}{4} - \frac{\psi}{2}\right) + a\frac{\sin\psi}{\cos^2\psi},$$

$\rho = 2a \sec^3\psi$; and hence

$$\frac{d^2y}{dx^2} = \frac{1}{2a}.$$

7.7 Radius of Curvature at the Origin

If any curve $y = f(x)$ passes through the origin (0, 0), then in order to use the formula $\rho = (1 + y_1^2)^{3/2}/y_2$, we shall have to evaluate y_1 and y_2 at the point (0, 0) i.e. we shall have to put $x = 0$, $y = 0$ in y_1 and y_2 separately. In this relation, we write

$$(y_1)_{x=0, y=0} = p \quad \text{and} \quad (y_2)_{x=0, y=0} = q.$$

Therefore, at the origin, $\rho = (1 + p^2)^{3/2}/q$. This is obtained by the method of substitutions.

Method of expansion by Maclaurin's theorem

Let the equation of the curve be

$$y = f(x) \tag{7.5}$$

By Maclaurin's theorem, we have

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''_{(0)} + \cdots \tag{7.6}$$

Since the curve passes through the origin (0, 0), from Eq. (7.5), we get

$$0 = f(0).$$

Differentiating (7.5) with respect to x, we get

$$\frac{dy}{dx} = f'(x) \tag{7.7}$$

At the origin (0, 0),

$$\left(\frac{dy}{dx}\right)_{x=0} = f'(0) = p\,(\text{say}) \tag{7.8}$$

Again differentiating Eq. (7.8), we get

$$\frac{d^2y}{dx^2} = f''(x)$$

At the origin (0, 0),

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = f''(0) = q \quad (\text{say}) \tag{7.9}$$

and so on.

Polar form. If the equation of the curve is given in polar form and x-axis (the initial line) is tangent at the origin, $x = r\cos\theta$, $y = r\sin\theta$. Therefore, writing the previous formula in polar coordinates

$$\rho \text{ (at the pole)} = \lim_{x\to 0}\frac{x^2}{2y}$$

$$= \lim_{\theta\to 0}\frac{r^2\cos^2\theta}{2r\sin\theta}, \quad \text{when } x\to 0, y\to 0, \text{ then } \theta\to 0$$

$$= \lim_{\theta\to 0}\frac{r}{2}\frac{\cos^2\theta}{\sin\theta}$$

$$= \lim_{\theta\to 0}\left(\frac{r}{2\theta}\frac{\theta}{\sin\theta}\cos^2\theta\right)$$

$$= \lim_{\theta\to 0}\frac{r}{2\theta} \quad \left[\text{as } \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1, \ \lim_{\theta\to 0}\cos\theta = 1\right]$$

Geometrical proof of Newtonian method

Let x-axis be the tangent to the curve at the origin (0, 0). Let P(x, y) be any arbitrary point on the curve, which is very near to 0. We draw a circle through O and P which touches the x-axis at the point O, i.e. the circle passes through two coincident points at O and the third point P (Fig. 7.8).

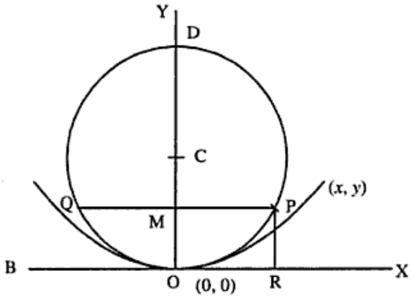


Fig. 7.8 Geometrical proof of Newtonian method.

It is obvious that when the circle moves along the curve to coincide with O, this circle becomes the circle of curvature at the point O.

Let OD be the diameter of the circle. From P, PQ perpendicular is drawn to OD. Therefore, the middle point M of PQ will be on the diameter OD. Now from geometry,

$$OM \cdot MD = PM \cdot MQ = PM \cdot PM \qquad (\text{as } PM = MQ)$$

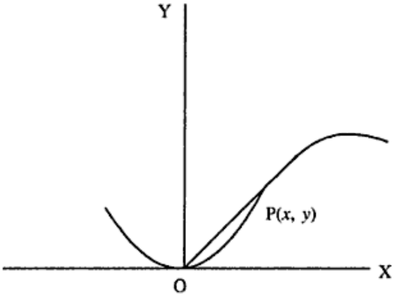


Fig. 7.9 Tangent at the origin.

Now, dividing Eq. (7.10) by x, we get

$$a_1 + b_1\frac{y}{x} + a_2x + b_2y + c_2y\frac{y}{x} + \cdots = 0$$

Taking the limit on both sides, when $x \to 0$, $y \to 0$, we find that $a_1 + b_1m = 0$, because all the succeeding terms become zero, when $x \to 0$, $y \to 0$. Therefore,

$$m = -\frac{a_1}{b_1} \tag{7.12}$$

Case I: Suppose $b_1 \neq 0$. Then from Eq. (7.11), the equation of the tangent at the origin is

$$Y = -\frac{a_1}{b_1}X \Rightarrow b_1Y = a_1X \Rightarrow a_1X + b_1Y = 0,$$

that is, in terms of (x, y),

$$a_1x + b_1y = 0 \tag{7.13}$$

But Eq. (7.13) is the collection of the lowest terms in Eq. (7.10).

Case II: Let $b_1 = 0$, then from Eq. (7.12) a_1 also vanishes, as m is finite. Therefore, Eq. (7.10) will be of the form

$$a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + \cdots + a_nx^n + \cdots + k_ny^n = 0$$

Now dividing by x^2, we get

$$a_2 + b_2\frac{y}{x} + c_2\left(\frac{y}{x}\right)^2 + \cdots = 0$$

Taking the limit of both sides when $x \to 0$, $y \to 0$, we find that

$$a_2 + b_2m + c_2m^2 = 0 \tag{7.14}$$

because all the succeeding terms become zero when $x \to 0$, $y \to 0$. But this is a quadratic equation in m. This means there will be two tangents at the origin. Now, putting $m = Y/X$ in Eq. (7.14), we get

$$a_2 + b_2\frac{Y}{X} + c_2\frac{Y^2}{X^2} = 0 \quad \text{or} \quad a_2X^2 + b_2XY + c_2Y^2 = 0.$$

But $a_1 = 0$, $b_1 = 0$, then the equation of the lowest degree terms in Eq. (7.10) is $a_2x^2 + b_2xy + c_2y^2 = 0$. Similarly, if $a_2 = 0$, $b_2 = 0$, then the truth of the theorem can be proved as before.

7.9 Chord of Curvature

Through the pole

Let O be the pole and OX the initial line. Let P be any point on the curve and C be the centre of the circle of curvature at the point P (Fig. 7.10). Join P with the origin O, which cuts the circle of the curvature at the point Q. Then PQ is the length of the chord of curvature passing through the origin (pole). We want to find the length PQ, the chord.

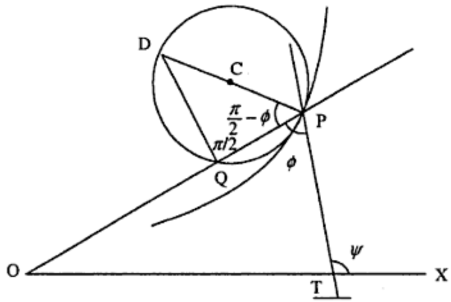


Fig. 7.10 Chord of curvature through the pole.

Draw a tangent at the point P of the curve which makes an angle ϕ with OP. Join P with C and produce it to meet the circle of curvature at the point D. Then $PD = 2\rho$. Join DQ.

Since the angle of the segment of a semi-circle is a right angle, $\angle PQD = \pi/2$. Also, since DP $\perp$ PT, $\angle DPT = \pi/2$. Therefore,

$$\angle DPQ = \frac{\pi}{2} - \phi.$$

Now, in ΔDPQ,

$$DQ \cos DPQ = \frac{PQ}{PD}$$

Therefore,

$$PQ = PD\cos\left(\frac{\pi}{2} - \phi\right) = 2\rho\sin\phi.$$

Thus the chord of curvature passing through the pole = $2\rho \sin \phi$.

Parallel to the coordinate axes

Let P(x, y) be any point on the curve $y = f(x)$. Let PQR be the circle of the curvature at the point P whose centre is C (Fig. 7.11).

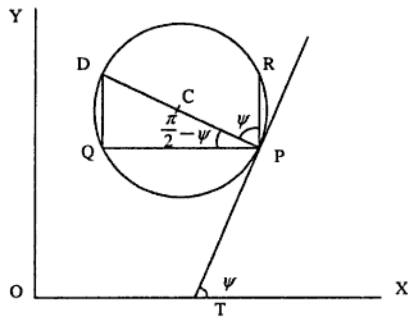


Fig. 7.11 Chord of curvature parallel to coordinate axes.

In the circle of curvature from P, draw a chord PQ parallel to x-axis. Then this will be called the *chord of curvature parallel to the x-axis.*

Similarly, from P we draw a chord PR parallel to the y-axis then this will be called the *chord of curvature parallel to y-axis.* We separately find out the lengths of PQ and PR. Since PT is the tangent, $\angle PTX = \psi$. Join PC and produce it to meet the circle of curvature at the point D. Then $PD = 2\rho$. Since PQ is parallel to OX,

$$\angle QPT = \angle PTX = \psi.$$

Again $\angle DPT = \pi/2$ as radius is perpendicular to tangent. Therefore,

$$\angle DPQ = \frac{\pi}{2} - \psi \quad \text{and} \quad \angle DPR = \angle QPR = \frac{\pi}{2} - \left(\frac{\pi}{2} - \psi\right) = \psi.$$

Again since angle in a semi-circle is a right angle,

$$\angle PQD = \frac{\pi}{2} \quad \text{and} \quad \angle PRD = \frac{\pi}{2}\cdot$$

Now, (i) from ΔPQD,

$$PQ = PD \cos DPQ = 2\rho \cos\left(\frac{\pi}{2} - \psi\right)$$

Then

$$PQ = \text{Chord of curvature} = 2\rho \sin\psi.$$

Hence chord of curvature parallel to x-axis $= 2\rho \sin\psi$.

Therefore,

$$\text{Chord of curvature through the pole} = 2\rho \sin\phi$$
$$= 2r \operatorname{cosec}\alpha \sin\alpha$$
$$= 2r.$$

Example 7.24 Find the chord of curvature through the pole of the curve $r = a(1 + \cos\theta)$.

Solution Here the equation of the curve is $r = a(1 + \cos\theta)$. Then

$$\frac{dr}{d\theta} = a(-\sin\theta) \quad \text{and} \quad \frac{d^2r}{d\theta^2} = -a\cos\theta.$$

Therefore,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2(1+\cos\theta)\cos\theta} = \frac{4}{3}a\cos\frac{\theta}{2}$$

Now,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2(\theta/2)}{-2\sin(\theta/2)\cos(\theta/2)} = -\cot\frac{\theta}{2}$$

or

$$\tan\phi = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

or

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}.$$

Therefore, chord of curvature through the pole is

$$2\rho\sin\phi = 2\left(\frac{4a}{3}\right)\cos\frac{\theta}{2}\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$
$$= \frac{8a}{3}\cos^2\frac{\theta}{2}$$
$$= \frac{8a}{3}\frac{1+\cos\theta}{2}$$
$$= \frac{4}{3}r.$$

Example 7.25 Show that the chord of curvature through the pole of the curve $r^m = a^m \cos m\theta$ is $2r/(m + 1)$.

Solution Here $r^m = a^m \cos m\theta$. Taking log both the sides, we get

$$m \log r = m \log a + \log \cos m\theta,$$

or

$$\frac{m}{r}\frac{dr}{d\theta} = -m \cot m\theta.$$

Therefore,

$$\tan\phi = -\cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right).$$

Hence

$$\phi = \frac{\pi}{2} + m\theta.$$

Now,

$$p = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta = \frac{r^{m+1}}{a^m}$$

Again,

$$\frac{dp}{dr} = \frac{(m+1)r^m}{a^m},$$

Then

$$\rho = r\frac{dr}{dp} = r\frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}.$$

Hence the chord of curvature passing through the pole is

$$\begin{aligned}
2\rho\sin\phi &= \frac{2a^m}{(m+1)r^{m-1}}\sin\left(\frac{\pi}{2} + m\theta\right) \\
&= \frac{2a^m}{(m+1)r^{m-1}}\cos m\theta \\
&= \frac{2a^m}{(m+1)r^{m-1}}\frac{r^m}{a^m} \\
&= \frac{2r}{m+1}
\end{aligned}$$

Example 7.26 In the catenary $y = c\cosh(x/c)$, prove that the chord of curvature parallel to the y-axis is double the ordinate.

Solution Here

$$y = c\cosh\frac{x}{c}, \qquad y_1 = \sinh\frac{x}{c}, \qquad y_2 = \frac{1}{c}\cosh\frac{x}{c}.$$

Now,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+\sinh^2(x/c)]^{3/2}}{(1/c)\cosh(x/c)} = \frac{[\cosh^2(x/c)]^{3/2}}{(1/c)\cosh(x/c)} = c\cosh^2\frac{x}{c}$$

Hence

Chord of curvature parallel to y-axis $= 2\rho \cos \psi$

$$= 2c \cosh^2 \frac{x}{c} \frac{1}{\sqrt{1+\tan^2 \psi}}$$

$$= 2c \cosh^2 \frac{x}{c} \frac{1}{\sqrt{1+y_1^2}}$$

$$= 2c \cosh \frac{x}{c} = 2y.$$

Therefore, chord of curvature parallel to y-axis is twice the ordinate.

Example 7.27 Show that in a parabola the chords of curvature: (i) through the focus (ii) parallel to x-axis, are each equal to four times the focal distance of the point.

Solution Let the equation of the parabola is $2a/r = 1 + \cos \theta$, with pole as a focus and initial line as axis. To find the pedal equation, taking log both the sides, we get

$$\log 2a - \log r = \log (1 + \cos \theta)$$

or

$$-\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1+\cos\theta} = -\tan\frac{\theta}{2}$$

We also have

$$\tan\phi = r\frac{d\theta}{dr} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

or

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}.$$

Now,

$$p = r\sin\phi = r\cos\frac{\theta}{2},$$

and

$$\frac{2a}{r} = 1 + \cos\theta = 2\cos^2\frac{\theta}{2} = 2\frac{p^2}{r^2}. \qquad (1)$$

Therefore,

$$p^2 = ar \quad \text{or} \quad p = \sqrt{ar}; \quad \frac{dp}{dr} = \frac{1}{2}\frac{\sqrt{a}}{\sqrt{r}}.$$

or

$$\lim_{x\to 0, y\to 0}\left(\frac{x^2}{2y}+\frac{1}{4}xy\frac{2y}{x^2}\right)=\frac{3a}{2}$$

Thus

$$\frac{1}{\rho}=\frac{3a}{2} \quad \text{or} \quad \frac{1}{\rho}=\frac{3a}{2}.$$

Case II: If the y-axis is tangent to the curve at the origin, then

$$\rho=\lim_{x\to 0, y\to 0}\frac{y^2}{2x}=\frac{3a}{2}.$$

Example 7.29 Show that the radii if curvature of the curve

$$y^2=\frac{x^2(a+x)}{a-x}$$

at the origin are $\pm a\sqrt{2}$.

Solution Here

$$y^2=x^2\frac{a+x}{a-x} \quad \text{or} \quad y^2(a-x)=x^2(a+x).$$

But

$$y=px+q\frac{x^2}{2}+\cdots$$

Then

$$(a-x)\left(px+q\frac{x^2}{2}+\cdots\right)^2=x^2(a+x)$$

or

$$(a-x)\left(p^2x^2+2p\frac{q}{2}x^3+\cdots\right)=x^2(a+x)$$

Equating the coefficients of x^2 and x^3 both sides, we get

$$ap^2=a \quad \text{or} \quad p^2=1 \quad \text{or} \quad p=\pm 1$$

and

$$apq-p^2=1 \quad \text{or} \quad apq=2$$

If $p=+1$, $q=2/a$ and if $p=-1$, $q=-2/a$, then

$$\rho=\frac{(1+p^2)^{3/2}}{q}=\frac{(1+1)^{3/2}}{\pm 2/a}=\pm\sqrt{2}a.$$

Example 7.30 Find the radius of curvature of the curve $4x^3 - 3xy + y^2 - 3y = 0$ at the origin.

Solution Here (0, 0) satisfies the equation of the curve and $y = 0$ is the tangent to the curve as the lowest-degree term to zero. Put $y = 0$ in the given curve, we get $x^2 = 0$, which shows that $y = 0$ cuts the curve at coincident points at (0, 0). So, we use Newtonian method

$$\rho = \lim_{x\to 0, y\to 0} \frac{x^2}{2y}$$

Now,

$$4x^3 - 3xy + y^2 - 3y = 0 \quad \text{or} \quad \frac{4x^2}{2y} - \frac{3}{2}x + \frac{y}{2} - \frac{3}{2} = 0.$$

Therefore,

$$\lim_{x\to 0, y\to 0} 4\frac{x^2}{2y} - \lim_{x\to 0}\frac{3}{2}x + \lim_{y\to 0}\frac{y}{2} - \lim_{x\to 0}\frac{3}{2} = 0$$

Thus $\rho = 3/8$.

Example 7.31 Find the radius of curvature at the origin of the curve

$$3x^2 + xy + y^2 - 4x = 0. \qquad (1)$$

Solution Obviously the curve passes through the origin. To get the tangent to the curve at the origin, equate the lowest-degree term to zero. We get

$$4x = 0 \quad \text{or} \quad x = 0$$

which is the equation to y-axis. Here y-axis is the tangent to the curve at the origin. Therefore,

$$\rho = \lim_{x\to 0, y\to 0} \frac{y^2}{2x} \qquad (2)$$

Dividing (1) by $2x$, we get

$$\frac{3x^2}{2x} + \frac{xy}{2x} + \frac{y^2}{2x} - \frac{4x}{2x} = 0$$

or

$$\frac{3}{2}x + \frac{1}{2}y + \frac{y^2}{2x} - 2 = 0$$

or

$$\lim_{x\to 0, y\to 0}\frac{3}{2}x + \lim_{x\to 0, y\to 0}\frac{1}{2}y + \lim_{x\to 0, y\to 0}\frac{y^2}{2x} - \lim_{x\to 0, y\to 0} 2 = 0$$

Thus

$$\rho = 2.$$

7.10 Centre of Curvature

The *centre of curvature* at any point P of a curve is the point which lies on the positive direction of the normal at P and is at a distance ρ from it.

Let the coordinates of P be (x, y) and those of C be (X, Y). Let the tangent PT to the curve makes an angle ψ with x-axis in positive direction so that the positive direction of the normal makes an angle $\psi + \pi/2$ with x-axis (Fig. 7.12).

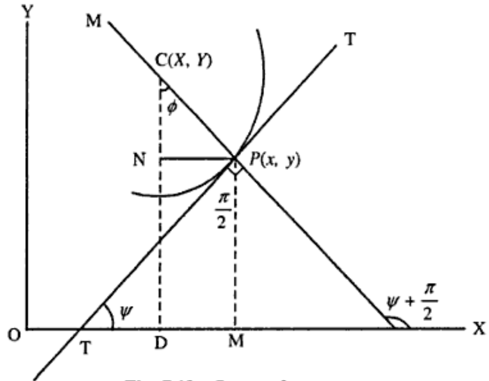


Fig. 7.12 Centre of curvature.

The equation of the normal at $P(x, y)$ is

$$\frac{X - x}{\cos(\psi + \pi/2)} = \frac{Y - y}{\sin(\psi + \pi/2)} = r$$

or

$$\frac{X - x}{-\sin\psi} = \frac{Y - y}{\cos\psi} = r$$

where X and Y are coordinates of any point on the normal and r the variable distance of the variable point (X, Y) from (x, y). Thus the coordinates of (X, Y) of the point on the normal at a distance, r from $P(x, y)$ are $(x - r\sin\psi, y + r\cos\psi)$.

For the centre of the curvature, we have $r = \rho$. Hence (x, y) be the centre of curvature, we have

$$X = x - \rho\sin\psi, \qquad Y = y + \rho\cos\psi.$$

But

$$\sin\psi = \frac{y_1}{\sqrt{1 + y_1^2}}, \qquad \cos\psi = \frac{1}{\sqrt{1 + y_1^2}}, \qquad \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}.$$

Therefore,

$$X = x - y_1 \frac{1 + y_1^2}{y_2}, \qquad Y = y + \frac{1 + y_1^2}{y_2}.$$

Note: For centre of curvature, we substitute the values of $\sin\psi$, $\cos\psi$ and ρ.

Evolute

The locus of the centres of curvature of a curve is called its *evolute* and a curve is said to be an *involute* of its evolute.

Example 7.32 Find the coordinates of centre of curvature at a point (x, y) of the parabola $y^2 = 4ax$. Hence obtain its evolute.

Solution Here

$$y^2 = 4ax, \qquad \frac{dy}{dx} = \frac{2a}{y}, \qquad \frac{d^2y}{dx^2} = \frac{-4a^2}{y^3}.$$

If (X, Y) be the centre of coordinates, we get

$$X = x - \frac{2a}{y}\frac{1 + 4a^2/y^2}{-4a^2/y^3} = x + \frac{y^2 + 4a^2}{2a} = \frac{2ax + 4ax + 4a^2}{2a} = 3x + 2a \tag{1}$$

and

$$Y = y + \frac{1 + 4a^2/y^2}{-4a^2/y^3} = y - \frac{y(y^2 + 4a^2)}{4a^2} = \frac{-y^3}{4a^2} = \pm\frac{(4ax)^{3/2}}{4a^2} = \pm\frac{2x}{a^{1/2}} \tag{2}$$

Eliminating x from (1) and (2), we get

$$Y^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{X - 2a}{3}\right)^3 \text{ or } 27aY^2 = 4(X - 2a)^3$$

is the required evolute.

Example 7.33 Find the evolute of the astroid $x = a\cos^3\theta$; $y = a\sin^3\theta$.

Solution We have

$$x = a\cos^3\theta, \quad \frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta)$$

$$y = a\sin^3\theta, \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = -\sec^2\frac{d\theta}{dx} = \frac{1}{3a}\sec^4\theta\,\mathrm{cosec}\,\theta.$$

Thus

$$X = a\cos^3\theta + \frac{\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\text{cosec}\,\theta}3a = a\cos^3\theta + 3a\sin^2\theta\cos\theta \qquad (1)$$

Also

$$Y = a\sin^3\theta + \frac{1+\tan^2\theta}{\sec^4\theta\,\text{cosec}\,\theta}3a = a\sin^3\theta + 3a\cos^2\theta\sin\theta \qquad (2)$$

Eliminating θ between (1) and (2), we get

$$X + Y = a(\cos\theta + \sin\theta)^3 \quad \text{or} \quad (X+Y)^{1/3} = a^{1/3}(\cos\theta + \sin\theta)$$

or

$$X - Y = a(\cos\theta - \sin\theta)^3 \quad \text{or} \quad (X-Y)^{1/3} = a^{1/3}(\cos\theta - \sin\theta)$$

Squaring and adding, we get

$$(X+Y)^{2/3} + (X-Y)^{2/3} = 2a^{2/3}$$

This is the required evolute.

Exercises 7.2

1. Find the chord of curvature through the pole of the curve $r = ae^{m\theta}$.
2. Find the chord of curvature parallel to x-axis for the curve $y = \log\sec x$.
3. Find the chord of curvature through the pole of the curves:
   (i) $r^2\cos 2\theta = a^2$, (ii) $r^2 = a^2\cos 2\theta$,
4. Show that the chord of curvature through the pole of the curve $p = f(r)$ is given by $2f(r)/f'(r)$.
5. Show that in any curve of the chord of curvature perpendicular to the radius vector is $2\rho(r^2 - p^2)^{1/2}/r$.
6. Show that the circle of curvature at the origin of the parabola $y = mx + x^2$ is $x^2 + y^2 = (1 + m^2)(y - mx)$.
7. Show that the circle of curvature at the point $(am^2, 2am)$ of the parabola $y^2 = 4ax$, has its equation $x^2 + y^2 - 6am^2x - 4ax + 4am^3y = 3a^2m^4$.
8. Find the equation of the circle of curvature at the point $(0, b)$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$.
9. Find the radius of curvature at any point P of the catenary $y = c\cosh(x/c)$ and show that $PC = PG$, where C is the centre of curvature at P, and G the point of intersection of the normal at P with x-axis.
10. For the lemniscate $r^2 = a^2\cos 2\theta$, show that the length of the tangent from the origin to the circle of curvature at any point is $\sqrt{3}r/3$.
11. The circle of curvature at any point P of the lemniscate $r^2 = a^2\cos 2\theta$ meets the radius vector OP at A, show that $OP{:}AP = 1{:}2$; O being the pole.

12. ρ_1, ρ_2 are the radii if curvature at the corresponding point if a cycloid and its evolute, prove that $(\rho_1^2 + \rho_2^2)$ is a constant.

13. Prove that the distance between the pole and the centre of curvature corresponding to any point on the curve $r^n = a^n \cos n\theta$ is

$$\frac{\left[a^{2n} + (n^2 - 1)r^{2n}\right]^{1/2}}{(n+1)r^{n-1}}.$$

14. If c_x and c_y be chords of curvature parallel to the axis at any point of the curve $y = ae^{x/a}$, prove that

$$\frac{1}{c_x^2} + \frac{1}{c_y^2} = \frac{1}{2ac_x}.$$

15. Show that (i) the chord of curvature at any point of the cardiod $r = a(1 + \cos\theta)$ is $(2/3\sqrt{(2ar)}$, (ii) ρ^2/r is constant.

16. If c_x and c_y be chord of curvature parallel to the axis of x at any point of the catenary $y = \cosh(x/c)$, prove that $4c^2(c_x^2 + c_y^2) = c_y^4$.

17. Find the centre of curvature for the following curves at the points indicated:

(i) $a(y^2 - x^2) = x^3$; (0, 0), (ii) $(y - x)(y - 2x) = x^3 + y^3$, (0, 0)
(iii) $4x^2 - 3xy + y^2 - 3y = 0$, (0, 0), (iv) $r = a \sin n\theta$, (0, 0)
(v) $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$, (0, 0),
(vi) $y^2 = x^2(a + x)/(a - x)$ at the point $(a, 0)$,
(vii) $y = x^4 - x^2$, (0, 0), (viii) $y = e^{-x^2}$, (0, 1).

18. Show that the centre of curvature of the curve $x^2/a^2 + y^2/b^2 = 1$ is

$$\bar{x} = \frac{a^2 - b^2}{a^4}x^3, \quad \bar{y} = \frac{b^2 - a^2}{b^4}y^3.$$

19. Prove that the centre of curvature at the point determined by t on the ellipse $x = a \cos t$, $y = b \sin t$ is given by

$$\bar{x} = \frac{a^2 - b^2}{a}\cos^3 t, \quad y = \frac{b^2 - a^2}{b}\sin^3 t.$$

20. Find the centre of curvature at the point determined by t on the astroid $x = a \cos^3 t$, $y = a \sin^3 t$.

21. Show that the circle of curvature at the origin of the parabola $y = mx + x^2/a^2$ is $x^2 + y^2 = a(1 + m^2)(y - mx)$.

22. Find the evolute of the following curves:

(i) $y = 4ax$, (ii) $x = a\cos\theta$, $y = b\sin\theta$, (iii) $x^2/a^2 - y^2/b^2 = 1$

23. Prove that the centre of curvature at points of a cycloid lies on an equal cycloid.

24. Show that in the curve $y = x + 3x^2 - x^3$, the radius of curvature at the origin is nearly at 0.4714 and that at the point (1, 3) is infinite.

25. Find the coordinates of the centre of curvature of the curves:

(i) $x^3 + y^3 = 3axy$ at $(3a/2, 3a/2)$ (ii) $x = a(t + \sin t)$, $y = a(1 + \cos t)$ at t (iii) $x^4 + y^4 = 2a^2xy$, at (a, a).

26. Show that the evolute of the ellipse $x = a \cos\theta$, $y = b \sin \theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

27. Show that for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ the equation of the evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$.

28. Prove that the evolute of the hyperbola $2xy = a^2$ is $(x + y)^{2/3} - (x - y)^{2/3} = 2a^{2/3}$.

29. Show that the evolute of the tractrix $x = a\,[\cos t + \log \tan (t/2)]$, $y = a \sin t$ is the catenary $y = a \cosh (x/a)$.

Chapter 8

Asymptotes

8.1 Introduction

A curve in a plane is either closed or open. Examples of closed curves are circle, ellipse, whose lengths are limited. Open curves are those whose graphs extend to infinity, such as parabola and hyperbola.

Thus a straight line touching a curve at infinity is called its *asymptotes*. Let a curve be given and a tangent be drawn at some point of the curve. If the point of contact of the tangent goes further away from the origin, then the distance of the tangent from the origin will also go on changing; somestimes it will increase continuously and sometimes it will decrease continuously. But it may be possible that when the point of contact tends to infinity, then the tangent takes up a definite position of a straight line. This is called 'asmyptotes'.

In other words, if P be a point on a branch of curves which extends to infinite and a straight line exists at a finite distance from the origin, from which the distance of P gradually diminishes and ultimately tends to zero as P tends to infinity, moving along the area, then such a straight line is called an 'asymptote' to the curve.

In a simple language, an asymptote is a straight line, which cuts a curve in two points at infinity (i.e. touches at infinity) but is not itself at infinity. In other words, an asymptote is a tangent whose points of contact are $x = \infty$, $y = \infty$.

This can be understood as follows: Let P(x, y) be any point on the curve $y = f(x)$. Then a straight line will be the asymptote of the curve if the perpendicular distance of the point P(x, y) from the straight line tends to zero as $x \to \infty$ or $y \to \infty$ or both x, $y \to \infty$.

For example, the straight line $x = 2a$ is an asymptote of the cissoid $y^2(2a - x) = x^3$. We find that as P(x, y) moves to infinity, its distance from the line $x = 2a$ tends to zero (Fig. 8.1).

Asymptote of a curve can be obtained in a number of ways and we shall discuss them one by one.

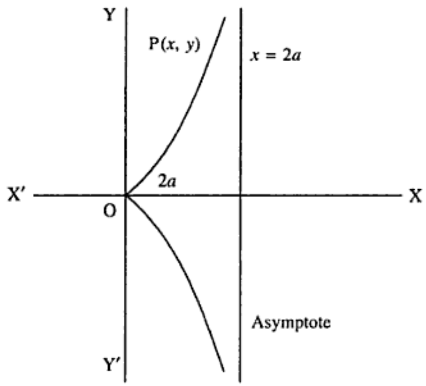


Fig. 8.1 An asymptote.

Solution to an equation with two infinite roots

If $y = mx + c$ be an asymptote of the curve $\phi(x, y) = 0$, we solve these equations for at least two infinite roots, such roots help find sets of suitable values of m and c for the asymptotes. This is the basic method to find the asymptotes.

Asymptotes of an algebraic curve

Let the equation of the curve be

$$(a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + b_3x^{n-3}y^2 + \cdots b_ny^{n-1}) +$$
$$(c_2x^{n-2} + c_3x^{n-3}y + c_4x^{n-4}y^2 + \cdots + c_ny^{n-2}) + \cdots = 0,$$

where a's, b's and c's, are constants. In the first parentheses, we put all those terms in which the sum of the indices of x and y is n, i.e. the term in the first parentheses is a homogeneous function of x and y of degree n. Similarly, the term in the second parentheses is a homogeneous function of degree $(n - 1)$, and so on. Therefore,

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots = 0 \qquad (8.1)$$

where $x^r\phi_r(y/x)$ is a homogeneous function of degree r in x and y.

Let the equation of the asymptote be

$$y = mx + c \qquad (8.2)$$

Now we want to find out the point of intersection of the line $y = mx + c$ with the given curve after simplification, we get

$$\frac{y}{x} = m + \frac{c}{x}$$

Putting this in Eq. (8.1), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\cdots=0 \tag{8.3}$$

which gives the abscissa of point of intersection of the line and the curve. Expanding Eq. (8.3) by Taylor's theorem, we get

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{x^2}\frac{1}{2!}\phi_n''(m)+\cdots\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\cdots\right]$$

$$+\,x^{n-2}\left[\phi_{n-2}(m)+\cdots\right]+\cdots=0$$

or

$$x^n\phi_n(m)+x^{n-1}\left[c\phi_n'(m)+\phi_{n-1}(m)\right]+x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]+\cdots=0 \tag{8.4}$$

This is the equation of the nth degree in x showing that the straight line cuts a curve of the nth degree in n points, in general (real or imaginary). If the straight line (8.2) is an asymptote to the curve, it cuts the curve at infinity. Therefore this equation has two infinite roots for which the coefficients of two highest-degree terms should be zero:

$$\phi_n(m)=0 \tag{8.5}$$

$$c\phi_n'(m)+\phi_{n-1}(m)=0 \tag{8.6}$$

If the roots of (8.5) be $m_1, m_2, m_3, \cdots, m_n$, the corresponding values of c, i.e. $(c_1, c_2, c_3, \cdots, c_n)$ are given by Eq. (8.6). Therefore,

$$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

Hence the n asymptotes are

$$y=m_1x+c_1,\qquad y=m_2x+c_2,\qquad y=m_3x+c_3\quad\cdots,\qquad y=m_nx+c_n.$$

Working rules:

(i) In the highest-degree terms, put $x = 1$ and $y = m$; then we get $\phi_n(m)$. Equating it to zero and solving it, we get $m = m_1, m_2, m_3, \cdots$

(ii) In the next lower-degree terms, put $x = 1$, $y = m$, then we get $\phi_{n-1}(m)$.

(iii) To get c put the values of m in the formula $c = -\phi_{n-1}(m)/\phi'_n(m)$.

(iv) If this formula takes the form 0/0 by the substitution of the value of m, then use

$$\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)=0$$

to get c.

Example 8.1 Find the asymptotes to the curve $y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x - 5y + 6 = 0$.

Solution Let the asymptote be $y = mx + c$. Here we put $x = 1$, $y = m$ in the highest-degree terms, i.e. in the third-degree terms of the given equation of the curve and equate it to zero, we get,

$$\phi_3(m) = m^3 - 3m + m^2 - 3 \tag{1}$$

Therefore, $\phi_3(m) = 0$ gives

$$m^3 - 3m + m^2 - 3 = 0 \quad \text{or} \quad (m + 1)(m^2 - 3) = 0.$$

We get

$$m = -1, \pm\sqrt{3}. \tag{2}$$

Again, put $x = 1$ and $y = m$ in the second-degree terms of the given equation of the curve, we get

$$\phi_2(m) = 2m^2 + 2m \tag{3}$$

Differentiating (1) with respect to m, we get $\phi_3' = 3m^2 - 3 + 2m$. Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m^2 + 2m}{3m^2 + 2m - 3}$$

Here, we put m from (2), we get $c = 0$, when $m = -1$, $c = 1$, when $m = \sqrt{3}$ and $c = 1$, when $m = -\sqrt{3}$. Hence the required asymptotes are $y = mx + 1$, that is

$$x + y = 0 \quad \text{and} \quad y = \pm\sqrt{3}x + 1$$

Oblique asymptotes of algebraic curve

Let the rational algebraic expression containing terms of the rth and lower, but of no higher degrees be denoted by P_r, F_r.

(a) Let the equation of the curve of nth degree can be put into the form

$$(ax + by + c)\, P_{n-r} + F_{n-1} = 0. \tag{8.7}$$

Then the straight line parallel to $ax + by = 0$, obviously cuts the curve (8.7) in one point at infinity. We are now to find out the particular member of this family of parallel straight lines which cuts the curve (8.7) in a second point at infinity. We will now examine the ultimate linear form to which the curve reaches infinity. We make x and y in the equation of the curve in the ratio by

$$\frac{x}{y} = -\frac{b}{a}.$$

Therefore, (8.7) becomes

$$ax + by + c \lim_{x\to\infty,\, y = -a/b} \frac{F_{n-1}}{P_{n-1}} = 0$$

$a_1y + b_1 = 0$, $y = -b_1/a_1$ is the asymptote parallel to x-axis. Similarly, rearranging the terms of the equation of the curve in descending powers of y, we get

$$a_ny^n + (a_{n-1}x + b_n)y^{n-1} + (a_{n-2}x^2 + b_{n-1}x + c_n)y^{n-2} + \cdots = 0 \qquad (8.12)$$

Hence if $a_n = 0$, and x be so chosen that $a_{n-1}x + b_n = 0$, the coefficient of the two highest powers of y in (8.12) vanish, and therefore two of its root are infinite.

Hence the straight line $a_{n-1}x + b_n = 0$ or $x = -b_n/a_{n-1}$ is an asymptote parallel to the axis of y.

Again if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$ and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, three roots of Eq. (8.11) be infinite and the lines represented by $a_2y^2 + b_2y + c_2 = 0$ represent a pair of asymptotes (real or imaginary) parallel to the x-axis.

Similarly, if $a_n = 0$, $a_{n-1} = 0$, $b_n = 0$ and x be so chosen that $a_{n-2}x^2 + b_{n-1}x + c_n = 0$, three roots of Eq. (8.12) be infinity and the lines represented by $a_{n-2}x^2 + b_{n-1}x + c_n = 0$ represent a pair of asymptotes (real or imaginary) parallel to the y-axis.

Working rules:

(i) In order to obtain the asymptotes parallel to the axis of x, equate to zero, the coefficient of the highest power of x. For example, if the curve be of the nth degree and term containing x^n be absent, the coefficient of x^{n-1} equated to zero will give the asymptotes parallel to the axis of x.

(ii) If both the terms containing x_n and x^{n-1} be absent, then the coefficient of x^{n-2} equated to zero will give two asymptotes parallel to the axis of x.

(iii) To get the asymptotes parallel to the axis of y, equated to zero, the coefficient of the highest power of y. For example, if the curve be of nth degree and the term containing y_n be absent then the coefficient of y_{n-1} equated to zero will give the asymptotes parallel to the axis of y.

(iv) If both the terms containing y^n and y^{n-1} be absent then the coefficients of y^{n-2} equated to zero will give two asymptotes parallel to the axis of y.

Corollary If the equation of the curve be the nth degree, and the coefficient of x^n is not zero, then there will be no asymptote parallel to the axis of x. Similarly, if the coefficient of y^n is not zero, then there will be no asymptote parallel to the axis of y. For example, the curve $x^3 + y^3$, $3axy$ will have no asymptote parallel either to x-axis or y-axis, as the coefficients of x^3 and y^3, the highest-degree terms, are not zero.

Example 8.2 Find the asymptotes to the curve $x^2y^2 = a^2y^2 + b^2x^2$.

Solution Here $x^2y^2 = a^2y^2 + b^2x^2$ be the equation of the curve. This is the 4th degree equation. The terms containing, x^4, x^3, y^4 and y^3 are absent. Hence equating to zero the coefficient of x^2 and y^2, will give the asymptotes parallel to the axis of x and the axis of y.

Here the coefficient of $x^2 = y^2 - b^2$ or $y = \pm b$. Hence the asymptotes to the axis of x are $y = b$ and $y = -b$. Again, the coefficient of $y^2 = x^2 - a^2$ to zero. Therefore $x = \pm a$.

Hence the asymptotes parallel to the axis of y are $x = a$ and $x = -a$. Thus the required asymptotes are $y - b = 0$, $y + b = 0$, $x - a = 0$ and $x + a = 0$.

8.3 Asymptotes by Inspection

(a) If the equation of an algebraic curve be put in the form $F_n + F_{n-2} = 0$, where F_n consists of nth degree and lower degree terms which cannot be expressed as a product of n linear factors, none of which is repeated; and F_{n-2} consists of terms of degree $(n-2)$ or lower degree terms. Then all the asymptotes to the given curve will be given by $F_n = 0$.

(b) If in the equation $F_n + F_{n-2} = 0$ of the curve, F_n consists of real linear factors (some repeated and some non-repeated factors). Then, the non-repeated factors equated to zero will definitely be the asymptotes to the curve. But the asymptotes corresponding to the repeated factors will however have to be obtained as in the general case.

Total number of asymptotes to a curve

Let $y = mx + c$ be the equation of an asymptote. We know that the value of m is found out by solving the equation $\phi_n(m) = 0$. Since the equations of nth degree has n roots, we shall get n values of m by solving $\phi_n(m) = 0$. We shall get an asymptote corresponding to each value of m. Hence the curve of nth degree has generally n asymptotes, real or imaginary.

Theorem 8.1 If $y = mx + c$ is an asymptote to a curve then

$$m = \lim_{x\to\infty} \frac{y}{x} \quad \text{and} \quad c = \lim_{x\to\infty}(y - mx)$$

Proof Let $y = mx + c$ be an asymptote to the curve, where m and c are to be obtained. Let P(x, y) be any point on the curve (Fig. 8.2). From P, draw a perpendicular on $y = mx + c$, whose length is p. Then

$$p = \frac{y - mx - c}{\sqrt{1 + m^2}} \tag{8.15}$$

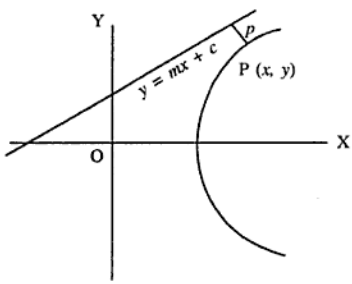


Fig. 8.2 Perpendicular from an asymptote to a curve.

From the figure, it is obvious that as the point tends to infinity along the curve, the distance between the curve and the line becomes lesser and lesser, i.e.

$p \to 0$. When $y = mx + c$ touches the curve at infinity, then $p \to 0$. Thus when $x \to \infty$, $p \to 0$. From this we get the values of m and c. Now taking the limit of Eq. (8.15), when $x \to \infty$,

$$\lim_{p\to 0} = \lim_{x\to\infty} \frac{y - mx - c}{\sqrt{1 + m^2}} \to 0 = \lim_{x\to\infty}(y - mx - c)$$

Therefore,

$$c = \lim_{x\to\infty} (y - mx) \tag{8.16}$$

Again,

$$y = mx + c + p\sqrt{1 + m^2}$$

or

$$\frac{y}{x} = m + \frac{c}{x} + \frac{p}{x}\sqrt{1 + m^2}$$

Therefore,

$$\lim_{x\to\infty} \frac{y}{x} = m + \lim_{x\to\infty}\frac{c}{x} + \lim_{x\to\infty}\frac{p}{x}\sqrt{1 + m^2}$$

Hence

$$m = \lim_{x\to\infty} \frac{y}{x} \tag{8.17}$$

Thus, we find out the values of c and m from Eqs. (8.16) and (8.17) and thereby the equation of the asymptotes $y = mx + c$ is found out.

Example 8.4 Find the asympotes of the curve $x^3 + y^3 = 3axy$.

Solution Let $y = mx + c$ be the equation of the asymptote. Here we divide the equation of the curve by x_1^3, we get

$$1 + \left(\frac{y}{x}\right)^3 = 3a\frac{y}{x^2} = 3a\frac{y}{x}\frac{1}{x}$$

Therefore,

$$\lim_{x\to\infty}\left[1 + \left(\frac{y}{x}\right)^3\right]^x = 3a \lim_{x\to\infty}\frac{y}{x} \lim_{x\to\infty}\frac{1}{x}$$

Then

$$1 + m^3 = 0 \quad \text{or} \quad (1 + m)(m^2 - m + 1) = 0.$$

Therefore, the real value of $m = -1$. Now

$$C = \lim_{x\to\infty} (y - mx) = \lim_{x\to\infty} (y + x)$$

Now put $(c - x)$ for y and take the limit $x \to \infty$. Therefore,

$$x^3 + (c - x)^3 = 3ax\,(c - x)$$

or

$$(3c+3a)-3c(c+a)\frac{1}{x}+\frac{c^3}{x^2}=0$$

Taking the limit when $x \to \infty$, we have

$$3c + 3a = 0 \quad \text{or} \quad c = -a.$$

Hence, the required equation of the asymptote is

$$y = -x - a \quad \text{or} \quad x + y + a = 0.$$

Example 8.5 Find asymptotes of the curve $x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0$.

Solution Putting $x = 1$, $y = m$ in the third-degree terms and equate to zero, we get

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2 = 0,$$

or

$$(1 + 2m) - m^2(1 + 2m) = 0$$

or

$$(1 + 2m)(1 + m)(1 - m) = 0$$

We get $m = 1, -1, -1/2$. Again,

$$\phi_2(m) = m - m^2 = m(1 - m)$$

Therefore,

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{m(1-m)}{-6m^2+2-2m}$$

when $m = 1$, $\quad c = 0$

when $m = 1$, $\quad c = -\frac{(-1)(1+1)}{-6.1+2+2}=-\frac{-2}{-2}=-1$

when $m = -\frac{1}{2}$, $\quad c=-\frac{(-1/2)(1+1/2)}{-6(1/4)+2+2(1/2)}=\frac{1}{2}.$

Therefore, putting the corresponding values of m and c in $y = mx + c$, we get the asymptotes as: $y = x$, $x + y + 1 = 0$, $x + 2y - 1 = 0$.

Example 8.6 Find the asymptotes of $x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$.

Solution Putting $x = 1$ and $y = m$ in the third-degree terms equating to zero $\phi_3(m) = 1 + m - m^2 - m^3 = 0$. Therefore, $m = -1, -1, 1$.

Again $\phi_2(m) = 2m + 2m^2 = 2m(m + 1)$. Then

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{2m(m+1)}{1-2m-3m^2}$$

$$\text{when } m = 1, \qquad c = -\frac{2.2}{1-2-3} = 1$$

$$\text{when } m = -1, \qquad c = \frac{0}{0}.$$

Now to find the value of c, we choose first-degree terms to zero

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or

$$\frac{c^2}{2}(-2-6m) + c(2+4m) + (-3+m) = 0$$

or

$$-c^2(1 + 3m) + c(2 - 4) + (-3 - 1) = 0$$

Solving, we get $c = -1, 2$.

When $m = 1$, then $c = 1$, when $m = -1$, then $c = -1$ or 2. Therefore, the equation to the asymptotes are: $y = x + 1$, $y + x + 1 = 0$, $y + x - 2 = 0$.

Example 8.7 Find the asymptotes to $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$.

Solution Putting $x = 1$ and $y = m$ in the third-degree terms and equating to zero, we get $\phi_3(m) = 4 - 3m^2 - m^3 = 0$. Solving, we get $m = 1, -2, -2$.

Again,

$$\phi_2(m) = 2 - m - m^2 = -(m^2 + m - 2) = -(m - 1)(m + 2)$$

Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-(m-1)(m+2)}{-6m-3m^2} = -\frac{m-1}{3m}$$

$$\text{when } m = 1, \qquad c = -\frac{1-1}{3} = 0$$

$$\text{when } m = -2, \qquad c = \frac{0}{0}$$

Therefore, using

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0.$$

we get

$$\frac{c^2}{2}(-6-6m) + c(-1-2m) + 0 = 0$$

Solving, we get

$$c = 0 \text{ or } 3c\,(1 + m) + (1 + 2m) = 0.$$

Thus when $m = 1$, $c = 0$; when $m = -2$, $c = -1$. Therefore, the equations of asymptotes are: $y - x = 0$, $y + 2x = 0$ and $y + 2x + 1 = 0$.

Example 8.8 Find the asymptotes to the curve

$$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0.$$

Solution Here the highest power term in x is x^3 and the coefficient of x^3 is $1(\neq 0)$. Therefore, there is no asymptote parallel to x-axis. Again, the highest power term in y is $xy^2 = 0$. We shall get one asymptote $x = 0$ from the coefficient x of y^2. For finding out the equation of the other two remaining systems, we follow the previous method.

Here the highest degree of the equation is 3. Therefore, putting $x = 1$ and $y = m$ in the third degree terms and equating to zero, we get

$$\phi_3(m) = 1 - 2m + m^2 = 0 \quad \text{or} \quad m = 1, 1$$

Again $\phi_2(m) = 1 - m$. Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{1-m}{-2+2m} = -\frac{1-m}{2(m-1)}.$$

When $m = 1$, then $c = 0/0$. Therefore, we get

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or

$$\frac{c^2}{2}(2) + c(-1) + 0 = 0.$$

Solving, we get $c = 0$ or $c = 1$. Thus when $m = 1$, $c = 0$ or $c = 1$, the equations of the asymptotes are: $y = x$, $y = x + 1$ and $x = 0$.

Alternative method: The equation of the curve can be written as

$$x(x^2 - 2xy + y^2) + x(x - y) + 2 = 0$$

or

$$x(x - y)^2 + x(x - y) + 2 = 0$$

or

$$x(x - y)(x - y + 1) + 2 = 0,$$

which is of the form of $F_n + F_{n-2} = 0$. Therefore, the three asymptotes are:

$$x = 0, x - y = 0 \quad \text{and} \quad x - y + 1 = 0.$$

Example 8.9 Find the asymptotes to the curve $y^2(x - 2a) = x^3 - a^3$.

Solution The equation of the given curve is of third order. Here the coefficient of x^3 is $1(\neq 0)$. Therefore, there is no asymptote parallel to x-axis.

Again the coefficient of $y^3 = 0$. Then equating to zero the coefficient of y^2, we get $x - 2a = 0$. Therefore, the equation of the asymptote parallel to y-axis is $x - 2a = 0$. Now, putting $x = 1$, $y = m$ in the third-degree terms and equating to zero, we get

$$\phi_3(m) = m^2 - 1 = 0 \quad \text{or} \quad m = 1, -1.$$

Now, by putting $x = 1/t$, $y = -1/t$ so that $t \to 0$, we have

$$x + y = \pm \lim_{t \to 0} \sqrt{\frac{1/t - 9/t + 2}{1/t - 2/t + 2}} = \lim_{t \to 0} \pm \sqrt{\frac{-8 + 2t}{-1 + 2t}} = \pm\sqrt{8} = \pm 2\sqrt{2}.$$

Hence the three asymptotes are: $x + 2y + 2 = 0$, $x + y = 2\sqrt{2}$ and $x + y = -2\sqrt{2}$.

Example 8.11 Find the asymptotes to the curve $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$, and prove that the asymptotes form a triangle of area a^2.

Solution Here the given equation of the curve can be written as $x^2y - y^3 - 2ay^2 + 5x - 7 = 0$. Here, equating the coefficient y of x^2, the highest power term of x, to zero, we get the equation of the asymptote parallel to x-axis as $y = 0$.

Now, putting $x = 1$, $y = m$ in the highest power, i.e. in the third-degree term in the equation of the curve and equating to zero, we get

$$\phi_3(m) = m - m^3 = 0 \quad \text{or} \quad m = 0, 1, -1.$$

Again, $\phi_2(m) = -2am^2$. Then

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-2am^2}{1 - 3m^2} = \frac{2am^2}{1 - 3m^2}.$$

Therefore, when $m = 0$, $c = 0$; when $m = 1$, $c = -a$; and when $m = -1$, $c = -a$.

Therefore, the equation of the asymptotes are

$$y = 0, \qquad y = x - a, \qquad y = -x - a.$$

Solving the three asymptotes, we get

$$x = a,\ \ y = 0; \qquad x = -a,\ \ y = 0; \qquad x = 0,\ \ y = -a.$$

Hence the coordinates of the three vertices of a triangle are: (a, 0), (–a, 0), and $(0, -a)$. Hence

$$\text{Area of } \Delta = \frac{1}{2}\,[a\,(0 + a) + (-a)(-a - 0) + 0(0 - 0)] = a^2.$$

Example 8.12 Find the asymptotes to the cubic $x^2y - xy^2 + xy + y^2 + x - y = 0$, and show that they cut the curve again in three points lying on the straight line.

Solution Putting $x = 1$, $y = m$, we get, $\phi_3(m) = m - m^2 = 0$. Therefore, $m = 0, 1$. Again $\phi_2(m) = m + m^2$. Hence.

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{m(1 + m)}{1 - 2m}.$$

When $m = 0$, $c = 0$; also when $m = 1$, $c = 2$. Therefore, the equation of two asymptotes are: $y = 0$ and $y = x + 2$.

Again equating to zero the higher powers in x and y, the equations to the asymptote parallel to the coordinate axes are: $y = 0$ and $1 - x = 0$, i.e. $x = 1$.

Thus the equations of three asymptotes are

$$y = 0, \quad x - 1 = 0, \quad x - y + 2 = 0.$$

Now the joint asymptotes of the three equations will be

$$y(x - 1)(x - y + z) = 0$$

or

$$x^2y - xy^2 + xy + y^2 - 2y = 0 \tag{1}$$

which we write as $P_3 = 0$.

But the equations to the curve is $x^2y - xy^2 + xy + y^2 - 2y + (x + y) = 0$, i.e. $P_3 + (x + y) = 0$. Hence the point of intersection of the curve and asymptotes satisfies $x + y = 0$, which represents a straight line.

But a straight line cuts a curve of third-degree in 3 points. But each asymptote passes through two points of infinity. Therefore, it will cut the given curve at (3 – 2) or 1 point more. Since the number of asymptotes = 3, the number of points of interesction of the curve and asymptotes = 3 × 1 = 3.

Thus all the three points of intersection of the curve and asymptotes lie on a straight line.

Example 8.13 Determine the asymptotes to the curve.

$$4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

and show that they pass through the point if intersection of the curve with the ellipse $x^2 + 4y^2 = 4$.

Solution The equation of the curve can be written as

$$(4x^4 + 4y^2 - 17x^2y^2) - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

or

$$(x^2 - 4y^2)(4x^2 - y^2) + 4x(x^2 - 4y^2) + 2(x^2 - 2) = 0.$$

Simplifying, we get

$$(x + 2y)(x - 2y)(2x + 1 + y)(2x + 1 - y) + (x^2 + 4y^2 - 4) = 0$$

Hence the required asymptotes are

$$(x - 2y) = 0, \quad x + 2y = 0, \quad 2x + y + 1 = 0, \quad 2x - y + 1 = 0. \tag{1}$$

From the equation of the curve and (1), it is evident that the points of intersection of the curve and the ellipse $x^2 + 4y^2 = 4$ satisfy the equations of the asymptotes.

Example 8.14 Find the asymptotes to the curve

$$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$$

Solution Here the equation of the curve is written as

$$x^3 - x^2y + 4x^2y - 4y^2 - x + y + 3 = 0$$

or

$$(x - y)(x + 2y)^2 = x - y - 3.$$

or

$$y - 2x + 4 = 0.$$

Thus the asymptotes are: $y + x + 1 = 0$, $y + x + z = 0$ and $y - 2x + 4 = 0$.

Example 8.16 Find the asymptotes of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$.

Solution The equation of the curve can be written as

$$y(y^2 + x^2 + 2xy) - y + 1 = 0,$$

or

$$y(y + x)^2 - y + 1 = 0.$$

Therefore, the curve has a pair of asymptotes

$$x + y = \pm\sqrt{\lim_{\substack{x,y\to\infty \\ x=-y}} \frac{y-1}{y}} = \pm 1$$

Thus the two asymptotes are $x + y = 1$ and $x + y = -1$.

The third asymptote is obtained by equating to zero the coefficient of x^2 which is $y = 0$. Thus the asympototes are $x + y = 1$, $x + y + 1 = 0$ and $y = 0$.

Example 8.17 Find the asymptotes of $x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 1 = 0$.

Solution The given equation can be written as

$$x^2(x + 2y) - 4y^2(x + 2y) - 4(x - 2y) - 1 = 0$$

or

$$(x + 2y)(x^2 - 4y^2) - 4(x - 2y) - 1 = 0$$

or

$$(x - 2y)(x + 2y + 2)(x + 2y - 2) - 1 = 0$$

which is in the from of $F_n + F_{n-2} = 0$. Therefore, the asymptotes are: $x - 2y = 0$, $x + 2y + 2 = 0$, and $x + 2y - 2 = 0$.

Example 8.18 Find the asymptotes of $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$.

Solution The given equation can be written as

$$y^2(y - x) - x^2(y - x) - (y - x)(y + x) - 1 = 0.$$

or

$$(y - x)[y^2 - x^2 - (y + x)] - 1 = 0$$

or

$$(y - x)(y + x)(y - x - 1) - 1 = 0$$

which is in the form of $F_n + F_{n-2} = 0$. Hence the asymptotes are: $y - x = 0$, $y + x = 0$ and $y - x - 1 = 0$.

Example 8.19 Show that there is an infinite series of parrallel asymptotes to the curve

$$r = \frac{a}{\theta \sin\theta} + b$$

and show that their distance from the pole are in harmonic progression.

Solution Here

$$r = \frac{a}{\theta \sin \theta} + b \quad \text{or} \quad r\,\theta \sin \theta - (a + b\theta \sin \theta) = 0.$$

It is of the form of $rf_1(\theta) + f_0(\theta) = 0$. The directions for asymptotes are

$$f_1(\theta) = 0 \quad \text{or} \quad \theta \sin \theta = 0 \quad \text{or} \quad \theta = n\pi,$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ Thus the asymptotes are

$$r \sin (n\pi + \theta) = \frac{f_0(n\pi)}{f_1'(n\pi)} = -\frac{a + bn\pi \sin n\pi}{\sin n\pi + n\pi \cos n\pi} \tag{1}$$

If $n \neq 0$, by (1) the asymptotes are

$$r(-1)^{n+1} \sin \theta = \frac{-a}{n\pi(-1)^n} \quad \text{or} \quad r \sin \theta = \frac{a}{n\pi} \tag{2}$$

When $n = 0$, the right-hand side of (1) is infinity so we get no asymptote in this case. These are infinite number of asymptotes corresponding to $n = \pm 1, \pm 2, \pm 3, \cdots$ and the perpendicular distance of these from the pole are

$$\frac{a}{\pi}, \frac{a}{2\pi}, \frac{a}{3\pi}, \cdots$$

which are clearly in harmonic progression.

Exercises 8.1

Find the asymptotes of the following:

1. $x^3 + y^3 = a^3$
2. $y^3 = x(a^2 - x^2)$
3. $x^2y + xy^2 = a^2$
4. $x^2y = x^3 + x + y$
5. $y^2(a - x) = x^2(x + a)$
6. $y^2(x^2 - y^2) = x^2(x^2 - 4a^2)$
7. $xy(x - y) + bx^2 - ay^2 = 0$
8. $y^2(a^2 + x^2) = x^2(a - x)^2$
9. $(y - a)^2(x^2 - a^2) = x^4 + a^4$
10. $9x^4 - 4x^2y^2 + x^2 + y^2 - 1 = 0$
11. $(a + x)^2(b^2 + x^2) = x^2y^2,$
12. $y^3 - xy^2 - x^2y + x^3 - x^2 - y^2 = 1$
13. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$
14. $2y^3 + 3y^2x + 3yx^2 + x^3 - y^2 - x^2 - x = 0$
15. $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0$
16. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0$
17. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$
18. $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + 5x - 6y + 7 = 0$

58. The asymptotes to the curve $x^2y^2 - x^2 - y^2 - x - y + 1 = 0$ form a square through two of whose angular points the curve passes.

59. The asymptotes to the curve $x^2y^2 - a^2(x^2 + y^2) - a^2(x + y) + a^4 = 0$ form a square two of whose angular points lie on the curve.

60. The four asymptotes to the curve

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle $x^2 + y^2 = 1$. [*Hint:* The given equation is $(x - y)(y - 2x)(x + y + 1)(y + 2x + 1) + x^2 + y^2 - 1 = 0$.]

61. All the asymptotes of the curve, $r \tan n\theta = a$ touch the circle $r = a/n$.

Chapter 9

Maxima and Minima

9.1 Introduction

Any function $f(x)$ admits a number of values of x varies within a certain interval. Some of these values may be greatest or least, when compared to other values of the function, are called *extreme values*.

We shall here be concerned with the application of differential calculus to the determination of the values of a function which are greatest or least in their immediate neighbourhood is known as relatively *greatest* and the *least* or *maximum* and *minimum* values.

The term 'maximum value' does not mean the absolute greatest value and neither the absolute least value of the function $y = f(x)$. Moreover, there may be several maxima values and several minima values of the function. Between two equal values of a function at least one maxinum and one minimum must lie (Fig. 9.1).

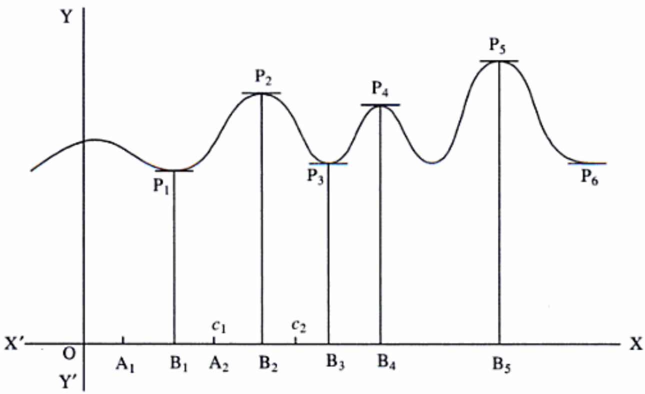


Fig. 9.1 Maximum and minimum points on a curve.

The function $y = f(x)$, represented graphically, has maximum values at P_2, P_4, $P_5, \cdots$, and has minimum values as P_1, P_3, P_6, ... For instance, at P_2, corresponding to $x = OB_2 (= c_2)$, the value of the function is not necessarily bigger than the value at P_5, but we can get a range, say $c_1B_2c_2$ in the neighbourhood of B_2 on either side of it ($\delta = c_1B_2 = B_2c_2$) such that the value of the function is less than P_2B_2 (i.e. the values at P_2). Hence, by definition, the function is maximum at $x = OB_2$. Similarly, in the interval $A_1B_1A_2$ ($A_1B_1 = \delta' = B_1A$, say) in the neighbourhood of B_1 within which for every value of x the function is greater than that of B_1. Hence the function at B_1 is a minimum.

At points P_1, P_2, P_3, P_4, $\cdots$, at which maximum and minimum ordinate occur, the tangents are parallel to one or the other of the coordinate axes. At points P_1, P_2, P_3, P_4, $\cdots$, the value of dy/dx vanishes, whilst of point P_5, dy/dx becomes infinite. The position of maxima and minima are given by the roots of the equations:

$$f'(x) = 0, \qquad f'(x) = \infty.$$

In Fig. 9.2(a), we observe that, at points A, B, the tangents are parallel to either of the axes but at the ordinates, they have neither a maximum nor a minimum value. In Fig. 9.2(b), in passing a maximum value of the ordinate, the angle ψ made by the tangent with OX changes from acute to obtuse and therefore $\tan \psi$ or dy/dx changes from positive to negative. While in passing, the maximum value, ψ changes from obtuse to acute. Therefore, dy/dx changes from negative to positive.

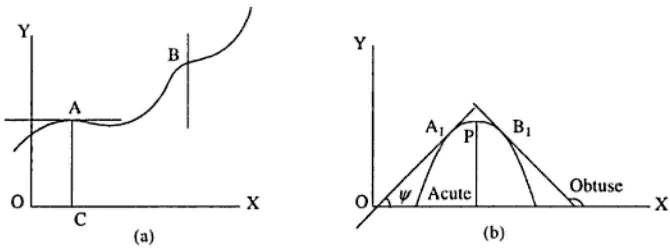


Fig. 9.2 Tangent at two points in a curve.

A function $f(x)$ is said to have a *maximum* at $x = c$ if there exists some $h > 0$ such that $f(x) < f(c)$ whenever $|x - c| < h$.

A function $f(x)$ is said to have a *minimum* at $x = c$ if there exists some $h > 0$, such that $f(x) > f(c)$ whenever $|x - c| < h$.

A function $f(x)$ is said to have an *extreme* value at $x = c$ if it has either a maximum or a minimum at that point.

For example, $y^2 = x^2 - 6x + 17 = (x - 3)^2 + 8$ has its minimum value 8, when $x = 3$ and has no maximum value. Similarly, $y = 5 - 2x - x^2 = 6 - (x + 1)^2$ has its maximum value 6, when $x = -1$, but has no minimum value. The function

$y = \sin x$ has both its maximum and minimum values. They are 1 and −1, when $x = (4n + 1)\pi/2$ and $(4n - 1)\pi/2$, respectively, where n is any integer. But $y = \tan x$ has no extreme value.

Explanation for maxima and minima

Let $f(x)$ be a function whose maximum corresponding to $x = c$ is at A. We consider the neighbourhood $|c - h, c + h|$ of $x = c$.

The tangent at A is parallel to the x-axis, i.e. at A, then $dy/dx = 0$. Now we take any point on the curve in the interval $[c - h, c]$ preceding A and draw a tangent to the curve at the point P (Fig. 9.3). We see that this tangent makes an acute angle with the x-axis. In other words, $f(x)$ is an increasing function in the interval $[c - h, c]$. Hence $f'(x) > 0$.

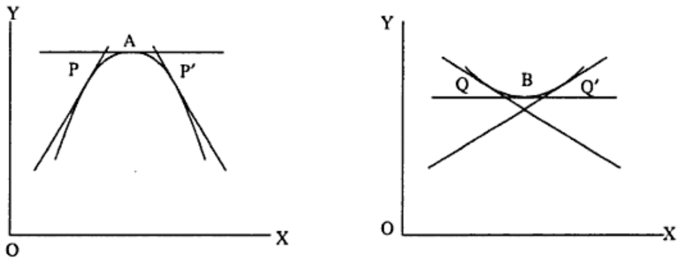


Fig. 9.3 Increasing and decreasing function.

Again, we take any point P′ on the curve in the interval $[c, c + h]$ succeeding A and draw a tangent to the curve at the point P′. Hence we see that this tangent makes an obtuse angle with the x-axis. In other words, $f(x)$ is a decreasing function in the interval $[c, c + h]$. Hence $f'(x) < 0$.

Thus we find that the value dy/dx is positive for every point before the maximum point and the value of dy/dx is negative for every point after the maximum point. Thus when the curve passes through $x = c$ and the sign of dy/dx changes from positive to negative in the neighbourhood of $x = c$ then we can say that the curve has a maximum value at $x = c$.

Since while passing through a maximum point, the sign of dy/dx changes from positive to negative, dy/dx is a decreasing function. Consequently, its derivative d^2y/dx^2 will be negative.

Again let the minimum of the curve corresponding to $x = d$ is at B. We consider the neighbourhood $[d - h, d + h]$ of $x = d$. The tangent at B is parallel to the axis of x, i.e. at B, $dy/dx = 0$.

Like before, we take any point Q on the curve in the interval $[d - h, d]$ preceding B and we draw a tangent to the curve at the point Q. We see that this tangent makes an obtuse angle with the x-axis, i.e. $f'(x)$ is decreasing function in the interval $[d - h, d]$. Hence $f'(x) < 0$. We take any point Q′ on the curve in the interval $[d, d + h]$ succeeding B and draw a tangent to the curve at the point Q′. Here we see that this tangent makes and acute angle with the x-axis, i.e. $f(x)$ is an increasing function in the interval $[d, d + h]$. Hence $f'(x) > 0$.

Thus we find that the value of dy/dx is negative for every point before the minimum point and the value of dy/dx is positive for every point after the minimum point. Thus if the sign of dy/dx changes from a negative to positive dy/dx is an increasing function. Consequently its derivatives d^2y/dx^2 will be positive.

Working rule I: Let $y = f(x)$ be a function of x.

(i) First find out dy/dx.
(ii) Then putting $dy/dx = 0$, we shall find out the value of x. Let one such value be $x = c$.
(iii) For every value of x (say $x = c$) we shall test whether the sign of dy/dx changes from positive to negative or for negative to positive, when x passes through the value (by putting $x = c - h$ and $c + h$ separately). If the sign of dy/dx changes from positive to negative then y is maximum for the value of x. But if the sign of dy/dx changes from negative to positive, then y is minimum for the value of x.
(iv) If for $x < \alpha$, $dy/dx > 0$ and for $x > \alpha$, $dy/dx < 0$, then y will be maximum at $x = \alpha$. If for $x > \alpha$, $dy/dx > 0$ and for $x < \alpha$, $dy/dx < 0$, then y will be minimum at $x = \alpha$.

Working rule II: Let $y = f(x)$ be a function of x.

(i) For maximum or minimum, we shall find out the roots of the equation $dy/dx = 0$. Let $x = c$ be one such root.
(ii) Next we shall find out the second derivative d^2y/dx^2 and we shall put $x = c$ therein.
(iii) If $d^2y/dx^2 < 0$, then y will be maximum at $x = c$. If $d^2y/dx^2 > 0$, then y will be minimum at $x = c$.

9.2 Extreme Values

Let $PN = f(c)$ be maximum value of any function $y = f(x)$ in the interval $[c - \delta, c + \delta]$ in Fig. 9.4(a).

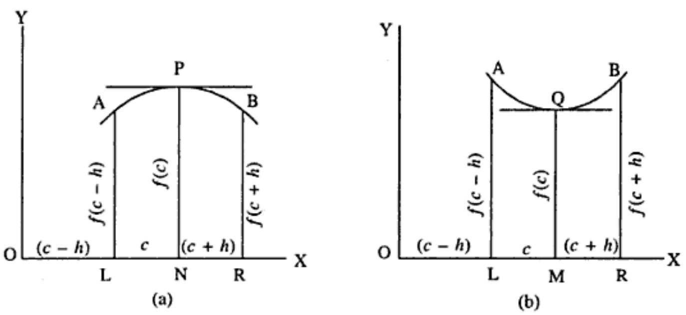


Fig. 9.4 Extreme values.

Let $-\delta < h < \delta$, and suppose that $[c - h, c + h]$ are abscissa if two points A and B on the left and right side of P respectively in its immediate neighbourhood. Then their ordinates are:

$$AL = f(c - h) < PN \quad \text{and} \quad BR = f(c + h) < PN.$$

Hence if $PN = f(c)$ be a maximum value of $f(x)$, then

$$f(c - h) - f(c) < 0 \quad \text{and} \quad f(c + h) - f(c) < 0 \tag{9.1}$$

Similarly, if $QM = f(c)$ be the minimum value of $y = f(x)$ in any interval in Fig. 9.4(b) then

$$AL = f(c - h) > QM \quad \text{and} \quad BR = f(c + h) > QM.$$

Thus

$$f(c - h) - f(c) > 0 \quad \text{and} \quad f(c + h) - f(c) > 0 \tag{9.2}$$

if $f(c)$ be the minimum value of $f(x)$.

Keeping Eqs. (9.1) and (9.2) in view, we have the following analytical, definition of extreme values of any function of one variable:

In any interval of a function, $f(x)$ has its maximum value $f(c)$ if $f(c + h) - f(c) < 0$ and minimum value $f(c)$ if $f(c - h) - f(c) > 0$, where h is any number positive or negative, but numerically very small.

Criteria for extreme values

Theorem 9.1 If $f(c)$ be an extreme value of $f(x)$ at $x = c$ and $f'(c)$ exists, then $f'(c) = 0$.

Proof Let $f(c)$ be an extreme value of $f(x)$ in any interval then if $|h|$ be very small,

$$f(c + h) - f(c) < 0 \quad \text{and} \quad f(c - h) - f(c) < 0$$

Therefore,

$$\frac{f(c+h) - f(c)}{h} < 0 \quad \text{and} \quad \frac{f(c-h) - f(c)}{-h} > 0.$$

Now Since $f(c)$ exists, by definition, we get

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$

Hence in the limit, $f'(c) \le 0$ and $f'(c) \ge 0$. This is possible only when we conclude $f'(c) = 0$.

Similarly, the theorem can be proved when $f(c)$ is minimum.

Note: The theorem provides only necessary condition for the existence of extreme values of $f(x)$. Its converse is not true, and $f'(x) = 0$ is possible without any extreme value for $f(x)$.

Theorem 9.2 If $f(x)$ be defined in any interval containing the point $x = c$, and $f'(c) = 0$, but $f''(c) \neq 0$, then $f(x)$ has its one maximum value of $f(c)$ if $f''(c) < 0$ and minimum value of $f(c)$ if $f''(c) > 0$.

Proof By Langrange's mean value theorem,

$$f(c + h) = f(c) + hf'(c + \theta h), \qquad \text{where } 0 < \theta < 1.$$

But since $f'(c) = 0$, we can write

$$hf'(c + \theta h) = h[f'(c + \theta h) - 0] = h[f'(c + \theta h) - f'(c)]$$

Therefore,

$$f(c+h) - f(c) = hf'(c+\theta h) = \theta h^2 \frac{f'(c+\theta h) - f'(c)}{\theta h} \tag{9.3}$$

Now, if we put $k = \theta h$, then $k \to 0$ as $h \to 0$

$$\lim_{h \to 0} \frac{f'(c+\theta h) - f'(c)}{\theta h} = \lim_{k \to 0} \frac{f'(c+k) - f'(c)}{k} = f''(c)$$

Hence from Eq. (9.3), we get

$$f(c + h) - f(c) = \theta h^2 f''(c) \tag{9.4}$$

In the limit, in the neighbourhood of $x = c$, making h as small as we please. Moreover, since $0 < \theta < 1$, for any positive or negative of h, $\theta h^2 > 0$. Also, $f''(c) \neq 0$.

From Eq. (9.4),

$$f(c + h) - f(c) < 0, \qquad \text{if } f''(c) < 0$$

and

$$f(c + h) - f(c) > 0, \qquad \text{if } f''(c) > 0.$$

Therefore, $f'(c) = 0$ and $f''(c) \neq 0$. $f(c)$ is a maximum or minimum value of $f(x)$ according as $f''(c) < 0$ or > 0.

Note: When both $f'(c) = f''(c) = 0$, the extreme values depend upon signs of derisatives of higher order.

Theorem 9.3 If $f(x)$ is defined in an interval containing $x = c$ and $f'(c) = f''(c) = f'''(c) = \cdots = f^{n-1}(c) = 0$, but $f^n(c) \neq 0$, then

(i) $f(c)$ is a maximum or minimum value of $f(c)$ according as $f^n(c) < 0$ or > 0 when n is even and

(ii) $f(c)$ is not an exterme value of $f(x)$, if n is odd.

Proof Since $f'(c) = f''(c) = f'''(c) = \cdots = f^{n-1}(c) = 0$, by mean value theorem of higher order, we find

$$f(c+h) - f(c) = \frac{h^{n-1}}{(n-1)!} f^{n-1}(c+\theta h) = \frac{\theta h^n}{(n-1)!} \frac{f^{n-1}(c+\theta h) - f^{n-1}(c)}{\theta h},$$

where $0 < \theta < 1$ and $\theta h \to 0$ when $h \to 0$. Then

$$\lim_{h \to 0} \frac{f^{n-1}(c+\theta h) - f^{n-1}(c)}{\theta h} = f^n(c)$$

9.3 Functions of Two Variables

Consider extreme values of $u = f(x, y)$ subject to the condition $\phi(x, y) = 0$. There are two possible cases, which are discussed in the following examples.

By solving equation $\phi(x, y) = 0$, we may find that $y = g(x)$ is satisfying the given condition.

Example 9.1 Find extreme values of $4x^3 - 15x^2 + 12x - 2$.

Solution Let $y = f(x) = 4x^3 - 15x^2 + 12x - 2$. Then

$$\frac{dy}{dx} = f'(x) = 12x^2 - 30x + 12 = 6(2x^2 - 5x + 2).$$

For maximum or minimum value, $f'(x) = 0$. Therefore,

$$6(2x - 5x + 2) = 0 \quad \text{or} \quad x = \frac{1}{2}, 2$$

Also

$$\frac{d^2y}{dx^2} = f''(x) = 6(4x - 5) = -18 < 0, \qquad \text{when } x = \frac{1}{2}$$

and

$$\frac{d^2y}{dx^2} = f''(x) = 18 > 0, \quad \text{when} \quad x = 2.$$

Therefore, $f(x)$ is maximum when $x = 1/2$ and minimum when $x = 2$. Hence the given equation has

$$\text{Maximum value} = f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 15\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) - 2 = \frac{3}{4}$$

and

$$\text{Minimum value} = f(2) = 4(2)^3 - 15(2)^2 + 12(2) - 2 = -6.$$

Example 9.2 Prove that x^x is minimum when $x = e^{-1}$.

Solution Let $\log y = \log x^x = x \log x$. Then

$$\frac{1}{y}\frac{dy}{dx} = \log x + x\frac{1}{x} = \log x + 1$$

and

$$\frac{1}{y}\frac{d^2y}{dx} - \frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x} \tag{1}$$

For maximum or minimum,

$$\frac{dy}{dx} = 0\cdot$$

Therefore,

$$\log x + 1 = 0 \quad \text{or} \quad \log x = -1 = \log e^{-1}$$

or

$$x = e^{-1}.$$

But from (1), when $x = e^{-1}$,

$$\frac{d^2y}{dx^2} = \frac{1}{y}\frac{dy}{dx} + \frac{y}{x} = \log(xe) + x^{x-1} > 0$$

Therefore, $y = x^x$ is minimum when $x = e^{-1}$.

Example 9.3 Prove that $x/(1 + x \tan x)$ is maximum when $x = \cos x$.

Solution Let

$$y = f(x) = \frac{x}{1 + x\tan x}$$

Then

$$\frac{dy}{dx} = f'(x) = \frac{(1 + x\tan x) - x(\tan x + x\sec^2 x)}{(1 + x\tan x)^2} = \frac{\cos^2 x - x^2}{(1 + x\tan x)^2}$$

Therefore, $f'(x) = 0$, when $x = \cos x$. Also $f'(x) > 0$ or < 0, according as $x < \cos x$ or $x > \cos x$. Hence the sign of $f'(x)$ changes from positive to negative as soon as the point $x = \cos x$ crosses from left to right. Therefore, $f(x)$ is maximum when $x = \cos x$.

Alternative method: Let

$$u = \frac{1 + x\tan x}{x} = \frac{1}{x} + \tan x$$

Then

$$\frac{du}{dx} = -\frac{1}{x^2} + \sec^2 x = \frac{x^2 - \cos^2 x}{x^2\cos^2 x} = 0, \quad \text{when } x = \cos x.$$

But when $x = \cos x$

$$\frac{d^2u}{dx^2} = \frac{2}{x^3} + 2\sec^2 x\tan x = \frac{2(1 + \sin x)}{\cos^3 x} = \frac{[\sin(x/2) + \cos(x/2)]^2}{[\cos(\cos x)]^3} > 0$$

Therefore,

$$\frac{1 + x\tan x}{x} \text{ is minimum} \quad \text{and} \quad \frac{x}{1 + x\tan x} \text{ is maximum.}$$

Example 9.4 Find the maximum value of $(\log x)/x$.

Solution Let $y = (\log x)/x$. Then

$$\frac{dy}{dx} = \frac{x(1/x) - \log x}{x^2} = \frac{1 - \log x}{x^2}.$$

For maximum or minimum value of y, we get $dy/dx = 0$. Then

$$\frac{1+\log x}{x^2}=0 \quad \text{or} \quad \log x = 1 = \log_e e \quad \text{or} \quad x = e$$

Now

$$\frac{d^2y}{dx^2}=\frac{x^2(-1/x)-(-\log x)2x}{x^4}=-\frac{1}{x^3}(3-2\log x)$$

When $x = e$ or $\log x = \log e = 1$. We also have

$$\frac{d^2y}{dx^2}=-\frac{1}{e^3}<0$$

Hence y is maximum when $x = e$. Therefore, the maximum value is

$$y=\frac{\log e}{e}=\frac{1}{e}.$$

Example 9.5 Find the maximum value of $x^p y^q$ when $x + y = a$.

Solution Le $z = x^p y^q = x^p(a - x)^q$. Then

$$\frac{dz}{dx}=px^{p-1}(a-x)^q-qx^p(a-x)^{q-1}=x^{p-1}(a-x)^{q-1}[p(a-x)-qx]$$

For maximum or minimum of z,

$$x^{p-1}(a-x)^{q-1}[p(a-x)-qx]=0$$

Therefore,

$$x=0, \quad x=a, \quad x=\frac{ap}{p+q} \qquad \text{if} \qquad p>1, \ q>1.$$

Now

$$\frac{d^2z}{dx^2}=(p-1)x^{p-2}(a-x)^{q-1}\left[p(a-x)-qx\right]-(q-1)x^{p-1}(a-x)^{q-z}\left[p(a-x)-qx\right]$$

$$+\,x^{p-1}(a-x)^{q-1}(-p-q)$$

Also,

$$\text{When } x = 0,\ x = a, \quad \frac{d^2z}{dx^2}=0$$

$$\text{When } x = \frac{ap}{p+q}, \quad \frac{d^2z}{dx^2}=\left(\frac{ap}{p+q}\right)^{p-q}\left(a-\frac{ap}{p+q}\right)^{q-1}(-p-q)<0$$

or

$$x = \frac{\pi}{3}, \quad x = \pi.$$

For $x = \pi/3$, we have

$$\frac{d^2y}{dx^2} = -\sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right) - 3\sin\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) - 3\frac{\sqrt{3}}{2}\frac{1}{2} = -\frac{3\sqrt{3}}{4} - \frac{3\sqrt{3}}{4}$$

$$= -\frac{3\sqrt{3}}{2} \qquad \text{(negative)}$$

Hence y is maximum for $x = \pi/3$.

Example 9.8 If $y = x/(\log x)$, show that y is minimum for $x = e$.

Solution We have $y = x/(\log x)$. Then

$$\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}.$$

For maximum or minimum of y, $dy/dx = 0$. So, $\log x - 1 = 0$. Therefore, $\log x = 1 = \log e$. Thus $x = e$. We also have

$$\text{when } \log x > 1, \quad \frac{dy}{dx} > 0$$

and

$$\text{when } \log x < 1, \quad \frac{dy}{dx} < 0$$

Hence dy/dx changes sign from negative to positive. Therefore, y is minimum for $x = e$ and the minimum value of y is $e/(\log e) = e$.

Example 9.9 What fraction exceed its pth power by the greatest number possible.

Solution Let x be the required fraction. Then $y = x - x^p$. Therefore,

$$\frac{dy}{dx} = 1 - px^{p-1}, \quad \frac{d^2y}{dx^2} = -p(p-1)x^{p-2}.$$

For maximum or minimum value of y, $dy/dx = 0$. Then, we get

$$px^{p-1} = 1 \quad \text{or} \quad x^{p-1} = \frac{1}{p} \quad \text{or} \quad x = \left(\frac{1}{p}\right)^{1/(p-1)}$$

Clearly for this value of d^2y/dx^2 is negative, and hence y is maximum for $x = (1/p)^{1/(p-1)}$.

For maxima and minima, $\cos x = 0$ and $\sin x = 1/4$. Thus x lies between 0 and 2π. Now, $\cos x = 0$ gives $x = \pi/2$ and $3\pi/2$ and $\sin x = 1/4$ gives $x = \sin^{-1}(1/4)$ and $\pi - \sin^{-1}(1/4)$. So $\sin^{-1}(1/4)$ lies between 0 and $\pi/2$.

We again have

$$\frac{d^2y}{dx^2} = -\sin x - 4\cos 2x$$

Now,

$$\text{at } x = \frac{\pi}{2}, \qquad \frac{d^2y}{dx^2} = 3 > 0$$

$$\text{at } x = \frac{3\pi}{2}, \qquad \frac{d^2y}{dx^2} = 5 > 0$$

$$\text{at } x = \sin^{-1}\frac{1}{4}, \qquad \frac{d^2y}{dx^2} = -\sin x - 4\left(1 - 2\sin^2 x\right) = -\frac{15}{7} < 0$$

$$\text{at } x = \pi - \sin^{-1}\frac{1}{4}, \qquad \frac{d^2y}{dx^2} = -\frac{15}{4} < 0$$

Thus y is maximum for $x = \sin^{-1}(1/4)$ and $\pi - \sin^{-1}(1/4)$, and is minimum for $x = \pi/2$, $3\pi/2$.

Example 9.15 If $y = a \log x + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$, then find a and b.

Solution Here

$$y = f(x) = a \log x + bx^2 + x$$

$$\frac{dy}{dx} = f'(x) = \frac{a}{x} + 2bx + 1$$

For extreme values, $f'(x) = 0$, if it exists. Given extreme values are $x = -1$ and $x = 2$. Therefore,

$$f'(-1) = 0 \quad \text{or} \quad -a - 2b + 1 = 0 \tag{1}$$

$$f'(2) = 0 \quad \text{or} \quad a + 8b + 2 = 0 \tag{2}$$

Solving them, we get $a = 2$, $b = 1/2$.

Example 9.16 Prove that $x - \sin x$ has neither maxima nor minima.

Solution Here

$$y = x - \sin x \tag{1}$$

$$\frac{dy}{dx} = 1 - \cos x \tag{2}$$

For maxima or minima, $dy/dx = 0$. Then $1 - \cos x = 0$. It gives $x = 2n\pi$.

Therefore, at $x = 2n\pi$,

$$\frac{d^2y}{dx^2} = \sin 2n\pi = 0, \quad \frac{d^3y}{dx^3} = \cos 2n\pi = 1.$$

Hence $f(x)$ has neither maxima nor minima.

Example 9.17 Find the greatest and the least value of 2 sin x + sin $2x$ in the interval $(0, 3\pi/2)$.

Solution Let

$$f(x) = 2\sin x + \sin 2x, \quad f'(x) = 2\cos x + 2\cos 2x$$

For maxima or minima, $f'(x) = 0$. So $\cos x + \cos 2x = 0$. Solving, we get $x = \pi/3$, π. Since.

$$f'(x) = 2(\cos x + \cos x) = 2(2\cos^2 x + cox - 1) = 2(\cos x + 1)(2\cos x - 1)$$

we get $f'(x) > 0$, for $\cos x > 1/2$. Then $f'(x)$ is increasing in $(0, \pi/3)$. Therefore, greatest value of

$$f(x) = f\left(\frac{\pi}{3}\right) = 2\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Also $f'(x) < 0$ for $\cos x < 1/2$. Therefore, $f'(x)$ is decreasing in $(\pi/3, 3\pi/2)$. The least value of $f(x)$ is $f(3\pi/2) = -2$.

Example 9.18 If $ax^2 + b/x \geq c$ for all positive x, where $a > 0$ and $b > 0$, show that $27ab^2 \geq 4c^3$.

Solution Here

$$f(x) = ax^2 + \frac{b}{x}$$

$$f'(x) = 2ax - \frac{b}{x^2} = \frac{2ax^3 - b}{x^2}.$$

$$f''(x) = 2a + \frac{2b}{x^3} > 0 \quad (\text{as } a, b, x > 0)$$

Now for maxima or minima, $f'(x) = 0$. Then, we get

$$x = \left(\frac{b}{2a}\right)^{1/3} > 0 \quad (\text{as } a > 0, b > 0)$$

Also, it has shown that $f''(x) > 0$. Therefore, $f(x)$ is minimum at $x = [b/(2a)]^{1/3}$ and the minimum value at the point is given by

$$f(x) = ax^2 + \frac{b}{x} - c = \frac{ax^3 + b}{x} - c = \frac{3b}{z}\left(\frac{2a}{b}\right)^{1/3} - c$$

According to the condition, $f(x) \geq 0$ for all the $x > 0$. Then

$$\frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \geq c$$

Thus we get a cubic,

$$\frac{27b^3}{8}\frac{2a}{b} \geq c^3 \text{ or } 27ab^2 \geq 4c^3.$$

9.4 Problems Involving Geometry

In this section, it will be shown as to how the method pertaining to maxima or minima is applied in the problems relating to geometry and solid geometry. In this connection, it is necessary to remember the formulae of areas and volumes of some important figures. These are the following:

Sphere. Refer to Fig. 9.6. If r be the radius of the sphere, then

$$\text{Volume} = \frac{4}{3}\pi r^3 \quad \text{and} \quad \text{Area of whole surface} = 4\pi r^3$$

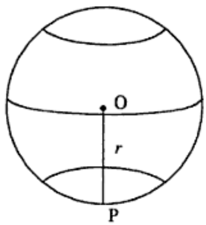


Fig. 9.6 A sphere.

Cylinder. Let h be the height and r be the radius of the base (Fig. 9.7). Then

$$\text{Volume} = \pi r^2 h$$

$$\text{Area of curved surface} = 2\pi rh$$

$$\text{Area of whole surface} = 2\pi rh + 2\pi r^2$$

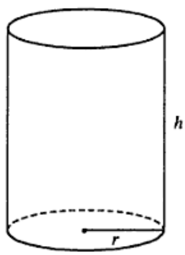


Fig. 9.7 A cylinder.

Cone. Let h be the height of the cone, r be its base radius and l the slant height (Fig. 9.8). Then

$$\text{Slant height} = \sqrt{h^2 + r^2}$$

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$\text{Area of curved surface } \pi rl = \pi r\sqrt{r^2 + h^2}$$

$$\text{Area of whole surface of a cylinder} = \pi r\sqrt{r^2 + h^2} + \pi r^2$$

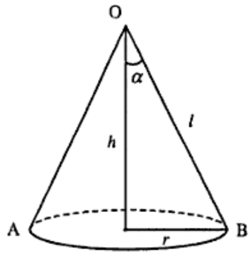


Fig. 9.8 A cone.

Example 9.19 The sum of the perimeters of a circle and a square is l. Show that when the sum of the areas is least, the side of the square is double of the radius of the circle.

Solution Let the radius of the circle = r and one side of a square = a. Therefore, the perimeter of the circle = $2\pi r$ and that of the square = $4a$. Then

$$2\pi r + 4a = l \tag{1}$$

and

$$\text{Sum of both the areas} = \pi r^2 + a^2.$$

From (1)

$$A = \pi r^2 + a^2 = \pi r^2 \left(\frac{l - 2\pi r}{4}\right)^2$$

we get

$$\frac{dA}{dr} = 2\pi r - \frac{\pi}{4}(l - 2\pi r) = 2\pi r - \pi a. \tag{2}$$

Now for maxima or minima, $dA/dr = 0$. Therefore,

$$2\pi r - \pi a = 0 \quad \text{or} \quad a = 2r$$

It is clear from (2), $dA/dr > 0$, if $2r > a$ and $dA/dr < 0$ if $2r < a$. Therefore, for $a = 2r$, A has a minimum value.

Solution Let a cone be given, whose radius of the base $CB = r$, height $OC = h$ and semi-vertical angle = α. Then from Fig. 9.8.

$$\text{Area of the surface of the cone} = \pi r^2 + \pi rl$$

and

$$\text{Volume} = \frac{1}{3}\pi r^2 h.$$

Given that $\pi r^2 + \pi rl = k$ (constant). Now, in ΔBOC,

$$h^2 = l^2 - r^2 = \left(\frac{k - \pi r^2}{\pi r}\right)^2 = \frac{k^2}{\pi^2 r^2} - \frac{2k}{\pi}$$

$$v^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4\left(\frac{k^2}{\pi^2 r^2} - \frac{2k}{\pi}\right) = \frac{1}{9}\pi^2\left(\frac{k^2 r^2}{\pi^2} - \frac{2kr^4}{\pi}\right)$$

Therefore, v is a function of r only. Differentiating with respect to r, we get

$$2v\frac{dv}{dr} = \frac{1}{9}k\left[k(2r) - 2\pi(4r^3)\right]$$

$$= \frac{1}{9}k(2kr - 8\pi r^3)$$

$$= \frac{1}{9}k\left[2r(\pi r^2 + \pi rl) - 8\pi r^3\right]$$

$$= \frac{1}{9}k2\pi r^2(l - 3r) \tag{1}$$

Now for maxima or minima, $dv/dr = 0$, i.e. $l - 3r = 0$ or $3r = l$. From (1), $dv/dr > 0$, $3r < 1$ and $dv/dr < 0$ if $3r > 1$. Therefore, the volume of the cone be maximum for $3r = l$. Now, in ΔOCB,

$$\sin\alpha = \frac{CB}{OB} = \frac{r}{l} = \frac{r}{3r} = \frac{1}{3} \quad \text{or} \quad \alpha = \sin^{-1}\frac{1}{3}.$$

Example 9.22 Prove that a conical tent of a given volume requires the least amount of material when its height is √2 times the radius of the base.

Solution Let r be the radius of the base of the circle, h the height and l the slant height (Fig. 9.8). Given $v = (1/3)\pi r^2 h$ (= constant), therefore,

$$r^2 h = k \text{ (constant)}. \tag{1}$$

The area of the surface

$$S = \pi rl = \pi r\sqrt{h^2 + r^2} \tag{2}$$

or

$$S^2 = \pi r^2(h^2 + r^2) = \pi r^2\left(\frac{k^2}{r^4} + r^2\right) = \pi^2\left(\frac{k^2}{r^2} + r^4\right)$$

Put $u = S^2$, so

$$u = \pi^2\left(\frac{k^2}{r^2} + r^4\right)$$

Differentiating with respect to r, we get

$$\frac{du}{dr} = \pi^2\left[k^2\left(-\frac{2}{r^3}\right) + 4r^3\right] \tag{3}$$

For maximum or minimum, $du/dr = 0$. Then from (1)

$$4r^3 - \frac{2k^2}{r^3} = 0 \quad \text{or} \quad 4r^3 = \frac{2}{r^3}r^4h^2$$

Solving, we get $h = \sqrt{2}r$. Again differentiating (3) with respect to r, we get

$$\frac{d^2u}{dr^2} = \pi^2\left[k^2\left(\frac{6}{r^4} + 12r^2\right)\right](>0)$$

Hence u is minimum when $h = r\sqrt{2}$ and consequently S is minimum. when $h = r\sqrt{2}$.

Example 9.23 Prove that the least perimeter of an isosceles triangle which can be circumscribed in a circle of radius a, is $(6\sqrt{3})a$.

Solution Let ABC be an isosceles triangle with $AB = AC$ circumscribing the circle whose centre is O and radius is a (Fig. 9.10).

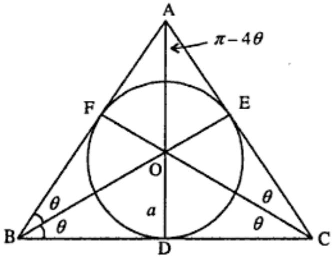


Fig. 9.10 Solution to Example 9.23.

Let

$$\angle B = \angle C = 2\theta, \text{ where } 0 < 2\theta < \pi/2.$$

Let O be the in centre of ΔABC. OA, OB and OC be the internal bisector of the angles of the triangle OD, OE, OF are perpendiculars to BC, CA and AB respectively D, E, and F be the points of contact with the circle. Then $BD = a \cot\theta$, $DC = a \cot\theta$. Therefore,

$$BC = BD + DC = 2a \cot \theta.$$

Also

$$AB = AC = AE + EC = a \cot\left(\frac{\pi}{2} - 2\theta\right) + a \cot\theta = a \tan 2\theta + a \cot\theta$$

Let P be the perimeter of the triangle, then

$$\begin{aligned} P &= BC + CA + AB \\ &= BC + 2AB \\ &= 2a\cot\theta + 2a \tan 2\theta + 2a \cot\theta \\ &= 4a \cot\theta + 2a \tan\theta, \qquad 0 < \theta < \pi/4 \end{aligned} \tag{1}$$

and

$$\frac{dP}{d\theta} = 4a(-\text{cosec}^2\theta) + 4a\sec^2(2\theta) \tag{2}$$

For maximum and minimum, $dP/d\theta = 0$. Therefore,

$$-4a \text{ cosec}^2\theta + 4a \sec^2\theta = 0$$

Solving, we get $\theta = \pi/6$.

Again, differentiating (2) with respect to θ, we get

$$\frac{d^2P}{d\theta^2} = 8a \text{ cosec}^2\theta \cot\theta + 16a \sec^2(2\theta) \tan 2\theta$$

At $\theta = \pi/6$, $d^2P/d\theta^2$ is positive. Therefore, P is minimum when $\theta = \pi/6$. Thus from (1), the least value of P is

$$P = 4a \cot\frac{\pi}{6} + 2a \tan\frac{\pi}{3} = 4a\sqrt{3} + 2a\sqrt{3} = 6\sqrt{3}a.$$

Example 9.24 Find the height of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius r.

Solution Let ABCD be a cylinder inscribed in a sphere with centre O and radius $OD = r$ (Fig. 9.11). Let the height of the cylinder $CD = 2x$. Draw OE ⊥ CD so that $CE = ED = 2x$. Then

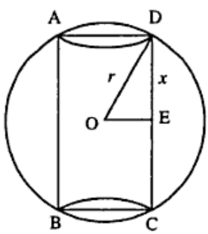


Fig. 9.11 Solution to Example 9.24.

$$\text{Radius of the cylinder, } OE = \sqrt{r^2 - x^2}$$

and

$$\text{Volume of the cylinder, } V = \pi r^2 h$$
$$= \pi(OE)^2\, CD$$
$$= \pi(r^2 - x^2)2x$$
$$= 2\pi(r^2x - x^3)$$

Therefore,

$$\frac{dV}{dx} = 2\pi(r^2 - 3x^2).$$

For maxima or minima, we have

$$\frac{dV}{dx} = 0 = r^2 - 3x^2 \quad \text{or} \quad x = \frac{r}{\sqrt{3}}$$

Again

$$\frac{d^2V}{dx^2} = 2\pi(-6x) = -12\pi x$$

At $x = r/\sqrt{3}$,

$$\frac{d^2V}{dx^2} = -12\pi\frac{r}{\sqrt{3}} < 0.$$

Hence V has maximum at $x = r/\sqrt{3}$ and height the cylinder = $2x = 2r/\sqrt{3}$.

Corollary Volume V of the cylinder is

$$V = 2\pi(r^2 - x^2) = 2\pi\left(r^2 - \frac{r^2}{3}\right)\frac{r}{\sqrt{3}} = \frac{4\pi r^2}{3\sqrt{3}}.$$

Example 9.25 Prove that the right circular cone of maximum volume which can be inscribed in a sphere of radius r has its altitude equal to $(4/3)r$.

Solution Let O be the centre and r be the radius of the sphere. Let AB be the diameter of the base of the cone (Fig. 9.12). Let VN is perpendicular to AB and let $ON = x$. Then

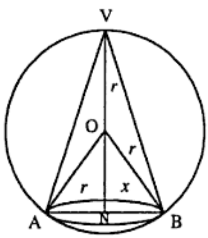


Fig. 9.12 Solution to Example 9.25.

Therefore,

$$\frac{dp}{d\theta}=(a^2-b^2)\frac{b^2\cos^4\theta-a^2\sin^4\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}$$

Ford maximum or minimum $dp/d\theta = 0$, and

$$\tan^4\theta=\frac{b^2}{a^2} \quad \text{or} \quad \tan\theta=\pm\sqrt{\frac{b}{a}}$$

If $p = 0$, $\theta = 0$, or $\pi/2$ and p is positive when θ lies between O and $\pi/2$. Therefore, p is maximum when $\tan\theta = \sqrt{(b/a)}$. Putting this value in (1), we get the maximum value $p = a - b$.

Example 9.28 Find a right circular cylinder of greatest volume that can be inscribed in a given right circular cone.

Solution Let h = the height of the cone α = the semi-angle and x = radius of the inscribed cylinder. From Fig. 9.14, we have $AO = h$, $AO = x \cot \alpha$.

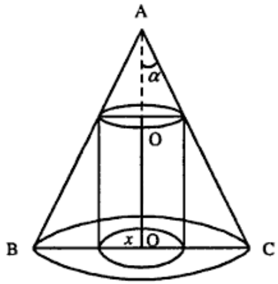


Fig. 9.14 Solution to Example 9.28.

Therefore,

$$\text{Height of the cylinder, } OO_1 = AO - AO_1 = h - x \cot \alpha \qquad (1)$$

The volume of the cylinder

$$V = \pi x^2(h - x \cot \alpha) \qquad (2)$$

Differentiating with respect to x, we get

$$\frac{dv}{dx}=\pi\left[2x(h-x\cot\alpha)+x^2(-\cot\alpha)\right]=\pi(2hx-3x^2\cot\alpha)$$

For maximum or minimum $dv/dx = 0$. Then

$$0 = \pi(2hx - 3x^2\cot\alpha) = \pi x(2h - 3x\cot\alpha)$$

Solving, we get

$$x=0, \quad \text{or} \quad x=\frac{2h}{3\cot\alpha}=\frac{2h}{3}\tan\alpha$$

Differentiating, we get

$$\frac{dv}{dh}=\frac{1}{3}\pi(l^2-3h^2) \quad (3)$$

For maximum and minimum, $dv/dh = 0$. Therefore, $l^2 - 3h^2 = 0$, or $h^2 + r^2 = 3h^2$ or $r^2 = 2h^2$ or $r = h\sqrt{2}$. Again differentiating (3), we get

$$\frac{d^2V}{dh^2}=\frac{1}{3}\pi(-6h) \quad \text{(negative)}$$

Therefore, V is maximum, when

$$r = h\sqrt{2} \quad \text{or} \quad \frac{r}{h}=\sqrt{2}$$

That is,

$$\tan\theta = \sqrt{2} \quad \text{or} \quad \theta = \tan^{-1}\sqrt{2}.$$

Example 9.31 Show that the radius of the right circular cylinder of the greatest curved surface, which can be inscribed in a given cone, is half that of the cone.

Solution Let O be the vertex, h be the height and r be the radius of the base of the cone (Fig. 9.15). Let us suppose that the cylinder be inscribed in a cone and CL be its radius $= x$ and $LM = y$. Therefore, the area of the curved surface of the cylinder,

$$S = 2\pi(CL)(LM) = 2\pi xy \quad (1)$$

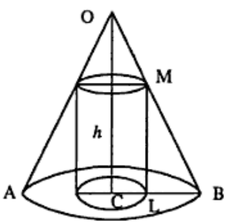


Fig. 9.15 Solution to Example 9.31.

Now triangles BLM and BCO are similar. Therefore,

$$\frac{LM}{OC}=\frac{BL}{BC} \quad \text{or} \quad \frac{LM}{h}=\frac{r-x}{r} \quad \text{or} \quad \frac{y}{h}=\frac{r-x}{r}$$

Then

$$ry = h(r-x) = rh - hx \quad (2)$$

From (1) and (2),

$$S=2\pi x\frac{h(r-x)}{r}=\frac{2\pi h}{r}(rx-x^2)$$

or

$$\frac{dS}{dx}=\frac{2\pi h}{r}(r-2x)$$

For maximum or minimum, $dS/dx = 0$. Then

$$\frac{2\pi h}{r}(r-2x)=0 \quad \text{or} \quad r=\frac{1}{2}r.$$

Again,

$$\frac{d^2S}{dx^2}=\frac{2\pi h}{r}(-2) \quad \text{(negative)}$$

Therefore, S be the greatest, when $x = r/2$.

Example 9.32 Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $(4/27)\pi h^3 \tan^2\alpha$.

Solution Let O be the vertex, $OC = h$, $BC = r$ and $\angle BOC = \alpha$, the semi-vertical angle of the cone (Fig. 9.15). Let the cylinder be inscribed in a core and $CL = x$ be the base and $LM = y$, the height of the cylinder. Therefore,

$$\text{Volume of the cylinder} = \pi x^2(LM) = \pi x^2 y \tag{1}$$

Here,

$$LM = \frac{h(r-x)}{r} \quad \text{or} \quad y = \frac{h(r-x)}{r} \quad \text{or} \quad r = h(r-x) = hr - hx.$$

Solving, we get

$$V=\pi x^2 h(r-x)=\frac{\pi h}{r}(rx^2-x^3).$$

Differentiating, we obtain

$$\frac{dV}{dx}=\frac{\pi h}{r}(2xr+3x^2) \tag{2}$$

Now, for maximum or minimum $dv/dx = 0$. Therefore,

$$2rx - 3x^2 = 0 \quad \text{or} \quad x = \frac{2}{3}r.$$

Again differentiating (2) with respect to x, we get

$$\frac{d^2V}{dx^2}=\frac{\pi h}{r}(2r-6x)$$

At $x = 2r/3$,

$$\frac{d^2V}{dx^2}=\frac{\pi h}{r}(2r-4r) \qquad \text{(negative)}$$

Then the volume of the cylinder is maximum when $x = 2r/3$. Thus,

$$\text{Maximum volume } = \pi x^2 y = \pi x^2 \frac{h(r-x)}{r}$$
$$= \frac{\pi h}{r}\frac{4}{9}r^2\left(r - \frac{2}{3}r\right)$$
$$= \frac{4\pi hr}{9}\frac{r}{3}$$
$$= \frac{4}{27}\pi hr^2$$

But

$$\tan\alpha = \frac{CB}{OC} = \frac{r}{h} \quad \text{or} \quad r = h\tan\alpha$$

Therefore,

$$\text{Maximum volume } = \frac{4}{27}\pi hr^2$$
$$= \frac{4}{27}\pi(h^2\tan^2\alpha)(h)$$
$$= \frac{4}{27}\pi h^3\tan^2\alpha$$

Example 9.33 An open tank is to be constructed with a square base and vertical side so as to contain a given quantity of water. Show that the cost of lining the tank with lead will be least if the depth is made half of the width.

Solution Let one side of the square base of the tank is x and the depth is y. Let the tank contains V cubic units of water. Then

$$V = x^2y \text{ (constant)} \tag{1}$$

is given. Let the area of the metal sheet used in the square base and the vertical side of the tank is S. Then

$$S = x^2 + 4xy = x^2 + 4x\frac{v}{x^2} = x^2 + \frac{4v}{x}$$

Then

$$\frac{dS}{dx} = 2x - \frac{4v}{x^2} \tag{2}$$

For maximum or minimum, $dS/dx = 0$. Then

$$2x - \frac{4v}{x^2} = 0 \quad \text{or} \quad x^3 = 2v.$$

45. The sum of the volume of the sphere and a cube is given. Show that the sum of the surface is greatest, the diameter of the sphere is equal to the side of the cube.

46. A wire of length 25 m is to be cut into two pieces. One of the pieces is to be made into a square and other into a circle. What should be the lengths of two pieces so that the combined area of square and circle in minimum?

47. An open box with a square box is to be made of given quantity of sheet of area a^2. Show that the maximum volume of the box is $a^3/(6\sqrt{3})$.

48. Prove that among all the triangles of a given hypotenuse the isosceles triangle has maximum area.

49. Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

50. Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c mile per hour is $(3/2)c$ mile per hour.

51. Divide 15 into two parts such that the square of one multiplied with the cube of the other is a maximum.

52. Show that of all the rectangle of a given area, the square has the smallest perimeter.

53. Prove that the area of the triangle formed by the tangent at any point of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and its axis is a minimum for the point $(a/\sqrt{2}, b/\sqrt{2})$.

54. A tangent to an ellipse meets the axes in P and Q; show that the least value of PQ is equal to the sum of the semi-axes of the ellipse, and also that PQ is divided at the point of contact in the ratio of its semi-axes.

55. Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse, the triangle having its vertex coincident with one extremity of the major axes.

56. The amount of fuel consumed per hour by a certain steamer varies as the cube of its speed. When the speed is 15 mile per hour, the fuel consumed is $4^1/_2$ ton of the coal per hour at Rs. 4 per ton. The other expenses are total Rs. 100 per hour. Find the most economical speed and the cost of a voyage of 1980 mile.

57. The strength of a beam varies as the product of its breadth and the square of its depth. Find the dimension of the strongest beam that can be cut from a circular log of wood of radius a unit.

58. The sum of the area of the surface of a cube and a sphere is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

$$r=\frac{\partial^2 f}{\partial x^2}, \quad s=\frac{\partial^2 f}{\partial x \partial y}, \quad t=\frac{\partial^2 f}{\partial y^2}$$

be the values of these partical derivatives at (a, b), then:

(i) $f(x, y)$ has a maximum value $f(a, b)$ if $r < 0$, $rt - s^2 > 0$

(ii) $f(x, y)$ has a minimum value $f(a, b)$, if $r > 0$, $rt - s^2 > 0$

(iii) $f(x, y)$ has no extreme value $f(a, b)$ if $rt - s^2 < 0$

(iv) The case in doubtful and deserves further investigation, if $rt - s^2 = 0$.

Thus (i) and (ii) provide sufficient conditions for the existence of extreme value. They are called *Lagrange's condition for extreme values.*

9.6 Taylor's Theorem for Two Independent Variables

The function $f(x, y)$ of independent variables x and y possesses continuous partial derivatives of order n in any domain of a point (a, b) and point (h, k) be such that the point $(a + h, b + k)$ may belong to the domain under consideration, then there exists a number θ, $0 < \theta < 1$, such that

$$f(a+h, b+k) = f(a, b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n$$

where

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+\theta h, b+\theta k).$$

Under similar condition, the Taylor's theorem for three independent variables x, y and z can be enunciated as follows:

$$f(a+h, b+k, c+l) = f(a, b, c) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(a, b, c)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(a, b, c) + \cdots$$

$$+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^{n-1} f(a, b, c) + R_n$$

where

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^n f(a+\theta h, b+\theta k, c+\theta l)$$

Similarly, the Taylor's theorem can be extended to functions of three or more independent variables.

The stationary point $(x, y, z) = (a, b, c)$ is obtained by solving these equations. Next if A, B, C, F, G, H be the values of partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y}$$

respectively at (a, b, c), we form inequalities if:

$$A > 0, \quad \begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0, \quad \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} > 0$$

then $f(a, b, c)$ is the minimum value of $f(x, y, z)$. In case they be alternatively negative and positive, i.e. if they be < 0, > 0 and < 0 respectively, then $f(a, b, c)$ is the maximum value of $f(x, y, z)$.

We have similar consideration for a function of more than three variables, too.

Subsidiary condition. Consider the extreme value of $f(x, y, z)$ when variables are connected by a given condition, $\phi(x, y, z) = 0$. We solve the equation $\phi(x, y, z) = 0$ for z and let $z = g(x, y)$. Therefore, $f(x, y, z) = f[x, y, g(x, y)]$ becomes a function.

Example 9.34 Find the maximum values of $x^3 + y^3 + 9xy$.

Solution Let $f = x^3 + y^3 + 9xy$. Then

$$\frac{\partial f}{\partial x} = 3x^2 + 9y = 3(x^2 + 3y)$$

$$\frac{\partial f}{\partial y} = 3y^2 + 9x = 3(y^2 + 3x) \tag{1}$$

When

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0,$$

we obtain $x^2 + 3y = 0$, $y^2 + 3x = 0$. Solving these equations, we find $(0, 0)$, $(-3, -3)$ are two stationary points of the function. Now, from (1) when $(x, y) = (-3, -3)$, we have

$$r = \frac{\partial^2 f}{\partial x^2} = 6x = -18 < 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 9$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y = 6(-3) = -18 < 0$$

Thus for the existence of extreme value, when $\phi(x, y) = 0$,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \quad \phi(x, y) = 0 \tag{9.10}$$

Since $u = f(x, y) + \lambda\phi(x, y)$, using partial derivatives $u_x, f_x, \lambda\phi_x, \ldots$, etc., these equations can be written as

$$u_x = f_x + \lambda\phi_x = 0, \quad u_y = f_y + \lambda\phi_y = 0, \quad u_\lambda = \phi(x, y) = 0 \tag{9.11}$$

Therefore,

$$-\lambda = \frac{f_x}{\phi_x} = \frac{f_y}{\phi_y}.$$

Solving

$$\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y}, \quad \phi(x, y) = 0$$

we get the finite values of x, y to determine the extreme value of Eq. (9.6). If x_1, y_1 satisfy these equations, then will also $\phi(x, y) = 0$. Hence from Eq. (9.6), the extreme value of $f(x, y)$ is given by

$$u = f(x_1, y_1) + \lambda\phi(x_1, y_1) = f(x_1, y_1) + \lambda \times 0 = f(x_1, y_1)$$

Thus we see that $f(x_1, y_1)$ does not depend upon the actual value of λ. Hence the theorem.

Working rule:

(i) $u = f(x, y) + \lambda\phi(x, y)$, where $\phi(x, y) = 0$.

(ii) $u_x = u_y$, $u_\lambda = 0$ for extreme values, we have $f_x + \lambda\phi_x = 0$, $f_y + \lambda\phi_y = 0$, $\phi(x, y) = 0$.

(iii) Finally solving equations, $f_x/\phi_x = f_y/\phi_y$ and $\phi(x, y) = 0$.

(iv) If x_1, y_1 satisfy them, the required extreme value $= f(x_1, y_1)$, then $f(x_1, y_1)$ is the maximum or minimum value of $f(x, y)$ according as $f(x_1, y_1) > f(x', y')$ or $f(x_1, y_1) < f(x', y')$, where (x', y') be an arbitrary point satisfying $\phi(x, y) = 0$.

Example 9.36 Find the maximum value of $x^2 + 5xy + 2y^2$ when $x + y = 4$.

Solution Here $f(x, y) = x^2 + 5xy + 2y^2$ and $\phi(x, y) = x + y - 4 = 0$. Let

$$u = f + \lambda\phi = x^2 + 5xy + 2y^2 + \lambda(x + y - 4) = 0. \tag{1}$$

For extreme value,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \lambda} = 0 \cdot$$

Therefore,

$$2x + 5y + \lambda = 0, \qquad 5x + 4y + 1 = 0, \qquad x + y - 4 = 0.$$

Solving these equations, $-\lambda = 2x + 5y = 5x + 4y$. That is, $3x - y = 0$. Solving $x + y - 4 = 0$ and $3x - y = 0$, we get $x = 1$, $y = 3$. From (1), the extreme value of $f(x, y)$ is $u = f(1, 3) = 34$.

Finally to verify it this is the maximum value, we take any arbitrary point $(x, y) = (4, 0)$, which satisfies the given condition $x + y = 4$. Since $f(x, y) = x^2 + 5xy + 2y^2$, $f(4, 0) = 4^2 + 0 + 0 = 16 < 34$.

Therefore, the required maximum value of $f(x, y) = 34$.

Example 9.37 Find extreme value of $x^a y^b$, when $x + y = c$.

Solution Here $f(x, y) = x^a y^b$ and $\phi(x, y) = x + y - c = 0$. Let

$$u = f + \lambda\phi = x^a y^b + \lambda (x + y - c) = 0 \tag{1}$$

For any extreme value,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \lambda} = 0.$$

Therefore,

$$ax^{a-1}y^b + \lambda = 0, \qquad bx^a y^{b-1} + \lambda = 0, \qquad x + y - c = 0.$$

From these equations

$$-\lambda = ax^{a-1}y^b = bx^a y^{b-1}$$

Then

$$ax^{a-1}y^b - bx^a y^{b-1} = 0 \quad \text{or} \quad x^{a-1}y^{b-1}(ay - bx) = 0.$$

Solving, we get $ay = bx$. Thus solving the equations $ay = bx$, $x + y = c$, we get

$$x = \frac{ac}{a+b}, \qquad y = \frac{bc}{a+b}.$$

Thus from (1), the extreme value of $x^a y^b$ is given by

$$u = \left(\frac{ac}{a+b}\right)^a \left(\frac{bc}{a+b}\right)^b + \lambda \times 0 \quad \text{or} \quad u = a^a b^b \left(\frac{c}{a+b}\right)^{a+b}.$$

Since $(x, y) = (c, 0)$ satisfies $x + y = c$ and

$$x^a y^b = 0 < a^a b^b \left(\frac{c}{a+b}\right)^{a+b},$$

it is a maximum value.

Example 9.38 Find the maximum or minimum value of

$$x^3 y^2 (a - x - y), \quad x \neq 0, \quad y \neq 0, \quad x + y \neq a.$$

Solution Let $u = x^3 y^2 (a - x - y)$. Then

$$\frac{\partial u}{\partial x} = 3x^2 y^2 (a - x - y) - x^3 y^2 \tag{1}$$

$$\frac{\partial u}{\partial y} = 2x^3 y(a - x - y) - x^3 y^2 \tag{2}$$

For maximum or minimum value of u, $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$. Hence

$$3x^2y^2(a - x - y) = x^3y^2 \tag{3}$$

$$2x^3y\,(a - x - y) = x^3y^2 \tag{4}$$

From (3) and (4), we get

$$3x^2y^2(a - x - y) = 2x^3y(a - x - y) \text{ or } y = \frac{2}{3}x$$

From (3), we get

$$3x^2\ \frac{4}{9}x^2\left(a - x - \frac{2}{3}x\right) = x^3\frac{4}{9}x^2$$

Solving

$$\frac{4}{3}x^4\left(a - \frac{5x}{3}\right) = \frac{4}{9}x^5 \quad \text{or} \quad a - \frac{5x}{3} = \frac{x}{3} \quad \text{or} \quad \frac{5x}{3} + \frac{x}{3} = a$$

Solving, we get

$$x = \frac{a}{2} \quad \text{and} \quad y = \frac{2}{3}x = \frac{2}{3}\frac{a}{2} = \frac{a}{3}$$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 3y^2\left[x^2(-1) + (a - x - y)2x\right] - 3x^2y^2$$

$$= -3x^2y^2 + 6xy^2(a - x - y) - 3x^2y^2$$

$$= 6xy^2(a - x - y - 6x^2y^2)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 3x^2[y^2(-1) + (a - x - y)2y] - 2x^3y$$

$$= 6x^2y(a - x - y) - 3x^2y^2 - 2x^3y$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3[y(-1) + (a - x - y)(1)] - x^3(2y)$$

$$= 2x^3(a - x - y) - 4x^3y$$

Hence at $(a/2, a/3)$, we get

$$r = -\frac{a^4}{9}, \quad s = -\frac{a^4}{12}, \quad t = -\frac{a^4}{8}$$

and

$$rt - s^2 = \frac{a^8}{144} > 0$$

Thus $r < 0$, and $rt - s^2 > 0$. Therefore, u is maximum at $(a/2, a/3)$.

The maximum value of the given function

$$f(x, y) = \left(\frac{a}{2}\right)^3\left(\frac{a}{3}\right)^2\left(a - \frac{a}{2} - \frac{a}{3}\right) = \frac{a^6}{432}.$$

Example 9.40 Prove that the function $x^2 + xy + 3x + 2y + 5$ has a stationary point, but it has no extreme value.

Solution Let $f = x^2 + xy + 3x + 2y + 5$. Then

$$\frac{\partial f}{\partial x} = 2x + y + 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 2.$$

Also,

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad t = \frac{\partial^2 f}{\partial y^2} = 0.$$

Now, $\partial f/\partial x = \partial f/\partial y = 0$ for the existence of any stationary point. Hence solving $2x + y + 3 = 0$ and $x + 2 = 0$, we get $x = -2$, $y = 1$. Therefore, $(-2, 1)$ is the stationary point but $rt - s^2 = 0 - 1^2 = -1 < 0$.

Thus the function cannot have any extreme value.

Example 9.41 Find the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = d$.

Solution Since $ax + by + cz = d$,

$$c^2z^2 = (ax + by - d)^2 \tag{1}$$

From the given function, we get

$$\frac{\partial f}{\partial x} = 2\left[x + \frac{a}{c^2}(ax + by - d)\right] = \frac{2}{c^2}\left[(a^2 + c^2)x + aby - ad\right]$$

$$\frac{\partial f}{\partial y} = 2\left[y + \frac{b}{c^2}(ax + by - d)\right] = \frac{2}{c^2}\left[(b^2 + c^2)y + abx - bd\right]$$

Now for maxima and minima,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

or

$$(a^2 + c^2)x + aby = ad \quad \text{and} \quad (b^2 + c^2)y + abx = bd$$

Solving them, we obtain

$$(a^2 + c^2)x + \frac{ab}{b^2 + c^2}(bd - abx) = ad$$

or

$$\left[(a^2 + c^2)(b^2 + c^2) - a^2b^2\right]x = ad\left[(b^2 + c^2 - b^2)\right]$$

or

$$(a^2 + b^2 + c^2)c^2x = adc^2$$

or

$$x = \frac{ad}{a^2 + b^2 + c^2}$$

Similarly

$$y = \frac{bd}{a^2 + b^2 + c^2} \quad \text{and} \quad z = \frac{cd}{a^2 + b^2 + c^2}. \tag{2}$$

We have

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2}{c^2}(a^2 + c^2) = 2\left(\frac{a^2}{c^2} + 1\right) > 0$$

$$s = \frac{\partial^2 f}{\partial x^2} = \frac{2ab}{c^2}$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2}{c^2}(b^2 + c^2) = 2\left(\frac{b^2}{c^2} - 1\right)$$

Then

$$rt - s^2 = 4\left[\left(\frac{a^2}{c^2} + 1\right)\left(\frac{b^2}{c^2} + 1\right) - \frac{a^2b^2}{c^4}\right] = 4\left(\frac{a^2}{c^2} + \frac{b^2}{c^2} + 1\right) > 0.$$

Thus $r > 0$ $rt - s^2 > 0$. Hence f is minimum for the values of x, y, z obtained in (2). Therefore, the required minimum value is

$$x^2 + y^2 + z^2 = \frac{(a^2 + b^2 + c^2)d^2}{(a^2 + b^2 + c^2)^2} = \frac{d^2}{a^2 + b^2 + c^2}. \tag{3}$$

Example 9.42 If $a > 0$, prove that $xy(a - x - y)$ is maximum when $x = y = a/3$.

Solution Here $f = axy - x^2y - xy^2$. Therefore,

$$\frac{\partial f}{\partial x} = ay - 2xy = y(a - 2x - y)$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy = x(a - x - 2y)$$

Since $x = y = a/3$ satisfies $\partial f/\partial x = \partial f/\partial y = 0$, $(x, y) = (a/3, a/3)$ is a stationary point. Now,

$$r = \frac{\partial^2 f}{\partial x^2} = -2y = -\frac{2}{3}a < 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2y - 2y = -\frac{1}{3}a$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x = -\frac{2}{3}a.$$

Then

$$rt - s^2 = -\phi\left(-\frac{2}{3}a\right)\left(-\frac{2}{3}a\right) - \frac{1}{9}a^2 = \frac{1}{3}a^2 > 0.$$

Thus $r < 0$, $rt - s^2 > 0$. Here the given function is maximum where $x = y = a/3$, where $a > 0$.

Example 9.43 Find the point within the triangle from which the sum of the squares of its perpendicular distances from the sides is least.

Solution Let ABC be a triangle and P be point inside the triangle from which the perpendiculars PL, PM, PN are drawn to the side BC, CA and AB respectively (Fig. 9.16). Let $PL = x$, $PM = y$, $PN = z$, also let

$$u = x^2 + y^2 + z^2$$

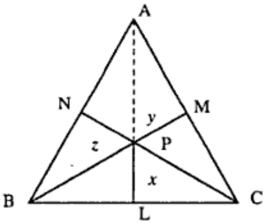


Fig. 9.16 Solution to Example 9.43.

Now, we have to find the minimum value of u. Here

$$\begin{aligned} \text{ar. } \Delta\text{ABC} &= \text{ar. } \Delta\text{PBC} + \text{ar. } \Delta\text{PCA} + \text{ar. } \Delta\text{PAB} \\ &= \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz \\ &= \frac{1}{2}(ax + by + cz) \end{aligned}$$

or

$$2\Delta = ax + by + cz. \tag{2}$$

Therefore, from (1),

$$u = x^2 + y^2 + \left(\frac{2\Delta - ax - by}{c}\right)^2 \tag{3}$$

Then

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x + 2\frac{2\Delta - ax - by}{c^2}(-a) \\ &= 2\left[x - \frac{a(2\Delta - ax - by)}{c^2}\right] \\ &= 2\frac{c^2x - 2a\Delta + a^2x + aby}{c^2} \\ &= 2\frac{(c^2 + a^2)x + aby - 2\Delta a}{c^2} \end{aligned}$$

and

$$\frac{\partial u}{\partial y}=2y+2\frac{2\Delta-ax-by}{c^2}(-b)$$

$$=2\left[y-\frac{b(2\Delta-ax-by)}{c^2}\right]$$

$$=2\frac{c^2y-2b\Delta+abx+b^2y}{c^2}$$

$$=2\frac{(c^2+b^2)y+abx-2b\Delta}{c^2}.$$

For maximum or minimum value of u, $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$. Therefore,

$$(c^2+a^2)x+aby-2a\Delta=0 \quad \text{and} \quad (c^2+b^2)y+abx-2b\Delta=0$$

Solving, we get

$$(c^2+a^2)x+aby=2a\Delta \tag{4}$$

$$(c^2+b^2)y+abx=2b\Delta \tag{5}$$

Multiplying (4) by $(c^2 + b^2)$ and (5) by ab and then subtracting, we obtain

$$[(c^2+a^2)(c^2+b^2)-a^2b^2]x=2a\Delta(c^2+b^2)-(2b\Delta)ab$$

or

$$[c^4+c^2(a^2+b^2)]x=2\Delta(ac^2+ab^2-ab^2)$$

or

$$c^2(c^2+a^2+b^2)\,x=2\Delta ac^2$$

or

$$x=\frac{2\Delta a}{a^2+b^2+c^2} \tag{6}$$

Similarly

$$y=\frac{2b\Delta}{a^2+b^2+c^2} \tag{7}$$

$$z=\frac{2c\Delta}{a^2+b^2+c^2} \tag{8}$$

Thus u will be the greatest and least for the point P whose perpendicular distance from the sides are given by (6), (7), (8).

For maximum or minimum,

$$r=\frac{\partial^2 u}{\partial x^2}=\frac{2}{c^2}(c^2+a^2)=2\left(1+\frac{a^2}{c^2}\right),$$

$$S=\frac{\partial^2 u}{\partial x\partial y}=\frac{2ab}{c^2},$$

$$t=\frac{\partial^2 u}{\partial y^2}=\frac{2}{c^2}(c^2+b^2)=2\left(1+\frac{b^2}{c^2}\right).$$

and

$$rt - S^2 = 2\left(1+\frac{a^2}{c^2}\right)2\left(1+\frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}+\frac{a^2b^2}{c^4}-\frac{a^2b^2}{c^4}\right)$$

$$= 4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)$$

$$= 4\left(\frac{a^2+b^2+c^2}{c^2}\right) > 0.$$

Thus we get $r > 0$ and $rt - S^2 > 0$. Hence u is minimum.

Example 9.44 Show that the maxima and minima of the fraction

$$u = \frac{ax^2+2hxy+by^2+2gx+2fy+c}{a'x^2+2h'xy+b'y^2+2g'x+2g'y+c'}$$

are given by the roots of the equation

$$\begin{vmatrix} a-a'u & h-h'u & g-g'u \\ h-h'u & b-b'u & f-f'u \\ g-g'u & f-f'u & c-c'u \end{vmatrix} = 0.$$

Solution We have,

$$u(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c') = ax^2+2hxy+by^2+2gx+2fy+c \quad (1)$$

Differentiating partially with respect to x and y respectively,

$$\frac{\partial u}{\partial x}(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c') + u(2a'x+2h'y+2g')$$
$$= 2ax+2hy+2g \quad (2)$$

and

$$\frac{\partial u}{\partial y}(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c^2) + u(2h'x+2b'y+2f')$$
$$= 2hx+2by+2f \quad (3)$$

For maxima and minima

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0\cdot$$

Then

$$u(a'x+h'y+zg') = ax+hy+g \quad (4)$$

$$u(h'x+b'y+f') = hx+by+f \quad (5)$$

Putting λ_1 in (4), (5) and (6), we get

$$x - uax + l\lambda_2 = 0 \quad \text{or} \quad x = \frac{l\lambda_2}{au - t}.$$

Similarly

$$y = \frac{m\lambda_2}{bu - l} \quad \text{and} \quad z = \frac{n\lambda_2}{cu - l}$$

Putting the values x, y and z in $lx + my + nz = 0$, we get

$$\frac{l^2}{au - l} + \frac{m^2}{bu - l} + \frac{n^2}{cu - 1} = 0.$$

This determines the values of u which are either maximum or minimum.

Example 9.46 Find the maximum and minimum value of u, when

$$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad lx + my + nz = 0.$$

Solution Here

$$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \tag{1}$$

has the maximum or minimum values with the two given conditions

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{2}$$

and

$$\psi(x, y, z) = lx + my + nz = 0. \tag{3}$$

Let $\omega = u + \lambda\phi + \mu\psi$, so that we obtain

$$\omega = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + \mu(lx + my + nz).$$

For the maximum and minimum values, $\omega_x = \omega_y = \omega_z = 0$. Therefore,

$$\frac{2x}{a^4} + \frac{2\lambda x}{a^2} + \mu l = \frac{2y}{b^4} + \frac{2\lambda y}{b^2} + \mu m = \frac{2z}{c^4} + \frac{2\lambda z}{c^2} + \mu n = 0 \tag{4}$$

or

$$x\left(\frac{2x}{a^4} + \frac{2\lambda x}{a^2} + \mu l\right) + y\left(\frac{2y}{b^4} + \frac{2\lambda y}{b^2} + \mu m\right) + z\left(\frac{2z}{c^4} + \frac{2\lambda z}{c^2} + \mu n\right) = 0$$

or

$$2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \mu(lx + my + nz) = 0$$

Differentiating (2), we find

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0\cdot \tag{4}$$

Multiplying (3) by 1 and (4) by λ and adding, we get

$$\left(\frac{1}{x}+\frac{\lambda x}{a^2}\right)dx+\left(\frac{1}{y}+\frac{\lambda y}{b^2}\right)dy+\left(\frac{1}{z}+\frac{\lambda z}{c^2}\right)dz=0$$

Equating to zero, the coefficient of dx, dy, dz, we get

$$\frac{1}{x}+\frac{\lambda x}{a^2}=0,\quad \frac{1}{y}+\frac{\lambda y}{b^2}=0,\quad \frac{1}{z}+\frac{\lambda z}{c^2}=0\cdot$$

Multiplying these by x, y, z respectively and adding, we get

$$3+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0 \quad \text{or} \quad \lambda=-3$$

Therefore,

$$x=\frac{a}{\sqrt{3}},\quad y=\frac{b}{\sqrt{3}},\quad z=\frac{c}{\sqrt{3}}$$

We also find

$$v=8xyz=\frac{8abc}{3\sqrt{3}}$$

Now, we have to find out whether this volume is maximum or minimum. Differentiating (2) partially with respect to x, regarding z as dependent variable, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial z}{\partial x}=-\frac{c^2x}{c^2z}\cdot$$

Now,

$$\frac{\partial V}{\partial x}=8yz+8xy\frac{\partial z}{\partial x}=8yz+8xy\left(-\frac{c^2x}{a^2z}\right).$$

Then

$$r=\frac{\partial^2 V}{\partial x^2}=8y\frac{\partial z}{\partial x}-\frac{16c^2x}{a^2z}+8\frac{x^2y}{z^2}\frac{c^2}{a^2}\frac{\partial z}{\partial x}$$

$$=8y\left(-\frac{c^2x}{a^2z}\right)-\frac{16c^2x}{a^2z}+\frac{8x^2yc^2}{a^2z^2}\left(-\frac{c^2x}{a^2z}\right)$$

Here $r<0$, when

$$x=\frac{a}{\sqrt{3}},\quad y=\frac{b}{\sqrt{3}},\quad z=\frac{c}{\sqrt{3}}.$$

Hence V is maximum and its value is

$$\frac{8}{3\sqrt{3}}abc.$$

Example 9.48 Find the maximum and minimum values of $x^2 + y^2$, where $ax^2 + 2hxy + by^2 = 1$ are given by the roots of the quadratic equation

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2.$$

Solution We have

$$r^2 = x^2 + y^2 \tag{1}$$

and

$$ax^2 + 2hxy + by^2 = 1 \tag{2}$$

For maximum and minimum of r, we have

$$dr = 0 \quad \text{or} \quad x\,dx + y\,dy = 0 \tag{3}$$

Differentiation of (2) gives

$$ax\,dx + hy\,dx + hx\,dy + by\,dy = 0$$

or

$$(ax + hy)\,dx + (hx + by)\,dy = 0 \tag{4}$$

Multiplying (3) by λ and (4) by 1 and adding, we get

$$(ax + hy + \lambda x)\,dx + (hx + by + \lambda y)\,dy = 0$$

Equating to zero, the coefficient of dx and dy, we get

$$ax + hy + \lambda x = 0 \tag{5}$$

$$hx + by + \lambda y = 0 \tag{6}$$

Multiplying (5) by x and (6) by y and adding, we get

$$(ax^2 + 2hxy + by^2) + \lambda(x^2 + y^2) = 0$$

Then

$$1 + \lambda r^2 = 0 \quad \text{or} \quad \lambda = -\frac{1}{r^2}.$$

Putting the value of λ in (5) and (6), we get

$$ax + hy - \frac{1}{r^2}x = 0 \quad \text{or} \quad hy = \left(\frac{1}{r^2} - a\right)x$$

and

$$hx + by - \frac{1}{r^2}y = 0 \quad \text{or} \quad hx = \left(\frac{1}{r^2} - b\right)y$$

28. Divide a number n into three parts x, y, z such that $ayz + bzx + cxy$ shall have maximum or minimum.

29. Find the minimum value of $u = x + y + z$, when $a/x + b/y + c/z = 1$.

30. Find the maximum value of u when $u = x^2y^3z^4$ and $2x + 3y + 4z = a$.

31. If $(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$ and $lx + my + nz = 0$, show that the maximum or minimum values of $r^2 = x^2 + y^2 + z^2$ are given by the equation

$$\frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0 \cdot$$

32. Prove that if the perimeter of a triangle is constant, its area is minimum when it is equilateral.

33. Find the triangle of maximum area which can be inscribed in a circle.

34. The sum of the three numbers is constant. Prove that their product is maximum when they are equal.

35. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

36. Find the lengths of the axes of the section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by the plane $lx + my + nz = 0$.

37. Find the maximum value of $yz + 2zx + 3xy$ if $x + y + z = 1$.

38. By Lagrange's method, prove that the triangle of maximum area inscribed in a circle is equilateral.

39. Find the maximum and minimum values of u for $u = a^2x^2 + b^2y^2 + c^2z^2$, $x^2 + y^2 + z^2 = 1$, $lx + my + nz = 0$.

40. By Lagrange's method of undetermined multiplier, find the maximum value of x^2y^3 when $x + y = 10$.

41. Find the minimum value of $x^2 + y^2 + z^2$ such that $x + y + z = 2x + 3y + 4z = 1$.

42. Determine the stationary values of $u = x^2 + y^2 + z^2$ under the condition $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = 0$.

43. If $x + y + z = 1$, prove that $ayz + bzx + cxy$ has an extreme value equal to

$$\frac{abc}{2(bc + ca + ab) - (a^2 + b^2 + c^2)} \cdot$$

44. If $ax^2 + 2hxy + ay^2 = d$, prove that the greatest and the least values of $x^2 + y^2$ are $d(a - h)$ and $d/(a + h)$ respectively.

45. Prove that the greatest rectangular solid inscribed in the sphere $x^2 + y^2 + z^2 = 24$ is a cube. Find its volume.

46. Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$.

Chapter 10

Envelopes

10.1 Introduction

Envelopes are the locus of points of intersection of any two curves for all values of the parameter when the parameter is eliminated.

In other words, the envelope of a family of curves is the locus of the limiting positions of the points of intersection of any two curves when one of them tends to coincide with the other which is kept fixed.

10.2 Equation of an Envelope

Let $f(x, y, c) = 0$ be any given family of curves. Consider the two curves:

$$f(x, y, c) = 0 \tag{10.1}$$

and

$$f(x, y, c + \delta c) = 0. \tag{10.2}$$

Expanding $f(x, y, c + \delta c) = 0$, we get

$$\phi(x, y, c) + \delta c \frac{\partial}{\partial c} f(x, y, c) + \cdots = 0$$

Hence in the limit when $\delta c \to 0$ and $f(x, y, c) = 0$, we have

$$\frac{\partial}{\partial c} f(x, y, c) = 0$$

as the equation of the curve passing through the point of intersection of the curves (10.1) and (10.2). If we eliminate c between the equations:

$$f(x, y, c) = 0 \tag{10.3}$$

and

$$\frac{\partial}{\partial c} f(x, y, c) = 0, \tag{10.4}$$

we get the locus of that point of intersection for all values of parameter c. Eliminating c between (10.3) and (10.4) leads to an equation

$$f(x, y) = 0,$$

which is the required envelope.

Note: To obtain the envelop of the family of curves $f(x, y, c) = 0$, eliminate c between $f(x, y, c) = 0$ and $f_c(x, y, c) = 0$, where $f_c(x, y, c)$ is the partial derivatives of $f(x, y, c)$ with respect to c.

Example 10.1 Find the envelope of the family of lines

$$y - cx - \frac{a}{c} = 0 \tag{1}$$

Solution We eliminate c between (1) and the partial derivatives of (1), i.e. $-x + a/c^2 = 0$ with respect to c. The eliminant is

$$y^2 = 4ax,$$

which is the envelope of the given family of lines.

Example 10.2 Find the envelope of the family of straight lines

$$y = mx + \frac{a}{m}, \tag{1}$$

where m is the parameter and a is any constant.

Solution Let there be two parametric values of m_1 and $m_1 + \delta m$ of the parameter m. Therefore, corresponding to these parametric values, the two members of the family are

$$y = m_1 x + \frac{a}{m_1} \tag{2}$$

and

$$y = (m_1 + \delta m)x + \frac{a}{m_1 + \delta m} \tag{3}$$

From these, we have

$$m_1 x + \frac{a}{m_1} = (m_1 + \delta m)x + \frac{a}{m_1 + \delta m}$$

or

$$m_1 x - (m_1 + \delta m)x = \frac{a}{m_1 + \delta m} - \frac{a}{m_1}$$

or

$$x(m_1 - m_1 - \delta m) = \frac{am_1 - a(m_1 + \delta m)}{m_1(m_1 + \delta m)}$$

or

$$x = \frac{a}{m_1(m_1 + \delta m)}$$

Now substituting the value of x in (2), we get

$$y = \frac{a(2m_1 + \delta m)}{m_1(m_1 + \delta m)}$$

Hence the two lines (2) and (3) intersect at

$$\left[\frac{a}{m_1(m_1 + \delta m)}, \frac{a(2m_1 + \delta m)}{m_1(m_1 + \delta m)}\right]$$

Since $\delta m \to 0$, in this case, this point of intersection is

$$\left(\frac{a}{m_1^2}, \frac{2a}{m_1}\right),$$

which lies on (2). This point is the limiting position of the point of intersection of the lines (2) with another line of the family when the latter tends to coincide with the former. Similarly, there will be a point on every line and so the locus of such points is called the envelope of the given family of lines.

Since

$$x = \frac{a}{m^2}, \quad y = \frac{2a}{m}$$

or

$$m^2 = \frac{a}{x}, \quad m = \frac{2a}{y},$$

we have

$$m^2 = \frac{a}{x} = \frac{4a^2}{y^2} \quad \text{or} \quad y^2 = 4ax.$$

This is the envelope of the given family of lines

Theorem 10.1 The evolute of a curve is the envelope of its normals.

Proof Let there be a curve, PR and QR are the normals to the curves, and PT and QL be the tangents at two points P, Q of a curve AB (Fig. 10.1). Let L be the point of intersection of the tangents PT and QL, such that $\angle PRQ = \angle TLQ = \delta\psi$ and arc $PQ = \delta S$.

From ΔPQR, using the sine formula, we get

$$\frac{PR}{PQ} = \frac{\sin RQP}{\sin PRQ}$$

or

$$PR = \sin RQP \frac{PQ}{\sin \delta\psi} = \sin RQP \frac{\text{chord } PQ}{\text{arc } PQ} \frac{\delta S}{\delta\psi} \frac{\delta\psi}{\sin \delta\psi}$$

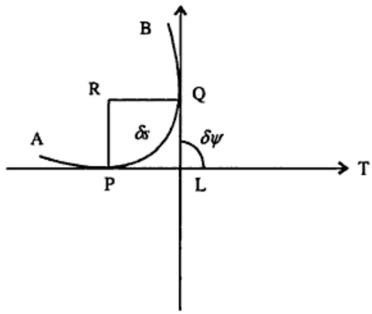


Fig. 10.1 Evolute of a curve.

Now let Q → P so that

$$\angle RQP \to \angle RPT = \frac{\pi}{2}$$

Therefore,

$$\lim_{Q \to P} PR = \sin\frac{\pi}{2}(1)\frac{dS}{d\psi}(1) = \frac{dS}{d\psi} = \rho\,.$$

Hence the limiting position of R, which is the intersection of the normals at P and Q, is the centre of the curvature at P.

Theorem 10.2 Prove that, in general, the envelope touches each of the intersecting members of the family.

Proof Let A, B, C be three consecutive intersecting members of the family. Let P be the point of intersection of A and B and Q be the point of intersection of B and C (Fig. 10.2). By the definition, P and Q are the points on the envelope. Hence the curve B and the envelope have two points common and so we have a common tangent and which touches each other.

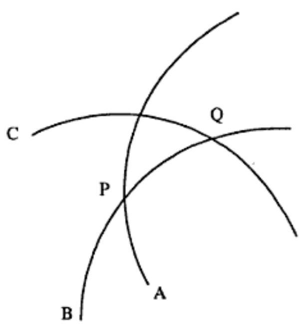


Fig. 10.2 Envelope.

Example 10.3 The evolute of a curve is the envelope of the normal. Find the evolute of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution We know that the equation of the normal at any point (a $\cos\theta$, $b\sin\theta$) on the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \tag{1}$$

where θ is the parameter. Differentiating (1) partially with respect to θ, we get

$$\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0 \tag{2}$$

Eliminating θ between (1) and (2), we get

$$\tan\theta=-\frac{(by)^{1/3}}{(ax)^{1/3}}$$

or

$$\frac{\sin\theta}{\cos\theta}=-\frac{(by)^{1/3}}{(ax)^{1/3}}$$

or

$$\frac{\sin\theta}{-(by)^{1/3}}=\frac{\cos\theta}{(ax)^{1/3}}=\frac{\sqrt{\sin^2\theta+\cos^2\theta}}{\sqrt{(by)^{2/3}+(ax)^{2/3}}}=\pm\frac{1}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Therefore,

$$\sin\theta=\pm\frac{(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}},\quad \cos\theta=\pm\frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substituting these values in (1), we get

$$\left[(ax)^{2/3}+(by)^{2/3}\right]\left[(ax)^{2/3}+(by)^{2/3}\right]^{1/2}=a^2-b^2$$

or

$$\left[(ax)^{2/3}+(by)^{2/3}\right]^{3/2}=a^2-b^2$$

or

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3},$$

which is the envelope of the normal required.

Example 10.4 Find the envelope of the straight line

$$\frac{x}{a}+\frac{y}{b}=1,$$

where a and b are variable parameters, connected by the relation $a + b = c$, c being a non-zero constant.

Solution Since $a + b = c$, $b = c - a$. Hence the equation of the line becomes

$$\frac{x}{a}+\frac{y}{c-a}=1 \tag{1}$$

Differentiating partially with respect to a, we get

$$-\frac{x}{a^2}+\frac{y}{(c-a)^2}=0$$

or

$$\frac{(c-a)^2}{a^2}=\frac{y}{x}$$

or

$$\frac{c-a}{a}=\frac{y^{1/2}}{x^{1/2}}$$

so that

$$a=\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}} \quad \text{and} \quad c-a=\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Substituting these values in (1), we get

$$\frac{x}{c\sqrt{x}/\left(\sqrt{x}+\sqrt{y}\right)}+\frac{y}{c\sqrt{y}/\left(\sqrt{x}+\sqrt{y}\right)}=1$$

or

$$\frac{x\left(\sqrt{x}+\sqrt{y}\right)}{c\sqrt{x}}+\frac{y\left(\sqrt{x}+\sqrt{y}\right)}{c\sqrt{y}}=1$$

or

$$\frac{\sqrt{x}+\sqrt{y}}{c}\left(\sqrt{x}+\sqrt{y}\right)=1$$

or

$$\sqrt{x}+\sqrt{y}=\sqrt{c}$$

or

$$\left(y-x-c\right)^2=4cx.$$

Substituting this in (1), we have

$$x^2 + y^2 - 2a\left(-\frac{y}{x}\right)^2 x - 4ay\left(-\frac{y}{x}\right) = 0$$

or

$$x^3 + xy^2 + 2ay^2 = 0$$

or

$$y^2(x + 2a) + x^3 = 0$$

which is the required envelope.

Example 10.7 Given that $x^{2/3} + y^{2/3} = c^{2/3}$ is the envelope of $x/a + y/b = 1$, find the necessary relation between a and b.

Solution Given that $x^{2/3} + y^{2/3} = c^{2/3}$ we have, after differentiation

$$\frac{dx}{x^{1/3}} + \frac{dy}{y^{1/3}} = 0 \tag{1}$$

Again, from $x/a + y/b = 1$, we have

$$\frac{dx}{a} + \frac{dy}{b} = 1 \tag{2}$$

Comparing the coefficients of dx and dy from (1) and (2), we have

$$\frac{1/x^{1/3}}{1/a} = \frac{1/y^{1/3}}{1/b}$$

or

$$\frac{x^{1/3}}{a} = \frac{y^{1/3}}{b} = \lambda \text{ (say)}$$

Therefore,

$$x^{1/3} = \lambda a \quad \text{and} \quad y^{1/3} = \lambda b$$

or

$$\frac{1}{a} = \frac{\lambda}{x^{1/3}}$$

or

$$\frac{x}{a} = x\frac{\lambda}{x^{1/3}} = \lambda x^{2/3}$$

Similarly,

$$\frac{y}{b} = \lambda y^{2/3}$$

Therefore,

$$\frac{x}{a}+\frac{y}{b}=\lambda\left(x^{2/3}+y^{2/3}\right)$$

or

$$\lambda=\frac{1}{c^{2/3}}$$

Then

$$a=\frac{1}{\lambda}x^{1/3}=c^{2/3}x^{1/3} \quad \text{and} \quad b=c^{2/3}y^{1/3}$$

Therefore,

$$a^2+b^2=c^{4/3}x^{2/3}+c^{4/3}y^{2/3}=c^{4/3}(x^{2/3}+y^{2/3})=c^{4/3}c^{2/3}$$

Then

$$a^2+b^2=c^2,$$

which is the required relation.

Example 10.8 Show that the pedal equation of the envelope of the line $x \cos m\alpha + y \sin m\alpha = a \cos n\alpha$, $(m \neq n)$, where α is a parameter, is

$$p^2=\frac{m^2r^2-n^2a^2}{m^2-n^2}.$$

Solution Let

$$F(\alpha) = x \cos m\alpha + y \sin m\alpha - a \cos n\alpha = 0. \tag{1}$$

Differentiating partially with respect to α, we have

$$\frac{\partial F}{\partial \alpha}=-(x \sin m\alpha)\, m+(y \cos m\alpha)m+(a \sin n\alpha)n=0$$

or

$$x \sin m\alpha - y \cos m\alpha - \frac{an}{m}\sin n\alpha = 0. \tag{2}$$

Every member of the family must touch the envelop. So (1) will be tangent to the envelope. Now (1) is of the form

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Then p and r for the envelope are given by

$$p = \alpha \cos n\alpha \quad \text{and} \quad r^2 = x^2 + y^2$$

in polar form, whose pole is F. Let the polar coordinate of P is (r, Q), then we have to construct a parabola with F as focus and P as vertex. The directrix of this parabola will be QR which is perpendicular to FP at Q such that $FP = PQ$.

Hence the equation will be

$$r\cos(\theta - \theta') = 2FP = 2\frac{l}{1+\cos\theta'} \tag{2}$$

Taking log both sides, we get

$$\log r + \log\cos(\theta - \theta') = \log 2l - \log(1+\cos\theta')$$

Differentiating this with respect to θ', we have

$$\frac{\sin(\theta-\theta')}{\cos(\theta-\theta')} = \frac{\sin\theta'}{1+\cos\theta'} = \frac{2\sin(\theta'/2)\cos(\theta'/2)}{2\cos^2(\theta'/2)} = \tan\frac{\theta'}{2}$$

or

$$\tan(\theta-\theta') = \tan\frac{\theta'}{2}$$

or

$$\theta - \theta' = \frac{\theta'}{2}$$

or

$$\theta = \frac{\theta'}{2} + \theta' = \frac{3}{2}\theta' \tag{3}$$

For the envelope of the directrix, we have to eliminate θ' between (3) and (2)

$$r\cos\left(\theta - \frac{2}{3}\theta\right) = \frac{2l}{1+\cos(2\theta/3)} = \frac{2l}{2\cos^2(\theta/3)}$$

or

$$r\cos\frac{1}{3}\theta = \frac{l}{\cos^2(\theta/3)}$$

or

$$r\cos^3\frac{\theta}{3} = l\,.$$

Example 10.10 Find the envelope of straight lines drawn at right angles to the radii vectors of the cardioide $r = a(1 + \cos\theta)$, through their extremities.

Solution Let P be any point on the curve. If α be its vectorial angle, then the radius vector is

$$OP = a(1 + \cos\alpha).$$

The equation of the line drawn through P at right angles to the radius vector OP is

$$r\cos(\theta - \alpha) = a(1 + \cos\alpha) \tag{1}$$

Differentiating (1) with respect to α, we get

$$r \sin (\theta - \alpha) = -a \sin \alpha \qquad (2)$$

Eliminating (1) and (2), we get the required envelope. Rewriting (1) and (2), we have

$$(r \cos\theta - \alpha)\cos \alpha + r\sin\theta \sin\alpha = a \qquad (3)$$

$$r \sin \theta \cos\alpha - (r \cos\theta - a)\sin\alpha = 0 \qquad (4)$$

Now from (4), we have

$$\tan\alpha = \frac{r \sin\theta}{r \cos\theta - a}$$

Therefore,

$$\sin\alpha = \frac{r\sin\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}, \qquad \cos\alpha = \frac{r\cos\theta - a}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

Substituting in (3), we get

$$\frac{(r \cos\theta - a)^2 + r^2 \sin^2\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = a$$

or

$$r^2 + a^2 - 2ar \cos\theta = a^2$$

or

$$r = 2a \cos\theta,$$

which is the required envelope and is a circle.

Example 10.11 On any radius vector of the curve $r^2 = a^2 \cos 2\theta$ as diameter, a circle is described. Find the envelope of all such circles.

Solution Let P be any point on the given curve $r^2 = a^2 \cos 2\theta$, whose coordinate in polar form be (r_1, θ_1). Therefore, we have

$$r_1^2 = a^2 \cos 2\theta_1 \qquad (1)$$

The circle on OP, as diameter, has its equation

$$r = r_1 \cos(\theta - \theta_1) = a\sqrt{\cos 2\theta_1} \cos(\theta - \theta_1) \qquad (2)$$

Differentiating partially with respect to θ_1, we get

$$O = \frac{a(-\sin 2\theta_1)}{\sqrt{\cos 2\theta_1}} \cos(\theta - \theta_1) + a\sqrt{\cos 2\theta_1} \sin(\theta - \theta_1)$$

or

$$-\sin 2\theta \cos (\theta - \theta_1) + \cos 2\theta_1 \sin (\theta - \theta_1)$$

or

$$\sin (\theta - \theta_1 - 2\theta_1) = 0$$

or

$$\sin(\theta - 3\theta_1) = 0$$

or

$$\theta - 3\theta_1 = 0$$

Thus

$$\theta_1 = \frac{1}{3}\theta \tag{3}$$

Eliminating θ, between (2) and (3), we have

$$r = a\sqrt{\cos\frac{2}{3}}\cos\frac{2}{3} \quad \text{or} \quad r^2 = a^2\cos^3\frac{2}{3}\theta.$$

Exercises 10.1

1. Find the envelope of the following families of lines:
   (i) $y = mx + \sqrt{(a^2m^2 + b^2)}$, m being the parameter,
   (ii) $x\cos^3\theta + y\sin^3\theta = a$, θ being the parameter,
   (iii) $x\sin\theta - y\cos\theta = a\theta$, θ being the parameter.

2. Find the envelope of the family

$$x^2(x - a) + (x + a)(y - m)^2 = 0,$$

   where a is constant and m is a parameter.

3. Find the envelope of the family of semi-cubical parabola $y^2 - (x + a)^3 = 0$.

4. Find the envelope of the family of straight line $A\alpha^2 + B\alpha + C = 0$, where α is the variable parameter and A, B, C are linear functions of x, y.

5. Find the envelope of the family of straight lines

$$\frac{x}{a} + \frac{y}{b} = 1$$

   where a, b are connected by the relation (i) $a^2 + b^2 = c^2$, (ii) $ab = c^2$.

6. Show that the relation between a and b so that the envelope of the line

$$\frac{x}{a} + \frac{y}{b} = 1$$

   may be the curve $x^py^q = k^{p+q}$ is

$$a^pb^qp^pq^q = (p+q)^{p+q}k^{p+q}.$$

7. Find the envelope of the family of curves

$$\frac{a^2\cos\theta}{x} - \frac{b^2\sin\theta}{y} = c$$

   for different values of θ.

8. Find the envelope of the family of trajectories

$$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{b^2\cos^2\theta}$$

9. Show that the envelope of the family of parabolas

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

under the condition (i) $ab = c^2$ is a hyperbola having its asymptotes coinciding with axes and (ii) $a + b = c$ is an astroid.

10. Find the envelope of the family of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the parameters a and b are connected by the relation: (i) $a + b = c$, (ii) $a^2 + b^2 = c^2$.

11. Show that the envelope of the ellipses

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1,$$

where the parameters α, β are connected by the relation

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1,$$

is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4.$$

12. Prove that the equation of the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ may be written in the form

$$x \sin \phi - y \cos \phi + a \cos 2\phi = 0$$

and find the envelope of the family of the normals.

13. Find the equation of the normal at any point of the curve

$$x = a\,(3\cos t - 2\cos^3 t), \qquad y = a(3\sin t - 2\sin^3 t)$$

and also find the equation of its evolute.

14. Show that the envelope of the family of ellipses

$$a^2x^2\sec^4\alpha + b^2y^2\operatorname{cosec}^4\alpha = (a^2 - b^2)^2,$$

where α is the parameter, is the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

24. Find the envelope of the circles described on the radii vectors of the following curves as diameter:

(i) $l/r = 1 + e\cos\theta$, (ii) $r^n = a^n \cos n\theta$.

25. Show that the pedal equation of the envelope of the line

$$x\cos 2\theta + y\sin 2\theta = 2\cos\theta \quad \text{is} \quad p^2 = \frac{4}{3}(r^2 - a^2).$$

26. Find the pedal equation of the envelope of the family of straight lines

$$x\cos\lambda t + y\sin\lambda t = a\cos\mu t,$$

where t is a variable parameter and λ, μ, a are constants.

27. An equilateral triangle moves so that two of its sides pass through two fixed points. Prove that the envelope of the third side is a circle.

28. Let circles are described having diameters equal to the radii vectors of the curve $x^3 + y^3 = 3at^2$. Prove that their envelope is the inverse of a semi-cubical parabola.

29. Show that the envelope of the common chords of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and its circle of curvature is the curve

$$\left(\frac{x}{a} + \frac{y}{b}\right)^{2/3} + \left(\frac{x}{a} - \frac{y}{b}\right)^{2/3} = 2.$$

30. If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate semi-diameters of an ellipse and a, b, semi-axes of the ellipse, prove that

$$(\rho_1^{2/3} + \rho_2^{2/3})\, a^{2/3} b^{2/3} = a^2 + b^2$$

31. Show that the family of circles $(x - a)^2 + y^2 = a^2$ has no envelope.

Chapter 11

Curve Tracing

11.1 Introduction

In this chapter, we will deal with the graphs of the curves of given equations in Cartesian or polar systems of coordinates. The main purpose of this chapter is to point out those rules which are used in tracing the graph of a curve. After describing the main rules of curve tracing and afterwards we will use them in tracing the graph of aforesaid curves. The graph of a given function is helpful in giving a visual presentation of the behaviour of the function involving the study of symmetries of asymptotes, the intervals of rising up or falling down and of the cavity upwards and downwards, etc. Curve tracing means that the equations of curves which we trace and are generally solvable for y, x or r. The case may come that some equations are not solvable for y or x, then we solve them for r by transforming from Cartesian to polar system.

11.2 Rules for Tracing Cartesian Curves

Curves passing through the origin

Firstly, we should see whether the curve passes through the origin or not. For this, we put $x = 0$, $y = 0$ in the equation of the given curve. If both the sides of the equation of the curve are satisfied then we say that the curve passes through the origin (0, 0).

In other words, we know that if any curve passes through the origin then there will be no constant term in the equations of the curve. For example, the curve $y^2 = 4ax$ passes through the origin

Symmetry of curves

The symmetry of a curve may occur in the following important ways:

(a) If the equation of the curve contains only even powers of y (that the equation of the curve does not change by putting $-y$ for y in it), then the curve is said to be symmetrical about the x-axis. It means that the portion of the curve above the x-axis is exactly the same below the x-axis.

(b) If the equation of the curve contains only even powers of x, then the curve is said to be symmetrical about the y-axis.

(c) If the equation of the curve contains only even powers of x and y, then the curve is said to be symmetrical about both the axes. For example, $x^2/a^2 + y^2/b^2 = 1$ is symmetrical about both the axes.

(d) If the equation remains unchanged by interchanging x and y, the curve is symmetrical about the line, $y = x$. For example, $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$ because the equation remains unchanged if we interchange x and y.

(e) If the equation remains unchanged by changing the signs of both x and y, the curve is symmetrical in opposite quadrant.

Equation of the tangent at the origin

The equation of the tangent at the origin can be found by equating the lowest term to zero. Since in the equation $y^2 = 4ax$, the lowest term is $4ax$ and so $4ax = 0$, i.e. $x = 0$.

Hence $x = 0$, i.e. y-axis is tangent to the curve at the origin.

Asymptotes to the curve, if any

Firstly, we shall find the asymptotes parallel to the axes, then the oblique asymptotes.

By equating the coefficients of the highest powers of x to zero, we get the asymptotes parallel to x-axis, i.e. if the curve is of nth degree and the term in x^n, is x^n, is absent, then the coefficient of x^{n-1} equated to zero, will give us the asymptotes parallel to x-axis. If both x^n and x^{n-1} are absent then the coefficient of x^{n-2}, equated to zero, will give us two asymptotes parallel to x-axis and so on. Similar is the case with the asymptotes parallel to y-axis

Intersection of the curve with the coordinates axes

Now we should see at what points the curve cuts the x-axis and y-axis. In order to determine the coordinates of the point of intersection with the x-axis we shall put $y = 0$ in the equations of the curve because the y coordinate of any point situtated on the x-axis is zero. Similarly, in order to find out the coordinates of the point of intersection with the y-axis we shall put $x = 0$ in the equation of the curve.

Region or regions of the plane such that no part of the curve lies in it

Such a region is generally obtained by solving the equation for one variable in terms of the other, and finding out the set of the values of one variable which makes the other imaginary. Such as we shall solve y from the equation of the given curve. If, for example, suppose that by giving to x, the values $x > a$ or $x < -a$, the values of y become imaginary, then it means that there would be no portion of the curve on the right-hand side of $x = a$ or there would be no portion of the curve on the left-hand side of $x = -a$. Sometimes it happens that the values of y increases corresponding to increasing values of x. In this case, the extent of the given curve will be up to infinity.

Curves possessing point of inflexion

If the curve as traced appears to possess a point of inflection that point can be accurately located by putting

$$\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{d^2x}{dy^2} = 0$$

and solving the equation thus obtained.

The method of tracing the curve will be clear from the following examples.

Example 11.1 Trace the curve $y^2(2a - x) = x^3$.

Solution

(i) The curve is symmetry about the x-axis, since the power of y is even.

(ii) It passes through the origin, since there is no constant terms. The tangent to the curve at the origin is obtained by equating to zero the lowest term of the equation. The tangents at the origin are $y^2 = 0$, i.e. the tangents are real and coincident so that (0, 0) is a cusp.

(iii) Since the equation of the curve is of 3rd degree and as y^3 is absent. So, equating the coefficient of y^2 to zero we get $2a - x = 0$, i.e. $x = 2a$ is the only asymptote of the curve.

(iv) If $x > 0$ but $x < 2a$, $y^2 > 0$ so that the curve lies between $x = 0$ and $x = 2a$.

(v) The curve does not cut the axis. Solving for y, we have

$$y = x\sqrt{\frac{x}{2a - x}}, \qquad \frac{dy}{dx} = \pm\frac{(3a - x)\sqrt{x}}{(2a - x)^{3/2}}$$

clearly $dy/dx \neq 0$, when $0 < x < 2a$.

(vi) No point of inflexion on the curve.

(vii) Framing the table of the values of y corresponding to the values of x, we get

x	0	$a/2$	a	$3a/2$	$2a$
y	0	$a/(2\sqrt{3})$	a	$3\sqrt{3}a/2$	∞

Hence the shape of the curve is shown in Fig. 11.1.

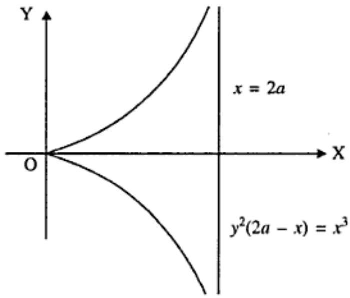


Fig. 11.1 Solution to Example 11.1.

Tracing the polar curves

(a) Symmetry:

(i) If we change θ by $-\theta$, the equation of the curve does not change, we say that the curve is symmetrical about the initial line. For example, $r = a(1 + \cos\theta)$ is symmetrical about the initial line because if we change θ by $-\theta$ the equation remains unchanged.

(ii) If by putting $\pi - \theta$ for θ, the equation of the curve does not change, we say that the curve is symmetrical about the line $\theta = \pi/2$, i.e. about the line OY.

(iii) If by putting $-r$ to r, the equation of the curve does not change, i.e. if the equation contains only the even powers of r, we say that the curve is symmetrical about the pole.

(iv) A curve symmetrical about the pole, if only even powers of r occur in the equation of the curve. For example, $r^2 = a^2 \cos 2\theta$.

(b) The curve passes through the pole: If $r = 0$ for any real value of θ, we say that the curve passes through the origin.

(c) Intersection of the curve with the initial line OX and also with OY: For this, we find out the value of r by putting $\theta = 0$ and $\theta = \pi/2$ in the equation of the curve.

(d) Region of the curve: We also determine those regions of the curve which do not contain any portion of the curve and they are determined by those values of θ for which r is imaginary.

(e) Tangent at the pole: If the curve passes through the origin, then we get the equation of the tangent at the origin corresponding to those values of θ for which $r = 0$.

(f) Asymptotes, if any: If the curve extends up to infinity, then we should also find out the asymptote to the curve from the formula in the polar form. Otherwise, it is convenient to determine the asymptote by changing the equation of the curve in Cartesian form.

(g) Form a table of values of r corresponding to values of θ, keeping attention to these values of θ for which r is zero to infinity.

(h) Sometimes it is convenient to trace out a curve by transforming the curve in the Cartesian form.

(i) As per our requirement, we find out the angle ϕ between the radius vector and the tangent by using the formula

$$\tan \phi = \frac{r\,d\theta}{dr}$$

so that we can determine the coordinates of those points, where $\phi = \pi/2$ is any specified angle.

The method of tracing the curve will be clear from the following example.

Example 11.2 Trace the curve $r = a\,(1 + \cos\theta)$.

Solution

(i) If we put $-\theta$ for θ in the equation of the curve, we find that $r = a[1 + \cos(-\theta)] = a(1 + \cos\theta)$, i.e. the equation of the curve does not change. Therefore, the given curve is symmetrical about the initial line.

(ii) If $r = 0$, the equation $1 + \cos\theta = 0$, i.e. $\cos\theta = -1$ or $\theta = \pi$. Hence the curve passes through the origin and the equation of the tangent at the pole is $\theta = \pi$, i.e. the initial line.

(iii) As r is not greater than $2a$ so that the curve wholly lies within the circle $r = 2a$.

(iv) The curve has no asymptote.

(v) Now we plot some of the points on the curve.

θ	0	$\pi/3$	$\pi/2$	π
r	$2a$	$3a/2$	a	0

We see that r continually decreases from $2a$ to 0 as θ increases from 0 to π and since there is symmetry about initial line. Hence the graph of the curve is shown as in Fig. 11.2.

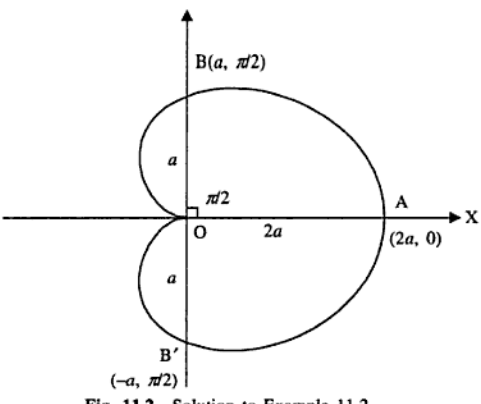


Fig. 11.2 Solution to Example 11.2.

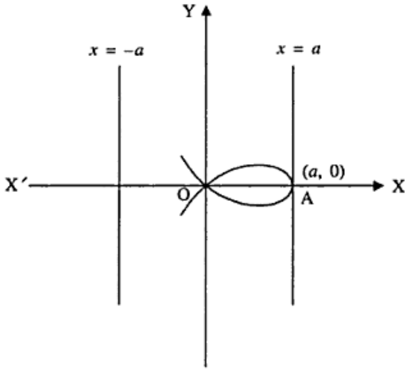


Fig. 11.3 Solution to Example 11.3.

Example 11.4 Trace the curve $x^4 + y^4 = a^2(x^2 - y^2)$.

Solution

(i) By putting $x = 0$, $y = 0$, in the equation of the curve we find both the sides are satisfied. Hence the curve passes through the origin.

(ii) Since the equation of the curve contains only the even powers of x and y, the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get $x^4 - a^2x^2 = 0 \Rightarrow x^2(x^2 - a^2) \Rightarrow x = 0, \pm a$. Thus the curve cuts the x-axis at three points (0, 0), (a, 0) and (0, $-a$).
Again, putting $x = 0$, we get $y^4 + a^2y^2 = 0$. Then $y = 0$, i.e. the curve cuts the y-axis at the origin only.

(iv) By equating to zero the terms of the lowest degree in the equation of the curve, we get $x^2 - y^2 = 0$. Then $y = \pm x$. Therefore, the equations of the tangents at the origin will be $y = x$, $y = -x$.

(v) There is no asymptotes to the curve.

(vi) Solving for y, from the equation of the curve, we get

$$y^4 + a^2y^2 = a^2x^2 - x^4 = x^2(a^2 - x^2).$$

If we give to x values $x > a$ or $x < -a$, then

$$a^2 - x^2 = \text{negative} = -k \text{ (suppose)}$$

So,

$$y^4 + a^2y^2 + k = 0.$$

Since there is no change in sign, the roots of y are neither positive nor negative and hence y is imaginary. That is, there is no portion of the curve on the right-hand side of $x = a$ or on the left-hand side of $x = -a$.

(vii) Differentiating the equation of the curve with respect to x, we get

$$4x^3 + 4y^3\frac{dy}{dx} = a^2\left(2x - 2y\frac{dy}{dx}\right)$$

or

$$\frac{dy}{dx}(4y^3 + 2a^2y) = 2a^2x - 4x^3$$

or

$$\frac{dy}{dx} = \frac{2a^2x - 4x^3}{4y^3 + 2a^2y}$$

Therefore, at the point $(a, 0)$, $dy/dx = \infty$ and also at $(-a, 0)$, we have $dy/dx = \infty$, i.e. at the point $(a, 0)$ and $(-a, 0)$, the tangent is perpendicular to the x-axis. Hence the graph of the curve will be as in Fig. 11.4.

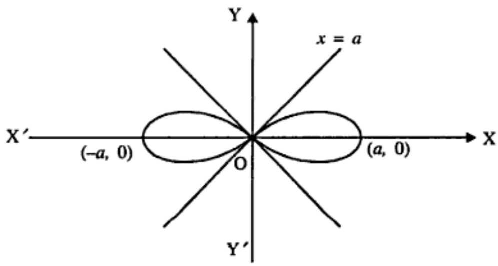


Fig. 11.4 Solution to Example 11.4.

Example 11.5 Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution

(i) Since by putting $x = 0$, $y = 0$, both the sides of the curve are not satisfied. Hence the curve does not pass through the origin.

(ii) The equation of the given curve does not change when we put $-x$ for x and $-y$ for y in it and therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get

$$x^{2/3} = a^{2/3} \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$$

Therefore, the curve cuts the x-axis at the point $(a, 0)$ and $(-a, 0)$. Similarly putting $x = 0$, we get

$$y^{2/3} = a^{2/3} \quad \text{or} \quad y = \pm a$$

That is, the curve cuts the y-axis at the point $(0, a)$ and $(0, -a)$.

(iv) Again from the equation of the curve $y^{2/3} = a^{2/3} - x^{2/3}$, if we put in this equation any quantity $x > a$ or $x < -a$, then $y^{2/3}$ = negative. Thus y^2 = negative.

Therefore, y becomes imaginary, i.e. there is no portion of the curve on the right-hand side of $x = a$ or on the left-hand side of $x = -a$. Similarly by solving for x from the given equation, it can be shown that there is no portion of the curve above $y = a$ or below $y = -a$. That is, the given curve is enclosed by four points $(a, 0)$, $(-a, 0)$, $(0, a)$ and $(0, -a)$.

Also, by comparing the curve $x^{2/3} + y^{2/3} = a^{2/3}$ with the circle $x^2 + y^2 = a^2$, we find that since $2/3 < 2$, the given curve will be inside the circle $x^2 + y^2 = a^2$.

Hence the graph of the curve will be as in Fig. 11.5.

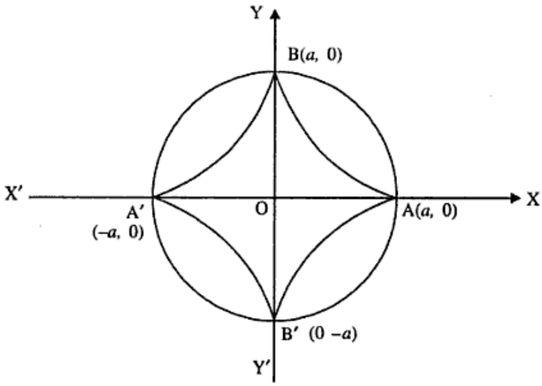


Fig. 11.5 Solution to Example 11.5.

Example 11.6 Trace the curve $x^3 + y^3 = 3axy$.

Solution

(i) The curve is symmetrical about the line $y = x$.

(ii) If the curve passes through the origin, as there is no constant term, the origin is a node.

(iii) It meets the coordinate axis only at the origin.

(iv) The tangents at the origin are $x = 0$, $y = 0$.

(v) Asymptote

$$x + y = \lim_{y/x \to -1} \frac{3axy}{x^2 - xy + y^2}$$

$$= 3a \lim_{y/x \to -1} \frac{y/x}{1 - y/x + y^2/x^2}$$

$$= 3a\left(\frac{-1}{1+1+1}\right)$$

$$= -a$$

So, $x + y + a = 0$ is the asymptote of the curve.

(vi) x and y cannot be both negative. Hence no part of the curve lies in the third quadrant.

(vii) No point of inflexion.

(viii) Polar form is obtained by putting $x = r \cos \theta$, $y = r \sin \theta$. Then

$$r^3\cos^3\theta + r^3\sin^3\theta = 3a\ r\cos\theta\ r\sin\theta$$

or

$$r^2(r\cos^3\theta + r\sin^3\theta - 3a\ \sin\theta\cos\theta) = 0.$$

But $r \neq 0$. Therefore,

$$r\cos^3\theta + r\sin^3\theta - 3a\ \sin\theta\cos\theta = 0$$

or

$$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

Frame a table of the value of r corresponding to the value of θ, as below

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$
r	0	$3a/\sqrt{2}$	0	$-\infty$

Here we see that when θ increases from 0 to $\pi/4$ then r increases from 0 to $3a/\sqrt{2}$ and before $\theta = \pi/4$ and $\theta = \pi/2$, decreases from $3a/\sqrt{2}$, too. Hence between $\theta = 0$ and $\theta = \pi/2$, we get a loop of the curve.

Again as θ increases from $\pi/2$ to $3\pi/4$, r is negative and increases negatively to infinity.

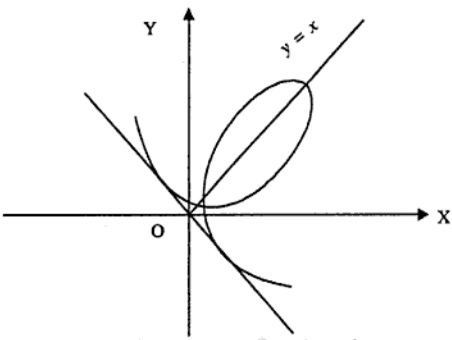


Fig. 11.6 Solution to Example 11.6.

Thus the shape of the curve will be as shown in Fig. 11.6.

9. $x^2y^2 = x^2 - 1$

10. $x^5 + y^5 = 5a^2xy^2$

11. $y^2(a^2 + x^2) = a^2x^2$

12. $y(a^2 + x^2) = a^2x$

13. $x^2 = y^2(x + 1)^3$

14. $x^2y = x + 1$

15. $y^2 = x^2(1 + x^2)/(1 - x^2)$

16. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$

17. $r = a \cos 3\theta$

18. $r = a \sin 2\theta$

19. $r = a + b \cos\theta$

20. $r^3 = a^3 \cos 3\theta$

21. $r = a(\sec\theta + \cos\theta)$

22. $r = \sin 2\theta$

23. $r = 2 + 4 \cos\theta$

24. $x = a\,(\theta + \sin\theta), -\pi < \theta \leq \pi$

25. $x = a\cos^3\theta + y = a \sin^3\theta$

26. $r \cos^3\theta = a \cos 2\theta$

27. $r \cos\theta = a \sin 3\theta$

Multiple-choice Questions

Each of the following questions is followed by alternative answers. Select the correct answer:

1. If $f(t)=\dfrac{1-t}{1+t}$ then $f(t^{-1})$ is

   (a) $f(t)$ (b) $f(-t)$
   (c) $-f(t)$ (d) $f^{-1}(t)$.

2. If $f(x) = \cos x + \sin x$, then $f(\pi/4)$ is

   (a) 2 (b) 1/2
   (c) $\sqrt{2}$ (d) 0.

3. If $f(x)=\dfrac{1+x}{1-x}$, then $f(\tan\ \theta)$ is

   (a) $\tan(\pi/4+\theta)$ (b) $\tan(\pi/4-\theta)$
   (c) $\cot(\pi/4+\theta)$ (d) $\cot(\pi/4-\theta)$.

4. $f(t)=\dfrac{t^4+t^2+1}{t^2}$ then $f^{-1}(t)$ is

   (a) $f(-t)$ (b) $f(1/t)$
   (c) $f(-t)$ (d) $f(t)$.

5. If $f(x)=\dfrac{\sin x-\cos x}{\sin x+\cos x}$ then $f\left(\dfrac{\pi}{3}\right)$ is

   (a) $\sqrt{3}$ (b) $2+\sqrt{3}$
   (c) $2-\sqrt{3}$ (d) $-\sqrt{3}$.

6. $\lim_{x \to 0} \dfrac{a^x - b^x}{x}$ is

(a) $\log(ab)$ (b) $\log(a/b)$

(c) $\dfrac{\log_e^a}{\log_e^b}$ (d) $\dfrac{\log_e^b}{\log_e^a}$.

7. $\lim_{x \to 2} \dfrac{x^{2/3} - 2^{2/3}}{x - 2}$ is

(a) $2^{2/3}/2$ (b) 1

(c) $2^{2/3}/3$ (d) -1.

8. $\lim_{\alpha \to \pi/4} \dfrac{\sin\alpha - \cos\alpha}{\alpha - \pi/4}$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) $\sqrt{5}$ (d) $\sqrt{6}$.

9. $\lim_{x \to 0} \dfrac{\sin m^\circ}{m}$ is

(a) 1 (b) m

(c) π (d) $\pi/180$.

10. $\lim_{x \to 1} \dfrac{(2x-5)(\sqrt{x}-1)}{2x^2+x-3}$ is

(a) $1/10$ (b) $-1/10$

(c) $1/5$ (d) $-1/5$.

11. $\lim_{x \to \infty} \dfrac{x+2}{x^2+2}$ is

(a) 1 (b) 0

(c) Indeterminate (d) ∞.

12. $\lim_{x \to \infty} \dfrac{\sin x}{x}$ is

(a) 1 (b) -1

(c) ∞ (d) 0.

13. $\lim_{x \to 0} \dfrac{e^{\sin x} - 1 - \sin x}{x^2}$ is

(a) $1/2$ (b) 1

(c) 0 (d) -1.

14. If $\lim_{x\to 0} \frac{\sin(x/3)}{x} = k$, the value of k is

(a) 0 (b) 3
(c) 1/3 (d) 1.

15. If $f(x) = \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$, $x \neq 2$

$= k$, $x = 2$

the value of k is

(a) 0 (b) 7
(c) 6 (d) 5.

16. If $f(x) = 2x - 1$, $x \le 0$
$= x^2$, $x > 0$

then $f(-1/2)$ is

(a) −2 (b) −1
(c) 1 (d) 2.

17. $\lim_{x\to 2} \frac{2x^2 - 4f(x)}{x-2}$, if $f(2) = 2$, is

(a) 1 (b) 4
(c) 2 (d) 3.

18. $\lim_{x\to 0} \frac{\log(1 + x + x^2) + (\log 1 - x + x^2)}{\sec x - \cos x}$ is

(a) 1 (b) 2
(c) 3 (d) None of these.

19. $\lim_{x\to 0} \frac{e^{\tan x} - e^x}{\tan x - x}$ is

(a) 2 (b) 1
(c) 3 (d) −1.

20. The period of the function $y = \sin\left(\frac{2t+3}{6\pi}\right)$ is

(a) 2π (b) 6π
(c) $6\pi^2$ (d) None of these.

21. The period of the function $y = |\sin x| + |\cos x|$ is

(a) $\pi/2$ (b) 2π
(c) 4π (d) π.

22. Which of the following is an odd function:

(a) $f(x) = \cos x$ (b) $f(x) = 2^{-x^2}$

(c) $f(x) = 2^{x^2-x^4}$ (d) None of these.

23. $\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$ is

(a) 0 (b) 1

(c) e^h (d) e^x.

24. If $f(x) = \frac{x^2 - 3x + 4}{(x+4)(x-1)}$ and domain = $R - \{1, -4\}$ then the range of $f(x)$ is

(a) $y \geq 7/25$ or $y \leq -1$ (b) $y \leq 7/25$ or $y \geq -1$,

(c) $y \geq 2/23$ or $y \leq 2$ (d) $y \leq 2/23$ or $y \geq -2$.

25. If $f(x) = \sin x - \cos x$, the range of the function is

(a) $-\sqrt{2} \leq y \leq \sqrt{2}$ (b) $-\sqrt{2} \geq y \geq \sqrt{2}$

(c) $\sqrt{2} \leq y \leq -\sqrt{2}$ (d) $\sqrt{2} \geq y \geq -\sqrt{2}$.

26. Range of the function $\sin^{-1}\left(\frac{x^2}{1+x^2}\right)$ is

(a) $[-\pi/2, \pi/2]$ (b) $[-\pi/2, 0]$

(c) $[0, \pi/2]$ (d) $[0, \pi/2]$.

27. If $f(x) = x^3 + 3x^2 + 10x + 2 \sin x$, the range of the function is

(a) $[-\infty, \infty]$ (b) $[0, \infty]$

(c) $[-\infty, 0]$ (d) None of these.

28. The range of the function $f(x) = \frac{|\sin x|}{1 + |\sin x|}$ is

(a) $0 < y < 1$ (b) $0 \leq y \leq 1$

(c) $0 \leq y < 1$ (d) $0 < y \leq 1$.

29. The periodic of the function $y = \sin\left(\frac{2x+3}{6\pi}\right)$ is

(a) 2π (b) 6π

(c) $6\pi^2$ (d) None of these.

30. The domain of $y = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4 - x)$ is

(a) $[-\infty, 4]$ (b) $[1, 4]$

(c) $[-\infty, 3]$ (d) None of these.

39. The points of continuity of the function $f(x) = \dfrac{1+\cos 5x}{1-\cos 4x}$ is

(a) $x = 0$ (b) $x = \pi/4$

(c) $x = \pi/2$ (d) $x = \pi$.

40. If $f(x) = -2 \sin x$, $x \leq -\pi/2$

$= A \sin x + B$ $-\pi/2 < x < \pi/2$

$= \cos x$ $x \geq -\pi/2$,

the values of A and B so that $f(x)$ is continuous everywhere are

(a) $A = 0,\ B = 1$ (b) $A = 1,\ B = 1$

(c) $A = -1, B = 1$ (d) $A = -1, B = 0$.

41. If $f(x) = (1+\sin^2\sqrt{x})^{1/(2x)}$, then the value of $f(0)$ that makes the function continuous everywhere is

(a) e (b) 1/2

(c) $\sqrt{e}$ (d) 0.

42. The function $f(x)$ is $f(x) = x^2$ if x is rational $= -x^2$ if x is irrational, then it is

(a) Continuous at $x = 0$, (b) Discontinuous at $x = 1$

(c) Discontinuous at $x = 0$, (d) Continuous at $x = 1/2$

43. If $f(x) = 1$, $x \leq 3$

$= ax + b$, $3 < x < 5$

$= 7$, $x \geq 5$

is continuous, then

(a) $a = 3, b = -8$, (b) $a = -3,\ b = 8$,

(c) $a = 5, b = 7$, (d) $a = 5,\ b = -7$.

44. If $x^y = e^{x-y}$, then dy/dx is

(a) $\dfrac{1}{1+\log x}$ (b) $\dfrac{1}{(1+\log x)^2}$

(c) $\dfrac{\log x}{(1+\log x)^2}$ (d) $\dfrac{\log x}{(1-\log x)^2}$.

45. If $y = x \log x$, then dy/dx is

(a) 1 (b) $\log x - 1$

(c) $\log x$ (d) $1 + \log x$.

46. The derivative of $x^2 \cos x$ is

(a) $2 \cos x - x^2 \sin x$ (b) $2x \cos x - x^2 \sin x$

(c) $2x^2 \cos x - x^2 \sin x$ (d) $2x^2\cos x - x^2 \sin x$.

47. The slope of the curve $y = (x^2 + 1)(1 - x^2)$ at $x = 1$ is

(a) $a = 0$ (b) -88

(c) 75/256 (d) None of these.

48. The equation of the tangent to the curve $y = x^2 + 1$ at the point (1, 2) is

(a) $y - 2 = -2(x - 1)$ (b) $y - 2 = 4(x - 1)$
(c) $y - 2 = 2(x - 1)$ (d) $y - 2 = 2(x + 1)$.

49. The displacement s of a particle moving in a straight line is given by $S = 4t^3 + 2t^2 - 3t + 1$, s being measured in cm and time t in second. The initial acceleration of the particle in cm/sec^2 is

(a) 1 (b) 4
(c) –3 (d) = 24

50. The differential coefficient of x^3 with respect to x^2 is

(a) $3x^2$ (b) $2x$
(c) $3x^2 + 2x$ (d) $3x/2$.

51. If $y = x^{1/x}$, then dy/dx vanishes, when x is equal to

(a) $x = 1$, (b) $x = e$
(c) $x = 1/x$ (d) $x = -1$.

52. If $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$ then $f(x) = \dfrac{g(x)f(a) - g(a)f(x)}{x - a}$, as $x \to a$, is

(a) 5 (b) 6
(c) –5 (d) None of these.

53. If $f(x) = \sin^{-1}\left(\dfrac{1-x}{1+x}\right)$, then $f'(x)$ with respect to $\sqrt{x}$ is

(a) $1/\sqrt{1-x^2}$ (b) $2/(1+x^2)$
(c) $-2/(1+x)$ (d) $2/\sqrt{1-x^2}$.

54. If $f(2) = 4$, and $f'(2) = 1$, then $f(x) = \lim\limits_{x \to 2} \dfrac{2f(2) - 2f(x)}{x - 2}$ is

(a) 1 (b) 2
(c) –2 (d) 3.

55. If $f(x) = x^2 \sin \dfrac{1}{x}, \quad x \neq 0$

$\qquad = 0, \quad x = 0$

then

(a) f and f' are continuous at $x = 0$
(b) f and f' are derivable at $x = 0$
(c) f is derivable at $x = 0$ and f' is not continuous at $x = 0$
(d) f is derivable at $x = 0$ and f' is continuous at $x = 0$

56. If $f(x) = \dfrac{x \log x}{\log(1+x^2)} \qquad x \neq 0$

$= 0, \qquad x = 0,$

then

(a) $f(x)$ is discontinuous at $x = 0$
(b) $f(x)$ is continuous but not differentiable at $x = 0$
(c) $f(x)$ is differntiable at $x = 0$
(d) $f(x)$ is not continuous at $x = 0$.

57. If $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ then

(a) $y'(0) = 1$ (b) $y'(0) = 1/2$
(c) $y'(0) = 0$ (d) $y'(0)$ does not exist.

58. If $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $z = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, then $\dfrac{dy}{dz}$ is

(a) 1 (b) 1/2
(c) –1 (d) None of these.

59. If $f(9) = 9$, $f'(9) = 4$, then $\lim\limits_{x \to 9} \dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ is

(a) 3 (b) 2
(c) 4 (d) None of these.

60. If $f(x) = (\log_{\cot x} \tan x)(\log_{\tan x} \cot x) + \tan^{-1}\left(\frac{4x}{4-x^2}\right)$, then $f'(2)$ is

(a) 1/3 (b) 1
(c) 2 (d) 1/2

61. If $a^x + a^y = a^{x+y}$, then dy/dx is

(a) $\dfrac{a^x + a^y}{a^x - a^y}$ (b) $\dfrac{a^{x-y}(a^y - 1)}{1 - a^x}$

(c) $\dfrac{a^{x+y} - a^x}{a^y}$ (d) None of these.

62. If $f(x) = \tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$, then $f'(\pi/2)$ is

(a) $\dfrac{1}{2(1+\cos x)}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{4}$ (d) None of these.

63. If $f(x) = x \tan^{-1}x$, then $f'(1)$ is

(a) $\frac{1}{2}+\frac{\pi}{4}$ (b) $-\frac{1}{2}+\frac{\pi}{4}$

(c) $-\frac{1}{2}-\frac{\pi}{4}$ (d) $\frac{1}{2}-\frac{\pi}{4}$.

64. If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=k$, then k is

(a) 0 (b) 1

(c) 2 (d) None of these.

65. $\lim_{x\to\pi/4}(2-\tan x)^{\log\tan 2x}$ equals to

(a) e (b) 1

(c) 0 (d) e^{-1}.

66. If $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant then $\frac{d^3f(x)}{dx^3}$ at $x = 0$ is

(a) p (b) $p + p^2$

(c) $p + p^3$ (d) Independent of p.

67. The acceleration of a moving particle, whose space–time equation is given by $s = 3t^2 + 2t - 5$, is

(a) 4 (b) 5

(c) 6 (d) 7.

68. The set of all points where the function $f(x) = \frac{x}{1+|x|}$ is differentiable is

(a) $(-\infty, \infty)$ (b) $(0, \infty)$

(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$.

69. If $f(x) = \frac{|x|}{\sin x}$ for $x \neq 0$

$f(0) = 1$, then

(a) $f(x)$ is continuous and differentiable at $x = 0$

(b) $f(x)$ is continuous not differentiable at $x = 0$

(c) $f(x)$ is discontinuous not differentiable at $x = 0$

(d) none of these.

70. If $\sin y = x \sin(a + y)$, then dy/dx is

(a) $\frac{\cos^2(a+y)}{\cos a}$ (b) $\frac{\sin^2(a+y)}{\sin a}$

(c) $\frac{\tan^2(a+y)}{\tan a}$ (d) None of these.

71. If $y = \sin(bx + c)$, then $\frac{d^n}{dx^n}\sin(bx + c)$ is

(a) $b^n \sin(bx + c + n\pi/2)$ (b) $b^n \cos(bx + c + n\pi/2)$

(c) $b^n \sin bx$ (d) $b^n \cos bx$.

72. If $x = a\cos^3 x$, $y = a\sin^3 x$, then d^2y/dx^2 at $t = \pi/2$ is

(a) $-3/4$ (b) $1/4$

(c) $3/4$ (d) $-3/2$

73. If $y = x^4 \log x$, then y_n, when $n \geq 5$ is

(a) $(-1)^{n-1}\frac{(n-5)!}{x^{n-4}}24$ (b) $(-1)^{n-1}\frac{(n-6)!}{x^{n-5}}48$,

(c) $(-1)^{n-1}\frac{(n-6)!}{x^{n-6}}64$ (d) None of these.

74. Find ξ of the mean value theorem given by $f(b) = f(a) + (b - a)f'(\zeta)$, where $f(x) = x(x-1)(x-2)$, $a = 0$, $b = 1/2$ is

(a) 1.253 (b) 2.362

(c) 0.236 (d) None of these.

75. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^2$, then du/dt is

(a) $1/\sqrt{t}$ (b) $3/\sqrt{1-t^2}$

(c) $3/\sqrt{1+t^2}$ (d) $-1/\sqrt{t}$.

76. If $x = \cot y$, then $\frac{d}{dx}\left(\frac{x}{1+x^2}\right)$ is

(a) $\sin y(\sin y - \cos y \sin 2y)$ (b) $\cos y(\cos y - \sin y \cos 2y)$

(c) $\cos^2 y[\sin^2 y - \cos y \sin^2(2y)]$ (d) None of these.

77. If $x^2 + 2x = 2\log\left(c\frac{dx}{dy}\right)$ and $y = a_0 + a_1 x + a^2\frac{x^2}{2!} + \cdots$ to ∞, then $a_{n+2} + a_{n+1} + a_n$ is

(a) 0 (b) n

(c) $n - 1$ (d) $n/2$.

78. If $u = x^y$ then $\frac{\partial^2 u}{\partial x^2 \partial y}$ is

(a) $\frac{\partial^2 u}{\partial x \partial^2 y}$ (b) $\frac{\partial^2 u}{\partial x^2 \partial y^2}$

(c) $\frac{\partial^2 u}{\partial x \partial y \delta z}$ (d) None of these.

79. Given that $F(u) = V(x, y, z)$, where V is a homogeneous function in x, y, z of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ is

(a) $n\frac{F(u)}{F'(x)}$ (b) $n\frac{F(u)}{F'(y)}$

(c) $n\frac{F(u)}{F'(z)}$ (d) $n\frac{F(u)}{F'(u)}$.

80. If $f(p, t, v) = 0$, then $\frac{dp}{dt}\frac{dt}{dv}\frac{dv}{dp}$ is

(a) 1 (b) –1

(c) 2 (d) –2.

81. The altitude h of a triangle ABC is computed from measuring the base a and the base angles B, C. If small errors δB, δC be made in the angles, then the consequent error in altitude is

(a) $\left(\frac{\sin C}{\sin A \sin B}\delta B + \frac{\sin B}{\sin A \sin C}\delta C\right)h$

(b) $\left(\frac{\sin B}{\sin A \sin C}\delta B + \frac{\sin C}{\sin B \sin A}\delta C\right)h\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(c) $\frac{\sin B \sin C}{\sin A}\delta B\,\delta C h$

(d) None of these.

82. With the usual meaning a, b, c and s if Δ be the area of the triangle, the error in the area of triangle resulting from a small error in the measurement of c is

(a) $\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)$

(b) $\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)$

(c) $\delta\Delta = \frac{\Delta}{4}\left(\frac{a+b+c}{s(s-a)(s-b)(s-c)}\right)$

(d) None of these.

101. The asymptote parallel to y-axis of the curve $x^2y^2 = a^2y^2 + b^2x^2$ is

(a) $y = \pm b$ (b) $x = \pm a$

(c) $x \pm y = a$ (d) None of these.

102. The asymptote parallel to x-axis to the curve $y^2(x^2 - a^2) = x$ is

(a) $x = \pm a$ (b) $x - y = \pm a$

(c) $x = 0, y = 0$ (d) None of these.

103. The real asymptote to the curve $x^3 + y^3 = 6xy$ is

(a) $x - y - 2 = 0$ (b) $x + y - 1 = 0$

(c) $x - y + 1 = 0$ (d) $x + y + 2 = 0$.

104. The real asymptote to the curve $x^3 + y^3 = a^2$ is

(a) $x - y = 0$ (b) $x + y = 0$

(c) $x + 2y = 0$ (d) $x - 2y = 0$.

105. The real asymptote to the curve $y^2 = x(a^2 - x^2)$ is

(a) $x - y = 0$ (b) $x + y = 0$

(c) $x + y = 0$ (d) $x + 2y = 0$.

106. The asymptote to the hyperbolic spiral $r\theta = a$ is

(a) $r \sin\theta = a$ (b) $r \cos\theta = a$

(c) $r \tan\theta = a$ (d) None of these.

107. The greatest triangle inscribed in a circle is

(a) Isosceles triangle (b) Equilaterial triangle

(c) Right-angled triangle (d) None of these.

108. The shortest distance from origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$ is

(a) 25 (b) 30

(c) 35 (d) 40.

109. The maximum value of the function $f(x, y) = x^3 + y^3 + 3xy$ at $(-1, -1)$ is

(a) 2 (b) 3

(c) 1 (d) None of these.

110. The absolute maximum of the function $y = x^3 + 1$, where $x \in |-2, 2|$ is

(a) $y = 2$ at $x = -1$ (b) $y = 9$ at $x = 2$

(c) $y = 6$ at $x = 3$ (d) None of these.

111. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

(a) ab (b) $(ab)/2$

(c) a^2b^2 (d) $(a^2b^2)/2$.

112. If P be a variable point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2. If A is the area of the ΔPF_1F_2, the maximum value of A is

(a) ab (b) $(ab)/2$

(c) abe (d) $(ab)^2e$.

113. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance is to be travelled by all 200 students is to be as small as possible, then the school should be built at

(a) Town B (b) 45 km from town A

(c) Town A (d) 45 km from town B.

114. The largest value of $2x^3 - 3x^2 - 12x + 5$ for $-2 \le x \le 4$ occurs at $x =$

(a) -2 (b) -1

(c) 2 (d) 4.

Answers

CHAPTER 1

Exercises 1.1

1. Continuous and differentiable at $x = 0$.

2. Continuous at $x = 2$, but not differentiable at $x = 2$.

3. Not differentiable at $x = 0, 1$.

4. $f(x)$ is continuous everywhere and is differentiable everywhere except $x = 2$ and -2.

5. Continuous but not differentiable at $x = 0$.

6. $f(x)$ is continuous everywhere and not differentiable at $x = 0$ only, differentiable at $x = \pi/2$.

8. Continuous at $x = 1$ and discontinuous at $x = 2$, and differentiable at both the points.

11. 5.

Exercises 1.2

1. (i) $20x^3$ (ii) $1/(2\sqrt{x})$ (iii) $2x + 15x^2$ (iv) $1/(3\sqrt[3]{x^2})$

(v) $(3/2x)^{1/2}$ (vi) $(-3/4x)^{-7/4}$ (vii) $-1/2(x + a)^{3/2}$ (viii) $-x/\sqrt{x^2 + 1}$

(ix) $1 - 1/x^2$ (x) $1/\sqrt{2x-1}$ (xi) $8x^7 + 3x^2$

2. (i) $4 \cos 4x$ (ii) $-2\sin(2x)$ (iii) $2\sec^2(2x)$ (iv) $k\sec^2(kx)$

(v) $\frac{\pi}{540}\sec^2\left(\frac{x}{3}\right)^0$ (vi) $-3 \operatorname{cosec} 3x \cot 3x$ (vii) $-2\operatorname{cosec}^2(2x)$

(viii) $3\sec^2(3x + 1)$ (ix) $2x \cos(x^2 + 1)$ (x) $\frac{\sec^2 x}{2\sqrt{\tan x}}$

(xi) $3\sin^2 x \cos x$ (xii) $-2x \sin x^2$ (xiii) $-2\sin x \cos x$

(xiv) $\sec^2(x + a)$ (xv) $\frac{\pi}{540}\sec\left(\frac{x}{3}\right)^0 \tan\left(\frac{x}{3}\right)^0$

(xvi) $2a \tan ax \sec^2(ax)$

3. (i) $-x^2 \sin x + 2x \cos x$ (ii) $\sin x + x \cos x$

(iii) $-(2ax + b) \sin (ax^2 + bx + c)$ (iv) $\tan x + x \sec^2 x$

(v) $\cos^2 x - \sin^2 x$ (vi) $\frac{1}{x}\cos x - \frac{1}{x^2}\sin x$

(vii) $-\frac{1}{x^2}, \sec^2 \frac{1}{x}$

4. (i) $\frac{3\sec^2(3\sqrt{x})}{2\sqrt{x}}$ (ii) $-3\sec^2(1 - 3x)$ (iii) $5 \sec 5x \tan 5x$

(iv) $-\frac{\sin x}{2\sqrt{\cos x}}$ (v) $\frac{\sec x \tan x}{2\sqrt{\sec x}}$ (vi) $-\frac{3\sin 3x}{2\sqrt{\cos 3x}}$

(vii) $\frac{\cos\sqrt{x}}{2\sqrt{x}}$ (viii) $-\frac{\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$

5. (i) ne^{nx} (ii) $2e^{2x+3}$ (iii) $2xe^{x^2}$

(iv) $-\sin x\, e^{\cos x}$ (v) $\sec^2 x e^{\tan x}$ (vi) $-\frac{e^x}{x^2}$

(vii) $\frac{1}{2}\frac{5e^{\sqrt{5x}}}{\sqrt{5x}}$ (viii) $-3e^{-3x}$

6. (i) $\frac{a}{ax+b}$ (ii) $\tan x$ (iii) $1 + \log x$

(iv) $\frac{1}{x}$ (v) $\frac{\sec^2 x}{\tan x}$ (vi) $\frac{1}{x}\cos(\log x)$

(vii) $\log \sin x + x \cot x$ (viii) $\frac{1}{a}\cot\frac{x}{a}$

(ix) $-\tan x$ (x) $4^x \log 4$

7. (i) $\frac{a}{a^2 + x^2}$ (ii) $\frac{1}{\sqrt{a^2 - x^2}}$ (iii) $\frac{1}{x\sqrt{x^2 - 1}}$

(iv) $\frac{1}{2x^2 - 6x + 5}$ (v) $\frac{3}{\sqrt{9x^2 + 30x + 26}}$

(vi) $\dfrac{a}{1+a^2x^2}$ (vii) $\dfrac{2}{\sqrt{1-4x^2}}$ (viii) $\dfrac{1}{x\sqrt{9x^2-1}}$

(ix) $-\dfrac{m}{1+m^2x^2}$

Exercises 1.3

1. (i) $1 - 1/x^2$ (ii) $2\cos x + 5\sin x$ (iii) $-ke^x - 9\sin x$

(iv) $-3/(2x)$ (v) $2x^2 + (5/2)x - 2$ (vi) $1 - 1/x^2$

(vii) $3x^2\cos x - x^2\sin x$ (viii) $5x^4 + 9x^2 + 10x + 2$

(ix) $15x^2 + 6 - 1/x^2$ (x) $2x\sin x + (x^2 + 1)\cos x$

(xi) $(\cos x)/x - (\sin x)/x^2$ (xii) $e^{\tan x}(\sec^2 x\cot x - \operatorname{cosec}^2 x)$

(xiii) $4x^3\log x + x^3$ (xiv) $(1/x^2)(1-\log x)$ (xv) $\dfrac{2x^3+3ax^2+a^3}{(x+a)^2}$

(xvi) $1/(2\sqrt{x})$ (xvii) $e^x + 1/x^2$ (xviii) $4/x - 1 - a^2/x^2$

2. (i) $\dfrac{x}{\sin^2 x}(2\sin x - x\cos x)$ (ii) $-\dfrac{1}{x^2}(x\operatorname{cosec}^2 x + \cot x)$

(iii) $\dfrac{x^{n-1}(n\log x-1)}{(\log x)^2}$ (iv) $-\dfrac{1}{e^x}(\operatorname{cosec}^2 x + \cot x)$

(v) $-\dfrac{2\sin x}{(1-\cos x)^2}$ (vi) $\dfrac{2\sec^2 x}{(1-\tan x)^2}$ (vii) $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$

(viii) $-\dfrac{x^2+1}{(x^2-1)^2}$ (ix) $-\dfrac{2}{(\sin x-\cos x)^2}$ (x) $\dfrac{x+\sin x(\sin x+\cos x)}{(\sin x+\cos x)^2}$

(xi) $\dfrac{(1+\tan x)(2x+\sec x\tan x)-\sec^2 x\,(x^2+\sec x)}{(1+\tan x)^2}$

(xii) $\dfrac{(1+\tan x)(x^2\cos x+2x\sin x)-x^2\sec x\tan x}{(1+\tan x)^2}$

(xiii) $\dfrac{(1+\log x)x\,(e^x+\cos x)-e^x-\sin x}{x(1+\log x)^2}$

(xiv) $\dfrac{2(x^3+2x^2-4x+4)}{(x+2)^2}$

(xv) $\dfrac{(1+\tan x)(2x+\sec x\tan x)-(x^2+\sec x)\sec^2 x}{(1+\tan x)^2}$

(xvi) $\dfrac{2x(1+\cot x)+\operatorname{cos}ecx(1-\cot x)+x^2(1+\cos^2 x)}{(1+\cot x)^2}$

(xvii) $\dfrac{2e^x \sec x \tan x - \sec^2 x}{(1+\tan x)^2}$ (xviii) $\dfrac{1}{2}\sec^2\dfrac{x}{2}$

(xix) $-\dfrac{1}{2}\sec^2\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$ (xx) $-\dfrac{x\sin x+4\cos x}{x^5}$

3. (i) $f'(u)=\dfrac{1}{4}$, $\dfrac{dy}{dx}=\infty$ at $x=0$ (ii) $\dfrac{(1+x^2)\cos x-2x\sin x}{(1+x^2)^2}-\dfrac{e^x(2x+1)}{2\sqrt{x}}$

(iii) $-\dfrac{2x}{(1+x^2)^2}+e^x\sec x\,(1+\tan x)$

(iv) $f'(0)=1$ (v) $f'\left(\dfrac{a}{2}\right)=\dfrac{11}{9}$

4. $2\pm\sqrt{5}i$ **5.** $1/e$ **6.** -1 **7.** 2

8. $1/\sqrt{2}$ **9.** 1

10. (i) $x^4\sqrt{1+\tan x}\,\sec^2 x\left(\dfrac{4}{x}+\dfrac{1}{2}\dfrac{\sec^2 x}{1+\tan x}+2\tan x\right)$

(ii) $x^3(x^2+4)^{1/2}(x^2+3)^{-1/2}\left(\dfrac{3}{x}+\dfrac{x}{x^2+4}-\dfrac{x}{x^2+3}\right)$

11. (i) $-\dfrac{\tan x\operatorname{cosec}^2 x+\cot x\sec^2 x}{(\tan x-\cot x)^2}$

(ii) $\dfrac{x\sec^2 x+\tan x(\sec x-1)-\sin x}{(x+\sin x)^2}$

(iii) $\dfrac{2x(x^4+2x^2-1)}{(x^2+1)^2}$ (iv) $\dfrac{2(x^2-1)}{(x^2+x+1)^2}$

(v) $-\dfrac{2(3x^2+4x-1)}{(2x^2+11x+8)^2}$ (vi) $\dfrac{x^2\cos x+x+x^2\sin x-x\sin x\cos x}{x^2\cos^2 x}$

(vii) $x^2\sec x\tan x+2x\sec x+\dfrac{(1+\sin x)2x-x^2\cos x}{(1+\sin x)^2}$

12. $a^2\cos a+2a\sin a$, **13.** $-\dfrac{1}{2}$

Exercises 1.4

1. (i) $3x^2(\sin 3x + x\cos 3x)$, (ii) $\dfrac{2x+1}{2\sqrt{x^2+x+1}}\cos\sqrt{x^2+x+1}$

(iii) $\dfrac{\cos\sqrt{\sin\sqrt{x}}\cos\sqrt{x}}{8\sqrt{x}\sqrt{\sin\sqrt{x}}\sqrt{\sin\sqrt{\sin\sqrt{x}}}}$ (iv) $\cos(\sin x)\cos x$

(v) $\cos(\tan x)\sec^2 x$ (vi) $-a\sin ax\cos(\cos ax)$

(vii) $\dfrac{\sqrt{a}}{2}\dfrac{\cos\sqrt{ax}}{\sqrt{ax}}$ (viii) $\dfrac{\cos x}{1+\sin^2 x}$

(ix) $\dfrac{\cos(\log x)}{x}$ (x) $\sec x \operatorname{cosec} x$

(xi) $-\tan x$ (xii) $\dfrac{1}{x}\dfrac{1}{\log x}$

(xiii) $\dfrac{1-10x^4}{\sqrt{5+2x-4x^5}}$ (xiv) $-\dfrac{4x+3}{2(2x^2+3x-10)^{3/2}}$

(xv) $\dfrac{\cos x}{2\sqrt{1+\sin x}}$ (xvi) $-\dfrac{2+\sqrt{x}}{2(1+\sqrt{x})^2}\sin\left(\dfrac{x}{1+\sqrt{x}}\right)$

(xvii) $\dfrac{\cos\sqrt{x}\cos\left(\sin\sqrt{x}\right)}{2\sqrt{x}}$ (xviii) $\dfrac{2}{x\log x}$,

(xix) $2axe^{ax^2}$, (xx) $\sec^2 x e^{\tan x}$,

(xxi) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ (xxii) $e^{-x}[2\sin(2x+3)+\cos(2x+3)]$,

(xxiii) $\dfrac{a}{ax+b}$ (xxiv) $\dfrac{x^2-1}{x(x^2+1)}$,

(xxv) $\dfrac{2ax+b}{ax^2+bx+c}$

2. (i) $2\cot x\log(\sin x)$ (ii) $x\tan x + \log(\sec x)$

(iii) $e^{ax}(a\sin bx + b\cos bx)$ (iv) $e^{ax}(a\cos bx - b\sin bx)$

(v) $e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$

(vi) $-\dfrac{4}{(e^x-e^{-x})^2}$ (vii) $\cos[\log(1+x^2)]\dfrac{1}{1+x^2}$

(viii) $\tan\dfrac{x}{a}$ (ix) $\sec x$

(x) $\dfrac{1}{2(\pi/4+x/2)}$ (xi) $\sec x$

(xii) $\frac{2}{1-x^2}$ (xiii) $-\frac{4}{1-x^4}$

(xiv) $\frac{2b(c-ax^2)}{(ax^2+bx+c)(ax^2-bx+c)}$ (xv) $2\sec x$

(xvi) $2\operatorname{cosec} x$ (xvii) $-\frac{2ab}{a^2x^2-b^2}$

(xviii) $\frac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$ (xix) $\frac{2\sec^2(x/2)}{4-\tan^2(x/2)}$

(xx) $\log\left(\frac{x}{a+bx}\right)+\frac{a}{a+bx}$ (xxi) $\frac{1}{(1-x)\sqrt{x}}$

(xxii) $\frac{2(\sin x+x\cos x)}{1-x^2\sin^2 x}$

3. (i) $-\frac{3}{2}\frac{1}{\sqrt{x}}\cos^2\sqrt{x}\sin\sqrt{x}$, (ii) $-\frac{\sin x}{2\sqrt{\cos x}}\cos(\sqrt{\cos x})$

(iii) $-\frac{\sin\sqrt{x}}{4\sqrt{x\cos\sqrt{x}}}$ (iv) $\frac{1}{2}\frac{\sec^2(\tan x)\sec^2 x}{\sqrt{\tan(\tan x)}}$

(v) $\frac{1}{4}\sqrt{\frac{a}{x}}\frac{\cos\sqrt{ax}}{\sqrt{\sin\sqrt{ax}}}$ (vi) $\frac{1}{2}(2x+a)\frac{\cos\sqrt{x^2+ax+1}}{\sqrt{x^2+ax+1}}$

(vii) $-\frac{x\sin(1+x^2)}{\sqrt{\cos(1+x^2)}}$ (viii) $-6\tan^2 x\ \sec^2 x\frac{1}{(1+\tan^3 x)^3}$

(ix) $-\frac{1}{\sqrt{1-x^2}}$ (x) $\frac{x\sec^2(1+x^2)}{\sqrt{\tan(1+x^2)}}$

(xi) $\frac{x\cos x^2}{\sqrt{\sin x^2}}$

(xii) $\frac{3}{2}\frac{2ax+b}{\sqrt{ax^2+bx+c}}\sin^2\sqrt{ax^2+bx+c}\cos\sqrt{ax^2+bx+c}$

(xiii) $-\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sin\left(\tan\sqrt{x}\right)\cos\left[\cos\left(\tan\sqrt{x}\right)\right]$

(xiv) $\operatorname{cosec}^2 x\ \sec^2(\cot x)\ \sin[\tan(\cot x)]\ \cos\ [\tan\ (\cot x)]$

(xv) $\sec x\ \tan x\ \sec^2(\sec x)\ (-1)[\tan(\sec x)]\ \cos\{\cos[\tan(\sec x)]\}$.

4. (i) $\sin^{m-1}(\alpha x)\ \cos^{n-1}(\beta x)\ (m\alpha\cos\beta x\ \cos\ \alpha x - n\beta\sin\ \alpha x\ \sin\beta x)$

(ii) $\frac{4}{\sqrt{4x^2+1}}(2x-3)(6x^2-3x+1)$, (iii) $e^{ax}a\left[\sin^{-1}(ax)+\frac{1}{\sqrt{1-a^2x^2}}\right]$

8. (i) $-x/\sqrt{1-x^2}$ (ii) $-x/\sqrt{1-x^2}$ (iii) $\frac{1}{2\sqrt{x}}\frac{1}{1+x}+\frac{1}{1+x^2}$

(iv) $2/(1+x^2)$ (v) $3/\sqrt{1-x^2}$ (vi) $2/(1+x^2)$

(vii) 1/2 (viii) $-1/2$ (ix) $1/\sqrt{1-x^2}$

(x) $-2/(1+x^2)$ (xi) $-2/(1+x^2)$ (xii) $4/(4+x^2)$

(xiii) $-1/2$ (xiv) $2/\sqrt{1-x^2}$ (xv) 1/2

(xvi) $-1/2$ (xvii) 0 (xviii) -1

(xix) 1/2 (xx) 1/2 (xxi) 1/2

(xxii) $-1/2$ (xxiii) $-1/2$ (xxiv) $\tan^{-1}\left(\frac{1}{x}\right)+\frac{1+x}{1+x^2}$

(xxv) $-\frac{\sin x}{1+\cos^2 x}$ (xxvi) $\frac{4}{5+3\cos x}$ (xxvii) $-\frac{\sqrt{b^2-a^2}}{b+a\cos x}$.

9. (i) $-2x^{3/2}\sec^2(5-2x^2)+(3/2)x^{1/2}\tan(5-2x^2)$

(ii) $\frac{3}{2}\frac{2ax+b}{\sqrt{ax^2+bx+c}}\sin^2\sqrt{ax^2+bx+c}\cos\sqrt{ax^2+bx+c}$

(iii) $\frac{1}{2\sqrt{x}}(\sin x+\cos\sqrt{x})+\sqrt{x}\cos x$

(iv) $\frac{1}{2\sqrt{x}}(\cos\sqrt{x}-\sin 2\sqrt{x})$

(v) $6x^2(x^2+1)^2+4x(2x^5+2x^3-x^2-1)$

(vi) $\frac{(2x^2+3)^{3/2}(18x^2+100x-3)}{3(x+5)^{4/3}}$

(vii) $2\cos(3x+4)\sin(2x+3)[2\cos(3x+4)\cos(2x+3)$
$-\sin(2x+3)\sin(3x+4)]$

(viii) $-(2ax+b)\sin(ax^2+bx+c)+\frac{3}{2}\frac{-(2ax+b)}{\sqrt{ax^2+bx+c}}$
$\times\sin^2\sqrt{ax^2+bx+c}\cos\sqrt{ax^2+bx+c}$

(ix) $\frac{-x\cos\sqrt{1-x^2}}{\sqrt{1-x^2}}+2x(\cos 4x-2x\sin 4x)$

(x) $\frac{2a(1-2a^2x^2)\cos\left(2ax\sqrt{1-a^2x^2}\right)}{\sqrt{1-a^2x^2}}$ (xi) $\frac{-4x^2}{(4x^3-1)^{4/3}}-5\sin 2(5x+8)$

(xii) $\frac{5(3-x)}{3(1-x)^{5/3}}-2\sin 2(2x+1)$ (xiii) $\frac{1}{1-x^4}$

(xiv) $\dfrac{2ax^2}{x^4 - a^4}$ (xv) $\dfrac{2 + a^2 - 2x^2}{\sqrt{a^2 - x^2}}$

(xvi) $\sin^{-1}x$

10. (i) $\dfrac{\sec^2 x}{2x}$ (ii) $-\cot x$ (iii) $\sin x$

(iv) $\dfrac{\cos x}{3x^2}$ (v) $\cos^3 x$ (vi) $\sec^3 x$

(vii) $\operatorname{cosec} x$ (viii) $-\tan^2 x$ (ix) $-\dfrac{1}{2}\operatorname{cosec} x$

(x) $\dfrac{2x + 1}{3x^2}$ (xi) $\dfrac{\cos^2 x}{(1 + x^2)^{3/2}}$ (xii) $\dfrac{2}{t - 1}$

(xiii) $\dfrac{2x}{\sqrt{1 - x4}}$ (xiv) $\dfrac{\sin x - x\cos x}{\sin^2 x\ \cos x}$ (xv) xe^x

(xvi) $\cos x^2$ (xvii) $2e^x$.

11. (i) $-4x \cos x^2 \sin x^2$ (ii) $\dfrac{\tan x}{(\log x)^2}$

(iii) $\dfrac{-\sec^2 x}{(1 + \tan x)^{3/2}(1 - \tan x)^{1/2}}$ (iv) $\dfrac{\cos(\sqrt{\sin x + \cos x})(\cos x - \sin x)}{2\sqrt{\sin x + \cos x}}$

(v) $\dfrac{1}{a^2 - b^2}\left(\dfrac{x}{\sqrt{x^2 + a^2}} - \dfrac{x}{\sqrt{x^2 + b^2}}\right)$ (vi) $\dfrac{4\sqrt{2}}{1 + x^4}$

(vii) $\dfrac{2\sqrt{x} + 1}{2\sqrt{x}}$ (viii) $8\left(\dfrac{4}{\pi^2 + 16} - \dfrac{1}{\log_e z}\right)$

(ix) 1/2 (x) 1 (xi) -1

(xii) -1 (xiii) $-\dfrac{1}{2\sqrt{1 - x^2}}$

(xiv) $\dfrac{2nx^{n-1}}{1 + x^{2x}}$ (xv) $\dfrac{1}{\sqrt{1 - x^2}} + \dfrac{1}{2\sqrt{x(1 - x)}}$

(xvi) $\dfrac{1}{2\sqrt{x}(x + 1)} - \dfrac{1}{1 + x^2}$ (xvii) $\dfrac{3}{1 + 9x^2} + \dfrac{2}{1 + 4x^2}$

(xviii) $-\dfrac{2e^{2x}}{1 + e^{4x}}$ (xix) $\dfrac{1}{1 + x^2}$

(xx) $\dfrac{2}{x[1 + 4(\log x)^2]}$ (xxi) $\dfrac{1}{3x^{2/3}(1 + x^{2/3})}$

(xxii) $-\dfrac{x}{\sqrt{1 - x^2}}$ (xxiii) $-\dfrac{1}{2(1 + x^2)}$

(xxiv) $\dfrac{4}{1+x^2}$ (xxv) $\dfrac{1}{a+b\cos x}$

12. (i) 1 (ii) $1/(1+\tan^{-1}x)^2$ (iii) $-2/(1+x)$

(iv) 1/2 (v) −1/2 (vi) 1

(vii) $e^{\sin^{-1}x}\left(\sin^{-1}x\dfrac{\sqrt{1-x^2}}{x}+\log x\right)$ (viii) −1/2

(ix) $-\tan x$

Exercises 1.5

1. (i) $-\dfrac{x^3}{y^3}$ (ii) $-\dfrac{x^2-ay}{y^2-ax}$ (iii) $-\dfrac{3x^2+10xy+12y^2}{5x^2+24xy+6y^2}$

(iv) $-\dfrac{y}{x}\left(\dfrac{2\sqrt{x}+\sqrt{y}}{2\sqrt{y}+\sqrt{x}}\right)$ (v) $-\dfrac{\sec^2(x+y)}{1-\sec^2(x+y)}$ (vi) $\dfrac{y\cos(xy)-2xy^2}{2x^2y-x\cos(xy)}$

(vii) $-\dfrac{\operatorname{cosec}^2(x+y^2)}{2y}$ (viii) $\dfrac{1-y\sec^2(xy)}{x\sec^2(xy)-1}$

(ix) $\dfrac{\sec^2(x+y)-\cos y+y\sin x}{\cos x-x\sin y-\sec^2(x+y)}$

(x) $-\dfrac{x}{y}$ (xi) $-\sqrt{\dfrac{y}{x}}$ (xii) $-\dfrac{x^{n-1}}{y^{n-1}}$

(xiii) $\dfrac{y}{x}$ (xiv) $\dfrac{y}{x}$ (xv) $-\dfrac{y}{x}$

(xvi) $\dfrac{1/\sqrt{x}-y^n}{2n\sqrt{x}/y-y^n}$ (xvii) $-\dfrac{24x^2+54y^2+70xy}{35x^2+81y^2+108xy}$

(xviii) $\dfrac{10(2x+y)^4-2xy^3}{3x^2y^2-5(2x+y)^4}$, (xix) $\dfrac{\cos(x+y)}{1-\cos(x+y)}$

(xx) $\dfrac{\sec(x+y)\tan(x+y)}{1-\sec(x+y)\tan(x+y)}$ (xxi) $-\dfrac{\operatorname{cosec}^2(x+y)}{1+\operatorname{cosec}^2(x+y)}$

(xxii) −1 (xxiii) 1

(xxiv) $\dfrac{1-\sec(x+y)\tan(x+y)}{1+\sec(x+y)\tan(x+y)}$ (xxv) $\dfrac{\cos(x+y)-y}{x-\cos(x+y)}$

(xxvi) $\dfrac{\sec^2(x+y)-y}{x-\sec^2(x+y)}$ (xxvii) $\dfrac{y\tan(x+y)}{1-y\tan(x+y)}$

(xxix) $-\dfrac{y}{x}$ (xxx) $-\dfrac{y}{x}$.

2. (i) $\frac{\cos y - \cos(x+y)}{x\sin y + \cos(x+y)}$ (ii) $\frac{\sin y + \sin(x+y)}{x\cos y + \sin(x+y)}$

(iii) $-\frac{2xy^3 + y\sin(xy)}{3x^3y^2 + x\sin(xy)}$ (iv) $\frac{y\cos(xy) - 2x}{2y - x\cos(xy)}$

(v) $\frac{y\cos(x+y) - 2xy^2}{2x^2y - x\cos(xy)}$ (vi) $\frac{1 - \frac{3}{2}\sin^2\frac{x}{2}\cos\frac{x}{2}y^2}{1 - 2y\sin^3\frac{x}{2}}$

(vii) $\frac{\sec^2(x+y) - \cos y + y\sin x}{\cos x - x\sin y - \sec^2(x+y)}$ (viii) $\frac{\sec^2(x+y) - \cos y - y\cos x}{\sin x - x\sin y - \sec^2(x+y)}$

16. $\frac{y - \sec^2(x+y)}{\sec^2(x+y) - x}$ **17.** $\frac{2x + (y\sin x - \cos y)(1+x^4)}{(1+x^4)(\cos x - x\sin y)}$

18. $-\frac{ye^{xy} + 2x\sin(x^2+y^2)}{xe^{xy} + 2y\sin(x^2+y^2)}$ **19.** $\frac{1 + (x+y)(e^x\cos e^x - 3x^2y^3)}{3y^2x^3(x+y) - 1}$

22. (i) $1/(2\sqrt{2})$ (ii) 2 (iii) 1

Exercises 1.6

1. (i) $-\frac{b}{a}\tan\theta$ (ii) -1, (iii) $-\tan\theta$ (iv) $\cot\frac{\theta}{2}$ (v) $\frac{a\cos t - b\sin t}{b\cos t - a\sin t}$

(vi) $\frac{t^2+1}{t^2-1}$ (vii) $-\sqrt{\frac{1+t^2}{1-t^2}}$ (viii) $\tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$ (ix) $-\frac{b}{a}e^{-2t}$

(x) $\frac{1}{t(e^t + \cos t)}$

2. (i) 1 (ii) $1\frac{1}{4}$ (iii) 2 **3.** (i) $\tan\theta$ (ii) $\tan\theta$ (iii) $\tan(\theta/2)$

4. 0 **5.** 0 **6.** 0

Exercises 1.7

1. (i) $x^x(1 + \log x)$ (ii) $\frac{y^2}{x(1 - y\log x)}$ (iii) $(\sin x)^x(x\cot x + \log\sin x)$

(iv) $(\sin x)^{\cos x}(\cos x\cot x - \sin x\log\sin x)$

(v) $(\sin x)^{\tan x}(1 + \sec^2 x\log\sin x)$

(vi) $(\sin x)^{\log x}[(1/x)\log\sin x + \cot x\log x]$

(vii) $(\cos x)^{\log x}[(1/x)\log\cos x - \tan x\log x]$

(viii) $-\sin x(\cos x)^{\cos x}(\log\cos x + 1)$

2. (i) $-\frac{1}{x^2}\left(1+\frac{1}{x}\right)^{1/x}\left[\log\left(1+\frac{1}{x}\right)+\frac{1}{1+x}\right]$ (ii) $1+x^{1/x-2}(1-\log x)$

(iii) $(1+x)^x\left(\log(1+x)+\frac{x}{1+x}\right)+x^{x+1}\left(\frac{1+x}{x}+\log x\right)$

(iv) $x^x(1+\log x)+x^{\sin x}\left(\frac{\sin x}{x}+\cos x\log x\right)$

(v) $(\tan x)^x\left(\log\tan x+x\frac{\sec^2 x}{\tan x}\right)+x(\sin x)^x(\log\sin x+x\cot x)$

(vi) $x^{\sin x}\left(\cos x\,\log x+\frac{\sin x}{x}\right)+(\sin x)^x\left(x\cos x+\log\sin x\right)$

(vii) $(\tan x)^{\cot x}(1-\log\tan x)\operatorname{cosec}^2 x+(\cot x)^{\tan x}(\log\cot x-1)\sec^2 x$

(viii) $x^x(1+\log x)+\sec^2 x\, e^{\tan x}$

(ix) $(\sin x)^{\cos x}\left(\cot x\cos^{-1}x-\frac{1}{\sqrt{1-x^2}}\log\sin x\right)$

3. (i) $\frac{y}{x}\frac{x\log y-y}{y\log x-x}$ (ii) $\frac{y}{x}\frac{x\log y\sec^2 x-\tan y}{\sec y(y\log x)-\tan x}$

(iii) $\frac{\log\tan x-y\tan x}{\log\sec x-\sec y\operatorname{cosec} y}$ (iv) $\frac{\cos x\log y-(1/x)\sin y}{\cos y\log x-(1/y)\sin x}$

(v) $-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

(vi) $x^{\tan x}\left(\frac{\tan x}{x}+\sec^2 x\log x\right)+(\tan x)^x(2x\operatorname{cosec}2x+\log\tan x)$

(vii) $\frac{y^2\sec^2 x}{\tan x(1-y\log\tan x)}$ (viii) $\frac{y^2}{x(2-y\log x)}$

(ix) $\frac{y(1-x\log y)}{x^2}$ (x) $\frac{y^2\cot x}{1-y\log\sin x}$

7. (i) $\frac{y}{x}\frac{\log y}{\log x}\frac{x\log x\log y+1}{1-x\log y}$ (ii) $e^{x^x}x^x(1+\log x)$

(iii) $y\log y\left(1+\log x+\frac{1}{x\log x}\right)$ (iv) $\frac{y}{x}\frac{\log y}{\log x}\frac{y\log x+1}{1-y\log x\log y}$

8. (i) $\frac{\log\sin y+y\tan x}{\log\cos x-x\cot y}$ (ii) $\frac{\log x}{(1+\log x)^2}$

(iii) $e^x x^{e^x}(\log x+1)$ (iv) $e^x e^{e^x}$

9. (i) $x^{\sin^{-1}x}\left(\dfrac{\log x}{\sqrt{1-x^2}}+\dfrac{\sin^{-1}x}{x}\right)$

(ii) $\dfrac{1}{x}(x\log x)^{\log(\log x)}\dfrac{1}{\log x}\log(\log x)(\log x+2)$

(iii) $(2x+3)^{3x+5}\left[3\log(2x+3)+\dfrac{2(3x+5)}{2x+3}\right]$

(iv) $x^x\sqrt{x}\left(\log ex+\dfrac{1}{2x}\right)$

10. $\dfrac{dy}{dx}=\dfrac{\sin^2 y}{\cos y},\left(\dfrac{dy}{dx}\right)_{x=-1}=\dfrac{3}{\pi\sqrt{\pi^2-3}}$

11. (i) $\dfrac{y^2\tan x}{y\log\cos x-1}$ (ii) $\dfrac{y^2\log y}{x[1-y(\log x)(\log y)]}$

12. (i) $\dfrac{1}{2y-1}$ (ii) $\dfrac{\sec^2 x}{2y-1}$ (iii) $\dfrac{y}{1-y}$ (iv) $\dfrac{y}{2y-x}$

14. $e^{x^{e^x}}\cdot x^{e^x}\left(\dfrac{e^x}{x}+e^x\log x\right)+x^{e^{e^x}}\cdot e^{e^x}\log x\left(e^x+\dfrac{1}{x\log x}\right)+x^{x^{e}}\cdot x^{x^e}\cdot e^{e-1}(1+e\log x)$

15. (i) $\sin^3 x\ \cos^5 x\,(3\cot x-5\tan x)$ (ii) $\dfrac{e^x(1+x)\cos x-x\sin x}{\cos^2 x}$

(iii) $\dfrac{x}{e^x\sin x}\left(\dfrac{1}{x}-1-\cot x\right)$

(iv) $\dfrac{e^x\tan x\,(1+\log x)-e^x x\log x(\sec^2 x+\tan x)}{(e^x+\tan x)^2}$

(v) $b\ \sin^m(bx)\ \cos^n(bx)\ (m\ \cot\ bx-n\ \tan\ bx)$

(vi) $2x+x^{\log x}\dfrac{1}{2x}\log x$

(vii) $x^x\log(ex)+(\log x)^x\left[\log(\log x)+\dfrac{1}{\log x}\right]$

(viii) $x^{\sin x}\left(\dfrac{\sin x}{x}+\cos x\log x\right)$

(ix) $e^{\sin 3x}\ (\sin x^3+3x^3\ \cos x^3)+(\tan x)^x\ (\log\ \tan x+x\ \cot x\ \sec^2 x)$

(x) $\dfrac{10^x(\cot x^2)\,x^{1/3}}{\sin 2x}\left[\log 10-4\,\mathrm{cosec}\,(2x^2)+\dfrac{1}{3}x-2\cot 2x\right]$

(xi) $\dfrac{(x+1)^2(x-1)^{1/2}}{(x+4)^3e^x}\left[\dfrac{2}{x+1}+\dfrac{1}{2(x-1)}-\dfrac{3}{x+4}-1\right]$

(xii) $\dfrac{1+x^2}{(1-x^2)^{1/3}}\left[\dfrac{2x}{1+x^2}+\dfrac{2x}{3(1-x^2)}\right]$

16. $\log a+1$ **17.** 0

18. (i) $x^{\sin x-\cos x}\left[(\cos x+\sin x)\log x+\dfrac{\sin x-\cos x}{x}\right]+\dfrac{4x}{(x^2+1)^2}$

(ii) $x^{\cot x}\left(\dfrac{\cot x}{x}-\operatorname{cosec}^2 x\log x\right)+\dfrac{2x^2+14x+3}{(x^2+x+2)^2}$

(iii) $\dfrac{2x^2\sqrt{4x+3}}{(3x+1)^2}\left(\dfrac{1}{x}+\dfrac{1}{4x+3}-\dfrac{3}{3x+1}\right)$

(iv) $\dfrac{2(x-\sin x)^{3/2}}{\sqrt{x}}\ \dfrac{2x-3x\cos x+\sin x}{2x(x-\sin x)}$

(v) $\dfrac{2x^4+15x^2+36}{3(x^2+3)^{2/3}(x^2+4)^{2/3}}$

19. (i) $\dfrac{2^{x+y}-2^x}{2^y-2^{x+y}}$ (ii) $\dfrac{y}{x}\dfrac{y^{x-1}\log y+x^{y-1}}{x^{y-1}\log x+y^{x-1}}$ (iii) $\dfrac{\log x}{(1+\log x)^2}$

(iv) $\dfrac{y}{x}\dfrac{x-y}{x+y}$ (v) $\dfrac{y}{x}\ \dfrac{1-x}{y-1}$ (vi) $\dfrac{x^2\sec^2(\log x)-2y}{x}$

(vii) $e^x(\log x)\sin x-\left(\cot x+\dfrac{1}{x\log x}+1+\log x\right)$

CHAPTER 2

Exercises 2.1

1. (i) $\dfrac{n!}{(a-x)^{n+1}}$ (ii) $\dfrac{(-1)^n n!}{a-b}\left[\dfrac{a}{(x-a)^{n+1}}-\dfrac{b}{(x-b)^{n+1}}\right]$

(iii) $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$ (iv) $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$

(v) $\dfrac{(-1)^n n!}{7}\left[\dfrac{1}{(x-3)^{n+1}}-\dfrac{3^{n+1}}{3(x-2)^{n+1}}\right]$

(vi) $(-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta$, where $\theta=\cot^{-1}x$

2. (i) $\dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

(ii) $e^{ax}\left[a^n\cos(bx)+{}^nC_2a^{n-1}\cos\left(bx+\dfrac{\pi}{2}\right)+\cdots\right]$

(iii) $x^n e^{3x}\sin\left(4x+n\tan^{-1}\dfrac{4}{3}\right)$,

(iv) $r^n e^{ax}\cos\left(br+c+\tan^{-1}\dfrac{b}{a}\right)$, where $r=(a^2+b^2)^{n/2}$

(v) $x^2\cos\left(x+\dfrac{n\pi}{2}\right)+2nx\cos\left(x+\dfrac{n-1}{2}\pi\right)+n(n-1)\cos\left(x+\dfrac{n-2}{2}\pi\right)$

(vi) $2(n-3)!\,\dfrac{(-1)^{n-3}}{x^{n-2}}$, (vii) $a^{n-2}e^{ax}\left[a^2x^2+2anx+n(n-1)\right]$

(viii) $\dfrac{(n-1)!}{x}$, (ix) $4^n x\sin\left(4x+\dfrac{n\pi}{2}\right)+n4^{n-1}\sin\left[4x+(n-1)\dfrac{\pi}{2}\right]$

(x) $-2^{n-3}\left[4x^2\cos\left(2x+\dfrac{n\pi}{2}\right)+4nx\cos\left(2x+\dfrac{n-1}{2}\pi\right)+n(n-1)\cos\left(2x+\dfrac{n-2}{2}\right)\right]$

(xi) $x^3\cos\left(x+\dfrac{n\pi}{2}\right)+3nx^2\cos\left(x+\dfrac{n-1}{2}\pi\right)+\dfrac{n(n-1)}{2!}6x\cos\left(x+\dfrac{n-2}{2}\pi\right)$

$+\dfrac{n(n-1)(n-2)}{3!}6\cos\left(x+\dfrac{n-3}{2}\pi\right)$

(xii) $n!\left(\log x+1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}\right)$ (xiii) $a^{n-2}e^{ax}\left[a^2x^2+2nax+n(n-1)\right]$

3. (i) $2(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$ (ii) $2(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$

(iii) $(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta,\ \theta=\cot^{-1}x$

(iv) $2(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta,\ \theta=\cot^{-1}x$

(v) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

26. $x^2y_2+xy_1+n^2y=0$

27. $(-1)^n(n-2)!\left[\dfrac{x-n}{(x-1)^n}-\dfrac{x+n}{(x+1)^n}\right]$

28. (i) $-4(-1)^n\dfrac{n!}{(x+2)^{n+1}}+\dfrac{9}{2}\dfrac{(-1)^n2^n}{n!(2x+3)^{n+1}}$

(ii) $(-1)^n n! a^{\frac{-n-1}{2}} \cos(n+1)\theta$, where $\theta = \tan^{-1}\dfrac{a}{x}$

29. $(-1)^{n-1}(n-1)!\ \text{cosec}^n\alpha \sin^n\theta \sin n\theta$, where $\theta = \tan^{-1}\left(\dfrac{x-\cos\alpha}{\sin\alpha}\right)$

30. $\dfrac{1}{8}\left[3e^x - 4\cdot 5^{n/2} e^x \cos(2x + n\tan^{-1}2) + 17^{n/2}\cos(4x + n\tan^{-1}7)\right]$

31. $\dfrac{1}{16}\left[2\cos\left(x+\dfrac{n\pi}{2}\right) - 3^n\cos\left(3x+\dfrac{n\pi}{2}\right) - 5^n\cos\left(5x+\dfrac{n\pi}{2}\right)\right]$

Exercises 2.2

1. (i) $e^x(x+n)$ (ii) $\dfrac{2(-1)^{n-3}(n-3)!}{x^{n-2}}$

(iii) $(1/2)e^x - (1/2)5^{n/2}e^x\cos(2x + n\tan^{-1}2)$

(iv) $2^{n-3}\left[4x^2\cos\left(2x+\dfrac{n\pi}{2}\right) + 4nx\cos\left[2x+\dfrac{n(n+1)}{2}\pi\right]\right.$

$\left.+ n(n-1)\cos\left(2x+\dfrac{n-1}{2}\pi\right)\right]$

(v) $e^{ax}a^{n-3}[a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)]$

10. $\dfrac{(-1)^n 6(n-4)!}{x^{n-3}}$ **11.** $\dfrac{n!}{(x+1)^{n+1}}$

14. $y_2 = -2ay, -(b^2+a^2)y, y_{n+2} + 2ay_{n+1} + (a^2+b^2)y_n$

27. $\dfrac{2(-1)^n n!}{(x+3)^{n+1}} - \dfrac{(-1)^n n!}{(x+2)^{n+1}}$ **28.** $\dfrac{(n+1)!(-1)^n}{3(x-1)^{n+2}} + \dfrac{5}{9}\dfrac{n!(-1)^n}{(x-1)^{n+1}} + \dfrac{4(-1)^n n!}{(x+2)^{n+1}}$

29. $\dfrac{1}{2}(-1)^n n!\left[\dfrac{1}{(x-1)^{n+1}} + \dfrac{1}{(x+1)^{n+1}}\right]$ **30.** $(-1)^n n!\left[\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right]$

31. $-\dfrac{(-1)^n n!}{(x-1)^{n+1}} - \dfrac{(-1)^n(n+1)!}{1!(x-1)^{n+2}} - \dfrac{(-1)^n(n+2)!}{2!(x-1)^{n+3}} + \dfrac{(-1)^n n!}{(x-1)^{n+1}}$

32. $\dfrac{1}{2}(-1)^n n!\left\{\dfrac{1}{(x+1)^{n+1}}\sin^{n+1}(\cot^{-1}x)\sin\left[(n+1)\cot^{-1}x\right]\right.$

$\left.-\cos^{n+1}(\cot^{-1}x)\cos\left[(n+1)\cot^{-1}x\right]\right\}$

33. $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \tan^{-1}\dfrac{1}{x}$

35. $5^{n/2} x^2 e^{2x} \sin\left(x + n\tan^{-1}\frac{1}{2}\right) + 2nx\, 5^{(n-1)/2} e^{2x} \sin\left[x + (n-1)\tan^{-1}\frac{1}{2}\right]$

$+ n(n-1)\, 5^{(n-2)/2} e^{2x} \sin\left[x + (n-2)\tan^{-1}\frac{1}{2}\right]$

CHAPTER 3

Exercises 3.1

1. $\dfrac{6 \pm 2\sqrt{3}}{8}$ **2.** (i) 1/7 (ii) $\sqrt{5}$ (iii) log(*e*–1) (iv) $\frac{3}{4}\log 2$

3. (i) $-2 \pm \sqrt{6}$ (ii) 1/2 (iii) $\frac{1}{h}\log\frac{e^h - 1}{h}$

21. $1 + ax + (a^2 - b^2)\frac{x^2}{2!} + (a^2 - 3ab^2)\frac{x^3}{3!} + \cdots$ **23.** $1 + \frac{1}{6}\frac{2^2 x^2}{2!} - \frac{1}{30}\frac{2^4 x^4}{4!} + \cdots$

24. $x - \frac{1}{2}\frac{x^2}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$ **25.** $x - \frac{3}{2}x^2 + \frac{4}{6}x^3 - \cdots$

26. $\log[1 - \log(1-x)] = x + \frac{x^3}{6} + \cdots,\ \log[1 + \log(1+x)] = x - x^2 + \frac{7x^3}{6} - \cdots$

27. $\sin\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = \sin\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right).f'\left(\frac{\pi}{2}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)f''\left(\frac{\pi}{2}\right) + \cdots$

CHAPTER 4

Exercises 4.1

1. (i) 2 (ii) log (*a*/*b*) (iii) 2 (iv) 1
(v) na^{n-1} (vi) 1 (vii) 2/3 (viii) –1/2
(ix) 2 (x) $\pi^2/2e$ (xi) 2*a*/*b* (xii) 3/2
(xiii) 1/2 (xiv) –2/3 (xv) 1/2

2. $a = -2$, limit is –1 as $x \to 0$ **3.** –1/2

4. (i) 1 (ii) –1/12 (iii) –1/2 (iv) 1/3
(v) 1 (vi) 4 (vii) $-2/\pi$ (viii) 2/3

5. (i) 3 (ii) 0 (iii) 1 (iv) 0
(v) 1 (vi) 5

6. (i) 0 (ii) 0 (iii) 0 (iv) 1
(v) 1 (vi) 1 (vii) $-1/\pi$ (viii) $2a/\pi$

7. (i) 1/2 (ii) 1 (iii) 1/2 (iv) –1/4
(v) 1/2 (vi) 0 (vii) 2/3 (viii) 1/6

40. $x=\frac{3a}{2}(\cosh u\sinh u+u)+a\sinh u\cosh^3 u$

44. $x+1=0$ at $(-1, 1)$

45. $x^2+y^2=a^2$, i.e. a circle

46. Tangent at $t=2a$
Normal at $t=9a$

47. Tangent: $x+y=3a$
Normal: $x+y+a=0$

48. Tangent: $31x+6y+9a=0$
Normal: $8x-31y+42a=0$

49. Tangent: $4x+2y-a=0$
Normal: $2x-4y+3a=0$

50. The tangent is parallel to y-axis at the origin and also at

$$\left(\frac{2^{2/3}}{3}a^{2/3}, b^{1/3}, \frac{2^{1/3}}{3}a^{1/3}b^{2/3}\right)$$

51. $mx(X-x)+xy(Y-y)=0$

52. $p=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}}$

58. $\left(\frac{1-2x_1}{2}, \frac{2x_1-7x_1^2+7x_1^3}{2}\right)$

59. $m^2h+m(a-b)-h=0$. The roots of the equation are on the slope of the axes.

Exercises 6.2

5. Angles of intersection is $(1/2)(\pi+\theta)$

10. $pr=a^2$

14. $r^2=2ap$

16. $r^3=2ap^2$

CHAPTER 7

Exercises 7.1

1. (i) a (ii) $4a\cos\psi$ (iii) $\operatorname{cosec}^2\psi$
(iv) $a\sec\psi$ (v) $c\tan\psi$

2. ae^{ψ}

3. $2a\sec^3\psi$

4. (i) $2a^{-1/2}(a+x)^{3/2}$ (ii) $\frac{(x^2+y^2)^3}{2c^2}$ (iii) $(4a+9x)^{3/2}\frac{\sqrt{x}}{6a}$

(iv) $\frac{a(1+e^{2x/a})^{3/2}}{e^{x/a}}$ (v) $\frac{b^4x^2+a^4y^2}{a^4b^4}$ (vi) $\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}$

(vii) $5\sqrt{5}/4$ (viii) $\pm 2a$ (ix) $1/2$

(x) at (xi) $\frac{125\sqrt{2}}{2}$ (xii) $\frac{a}{\sqrt{2}}$

(xiii) $\frac{2a^2}{b}$ (xiv) $\frac{1}{\sqrt{2}}$ (xv) $\frac{a\sqrt{2}}{3}$

5. 2 **6.** $-\frac{3a}{8\sqrt{2}}$ **8.** $\pm 2\sqrt{2}$ **11.** y^2/c

20. (i) $\frac{1}{z}(a^2 + b^2)^{1/2}$ (ii) $\frac{a(1+\theta^2)^{3/2}}{2+\theta^2}$ (iii) $\frac{2x}{(2a-x)\sqrt{a}}$

(iv) $\frac{a}{2}$ (v) $\frac{a^2}{3r}$ (vi) $\frac{a^n}{(n+1)r^{n-1}}$

(vii) $\frac{2}{3}(2ar)^{1/2}$ (viii) $\frac{a}{3}$ (ix) $\frac{a}{2}$

21. (i) a (ii) $\frac{2\sqrt{2}}{3}ar$ (iii) r cosec α

(iv) $\frac{2r^{3/2}}{a^{1/2}}$ (v) $-3p$ (vi) $\frac{a^2b^2}{p^3}$

24. (i) a (ii) $2a$ cosec$^2\psi$

27. $\frac{a}{4}$ **32.** $x=\frac{2}{3},\ y=\pm\frac{2\sqrt{2}}{\sqrt{3}}$

33. $\frac{(x^4+a^4)^{3/2}}{2a^4y}, \frac{(x^4+a^4)^{3/2}}{2a^4x}$

Exercises 7.2

1. $2r$ **2.** 2
3. (i) $2r$ (ii) $2r/3$

8. $x^2 + y^2 - \frac{2(b^2-a^2)}{b}y + (b^2 - 2a^2) = 0$ **9.** y^2/c

17. (i) $\pm 2a\sqrt{2}$ (ii) $5\sqrt{5}/18$ (iii) 3/8

(iv) $na/2$ (v) $5\sqrt{5}/2$, -2 (vi) $a/4$

(vii) (0, −1/2) (viii) (0,1/2)

20. $x = a \cos t\,(\cos^2 t + 3\sin^2 t)$, $y = a \sin t\,(\sin^2 t + 3\cos^2 t)$.

22. (i) $\sqrt{2}, \frac{5}{2}\sqrt{5}$ (ii) 1 (iii) $\frac{85\sqrt{17}}{2}, 5\sqrt{2}$

25. (i) $\left(\frac{21}{16}a, \frac{21}{16}a\right)$ (ii) $x = a(t - \sin t)$, $y = a(1 + \cos t)$ (iii) $\left(\frac{6}{7}a, \frac{6}{7}a\right)$

CHAPTER 8

Exercises 8.1

1. $x + y = 0$
2. $x + y = 0$
3. $x = 0, \quad y = 0, \quad x + y = 0$
4. $x = \pm 1, \quad x = y$
5. $x = a$
6. $x = \pm a, \quad y = \pm x$
7. $x - y = a - b, \quad x + a = 0, \quad y + b = 0$
8. $x - y = a, \quad x + y = a$
9. $x = \pm a, \quad y \pm x = a$
10. $2x = \pm 1, \quad 2y = 3x, \quad 2y + 3x = 0$
11. $x = 0, \quad y - x = a, \quad x + y + a = 0$
12. $y + x = 0, \quad y = x, \quad y - x = 1$
13. $x = 0, \quad y = 0, \quad x = 1, \quad y = 1$
14. $x + 2y = 5/3$
15. $x = y, \quad x + y = 0, \quad x + 2y + 1 = 0$
16. $x + 3 = 0, \quad x = 2y, \quad x + 2y = 6$
17. $x - y + 1 = 0, \quad x + y = 1, \quad x + 2y = 0$
18. $x - y + 2 = 0, \quad 2x - 3y + 4 = 0, \quad 4x - 5y + 6 = 0$
19. $y = x - 1, \quad y + x + 2 = 0, \quad y = 2x$
20. $2x + 5 = 0, \quad 2x + 4y + 1 = 0, \quad x + y + 1 = 0$
21. $y + x = 2, \quad y = x + 2, \quad y = 2x - 4$
22. $2x = y, \quad x = y, \quad 2x - y + 3 = 0$
23. $x = \pm a, \quad y = \pm a$
24. $x = 0, \quad y = 0, \quad 2y - 4x + 3 = 0, \quad 2y + 4x = 15$
25. $x - y = \pm 1, \quad x + y = \pm 1$
26. $x = 0, \quad y = 2, \quad 2y - 3x - 10 = 0$
27. $x = 0, \quad y = 0, \quad x + y = 0$
28. $x + y - a = 0, \quad x + y + a = 0$
29. $x + y = \pm a, \quad x - y = \pm a$
30. $y - x = 0, \quad y - x - 1 = 0, \quad y - x - 2 = 0, \quad y + x = 0$
31. $y - 2x + 2 = 0, \quad y - 2x + 3 = 0, \quad x - y + 4 = 0$
32. $y = 0, \quad x = \pm a$
33. $x + 3 = 0, \quad x - y - 2 = 0, \quad x - y - 4 = 0$
34. $2x = \pm 1, \quad 3x = \pm 2y$
35. $x = y, \quad x = 2y, \quad x = 3y$
36. $y = x, \quad y = 2x, \quad y = 3x$
37. $x + a = 0, \quad y + b = 0, \quad x - a = a - b$
38. $x = 0, \quad y - x = m/2, \; y + x + m/2 = 0$
39. $x = 0, \quad y - x - a = 0, \quad y + x + a = 0$
40. $x + 2y + 2 = 0, \quad 2x + 4y + 1 = 0, \quad 2x + 5 = 0$
41. $x = 0, \quad y = 0, \quad 2y = 4x + 3, \quad 2y + 4x = 15$
42. $x = 0, \quad y = 2, \quad 2y - 3x - 10 = 0$
43. $x + y = \pm 1/\sqrt{2}, \quad x - y = \pm 1$
44. $x + y = 0, \quad 2x - 3y - 1 = 0, \quad 2x - 3y + 3 = 0$
45. $x = 3, \quad y = x + 1, \quad y = x + 2$
46. $y = 0, \quad x - y = 0, \quad x - y - 1 = 0, \quad x - y + 1 = 0$
47. $x - y = a/2, \qquad x + y = a/2$
48. $x + y = 0, \quad x + y + 2 = 0, \quad 4x - 8y - 7 = 0, \quad 4x + 8y - 9 = 0$
49. $r \cos\theta = \pm a$
50. $r(\sqrt{3}\sin\theta - 3\cos\theta) = \pm 4a$
51. $r \sin\left(\theta - m\pi/n\right)$
52. $r \cos\theta = a \pm b$
53. $r \cos\theta = \pm 1$
54. $r \sin\theta = 2$
55. $a + 2r(\cos\theta + \sin\theta) = 0, \; a - 2r(\cos\theta - \sin\theta) = 0$

46. One piece = $25\pi/(\pi + 4)$ metre, other piece = $100/(\pi + 4)$ metre
51. 9, 6
55. $3\sqrt{3}/4\,ab$ sq.m
56. $15(8/3)^{2/3}$ miles per hour
57. Breadth = $\left(\sqrt{4/3}\right)a$ depth $=\left(\sqrt{8}/9\right)a$

Exercises 9.2

1. (i) Minima at (1, 0, 0); neither maxima nor minima at (0, 0, 0)
   (ii) Maxima at (–2, –1)
   (iii) Maxima at (3, –1)
   (iv) When $x = y \neq 0$, $x + y = a$ at $(a/3, a/3)$; max value is $a^3/27$
   (v) Min. at (2, 1/2)
   (vi) $(108/77)a^7$, min. value at $3a^2$
   (vii) Maxima and minima at (a, a), neither max. nor min. at (0, 0)
   (viii) Minima at (2/3, –4/3); neither maxima nor minima at (0, 0)
   (ix) Max. value at $x = 3$, $y = -1$
   (x) Max. at (9, 1).
   (xi) Max. at (–2, –1), min at (2, 1), but the function has neither min. nor max. at (–2, 1) and (2, –1) respectively
   (xii) Max. value at $(\sqrt{2}, -\sqrt{2})$ also $(-\sqrt{2}, \sqrt{2})$
2. Neither maxima nor minima at (0, 0)
7. (i) $3\sqrt{3}/8$ (ii) $3\sqrt{3}/8$
8. $x = y = n\pi + (-1)^n\pi/6$
10. $\dfrac{2a\Delta}{a^2+b^2+c^2}, \dfrac{2b\Delta}{a^2+b^2+c^2}, \dfrac{2c\Delta}{a^2+b^2+c^2}$ are perpendicular distance of the point from the sides of the triangle
11. Min. value = 4
13. $108(a/7)^{7\cdot}$ min $3a^2$
17. $\dfrac{1}{2e}\sum\dfrac{a^2}{\alpha^2}$
18. (i) Min. value = $3a^2$, (ii) Min. value = $3a^2$, (iii) Min. value = $3a^2$
19. (i) (0, 0), (ii) Min. value = 1/3; extreme point
20. $$\begin{vmatrix} u & 1 & 1 \\ 1 & a^2+b^2+c^2 & aa'+bb'+cc' \\ 1 & aa'+bb'+cc' & a'^2+b'^2+c'^2 \end{vmatrix}$$
21. Min value = $3a^2$ at point (a, a, a), $(-a, -a, -a)$
23. Min value = 9
24. $A = B = C = \pi/3$, max. value = $(1/2)^3$
26. $[\log(Aabc)]^3/(\log a^3 \log b^3 \log c^3)$
27. r_1 and r_2 the maximum and minimum values, where r_1^2 and r_2^2 are the roots of the equation; are $a = \pi c/\sqrt{(ab - h^2)}$

28. Solving the following determinant, we get the required parts of n:

$$\begin{vmatrix} 0 & c & b & 2u/n \\ c & 0 & a & 2u/n \\ b & a & 0 & 2u/n \\ 1 & 1 & 1 & n \end{vmatrix} = 0$$

29. $x/\sqrt{a} = y/\sqrt{b} = z/\sqrt{c} = \sqrt{a} + \sqrt{b} + \sqrt{c}$

30. Max. value = $(a/9)^9$

33. Equilateral triangle

35. Min. value = 12, max. value = $14e$

36. Quadratic in r and the equation is in the form of r^2

37. Max. value = 3/4

39. $\sum \dfrac{l^2}{au-1}$

40. Max. value = 3456

41. Min. value = 7/3

42. Stationary values are $x = \dfrac{l\lambda_2}{au-1}, y = \dfrac{m\lambda_2}{bu-1}, z = \dfrac{n\lambda_2}{cu-1}$ where $u = x^2 + y^2 + z^2$

46. Max. value = $64/\sqrt{3}$

48. Max. value = 24, min. value = –24

49. Min. value = 2/27

50. Max. value = 5

51. Max. value = 12

52. Max. value = 12

54. Min. value = 8

CHAPTER 10

Exercises 10.1

1. (i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (ii) $c^2(x^2 + y^2) = x^2y^2$

(iii) $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\,\theta\cos\theta$

2. $x^2(x - a) = 0$

3. $y = 0$

4. $B^2 = 4AC$

5. (i) $x^{2/3} + y^{2/3} = c^{2/3}$ (ii) $4xy = c^2$

7. $\dfrac{a^4}{x^2} + \dfrac{b^4}{y^2} = c^2$

8. $y + \dfrac{1}{2}g\dfrac{x^2}{b^2} = \dfrac{b^2}{2g}$

10. (i) $x^{2/3} + y^{2/3} = c^{2/3}$ (ii) $x \pm y \pm c = 0$

12. $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$

13. $x^{2/3} + y^{2/3} = (4a)^{2/3}$

19. (i) $r^2 - 2\,br\cos\theta + (b^2 - a^2) = 0$ (ii) $r^{n/(1-n)} = a^{n/(1-n)}\cos\dfrac{n\theta}{1-n}$

(iii) $r\sin\alpha = ae^{\alpha-\pi/2}\cot\alpha e^{\theta\cot\alpha}$

21. $p^2x - py + (1 + ap^2 + pq) = 0$

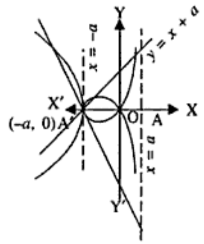
24. (i) $r^2(e^2 - 1) - 2le^r\cos\theta + 2l^2 = 0$ (ii) $r^{n/(n+1)} = a^{n/(n+1)}\cos\dfrac{n\theta}{n+1}$

26. $\lambda^2r^2 = a^2\mu^2 + (\lambda^2 - \mu^2)p^2$

CHAPTER 11

Exercises 11.1

1.



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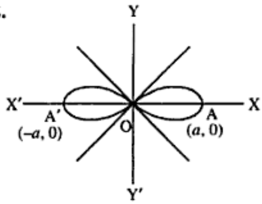

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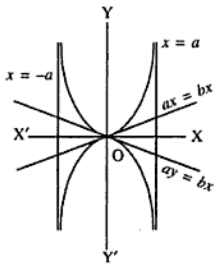

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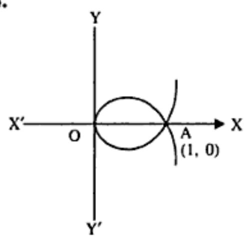

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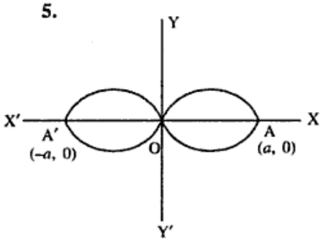

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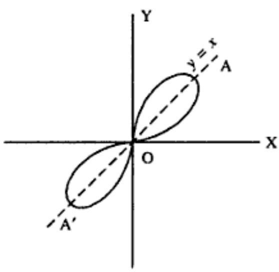

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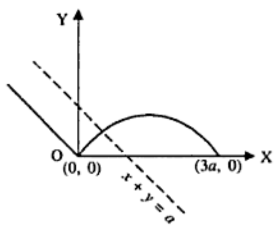

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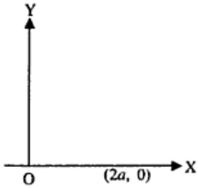

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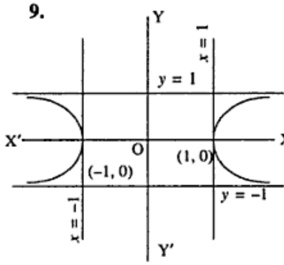

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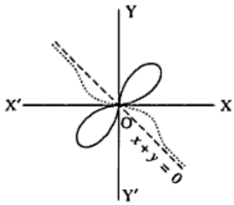

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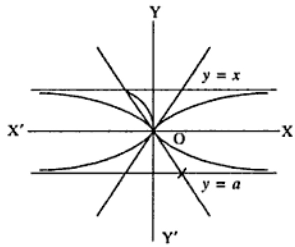

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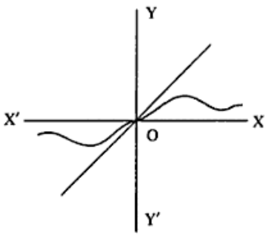

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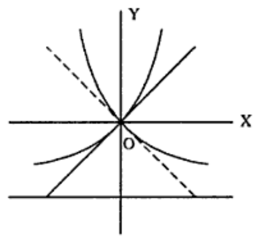

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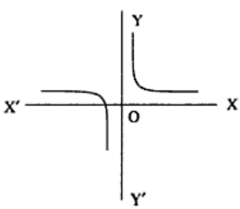

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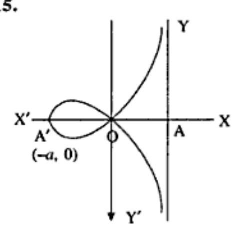

MULTIPLE-CHOICE QUESTIONS

1. (c)	2. (c)	3. (a)	4. (b)
5. (c)	6. (b)	7. (c)	8. (a)
9. (d)	10. (b)	11. (b)	12. (d)
13. (a)	14. (c)	15. (b)	16. (a)
17. (b)	18. (a)	19. (b)	20. (c)
21. (a)	22. (d)	23. (a)	24. (a)
25. (a)	26. (a)	27. (a)	28. (c)
29. (c)	30. (b)	31. (d)	32. (c)
33. (a)	34. (c)	35. (b)	36. (b)
37. (b)	38. (d)	39. (b)	40. (c)
41. (c)	42. (a)	43. (a)	44. (c)
45. (d)	46. (b)	47. (c)	48. (c)
49. (b)	50. (d)	51. (b)	52. (a)
53. (c)	54. (b)	55. (c)	56. (c)
57. (b)	58. (b)	59. (c)	60. (d)
61. (b)	62. (b)	63. (a)	64. (a)
65. (b)	66. (d)	67. (c)	68. (a)
69. (c)	70. (b)	71. (a)	72. (d)
73. (a)	74. (c)	75. (b)	76. (a)
77. (a)	78. (c)	79. (d)	80. (b)
81. (a)	82. (b)	83. (a)	84. (c)
85. (d)	86. (d)	87. (b)	88. (b)
89. (c)	90. (b)	91. (a)	92. (a)
93. (a)	94. (c)	95. (a)	96. (c)
97. (a)	98. (a)	99. (a)	100. (a)
101. (b)	102. (c)	103. (d)	104. (b)
105. (c)	106. (a)	107. (b)	108. (a)
109. (c)	110. (b)	111. (b)	112. (c)
113. (c)	114. (d)		

Index

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ISBN:978-81-203-3853-1

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