DIFFERENTIAL EQUATIONS

USEFUL FOR

CSIR UGC NET, GATE, IIT-JAM, NBHM, TIFR & other exams with similar syllabus

First Edition

By

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Preface

This book is designed for the students who are preparing for various National Level competitive examinations and also inspires to enter into Ph.D. Programs by qualifying the various entrance exams.

The content of the book is divided into two parts: first part introduces the ordinary differential equations while the second one introduces partial differential equations. The chapter one gives a brief introduction of differential equations and elaborates various methods to solve the ordinary differential equations of first order. The second chapter named " General Linear Differential Equations" explains how to solve the differential equations of higher order and the methods to find their general, particular and singular solutions. In Chapter three, the solutions of initial value problems are being explained by using Lipschitz condition and by general methods. It also explains Green's function and Sturm Liouville's problem which gives non-trivial solutions of a boundary value problem. The fourth chapter introduces the partial differential equations and various kind of methods to solve them. Chapter five includes the behaviours of second order differential equations and the transformation to reduce them into the canonical form. Apart from this, the solutions of Heat and Laplace equations are also discussed.

The practice sets are introduced at the end of the topics which includes a variety of questions from previous year papers of CSIR UGC NET, IIT-JAM, TIFR, NBHM and GATE. These questions are carefully selected so that the students can apply mathematical knowledge in solving the questions. In addition to it, the solved examples are also given at the end of every chapter which will help in deep understanding of the topics discussed. The key points provides the quick revision of every chapter. Also, a well-thought question bank, in the form of various assignments is given at the end of each chapter which covers entire prescribed topics, so as to facilitate students to do more and more practice and hence secure good results.

While compiling this book, more stress is given on problem solving technique rather than language or exact mathematical symbols. Any suggestions for the improvement of the book will be highly appreciated.

Dr A.P. Singh

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CHAPTER – 1
INTRODUCTION TO DIFFERENTIAL EQUATIONS AND DIFFERENTIAL EQUATIONS OF FIRST ORDER

INTRODUCTION

Differential equations arise from many problems in Algebra, Geometry, Mechanics, Physics and Chemistry. In this chapter we shall study how a differential equation look like, what are its different types, what is the order and the degree of the differential equation and we shall learn different kinds of solutions of the differential equations and also the method of solving first order differential equation.

§ 1.1. DIFFERENTIAL EQUATION

An equation $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$, *which expresses a relation between dependent and independent variables and their derivatives of any order, is called a differential equation. We may also define it as, "An equation involving unknown functions and their derivatives w.r.t. one or more independent variables".*

e.g. *(i)* $\frac{dy}{dx} = x + 3$ *(ii)* $\frac{dy}{dx} = \sin x$ *(iii)* $\frac{d^2y}{dx^2} + \mu x = 0$

(iv) $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$ *(v)* $\frac{\partial z}{\partial x} = z + x\frac{\partial z}{\partial y}$ *(vi)* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$

There are two types of differential equations:

(i) Ordinary *(ii) Partial*

DEFINITIONS:

(i) ***Ordinary Differential Equation:*** *An ordinary differential equation is the equation which involves one independent variable and differential co-efficients w.r.t. it. Thus (i) to (iv) are ordinary differential equations.*

(ii) ***Partial Differential Equation:*** *A partial differential equation is the equation which involves two or more independent variables and partial derivatives w.r.t any of them. Thus (v) and (vi) are partial differential equations.*

(iii) ***Order of Differential Equation:*** *The order of a differential equation is the order of the highest derivative it contains. In the above examples the order of (i) and (ii) is 1, the order of (iii) is 2, the order of (iv) is 3.*

(iv) ***Degree of differential equation:*** *The degree of the differential equation, which can be written as a polynomial in the derivatives is the degree of the highest ordered derivative which then occurs.*

e.g. $2y=\sqrt{3x}\frac{dy}{dx}+\frac{7}{\frac{dy}{dx}}$

This differential equation is of degree 2 (obtained by making it free from fractions)

(v) **General Forms**. *An ordinary differential equation of order one is of the form* $f\left(x,y,\frac{dy}{dx}\right)=0$.

An ordinary differential equation of order two is of the form $f\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0$, *etc. In general, an ordinary differential equation of order n is of the form* $f\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n}\right)=0$ *or* $f(x,y,y',y'',\ldots,y^n)=0$.

§ 1.2. LINEAR DIFFERENTIAL EQUATION

A differential equation $f\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n}\right)=0$ *is said to be linear iff the function f is a linear function of the variables* $y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n}$, *i.e., iff the dependent variables and its derivatives occur in the first degree and are not multiplied together.*

Thus $P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+P_ny=Q$, *where* $P_0,P_1,\ldots,P_n$ *and Q are functions of x only and* $P_0\neq0$ *is the general linear differential equation of order n.*

The above equation is said to be **homogeneous** *iff Q is a zero function and* **non-homogeneous** *iff Q is a non-zero function.*

Example 1. *Obtain the differential equation of all circles of radius r.*

Solution: *The equation of the family of circles of radius r is* $(x-a)^2+(y-b)^2=r^2$, ... (i)

where a and b are arbitrary constants.

Since equation (i) contains two arbitrary constants, we differentiate it two times w.r.t. x and the differential equation will be of second order.

Differentiating (i) w.r.t. x, we get $2(x-a)+2(y-b)\frac{dy}{dx}=0$

$\Rightarrow (x-a)+(y-b)\frac{dy}{dx}=0$ ...(ii)

Differentiating (ii) w.r.t. x, we get $1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$ ...(iii)

From (iii), we have $y-b = -\left(\dfrac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}\right)$ ...(iv).

Putting the value of $(y-b)$ *in (ii), we obtain* $x-a = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]\dfrac{dy}{dx}}{d^2y/dx^2}$ ...(v)

Substituting the values of $(x-a)$ *and* $(y-b)$ *in (i), we get*

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2}{(d^2y/dx^2)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{(d^2y/dx^2)^2}=r^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

This is the required differential equation.

§ 1.3. *TYPES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS*

(i) ***Solution.*** *The solution of a differential equation is a relation between the variables involved such that this relation and differential co-efficients obtained from there satisfy the given differential equation. This is also called* ***primitive*** *or* ***integral*** *of the differential equation.*

(ii) ***General Solution.*** *The solution of a differential equation, which contains as many arbitrary constants as the order of the differential equation, is said to be general solution. This is also called* ***complete primitive*** *or* ***complete solution*** *of the differential equation.*

e.g. *the general solution of* $\dfrac{d^2y}{dx^2} + y = 0$ *is* $y = c_1 \cos x + c_2 \sin x$.

(iii) ***Particular Solution.*** *The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to arbitrary constants.*

e.g. *the particular solution of* $\dfrac{d^2y}{dx^2} + y=0$ *is* $y = \cos x$

(iv) ***Singular Solution.*** *A solution which cannot be derived from the general solution by giving particular values to the arbitrary constants and has no arbitrary constant is called singular solution.*

§ 1.4. *EQUATIONS OF FIRST ORDER AND FIRST DEGREE*

A general differential equation of first order and first degree is of the form $\dfrac{dy}{dx} = f(x,y)$ *or* $Mdx + Ndy = 0$, *where M, N are functions of x and y both.*

- ***Existence and Uniqueness Theorem Statement:*** *If f(x, y) and* $\frac{\partial f}{\partial y}$ *are continuous functions of x and y in a region D of the xy-plane and if* $P(x_0, y_0) \in D$, *then there exists one and only one function; say* $\phi(x)$, *which in some neighbourhood of P (contained in D) is the solution of the differential equation*

  $\frac{dy}{dx} = f(x,y)$ *and* $\phi(x_0) = y_0$.

Following are some of the Standard methods to solve differential equations:

1. Variable separable form
2. Reducible to variable separable.
3. Homogeneous equations
4. Reducible to homogeneous form
5. Linear differential equations of first order
6. Reducible to linear form
7. Exact differential equations.

Case I. Variable Separable: *An equation whose variables are separable and can be put in the form* $g(x)dx + h(y)\,dy = 0$ *is called an equation of variable separable form. Integrating,* $\int g(x)\,dx + \int h(y)\,dy = c$, *where c is an arbitrary constant. This is the general solution of the differential equation.*

Case II. Equations Reducible to variable separable: *To solve the equation* $\frac{dy}{dx} = f(ax + by + c)$ ... (1)

Put $ax + by + c = z$ *so that* $a + b\frac{dy}{dx} = \frac{dz}{dx}$, *i.e.,* $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$

Putting in (1), $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z) \Rightarrow \frac{dz}{dx} = a + bf(z)$, *which is of the type "Variable Separable" and hence can be solved.*

Method to Solve:

(i) Put $ax + by + c = z$ *(ii) Separate the variables z and x.* *(iii) Integrate both sides.*

Case III. Homogeneous Equations:

Definition: Homogeneous Function*. A function is said to be homogeneous of the nth degree in x and y if it can be put in the form* $x^n f\left(\frac{y}{x}\right)$.

e.g. $\phi(x, y) = x^4 + x^2y^2 + xy^3 + y^4$.

Definition: Homogeneous Equation*. An equation in x and y is said to be homogeneous equation if it can be put in the form* $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, *where f(x, y) and* $\phi(x, y)$ *are both homogeneous functions of the same degree in x and y.*

(i) *To solve the equation* $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, *where f(x,y) and* $\phi(x, y)$ *are both homogeneous functions of the same degree in x and y.*

(i) Put $y = vx$ *so that* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *(ii) Proceed as in Case I.*

(ii) *Sometimes the equation is not homogeneous, even then it can be solved by the same method as for homogeneous equations.*

Example 1. *Solve:* $x^2y\,dx - (x^3 + y^3)dy = 0$

Solution: *The given differential equation is* $x^2ydx-(x^3+y^3)dy=0 \Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$ ...(1)

Since each of the functions x^2y *and* $x^3 + y^3$ *are homogeneous functions of degree 3, so the given differential equation (1) is homogeneous.*

Putting $y=vx$ *and thus* $\frac{dy}{dx}=v+x\frac{dv}{dx}$ *in (1), we get* $v+x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$

$\Rightarrow v+x\frac{dv}{dx} = \frac{v}{1+v^3} \quad \Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v \quad \Rightarrow x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3}$

$\Rightarrow x\frac{dv}{dx} = -\frac{v^4}{1+v^3} \quad \Rightarrow x(1+v^3)dv = -v^4dx \Rightarrow \frac{1+v^3}{v^4}dv = -\frac{dx}{x}$

$\Rightarrow \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\frac{dx}{x}$

Integrating both sides, we get $\frac{v^{-3}}{-3} + \log v = -\log x + C$

$\Rightarrow -\frac{1}{3v^3} + \log v + \log x = C$

$\Rightarrow -\frac{1}{3}\frac{x^3}{y^3} + \log\left(\frac{y}{x}.x\right) = C \quad [\because v = y/x]$

$\Rightarrow -\frac{1}{3}.\frac{x^3}{y^3} + \log y = C$, *which is the required solution.*

Case IV. Equations Reducible to Homogeneous equations

To solve the equation $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Method to solve:

(i) *Put* $x = X + h$, $y = Y + k$.

(ii) *Equate the constant terms of numerator and denominator to zero and find the values of h and k.*

(iii) *Proceed as in case III.*

Example 2. $\frac{dy}{dx} = \frac{x+y+2}{x-y-3}$

Solution: *Given* $\frac{dy}{dx}=\frac{x+y+2}{x-y-3}$

Let $x=X+h,\ y=Y+h \Rightarrow \frac{dy}{dx}=\frac{dY}{dX} \Rightarrow \frac{dY}{dX}=\frac{X+Y+(h+k+2)}{X-Y+(h-k-3)}$

i.e., choose h, k such that

$h+k+2=0$ ......(i)

$h-k-3=0$ ....(ii)

Solving equation (i) and (ii), we get

$\Rightarrow h=\frac{1}{2};\ k=-\frac{5}{2}$

$\Rightarrow \frac{dY}{dX}=\frac{X+Y}{X-Y}=\frac{1+\frac{Y}{X}}{1-\frac{Y}{X}}$

Putting $Y=Xv \quad \Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX}$

$\Rightarrow v+X\frac{dv}{dX}=\frac{1+v}{1-v} \quad \Rightarrow \frac{dX}{X}=\frac{1-v}{1+v^2}dv \quad \Rightarrow \frac{dv}{1+v^2}-\frac{vdv}{1+v^2}=\frac{dX}{X}$

$\Rightarrow 2\log X+\log\left(1+\frac{Y^2}{X^2}\right)-\log c=2\tan^{-1}\left(\frac{Y}{X}\right)$ *as* $v=\frac{Y}{X}$

$\Rightarrow \log\{(X^2+Y^2)/c\}=2\tan^{-1}\left(\frac{Y}{X}\right) \Rightarrow X^2+Y^2=ce^{2\tan^{-1}\left(\frac{Y}{X}\right)}$

$\Rightarrow \left(x-\frac{1}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=ce^{2\tan^{-1}\left(y+\frac{5}{2}/x-\frac{1}{2}\right)}$, *where c is arbitrary constant.*

Example 3. *Solve* $\frac{dy}{dx}$ *= cos (x+y) + sin(x+y).*

Solution: *Put x+y=v, so that* $1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1.$

So, the given differential equation reduces to $\frac{dv}{dx}-1=\cos v+\sin v$

$\Rightarrow \frac{dv}{dx}=1+\cos v+\sin v$

By separating the variables we have $\frac{1}{1+\cos v+\sin v}dv=dx$

On integration, we get $\int\frac{1}{1+\cos v+\sin v}dv=\int 1.dx+C$

$\Rightarrow \int\frac{1}{1+\frac{1-\tan^2(v/2)}{1+\tan^2(v/2)}+\frac{2\tan(v/2)}{1+\tan^2(v/2)}}dv=x+C \quad \left[\because \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta},\ \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \int \frac{\sec^2(v/2)}{2(1+\tan(v/2))} dv = x + C \quad \Rightarrow \log|1 + \tan(v/2)| = x + C$

$\Rightarrow \log\left|1 + \tan\left(\frac{x+y}{2}\right)\right| = x + C$, *which is the required solution.*

Case V: Linear Differential Equations of first order

Definition: *A differential equation is said to be linear differential equation of first order if the dependent variable and its differential co-efficients occur in the first degree only and are not multiplied together.*

Thus $\frac{dy}{dx} + Py = Q$, *where P and Q are functions of x only is called linear differential equation of the first order and thus* $\frac{dx}{dy} + Px = Q$, *where P and Q are functions of y only is also called linear differential equation of the first order.*

Another Definition: *The differential equation of the form* $y' = p(x)y + q(x)$ ...(1) *is said to be linear differential equation of first order, where y is the dependent variable and x is the independent variable. If q(x)=0, then (1) is called* **homogeneous linear differential equation**, *otherwise it is called* **non-homogeneous linear differential equation**.

The solution of $\frac{dy}{dx} + Py = Q$, *where P and Q are functions of x only, is given by*

$y(I.F.) = \int (I.F)Q\,dx + C$, *where I.F = integrating factor and C is constant and* $I.F. = e^{\int P dx}$

Example 4: *Solve :* $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ *subject to the initial condition* $y(0) = 0$.

Solution: *The given differential equation can be written as* $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$ ...(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, *where* $P = \frac{2x}{1+x^2}$ *and* $Q = \frac{4x^2}{1+x^2}$

We have, $I.F. = e^{\int P dx} = e^{\int 2x/(1+x^2)dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of (i) by I.F. $(1 + x^2)$, *we get* $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Integrating both sides w.r.t. x, we get $y(1 + x^2) = \int 4x^2\,dx + C \Rightarrow y(1 + x^2) = \frac{4x^3}{3} + C$

It is given that y=0, when x=0. So, $0 = 0 + C \Rightarrow C = 0$.

Hence, $y(1+x^2) = \frac{4x^3}{3}$ *is the required solution.*

Case VI: Equations reducible to Linear differential Equation

To Solve the equation $\frac{dy}{dx} + Py = Qy^n$, *where P and Q are functions of x.*

Method to solve

(i) Dividing throughout by y^n.

(ii) Putting $y^{1-n}=z$ and get the linear equation in z.

(iii) Proceed as in Case V.

Example 5. *The solution of* $(dy/dx)+2xy=2xy^2$, *is*

(a) $y=(cx)/(1+e^{-x^2})$ (b) $y=1/(1-ce^{x})$ (c) $y=1/(1+ce^{x^2})$ (d) $y=(cx)/(1+e^{x^2})$

Solution: Ans. (c). *Dividing by* y^2, *we get* $y^{-2}\dfrac{dy}{dx}+2xy^{-1}=2x$

Putting $y^{-1}=v$, so that $-y^{-2}(dy/dx)=dv/dx$, or $(dv/dx)-2xv=-2x$, which is linear whose I.F. $=e^{-2\int x dx}=e^{-x^2}$.

So, solution is $ve^{-x^2}=\int(-2x)e^{-x^2}dx=\int e^t dt$, putting $(-x^2)=t$ ...(1)

or $y^{-1}e^{-x^2}=e^t+c=e^{-x^2}+c$, (using (1))

or $y^{-1}=1+ce^{x^2}$ or $y=1/(1+ce^{x^2})$, which is the required solution.

PRACTICE SET – 1

1. The solution of the ordinary differential equation $\dfrac{dy}{dx}=y,\ y(0)=0$ ***(TIFR-2010)***

(A) is unbounded. (B) is positive. (C) is negative. (D) is zero.

2. Find the general solution of the following differential equation $\dfrac{dy}{dx}=\dfrac{x+2y+8}{2x+y+7}$ ***(IIT-JAM 2011)***

3. Find the general solution of the following differential equation $y-x\dfrac{dy}{dx}=\dfrac{dy}{dx}y^2e^y$ ***(IIT-JAM 2011)***

4. Solve the differential equation $\dfrac{dy}{dx}+\dfrac{5y}{6x}=\dfrac{5x^4}{y^5}$ subject to the condition y(1) = 1. ***(IIT-JAM 2012)***

5. The equation of the curve passing through the point $\left(\dfrac{\pi}{2},1\right)$ and having slope $\dfrac{\sin(x)}{x^2}-\dfrac{2y}{x}$ at each point (x, y) with $x\neq 0$ is ***(IIT-JAM 2014)***

(A) $-x^2y+\cos(x)=\dfrac{-\pi^2}{4}$ (B) $x^2y+\cos(x)=\dfrac{\pi^2}{4}$

(C) $x^2y-\sin(x)=\dfrac{\pi^2}{4}-1$ (D) $x^2y+\sin(x)=\dfrac{\pi^2}{4}+1$

Case VII: Exact Differential Equation

The equation Mdx + Ndy = 0 (where M, N are functions of x and y) is exact if Mdx + Ndy is the exact differential of a function of x and y; say F, i.e., dF = Mdx + Ndy.

The equation Mdx + Ndy = 0 is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *and its solution is*

$$\underset{(\text{treating } y \text{ as constant})}{\int M dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c, \text{ a constant.}$$

Integrating Factors

If a differential equation is multiplied by a function μ(x,y) so that the resulting equation becomes exact, then μ(x, y) is called an integrating factor and is denoted by I.F.

The number of integrating factors of Mdx + Ndy = 0 is infinite.

Integrating factors by inspection of Terms:

(i) $xdy - ydx : \frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{xy}$ or $\frac{1}{x^2+y^2}$

(ii) $xdy + ydx : \frac{1}{(xy)^n}$

(iii) $xdx + ydy : \frac{1}{(x^2+y^2)^n}$

Five Rules for finding integrating factors

Rule I: *If the equation Mdx+Ndy=0 is homogeneous in x and y i.e. if M, N are homogeneous functions of the same degree in x and y, then* $\frac{1}{Mx+Ny}$ *is an integrating factor provided* $Mx + Ny \neq 0$.

Example 6. *Solve:* $x^2ydx-(x^3+y^3)dy=0$

Solution: *Here* $M = x^2y$, $N= -x^3-y^3$

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{x^3y - x^3y - y^4} = -\frac{1}{y^4}$$

Now, multiplying by integrating factor, the equation becomes $-\frac{x^2}{y^3}dx + \frac{x^3+y^3}{y^4}dy = 0,$

or $\frac{dy}{y} = \frac{x^2}{y^3}dx - \frac{x^3}{y^4}dy$ or $\frac{dy}{y} = d\left(\frac{1}{3}\frac{x^3}{y^3}\right)$

Integrating, we get $\log y = \frac{1}{3}\frac{x^3}{y^3} + k.$ $\Rightarrow y = ce^{\frac{x^3}{3y^3}}$, *where c is an arbitrary constant.*

Rule II : *If the equation Mdx + Ndy = 0 is of the form* $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$*, then* $\frac{1}{Mx-Ny}$ *is an integrating factor provided* $Mx - Ny \neq 0$.

Example 7. *Solve* $(1+xy)ydx+(1-xy)xdy=0$

Solution: *The given equation is of the form* $f_1(xy)ydx + f_2(xy)xdy = 0$

Here $M=(1+xy)y$, $N=(1-xy)x$.

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{(1+xy)yx-(1-xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $\frac{1}{2x^2y^2}$, *it becomes* $\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0,$

which is an exact differential equation.

$\therefore$ *The solution is* $\frac{1}{2y}\left(-\frac{1}{x}\right)+\frac{1}{2}\log x-\frac{1}{2}\log y=c$ *or* $\log\frac{x}{y}-\frac{1}{xy}=c'$.

Rule III: *If in the equation Mdx + Ndy = 0,* $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ *is a function of x only = f(x) (say), then* $e^{\int f(x)dx}$ *is an integrating factor.*

Example 8. *Solve* $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(x+xy^2)dy=0.$

Solution: $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(x+xy^2)dy=0.$

$$M = y+\frac{1}{3}y^3+\frac{1}{2}x^2,\ N=\frac{1}{4}(x+xy^2)$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{4}{(x+xy^2)}\left(1+y^2-\frac{1}{4}-\frac{y^2}{4}\right)=\frac{4}{x(1+y^2)}\frac{3}{4}(1+y^2)=\frac{3}{x}$$

which is a function of x only, hence $I.F=e^{\int(3/x)dx}=x^3$.

The equation becomes $(4x^3y+\frac{4}{3}x^3y^3+2x^5)dx+(x^4+x^4y^2)dy=0$

The solution is $x^4y+\frac{1}{3}x^4y^3+\frac{1}{3}x^6=k$ *or* $3x^4y+x^4y^3+x^6=c$, *where c is an arbitrary constant*

Rule IV: *If in the equation Mdx + Ndy = 0,* $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ *is a function of y only = f(y) (say), then* $e^{\int f(y)dy}$ *is an integrating factor.*

Example 9. *Solve* $(xy^3+y)dx+2(x^2y^2+x+y^4)\,dy=0$

Solution: *Here* $M=xy^3+y$, $N=2(x^2y^2+x+y^4)$

$$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y(xy^2+1)}(4xy^2+2-3xy^2-1)=\frac{1}{y},$$ *which is a function of y only.*

$$\therefore I.F.=e^{\int \frac{1}{y}dy}=e^{\log y}=y$$

Multiplying throughout by y, it becomes $(xy^4+y^2)\,dx+(2x^2y^3+2xy+2y^5)\,dy=0$, *which is an exact differential equation*

$\therefore$ *The solution is* $\frac{1}{2}x^2y^4+xy^2+\frac{1}{3}y^6=c$

$\therefore$

Rule V: *If the equation is* $x^a y^b\,(mydx+nxdy)+x^r y^s\,(pydx+qxdy)=0$, *then* $x^h y^k$ *is an integrating factor, where* $\frac{a+h+1}{m}=\frac{b+k+1}{n},\ \frac{r+h+1}{p}=\frac{s+k+1}{q}$

PRACTICE SET – 2

1. *If* y^a *is an integrating factor of the differential equation* $2xy\,dx-(3x^2-y^2)\,dy=0$, *then the value of a is* **(IIT-JAM 2011)**

(A) -4 (B) 4 (C) -1 (D) 1

2. *The differential equation* $(1+x^2y^3+\alpha x^2y^2)\,dx+(2+x^3y^2+x^3y)\,dy=0$ *is exact, if* α *equals* **(IIT-JAM 2012)**

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) 2 (D) 3

3. *For* $a, b, c \in \mathbb{R}$, *if the differential equation* $(ax^2+bxy+y^2)\,dx+(2x^2+cxy+y^2)\,dy=0$ *is exact, then* **(IIT-JAM 2014)**

(A) $b=2,\ c=2a$ (B) $b=4,\ c=2$ (C) $b=2,\ c=4$ (D) $b=2,\ a=2c$

4. *An integrating factor of the differential equation* $\frac{dy}{dx}=\frac{2xy^2+y}{x-2y^3}$ *is* **(IIT-JAM 2015)**

(A) $\frac{1}{y}$ (B) $\frac{1}{y^2}$ (C) y (D) y^2

5. *If* x^3y^2 *is an integrating factor of* $(6y^2+a\,xy)\,dx+(6xy+b\,x^2)\,dy=0$, *where* $a, b \in \mathbb{R}$, *then* **(GATE- 2017)**

(A) $3a-5b=0$ (B) $2a-b=0$ (C) $3a+5b=0$ (D) $2a+b=0$

6. *The differential equation* $(3a^2x^2+by\cos x)\,dx+(2\sin x-4ay^3)\,dy=0$ *is exact for* **(GATE-1999)**

(A) *any value of a, but* $b\neq 2$ (B) *any value of a, but* $b=2$

(C) *any value of b, but* $a=2$ (D) *for any value of a and b*

§ 1.5. EQUATION OF FIRST ORDER AND HIGHER DEGREE

A differential equation $f\left(x, y, \frac{dy}{dx}\right) = 0$, *where degree of* $\frac{dy}{dx} > 1$ *is said to be non-linear differential equation of first order and higher degree. It is generally of the type*

$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots + P_{n-1}\frac{dy}{dx} + P_n = 0$, *where* $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ *are functions of x and y. For convenience, we write* $\frac{dy}{dx} = p$ *and the above equation takes the form*

$$P_0p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_{n-1}p + P_n = 0.$$

Type I : Equations solvable for p

Let $f(x, y, p) = 0$ ...(1) *be the given differential equation of first order and degree* $n > 1$.

$\because$ (1) *is an equation of degree n solvable for p,* $\therefore$ *L.H.S. of (1) can be expressed as the product of n linear factors in p.*

Let (1) be written as $(p-f_1)(p-f_2)\ldots(p-f_n) = 0$, *where* $f_1, f_2, \ldots, f_n$ *are functions of x and y.*

Now, solutions of given equation (1) are given by n equations $p - f_1 = 0, p - f_2 = 0, \ldots, p - f_n = 0$ *or*

$$\frac{dy}{dx} = f_1(x, y), \frac{dy}{dx} = f_2(x, y), \ldots, \frac{dy}{dx} = f_n(x, y)$$

These equations can be solved by the methods already known to us. Let the solutions of these n equations be $F_1(x, y, c_1) = 0, F_2(x, y, c_2) = 0, \ldots, F_n(x, y, c_n) = 0$

Since the given equation is of the first order, $\therefore$ *it cannot have more than one independent arbitrary constant.*

Let $c_1 = c_2 = \ldots = c_n = c$, *say.*

$\therefore$ *The general solution of given equation (1) is* $F_1(x, y, c).F_2(x, y, c)\ldots F_n(x, y, c) = 0$

Type II: Equations solvable for y

Let the given differential equation be $f(x, y, p) = 0$ ... (1)

Since (1) is solvable for y,

$\therefore$ (1) *can be expressed as* $y = g(x, p)$ ... (2)

Differentiating (2). w.r.t x, $\frac{dy}{dx} = p = h\left(x, p, \frac{dp}{dx}\right)$ ... (3)

which is an equation in two variables x and p. Integrating (3), let its solution be $F(x, p, c) = 0$ ... (4)

The elimination of p between (2) and (4) gives the general solution of (1).

If the elimination of p between (2) and (4) is not possible, the values of x and y may be obtained in terms of the parameter p; say $x = f_1(p, c)$, $y = f_2(p, c)$.

These two equations together constitute the solution of (1) in the parametric form.

Type III: Equations solvable for x

Let the given differential equation be $f(x, y, p) = 0$ ... (1)

Since (1) is solvable for x,

$\therefore$ *(1) can be expressed as* $x = g(y, p)$ ... (2)

Differentiating (2) w.r.t. y, $\frac{dx}{dy} = \frac{1}{p} = h\left(y, p, \frac{dp}{dy}\right)$ ... (3)

which is an equation in two variables y and p.

Integrating (3), let its solution be $F(y,p,c)=0$ ... (4)

The elimination of p between (2) and (4) gives the general solution of (1).

Type IV: Clairaut's Equation

An equation of the form $y=px+f(p)$ *...(1) is known as Clairaut's equation*

Differentiating with respect to x, we have $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$ $\left[\because \frac{dy}{dx} = p\right]$

or $[x + f'(p)]\frac{dp}{dx} = 0$

$\therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$

$\frac{dp}{dx} = 0$, *gives* $p = c$ ...(2)

Thus eliminating p from (1) and (2), we get $y=cx+f(c)$ *...(3), as the general solution of (1).*

Hence, the solution of the Clairaut's equation is obtained on replacing p by c.

Remark 1: *If we eliminate p from* $x + f'(p) = 0$ *and (1), we get an equation involving no constant. This is the singular solution of (1) which gives the envelope of the family of straight lines (3).*

Remark 2: Equations reducible to Clairaut's form: *Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.*

Example 1. *Solve* $(px - y)(py + x) = a^2p$.

Solution: *Put* $x^2=u$ *and* $y^2=v$, *s.t.,* $2xdx=du$ *and* $2ydy=dv$

$\therefore p = \frac{dy}{dx} = \frac{dv}{y} / \frac{du}{x} = \frac{x}{y}P$, *where* $P = \frac{dv}{du}$

Then the given equation becomes $\left(\frac{xP}{y}.x - y\right)\left(\frac{xP}{y}.y + x\right) = a^2\frac{xP}{y}$

or $(uP-v)(P+1)=a^2P$ *or* $uP-v = \frac{a^2P}{P+1}$ *or* $v = uP - a^2P/(P+1)$, *which is Clairaut's form.*

$\therefore$ *Its General solution is* $v = uc - a^2c/(c+1)$, *i.e.,* $y^2 = cx^2 - a^2c/(c+1)$.

Type V. Equations not containing x

The differential equation, not containing x, is of the form $f(y, p) = 0$ ...(1)

Two cases arise:

Case I. *When (1) is solvable for p. Then p = g(y), i.e.,* $\frac{dy}{dx}=g(y) \Rightarrow \frac{dy}{g(y)}=dx$

Integrating, $\int \frac{dy}{g(y)} = x + c$, *which is the General solution.*

Case II. *When (1) is solvable for y. Then y = h(p), and we can proceed as earlier.*

Example 2. *Solve* $y(1-\log y)\frac{d^2y}{dx^2}+(1+\log y)\left[\frac{dy}{dx}\right]^2=0$

Solution: *The given equation does not contain x directly.*

$\therefore$ *Putting* $\frac{dy}{dx}=p$, *so that* $\frac{d^2y}{dx^2}=\frac{dp}{dx}=p\frac{dp}{dy}$.

The given equation becomes $y(1-\log y)p\frac{dp}{dy}+(1+\log y)p^2=0$ *or* $\frac{dp}{p}+\frac{1}{y}\frac{(1+\log y)}{(1-\log y)}dy=0$

$\therefore \log p=-\int\frac{1+\log y}{y(1-\log y)}dy+c_1$ *(put log y=t)*

$\log p=-\int\frac{1+t}{1-t}dt+c_1$

$\therefore \log p=\int\left(1+\frac{2}{t-1}\right)dt+c_1=t+2\log(t-1)+c_1=\log y+2\log(\log y-1)+\log c_1$

$\Rightarrow \log p=\log(c_1y(\log y-1)^2)$.

$\Rightarrow p=\frac{dy}{dx}=c_1y(\log y-1)^2 \quad \Rightarrow \frac{dy}{y(\log y-1)^2}=c_1dx$

Integrating, $-\frac{1}{\log y-1}=c_1x+c_2$ *or* $1-\log y=\frac{1}{c_1x+c_2}$, *which is the required solution.*

Type VI. Equations not containing y.

The differential equation, not containing y, is of the form f(x, p) = 0 ... (1)

Two cases arise:

Case I. *When (1) is solvable for p. Then p = g(x), i.e.* $\frac{dy}{dx}=g(x) \Rightarrow dy=g(x)\,dx$

Integrating, $y+c=\int g(x)\,dx$, *which is the General solution.*

Case II. *When (1) is solvable for x. Then x = h(p), and we can proceed as earlier.*

Example 3. *Solve* $(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0$.

Solution: *The given equation does not contain y- directly.*

$\therefore$ *Putting* $\frac{dy}{dx}=p$ *so that* $\frac{d^2y}{dx^2}=\frac{dp}{dx}$ *the given equation reduces to* $(1+x^2)\frac{dp}{dx}+1+p^2=0$

or $\dfrac{dp}{1+p^2}+\dfrac{dx}{1+x^2}=0$

Integrating, $\tan^{-1}p+\tan^{-1}x=\tan^{-1}c_1$ *or* $\tan^{-1}\dfrac{p+x}{1-px}=\tan^{-1}c_1$ *or* $\dfrac{p+x}{1-px}=c_1$

or $p+x=c_1(1-px)$ or $p=\dfrac{dy}{dx}=\dfrac{c_1-x}{1+c_1x}=\dfrac{1}{c_1}\left(\dfrac{1+c_1^2}{1+c_1x}-1\right)$.

Integrating, $y=\dfrac{(1+c_1^2)}{c_1^2}\log(1+c_1x)-\dfrac{1}{c_1}x+c_2$, *which is the required solution.*

Example 4. *Solve:* $ydx+(x-y^3)dy=0$

Solution: *The given differential equation is* $ydx+(x-y^3)dy=0 \Rightarrow \dfrac{dx}{dy}+\dfrac{x}{y}=y^2$ ...(1)

This is a linear differential equation of the form $\dfrac{dx}{dy}+Px=Q$, *where* $P=\dfrac{1}{y}$ *and* $Q=y^2$.

$\therefore$ I.F. $= e^{\int P dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$.

Multiplying both sides of (i) by I.F. = y, we obtain $y\dfrac{dx}{dy}+x=y^3$

Integrating both sides w.r.t. y, we get $xy=\int y^3\,dy+c$ [Using: $x(I.F.)=\int Q\,(I.F.)\,dy+C$]

$\Rightarrow xy=\dfrac{y^4}{4}+C$, *which is the required solution.*

Example 5. *Solve* $x\dfrac{dy}{dx}+y=x^3y^6$.

Solution: *The given differential equation can be written as* $\dfrac{1}{y^6}\dfrac{dy}{dx}+\dfrac{1}{xy^5}=x^2$

Putting $y^{-5}=v$ *so that* $-5y^{-6}\dfrac{dy}{dx}=\dfrac{dv}{dx}$ *or* $y^{-6}\dfrac{dy}{dx}=-\dfrac{1}{5}\dfrac{dv}{dx}$ *we get* $-\dfrac{1}{5}\dfrac{dv}{dx}+\dfrac{1}{x}v=x^2$

$\Rightarrow \dfrac{dv}{dx}-\dfrac{5}{x}v=-5x^2$ ...(i)

This is the standard form of the linear differential equation having integrating factor

I.F. $= e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \dfrac{1}{x^5}$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x, we get $v\dfrac{1}{x^5}=\int -5x^2\cdot\dfrac{1}{x^5}dx$

$\Rightarrow \dfrac{v}{x^5}=\dfrac{5}{2}x^{-2}+C \Rightarrow y^{-5}x^{-5}=\dfrac{5}{2}x^{-2}+C$, *which is the required solution.*

PRACTICE SET – 3

1. *One of the points which lies on the solution curve of the differential equation $(y-x)\,dx+(x+y)\,dy=0$, with the given condition $y(0)=1$, is* **(IIT-JAM 2016)**
   (A) (1, -2) (B) (2, -1) (C) (2, 1) (D) (-1, 2)

2. *Consider the ODE on $\mathbb{R}$ $y'(x)=f(y(x))$. If f is an even function and y is an odd function, then* **(CSIR UGC NET DEC-2015)**
   (A) $-y(-x)$ is also a solution (B) $y(-x)$ is also a solution.
   (C) $-y(x)$ is also a solution. (D) $y(x)\,y(-x)$ is also a solution.

3. *Let y be a solution of $y'=e^{-y^2}-1$ on [0,1] which satisfies $y(0)=0$. Then* **(GATE-2008)**
   (A) $y(x)>0$ for $x>0$ (B) $y(x)<0$ for $x>0$
   (C) y changes sign in [0,1] (D) $y\equiv 0$ for $x>0$

4. *Consider the equation $\frac{dy}{dt}=(1+f^2(t))y(t),\ y(0)=1: t\geq 0$ where f is a bounded continuous function on $[0,\infty)$. Then* **(CSIR UGC NET DEC-2011)**
   (A) This equation admits a unique solution y(t) and further $\lim_{t\to\infty} y(t)$ exists and is finite
   (B) This equation admits two linearly independent solutions
   (C) This equation admits a bounded solution for which $\lim_{t\to\infty} y(t)$ does not exist
   (D) This equation admits a unique solution y(t) and further, $\lim_{t\to\infty} y(t)=\infty$

KEY POINTS

- *A Differential Equation expresses a relation between dependent and independent variables and their derivatives of any order.*
- *The order of a differential equation is the highest order derivative it contains.*
- *The degree of the differential equation, which can be written as a polynomial in the derivatives is the degree of the highest ordered derivative which then occurs.*
- *A differential equation is linear iff the function f is a linear function of the variables $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$.*
- *General solution is the solution which contains as many arbitrary constants as the order of the differential equation.*

- *The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to arbitrary constants.*

- *The solution which cannot be derived from the general solution and has no arbitrary constants is called singular solution.*

- *Homogeneous Equation* $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, *where f(x, y) and* ϕ *(x, y) are both homogeneous functions of the same degree in x and y. Put* $y = vx$ *so that* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *and solve.*

- *For linear differential equations of first order* $\frac{dy}{dx} + Py = Q$, *where P and Q are functions of x only, thesolution is given by* $y(I.F.) = \int (I.F)Q\,dx + C$, *where I.F = integrating factor and C is constant and* $I.F. = e^{\int P dx}$

- *For exact differential equation* $Mdx + Ndy = 0$ *we have* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *and its solution is*

$$\underset{(\text{treating } y \text{ as constant})}{\int M dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c, \text{ a constant.}$$

If a differential equation is not exact then we make it exact by multiplying it with the integrating factor.

- *The number of integrating factors of* $Mdx + Ndy = 0$ *is infinite.*

- *The Clairaut's Equation is* $y=px + f(p)$ *and its general solution is given by* $y=cx + f(c)$.

SOLVED QUESTIONS FROM PREVIOUS PAPERS

1. *The minimum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having* $x^2 \sin(x)$ *as a solution is equal to______.* ***(GATE-2015)***

Solution: (Ans: 6) *As we know* $(D^2 + 1)y=0$ *differential equation has solution* $y=c_1 \cos x + c_2 \sin x$ *and a factor x is multiplied to linearly independent solutions, when we have repeated roots and if again the roots repeats we multiply* x^2 *to the same.*

$\Rightarrow$ *for solution* $x^2 \sin x$, *roots must be repeated 3 times*

Thus the linear and homogeneous differential equation must be $(D^2 + 1)^3 y = 0$

It is of order 6.

2. *The equation of the curve satisfying* $\sin y\dfrac{dy}{dx}=\cos y(1-x\cos y)$ *and passing through the origin is* ________ **(IIT-JAM-2013)**

Solution: (Ans: sec y = x + 1) *Given* $\sin y\dfrac{dy}{dx}=\cos y(1-x\cos y)$

$$\Rightarrow \frac{dy}{dx}=\cot y-x\cos y.\cot y \quad \Rightarrow \frac{dy}{dx}-\cot y=-x\cos y.\cot y$$

$$\Rightarrow \sec y\ \tan y\frac{dy}{dx}-\sec y=-x \qquad \ldots (1)$$

Put v = sec y

$$\Rightarrow \frac{dv}{dx}=\sec y\tan y\frac{dy}{dx}$$

$\therefore$ *(1) becomes* $\dfrac{dv}{dx}-v=-x$

$I.F=e^{-\int dx}=e^{-x}$

$\therefore$ *Complete solution is* $v\cdot e^{-x}=-\int e^{-x}\cdot x\,dx+c$

$$\Rightarrow v\cdot e^{-x}=x\cdot e^{-x}+e^{-x}+c \ \Rightarrow v=x+1+ce^{x} \ \Rightarrow \sec y=(x+1)+ce^{x}$$

As the curve passes through origin. i.e. (0,0)

So, $\sec 0=(0+1)+ce^{0}$

$\Rightarrow 1=1+c \Rightarrow c=0$

Hence the required equation of curve satisfying the given differential equation is sec y = x + 1.

3. *Which of the following is an integrating factor of* $xdy-ydx=0$? **(GATE – 2001)**

(A) $\dfrac{1}{x^2}$ (B) $\dfrac{1}{x^2+y^2}$ (C) $\dfrac{1}{xy}$ (D) $\dfrac{x}{y}$

Solution: (A), (B), (C)

Here M = -y; N=x

Clearly $\dfrac{\partial M}{\partial y}=-1,\ \dfrac{\partial N}{\partial x}=1 \Rightarrow \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

$\Rightarrow$ *This eqution is not exact.*

By option (A), $\dfrac{-y}{x^2}dx+\dfrac{1}{x}dy=0$

$M'=\dfrac{-y}{x^2}, N'=\dfrac{1}{x}$

$$\frac{\partial M'}{\partial y}=\frac{-1}{x^2},\ \frac{\partial N'}{\partial x}=\frac{-1}{x^2} \Rightarrow \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}=-\frac{1}{x^2}$$

$\Rightarrow$ *now equation is exact.*

Hence, option (a) is correct.

By option (b), $M' = \dfrac{-y}{x^2+y^2}, N' = \dfrac{x}{x^2+y^2}$

$$\frac{\partial M'}{\partial y} = \frac{(x^2+y^2)(-1)-(2y)(-y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N'}{\partial x} = \frac{(x^2+y^2)(1)-(2x)(x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$\Rightarrow \dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}$, *hence equation is exact*

$\Rightarrow$ *option (b) is correct.*

By option (c), $M' = \dfrac{-1}{x}, N' = \dfrac{1}{y}$

$\dfrac{\partial M'}{\partial y} = 0 = \dfrac{\partial N'}{\partial x} \Rightarrow$ *equation is exact*

$\Rightarrow$ *option (c) is correct.*

By option (d), $M' = -x; N' = \dfrac{x^2}{y}$

$\dfrac{\partial M'}{\partial y} = 0;\ \dfrac{\partial N'}{\partial x} = \dfrac{2x}{y} \quad \Rightarrow \dfrac{\partial M'}{\partial y} \neq \dfrac{\partial N'}{\partial x}$

$\Rightarrow$ *It will not act as integrating factor*

$\Rightarrow$ *Option (D) is incorrect*

Hence, options (A), (B) and (C) are correct.

4. *The initial value problem* $(x^2-x)\dfrac{dy}{dx} = (2x-1)y,\ y(x_0) = y_0$ *has a unique solution if* (x_0, y_0) *equals*

(GATE-2002)

(A) (2, 1) *(B) (1, 1)* *(C) (0, 0)* *(D) (0,1)*

Solution: *(a) The given equation is* $(x^2-x)\dfrac{dy}{dx} = (2x-1)y,\ y(x_0) = y_0$

$$\Rightarrow \frac{dy}{dx} = \frac{(2x-1)}{x^2-x}y$$

$$\frac{dy}{dx} = \frac{(2x-1)}{x(x-1)}y$$

$$\frac{dy}{dx} = \left(\frac{x+x-1}{x(x-1)}\right)y = \left[\frac{x}{x(x-1)} + \frac{(x-1)}{x(x-1)}\right]y$$

$$\frac{dy}{dx} = \left(\frac{1}{x-1} + \frac{1}{x}\right)y$$

Integrating, $\frac{dy}{y}=\frac{dx}{x-1}+\frac{dx}{x}$, *we get*

$\log y=\log[x(x-1)]+\log k \quad \Rightarrow y=kx(x-1)$

From option;

By option (A), $y(2)=1$

$1=k2.1 \Rightarrow k=\frac{1}{2}$

$\Rightarrow$ *It has unique solution in this case*

For other options (B) and (D) initial value problem has no solution, whereas for option (C) it has infinitely many solutions.

5. *Integrating factor of* $(x^7y^2+3y)dx+(3x^8y-x)dy=0$ *is* $x^m y^n$, *where* **(GATE-2002)**

(A) $m=-7,n=5$ *(B)* $m=-1,n=5$ *(C)* $m=-7,n=1$ *(D)* $m=-7,n=-1$

***Solution:* (C)** *The given equation is* $(x^7y^2+3y)dx+(3x^8y-x)dy=0$

Here $M=x^7y^2+3y \quad N=3x^8y-x$

By options, for option (A), if $x^{-7}y^5$ *is integrating factor* $\Rightarrow$ *(A) is incorrect*

$M'=x^{-7}y^5(x^7y^2+3y); N'=x^{-7}y^5(3x^8y-x)$

$M'=y^7+3x^{-7}y^6; N'=3xy^6-x^{-6}y^5$

$\frac{\partial M'}{\partial y}=7y^6+3x^{-7}(6y^5); \ \frac{\partial N'}{\partial x}=3y^6-(-6x^{-7})y^5$

$\Rightarrow \frac{\partial M'}{\partial y}\neq\frac{\partial N'}{\partial x}$, *it is not exact*

$\Rightarrow x^{-7}y^5$ *is not integrating factors*

For option (B), $m=-1,n=5$, *i.e. if* $x^{-1}y^5$ *is an integrating factor*

$M'=x^{-1}y^5(x^7y^2+3y); N'=x^{-1}y^5(3x^8y-x)$

$M'=x^6y^7+3x^{-1}y^6; N'=3x^7y^6-y^5$

$\frac{\partial M'}{\partial y}=7x^6y^6+3x^{-1}(6y^5); \frac{\partial N'}{\partial x}=21x^6y^6$

$\frac{\partial M'}{\partial y}\neq\frac{\partial N'}{\partial x}$, *it is not exact*

$\Rightarrow$ *it is not an integrating factor*

For option (c), m=-7, n=1, if $x^{-7}y$ *is integrating factor* $\Rightarrow$ *(B) is incorrect.*

$M'=x^{-7}y(x^7y^2+3y); N'=x^{-7}y(3x^8y-x)$

$M'=y^3+3x^{-7}y^2; N'=3xy^2-x^{-6}y$

$\frac{\partial M'}{\partial y}=3y^2+3x^{-7}(2y); \frac{\partial N'}{\partial x}=3y^2+6x^{-7}y \Rightarrow \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$, *hence it is exact*

For option (D), m=-7, n=-1, if $x^{-7}y^{-1}$ is an integrating factor

$M' = x^{-7}y^{-1}(x^7y^2+3y); N' = x^{-7}y^{-1}(3x^8y-x)$

$M' = y+3x^{-7}; N' = 3x - x^{-6}y^{-1}$

$\frac{\partial M'}{\partial y} = 1, \frac{\partial N'}{\partial x} = 3-(-6)x^{-7}y^{-1}$

It is not exact

Hence, option (C) is correct.

6. *The solution of $\frac{dy}{dx} = y^2$, y(0)=1 does not exist for all* **(GATE-1995)**

(A) $x \in (-\infty, 1)$ *(B) $x \in [0,a]$ where a > 1* *(C) $x \in (-\infty, \infty)$* *(D) $x \in [1,a]$ where a > 1*

Solution: (B), (C), (D) *The given differential equation* $\frac{dy}{dx} = y^2$ *and* $y(0) = 1$

$\frac{dy}{y^2} = dx, \quad \frac{y^{-2+1}}{-1} = x+c, \quad \frac{-1}{y} = x+c$

Applying initial condition, -1=0+c $\Rightarrow$ c =-1

$\frac{-1}{y} = x-1$

$y = \frac{1}{1-x}$

The solution can't exist at x=1 and as option (B), (C), (D) includes x=1

$\Rightarrow$ *Solution does not exists for $\forall$ x given in option (B), (C), (D).*

7. *The general solution of the differential equation $dy/dx + \tan y \tan x = \cos x \sec y$, is* **(GATE-2001)**

(A) 2sin y=(x+ c- sin x cos x) sec x *(B) sin y = (x + c) cos x*

(C) cos y = (x + c) sin x *(D) sec y = (x + c) cos x*

Solution. *The given differential equation is* $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$

$\cos y \frac{dy}{dx} + \sin y \tan x = \cos x$

put sin y = t $\Rightarrow \cos y \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dt}{dx} + t(\tan x) = \cos x$

It is a linear differential equation of first order

$I.F. = e^{\int \tan x dx} = e^{-\log \cos x} = \sec x$

The solution is given by $t(I.F.) = \int (I.F.)\cos x \, dx + c$

$(\sec x)t = \int \sec x \cos x \, dx + c$

$(\sec x)t = x + c \Rightarrow \sin y = (x+c)\cos x$

$\Rightarrow$ *option (B) is correct.*

ASSIGNMENT - 1.1

NOTE: CHOOSE THE BEST OPTION

1. $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$ *is of degree*

(A) zero *(B) two* *(C) three* *(D) one*

2. *The differential equation* $\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$ *is of*

(A) first order *(B) zero order* *(C) second order* *(D) none of these*

3. *The degree of the differential equation* $\frac{d^2y}{dx^2} + n^2x = 0$ *is–*

(A) zero *(B) one* *(C) two* *(D) three*

4. *The differential equation* $f(x, y)\left(\frac{d^m y}{dx^m}\right)^p + \phi(x, y)\left(\frac{d^{m-1} y}{dx^{m-1}}\right)^q + \ldots = 0$ *is of order*

(A) p *(B) q* *(C) m* *(D) none of these*

5. *The number of arbitrary constants a general solution of first order differential equation contains is*

(A) zero *(B) one* *(C) two* *(D) three*

6. *The differential equation* $\frac{dy}{dx} + Py = Q$*, where P and Q are functions of x only, have the integrating factor–*

(A) $e^{\int P dx}$ *(B)* $e^{\int Q dy}$ *(C)* $e^{\int P dy}$ *(D)* $e^{\int Q dx}$

7. *A solution of a differential equation which contains no arbitrary constants is–*

(A) particular solution *(B) general solution*

(C) primitive solution *(D) none of these*

8. *A general solution of a linear differential equation with constant coefficients is–*

(A) sum of particular solution and complementary function.

(B) product of particular solution and complementary function.

(C) quotient of particular solution and complementary function.

(D) none of these

9. *Given a differential equation of order n, then its complete primitive contains–*

(A) n–arbitrary constants *(B) more than n–arbitrary constants*

(C) less than n–arbitrary constants *(D) no arbitrary constant*

10. *The necessary condition for the differential equation M(x, y)dx + N(x, y)dy = 0 to be exact is–*

(A) $\frac{\partial N}{\partial y}=\frac{\partial M}{\partial x}$ (B) $\frac{\partial N}{\partial y}=-\frac{\partial M}{\partial x}$ (C) $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ (D) $\frac{\partial M}{\partial y}=-\frac{\partial N}{\partial x}$

11. *If the solution of differential equation contains as many arbitrary constants as the order of a differential equation, then the solution is called*

(A) particular solution (B) complete primitive
(C) singular solution (D) none of these

12. *In linear ordinary differential equation, the dependent variable and its differential coefficients are not multiplied together and occurs only in–*

(A) first degree (B) second degree (C) third degree (D) fourth degree

13. *If M(x,y)dx + N(x,y)dy = 0 and* $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ *then the differential equation is*

(A) exact (B) not exact (C) linear (D) non–linear

14. *If a differential equation Mdx + Ndy = 0 is not exact and let F(x, y) be such that FM dx + FN dy = 0 is exact, then the function F is called a/an*

(A) differentiable function (B) arbitrary function
(C) integrating factor (D) none of these

15. *If differential equation* $Mdx + Ndy = 0$ *is of the form* $f_1(xy)\,y\,dx + f_2(xy)\,xdy = 0$ *and* $Mx - Ny \neq 0$, *then an integrating factor is*

(A) $Mx - Ny$ (B) $My - Nx$ (C) $\frac{1}{Mx - Ny}$ (D) None of the above

16. *The differential equation* $\frac{dy}{dx} + Py = Q$ *is linear differential equation of first order if–*

(A) P, Q are functions of x only (B) P, Q are functions of y only
(C) P, Q are functions of x and y (D) None of these

17. *The order of the differential equation is defined as the*

(A) power of the highest order derivative occurring in the equation
(B) highest order derivative occurring in the equation
(C) highest power among the powers of the derivatives occurring in the equation.
(D) highest power of the variable occurring in the equation

18. *The degree of a differential equation is defined as the*

(A) highest of the orders of the differential coefficients occurring in it.
(B) highest power of the highest order differential coefficient occurring in it.
(C) any power of the highest order differential coefficient occurring in it.
(D) highest power among the powers of the differential coefficients occurring in it.

19. *A linear differential equation*
   (A) *is necessarily of first order*
   (B) *is necessarily of first degree*
   (C) *may or may not be of first degree but is of first order*
   (D) *is either of first order or of first degree.*

20. *The degree of differential equation satisfying the relation* $\sqrt{1+x^2}+\sqrt{1+y^3}=\lambda\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right)$ *is*
   (A) 1 (B) 2 (C) 3 (D) none of these

21. *Determine the type of the following differential equation* $\frac{d^2y}{dx^2}+\sin(x+y)=\sin x$
   (A) Linear, homogeneous (B) Nonlinear, homogeneous
   (C) Linear, non homogeneous (D) Non linear, non homogeneous

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

22. *The equation* $e^x\,dx + e^y\,dy = 0$ *is not of order–*
   (A) zero (B) one (C) two (D) three

23. *The equation* $\frac{d^2y}{dx^2} + n^2x = 0$ *is of–*
   (A) order zero (B) order two (C) degree two (D) degree one

24. *The differential equation* $\left(\frac{d^2y}{dx^2}\right)^{3/2}-\left(\frac{dy}{dx}\right)^{1/2}-4=0$ *is of*
   (A) order 2 (B) degree 3 (C) order 4 (D) degree 6

25. *Which of the following differential equations is not of first order and second degree?*
   (A) $\left(\frac{d^2y}{dx^2}\right)^{1/2}+\sin x\left(\frac{dy}{dx}\right)^3+xy=x$ (B) $x^3\left(\frac{dy}{dx}\right)^2+xy^3=e^x$
   (C) $\frac{dy}{dx}+x^2y^2+xy=e^x$ (D) $(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0$

ASSIGNMENT - 1.2

NOTE: CHOOSE THE BEST OPTION

1. What is the order and degree of the differential equation $[1 + (y')^2]^{3/2} - \rho y'' = 0$?
(A) First order, second degree (B) Second order, first degree
(C) Second order, second degree (D) Third order, second degree

2. Order and degree of the differential equation $\frac{d^2\rho}{d\theta^2} = \left(\rho + \left(\frac{d\rho}{d\theta}\right)^2\right)^{\frac{1}{4}}$ are respectively
(A) 2, 1 (B) 1,1 (C) 2, $\frac{1}{4}$ (D) 2,4

3. If $Mdx + Ndy = 0$ is a differential equation and $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say $f(y)$ then integrating factor is
(A) e^x (B) e^{x+y} (C) $e^{\int f(y)dy}$ (D) None of the above

4. The integrating factor for the Leibnitz linear equation $\frac{dy}{dx} + Py = Q$ is
(A) $\int Pdx$ (B) $\int Qdx$ (C) exp $\left(\int Qdx\right)$ (D) exp $\left(\int Pdx\right)$

5. The homogeneous differential equation M (x, y) dx + N(x, y) dy = 0 can be reduced to a differential equation, in which the variables are separated by the substitution
(A) $y = vx$ (B) $xy = v$ (C) $x + y = v$ (D) $x - y = v$

6. The differential equation $\frac{dy}{dx} = \frac{y^3}{x^2}$ is
(A) homogeneous (B) non-homogeneous
(C) exact equation (D) none of these

7. Consider the following differential equations
(1) $x^2(y'')^6 + y^{-2/3}\{1+(y''')^5\}^{\frac{1}{2}} + \frac{d^2}{dx^2}\{(y'')^{-2/3}\} = 0$
(2) $y' - 6x = \{ay + bx y'\}^{-3/2},\ b \neq 0$
The sum of order of first differential equation and degree of second differential equation is,
(A) 6 (B) 7 (C) 8 (D) 9

8. For non homogeneous equation $y'+p(x)y = r(x)$, if y_1 and y_2 are its solutions, then the solution of homogenous equation $y'+p(x)y = 0$, is

(A) $y = y_1 - y_2$ (B) $y = \frac{y_1}{y_2}$ (C) $y = \frac{y_2}{y_1}$ (D) None of these

9. The integrating factor of $y^2dx + (1 + xy)dy = 0$ is
(A) e^y (B) e^x (C) e^{xy} (D) e^{-xy}

10. An integrating factor for $ydx - xdy = 0$ is
(A) x/y (B) y/x (C) $\frac{1}{x^2 + y^2}$ (D) $\frac{1}{x^2 + y}$

11. For the differential equation $y' + p(x)y = r(x)$ to be homogeneous,
(A) $r(x) \neq 0$ (B) $r(x) = 0$ (C) $r(x) = p(x)$ (D) $p(x) = 0$

12. The general solution of the differential equation $ydx - xdy = 0$, is
(A) $\frac{x}{y} = c$ (B) $x+y=c$ (C) $xy=c$ (D) $x-y=c$

13. (1) The solution of ordinary differential equation of order n have n arbitrary constants.
(2) The solution of partial differential equation of order n have n arbitrary functions.
Which of the following statements is true?
(A) 1 is true 2 is false (B) 1 is false 2 is true
(C) 1 and 2 both are true (D) 1 and 2 both are false

14. The solution of differential equation $ydx+xdy=0$ is
(A) $xy = c$ (B) $x = y + c$ (C) $x - y = c$ (D) none of these

15. The general solution of first order differential equation $\frac{dy}{dx} = \cos x$, is given by
(A) $y = \cos x$
(B) $y = \sin x$
(C) $y = \cos x + c$, c an arbitrary constant
(D) $y = \sin x + c$, c an arbitrary constant

16. The general solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ is
(A) $x^2 + y^2 = a^2$ (B) $x^2 - y^2 = a^2$ (C) $x = -y$ (D) $x + y = a$

17. The particular solution of the initial value problem $xdx + ydy = 0$, $x_0 = 4$, $y_0 = -3$ is
(A) $y = \pm \sqrt{25 - x^2}$ (B) $y = \pm\sqrt{x^2}$ (C) $y = -4x$ (D) $y^2 = 3x$

18. The equation $x=A\cos(pt-\alpha)$ can be expressed as

(A) $\frac{d^2x}{dt^2} = x$ (B) $\frac{d^2x}{dt^2} = -p^2x$ (C) $\frac{d^2x}{dt^2} = 0$ (D) none of these

19. The general solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2$ is

(A) $xy = \frac{1}{4}x^4 + c$ (B) $xy = c$ (C) $\frac{x}{y} = c$ (D) none of these

20. Solution of $\frac{dy}{dx} + \frac{1}{x}y = x^3$ is

(A) $e^{\log y} = x$ (B) $y = x$ (C) $\log x = y$ (D) none of these

21. The equation of the curve, for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = A\sin 2\theta$. This satisfies the differential equation

(A) $r(dr/d\theta) = \tan 2\theta$ (B) $r(d\theta/dr) = \tan 2\theta$
(C) $r(dr/d\theta) = \cos 2\theta$ (D) $r(d\theta/dr) = \cos 2\theta$

22. Let m be the order of a differential equation. Then

(A) m is not unique (B) m is unique
(C) m may or may not be unique (D) m may be infinite

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

23. The differential equation $\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0$ is of

(A) second degree (B) first degree (C) first order (D) second order

24. The equation $y\,dx + x\,dy = 0$ is

(A) exact differential equation (B) non-exact differential equation
(C) partial differential equation (D) of first order

25. Which of the following is not true for the statement
"The complete solution of a differential equation contains arbitrary constants" ?

(A) more than the order of equation (B) can't say
(C) equal to the order of equation (D) less than the order of equation

26. Which of the following differential equations is not linear ?

(A) $(1+y)\frac{dy}{dx} + xy = \sin x$ (B) $y\frac{dy}{dx}x + x = 0$
(C) $(1+y)\frac{dy}{dx} + \sin x = 0$ (D) $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + (1+x)^3 y = \sin x + e^x$

27. *Consider differential equation $|y'| + |y| = 0$. Which of the following statements is true?*
(A) *General solution exists*
(B) *General solution exists but contains no arbitrary constants*
(C) *It has particular solution which is bounded*
(D) *None of the above*

28. *Given,* $\frac{dy}{dx} = 2xy$. *Solving the above differential equation gives*
(A) $\frac{dy}{y} = 2xdx$ (B) $\log y = x^2 + \log a$ (C) $\log y = \frac{x}{\log a}$ (D) $y = ae^{x^2}$

29. *Given,* $\frac{dx}{x} = \tan y dy$. *Solving the above differential equation gives*
(A) $\log x = -\log \cos y + C$ (B) $\log(x \cos y) = C$
(C) $\log(x \sin y) = C$ (D) $x \cos y = e^C$

30. *Consider a family of parabolas $y^2 = 4a(x + a)$, which of the following statements is true?*
(A) *The differential equation satisfied by the given system of parabolas $y^2=4a(x+a)$ is* $y\left(\frac{dy}{dx}\right)^2 - y + 2x\frac{dy}{dx} = 0$
(B) *Differentiating the equation with respect to x, we have* $2y\frac{dy}{dx} = 4a$
(C) *No differential equation for the given equation exists*
(D) *None of these*

ASSIGNMENT - 1.3

NOTE: CHOOSE THE BEST OPTION

1. The solution of the differential equation $(2ax + by) y dx + (ax+2by) x dy = 0$ is
   (A) $axy^2 + bxy = c$
   (B) $ax^2y^2 + bxy = c$
   (C) $ayx^2 + bxy^2 = c$
   (D) $axy + by^2 = c$

2. Differential equation $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$ has an integrating factor
   (A) $e^{\int \frac{2x^2-1}{1-x}dx}$
   (B) $e^{\int \frac{2x^2-1}{x}dx}$
   (C) $e^{\int \frac{2x^2-1}{x(1-x^2)}dx}$
   (D) none of the above

3. The solution of $\frac{dy}{dx} = \frac{1-x}{y}$ represents
   (A) a family of circles centered at (1,0)
   (B) a family of circles centered at (0,0)
   (C) a family of circles centered at (-1,0)
   (D) a family of straight lines with slope -1.

4. The solution of $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ is
   (A) $y \sin x + (\sin y + y) x = c$
   (B) $y \sin x + (\sin x + x) = c$
   (C) $y = \sin x + y \sin y + c$
   (D) none of these

5. The solution $y = A\cos(x+B)$ is equivalent to $\alpha \sin x + \beta \cos x$, where
   (A) $\alpha = A\sin B, \beta = A\cos B$
   (B) $\alpha = B\sin B, \beta = B\cos B$
   (C) $\alpha = -A\sin B, \beta = A\cos B$
   (D) none of the above

6. General solution of $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ is
   (A) $y = (2x + c)e^{-x^2}$
   (B) $y = 2xe^{-x}$
   (C) $y = e^{-x}$
   (D) none of these

7. The solution of the differential equation $(x + y - 2) dy/dx = (x + y)$ is
   (A) $(y + x) = \log(y - x + 1) + c$
   (B) $(y - x) = \log(x + y - 1) + c$
   (C) $y - 2x = \log(x + y - 1) + c$
   (D) $y + 2x = \log(x + y + 1) + c$

8. The solution of the differential equation $y = px + \sqrt{4 + p^2}$ is
   (A) $(y - cx)^2 - c^2 = 4$
   (B) $(y - cx)^2 - 4c^2 = 0$
   (C) $(y - cx)^2 + c^2 = 0$
   (D) $(y - cx)^2 + 4c^2 = 0$

9. *The solution of* $(x-y^2)dx+2xy\,dy=0$ *is*

(A) $ye^{\frac{y^2}{x}}=A$ (B) $xe^{\frac{y^2}{x}}=A$

(C) $ye^{\frac{x}{y^2}}=A$ (D) $ye^{\frac{x}{y^2}}=A$

10. *The general solution of* $\left(\frac{dy}{dx}\right)=\left(\frac{y}{x}\right)+\tan\left(\frac{y}{x}\right)$ *is*

(A) $\sin\left(\frac{y}{x}\right)=c$ (B) $\cos\left(\frac{y}{x}\right)=cx$

(C) $\sin\left(\frac{y}{x}\right)=cx$ (D) $\cos\left(\frac{y}{x}\right)=c.$

11. *Differential equation* $xdy-ydx-2x^3dx=0$ *has the solution*

(A) $y+x^2=c_1x$ (B) $-y+x^3=c_2x$

(C) $y-x^2=c_3x$ (D) $y^3-x^3=c_4x$

12. *The differential equation of the family of circles of radius 'r' whose centers lie on the x-axis is*

(A) $y(dy/dx)+y^2=r^2$ (B) $y[(dy/dx)+1]=r^2$

(C) $y^2[(dy/dx)+1]=r^2$ (D) $y^2[(dy/dx)^2+1]=r^2$

13. *The primitive of the differential equation* $(2xy^4e^y+2xy^3+y)dx+(x^2y^4e^y-x^2y^2-3x)dy=0$ *is*

(A) $x^2e^y+(x^2/y)+(x/y^3)=c$ (B) $x^2e^y-(x^2/y)+(x/y^3)=c$

(C) $x^2e^y+(x^2/y)-(x/y^3)=c$ (D) $x^2e^y-(x^2/y)-(x/y^3)=c$

14. *Let* $(y-c)^2=cx$ *be the primitive of* $4x(dy/dx)^2+2x(dy/dx)-y=0$. *The number of integral curves which will pass through (1,2) is*

(A) one (B) two (C) three (D) four.

15. *A solution curve of the equation* $xy'=2y$, *passing through (1, 2), also passes through*

(A) (2, 1) (B) (0, 0) (C) (4, 24) (D) (24, 4)

16. *The solution of the differential equation* $(dy/dx)+(y/x)=x^2$ *under the condition that* $y=1$ *when* $x=1$ *is*

(A) $4xy=x^4+3$ (B) $4xy=y^4+3$ (C) $4xy=x^2+3$ (D) $4xy=y^3+3.$

17. *The family of straight lines passing through the origin is represented by the differential equation*

(A) $ydx+xdy=0$ (B) $xdx+ydy=0$

(C) $xdy-ydx=0$ (D) $ydy-xdx=0.$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

18. *Which of the following is not the integrating factor for the differential equation*

$(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$

(A) $\frac{1}{x+1}$ (B) $x+1$

(C) $\frac{1}{x^2+1}$ (D) x

19. *Which of the following is a solution of* $y''+y= 0$?

(A) $y = \sin x$ (B) $y = \cos x$

(C) $y = 3\cos x$ (D) $y = \sin x + \frac{1}{2}$

20. *Which of the following is a solution of* $xy'+y=0$?

(A) $xy = \sqrt{3}$ (B) $xy = -2$

(C) $x = \sqrt{3}\, y$ (D) $xy = \frac{\pi}{2}$

21. *Which of the following is not the general solution of the differential equation* $\sin x dx + \frac{dy}{\sqrt{y}} = 0$?

(A) $2\sqrt{y} + \cos x = c$ (B) $2\sqrt{y} - \cos x = c$

(C) $2\sqrt{y} = \sin x$ (D) $\sin x + \cos x = \sqrt{y}$

22. *The general solution of the differential equation* $dy/dx + y\frac{d\phi}{dx} = \phi(x)\frac{d\phi}{dx}$, *where* ϕ *is a function of x alone, is not given by*

(A) $y = \phi + ce^{-\phi}$ (B) $y = \phi + 1 - ce^{-\phi}$

(C) $y = \phi - 1 + ce^{\phi}$ (D) $y = \phi - 1 + ce^{-\phi}$

23. *The solution of* $(x+y)^2(dy/dx) = a^2$ *is not given by*

(A) $y + x = a\tan\{(y-c)/a\}$ (B) $y - x = a\tan(y-c)$

(C) $y - x = \tan\{(y-c)/a\}$ (D) $a(y-x) = \tan\{(y-c)/a\}$

24. *Which of the following information are true for the differential equation?*

$E : (x^3 + y^3)dx - (xy^2)dy = 0$

(A) *E is an exact differential equation.* (B) *Integrating factor of E is* x^{-4}

(C) *Solution of E is* $\log x - \frac{y^3}{3x^3} = C$ (D) *Solution of E is* $\log x - \frac{x^3}{3y^3} = C$

ANSWERS TO EXERCISES

(PRACTICE SET – 1)

1. (D)   2. $x+y+5=c(x-y-1)^3$ 3. $x = y(e^y + c)$ 4. $y^5 = 3x^5 - 2x^{-5}$ 5. (B)

(PRACTICE SET – 2)

1. (A)   2. (B)   3. (B)   4. (B)   5. (A)   6. (B)

(PRACTICE SET – 3)

1. (C)   2. (A)   3. (B)   4. (D)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 1.1

1. (B)   2. (C)   3. (B)   4. (C)   5. (B)   6. (A)
7. (D)   8. (A)   9. (A)   10. (C)   11. (B)   12. (A)
13. (A)   14. (C)   15. (C)   16. (A)   17. (B)   18. (B)
19. (B)   20. (A)   21. (D)
22. (A ,C, D)   23. (B, D)   24. (A, D)   25. (A, C, D)

ASSIGNMENT - 1.2

1. (C)   2. (D)   3. (C)   4. (D)   5. (A)   6. (B)
7. (D)   8. (A)   9. (C)   10. (C)   11. (B)   12. (A)
13. (C)   14. (A)   15. (D)   16. (A)   17. (A)   18. (B)
19. (A)   20. (D)   21. (B)   22. (B)
23. (A, D)   24. (A, D)   25. (A, B, D)   26. (A, B, C)   27. (C)   28. (A, B, D)
29. (A, B, D)   30. (A,B)

ASSIGNMENT - 1.3

1. (C)   2. (C)   3. (A)   4. (A)   5. (C)   6. (A)
7. (B)   8. (A)   9. (B)   10. (C)   11. (B)   12. (D)
13. (A)   14. (B)   15. (B)   16. (A)   17. (C)
18. (B, C, D)   19. (A, B, C)   20. (A, B, D)   21. (A, C, D)   22. (A, B, C)   23. (B, C, D)
24. (B, C)

CHAPTER – 2
LINEAR DIFFERENTIAL EQUATIONS

INTRODUCTION

In this chapter, we will learn how to solve a homogeneous and non-homogeneous differential equation i.e. how to find particular integral and general solution. We will also learn to find the envelop of general solution, i.e., singular solutions. The method of undetermined coefficients and variation of parameters are another ways of solving a differential equation. Here, we learn what are regular points, irregular points and ordinary points and also to solve the simultaneous differential equation. In the end, we will discuss e^{At}.

§ 2.1. DEFINITIONS

(i) **Linear Combination.** *If* $f_1, f_2, ..., f_n$ *are n functions defined on the interval I and* $c_1, c_2, ..., c_n$ *are n arbitrary constants, then the function* $c_1 f_1 + c_2 f_2 + ... + c_n f_n$ *is called linear combination of* $f_1, f_2, ..., f_n$ *over I.*
e.g. $e^{-x} + 2e^x - 4e^{2x}$ *is a linear combination of* e^{-x}, e^x, e^{2x} *[Here* $c_1 = 1, c_2 = 2, c_3 = -4$*].*

(ii) **Linearly Dependent.** *The functions* $f_1, f_2, ..., f_n$ *of x are said to be linearly dependent over an interval I iff there exist constants* $c_1, c_2, ..., c_n$ *(not all zero) such that* $c_1 f_1 + c_2 f_2 + ... + c_n f_n = 0$ *for all x in I.*
In particular, *two functions* f_1 *and* f_2 *are linearly dependent on [a, b] iff there exist two constants* c_1, c_2 *(not both zero) such that* $c_1 f_1 + c_2 f_2 = 0$ *for all x in [a, b].*
e.g. *The functions x and 2x are linearly dependent on [0,1] because there exist constants* $(c_1 = 2, c_2 = -1)$ *s.t.* $c_1 f_1 + c_2 f_2 = 0$ *i.e.* $(2)(x) + (-1)(2x) = 0$.

(iii) **Linearly Independent.** *The functions* $f_1, f_2, ..., f_n$ *of x are said to be linearly independent over an interval I iff there exist constants* $c_1, c_2, ..., c_n$ *such that* $c_1 f_1 + c_2 f_2 + ... + c_n f_n = 0$ *for all x in I; then* $c_i = 0 \ \forall \ i = 1, 2, ..., n$.
e.g. *The functions x and* x^2*, are linearly independent on [0, 1]*

Conclusions:

(i) *If* $f_1, f_2, ..., f_n$ *are linearly dependent, then at least one of them is a linear combination of others.*

(ii) *If* $f_1, f_2, ..., f_n$ *are linearly independent, then none of them is linear combination of others.*

(iii) **Wronskian.** *Let* $f_1, f_2, ..., f_n$ *be n real functions over I each of which has a derivative of order (n – 1) over I, then the determinant.*

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(f_1, f_2, ..., f_n)$$

is called Wronskian of $f_1, f_2, ..., f_n$ *over I. Thus the Wronskian* $W(f_1, f_2, ..., f_n)$ *is itself a real valued function on I. Its value at x is denoted by* $W(f_1, f_2, ..., f_n)(x)$ *or by* $W[f_1(x), f_2(x), ... f_n(x)]$.

(iv) *If $f_1, f_2, ..., f_k$ are linearly dependent on an interval, they are linearly dependent at each point in the interval. However, if $f_1, f_2, ..., f_k$ are linearly independent on an interval, they may or may not be linearly independent at each point; they may, in fact, be linearly dependent at each point, but with different sets of constants at different points.*

e.g. $f_1(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, f_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$. *$f_1(t)$ and $f_2(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. But $f_1(t)$ and $f_2(t)$ are linearly independent on $0 \le t \le 1$.*

Conclusion: *If $W(f_1, f_2, ..., f_n)$ over I is non-zero, then $f_1, f_2, ..., f_n$ are linearly independent over I (provided they are solutions of same differential equation)*

Example. *Show that the following functions are linearly dependent:*

(i) 1, $\sin^2 x$, $\cos^2 x$ *(ii) e^x, e^{-x}, $\cosh x$* *(iii) x, e^x, xe^x, $(2 - 3x)e^x$.*

§ 2.2. GENERAL LINEAR DIFFERENTIAL EQUATION

Definition. *A general linear differential equation of order n is*

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_n y = Q, \quad ... (1)$$

where $P_0, P_1, P_2, ..., P_n$ and Q are functions of x defined on some interval I. When $Q = 0$, then (1) is said to be homogeneous. When $Q \neq 0$, then (1) is said to be non-homogeneous.

Existence and Uniqueness Theorem

Statement: *If $P_1, P_2, ..., P_n$ and Q are continuous functions of x over an open interval I and if $x_0 \in I$ be the real number and $y_0, y_1, ..., y_{n-1}$ are arbitrary real numbers, then there exists one and only one function; say $\phi(x)$, which in some neighbourhood of x_0, is a solution of the differential equation*

$$y^{(n)} + P_1 y^{(n-1)} + P_2 y^{(n-2)} + ... + P_n y = Q \text{ s.t. } \phi(x_0) = y_0, \phi'(x_0) = y_1, ..., \phi^{n-1}(x_0) = y_{n-1}.$$

2.2.1. Homogeneous Linear Differential Equation.

Theorem 1. *Let $f_1(x)$ and $f_2(x)$ be two linearly independent solutions of* $P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad ... (1)$

over an open interval I, where P_0, P_1, P_2 are all continuous functions of x and $P_0(x) \neq 0$ on I. If $f(x)$ is any solution of (1), then $f = \alpha f_1 + \beta f_2$, where α, β are some constants.

Conclusion: *Each solution of $P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$ is linear combination of two linearly independent solutions.*

Theorem 2. *Let $P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$, where P_0, P_1, P_2 are all continuous functions of x over an open interval I and $P_0(x) \neq 0$, then there exist linearly independent solutions $f_1(x)$ and $f_2(x)$ such that*

$f_1(x_0)=1, f_2(x_0)=0; f_1'(x_0)=0, f'_2(x_0)=1$; where $x_0 \in I$.

Conclusion: *Above two results assure that a homogeneous linear differential equation of order 2 has two L.I. solutions and any other solution is a linear combination of these two solutions.*

Extension: *Let $f_1(x), f_2(x), ..., f_n(x)$ be n linearly independent solutions of*

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+...+P_ny=0 \qquad ... (1)$$

over an open interval I, where $P_0, P_1, ..., P_n$ are all continuous functions of x and $P_0(x) \neq 0$ on I. If $f(x)$ is any solution of (1), then $f = c_1f_1 + c_2f_2 + ... + c_nf_n$, where $c_1, c_2, ..., c_n$ are constants.

Here $c_1f_1+c_2f_2+...+c_nf_n$ is the general solution (or complete primitive) of the given differential equation.

2.2.2. Non-homogeneous Linear Equation.

Let $f_p(x)$ be any particular solution of $P_0y^{(n)} + P_1y^{(n-1)} + ... + P_ny = Q$, ... (1)

where $P_0, P_1, ..., P_n, Q$ are all continuous functions over an open interval I and $P_0(x)\neq0$ on I and $f_0(x)$ be any solution of $P_0y^{(n)} + P_1y^{(n-1)} + ...+ P_ny = 0$, ... (2)

then $f(x) = f_0(x) + f_p(x)$ is also a solution of the given equation.

* *If $f_1, f_2, ..., f_n$ are L.I. solutions of (2), then $c_1f_1 + c_2f_2 + ...+ c_nf_n$, where $c_1, c_2, ..., c_n$ are arbitrary constants, is called the complementary function of (1).*

* *Let f_0 be the complementary function and f_p be the particular solution of the non-homogeneous linear equation $P_0y^{(n)}+P_1y^{(n-1)}+...+P_ny = Q$, where $P_0,P_1,...,P_n$, Q are all continuous functions over an open interval I and $P_0(x)\neq0$ on I. If f is any solution of this equation, then $f=f_0+f_p$ for some particular values of $c_1, c_2, ..., c_n$.*

Remember: *General solution of a non-homogeneous equation is Complementary Function + Particular Solution.*

Differential Operators: *D denotes $\frac{d}{dx}$, D^2 denotes $\frac{d^2}{dx^2}$, ..., D^k denotes $\frac{d^k}{dx^k}$.*

Thus the polynomial $P_0D^n + P_1D^{n-1} + ... + P_n$ in D is said to be differential operator of order n, where $P_0, P_1, ..., P_n$ are functions of x. It is usually denoted by L.

Thus $L = P_0D^n + P_1D^{n-1} + ... + P_n$.

e.g. *Consider the equation $3\frac{d^2y}{dx^2}-5\frac{dy}{dx}+7y = e^x$. In the operator form, the equation is $(3D^2-5D+7)y= e^x$, where the operator is $L = 3D^2 - 5D+ 7$.*

Basic Laws: *If L_1, L_2 and L_3 are any three differential operators, then the following laws hold.*

Closed:

(i) Under addition: $L_1 + L_2$ is a differential operator.

(ii) Under multiplication: L_1L_2 is differential operator.

Commutative:

(i) Under addition : $L_1 + L_2 = L_2 + L_1$ *(ii) Under multiplication : $L_1L_2 = L_2L_1$*

Associative:

(i) Under addition : $(L_1+L_2)+L_3 = L_1+(L_2+L_3)$ *(ii) Under multiplication :$(L_1L_2)L_3=L_1(L_2L_3)$*

Distributive:

(i) $L_1(L_2 + L_3) = L_1L_2 + L_1L_3$ *(ii) $(L_1 + L_2)L_3 = L_1L_3 + L_2L_3$*

Exponential Shift: *If $f(D)$ is any polynomial in D with constant co-efficients, then $e^{\alpha x} f(D)y = f(D-\alpha)(e^{\alpha x})$, where y is any function of x.*

§ 2.3. LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Definition: *A linear differential equation with constant coefficients is one in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together.*

Thus $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = Q$

(where $P_0, P_1, P_2, \dots, P_n$ are constants and Q is a function of x) is the linear equation of the nth order.

* *If $y=y_1, y=y_2, \dots, y=y_n$ are n linearly independent particular solutions of the equation $P_0\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = 0$, where $P_0, P_1, P_2, \dots, P_n$ are constants, then the complete solution is $y=c_1y_1+c_2y_2+\dots+c_ny_n$.*

Auxiliary Equation (A.E.)

Auxiliary equation is obtained by equating to zero the symbolic co-efficients of y.

Thus, $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 0$ is the auxiliary equation.

Method to solve the equation:

$P_0\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = 0$, *where $P_0, P_1, P_2, \dots, P_n$ are constants*

(i) *Write the equation in the symbolic form as: $(P_0D^n+P_1D^{n-1}+P_2D^{n-2}+\dots+P_n)y=0$.*

(ii) *Write down the auxiliary equation (A.E) as: $P_0m^n + P_1m^{n-1} + P_2m^{n-2} + \dots + P_n=0$. Solve it for m*

(iii) *From the roots of A.E., write down the corresponding part of the complete solution (C.S) as follows:*

No.	Roots of A.E.	Corresponding parts of C.S
(i)	One real root m_1.	$c_1 e^{m_1x}$
(ii)	Two real and distinct roots m_1, m_2.	$c_1 e^{m_1x} + c_2 e^{m_2x}$
(iii)	Two real and equal roots m_1, m_1.	$e^{m_1x}(c_1 + c_2x)$
(iv)	Three real and equal roots m_1, m_1, m_1.	$e^{m_1x}(c_1 + c_2x + c_3x^2)$
(v)	One pair of imaginary and different roots $\alpha \pm i\beta$.	$e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$
(vi)	Two pairs of imaginary and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$.	$e^{\alpha x}[\cos\beta x(c_1+c_2x)+\sin\beta x(c_3+c_4x)]$

* If $y=Y$ is the complementary solution of the equation $P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+...+P_ny=0$ ...(1) and $y = u$ is a particular solution (containing no arbitrary constants) of the equation $P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+...+P_ny=Q$ ... (2), where Q is a function of x, then the complete solution of (2) is $y = Y + u$.

(i) **Complementary Function (C.F.):** The part Y is known as complementary function.

(ii) **Particular Integral (P.I):** The part u is known as particular integral.

Remark: Particular integral (P.I) is not unique for a differential equation

Inverse Operator: $\frac{1}{f(D)}$: $\frac{1}{f(D)}Q$ is that function of x which is independent of arbitrary constants and which when operated on by $f(D)$ gives Q. Thus $f(D).\frac{1}{f(D)}Q = Q$. Hence $\frac{1}{f(D)}$ is the inverse operator of $f(D)$.

- $\frac{1}{f(D)}Q$ is the particular integral of the equation $f(D)\,y = Q$.
- The symbol $\frac{1}{D}$ stands for integration.
- $\frac{1}{D}Q = \int Q\,dx$, no arbitrary constant is being added.
- $\frac{1}{D-a}Q = e^{ax}\int Q \cdot e^{-ax}\,dx$, no arbitrary constant is being added.

Method to solve the equation:

$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+...+P_ny=Q$, where $P_0, P_1, P_2, ..., P_n$ are constants and Q is a function of x

Remember:

(i) Write the equation in the symbolic form as: $(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ... + P_n)y = Q$.

(ii) Write down auxiliary equation (A.E) as : $P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ... + P_n = 0$. Solve it for D.

(iii) Write down complementary function (C.F.) by the same method as for writing down C.S. if R.H.S is zero instead of Q.

(iv) Find particular integral (P.I.) [For P.I, see 1.5]

(v) Then C.S. is y = C.F. + P.I.

§ 2.4. FIVE STANDARD CASES OF PARTICULAR INTEGRALS

(a) **Method to evaluate** $\frac{1}{f(D)} e^{ax}$ **:** Put D = a provided $f(a) \neq 0$.

Case of Failure.

Method to evaluate $\frac{1}{f(D)} e^{ax}$, when $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x. \frac{1}{\frac{d}{dD}[f(D)]} e^{ax}.$$

Note: If by using the said rule, the denominator again vanishes, repeat the rule and so on.

(b) **Method to evaluate** $\frac{1}{f(D^2)}\sin ax$ or $\frac{1}{f(D^2)}\cos ax$ **:** Put $D^2 = -a^2$ provided $f(-a^2) \neq 0$.

Case of failure

Method to evaluate $\frac{1}{f(D^2)}\sin ax$ or $\frac{1}{f(D^2)}\cos ax$ when $f(-a^2) = 0$ **:** $\frac{1}{f(D^2)}\sin ax = x. \frac{1}{\frac{d}{dD}[f(D^2)]}\sin ax$.

$$\frac{1}{f(D^2)}\cos ax = x. \frac{1}{\frac{d}{dD}[f(D^2)]}\cos ax.$$

Note: If by using the said rule, the denominator again vanishes, repeat the rule and so on.

(c) **Method to evaluate** $\frac{1}{f(D)} x^m$, where m is positive integer, is as follows:

(i) From f(D), take the lowest degree term outside. Then the remaining factor will be of the type $[1 \pm \phi(D)]$.

(ii) Take $[1 \pm \phi(D)]$ to the numerator and expand it by Bionomial Theorem ($\because D^{m+1}(x^m) = 0$)

(iii) Operate each term on x^m.

Remember:

(i) $(1-D)^{-1} = 1 + D + D^2 + ...$ to ∞ (ii) $(1+D)^{-1} = 1 - D + D^2 - ...$ to ∞

(iii) $(1-D)^{-2} = 1 + 2D + 3D^2 + ...$ to ∞ (iv) $(1-D)^{-3} = 1 + 3D + 6D^2 + ...$ to ∞

(d) ***Method to evaluate*** $\frac{1}{f(D)}$ ***$(e^{ax} X)$**, where X is any function of x.*

$$\frac{1}{f(D)}(e^{ax}X) = e^{ax}.\frac{1}{f(D+a)}X$$

In other words, take e^{ax} outside and in f(D) write (D + a) for every D so that f(D) becomes f(D+a) and operate $\frac{1}{f(D+a)}$ *with x alone by the previous methods.*

(e) ***Method to evaluate*** $\frac{1}{f(D)}$ ***(x X)**, where X is any function of x.*

$$\frac{1}{f(D)}(xX) = x\frac{1}{f(D)}X + \frac{d}{dD}\left[\frac{1}{f(D)}\right]X.$$

Example 1. *Solve* $(D^4+2D^3-3D^2)\ y = 3e^{2x}+4\sin x$

Solution: *A.E is* $m^4+2m^3-3m^2=0$

$\Rightarrow m = 0,0,1,-3$

C.F is $y = (c_1+c_2x)+c_3e^x+c_4e^{-3x}$

on putting D=2, P.I of $3e^{2x}$ *is* $\frac{3}{20}e^{2x}$

P.I of $4\sin x = \frac{4\sin x}{D^2(D^2+2D-3)}$

$$=\frac{4\sin x}{(-1)(-1+2D-3)} = -2\frac{\sin x}{(D-2)}$$

$$=-2\frac{(D+2)\sin x}{D^2-4} - \frac{-2}{-1-4}(\cos x+2\sin x)$$

$$=\frac{2}{5}(2\sin x+\cos x)$$

$\therefore$ *General Solution is y = C.F. + P.I.*

$$y = (c_1+c_2x)+c_3e^x+c_4e^{-3x}+\frac{2}{5}(2\sin x+\cos x)+\frac{3}{20}e^{2x}$$

Example 2. *Find P.I of* $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y = x^2+x$

Solution: *P.I for* x^2+x *is* $\frac{1}{D^2-3D+2}(x^2+x)$

$$=\frac{1}{2\left(1+\frac{D^2-3D}{2}\right)}(x^2+x) = \frac{1}{2}\left(1+\frac{D^2-3D}{2}\right)^{-1}(x^2+x)$$

$$=\frac{1}{2}\left[1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2\ldots\right](x^2+x)=\frac{1}{2}\left(1+\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}\ldots\right)(x^2+x)$$

$$=\frac{1}{2}\left[(x^2+x)+\frac{3}{2}D(x^2+x)+\frac{7}{4}D^2(x^2+x)\right]=\frac{1}{2}\left[x^2+x+\frac{3}{2}(2x+1)+\frac{7}{4}(2)\right]=\frac{1}{2}(x^2+4x+5)$$

Example 3. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2e^{3x}$

Solution: *A.E is* $m^2-2m+1=0 \Rightarrow m=1,1$

$\therefore$ *C.F is* $y=(c_1+c_2x)e^x$

P.I for x^2e^{3x} *will be* $\frac{x^2e^{3x}}{(D-1)^2}$

$$=e^{3x}\left\{\frac{1}{(D+3-1)^2}\right\}x^2=e^{3x}\left\{\frac{1}{(D+2)^2}\right\}x^2$$

$$=\frac{e^{3x}}{4}\left\{\left(1+\frac{D}{2}\right)^{-2}\right\}x^2=\frac{e^{3x}}{4}\left\{1-2\left(\frac{D}{2}\right)+3\left(\frac{D^2}{4}\right)\ldots\right\}x^2$$

$$=\frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{4}\cdot 2\right)=\frac{e^{3x}}{8}\left(2x^2-4x+3\right)$$

$\therefore$ *General solution is* $y=(c_1+c_2x)e^x+\frac{e^{3x}}{8}(2x^2-4x+3)$

§ 2.5. LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Definition. *A differential equation of the form* $P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_ny=Q$, *where* $P_0,P_1,P_2,\ldots,P_n$ *and Q are functions of x is called linear differential equation with variable coefficients.*

2.5.1. Cauchy's Homogeneous Linear Equation

Definition. *A homogeneous linear equation is of the form* $P_0x^n\frac{d^ny}{dx^n}+P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+P_ny=Q(x)$, *where* $P_0, P_1,\ldots, P_n$ *are all real constants and Q(x) is a function of x; is called Cauchy's linear equation.*

Method to solve :

(i) *Put* $x=e^z$, *i.e.,* $z=\log x, x>0$

(ii) *Put* $\frac{d}{dz}=\theta$ *so that* $xD=\theta, x^2D^2=\theta(\theta-1),\ldots, x^nD^n=\theta(\theta-1)\ldots(\theta-n+1)$.

Putting these in the given equation, we get

$[P_0\theta(\theta-1)\ldots(\theta-n+1)+P_1\theta(\theta-1)\ldots(\theta-n+2)+\ldots+P_n]y=Q(e^z)$, *which is linear equation with constant coefficients and solve for y in terms of z.*

(iii) *Put* $z=\log x$, *and we get the required solution.*

Example 1. Solve $x^4\frac{d^3y}{dx^3}+2x^3\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=1$

Solution: *The given equation may be written as* $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\frac{1}{x}$.

Putting $x=e^z$, *the equation becomes* $[\theta(\theta-1)(\theta-2)+2\theta(\theta-1)-\theta+1]y=e^{-z}$ *or* $(\theta-1)^2(\theta+1)y=e^{-z}$

A.E is $(m-1)^2(m+1)=0$ *or* $m=1,1,-1$

$\therefore C.F=(c_1+c_2z)e^z+c_3e^{-z}=(c_1+c_2\log x)x+c_3x^{-1}$

An $P.I=\frac{1}{(D-1)^2(D+1)}e^{-z}=\frac{1}{(D+1)}\frac{1}{(D-1)^2}e^{-z}=\frac{1}{(D+1)}\frac{1}{4}e^{-z}=\frac{1}{4}e^{-z}\frac{1}{D-1+1}1$

$=\frac{1}{4}ze^{-z}=\frac{1}{4x}\log x.$

Hence, the solution is $y=(c_1+c_2\log x)x+\frac{c_3}{x}+\frac{1}{4x}\log x.$

2.5.2. Legendre's Linear Equation.

An equation of the form $P_0(a+bx)^n\frac{d^ny}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+.....+P_n(y)=Q(x)$ ... (1)

where $P_0, P_1, ..., P_n$ *are all real constants and* $Q(x)$ *is a function of x is called Legendre's linear equation.*

Method to solve:

(i) *Put* $a+bx=e^z$, *i.e.,* $z=\log(a+bx)$, $a+bx>0$.

(ii) *Put* $\frac{d}{dz}=\theta$, *so that* $(a+bx)D=b\theta$, $(a+bx)^2D^2=b^2\theta(\theta-1),...,(a+bx)^nD^n=b^n\theta(\theta-1)...(\theta-n+1)$.

(iii) *Putting these in the given equation, we get*

$[P_0b^n\theta(\theta-1)...(\theta-n+1)+P_1b^{n-1}\theta(\theta-1)...(\theta-n+2)+...+P_n]y=Q\left(\frac{e^{z-a}}{b}\right)$,

which is linear equation with constant co-efficients and solve for y in terms of z.

(iv) *Put* $z=\log(a+bx)$, *and, we get the required solution.*

Example 2. Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$

Solution: *Putting* $1+x=e^z\Rightarrow z=\log(1+x)$ *equation becomes* $[\theta(\theta-1)+\theta+1]y=4\cos z$

or $(\theta^2+1)y=4\cos z$

A.E is $m^2+1=0$ *or* $m=\pm i$ $\therefore C.F=c_1\cos(z+c_2)=c_1\cos[\log(x+1)+c_2]$.

$P.I=\frac{1}{D^2+1}4\cos z=\frac{z}{2D}(4\cos z)=4\left(\frac{z}{2}\sin z\right)=2z\sin z=2\log(x+1)\sin\log(x+1).$

Hence the solution is $y=c_1\cos[\log(x+1)+c_2]+2\log(x+1)\sin\log(x+1)$

§ 2.6. EXACT DIFFERENTIAL EQUATION

Definition: *A differential equation* $f\left(\frac{d^n y}{dx^n},\ldots,\frac{dy}{dx},y\right)=X$, *is said to be exact if it can be derived by differentiation merely, and without any further process from an equation of the next lower order*

The condition that the differential equation $P_0\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n y=X$, *where* $P_0, P_1, P_2, \ldots, P_n$ *are functions of x, is exact is* $P_n-P'_{n-1}+P''_{n-2}-P'''_{n-3}+\ldots+(-1)^n P_0^{(n)}=0$

PRACTICE SET – 1

1. *Given that, there is a common solution to the following equations* $P: y'+2y=e^x y^2, y(0)=1$, $Q: y''+2y'+\alpha y=0$, *Find the value of* α *and hence find the general solution of Q.* **(IIT JAM 2014)**

2. *The general solution of the differential equation with constant coefficients* $\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ *approaches zero as* $x\to\infty$, *if* **(IIT JAM 2016)**
(A) b is negative and c is positive
(B) b is positive and c is negative
(C) both b and c are positive
(D) both b and c are negative

3. *Consider the equation of an ideal planar pendulum given by* $\frac{d^2x}{dt^2}=-\sin x$, *where x denotes the angle of displacement. For sufficiently small angles of displacement, the solution is given by (where a, b are constants)* **(CSIR UGC NET JUNE-2013)**
(A) $x(t)=a\cosh t+b\sinh t$
(B) $x(t)=a+bt$
(C) $x(t)=ae^t+be^{2t}$
(D) $x(t)=a\cos t+b\sin t$

4. *If* $D\equiv\frac{d}{dx}$, *then the value of* $\frac{1}{(xD+1)}(x^{-1})$ *is* **(GATE-2009)**
(A) $\log x$
(B) $\frac{\log x}{x}$
(C) $\frac{\log x}{x^2}$
(D) $\frac{\log x}{x^3}$

5. *Let* $y_1(x)=x^3$ *and* $y_2(x)=x^2|x|$ *for* $x\in\mathbb{R}$. *Consider the following statements:*
(P) : $y_1(x)$ *and* $y_2(x)$ *are linearly independent solutions of* $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0$ *on* $\mathbb{R}$
(Q) : *The Wronskian* $y_1(x)\frac{dy_2}{dx}(x)-y_2(x)\frac{dy_1}{dx}(x)=0$ *for all* $x\in\mathbb{R}$.
Which of the above statements hold TRUE? **(GATE-2017)**
(A) Both P and Q
(B) Only P
(C) Only Q
(D) Neither P nor Q

§ 2.7. INTEGRATING FACTOR

Definition: *An integrating factor of a differential equation is that factor such that if the equation is multiplied by it, the resulting equation is exact.*

Rules For Finding The Integrating Factor

Rule 1. *When the coefficients $P_0, P_1, P_2, \ldots, P_n$ are of the form kx or the sum or difference of the terms of the said form, then x^m is the integrating factor.*

Rule 2. *When the coefficients $P_0, P_1, P_2, \ldots, P_n$ are trigonometric functions, then by trial we shall obtain some trigonometric function as the integrating factor.*

§ 2.8. SIMULTANEOUS DIFFERENTIAL EQUATIONS

A pair of linear differential equations $\dfrac{dx}{dt}+ax+by=f(t)$ *and* $\dfrac{dy}{dt}+cx+dy=g(t)$ *(a, b, c, d are constants) is called simultaneous linear differential equation with constant coefficients, here x and y are dependent variables and t is an independent variable.*

2.8.1. Working Rule:

(i) *Eliminate one of the dependent variable say y from given equations.*

(ii) *After eliminating y, we get a linear differential equation in x and t, which can be solved to get $x = \phi(t)$.*

(iii) *Now put x and $\frac{dx}{dt}$ in any of the given equations, to obtain $y = \Psi(t)$.*

(iv) *$x = \phi(t)$ and $y = \Psi(t)$ gives the required solution.*

Example 1. *Solve* $\dfrac{dx}{dt}=-wy$ *and* $\dfrac{dy}{dt}=wx$. *Also, show that the point (x, y) lies on a circle.*

Solution: *We are given the system of equations:-*

$$\frac{dx}{dt}+wy=0$$

$$\frac{dy}{dt}-wx=0$$

We can write this system as

$$Dx + wy = 0 \quad \ldots(i)$$

$$wx - Dy = 0 \quad \ldots(ii) \qquad \text{where } D=\frac{d}{dt}$$

We operate D on (i) and multiply (ii) by W, we get $\left.\begin{array}{l} D^2x+wDy=0 \\ w^2x-wDy=0 \end{array}\right] \quad \ldots(iii)$

On adding these two equations we get $(D^2+w^2)x=0 \quad \ldots(iv)$

It's auxiliary equation is $m^2+w^2=0 \Rightarrow m=\pm wi$

So, solution is $x=c_1\cos wt+c_2\sin wt$ *...(v) where c_1 and c_2 are arbitrary constants*

Now $\frac{dx}{dt}=-c_1 w\sin wt+c_2 w\cos wt$ ...(vi)

Put (v) and (vi) in the equation $\frac{dx}{dt}=-wy$ then $-c_1 w\sin wt+c_2 w\cos wt=-wy$

or $y=c_1\sin wt-c_2\cos wt$ ...(vii)

(v) and (vii) gives the required solution.

On squaring and adding (v) and (vii), we get

$x^2+y^2=(c_1\cos wt+c_2\sin wt)^2+(c_1\sin wt-c_2\cos wt)^2$

$x^2+y^2=c_1^2+c_2^2=\left[(c_1^2+c_2^2)^{1/2}\right]^2$, which is a circle

So, the point (x, y) lies on a circle.

Example 2. Solve $\frac{dy}{dt}=y$ and $\frac{dx}{dt}=2y+x$

Solution: We are given $\frac{dy}{dt}=y$ ...(i) and $\frac{dx}{dt}=2y+x$ ...(ii)

From equation (i) $\frac{dy}{y}=dt \Rightarrow \int\frac{dy}{y}=\int dt$

$\log y=t+\log c \Rightarrow \log y=\log e^t+\log c \Rightarrow y=ce^t$ ...(iii)

Now from equation (ii), $\frac{dx}{dt}=2y+x \Rightarrow \frac{dx}{dt}=2ce^t+x$

$\Rightarrow \frac{dx}{dt}-x=2ce^t$, which is a linear differential equation.

$I.F.=e^{-\int dt}=e^{-t}$, so the required solution is $x.e^{-t}=\int 2ce^t.e^{-t}dt+c_1$

$x.e^{-t}=2ct+c_1 \Rightarrow x=2cte^t+c_1e^t$

So $x=2cte^t+c_1e^t$ and $y=ce^t$ are the required solutions.

Example 3. Solve $\frac{dx}{dt}+\frac{dy}{dt}+2x+y=0,\ \frac{dy}{dt}+5x+3y=0$

Solution: We can express the given system of equations as $(D+2)x+(D+1)y=0$ ...(i)

$5x+(D+3)y=0$ ...(ii)

Multiply (i) by (D + 3) and (ii) by (D + 1) and then on subtracting

$(D+3)(D+2)x-5(D+1)x=0 \Rightarrow (D^2+1)x=0$ ...(iii)

Its A.E is $m^2+1=0 \Rightarrow m=\pm i$

So $x=c_1\cos t+c_2\sin t$ ...(iv)

$\frac{dx}{dt}=-c_1\sin t+c_2\cos t$ ...(v)

On subtracting (ii) from (i) we get $\frac{dx}{dt}-3x-2y=0 \Rightarrow 2y=\frac{dx}{dt}-3x \Rightarrow y=\frac{1}{2}\left[\frac{dx}{dt}-3x\right]$

Put the values of x and $\frac{dx}{dt}$ *from (iv) and (v)*

$$y=\frac{1}{2}[-c_1\sin t+c_2\cos t-3(c_1\cos t+c_2\sin t)]$$

$$y=-\frac{1}{2}[(c_2-3c_1)\cos t-(c_1+3c_2)\sin t] \quad ...(vi)$$

Hence (iv) and (vi) gives the required solution.

Example 4. *Solve* $\frac{d^2x}{dt^2}-3x-4y+3=0$ *and* $\frac{d^2y}{dt^2}+y+x+5=0$

Solution: *The given system of equations can be written as* $(D^2-3)x-4y=-3$ *...(i)*

and $x+(D^2+1)y=-5$ *...(ii)*

multiply equation (i) by (D^2+1) *and second by 4 and on adding* $(D^2+1)(D^2-3)x+4x$

$=-(D^2+1)3-20$

$\Rightarrow (D^4-2D^2+1)x=-3-20=-23$

$\Rightarrow (D^4-2D^2+1)x=-23$ *...(iii)*

Its auxiliary equation is $(m^2-1)^2=0 \Rightarrow m=1,1,-1,-1$

$C.F=(c_1+c_2t)e^t+(c_3+c_4t)e^{-t}$

$$PI=\frac{-23}{(D^2-1)^2}e^{0t}=\frac{-23}{1}=-23$$

So, solution of (iii) will be $x=(c_1+c_2t)e^t+(c_3+c_4t)e^{-t}-23$ *...(iv)*

$$\frac{dx}{dt}=(c_1+c_2t)e^t+c_2e^t-(c_3+c_4t)e^{-t}+c_4e^{-t} \quad ...(v)$$

$$\frac{d^2x}{dt^2}=(c_1+c_2t)e^t+2c_2e^t+(c_3+c_4t)e^{-t}-2c_4e^{-t} \quad ...(vi)$$

Now from the given first equation $y=\frac{1}{4}\left[\frac{d^2x}{dt^2}-3(x-1)\right]$

Now, put x and $\frac{d^2x}{dt^2}$ *from (iv) and (vi)*

$$y=\frac{1}{4}\left[(c_1+c_2t)e^t+2c_2e^t+(c_3+c_4t)e^{-t}-2c_4e^{-t}-3\{(c_1+c_2t)e^t+(c_3+c_4t)e^{-t}-24\}\right]$$

$$y=-\frac{1}{2}\left[(c_1+c_2t)e^t+(c_3+c_4t)e^{-t}\right]+\frac{1}{2}(c_2e^t-c_4e^{-t})+18 \quad ...(vii)$$

So, (iv) and (vii) gives the required solution.

2.8.2. *Solving Homogeneous system of differential equations with constant coefficients by using Eigen Values.*

The homogeneous system of differential equations of the type

$$x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$
$$\vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots$$
$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$, *where* a_{ij}*'s are constants* $\forall\, i, j$

It is equivalent to system $X' = AX$, *where* $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$ *and* $X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, *where* $x_i' = \dfrac{dx_i}{dt}$

It can be solved by using eigenvalues and eigenvectors as:

Method to solve:

1. *Form simultaneous homogeneous system of equations to the form* $X' = AX$
2. *Find eigen values* λ_i *of matrix A*
3. *If* $\forall\, \lambda_i$ *algebraic multiplicity = geometric multiplicity, then find eigen vector* X_i *of each eigen value* λ_i
   The solution of the system is $X = c_1e^{\lambda_1 t}X_1 + c_2e^{\lambda_2 t}X_2 + \dots + c_ne^{\lambda_n t}X_n$.

Or

if for a eigenvalue λ_k, *algebraic multiplicity > geometric multiplicity.*
Then

(i) *Find possible linearly independent eigen vectors of* λ_k.
Let algebraic multiplicity is m and geometric multiplicity is n, where $m - n = p$, $m > n$, *or* $p > 0$, $m, n, p \in \mathbb{N}$
Find m linearly independent eigen vectors then we are to find remaining p vectors

(ii) *Solve* $|A - \lambda_k I| Y_1 = X_k$, *where* X_k *is eigen vector of* λ_k *find* Y_1

(iii) *Solve* $|A - \lambda_k I|\, Y_2 = Y_1$ *and find* Y_2 *and so on*
So, we get $Y_1, \dots, Y_p$ *vectors*
Now, the solution of the system is

$$X = \sum_{i=1}^{n} c_i e^{\lambda_i t} X_i + b_1 e^{\lambda_k t}(tX_k + Y_1) + b_2 e^{\lambda_k t}\left(\frac{t^2}{2!}X_k + tY_1 + Y_2\right) + \dots + b_p e^{\lambda_k t}\left(\frac{t^p}{p!}X_k + \frac{t^{p-1}}{(p-1)!}Y_1 + \dots + Y_p\right),$$

where c_i *and* $b_1, \dots, b_p$ *are all constants*

Example 5. *Solve the system of equations* $x_1' = x_1 + x_2$ *and* $x_2' = 2x_2$

Solution: *It is equivalent to the form* $X' = AX$ *i.e.* $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Consider matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

The eigen values of A are 1, 2. (by solving $|A - \lambda I| = 0$*)*

Let $\lambda_1 = 1, \lambda_2 = 2$*, let* X_1 *be its eigen vector*
For $\lambda_1 = 1$*, solving* $(A - I)X_1 = 0$

$$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow y_2 = 0$$

Eigen vector is $X_1 = \begin{bmatrix}1\\0\end{bmatrix}$*, corresponding to* $\lambda_1 = 1$

For $\lambda_2 = 2$*, let* X_2 *be eigen vector. Then* $(A - 2I)X_2 = 0$

$$\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow -y_1 + y_2 = 0 \Rightarrow y_1 = y_2$$

Eigen vector is $X_2 = \begin{bmatrix}1\\1\end{bmatrix}$ *corresponding to eigen value* $\lambda_2 = 2$

hence, the solution is $X = c_1e^tX_1 + c_2e^{2t}X_2 = c_1e^t\begin{bmatrix}1\\0\end{bmatrix} + c_2e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$

$\Rightarrow x_1 = c_1e^t + c_2e^{2t}$ *and* $x_2 = c_2e^{2t}$ *are required solution*

Example 6. *Consider the system of differential equations* $x_1' = x_2,\ x_2' = x_3,\ x_3' = x_1 - 3x_2 + 3x_3$

Solution: *It is equivalent to the form* $X' = AX$*, where* $X' = \begin{bmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{bmatrix} \Rightarrow A = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & -3 & 3\end{bmatrix}, X' = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

For matrix A, the eigen values of A are given by equation $|A - \lambda I| = 0$
on solving, we get $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$
For finding eigen vectors X, corresponding to $\lambda_1 = 1$ *put* $(A - I)X_1 = 0$

$$\begin{bmatrix}-1 & 1 & 0\\0 & -1 & 1\\+1 & -3 & 2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\ \text{where rank } (A-I) = 2$$

Geometric multiplicity of $\lambda_1 = 1$ *is* $3 - 2 = 1$

$\because$ *(By Rank Nullity Theorem,* $\rho(A)$*+nullity A = no. of columns)*
Geometric multiplicity < Algebraic multiplicity

$\Rightarrow$ *only 1 linearly independent eigen vector exists corresponding to* $\lambda_1 = 1$.
$(A - \lambda_1 I)X = 0$

on solving, we get $X_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ *i.e. eigen vector of* $\lambda_1 = 1$

To find other two vectors solve $(A - I)Y_1 = X_1$*, where* $X_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, Y_1 = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ *is to be find out* on solving, we get $Y_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ {on solving $Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, ignoring 1^{st} part }

For Y_2, solve $(A - \lambda_1 I)Y_2 = Y_1$, where $\lambda_1 = 1$

on solving, we get $Y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The solution is $c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^t \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} + c_3 e^t \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

This is the required solution.

§ 2.9. VARIATION OF PARAMETERS

2.9.1. Let $y'' + Ry' + Sy = Q$ ... (1)

where R, S and Q are functions of x, be any given linear equation of the second order. When the complementary solution of (1) is known or can be found, we proceed as under:

The corresponding homogeneous equation is $y'' + Ry' + Sy = 0$ ... (2)

Let $y = c_1 u + c_2 v$ be the general solution of (2) and hence the complementary solution of (1) where u, v are linearly independent functions of x over an interval, say I.

Let us seek a particular solution of (1) by considering $y = Au + Bv$ ... (3)

where A, B are functions of x, and determine the functions A, B so that (3) is a solution of (1).

Differentiating (3) w.r.t. x, we get $y' = Au' + Bv' + A'u + B'v$ ... (4)

Now, instead of being involved in the derivatives of order higher than one of the functions A and B, we choose some particular functions of $A'u + B'v$.

For simplicity, we choose $A'u+B'v=0$ ... (5)

Then (4) becomes $y' = Au' + Bv'$ ... (6)

Differentiating (6) w.r.t. x, we get, $y'' = Au'' + Bv'' + A'u' + B'v'$ ... (7)

Substituting the values of y, y' and y'' from (3), (6) and (7) in (1), we get

$(Au'' + Bv'' + A'u' + B'v') + R(Au' + Bv') + S(Au + Bv) = Q$

or $A(u'' + Ru' + Su) + B(v''+Rv'+Sv)+(A'u'+B'v')=Q$ ... (8)

But u, v are (particular) solutions of (2).

$\therefore$ $u'' + Ru' + Su = 0$ and $v'' + Rv' + Sv = 0$

Therefore, (8) reduces to $A'u' + B'v'=Q$ ...(9)

Now, the equations (5) and (9) will have solutions for A' and B' provided $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \neq 0$.

But this determinant is precisely the wronskian of the functions u, v. Since the functions u, v are linearly independent over I, therefore, wronskian of the functions u, v is non-zero over I. Therefore, the equations (5) and (9) are solvable for the functions A', B', and by integration we can obtain the functions A and B.

Thus, the equation (3) will give us a particular solution of (1) and hence we can find the general solution of (1).

Remark: *The above procedure of finding a particular solution is called the method of variation of parameters.*

Example 1. *Solve* $\frac{d^2y}{dx^2}+y=\tan x$ *by the method of variation of parameters.*

Solution: *The given equation can be re-written as* $(D^2+1)\,y=\tan x$ ... (1)

The corresponding homogeneous equation is $(D^2+1)y=0$ ... (2)

It's A.E. is $m^2+1=0 \Rightarrow m=\pm i$

$\therefore$ *the complementary function of (1) is* $y_c=C_1\cos x+C_2\sin x$

Now, we seek a particular solution of (1) by variation of parameters.

Let $y=A\cos x+B\sin x$ ... (3)

Differentiate (3) w.r.t. x, we get $y'=A'\cos x+B'\sin x-A\sin x+B\cos x$ ...(4)

We choose $A'\cos x+B'\sin x=0$ ... (5)

$\therefore$ *(4) becomes,* $y'=-A\sin x+B\cos x$ ...(6)

Differentiate w.r.t. x; $y''=-A'\sin x+B'\cos x-A\cos x-B\sin x$ ... (7)

Substituting the values of y, y'' from (3) and (7) in (1), we get

$-A'\sin x+B'\cos x-A\cos x-B\sin x+A\cos x+B\sin x=\tan x$ ... (8)

or $-A'\sin x+B'\cos x=\tan x$

Next, we find values of A' and B' from (5), (8)

Multiplying (5) by cos x and (8) by sin x and subtracting, we get $A'(\cos^2x+\sin^2x)=0-\sin x\tan x$

$\therefore A'=-\sin x\tan x$

Again, multiplying (5) by sin x and (8) by cos x and adding, we get

$B'(\sin^2x+\cos^2x)=\cos x\tan x$ *or* $B'=\sin x$

Now, $A'=-\sin x\tan x=-\frac{\sin^2 x}{\cos x}=-\frac{1-\cos^2 x}{\cos x}=-\left(\frac{1}{\cos x}-\cos x\right)$

$A'=\cos x-\sec x.$

On integration, we get $A=\sin x-\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

Also, $B'=\sin x \Rightarrow B=-\cos x$

Putting values of A and B is (3), we get $y=\left[\sin x-\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right]\cos x-\cos x\sin x$

which is a particular solution of (1).

The general solution of (1) is $y=C_1\cos x+C_2\sin x+\left[\sin x-\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right]\cos x-\cos x\sin x$

2.9.2. An alternative approach

Method of variation of parameters is quite general and applies to $y''+py'+qy=X$ ... (1)

where p, q, and X are functions of x. It gives $P.I.=-y_1\int\frac{y_2X}{W}dx+y_2\int\frac{y_1X}{W}dx$, $(W\neq0)$ ...(2)

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ ... (3)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Proof: Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as parameters) by unknown functions $u(x)$ and $v(x)$, let the P.I.be

$y = uy_1 + vy_2$ ...(4)

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$ ... (5)

$y' = uy'_1 + vy'_2$ ... (6)

On assuming that $u'y_1 + v'y_2 = 0$ ... (7)

Differentiate (5) and substitute in (1), then noting that y_1 and y_2, satisfy (3),

we obtain $u'y'_1 + v'y'_2 = X$ ... (8)

Solving (7) and (8), we get $u' = -\frac{y_2X}{W}$, $v' = \frac{y_1X}{W}$, where $W = y_1y'_2 - y_2y'_1$

On integrating, we get $u = -\int \frac{y_2X}{W}dx$, $v = \int \frac{y_1X}{W}dx$.

Substituting these in (4), we get (2).

Example 2. Using the method of variation of parameters, solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Solution: Since given equation is $(D^2+4)y = \tan 2x$.

(i) To find C.F. It's A.E. is $m^2 + 4 = 0$ $\therefore$ $m = \pm 2i$. Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(ii) To find P.I, here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus, P.I. $= -y_1 \int \frac{y_2X}{W}dx + y_2 \int \frac{y_1X}{W}dx = -\cos 2x \int \frac{\sin 2x \tan 2x}{2}dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}dx$

$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)\,dx + \frac{1}{2}\sin 2x \int \sin 2x\,dx$

$= -\frac{1}{4}\cos 2x\,[\log|\sec 2x + \tan 2x| - \sin 2x] - \frac{1}{4}\sin 2x \cos 2x = -\frac{1}{4}\cos 2x \log|\sec 2x + \tan 2x|$

Hence, the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log|\sec 2x + \tan 2x|$

Example 3. Solve, by the method of variation of parameters, $y'' - 2y' + y = e^x \log x$.

Solution: Since given equation is $(D^2 - 2D + 1)\,y = e^x \log x$

(i) To find C.F. It's A.E. is $(m-1)^2=0$,

$\Rightarrow m=1,1$. Thus C.F. is $y = (c_1 + c_2x)\,e^x$

(ii) To find P.I.

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$\therefore W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}=\begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix}=e^{2x}$

thus $P.I. = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx = -e^x\int\frac{xe^x.e^x\log x}{e^{2x}}dx + xe^x\int\frac{e^x.e^x\log x}{e^{2x}}dx$

$= -e^x\int x\log x\,dx + xe^x\int\log x\,dx = -e^x\left(\frac{x^2}{2}\log x - \int\frac{1}{x}.\frac{x^2}{2}dx\right) + x.e^x\left(x\log x - \int\frac{1}{x}.x\,dx\right)$

$= -e^x\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + xe^x(x\log x - x) = \frac{1}{4}x^2e^x(2\log x - 3)$

Hence, C.S. is $y=(c_1+c_2x)e^x+\frac{1}{4}x^2e^x(2\log x -3)$.

2.9.3. *Method of undetermined coefficients:*

To Find the P.I. of f (D) y = X, We assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. thus when

(i) $X = 2e^{3x}$, *trial solution* $= ae^{3x}$

(ii) $X = 3\sin 2x$, *trial solution* $=a_1\sin 2x + a_2\cos 2x$

(iii) $X=2x^3$, *trial solution* $= a_1x^3 + a_2x^2 + a_3x + a_4$

However, when X=tan x or sec x, this method fails, since the number of terms obtained by differentiating x = tan x or sec x is infinite.

The above method holds as long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x, which is large enough so that none of the terms, which are then present, appear in the C.F.

2.9.4. *Linear Ordinary Differential Equations of second order with variable coefficients:*

An equation of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ ...(1)

where P, Q, R are functions of x only, is called Linear Ordinary Differential Equation of second order

2.9.4. *(i) The complete solution when one integral is known:*

Let y = u be a known integral in the C.F of (1) i.e., y = u is a solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \quad ...(2)$$

$$\Rightarrow \frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0 \quad ...(3)$$

Let y = uv be the solution of (1)

$$y = uv \Rightarrow \frac{dy}{dx} = v\frac{du}{dx}+u\frac{dv}{dx} \Rightarrow \frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}$$

hence (1) becomes $v\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}+P\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+Quv=R$

$$\Rightarrow u\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{du}{dx}+Pu\right)+v\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)=R \Rightarrow u\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{du}{dx}+Pu\right)=R \text{ (from (3))}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u} \quad ...(4)$$

Now putting $\frac{dv}{dx}=p \Rightarrow \frac{d^2v}{dx^2}=\frac{dp}{dx}$

Equation (4) $\Rightarrow \frac{dp}{dx}+\left(P+\frac{2}{u}\frac{du}{dx}\right)p=\frac{R}{u}$ ...(5)

which is linear with p as dependent variable

$$I.F. = e^{\int\left(P+\frac{2}{u}\frac{du}{dx}\right)dx}=e^{\int Pdx}.e^{\int\frac{2}{u}du}=u^2e^{\int Pdx}$$

hence, solution of (5) is $pu^2e^{\int Pdx}=\int\left[\frac{R}{u}.u^2e^{\int Pdx}\right]dx+c_1$

$$\Rightarrow p=\frac{c_1}{u^2}e^{-\int Pdx}+\frac{e^{-\int Pdx}}{u^2}\left[\int Rue^{\int Pdx}dx\right]$$

Integrating, $v=c_2+c_1\int\frac{e^{\int -Pdx}}{u^2}dx+\int\frac{e^{-\int Pdx}}{u^2}\left[\int Rue^{\int Pdx}dx\right]dx$

y = uv is a solution of (1)

Second part of the complementary function is $u\int\frac{e^{-\int Pdx}}{u^2}dx$ *and particular integral is*

$$=u\int\frac{e^{-\int Pdx}}{u^2}\left[\int Rue^{\int Pdx}dx\right]dx$$

2.9.4. (ii) To find one integral in C.F. by inspection, *i.e., to find a solution of* $D^2y+PDy+Qy=0$ ...(1)

1. $y=x$ *is a solution of (1) if* $P+Qx=0$
2. $y=x^2$ *is a solution of (1) if* $2+2Px+Qx^2=0$
3. $y=e^x$ *is a solution of (1) if* $1+P+Q=0$
4. $y=e^{-x}$ *is a solution of (1) if* $1-P+Q=0$
5. $y=e^{mx}$ *is a solution of (1) if* $m^2+mP+Q=0$
6. $y=x^m$ *is a part of C.F. if* $m(m-1)+Pmx+Qx^2=0$

If in a linear equation $A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=Q(x)$, $A+B+C=0$ *then* $y=e^x$ *is a part of C.F.*

Example 4. $x\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}+(x+2)y=(x-2)e^x$

Solution: $\frac{d^2y}{dx^2}-\frac{2(x+1)}{x}\frac{dy}{dx}+\frac{x+2}{x}y=\left(\frac{x-2}{x}\right)e^x$ ...(1)

Here $1+P+Q=1-\frac{2(x+1)}{x}+\frac{x+2}{x}=0 \Rightarrow y=e^x$ *is a part of the C.F.*

Putting $y = ve^x \Rightarrow \frac{dy}{dx} = e^x\left(\frac{dv}{dx} + v\right) \Rightarrow \frac{d^2y}{dx^2} = e^x\left[\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right]$

Equation (1) $\Rightarrow e^x\left[\frac{d^2v}{dx^2} + \frac{2dv}{dx} + v\right] - \frac{2(x+1)}{x}e^x\left[\frac{dv}{dx} + v\right] + \frac{(x+2)}{x}ve^x = \frac{(x-2)}{x}e^x$

$\Rightarrow \frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2 - \frac{2(x+1)}{x}\right) + v\left(1 - \frac{2(x+1)}{x} + \frac{x+2}{x}\right) = \frac{x-2}{x}$

$\Rightarrow \frac{dp}{dx} + \left(-\frac{2}{x}\right)p = \frac{x-2}{x} \left[\because \frac{dv}{dx} = p\right]$

I.F. $= e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

$\therefore \frac{p}{x^2} = \int \frac{x-2}{x^3}dx = \int \frac{dx}{x^2} - \int \frac{2dx}{x^3} = -\frac{1}{x} + \frac{1}{x^2} + c_1$

$\Rightarrow p = c_1x^2 - x + 1 \quad \Rightarrow v = c_2 + \frac{c_1x^3}{3} - \frac{x^2}{2} + x$

Hence, complete solution is $y = ve^x = e^x\left(x - \frac{x^2}{2} + \frac{c_1}{3}x^3 + c_2\right)$

Example 5. *Solve* $x\frac{d^2y}{dx^2} + (x-2)\frac{dy}{dx} - 2y = x^3$

Solution: *Given equation can be re-written as* $\frac{d^2y}{dx^2} + \left(\frac{x-2}{x}\right)\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x^2$ *...(1)*

We have 1-P+Q=1-$\frac{x-2}{x} - \frac{2}{x} = 0$

$\Rightarrow y = e^{-x}$ *is a part of C.F.*

$\therefore$ $y = ve^{-x}$ *is complete solution of (1)*

$\frac{dy}{dx} = e^{-x}\left(\frac{dv}{dx} - v\right)$ *and* $\frac{d^2y}{dx^2} = e^{-x}\left(\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right)$

Equation (1)$\Rightarrow e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right] + \frac{x-2}{x}e^{-x}\left(\frac{dv}{dx} - v\right) - \frac{2}{x}ve^{-x} = x^2$

$\Rightarrow \frac{d^2v}{dx^2} + \frac{dv}{dx}\left(\frac{x-2}{x} - 2\right) + v\left(1 - \frac{x-2}{x} - \frac{2}{x}\right) = x^2e^x \Rightarrow \frac{dp}{dx} - \frac{(x+2)}{x}p = x^2e^x \left[\therefore \frac{dv}{dx} = p\right]$

I.F. $= e^{\int\left(-1-\frac{2}{x}\right)dx} = \frac{e^{-x}}{x^2}$

$p\frac{e^{-x}}{x^2} = \int x^2e^x.\frac{e^{-x}}{x^2}dx + c_1$

$$\Rightarrow p = c_1x^2e^x + x^3e^x \quad \Rightarrow v = c_2 + \int x^3e^xdx + c_1\int x^2e^xdx$$
$$= c_2 + x^3e^x - \int 3x^2e^xdx + c_1\int x^2e^xdx = c_2 + x^3e^x + (c_1-3)[x^2e^x - \int 2xe^xdx]$$
$$= c_2 + x^3e^x + (c_1-3)[x^2e^x - 2xe^x + 2e^x]$$
$$= x^3e^x + (c_1-3)x^2e^x - 2(c_1-3)xe^x + 2(c_1-3)e^x + c_2$$
$$\therefore \quad y = ve^{-x} = x^3 + (c_1-3)x^2 - 2(c_1-3)x + 2(c_1-3) + c_2e^{-x}$$

2.9.5. *Removal of the first derivative (Reduction to Normal form)*

When unable to obtain a part of C.F. of solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ...(1)

Put $y = uv$ *where u is some function of X.*

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \text{ and } \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$(1) \Rightarrow \left[v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right] + P\left[v\frac{du}{dx} + u\frac{dv}{dx}\right] + Quv = R$$

$$\Rightarrow u\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]u\frac{dv}{dx} + v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = R \quad \text{...(2)}$$

Now to remove the term of the first derivative in (2) we choose u s. t.

$$P + \frac{2}{u}\frac{du}{dx} = 0 \Rightarrow \frac{du}{u} + \frac{P}{2}dx = 0 \quad \Rightarrow u = e^{-\frac{1}{2}\int Pdx} \quad \text{...(3)}$$

$$(2) \Rightarrow \frac{d^2v}{dx^2} + \frac{v}{u}\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = \frac{R}{u} \quad \text{...(4)}$$

$$\therefore \frac{du}{dx} = \frac{-1}{2}Pu \quad \Rightarrow \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right] \Rightarrow \frac{d^2u}{dx^2} = -\frac{1}{2}\left[\frac{-1}{2}P^2u + u\frac{dP}{dx}\right] = \frac{1}{4}P^2u - \frac{u}{2}\frac{dP}{dx}$$

$$(4) \Rightarrow \frac{d^2v}{dx^2} + v\left[\frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} - \frac{1}{2}P^2 + Q\right] = \frac{R}{e^{-\frac{1}{2}\int Pdx}}$$

$$\Rightarrow \frac{d^2v}{dx^2} + v\left[Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right] = \text{Re}^{\frac{1}{2}\int Pdx} \quad \Rightarrow \frac{d^2v}{dx^2} + Xv = z \quad \text{...(5)}$$

which is in normal form where $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$, $z = \text{Re}^{\frac{1}{2}\int Pdx}$

Example 6. Solve $\frac{d^3y}{dx^3} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} + xy = 0$ ...(1)

Solution: *Since sum of the coefficients is zero* $\Rightarrow$ *One of the solution of ODE is* $y = e^x$

Let v be another solution of (1)

Then putting $y = ve^x$ *in (1), we get*

$$\frac{dy}{dx}=e^x\left[\frac{dv}{dx}+v\right],\frac{d^2y}{dx^2}=e^x\left[\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right]\Rightarrow\frac{dy^3}{dx^3}=e^x\left[\frac{d^3v}{dx^3}+3\frac{d^2v}{dx^2}+3\frac{dv}{dx}+v\right]$$

equation (1)

$$\Rightarrow e^x\left[\frac{d^3v}{dx^3}+3\frac{d^2v}{dx^2}+3\frac{dv}{dx}+v\right]-x\left[\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right]e^x-e^x\left[v+\frac{dv}{dx}\right]+xve^x=0$$

$$\Rightarrow\frac{d^3v}{dx^3}+(3-x)\frac{d^2v}{dx^2}+(3-2x-1)\frac{dv}{dx}=0\Rightarrow\frac{d^2p}{dx^2}+(3-x)\frac{dp}{dx}+(2-2x)p=0 \qquad ...(2)$$

(where $p\equiv\frac{dv}{dx}$*)*

hence $P = 3-x$, $Q = 2-2x$ *and* $(-2)^2+(-2)(3-x)+(2-2x)=0\Rightarrow p=e^{-2x}$ *is a solution of (2)*

Putting $p=qe^{-2x}$ *in (2)* $\Rightarrow\frac{dp}{dx}=e^{-2x}\left[\frac{dq}{dx}-2q\right]$ *and* $\frac{d^2p}{dx^2}=e^{-2x}\left[\frac{d^2q}{dx^2}-4\frac{dq}{dx}+4q\right]$

$$\Rightarrow e^{-2x}\left[\frac{d^2q}{dx^2}-4\frac{dq}{dx}+4q\right]+(3-x)e^{-2x}\left[\frac{dq}{dx}-2q\right]+(2-2x)qe^{-2x}=0$$

$$\Rightarrow\frac{d^2q}{dx^2}-(1+x)\frac{dq}{dx}=0$$

Let $u=\frac{dq}{dx}$

Then $\frac{du}{dx}=(1+x)u\Rightarrow\log u=x+\frac{x^2}{2}+\log c_1\Rightarrow u=c_1e^{x+\frac{x^2}{2}}\Rightarrow q=\int udx=\int c_1e^{x+\frac{x^2}{2}}+c_2$

$$\therefore p=e^{-2x\int c_1e^{x+\frac{x^2}{2}}dx}+c_2e^{-2x}\Rightarrow v=\int pdx=\int\left(e^{-2x}\int c_1e^{x+\frac{x^2}{2}}dx\right)dx+\int c_2e^{-2x}dx+c_3$$

$$=c_1\left[-\frac{1}{2}e^{-2x}\int e^{x+\frac{x^2}{2}}dx-\int-\frac{1}{2}e^{-2x}e^{x+\frac{x^2}{2}}dx\right]+c_2\int e^{-2x}dx+c_3$$

$$=c_1\left[-\frac{1}{2}e^{-2x}\int e^{x+\frac{x^2}{2}}dx+\frac{1}{2}\int e^{-x+\frac{x^2}{2}}dx\right]+c_2\int e^{-2x}dx+c_3$$

Hence $y=ve^x\Rightarrow y=c_1\left[-\frac{1}{2}e^{-x}\int e^{x+\frac{x^2}{2}}dx+\frac{1}{2}e^x\int e^{-x+\frac{x^2}{2}}dx\right]-\frac{1}{2}c_2e^{-x}+c_3e^x,$

which is the required solution.

PRACTICE SET – 2

1. *Suppose* $y_p(x)=x\cos(2x)$ *is a particular solution of* $y''+\alpha y=-4\sin(2x)$. *Then the constant* α *equals* **(GATE-2007)**

(A) -4 *(B)* -2 *(C)* 2 *(D)* 4

2. Let $Y(x) = (y_1(x), y_2(x))^T$ and let $A = \begin{bmatrix} -3 & 1 \\ k & -1 \end{bmatrix}$. Further, let S be the set of values of k for which all the solutions of the system of equations $Y'(x) = AY(x)$ tend to zero as $x \to \infty$. Then S is given by **(GATE-2007)**

(A) $\{k: k = 5\}$ (B) $\{k: k \geq 3\}$ (C) $\{k: k \geq 5\}$ (D) $\{k: k \leq 3\}$

3. If $x(t)$ and $y(t)$ are the solutions of the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$ with the initial conditions $x(0) = 1$ and $y(0) = 1$, then $x(\pi/2) + y(\pi/2)$ equals ______. **(GATE-2017)**

4. Consider the system of ODE $\frac{d}{dx}Y = AY, Y(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $Y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$. Then **(CSIR UGC NET JUNE-2012)**

(A) $y_1(x) \to \infty$ and $y_2(x) \to 0$ as $x \to \infty$ (B) $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$

(C) $y_1(x) \to \infty$ and $y_2(x) \to -\infty$ as $x \to -\infty$ (D) $y_1(x), y_2(x) \to -\infty$ as $x \to -\infty$

5. For the system of ordinary differential equations: $\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ **(CSIR UGC NET SAMPLE PAPER)**

(A) every solution is bounded (B) every solution is periodic
(C) there exists a bounded solution (D) there exists a non periodic solution.

§ 2.10. WRONSKIAN.

2.10.1. Definitions:

(i) $f_1, f_2, \ldots, f_m$ are m given functions and $c_1, c_2, \ldots, c_m$ are m constants, then the expression $c_1f_1 + c_2f_2 + \ldots + c_mf_m$, is called a **linear combination of the functions** $f_1, f_2, \ldots, f_m$.

(ii) **Wronskian of n Vector Functions:**

Consider n vector functions $\vec{\phi}_1, \vec{\phi}_2, \ldots, \vec{\phi}_n$ defined by

$$\vec{\phi}_1(t) = \begin{bmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{bmatrix}, \ \vec{\phi}_2(t) = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{bmatrix}, \ldots, \vec{\phi}_n(t) = \begin{bmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{bmatrix}, \quad \ldots(1)$$

then, the n X n determinant defined by $\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$, ...(2)

is called the ***Wronskian of n vector functions*** $\vec{\phi}_1, \vec{\phi}_2, ..., \vec{\phi}_n$. *It is denoted by* $W(\vec{\phi}_1, \vec{\phi}_2, ..., \vec{\phi}_n)$, *and its value at t be denoted by* $W(\vec{\phi}_1, \vec{\phi}_2, ..., \vec{\phi}_n)(t)$.

(iii) ***Wronskian of n Functions***

Let $f_1, f_2, ..., f_n$ *be n real valued functions, each of which is differentiable at least (n – 1) times in the interval* $a \le x \le b$, *then the determinant* $\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-2)} & \cdots & f_n^{(n-1)} \end{vmatrix}$, *is called the Wronskian of the n functions and it is denoted by* $W(f_1, f_2, ..., f_n)$.

Theorem 1. *Consider the LDE of* 2^{nd} *order* $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ ...(1)

where $P_0(x) \neq 0$ *and* $P_0(x), P_1(x), P_2(x)$ *are* ***continuous*** *functions of* $x \in [a, b]$.

(A) Then two solutions y_1 *and* y_2 *of equation (1) are L.I. iff* $W(y_1, y_2) \neq 0 \ \forall \ x \in [a, b]$.

(B) Two solutions y_1 *and* y_2 *of equation (1) are L.D. iff* $W(y_1, y_2) = 0 \ \forall \ x \in [a, b]$.

Note: *(1) If given that* y_1 *and* y_2 *are two solutions of a differential equation and D.E. is not given then*

if $W(y_1, y_2) \neq 0 \Rightarrow y_1$ *and* y_2 *are L.I. and*

if $W(y_1, y_2) = 0$, *then we can't say anything*

Remark: *A set of functions can be independent even when their Wronskian is identically zero.*

e.g. $x^3, x^2|x|$ *in* $[-1,1]$

Theorem 2. *If* $y_1(x)$ *and* $y_2(x)$ *are two functions for which* $W(y_1, y_2) = 0$ *for each x in an interval I then each sub interval* I_1 *of I contains a sub interval* I_2 *over which* $y_1(x)$ *and* $y_2(x)$ *are dependent.*

Theorem 3. *The Wronskian of the two solutions of the equation* $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ *is either identically zero or never zero* $\forall \ x \in [a, b]$ *[where* $P_0(x) \neq 0$ *and* $P_0(x), P_1(x), P_2(x)$ *are continuous functions on the given interval]*

Results: *Consider the differential equation* $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ *on* $x \in [a,b]$ ...(1)

where $P_0(x) \neq 0$ *and* $P_0(x), P_1(x), P_2(x)$ *are continuous functions on [a, b]*

Let y_1 *and* y_2 *are the solutions of the differential equation then*

(1) *If y_1 and y_2 have common zero at a point $x_0 \in [a, b]$ then y_1 and y_2 are L.D.*

(2) *If y_1 and y_2 have relative maxima or minima at a common point, $x_0 \in [a, b]$ then y_1 and y_2 are L.D.*
$\Rightarrow y_1$ and y_2 have maxima/minima at common point.

(3) *If y_1 and y_2 are L.I. solutions of (1) and $y_1''(x_0)=0$ and $y_2''(x_0)=0$ then $P_1(x_0)=P_2(x_0)=0$.*

(4) *Let $\{f_1, f_2\}$ be the one set of two L.I. solutions of equation (1) and let $\{g_1, g_2\}$ be the another set of L.I. solutions of (1), then there exists a constant $c \neq 0$ such that $W(f_1,f_2) = cW(g_1, g_2)$*

(5) *If y_1 and y_2 are L.I. solutions of equation (1) on [a, b]. A function $f(x)=\frac{y_1}{y_2}$ defined on [a, b] such that $y_2 \neq 0$ on [a,b], then f is monotonic on [a, b].*

(6) *If y_1 and y_2 form a fundamental set of solutions of (1) on $-\infty < x < \infty$ then there is one and only one (i.e. unique) zero of y_1 between the consecutive zeros of y_2 and vice-versa.*

2.10.2. Abel's formula

This formula is applied when Wronskian is given at one point and we have to find Wronskian at any another point. If y_1 and y_2 are two solutions of the equation $P_0(x)\, y'' + P_1(x)\, y' + P_2(x)\, y = 0$ ...(1)
[where $P_0(x) \neq 0$ and $P_0(x)$, $P_1(x)$, $P_2(x)$ are continuous functions on the given interval]

then $W(y_1, y_2) = ce^{\int \frac{P_1(x)}{P_0(x)} dx}$ where $c = W(y_1,y_2)(x_0)$ and $x_0 \in$ interval

Examples 1. *Let y_1 and y_2 are two L.I. solutions of $xy''-2x^2y' + e^x y = 0$ and $y_1(0)=1$, $y_2(0)=-1$, $\left.\frac{dy_1}{dx}\right|_{x=0}=1$, $\left.\frac{dy_2}{dx}\right|_{x=0}=1$, then $W(2) = ?$*

Solution: *Given D.E. is $xy'' - 2x^2y' + e^x y = 0$*

By Abel's formula $W(x) = ce^{-\int \frac{P_1(x)}{P_0(x)} dx}$

$$W(x) = ce^{-\int \frac{-2x^2}{x} dx} = ce^{\int 2x\,dx} = ce^{x^2} \Rightarrow W(0) = c \quad \text{... (1)}$$

$$\text{Also, } W(0) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1=2 \quad \text{...(2)}$$

from (1) and (2) we get $c = 2$

$$\Rightarrow W(x) = ce^{x^2} \Rightarrow W(x) = 2e^{x^2} \Rightarrow W(2) = 2e^4.$$

Example 2. *Let y_1 and y_2 be two L.I. solutions of $y''+\sin xy = 0$; $0 \leq x \leq 1$ and let $g(x) = w(y_1, y_2)(x)$ then*

(A) $g(x) \neq 0$ on [0, 1] *(B) $g'(x) < 0$ on [0, 1]*
(C) g' vanish at only one point of [0, 1] *(D) g' vanish at all points of [0, 1]*

Solution: *As y_1 and y_2 are L.I. solutions, so theorem is applicable here.*

Since $W(y_1, y_2) \neq 0 \ \forall x \in [0, 1] \Rightarrow g(x) \neq 0$ on [0, 1]

$\Rightarrow$ Option (A) is correct

Also, by Abel's formula $W(x) = ce^{-\int \frac{P_1(x)}{P_0(x)} dx} = c$ *(say)*

$\Rightarrow W(x) = c \ \forall x \in [0, 1] \Rightarrow W'(x) = 0 \ \forall x \in [0, 1] \Rightarrow g'(x) = 0 \ \forall x \in [0, 1].$

$\Rightarrow$ Option (D) is correct.

PRACTICE SET – 3

1. *Consider the ODE* $u''(t) + P(t)u'(t) + Q(t)u(t) = R(t), t \in [0,1]$
There exist continuous functions P, Q and R defined on [0, 1] and two solutions u_1 and u_2 of this ODE such that the Wronskian W of u_1 and u_2 is **(CSIR UGC NET JUNE-2011)**

(A) $W(t) = 2t - 1, \ 0 \le t \le 1$ (B) $W(t) = \sin 2\pi t, \ 0 \le t \le 1$
(C) $W(t) = \cos 2\pi t, \ 0 \le t \le 1$ (D) $W(t) = 1, \ 0 \le t \le 1$

2. *Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of* $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, a \le x \le b$, *where p(x) and q(x) are real valued continuous functions on [a, b]. If x_0 and x_1, with $x_0 < x_1$, are consecutive zeros of $y_1(x)$ in (a, b), then* **(CSIR UGC NET DEC-2011)**

(A) $y_1(x) = (x - x_0)q_0(x)$ *is continuous on [a, b] with $q_0(x_0) \neq 0$*
(B) $y_1(x) = (x - x_0)^2 p_0(x)$, *where $p_0(x)$ is continuous on [a,b] with $p_0(x_0) \neq 0$*
(C) *$y_2(x)$ has no zeros in (x_0, x_1)*
(D) *$y_2(x_0) = 0$ but* $y_2'(x_0) \neq 0$

3. *Let W be the Wronskian of two linearly independent solutions of ODE* $2y'' + y' + t^2 y = 0; t \in \mathbb{R}$. *Then, for all t, there exists a constant $C \in \mathbb{R}$ such that W(t) is* **(CSIR UGC NET DEC-2013)**

(A) Ce^{-t} (B) $Ce^{-t/2}$ (C) Ce^{2t} (D) Ce^{-2t}

4. *Let $W(y_1, y_2)$ be the Wronskian of two linearly independent solutions y_1 and y_2 of the equation* $y'' + P(x)y' + Q(x)y = 0$. *If $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, then the value of P(0) is* **(GATE-2013)**

(A) 4 (B) -4 (C) 2 (D) -2

5. *Let $W(y_1, y_2)$ be the Wronskian of two linearly independent solutions y_1 and y_2 of the equation* $y'' + p(x)y' + Q(x)y = 0$. *The product $W(y_1,y_2)P(x)$ equals* **(GATE-2013)**

(A) $y_2 y_1'' - y_1 y_2''$ (B) $y_1 y_2' - y_2 y_1'$ (C) $y_1' y_2'' - y_2' y_1''$ (D) $y_2' y_1' - y_1'' y_2''$

§ 2.11. ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal trajectories of each other.

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and vice versa. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

2.11.1. *To find the orthogonal trajectories of the family of curves F(x,y,c)=0.*

(i) *Form its differential equation of the form f(x,y,dy/dx)=0 by eliminating c.*

(ii) *Replace, in this differential equation, dy/dx by – dx/dy, (so that the product of their slopes at each point of intersection is -1) such that* $\frac{dy}{dx} \neq 0$

(iii) *Solve the differential equation of the orthogonal trajectories, i.e., f(x,y, - dx/dy) = 0.*

Example 1. *If the stream lines (paths of fluid particles) of a flow around a corner are xy = constant, find their orthogonal trajectories.*

Solution: *Taking the axes as the walls, the stream lines of the flow around the corner of the walls is*

$xy=c$ ...(i)

Differentiating, we get, $x\frac{dy}{dx} + y = 0$ ...(ii) *as the differential equation of the given family (i).*

Replacing $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ *in (ii), we obtain*

$x\left(-\frac{dx}{dy}\right) + y = 0$ *or* $xdx - ydy = 0$ ...(iii)

as the differential equation of the orthogonal trajectories.

Integrating (iii), we get $x^2 - y^2 = c'$ *as the required orthogonal trajectories of(i), i.e., the equipotential lines,*

Example 2. *Find the orthogonal trajectories of the family of confocal conics* $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$*, where λ is the parameter.*

Solution: *Differentiating the given equation, we get* $\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$ *or* $\frac{y}{b^2+\lambda} = -\frac{x}{a^2(dy/dx)}$ *or*

$$\frac{y^2}{b^2+\lambda} = -\frac{xy}{a^2(dy/dx)}$$

Substituting this in the given equation, we get, $\frac{x^2}{a^2} - \frac{xy}{a^2(dy/dx)} = 1$ *or* $(x^2 - a^2)\frac{dy}{dx} = xy$ ...(i)

which is the differential equation of the given family.

Changing dy/dx to –dx/dy in (i), we get $(a^2-x^2)\,dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain $\int y\,dy = \int \frac{a^2-x^2}{x}dx + c$

or $\frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$ *or* $x^2 + y^2 = 2a^2 \log x + c'$ $[\because c' = 2c]$

which is the equation of the required orthogonal trajectories.

Example 3. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution: *The equation of the family of confocal parabolas having x- axis as their axis, is of the form*

$$y^2 = 4a(x + a) \qquad \text{...(i)}$$

Differentiating, $y\frac{dy}{dx} = 2a$ ...(ii)

Substituting the value of a from (ii) in (i), we get $y^2 = 2y\frac{dy}{dx}\left(x + \frac{1}{2}y\frac{dy}{dx}\right)$

,i.e., $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ ...(iii)

as the differential equation of the family. Replacing $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ *in (iii),*

we obtain $y\left(\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0$ *or* $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$*, which is the same as (iii)*

Thus we see that a system of confocal and coaxial parabolas is self- orthogonal, i.e., each member of the family (i) cuts every other member of the same family orthogonally.

2.11.2. To find the orthogonal trajectories of the curves $F(r,\theta,c) = 0$.

(i) *Form its differential equation of the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.*

(ii) *Replace in this differential equation,* $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$

[$\because$ for the given curve through $P(r,\theta)$ $\tan\phi = r\,d\theta/dr$ and for the orthogonal trajectory through p

$\tan\phi' = \tan(90^0 + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$

Thus for getting the differential equation of the orthogonal trajectories.

$r\frac{d\theta}{dr}$ *is to be replaced by* $-\frac{1}{r}\frac{dr}{d\theta}$

$\frac{dr}{d\theta}$ *is to be replaced by* $-r^2\frac{d\theta}{dr}$ *]*

(iii) *Solve the differential equation of the orthogonal trajectories i.e.,* $f\left(r,\theta,-r^2\dfrac{d\theta}{dr}\right)=0$

Example 4. *Find the orthogonal trajectories of the cardioids $r = a(1\text{-}\cos\theta)$.*

Solution: *Differentiating $r = a(1\text{-}\cos\theta)$ w.r.t θ* ...(i)

we get $\dfrac{dr}{d\theta} = a\sin\theta$ ...(ii)

Eliminating a from (i) and (ii), we obtain $\dfrac{dr}{d\theta}\cdot\dfrac{1}{r}=\dfrac{\sin\theta}{1-\cos\theta}=\cot\dfrac{\theta}{2}$ *which is the differential equation of the given family.*

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain $\dfrac{1}{r}\left(-r^2\dfrac{d\theta}{dr}\right)=\cot\dfrac{\theta}{2}$ *or* $\dfrac{dr}{r}+\tan\dfrac{\theta}{2}d\theta=0$

as the differential equation of orthogonal trajectories. It can be rewritten as $\dfrac{dr}{r}=-\dfrac{(\sin(\theta/2))d\theta}{\cos(\theta/2)}$

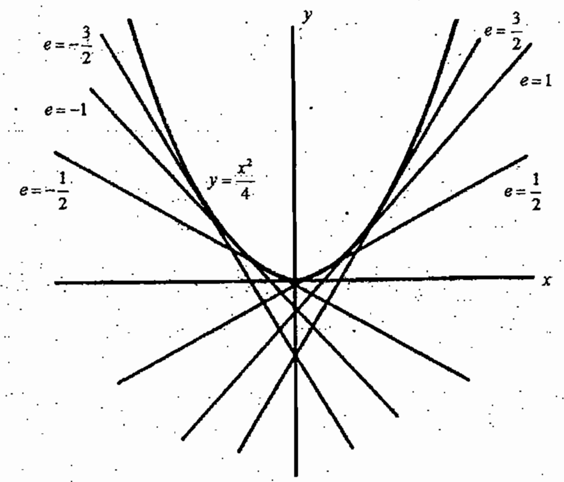
Integrating, $\log r = 2\log\cos(\theta/2)+\log c$ *or* $r = c\cos^2(\theta/2) = \dfrac{1}{2}c(1+\cos\theta)$ *or* $r = a'(1+\cos\theta)$

which is the equation of required orthogonal trajectories.

§ 2.12. SINGULAR SOLUTIONS

Singular Solution: *A solution that contains no arbitrary constant and cannot be derived from complete primitive by substituting particular values to arbitrary constants.*

e.g. *The general solution of $y=xy'-y'^2$ is $y=cx-c^2$. However, another solution is $y = x^2/4$ which cannot be obtained from the general solution by substituting any value for constant c. This second solution is a Singular Solution. For a relationship between the general and singular solutions consider the following graphical relationship.*



Referring to fig., it is seen that $y = cx - c^2$ represents a family of straight lines tangent to the parabola $y = x^2/4$. The parabola is the envelope of the family of straight lines.

The envelope of a family of curves $G(x, y, c) = 0$, if it exists, can be found by solving simultaneously the equations $\partial G/\partial c = 0$ and $G = 0$. In this example $G(x, y, c) = y - cx + c^2$ and $\partial G/\partial c = -x + 2c$. Solving simultaneously $-x + 2c = 0$ and $y - cx + c^2 = 0$, we get $x = 2c$, $y = c^2$ or $y = x^2/4$.

Envelope: *A curve which touches each member of a family of curves and each point is touched by some member of the family is called the envelope of the family of curves.*

Complete primitive (General solutions) and Singular solutions: *If family of curves represents the complete primitive of a differential equation of first order, then envelope represents a singular solution of a differential equation.*

Discriminant:

(a) *A discriminant of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.*

(b) *If $\phi(x, y, c)$ is the solution of the differential equation $f(x, y, p) = 0$ then*

(i) p- discriminant is obtained by eliminating p between $f(x, y, p) = 0$ and $\frac{\partial f}{\partial p} = 0$

(ii) c- discriminant is obtained by eliminating c between $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$

Tac-locus: *p- discriminant gives equal value of p, these values may belong to two curves of the system that are not consecutive. The locus of such points is called Tac-locus.*

Nodal-locus: *c-discriminant gives equal value of c, but these values may belong to nodes (double point with distinct tangents) which are also ultimate point of intersection of consecutive curves. Locus of such points is called Nodal-locus.*

Cusp-locus: *c- discriminant gives equal values of c but these may belong to the cusp (i.e. double point with coincident tangents), which are also ultimate points of intersection of the consecutive curves. This locus is called cusp locus.*

- *The p- discriminant equated to zero may include the envelope as a factor once, the cusp locus once and tac-locus twice. ($p \sim ECT^2 \equiv 0$)*
- *The c- discriminant equated to zero may include the envelope once, the cusp-locus thrice and nodal locus twice. ($c \sim EC^3N^2 \equiv 0$)*
- *If $\phi(x,y) = 0$ is the singular solution, then $\phi(x,y)$ is also a factor of both p- discriminant and c- discriminant.*

To obtain the singular solution of the Clairaut's equation i.e. $y = px + f(t)$, we proceed as follows:-

(i) Find the general solution by replacing p by c i.e. $y = cx + f(c)$

(ii) Differentiate this w.r.t. c giving we get $x + f'(p) = 0$.

(iii) Eliminate c from these equations to get singular solution.

Example 1. *Solve $p = \sin(y-xp)$. Also find its singular solution.*

Solution: *Given equation can be written as $\sin^{-1}p = y - xp$ or $y = px + \sin^{-1}p$, which is a Clairaut's equation.*

∴ *its solution is $y = cx + \sin^{-1}c$.* ...(i)

To find the singular solution, differentiate (i) w.r.t. c we get, $0 = x + \frac{1}{\sqrt{1-c^2}}$ ...(ii)

To eliminate c from (i) and (ii), we re-write (ii) as $c = [(x^2-1)]^{1/2}/x$

Now, substituting this value of c in (i), we get $y = [(x^2-1)]^{1/2} + \sin^{-1}\{[(x^2-1)]^{1/2}/x\}$,

which is the required singular solution.

Example 2. *Find singular solution of $y^2 - 2pxy + p^2(x^2-1) = m^2$*

Solution: *The given equation can be written as* $(x^2-1)p^2 - 2xyp + (y^2 - m^2) = 0$ ...(1)

or $(px-y)^2 = p^2 + m^2$ *or* $y = px \pm \sqrt{(p^2+m^2)}$, *which is the Clairaut's form*

Hence, the general solution is

$y = cx \pm \sqrt{(c^2+m^2)}$

$(y-cx)^2 = c^2 + m^2$

or $c^2(x^2-1) - 2xyc + (y^2 - m^2) = 0$

Hence from (i) and (ii) both x and c-discriminant are $4x^2y^2 - 4(x^2-1)(y^2-m^2) = 0$

or $y^2 + m^2x^2 = m^2$ *which is the singular solution.*

PRACTICE SET – 4

1. *The orthogonal trajectories to the family of straight lines $y = k(x-1)$, $k \in \mathbb{R}$ are given by* **(GATE – 2004)**

   (A) $(x-1)^2 + (y-1)^2 = c^2$ (B) $x^2 + y^2 = c^2$

   (C) $x^2 + (y-1)^2 = c^2$ (D) $(x-1)^2 + y^2 = c^2$

2. *If the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 = c_1$, $c_1 > 0$, are given by $y = c_2x^{\alpha}$, $c_2 \in \mathbb{R}$, then α = __________* **(IIT-JAM-2017)**

3. *Let γ be the curve which passes through (0,1) and intersects each curve of the family $y = cx^2$ orthogonally. Then γ also passes through the point* **(GATE-2016)**

   (A) $(\sqrt{2}, 0)$ (B) $(0, \sqrt{2})$ (C) (1,1) (D) (-1, 1)

4. *The singular integral of the ODE $(xy' - y)^2 = x^2(x^2 - y^2)$ is* **(CSIR UGC NET JUNE-2015)**

   (A) $y = x\sin x$ (B) $y = x\sin\left(x + \frac{\pi}{4}\right)$

   (C) $y = x$ (D) $y = x + \frac{\pi}{4}$

§ 2.13. REGULAR AND IRREGULAR SINGULAR POINTS

A point x_0 is a singular point of the D.E. $y'' + P(x)\, y' + Q(x)y=0$ ...(1) if one or the other (or both) of the co-efficient functions P(x) and Q(x) fails to be analytic at x_0. A point x_0 is a regular singular point of the D.E. $y'' + P(x)\, y' + Q(x)y = 0$ if the functions $(x-x_0)\, P(x)$ and $(x-x_0)^2\, Q(x)$ are analytic and irregular otherwise.

Consider Legendre's equation of the form $y'' - \frac{2x}{1-x^2}\, y' + \frac{p(p+1)}{1-x^2} y = 0$

it is clear that x=1 and x = -1 are singular points. Now, x=1 is regular because

$(x-1)P(x)= \frac{2x}{x+1}$ *and* $(x-1)^2\, Q(x)= \frac{(x-1)p(p+1)}{x+1}$ *are analytic at x = 1.*

x = -1 is also regular for similar reasons.

Consider Bessel's equation of order p as $x^2\, y'' + x\, y' + (x^2-p^2)y=0$ *or* $y'' + \left(\frac{1}{x}\right) y' + \left(\frac{x^2-p^2}{x^2}\right) y = 0$

Also, x=0 is a regular singular point because $xP(x)=1$ *and* $x^2\, Q(x)=x^2-p^2$ *are analytic at x=0.*

Example 1. *For the differential equation* $4x^3 y'' + 6x^2 y' + y = 0$, *point at infinity is* **(GATE-1997)**

(A) an ordinary point (B) a regular singular point

(C) an irregular singular point (D) a Critical point

Solution: (B)

Given , $4x^3\, y'' + 6x^2\, y' + y = 0$ ...(i)

We transform the independent variable x to t by the relation $t = \frac{1}{x}$ *or* $x = \frac{1}{t}$

Now $y' = \frac{-1}{x^2}\frac{dy}{dt} \Rightarrow y'' = \frac{1}{x^4}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}$ *and given differential equation (i) transform to 4t*

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \qquad ...(ii)$$

The point at ∞ is transformed to the origin

From equation (ii), we note that the origin is regular singular point.

Hence, the point at ∞ is a regular singular point of the given equation.

§ 2.14. e^{At}

Definition: *Let A be a square matrix, then the infinite series* $e^{At} \equiv I + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + ... = \sum_{n=0}^{\infty}\frac{1}{n!}A^n t^n$ ...(1)

converges for every A and t, so that e^{At} is defined for all square matrices.

Computation of e^{At}

To compute, the elements of e^{At}, (1) is not generally useful. However, it follows from Cayley-Hamilton Theorem, applied to the matrix At, that the infinite series can be reduced to a polynomial in t. Thus we have:

Theorem 1. *If A is a matrix having n rows and n columns, then*

$$e^{At} = \alpha_{n-1}A^{n-1}t^{n-1} + \alpha_{n-2}A^{n-2}t^{n-2} + ... + \alpha_2 A^2t^2 + \alpha_1 At + \alpha_0 I, \quad ...(2)$$

where $\alpha_0, \alpha_1, ..., \alpha_{n-1}$ *are functions of t which must be determined for each A.*

e.g. *If A is a matrix having n rows and n columns, then for n=2,* $e^{At} = \alpha_1 At + \alpha_0 I$ *...(3)*

when A has three rows and three columns, then n=3 and $e^{At} = \alpha_2 A^2t^2 + \alpha_1 At + \alpha_0 I$ *...(4)*

Theorem 2. *For a matrix A having n rows and n columns, define*

$$r(\lambda) \equiv \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + ... + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \quad ...(5)$$

Then, if λ_i *is an eigenvalue of At,* $e^{\lambda_i} = r(\lambda_i)$ *...(6)*

Furthermore, if λ_i *is an eigenvalue of multiplicity k, k>1, then the following equations are also valid*

$$\left.\begin{aligned} e^{\lambda_i} &= \frac{d}{d\lambda} r(\lambda)\Big|_{\lambda=\lambda_i} \\ e^{\lambda_i} &= \frac{d^2}{d\lambda^2} r(\lambda)\Big|_{\lambda=\lambda_i} \\ &\dots \\ e^{\lambda_i} &= \frac{d^{k-1}}{d\lambda^{k-1}} r(\lambda)\Big|_{\lambda=\lambda_i} \end{aligned}\right\} \quad ...(7)$$

Note that theorem 7.2 involves the eigenvalues of At; which are t times the eigenvalues of A. When computing the various derivatives in (7), one first calculates the appropriate derivatives of the expression (5) with respect to λ*, and then substitutes* $\lambda = \lambda_i$*. The reverse procedure of first substituting* $\lambda = \lambda_i$ *(a function of t) into (5), and then calculating the derivatives with respect to t, can give errored result.*

e.g. : *Let A have four rows and four columns and let* $\lambda = 5t$ *and* $\lambda = 2t$ *be eigenvalues of At of multiplicities three and one, respectively. Then n = 4 and* $r(\lambda) = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + a_0$,

$r'(\lambda) = 3\alpha_3\lambda^2 + 2\alpha_2\lambda + \alpha_1$, $r''(\lambda) = 6\alpha_3\lambda + 2\alpha_2$

Since $\lambda = 5t$ *is an eigenvalue of multiplicity three, it follows that* $e^{5t} = r(5t), e^{5t} = r'(5t)$*, and* $e^{5t} = r''(5t)$*. Thus,* $e^{5t} = \alpha_3(5t)^3 + \alpha_2(5t)^2 + \alpha_1(5t) + \alpha_0$

$e^{5t} = 3\alpha_3(5t)^2 + 2\alpha_2(5t) + \alpha_1$

$e^{5t} = 6\alpha_3(5t) + 2\alpha_2$

Also, since $\lambda = 2t$ *is an eigenvalue of multiplicity one, it follows that* $e^{2t} = r(2t)$,

or $e^{2t} = \alpha_3(2t)^3 + \alpha_2(2t)^2 + \alpha_1(2t) + \alpha_0$

Thus, we have four equations in the four unknowns α_i*'s*

Method of Computation: *For each eigenvalue* λ_i *of At, apply Theorem (2) to obtain a set of linear equations. When this is done for each eigenvalue, the set of all equations so obtained can be solved for* $\alpha_0, \alpha_1, ..., \alpha_{n-1}$*. These values are then substituted into equation (3), which is then used to compute* e^{At}*.*

Example 1. *Find* e^{At} *for* $A=\begin{bmatrix}1 & 1\\ 9 & 1\end{bmatrix}$.

Solution: *Since* $e^{At}=\alpha_1 At+\alpha_0 I=\begin{bmatrix}\alpha_1 t+\alpha_0 & \alpha_1 t\\ 9\alpha_1 t & \alpha_1 t+\alpha_0\end{bmatrix}$ ...(8)

and from equation (5), $r(\lambda)=\alpha_1\lambda+\alpha_0$. *The eigenvalues of At are* $\lambda_1=4t$ *and* $\lambda_2=-2t$, *which are both of multiplicity one. Substituting these values successively into equation (6), we obtain two equations* $e^{4t}=4t\alpha_1+\alpha_0$ *and* $e^{-2t}=-2t\alpha_1+\alpha_0$

Solving these equations for α_1 *and* α_0, *we find that* $\alpha_1=\frac{1}{6t}(e^{4t}-e^{-2t})$ *and* $\alpha_0=\frac{1}{3}(e^{4t}+2e^{-2t})$

Substituting these values into (8) and simplifying, we have $e^{At}=\frac{1}{6}\begin{bmatrix}3e^{4t}+3e^{-2t} & e^{4t}-e^{-2t}\\ 9e^{4t}-9e^{-2t} & 3e^{4t}+3e^{-2t}\end{bmatrix}$

Example 2. *Find* e^{At} *for* $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

Solution: *Since* $e^{At}=\alpha_1 At+\alpha_0 I=\begin{bmatrix}\alpha_0 & \alpha_1 t\\ -\alpha_1 t & \alpha_0\end{bmatrix}$ ...(9)

and from equation (5) $r(\lambda)=\alpha_1\lambda+\alpha_0$. *The eigenvalues of At are* $\lambda_1=it$ *and* $\lambda_2=-it$, *which are both of multiplicity one. Substituting these values successively into equation (6), we obtain*

$e^{it}=\alpha_1(it)+\alpha_0$ $e^{-it}=\alpha_1(-it)+\alpha_0$

Solving these equations for α_1 *and* α_0 *and using Euler's relations, we find that*

$\alpha_1=\frac{1}{2it}(e^{it}-e^{-it})=\frac{\sin t}{t}$, $\alpha_0=\frac{1}{2}(e^{it}+e^{-it})=\cos t$

Substituting these values into (9), we obtain $e^{At}=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$.

PRACTICE SET – 5

1. *For the differential equation* $x^2(1-x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ **(GATE-2005)**

   (A) x = 1 is an ordinary point (B) x = 1 is a regular singular point
   (C) x = 0 is an irregular singular point (D) x = 0 is an ordinary point

2. *Consider the initial value problem in* $\mathbb{R}^2$ $Y'(t)=AY+BY;Y(0)=Y_0$, *where* $A=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$. *Then Y(t) is given by* **(CSIR UGC NET JUNE – 2014)**

   (A) $e^{tA}e^{tB}Y_0$ (B) $e^{tB}e^{tA}Y_0$ (C) $e^{t(A+B)}Y_0$ (D) $e^{-t(A+B)}Y_0$

3. *For the ordinary differential equation* $(x-1)\frac{d^2y}{dx^2}+(\cot \pi x)\frac{dy}{dx}+(\operatorname{cosec}^2 \pi x)y=0$, *which of the following statement is true?* **(GATE-2006)**
(A) 0 is regular and 1 is irregular
(B) 0 is irregular and 1 is regular
(C) Both 0 and 1 are regular
(D) Both 0 and 1 are irregular

4. *For the differential equation* $t(t-2)^2 y'' + ty' + y = 0$, $t = 0$ *is* **(GATE-1996)**
(A) an ordinary point
(B) a branch point
(C) an irregular point
(D) a regular singular point

KEY POINTS

- *The functions* $f_1, f_2, \ldots, f_n$ *of x are said to be linearly dependent over an interval I iff there exist constants* $c_1, c_2, \ldots, c_n$ *(not all zero) such that* $c_1f_1 + c_2f_2 + \ldots + c_nf_n = 0$ *for all x in I and if all* $c_i = 0\ \forall\ x \in I$ *then linearly independent.*

- *If* $W(f_1 f_2 \ldots f_n)$ *over I is non-zero, then* $f_1 f_2 \ldots f_n$ *are linearly independent over I (provided they are solutions of same differential equation)*

- *For a homogenous differential equation of order n, and* $f_1, f_2, \ldots, f_n$ *are linearly independent solution of it then their linear combination* $c_1f_1 + \ldots + c_nf_n$ *is also its solution.*

- *For non-homogeneous linear equation if* $f_1, f_2, \ldots, f_n$ *are L.I. solutions of (2), then* $c_1f_1 + c_2f_2 + \ldots + c_nf_n$ *where* $c_1, c_2, \ldots, c_n$ *are arbitrary constants, is called the complementary function of (1). General sol. of a non-homogeneous equation is complementary function + particular solution.*

- *For particular integral of* $\frac{1}{f(D)}e^{ax}$; *put* $D = a$, *provided* $f(a) \neq 0$. *When* $f(a) = 0$ *then* $\frac{1}{f(D)}e^{ax} = x\frac{1}{\frac{d}{dD}[f(D)]}e^{ax}$.

- *For P.I. of* $\frac{1}{f(D^2)}\sin ax$ *or* $\frac{1}{f(D^2)}\cos ax$; *Put* $D^2 = -a^2$ *provided* $f(-a^2) \neq 0$.

- *For P.I. of* $\frac{1}{f(D)}x^m$, *take the lowest degree term outside. Remaining factor will be of the type* $[1 \pm \phi(D)]$. *Take* $[1 \pm \phi(D)]$ *to the numerator and expand it by Binomial Theorem.*

- *For P.I. of* $\frac{1}{f(D)}(e^{ax}X)$, *where X is any function of x, use* $\frac{1}{f(D)}(e^{ax}X) = e^{ax}\cdot\frac{1}{f(D+a)}X$.

- P.I. of $\frac{1}{f(D)}(xX)$, where X is any function of x, use $\frac{1}{f(D)}(xX) = x\frac{1}{f(D)}X + \frac{d}{dD}\left[\frac{1}{f(D)}\right]X$.

- For Cauchy's homogeneous linear equation, $P_0x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = Q(x)$, where $P_0, P_1, \ldots, P_n$ are all real constants and $Q(x)$ is a function of x. Put $x = e^z$, i.e., $z = \log x$, $x > 0$ and $\frac{d}{dz} = \theta$ so that $xD = \theta$, $x^2D^2 = \theta(\theta-1), \ldots, x^nD^n = \theta(\theta-1)\ldots(\theta-n+1)$ and solve.

- For Legendre's linear equation $P_0(a+bx)^n\frac{d^ny}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n(y) = Q(x)$. Put $a+bx = e^z$, i.e., $z = \log(a+bx)$, $a+bx > 0$ and $\frac{d}{dz} = \theta$, so that $(a+bx)D = b\theta$, $(a+bx)^2D^2 = b^2\theta(\theta-1), \ldots,$

  $(a+bx)^nD^n = b^n\theta(\theta-1)\ldots(\theta-n+1)$ and solve.

- For $\frac{dx}{dt} + ax + by = f(t)$ and $\frac{dy}{dt} + cx + dy = g(t)$. Eliminate one of the dependent variable say y from given equations and solve linear differential equation in x and t then put x and $\frac{dx}{dt}$ in any of the given equations, to obtain $y = \Psi(t)$.

- For differential equation $y'' + py' + qy = X$, P.I. $= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$, $(W \neq 0)$, where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

- Wronskian of the n functions is $\begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-2)} & \ldots & f_n^{(n-1)} \end{vmatrix}$.

- The Wronskian of the solutions is either identically zero or never zero.

- Abel's formula for $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ is $W(y_1, y_2) = ce^{-\int\frac{P_1(x)}{P_0(x)}dx}$, where

  $c = W(y_1, y_2)(x_0)$ and $x_0 \in$ interval.

- *To find the orthogonal trajectories of the family of curves F(x,y,c)=0, find its differential equation and replace, in this differential equation, dy/dx by – dx/dy and solve. For F(r,θ,c) = 0, replace* $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$

- *Singular solution is the envelope of general solutions of a differential equation.*

- *The p- discriminant of a differential equation is of form* $ECT^2 \equiv 0$ *and the c- discriminant is of form* $EC^3N^2 \equiv 0$

- *For the D.E.* $y'' + P(x)\ y' + Q(x)y = 0$ *if the functions* $(x-x_0)\,P(x)$ *and* $(x-x_0)^2\,Q(x)$ *are analytic at* $x = x_0$ *then* x_0 *is a regular singular point otherwise irregular.*

- *A point which is not a singular point is called an ordinary point.*

- $e^{At} \equiv I + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + ... = \sum_{n=0}^{\infty}\frac{1}{n!}A^nt^n$, *where A is a square matrix.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

1. *"For homogeneous linear ordinary differential equation, the linear combination of two solutions is again a solution of the equation". The statement is* ***(GATE-2002)***

(A) true (B) false (C) neither true nor false (D) can't say

Solution. (A) *For homogenous linear ordinary differential equation of order n, if* $y_1, y_2, ..., y_n$ *are solutions of it Then,* $\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n$ *is also solution of it*

$L[y_1] = 0$

$L[y_2] = 0$

⋮

$L[y_n] = 0 \Rightarrow L\,[\alpha_1 y_1 + ... + \alpha_n y_n] = \alpha_1 L[y_1] + ... + \alpha_n L[y_n] = 0$

Hence, the given statement is true.

Option (A) is correct.

2. *Let* $y = \phi(x)$ *and* $y = \psi(x)$ *be solutions of* $y'' - 2xy' + (\sin x^2)\,y = 0$ *such that* $\phi(0) = 1$, $\phi'(0) = 1$ *and* $\psi(0) = 1$, $\psi'(0) = 2$. *Then the value of the Wronskian* $W(\phi, \psi)$ *at* $x = 1$ *is* ***(GATE-2004)***

(A) 0 (B) 1 (C) e (D) e^2

Solution. (C) *The given differential equation is* $y'' - 2xy' + (\sin x^2)y = 0$

$\phi(x)$ *and* $\psi(x)$ *are its solutions*

Now $W(\phi,\psi)=\begin{vmatrix}\phi(t) & \psi(t)\\ \phi'(t) & \psi'(t)\end{vmatrix}$ *at* $x = 0$, $W(\phi(0),\psi(0))=\begin{vmatrix}\phi(0) & \psi(0)\\ \phi'(0) & \psi'(0)\end{vmatrix}=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1$

Now, $W(x)=W(0)\,e^{-\int\left(\frac{-2x}{1}\right)dx}=W(0)\,e^{x^2}$

$W(1)=W(0)\,e^{1}=e$

$\Rightarrow$ *Option (C) is correct answer*

3. *The value of the Wronskian of the functions* $x^2,\ 3x+2,\ 2x+3$ *is.* **(GATE-1999)**

(A) 0 (B) $2+x$ (C) $10+x^2$ (D) -10

Solution. *Given functions are* x^2, $3x+2$, $2x+3$

$$W=\begin{vmatrix}x^2 & 3x+2 & 2x+3\\ 2x & 3 & 2\\ 2 & 0 & 0\end{vmatrix}=2\,[6x+4-6x-9]=-10$$

$\Rightarrow$ *Option (D) is correct.*

4. *Which of the following pair of functions is a linearly independent pair of solutions of* $y''+9y=0$? **(GATE-2001)**

(A) $\sin 3x,\ \sin 3x-\cos 3x$ (B) $\sin 3x+\cos 3x,\ 3\sin x-4\sin^3 x$

(C) $\sin 3x,\ \sin 3x\cos 3x$ (D) $\sin 3x+\cos 3x,\ \sin 3x\cos 3x$

Solution. (A, B) *The given differential equation is* $y''+9y=0$, *A.E. is* $m^2+9=0$

Its roots are $\pm 3i$

The solution of equation is $y=c_1\cos 3x+c_2\sin 3x$

The linearly independent solutions are $\cos 3x$ *and* $\sin 3x$

For option (A),

Clearly, $\sin 3x$ *and* $\sin 3x-\cos 3x$ *are its solutions*

Now, $c_1(\sin 3x)+c_2(\sin 3x-\cos 3x)=0$

$(c_1+c_2)\sin 3x-c_2(\cos 3x)=0\Rightarrow c_1+c_2=0$ *and* $-c_2=0\Rightarrow c_1=0=c_2$

Both are linearly independent

By option (B), $c_1(\sin 3x+\cos 3x)+c_2(3\sin x-4\sin^3 x)=0$

$c_1(\sin 3x+\cos 3x)+c_2(\sin 3x)=0$

$(c_1+c_2)\sin 3x+c_1\cos 3x=0\Rightarrow c_1+c_2=0,\ c_1=0\Rightarrow c_1=c_2=0$

Both solutions are linearly independent

For option (C), $\sin 3x\ \cos 3x=\dfrac{1}{2}\sin 6x$

It is not solution of differential equation

$\Rightarrow$ *option (C) is incorrect*

For option (D), sin 3x cos 3x again not a solution of differential equation

$\Rightarrow$ *Option (D) is incorrect.*

$\Rightarrow$ *Options (A), (B) are correct.*

5. *The differential equation whose linearly independent solutions are cos 2x, sin 2x and e^x, is* **(GATE-2001)**

(A) $(D^3 + D^2 + 4D)\,y = 0$ *(B)* $(D^3 - D^2 + 4D - 4)\,y = 0$

(C) $(D^3 + D^2 - 4D - 4)\,y = 0$ *(D)* $(D^3 - D^2 - 4D + 4)\,y = 0$

Solution. *For cos 2x, sin 2x and e^x to be solution of differential equation, the A.E. must have roots $\pm$ 2i and 1.*

i.e. $(D^2 + 4)(D - 1)\,y = 0$

$(D^3 + 4D - D^2 - 4)\,y = 0$

$(D^3 - D^2 + 4D - 4)\,y = 0$

$\Rightarrow$ *Option (B) is correct.*

6. *If* $y = \sum_{m=0}^{\infty} c_m x^{r+m}$ *is assumed to be a solution of the differential equation* $x^2 y'' - xy' - 3(1 + x^2)\,y = 0$, *then the values of r are* **(GATE-2012)**

(A) 1 and 3 *(B) -1 and 3* *(C) 1 and -3* *(D) -1 and -3*

Solution. (B) $y = \sum_{m=0}^{\infty} c_m x^{r+m}$

$$y' = \sum_{m=0}^{\infty} c_m (r+m)\, x^{r+m-1}$$

$$y'' = \sum_{m=0}^{\infty} c_m (r+m)(r+m-1)\, x^{r+m-2}$$

Substituting the values in the differential equations $x^2 y'' - xy' - 3(1+x^2)\,y = 0$

$$\Rightarrow x^2 \sum_{m=0}^{\infty} c_m (r+m)(r+m-1)\, x^{r+m-2} - x \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} - 3(1+x^2) \sum_{m=0}^{\infty} c_m x^{r+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \left[(r+m)(r+m-1) - (r+m) - 3\right] x^{r+m} + \sum_{m=0}^{\infty} -3 c_m x^{r+m+2} = 0$$

$\Rightarrow (r + m)(r + m - 1) - (r + m) - 3 = 0$

Take $m = 0 \Rightarrow r(r - 1) - r - 3 = 0 \Rightarrow r^2 - r - r - 3 = 0 \Rightarrow r^2 - 2r - 3 = 0$

$\Rightarrow r = -1, 3.$

ASSIGNMENT - 2.1

NOTE: CHOOSE THE BEST OPTION

1. *Given an equation* $\frac{d^3y}{dx^3} = \frac{dy}{dx}$ *and a solution of it is* $y = a_0 + a_1 \sinh x + a_2 \cosh x$, *where* a_0, a_1, a_2 *are arbitrary constants then this solution is*
   (A) particular solution   (B) complete primitive
   (C) singular solution   (D) none of these

2. *In general solution, the arbitrary constants are*
   (A) dependent   (B) independent   (C) dependent variables   (D) none of these

3. *The equation of the envelope of the family of curves represented by the general solution of the differential equation is called*
   (A) complementary solution   (B) particular solution
   (C) singular solution   (D) none of these

4. *General solution of nth order ordinary linear (homogeneous) differential equation contains*
   (A) every solution   (B)some solutions   (C) no solution   (D) None of these

5. *"The nth order ordinary linear homogeneous differential equation has no solution other than general solution". The statement is*
   (A) always true   (B) always false   (C) partially true   (D) partially false

6. *Consider the differential equation* $f(D)y = e^{ax}$, *where* $f(a)=0$ *and* $f(D)=0$ *is the corresponding auxiliary equation, then*
   (A) a particular integral may or may not be obtained
   (B) no particular integral can be obtained
   (C) a particular integral can always be obtained
   (D) none of the above is true in this case

7. $\frac{1}{D-a}Q(x)$ *equal to*
   (A) $e^{ax}\int Q(x)dx$ (B) $e^{-ax}\int Q(x)dx$ (C) $e^{ax}\int e^{-ax}Q(x)dx$ (D) $e^{-ax}\int e^{ax}Q(x)dx$

8. *The c-discriminant when equated to zero include nodal locus*
   (A) once   (B) twice   (C) thrice   (D) none of these

9. *The c-discriminant and the p-discriminant both contain*
   (A) envelope   (B) tac-locus   (C) node-locus   (D) none of these

10. *If any equation contains n-arbitrary constants, then the order of differential equation derived from it is*
(A) n (B) n – 1 (C) 2 (D) n + 1

11. *The solution* $y = A\sin x + B\cos(x+c)$ *contains three arbitrary constants, they are really equivalent to*
(A) only four constants (B) only three constants
(C) two only (D) none of these

12. *The set of orthogonal trajectories to a family of curves whose differential equation is* $\phi(r,\theta,dr/d\theta)=0$, *is found by the differential equations*
(A) $\phi\left(r,\theta,r\frac{dr}{d\theta}\right)=0$ (B) $\phi\left(r,\theta,r\frac{d\theta}{dr}\right)=0$
(C) $\phi\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$ (D) $\phi\left(r,\theta,-\frac{1}{r}\frac{dr}{d\theta}\right)=0$

13. *The linearity principle for ordinary differential equation holds for*
(A) non – homogeneous equation (B) non linear equation
(C) linear differential equation (D) none of the above

14. *General solution of* $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}+2y=0$ *is*
(A) $y = Ae^{-2x}$ (B) $y = Be^{\frac{1}{2}x}$ (C) $y = Ae^{-2x} + Be^{-\frac{1}{2}x}$ (D) none of these

15. *The differential equation, derived from* $y = Ae^{2x} + Be^{-2x}$ *have the order, (where A, B are constants)*
(A) 3 (B) 2 (C) 1 (D) none of these

16. *The differential equation associated with the primitive* $y = Ax^2 + Bx + C$ *is given by*
(A) $\frac{d^3y}{dx^3}=0$ (B) $\frac{d^2y}{dx^2}=2A$ (C) $\frac{dy}{dx}-2Ax-B=0$ (D) none of these

17. *General solution of* $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$ *is*
(A) $y = Ae^{x} + Be^{2x} + Ce^{3x}$ (B) $y = 3e^{x}$
(C) $y = A + Be^{2x}$ (D) none of these

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

18. *The number of arbitrary constants in the complete primitive of differential equation* $\frac{d^5y}{dx^5}+2\frac{d^4y}{dx^4}=0$ *is / are not*
(A) 5 (B) 4 (C) 1 (D) 6

19. *The nth order ordinary linear homogeneous differential equation donot have*
(A) *n-singular solutions* (B) *no singular solution*
(C) *one singular solution* (D) *two singular solutions*

20. *Singular solution of differential equation contains*
(A) *arbitrary constants* (B) *can be obtained from general solution*
(C) *do not contain arbitrary constants* (D) *cannot be obtained from general solution*

21. *Consider two statements*
(a) *Singular solution contains no arbitrary constants.*
(b) *Singular solution can be obtained from complete primitive.*
Which of the following statements is/are true?
(A) *(a) is true* (B) *(b) is false*
(C) *(a) and (b) both true* (D) *(a) and (b) both false*

22. *If* $\frac{xdy}{dx} = 2y$ *has a solution* $y = 2x^2$ *then the solution is not*
(A) *general solution* (B) *complete primitive*
(C) *particular solution* (D) *singular solution*

23. *For a given differential equation which of the following is false*
(A) *an envelope gives a singular solution* (B) *node locus gives a solution*
(C) *cusp-locus gives a solution* (D) *envelope does not give a singular solution*

ASSIGNMENT - 2.2

NOTE: CHOOSE THE BEST OPTION

1. The general solution of the differential equation $\frac{d^3y}{dx^3}-\frac{2d^2y}{dx^2}-\frac{dy}{dx}+2y=0$, is
   (A) $\left(c_1e^x+c_2e^{2x}+c_3e^{3x}\right)$
   (B) $\left(c_1e^{-x}+c_2e^x+c_3e^{2x}\right)$
   (C) $(c_1+c_2x)e^x+c_3e^{2x}$
   (D) None of the above

2. $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$, has the solution
   (A) $y=c_1e^{-2x}+c_2e^x$
   (B) $y=ce^{-2x}$
   (C) $y=c_1e^{-2x}+c_2e^{-x}+c_3$
   (D) None of these

3. The differential equation $\frac{d^2y}{dx^2}-y=0$, has the solution
   (A) e^x+C
   (B) e^x
   (C) e^{2x}
   (D) none of the above

4. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$, has the solution
   (A) $y=c_1e^{-x}+xc_2e^{-x}+x^2c_3e^{-x}$
   (B) $y=c_1\cos 2x+c_2\sin 2x$
   (C) $y=c_1e^{-x}+c_2\cos 2x+c_3\sin 2x$
   (D) None of these

5. The solution of $(D^2+1)^2y=0, D=d/dx$, is
   (A) $A\cos x+B\sin x$
   (B) $e^x(A\cos x+B\sin x)$
   (C) $(A_1+A_2)\cos x+(A_3+A_4)\sin x$.
   (D) $(A_1+A_2x)\cos x+(A_3+A_4x)\sin x$.

6. The particular integral of $(D^3-D)y=e^x+e^{-x}$, is
   (A) $(1/2)(e^x+e^{-x})$
   (B) $(1/2)x(e^x+e^{-x})$
   (C) $(1/2)x(e^x-e^{-x})$
   (D) $(1/2)x^2(e^x-e^{-x})$.

7. The solution of $(d^2y/dx^2)+y=0$, satisfying the condition $y(0)=1$, $y(\pi/2)=2$, is
   (A) $\cos x+2\sin x$
   (B) $\cos x+\sin x$
   (C) $2\cos x+\sin x$
   (D) $2(\cos x+\sin x)$

8. The primitive of $(D^2-2D+5)^2y=0, D=d/dx$ is
   (A) $e^x\{(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x\}$
   (B) $e^{2x}\{(c_1+c_2x)\cos x+(c_3+c_4x)\sin x\}$
   (C) $e^x\{(c_1+c_2x)\cos x+(c_3+c_4)\sin 2x\}$
   (D) $e^x\{(c_1\cos x+c_2\cos 2x+c_3\sin x+c_4\sin 2x)\}$

9. *Let* $p=\dfrac{dy}{dx}$, *then* $y=px+\sqrt{1-p^2}$ *has the solution*

(A) $y=cx$ (B) $y=cx+\sqrt{1-c^2}$ (C) $y=x^2+c$ (D) $y=c_1+c_2x$

10. *The difference of two solutions of non homogeneous nth order ordinary differential equation is a solution of corresponding homogeneous equation. The statement is*

(A) always true (B) always false (C) partially true (D) partially false

11. *If* y_1 *is the solution of non-homogeneous nth order ordinary differential equation and* y_2 *is the solution of corresponding homogeneous equation. Then the solution* y_1+y_2 *is also a solution of*

(A) homogeneous equation (B) non-homogeneous equation
(C) both (A) and (B) (D) none of these

12. *The orthogonal trajectories of the parabola* $y^2=4a(x+a)$, a *being the parameter are the curves given by*

(A) $y^2=4b(x+b)$ (B) $y^2=4b(x-b)$ (C) $y^2=4bx$ (D) $y^2=4by$.

13. *The singular solution of the differential equation* $y=px+p^3$, $(p=dy/dx)$ *is*

(A) $4y^3+27x^2=0$ (B) $4x^2+27y^3=0$ (C) $4y^2-27x^3=0$ (D) $4x^3+27y^2=0$

14. *Which one of the following is not a solution of* $(dy/dx)^2+x(dy/dx)-y=0$:

(A) $x^2+4y=0$ (B) $y=x+1$ (C) $y+x=1$ (D) $y^2-4x=0$.

15. *If* $D\equiv d/dz$ *and* $z=\log x$, *then the differential equation* $x(d^2y/dx^2)+2(dy/dx)=6x$ *becomes*

(A) $D(D-1)y=6e^z$ (B) $D(D-1)y=6e^z$ (C) $D(D+1)y=6e^{2z}$ (D) $D(D+1)y=6e^z$.

16. *The singular solution of the differential equation* $y^2\{1+(dy/dx)^2\}=R^2$ *is*

(A) $y=R/2$ (B) $y=R$ (C) $y=3R/2$ (D) $y=2R$

17. *The c-discriminant of the equation* $(y-c)^2=x(x-a)^2$ *is*

(A) $x(x-a)^2=0$ (B) $x-a^2=0$ (C) $x=a$ (D) none of these

18. *The solution of the differential equation* $\left(x\sin\left(\dfrac{y}{x}\right)\right)dy-\left(y\sin\left(\dfrac{y}{x}\right)-x\right)dx=0$ *is*

(A) $\cos\left(\dfrac{y}{x}\right)=0$ (B) $\sin\left(\dfrac{y}{x}\right)=0$

(C) $\cos\left(\dfrac{y}{x}\right)-\log x=constant$ (D) $\sin\left(\dfrac{y}{x}\right)-\log x=constant$

19. *For the differential equation* $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0$

(A) *every solution converges to 0 as* $x \to \infty$

(B) *every solution is bounded on* $[0, \infty)$

(C) *every solution has countable number of zeros in* $[0, \infty)$

(D) *there exists a solution, which is not bounded on* $[0, \infty)$

20. *Integrating factor of the D.E.* $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$, *is*

(A) $x^{1/2}y^{-5/2}$ (B) $x^{3/2}y^{-5/2}$ (C) $x^{-5/2}y^{-1/2}$ (D) $x^{-5/2}y^{-5/2}$

21. *If* $y_1(x)$ *and* $y_2(x)$ *are solutions of* $y'' + x^2y' + (1-x)y = 0$ *such that* $y_1(0) = 0, y_1'(0) = -1$ *and* $y_2(0) = -1, y_2'(0) = 1$ *then the Wronskian* $W(y_1, y_2)$ *on R*

(A) is never zero (B) is identically zero

(C) is zero only at finite number of points (D) is zero at countably infinite number of points

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

22. *Which of the following cannot be the solution of differential equation?*

(A) Envelope (B) Cusp locus (C) Node-locus (D) Tac-locus

23. *A particular integral of the given differential equation* $f(D^2)y = \sin ax$ *is not given by*

(A) $y = \frac{\sin ax}{f(a^2)}$ *if* $f(a^2) \neq 0$ (B) $y = \frac{\sin ax}{f(-a^2)}$ *if* $f(-a^2) \neq 0$

(C) $y = \frac{\sin ax}{f(a)}$ *if* $f(a) \neq 0$ (D) $y = \frac{\cos ax}{f(a)}$ *if* $f(a) \neq 0$

24. *The c-discriminant contains which of the following?*

(A) The envelope (B) The tac-locus (C) The cusp-locus (D) node-locus

25. *The general solution of* $\frac{d^2y}{dx^2} - y = \sin x$ *is* $Ae^x + Be^x - \frac{1}{2}\sin x$, *(A, B are constants). Which part of this solution is not a particular integral?*

(A) Ae^x (B) Be^{-x} (C) $Ae^x + Be^{-x}$ (D) $-\frac{1}{2}\sin x$

26. *The p-discriminant contains which of the following?*

(A) The envelope (B) The tac-locus (C) The cusp-locus (D) node-locus

27. *If y_1 and y_2 are two solutions of $y'' + p(x)\, y' + q(x)y = 0$, then for general solution of this given equation, y_1 and y_2 are not*
(A) linearly independent (B) linearly dependent (C) proportional (D) dependent

28. *The singular solution(s) of the differential equation $4xp^2 = (3x-a)^2$ is/are not given by*
(A) $x = 0$ only (B) $3x - a = 0$ only
(C) $x - a = 0$ only (D) $x = 0, 3x - a = 0$ & $x - a = 0$

29. *Given $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$, then its*
(A) auxiliary equation is $m^2 - 6m + 13 = 0$ (B) auxiliary equation is $m^2 + 6m - 13 = 0$
(C) general solution is $y = e^{ax} + A\cos x + Bx$ (D) general solution $y = Ae^{(3+2i)x} + Be^{(3-2i)x}$

30. *Given, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 50e^{2x}$, then*
(A) $(D-2)^2 y = 50e^{2x}$ (B) $(D+2)^2 y = e^{5x}$
(C) general solution is $y = 25x^2e^{2x} + (A+Bx)e^{2x}$ (D) no solution exist

31. *Given, $(x + c)^2 + y^2 = r^2$, then*
(A) this equation is equivalent to $(x+c) = \pm\sqrt{r^2 - y^2}$
(B) this equation is equivalent to $x + c = r^2 - y^2$
(C) the p-discriminant is $y^2(y^2 - r^2) = 0$
(D) the p-discriminant is $y(y-r)=0$

32. *Given, $cy=c^2x+1$, then*
(A) elimination of c gives $y = \pm 2\sqrt{x}$ (B) elimination of c gives $y^2 = 4x$
(C) elimination of c gives $y = 4x$ (D) elimination of c gives $y=3x+2$

33. *For the equation $\frac{dy}{dx} + 3y = e^{2x}$*
(A) Integrating factor is e^{3x} (B) Integrating factor is cosx
(C) Solution of the equation is $y = \frac{1}{5}e^{2x} + Ce^{-3x}$ (D) Solution of the equation is $y = e^x + \frac{1}{x}$

34. *Given, $y = px \pm \sqrt{(p^2 + m^2)}$, then*
(A) The p-discriminant is $4x^2y^2 - 4(x^2-1)(y^2-m^2) = 0$
(B) The c-discrimininant is $4xy + 6x^2 + 2y = 0$
(C) The singular solution is $y^2 + m^2x^2 = m^2$
(D) No singular solution exist

35. For the equation $\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0$

(A) $v=\frac{A}{r}+B$ is a solution of given equation

(B) $v=\frac{A}{r}+B$ is not a solution of given equation

(C) $\frac{dv}{dr}=-\frac{A}{r^2}$ is a solution of given equation

(D) $\frac{dv}{dr}=-\frac{A}{r^2}$ is not a solution of given equation

36. The general solution $\frac{dy}{dx}+y=f(x)$, where $f(x)=\begin{cases}x, & 0\le x\le 1\\ 0, & x\ge 1\end{cases}$ and $y(0)=0$ is

(A) $y=\begin{cases}2(1-e^{-x}), & 0\le x\le 1\\ 2e^{-x}(e-1), & x\ge 1\end{cases}$

(B) $y=\begin{cases}2(e^{-x}-1),, & 0\le x\le 1\\ 2e^{-x}(e-1), & x\ge 1\end{cases}$

(C) $y=\begin{cases}2(1-e^{-x}), & 0\le x\le 1\\ 2e^{-x}(1-e), & x\ge 1\end{cases}$

(D) none of the above

37. If $Ly=xe^x\ell nx\ \ (x>0)$ when $L=\left(\frac{d^2}{dx^2}+P\frac{d}{dx}+Q\right)$, and two L.I. solutions of $Ly=0$ are xe^x and e^x, then P.I. is

(A) $xe^x\left[\frac{x}{2}\ell n\frac{x-x^2}{4}\right]+e^{-x}\left[\frac{-x^3}{3}\ell n\frac{x^3}{9}\right]$

(B) $x^2e^x\left[\frac{x^2}{2}\ell n\frac{x-x^2}{4}\right]+e^x\left[\frac{-x^2}{2}\ell n\frac{x+x^3}{9}\right]$

(C) $x^2e^x\left[\frac{x^2}{2}\ell n\frac{x-x^2}{4}\right]+e^x\left[\frac{-x^2}{3}\ell n\frac{x+x^3}{9}\right]$

(D) None of these

38. Let y_1 and y_2 be solutions of Bessel's equation $t^2y''+ty'+(t^2-n^2)y=0$ on the interval $0<t<\infty$ with $y_1(1)=1$, $y_1'(1)=0$, $y_2(1)=0$ and $y_2'(1)=1$ then value of $W(y_1, y_2)(t)$ is

(A) $1/t^2$

(B) $1/t$

(C) $\frac{(\log t)^2}{t}$

(D) 0

ASSIGNMENT - 2.3

NOTE: CHOOSE THE BEST OPTION

1. The differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=50e^{2x}$ have particular integral

(A) $\frac{2e^{2x}}{3}$ (B) $2e^{2x}$ (C) e^{2x} (D) none of these

2. The complementary function of the differential equation $\frac{dy}{dx}+4y=\cos \pi x$ is

(A) ce^{4x} (B) $(c_1+c_2x)e^{4x}$ (C) ce^{-4x} (D) None of the above

3. If $\phi(x, y) = 0$ is a singular solution, then $\phi(x, y)$ is a factor of

(A) p - discriminant only (B) c - discriminant only
(C) p - and c - discriminant both (D) none of these

4. If y_1 and y_2 are two solutions of initial value problem $y''+p(x)y'+q(x)y=0$, $y(x_0)=y_0$, $y'(x_0)=y'_0$ and the Wronskian $W(y_1,y_2)=0$, then y_1 and y_2 are

(A) linearly dependent (B) linearly independent
(C) proportional (D) none of these

5. The value of Wronskian $W(x, x^2, x^3)$ is

(A) $2x^4$ (B) $2x^2$ (C) $2x^3$ (D) none of these

6. The equation $y = Ae^{3x} + Be^{5x}$ is solution of the differential equation given by

(A) $y'' - 8y' + 15y = 0$ (B) $y'' + 8y' = 0$ (C) $y' + 8y' = 0$ (D) $y''+8y'+15y= 0$

7. The particular solution of the given differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=e^x$ is given by

(A) e^x (B) $1/2e^x$ (C) $1/3e^x$ (D) does not exist

8. If $P=\frac{dy}{dx}$, then solution of differential equation $P^2-5P+6=0$ is

(A) $(y-4x-c)(y-3x-c)=0$ (B) $(y-2x-c)(y-3x-c)=0$
(C) $(y-x-c)(y-5x-c)=0$ (D) none of the above

9. For $\frac{d^2y}{dx^2}+4y=\tan 2x$ solving by variation of parameters, the value of Wronskian W is

(A) 1 (B) 2 (C) 3 (D) 4

10. Solving by variation of parameters $y''-2y'+y=e^x \log x$, the value of Wronskian can be

(A) e^{2x} (B) 2 (C) e^{-2x} (D) none of these

11. *Solving by variation of parameters the equation $y''+y=\sec x$, the value of Wronskian is*
(A) 1 (B) 2 (C) 3 (D) 4

12. *The solution of the differential equation* $\frac{dy}{dx}=\frac{3x^2y^4+2xy}{x^2-2x^3y^3}$, *is*
(A) $x^3y^2+\frac{x^2}{y}=\frac{x}{y}$ (B) $x^3y^2+\frac{x^2}{y^2}=x$
(C) $x^3y^2+\frac{x^2}{y}=c$ (D) none of these

13. *If a differential equation has the general solution* $(x+c)^2+y^2(3-y)=0$, *the singular solution is*
(A) $y=0$ (B) $y=1$ (C) $y=3$ (D) none of these

14. *Let* $y=px-2p^2$, *then its singular solution is*
(A) $8y=x^2$ (B) $x^2-4y=0$ (C) $x^2=2y$ (D) $x^2=y$

15. *The solution of the differential equation* $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x, y=3$ *and* $\frac{dy}{dx}=3$ *when* $x=0$, *is*
(A) $y=2e^x+e^{-2x}-xe^{-x}$ (B) $y=2e^x+e^{2x}-xe^x$
(C) $y=e^x+2e^{2x}-x^2e^x$ (D) $y=e^x-e^{2x}+xe^{-x}$

16. *The number of singular solutions of differential equation* $p^2+y^2=1$ *is*
(A) zero (B) one (C) two (D) three

17. *The solution of the differential equation* $(x^2y^3+xy)\frac{dy}{dx}=1$ *is*
(A) $\frac{1}{x}=2-y^2+Ae^{y^2/2}$ (B) $x=2-y^2+Ae^{y^2/2}$
(C) $\frac{1}{x}=2-y^2+Ae^{-y^2/2}$ (D) $x=2+y^2+Ae^{-y^2/2}$

18. *General solution of the differential equation* $(\cos x-\sin x)\frac{d^2y}{dx^2}+2\sin x\frac{dy}{dx}-(\cos x+\sin x)y=0$, *(given that $y=\sin x$ is a solution), is*
(A) $c_1\cos x+c_2x\sin x$ (B) $c_1\sin x+c_2x\sin x$
(C) $c_1\sin x+c_2e^x$ (D) $c_1\sin x+c_2x$

19. *The solution of differential equation* $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=0$, *is*
(A) $y^2=A\ln x+B$ (B) $y^2=A\ln^2x+B$
(C) $y=A\ln x+B$ (D) $y=A\ln^2x+B$

20. The particular integral corresponding to the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x$, is given by

(A) xe^x (B) $1/6e^x$ (C) e^x (D) $-xe^x$

21. If the differential equation is of the type f(D).y = sin ax, where f(D) is a polynomial in D containing the odd powers in D only, then

(A) A particular integral can be obtained
(B) No particular integral can be obtained
(C) Particular integral in this case is constant
(D) Particular integral is given by $\frac{\sin ax}{f(-a^2)}$ if $f(-a^2)\neq 0$

22. The general solution of $(d^2y/dx^2)-(dy/dx)-2y=10\cos x$ is

(A) $y=c_1e^{-x}+c_2e^{2x}-3\cos x-\sin x$
(B) $y=c_1e^{x}+c_2e^{2x}-3\cos x$
(C) $y=c_1e^{-x}+c_2e^{2x}-3\cos x+\sin x$
(D) $y=c_1e^{-x}+c_2e^{2x}+3\cos x+\sin x$.

23. The solution of $x^2(d^2y/dx^2)-3x(dy/dx)+4y=0$ is

(A) $y=(c_1+c_2x)e^{2x}$ (B) $y=(c_1-c_2x)e^{2x}$
(C) $y=(c_1+c_2x)\log x$ (D) $y=(c_1+c_2\log x)x^2$.

24. The particular integral of the differential equation $\frac{d^2y}{dx^2}-13\frac{dy}{dx}+12y=36$, is given by

(A) 6 (B) 3 (C) e^{3x} (D) -36/13

25. The p - discriminant of the differential equation $y=px+\frac{1}{p}$, is

(A) $y^2=x$ (B) $y=x$ (C) $y^2=4x$ (D) none

26. The solution of the differential equation $y''+(3i-1)y'-3iy=0$, is

(A) $y=c_1e^{x}+c_2e^{3ix}$ (B) $y=c_1e^{-x}+c_2e^{3ix}$ (C) $y=c_1e^{x}+c_2e^{-3ix}$ (D) $y=c_1e^{-x}+c_2e^{-3ix}$

27. The equation whose solution is self orthogonal, is

(A) $p-(1/p)=p^2$
(B) $(px+y)(x+yp)-\lambda p=0, p=dy/dx, p\neq 0$
(C) $(px-y)(x+yp)-\lambda p=0$
(D) $(px+y)(x-yp)-\lambda p=0, p\neq 0, p=dy/dx$

28. *The solution of* $(d^2y/dx^2)-y=k$, *(k = a non – zero constant) which vanishes when* $x = 0$ *and which tends to finite limit as x tends to infinity is*

(A) $y=k(1+e^{-x})$ (B) $y=k(e^{-x}-1)$

(C) $y=k(e^x+e^{-x}-2)$ (D) $y=k(e^x-1)$.

29. *The general solution of* $d^2y/dx^2+2(dy/dx)+y=e^{-x}\cos x$, *is*

(A) $(C_1+C_2x+\sin x)e^{-x}$ (B) $(C_1+C_2x-\sin x)e^{-x}$

(C) $(C_1+C_2x+\cos x)e^{-x}$ (D) $(C_1+C_2x-\cos x)e^{-x}$

30. *If* $y_1(x)=x$ *and* $y_2(x)=xe^x$ *are two linearly independent solutions of* $x^2\frac{d^2y}{dx^2}-x(x+2)\frac{dy}{dx}+(x+2)y=0$, *then the interval on which they form a fundamental set of solutions is*

(A) $x > 0$ or $x < 0$ (B) $-1 < x < \infty$

(C) $-1 < x < 2$ (D) $-\infty < x < \infty$

31. *Suppose that* y_1 *and* y_2 *form a fundamental set of solutions of a second order ordinary differential equation on the interval* $-\infty < t < +\infty$, *then*

(A) there is only one zero of y_1 between consecutive zeros of y_2

(B) there are two zeros of y_1 between consecutive zeros of y_2

(C) there are finite number of zeros of y_1 between consecutive zeros of y_2

(D) none of the above

32. *If* $y_1(x)$ *and* $y_2(x)$ *are solutions of* $y'' + xy' + (1-x^2)y = x\sin x$, *then which of the following is also its solution?*

(A) $y_1(x)+y_2(x)$ (B) $y_1(x)-y_2(x)$

(C) $2y_1(x)-y_2(x)$ (D) $y_1(x)-2y_2(x)$

33. *For which of the following pair of functions* $y_1(x)$ & $y_2(x)$, *continuous functions p(x) and q(x) can be determined on [-1, 1] such that* $y_1(x)$ *and* $y_2(x)$ *give two linearly independent solutions of* $y''+p(x)y'+q(x)y=0,\ x\in[-1,1]$

(A) $y_1(x)=x\sin(x),\ y_2(x)=\cos(x)$ (B) $y_1(x)=xe^x,\ y_2(x)=\sin(x)$

(C) $y_1(x)=e^{x-1},\ y_2(x)=e^x-1$ (D) $y_1(x)=x^2,\ y_2(x)=\cos(x)$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

34. $\frac{1}{D-a}(e^{ax}x)$ *is not equal to*

(A) x^2e^{ax} (B) $(x^2/2)e^{ax}$ (C) x^2e^{-ax} (D) xe^{-ax}

35. *A particular integral of* $(d^2y/dx^2)-(dy/dx)-2y=\cos x+3\sin x$ *is not*
(A) $\sin x$ (B) $\cos x$ (C) $-\sin x$ (D) $-\cos x$

36. *The solution of* $(D^4+8D^2+16)y=0$ *is not given by*
(A) $c_1e^{2x}+c_2e^{-2x}+c_3e^{x}+c_4e^{-x}$
(B) $(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-2x}$
(C) $(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$
(D) $(c_1+c_2x)\cosh 2x+(c_3+c_4x)\sinh 2x$

37. *The p-discriminant of equation* $y^2p^2+y^2-r^2=0$, *is not*
(A) $y^2=0$ (B) $y^2=r^2$ (C) $y^2(y^2+r^2)=0$ (D) $y^4=0$

38. *If* y_1 *and* y_2 *are linearly independent solutions of the homogeneous equation* $L(y)=y''+p_1(x)y'+p_2(x)y=0$ *Then* $p_1(x)$ *and* $p_2(x)$ *are not given by*
(A) $p_1(x)=\dfrac{y_1y_2''-y_1''y_2}{w(x)}$, $p_2(x)=\dfrac{y_1'y_2''-y_1''y_2'}{w(x)}$
(B) $p_1(x)=-\left[\dfrac{y_1y_2''-y_1''y_2}{w(x)}\right]$, $p_2(x)=\dfrac{y_1'y_2''-y_1''y_2'}{w(x)}$
(C) $p_1(x)=\dfrac{y_1y_2''-y_1''y_2}{w(x)}$, $p_2(x)=-\left[\dfrac{y_1'y_2''-y_1''y_2'}{w(x)}\right]$
(D) $p_1(x)=-\left[\dfrac{y_1y_2''-y_1''y_2}{w(x)}\right]$, $p_2(x)=-\left[\dfrac{y_1'y_2''-y_1''y_2'}{w(x)}\right]$

39. *The equation* $\dfrac{d^2y}{dx^2}-2\left(\dfrac{dy}{dx}\right)+10y=0$ *has characteristic roots.*
(A) $1+3i$ (B) $1-3i$ (C) $1+\dfrac{3}{2}i$ (D) $1-\dfrac{3}{2}i$

40. *Given, the equation* $x=A\cos(pt-\alpha)$, *then*
(A) $\dfrac{d^2x}{dt^2}=-p^2x$
(B) $\dfrac{d^3x}{dt^3}=-p^2\dfrac{dx}{dt}$
(C) $\dfrac{d^2x}{dt^2}=px^2$
(D) *elimination of A, p and x gives* $x\dfrac{d^3x}{dt^3}-\dfrac{dx}{dt}\cdot\dfrac{d^2x}{dt^2}=0$

41. *Given,* $\frac{dx}{x} = \tan y dy$ *Solving the above equation gives*

(A) $\log x = -\log \cos y + C$
(B) $\log (x \cos y) = C$
(C) $\log (x \sin y) = C$
(D) $x \cos y = e^C$

42. *Given, sinx, cosx, sin2x, then*

(A) the wronskian of given functions is 3sin2x
(B) the wronskian of given functions is zero
(C) the functions are linearly independent in $\left(0, \frac{\pi}{2}\right)$
(D) the functions are linearly dependent in $(0, \pi)$

43. *Particular Integral of the following differential equation* $(D^2 - 4D + 4)y = \frac{e^{2x} \log x}{x}$ *is obtained by integrating*

(A) $\frac{x \log x}{2}$
(B) $\frac{x(\log x)^2}{2}$
(C) $\frac{(\log x)^2}{2}$
(D) $\left(\frac{x \log x}{2}\right)^2$

44. *An integral curve of* $x(4ydx + 2xdy) + y^3(3ydx + 5xdy) = 0$ *is not*

(A) $x^4y^2 + x^3y^5 = 1$
(B) $x^4y^2 + x^3y^4 = 1$
(C) $x^3y^3 + x^4y^3 = 1$
(D) $x^2y^4 + x^3y^4 = 1$.

ASSIGNMENT - 2.4

NOTE: CHOOSE THE BEST OPTION.

1. The c-discriminant relation of the differential equation $x^3p^2+x^2yp=1$, is given by (where $p=\frac{dy}{dx}$)

(A) $x^3(xy^2+4)=0$ (B) $x(xy-2)=0$
(C) $y(x-y)=2$ (D) $y(x+y)=2$

2. The particular solution of $\frac{d^3y}{dx^3}+y=\cos(2x-1)$, is

(A) $\frac{1}{65}[\cos(2x-1)-8]$ (B) $\frac{1}{65}[\cos(2x-1)-8\sin(2x-1)]$
(C) $\frac{1}{65}[\cos(2x+1)+8]$ (D) None of these

3. P.I. of $(D+2)(D-1)^3y=e^x$, is

(A) $\frac{1}{18}e^x$ (B) $\frac{1}{18}e^xx^2$
(C) $\frac{1}{18}x^3e^x$ (D) None of the above

4. The solution of the differential equation $\frac{d^2y}{dx^2}+4y=\sec^2 2x$, using the method of variation of parameters, is

(A) $y=c_1\cos 2x+c_2\sin 2x-\frac{1}{4}+\frac{1}{4}\sin 2x\log|\sec 2x+\tan 2x|$

(B) $y=c_1\cos 2x-c_2+\sin 2x+\frac{1}{4}\sin 2x\log|\sec 2x+\tan 2x|$

(C) $y=(c_1+c_2x)e^{2x}+\frac{1}{4}\log|\sec 2x+\tan 2x|$

(D) $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}-\frac{1}{4}\sin 2x$

5. The solution of the differential equation $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{3x}$, is

(A) $y=Ae^x+Be^{3x}+Ce^{-2x}+\frac{e^{3x}}{10}$ (B) $y=Ae^x+Be^{-3x}+Ce^{-2x}+\frac{e^{3x}}{10}$
(C) $y=Ae^x+Be^{3x}+Ce^{-2x}+\frac{xe^{3x}}{10}$ (D) $y=Ae^x+Be^{3x}+Ce^{-2x}-\frac{xe^{3x}}{10}$

6. *The solution of the differential equation* $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ *for* $y\left(\frac{2}{3}\right) = \frac{1}{3}$, *is*

(A) $y = x + \log(x+y) - \frac{1}{3}$

(B) $y = x - \log(x+y) + \frac{2}{3}$

(C) $2y = x - \log(x+y) + \frac{1}{3}$

(D) $2y = x + \log(x+y) + 1$

7. *If* $e^{ax}\,u(x)$ *is a P.I of* $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = f(x)$ *where a is any constant then* $\frac{d^2u}{dx^2}$ *is equal to*

(A) $f(x)$

(B) $f(x)\,e^{ax}$

(C) $f(x)\,e^{-ax}$

(D) $f(x)\,(e^{ax} + e^{-ax})$

8. *The general solution of* $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\,\frac{dy}{dx}$, *is*

(A) $ax + by = c$

(B) $ax^2 + by = 0$

(C) $ax^2 + by^2 = 1$

(D) $ax + by^2 = 0$

9. *The complementary function for the differential equation* $(D^4+D^2+1)y = ax^2 + be^{-x}\sin 2x$, *is*

(A) $e^{x} + e^{-x}$

(B) $e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$

(C) $e^{x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{-x}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$

(D) *none of these*

10. *The linear differential equation of order n* $P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = Q(x)$, *where* $P_0, P_1 \ldots, P_n$ *and Q are functions of x alone, is exact if*

(A) $P_n - P'_{n-1} + P''_{n-2} + \ldots - P_0^n = 0$

(B) $P_n - P'_{n-1} + P''_{n-2} + \ldots + P_0^n = 0$

(C) $P_n - P'_{n-1} + P''_{n-2} + \ldots + (-1)^n P_0^n = 0$

(D) $P_n - P'_{n-1} + P''_{n-2} - \ldots + (-1)^{n-1} P_0'' = 0$

11. *The orthogonal trajectories of the given family of curves* $y = cx^k$ *are given by*

(A) $x^2 + y^2 =$ *constant*

(B) $x^2 - ky^2 =$ *constant*

(C) $kx^2 + y^2 =$ *constant*

(D) $x^2 + ky^2 =$ *constant*

12. *The particular integral of the differential equation* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos 3x$ *is given by*

(A) $-3/25 \sin 3x$

(B) $-3/25 \cos 3x$

(C) 1/50 [-3 sin 3x - 4 cos 3x] (D) -1/25 [3 sin 3x - 4cos 3x]

13. The particular integral of differential equation $\frac{d^2y}{dx^2}+a^2y=\sin ax$, is

(A) (–x/2a) sin ax
(B) $(-x/2a)\cos ax$
(C) (x/2a) sin ax
(D) (x/2a) cos ax

14. The particular integral of the differential equation $(D^2-4)y=\sin 4x$ is given by

(A) $y=\frac{x\sin 4x}{20}$
(B) $y=\frac{-\sin 4x}{20}$
(C) $y=\frac{-x\cos 4x}{8}$
(D) $y=\frac{-x\cos 4x}{20}$

15. The particular integral of the differential equation. $(D^2+9)y=\cos 2x+\sin 2x$ is given by

(A) $y=\frac{1}{13}(\cos 2x+\sin 2x)$
(B) $y=\frac{1}{25}(\cos 2x+\sin 2x)$
(C) $y=\frac{1}{5}(\cos 2x+\sin 2x)$
(D) $y=\frac{1}{13}(\cos 2x-\sin 2x)$

16. The particular integral of the differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=2e^{3x}\sin 2x$, is given by

(A) $-\frac{1}{2}xe^{3x}\sin 2x$
(B) $-\frac{1}{2}xe^{3x}\cos 2x$
(C) $-\frac{1}{2}xe^{-3x}\sin 2x$
(D) $-\frac{1}{2}xe^{-3x}\cos 2x$

17. The particular integral of the differential equation $(D^2-2D+1)y=xe^x\sin x$, is given by

(A) $e^x(x\cos x+\sin x)$
(B) $x(e^x\cos x+\sin x)$
(C) $e^x\sin x(x+1)$
(D) $-e^x(x\sin x+2\cos x)$

18. The differential equation $x(dy/dx)^2-(x-3)^2=0$ has p - discriminant as x (x - 3)^2 and c - discriminant as $x(x-9)^2=0$. The singular solution is

(A) $x-3=0$
(B) $x-9=0$
(C) $x=0$
(D) $x(x-3)(x-9)=0$

19. The equation $8ap^3=27y$, where $p=dy/dx$, has singular solution

(A) $y=0$
(B) $y=c$
(C) $y^2=(x-c)^2/a$
(D) $y=(x-c)^2/a$

20. *The differential equation of the orthogonal trajectories of the system of parabolas* $y=ax^2$, *is*

(A) $y'=x^2+y$ (B) $y'=x-y^2$
(C) $y'=-(x/2y)$ (D) $y'=x/(2y)$

21. *The general and singular solutions of* $(dy/dx)^2+x(dy/dx)-y=0$, *are*

(A) $(y-c_1x)(y-x^2/4-c_2)=0,\ x^2+4y=0$ (B) $y=cx+c^2;\ x^2+4y=0$
(C) $x^2+y^2=cxy+c^2$; $(xy)^2-4(x^2+y^2)=0$ (D) None of these

22. *The singular solution of* $p=\log(px-y)$, *is*

(A) $y=x(\log x-1)$ (B) $y=x\log x-1$
(C) $y=\log x-1$ (D) $y=x\log x$

23. *The general solution of* $\frac{d^4y}{dx^4}+y=0$, *is*

(A) $y=exp\left(\frac{x}{\sqrt{2}}\right)\left[c_1\cos\frac{x}{\sqrt{2}}+c_2\sin\frac{x}{\sqrt{2}}\right]+exp\left(\frac{-x}{\sqrt{2}}\right)\left[c_3\cos\frac{x}{\sqrt{2}}+c_4\sin\frac{x}{\sqrt{2}}\right]$
(B) $y=c_1\sin x+c_2\cos x+c_3\sinh x+c_4\cosh x$
(C) $y=c_1e^x+c_2e^{-x}+c_3e^{-2x}+c_4e^2$
(D) $y=c_1\sin 2x+c_2\cos 2x+c_3\sinh 2x+c_4\cosh x$

24. *The singular solution of the differential equation* $(px-y)^2=p^2-1$, *is*

(A) $x^2+y^2=1$ (B) $x^2-y^2=1$
(C) $x^2+2y^2=1$ (D) $x^2-2y^2=1$

25. *The number of linearly independent solutions of*

$(d^4y/dx^4)-(d^3y/dx^3)-3(d^2y/dx^2)+5(dy/dx)-2y=0$ *of the form* e^{ax} *(a being a real number), is*

(A) one (B) two
(C) three (D) four

26. *The singular solution of* $y=px+a(1+p^2)^{1/2}$ *is*

(A) parabola (B) hyperbola
(C) circle (D) straight line

27. *Let* $y_1(x)$ *and* $y_2(x)$ *be solutions of* $y''x^2+y'+(\sin x)y=0$, *which satisfy the boundary conditions* $y_1(0)=0, y_1'(0)=1$ *and* $y_2(0)=1, y_2'(0)=0$ *respectively. Then*

(A) y_1 and y_2 do not have common zeros (B) y_1 and y_2 have common zeroes
(C) either y_1 or y_2 has a zero of order 2 (D) both y_1 and y_2 have zeroes of order 2

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

28. $e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+c_3e^{2x}$ *is not the general solution of*

(A) $(d^3y/dx^3)+4y=0$

(B) $(d^3y/dx^3)+8y=0$

(C) $(d^3y/dx^3)-8y=0$

(D) $(d^3y/dx^3)-2(d^2y/dx^2)+(dy/dx)-2=0$

29. *Which of the following transformation cannot reduce the differential equation* $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$ *into the form* $\frac{du}{dx}+P(x)(u)=Q(x)$?

(A) $u=\log z$

(B) $u=\frac{1}{\log z}$

(C) $u=e^z$

(D) $u=(\log z)^2$

30. *The largest value of c such that there exists a function* $h(x)$ *for* $-c<x<c$ *such that h(x) is a solution of* $\frac{dy}{dx}=1+y^2$ *with* $h(0)=0$ *is not given by*

(A) $\frac{\pi}{3}$

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$

(D) π

31. *The singular solution of* $x(dy/dx)^2-2y(dy/dx)+4x=0,(x>0)$ *is*

(A) $y=x^2$

(B) $y=-x^2$

(C) $(y-2x)(y+2x)$

(D) None of the above

32. *Consider the Assertion (A) and Reason (R) given below;*

Assertion (A): *the singular solution of* $y=2xp+p^2$ *is given by* $x^2+y=0$.

Reason (R): *The p and c discriminants are equal and given by* $x^2+y=0$. *The correct answer is*

(A) A is true

(B) R is true

(C) A is false

(D) R is false

33. *Consider the Assertion (A) and Reason (R) given below:*

Assertion (A) : *The curves* $y=ax^3$ *and* $x^2+3y^2=c^2$ *form orthogonal trajectories.*

Reason (R) : *The differential equation of the first is obtained by replacement of* (dy/dx) *by* $-(dx/dy)$.

The correct answer is

(A) A is true

(B) R is true

(C) A is false

(D) R is false

ANSWERS TO EXERCISES

(PRACTICE SET – 1)

1. $\alpha = 1, y = c_1e^{x} + c_2e^{x}$	2. (C)	3. (D)	4. (B)	5. (A)

(PRACTICE SET – 2)

1. (D)	2. (D)	3. (0)	4. (A,C)	5. (A,B,C)

(PRACTICE SET – 3)

1. (D)	2. (A)	3. (B)	4. (B)	5. (A)

(PRACTICE SET – 4)

1. (D)	2. (2)	3. (A)	4. (C)

(PRACTICE SET– 5)

1. (B)	2. (C)	3. (A)	4. (D)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT – 2.1

1. (B)	2. (B)	3. (C)	4. (A)	5. (A)	6. (C)	7. (C)
8. (B)	9. (A)	10. (A)	11. (C)	12. (C)	13. (C)	14. (C)
15. (B)	16. (A)	17. (A)				
18. (B,C,D)	19. (A,C,D)	20. (C,D)	21. (A,B)	22. (A,B,D)	23. (B,C,D)	

ASSIGNMENT – 2.2

1. (B)	2. (A)	3. (B)	4. (C)	5. (D)	6. (B)	7. (A)
8. (A)	9. (B)	10. (A)	11. (B)	12. (A)	13. (D)	14. (D)
15. (C)	16. (B)	17. (A)	18. (C)	19. (D)	20. (C)	21. (A)
22. (B,C,D)	23. (A,C,D)	24. (A,C,D)	25. (A,B,C)	26. (A,B,C)	27. (B,C,D)	28. (A)
29. (A,D)	30. (A,C)	31. (A,C)	32. (A,B)	33. (A,C)	34. (A,C)	35. (A,C)
36. (D)	37. (D)	38. (B)				

ASSIGNMENT – 2.3

1. (B)	2. (C)	3. (C)	4. (A)	5. (C)	6. (A)	7. (A)
8. (B)	9. (B)	10. (A)	11. (A)	12. (C)	13. (C)	14. (A)
15. (B)	16. (B)	17. (C)	18. (C)	19. (A)	20. (D)	21. (A)
22. (A)	23. (D)	24. (B)	25. (C)	26. (C)	27. (D)	28. (B)
29. (D)	30. (A)	31. (A)	32. (C)	33. (C)		
34. (A,C,D)	35. (A,B,D)	36. (A,B,D)	37. (A,C,D)	38. (A,C,D)	39. (A,B)	40. (A,B,D)
41. (A,B,D)	42. (A,C)	43. (C)	44. (B,C,D)			

ASSIGNMENT – 2.4

1. (A)	2. (B)	3. (C)	4. (A)	5. (C)	6. (A)	7. (C)
8. (C)	9. (B)	10. (C)	11. (D)	12. (C)	13. (B)	14. (B)
15. (C)	16. (B)	17. (D)	18. (C)	19. (A)	20. (C)	21. (B)
22. (A)	23. (A)	24. (B)	25. (B)	26. (C)	27. (A)	
28. (A,B,D)	29. (A,C,D)	30. (A,C,D)	31. (C)	32. (A,B)	33. (A,B)	

CHAPTER – 3
INITIAL AND BOUNDARY VALUE PROBLEMS

INTRODUCTION

In this chapter, we will solve the differential equations with some given conditions either we have knowledge of the system in the starting, i.e., the system called initial value problems or we will be given the behaviour of the problem at the boundaries, i.e., boundary value problem. Lipschitz condition is useful in solving initial value problems.

§ 3.1. INITIAL VALUE PROBLEM

3.1.1. Fundamental Existence Theorem

Theorem 1. *Consider the differential equation*

$$P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + P_2(x)\frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1}(x)\frac{dy}{dx} + P_n y = F(x) \quad ...(1)$$

This is a linear differential equation in y of order n and $P_0(x) \neq 0, P_0, P_1, P_2, ..., P_n$, $F(x)$ are continuous functions of x in [a, b] and if there exists constants C_0, C_1,..., C_{n-1} such that

$$y(x_0) = C_0,\ y'(x_0) = C_1,\ y''(x_0) = C_2,\ ...,\ y^{n-1}(x_0) = C_{n-1}\ , \quad ...(2)$$

(Here C_i's may or may not be distinct)

then equation (1) has unique solution.

Remark: *Fundamental Existence Theorem gives sufficient condition for the existence of unique solution.*

Also, if $P_0(x) = 0$ for some $x \in [a, b]$ then the solution of the given solution may not exists or may not be unique.

Theorem 2. *If $F(x) = 0$ and $y(x_0) = y'(x_0) = ... = y^{n-1}(x_0) = 0$, then equation (1) has one and only one (unique) solution which is $y(x) = 0\ \forall\ x \in [a, b]$*

Note :

(1) In IVP, the solution of D.E. $f(x, y) = \frac{dy}{dx}$ at $y(x_0) = y_0$ must satisfy the two supplementary conditions.

(i) Solution satisfy the D.E.

(ii) Solution satisfy the initial condition i.e. $y(x_0) = y_0$

(2) The D.E. $f(x, y) = \frac{dy}{dx}$ at $y(x_0) = y_0$ has a unique solution that is valid in some interval about the initial point x_0, if

(i) The function f(x, y) is continuous in some domain D of xy-plane.

(ii) The partial derivative $\frac{\partial f}{\partial y}$ *is also continuous function of x and y in D and* (x_0, y_0) *be a point in D.*

Remark: *This condition is sufficient.*

(3) *Also If* $f(x) = \frac{dy}{dx}$, *then continuity of f(x, y) is a sufficient condition for the existence of the solution in some domain D. If* $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ *and* $|x - x_0| \le a$, $|y - y_0| \le b$ *where a, b are finite, then*

(i) If f(x, y) is continuous over the given interval and

(ii) f(x, y) is bounded over the given interval then IVP has atleast one solution in the interval $|x - x_0| \le h$, *where* $h = \min(a, b/m)$ *and* $|f(x)| \le m$ *over the region given R.*

(iii) If f(x, y) satisfies the Lipschitz condition then IVP has a unique solution in the given interval.

Remark: *Condition (3) is sufficient condition but not necessary.*

(4) *Also, if by putting the given condition (i.e.* $y(x_0) = y_0$*) in the general solution of the given equation, if the value of*

(i) c = 0, then unique solution.

(ii) c . 0 = 1, no solution

(iii) c.0 = 0, then infinitely many solutions.

3.1.2. Lipschitz Condition

Let f be defined on D, where D is either a domain or closed domain of the xy-plane. Then, the function f is said to satisfy a Lipschitz condition (with respect to y) in D if ∃ a positive constant k such that

$$|f(x, y_1) - f(x, y_2)| \le k\,|y_1 - y_2| \qquad \ldots(1)$$

for every pair of points (x, y_1) *and* (x, y_2) *which belong to D. The constant k being independent of* x, y_1, y_2 *and is called the* ***Lipschitz constant.*** *The class of all functions satisfying the Lipschitz condition (1) with the Lipschitz constant k in a domain* $D \subset \mathbb{R}^2$ *is denoted by Lip (D, k).*

From the definition, we note that if $f \in$ *Lip (D, k), then* $\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \le k$ *for all* $y_1 \ne y_2$.

Hence, to show that f(x,y) satisfies Lipschitz condition with respect to y in $D \subset \mathbb{R}^2$, *it is enough if we prove that* $\left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right|$ *is bounded for all* $(x, y) \in D \subset \mathbb{R}^2$. *The least upper bound of the expressions on the left hand side of the above inequality as* $(x,y) \in D$ *gives the Lipschitz constant k.*

Note: *In the above definition, by a domain D we mean a non-empty connected open set in $\mathbb{R}^2$. Hence, the line segment joining any two points of D lies entirely in D.*

Theorem 1. *Let f(x, y) be a continuous function defined over a rectangle* $R=\{(x, y): |x-x_0| \le a,\ |y-y_0| \le b\}$ *where a, b > 0. If* $\frac{\partial f}{\partial y}$ *exists and continuous on R, then f(x, y) satisfies Lipschitz condition with respect to y in R and the Lipschitz constant k is given by* $k = \underset{(x,y)\in R}{\text{lub}} \left|\frac{\partial}{\partial y} f(x, y)\right|$.

Proof: *Since* $\frac{\partial f}{\partial y}$ *is continuous in a closed rectangle R, it is bounded in R so that its least upper bound exists in R. Let* $k = \underset{(x,y)\in R}{\text{lub}} \left|\frac{\partial}{\partial y} f(x, y)\right|$ ...(2)

Let (x, y_1) *and* (x, y_2) *be any two points of R. Then by the mean value theorem of differential calculus, there exists a point ξ, where ξ is between* y_1 *and* y_2 *such that*

$$f(x, y_1) - f(x, y_2) = \left[\frac{\partial}{\partial y} f(x,\xi)\right](y_1 - y_2), (x,\xi) \in R \qquad ...(3)$$

Using (2) and (3), we obtain $|f(x, y_1) - f(x, y_2)| \le k\,|y_1 - y_2|$ *for all* (x, y_1) *and* (x, y_2) *in R. This proves that f(x, y) satisfies Lipschitz condition with Lipschitz constant k in R.*

Note: *The condition given in the above theorem is only sufficient but not necessary for a function f(x, y) to satisfy a Lipschitz condition in R as illustrated by the following example:*

Example 1. *The function* $f(x, y) = x^2 |y|$ *satisfies the Lipschitz condition on* $R = \{(x, y) : |x| \le 1, |y| \le 1\}$ *for which* $\frac{\partial f}{\partial y}$ *does not exist in R.*

Solution: *Now* $f(x, y_1) - f(x, y_2) = x^2 [|y_1| - |y_2|] \le x^2 |y_1 - y_2|$.

Since $|x| \le 1$, *we get* $|f(x, y_1) - f(x,y_2)| \le |y_1 - y_2|$ *which show that* $f(x, y) \in L(R, 1)$.

Now $\frac{\partial f}{\partial y} = x^2$, *if* $y>0$ *and* $\frac{\partial f}{\partial y} = -x^2$ *if* $y < 0$.

Hence, $\frac{\partial f}{\partial y}$ *does not exist at any point* $(x, 0) \in R$ *for which* $x \ne 0$.

Example 2. *Check whether the function* $f(x, y) = y^{1/2}$ *satisfies Lipschitz condition in*

(i) $R_1 = \{(x, y) : |x| \le 1, 0 \le y \le 2\}$.

(ii) $R_2 = \{(x, y) : |x| \le a, b \le y \le c, a, b, c > 0\}$.

Solution: *To prove*

(i) *we first note that f(x, 0) = 0*

Now $\left|\frac{f(x,y)-f(x,0)}{y-0}\right| = \frac{1}{|y^{1/2}|}, y \neq 0$ ...(1)

Since (x, 0) $\in R_1 \subset \mathbb{R}^2$, *therefore (4) tends to ∞ so that the left hand side of (1) is unbounded as y→0. Hence, the function does not satisfy the Lipschitz condition in* R_1.

(ii) *Now* $\frac{\partial f}{\partial y} = \frac{1}{2} y^{-1/2}$. *Since* $y \in [b,c], \left|\frac{\partial f}{\partial y}\right| \leq \frac{1}{2\sqrt{b}}, b \neq 0.$

Therefore, f(x,y) satisfies Lipschitz condition in R_2.

Example 3. *Show that the function* $f(x,y) = (y+y^2)\frac{\cos x}{x^2}$ *satisfies Lipschitz condition in* $|y| \leq 1$ *and* $|x-1| < \frac{1}{2}$ *and find the Lipschitz constant.*

Solution: *Now* $f(x,y_1) - f(x,y_2) = (y_1 + y_1^2)\frac{\cos x}{x^2} - (y_2 + y_2^2)\frac{\cos x}{x^2}$

$= \frac{\cos x}{x^2}[(y_1 - y_2) + (y_1 - y_2)(y_1 + y_2)]$

Hence, $|f(x,y_1) - f(x,y_2)| \leq \frac{|\cos x|}{x^2}|y_1 - y_2||(1 + y_1 + y_2)|$ ...(1)

Since $|x-1| < \frac{1}{2}$, *we have* $\frac{1}{2} < x < \frac{3}{2}$ ...(2)

Maximizing the right hand side of (1) using (2), we get

$$|f(x,y_1) - f(x,y_2)| \leq \frac{1}{\left(\frac{1}{2}\right)^2}[|y_1 - y_2|.3] = 12|y_1 - y_2|,$$

Therefore, $|f(x,y_1) - f(x,y_2)| \leq 12\,|y_1 - y_2|$, *which shows that f(x,y) satisfies Lipschitz condition in the given region with the Lipschitz constant 12.*

Example 4. *Find the largest interval in which the solution of IVP exists.*

$y' = 5x^2 + 9y^2$, $y(0) = 0$; $|x| \leq 1$, $|y| \leq 1$

Solution: *The given D.E. is* $y' = 5x^2 + 9y^2$

Since f (x, y) is a continuous function over the given interval

(i) *f(x, y) is bounded over the given interval*

So, there exists a solution of the problem.

Now $|x - 0| \leq h$, $h = \min(a, b/m)$

$\Rightarrow h = \min(1, \frac{1}{14})$, where max f(x, y) = 14 $\Rightarrow |x-0| \le \frac{1}{14} \Rightarrow |x| \le \frac{1}{14}$

$\Rightarrow -\frac{1}{14} \le x \le \frac{1}{14}$

Thus the largest interval is $\left[-\frac{1}{14}, \frac{1}{14}\right]$.

Example 5. *Consider the* $\left|\frac{dy}{dx}\right| + |y| = 0$ *and initial condition y(0) = 0*

Solution: *Sum of two non – negative terms equal to zero iff they are separately equal to zero.*

$\Rightarrow$ *y(x) = 0 is a solution of given ODE which also satisfies the initial condition.*

Note: *Before separating the variables, check if R.H.S. becomes zero by putting y=0 as well as L.H.S. and y = 0 satisfies the initial condition then y = 0 is a solution of the given D.E.*

Example 6. *Consider IVP* $\frac{dy}{dx} = y^{\frac{1}{3}}$; *y(0) = 1 then how many solutions does the IVP has?*

Solution: *The given differential equation is* $\frac{dy}{dx} = y^{\frac{1}{3}} \Rightarrow \frac{dy}{y^{\frac{1}{3}}} = dx$

$\Rightarrow y^{\frac{2}{3}} = \frac{2}{3}(x+c)$

Here initial condition is y (0) = 1.

which gives $c = \frac{3}{2}$

so in this problem 'c' is unique

$\Rightarrow y^{\frac{2}{3}} = \frac{2}{3}\left(x + \frac{3}{2}\right)$ *is the unique solution of the given IVP.*

Example 7. *Consider IVP* $xy' - y = 2x^2$, *y(0) = 0* ...(1)

then how many solutions does the given IVP has?

Solution: *Given equation is* $x\frac{dy}{dx} - y = 2x^2 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = 2x$, *which is a linear in y.*

Thus I.F. $= e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

Solution of (1) is $y.\frac{1}{x} = \int 2x.\frac{1}{x}dx$

$\Rightarrow \frac{y}{x} = 2x + c \quad \Rightarrow y = 2x^2 + cx$

Since initial condition is y(0) = 0

$\Rightarrow$ *0 = 0, so here we can't find 'c'*

$\Rightarrow$ *Given IVP has infinite solutions.*

PRACTICE SET – 1

1. *The initial value problem;* $\dot{x}(t) = 3x^{2/3}, x(0) = 0$ *in an interval around t = 0 has*
(CSIR UGC NET SAMPLE PAPER)
(A) no solution
(B) a unique solution
(C) finitely many linearly independent solutions
(D) infinitely many linearly independent solutions.

2. *The initial value problem* $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0;\ y(0)=1,\ \left(\frac{dy}{dx}\right)_{x=0} = 0$ *has* **(GATE-2006)**
(A) a unique solution (B) no solution
(C) infinitely many solutions (D) two linearly independent solutions

3. *The initial value problem* $x\frac{dy}{dx} = y + x^2, x > 0; y(0) = 0,$ *has* **(GATE-2011)**
(A) infinitely many solutions (B) exactly two solutions
(C) a unique solution (D) no solution

4. *For the boundary value problem* $y'' + \lambda y = 0;\ y(0) = 0,\ y(1) = 0$. *There exists an eigen value* λ *for which there corresponds an eigen function in* $(0,1)$ *that* **(CSIR UGC NET JUNE-2012)**
(A) does not change sign (B) changes sign
(C) is positive (D) is negative

5. *If* $y = 3e^{2x} + e^{-2x} - \alpha x$ *is the solution of the initial value problem* $\frac{d^2y}{dx^2} + \beta y = 4\alpha x$, *y(0) = 4 and* $\frac{dy}{dx}(0) = 1$, *where* $\alpha, \beta \in \mathbb{R}$, *then* **(GATE-2017)**
(A) $\alpha = 3$ and $\beta = 4$ (B) $\alpha = 1$ and $\beta = 2$
(C) $\alpha = 3$ and $\beta = -4$ (D) $\alpha = 1$ and $\beta = -2$

§ 3.2. STURM – LIOUVILLE BOUNDARY VALUE PROBLEMS

Definition. *Consider a boundary value problem which consists of*

(i) *A second order homogeneous linear differential equation of the form*

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda r(x)]y=0, \qquad ...(5)$$

where p(x), q(x) and r (x) are real functions such that p(x) has a continuous derivative, q(x) and r(x) are continuous on $a \le x \le b$*; and p(x)>0 and r(x) > 0 for all values of x on a real interval* $a \le x \le b$*;* λ *is a parameter independent of x,*

(ii) *Two supplementary homogenous boundary conditions*

$$\left.\begin{aligned} A_1 y(a)+A_2 y'(a)=0,\\ B_1 y(b)+B_2 y'(b)=0,\end{aligned}\right\} \qquad ...(6)$$

where A_1, A_2, B_1 *and* B_2 *are real constants, such that* A_1 *and* A_2 *are not both zero; and* B_1 *and* B_2 *are not both zero.*

This type of boundary value problem is called a ***Sturm – Liouville Problem or Sturm – Liouville System***

Note : *Two important particular forms of the supplementary homogenous conditions (6) are*

(i) $y(a)=0,\ y(b)=0$ ...(7)

(ii) $y'(a)=0,\ y'(b)=0$ ...(8)

Example 1. *Consider the boundary value problem* $\dfrac{d^2y}{dx^2}+\lambda y=0,$ ...(9)

with $y(0)=0,\ y(\pi)=0$ ...(10)

Now, equation (9) may be written as $\dfrac{d}{dx}\left[1.\dfrac{dy}{dx}\right]+[0+\lambda.1]y=0.$ ...(11)

Recalling the Sturm – Liouville problem $\dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right]+[q(x)+\lambda r(x)]y=0,$

$y(a)=0,\ y(b)=0,$ *(Particular case)*

Now, equation (11) is of the same form as Sturm-Liouville problem, where p(x)=1, q(x)=0 and r(x)=1.

Example 2. *Find the non –trivial solutions of the Sturm Liouville problem*

$$\frac{d^2y}{dx^2}+\lambda y=0, \qquad ...(1)$$

$$y(0)=0,\ y(\pi)=0 \qquad ...(2)$$

Solution. *For finding the non – trivial solutions, we consider separately the three cases, namely $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. For each case, we first find the general solution of the differential equation (1) and then determine the two arbitrary constants involved, so that the supplementary conditions (2) are also satisfied. Now, we proceed as follows:*

Case I : When $\lambda = 0$

For the case when $\lambda = 0$, then differential equation (1) reduces to

$$\frac{d^2y}{dx^2} = 0, \qquad ....(3)$$

The general solution of equation is $y = c_1 + c_2x$. ...(4)

Applying given conditions (2) to the solution (4).

On applying $y(0)=0$, we get $c_1= 0$. ...(5)

On applying $y(\pi) = 0$, we get $c_1 + c_2\pi = 0$.

Using (5) i.e. $c_1 = 0$, we get $c_2 = 0$.

Thus, for the solution (4) to satisfy the conditions (2), we must have $c_1=0$ and $c_2=0$.

Therefore, solution (4) becomes $y(x) = 0$, for all values of x.

Thus, for the case when the parameter $\lambda= 0$, we get only the trivial solution of the given problem.

Case II : When $\lambda < 0$

Let $\lambda = -\alpha^2$, $\alpha \neq 0$

Then equation (1) gives $\frac{d^2y}{dx^2} - \alpha^2 y = 0$.

The auxiliary equation is $m^2 - \alpha^2 = 0$, which gives , $m = \alpha, -\alpha$

Therefore, the general solution of the equation is of the form

$$y = c_1e^{\alpha x} + c_2e^{-\alpha x} \qquad ...(6)$$

Applying the conditions (2) to the solution (6).

On applying $y(0) = 0$, we get $c_1 + c_2 = 0$ ...(7)

On applying $y(\pi) = 0$, we get $c_1e^{\alpha\pi} + c_2e^{-\alpha\pi} = 0$. ...(8)

Now, we must determine c_1 and c_2 such that the system consisting of (7) and (8) is satisfied. Obviously $c_1 = c_2 = 0$ is a solution of this system, but these values of constants give only the trivial solution of the given system. Therefore, we must seek non – zero values of c_1 and c_2 which satisfy (7) and (8). From the theory of linear equations, this system has non-zero solutions only if the determinant of the coefficients is zero.

Therefore, from equations (7) and (8), we must have $\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{vmatrix} = 0,$

which gives $e^{-\alpha\pi} - e^{\alpha\pi} = 0$,

or $e^{\alpha\pi} = e^{-\alpha\pi}$ or $e^{2\alpha\pi} = 1$, *or* $2\alpha\pi = 0$,

or $\alpha = 0$.

Therefore, the solution (6) is non –trivial if $\alpha=0$, *which is a contradiction because* $\alpha \neq 0$.

Thus, for the case when $\lambda < 0$, *there are no non –trivial solutions of the given problem.*

Case III : When $\lambda > 0$

Let $\lambda = \alpha^2, \alpha \neq 0$,

Then equation (1) gives $\frac{d^2y}{dx^2}+\alpha^2 y=0$.

The auxiliary equation is $m^2 + \alpha^2 = 0$, *which gives* $m = \pm i\alpha$.

Therefore, the general solution is of the form $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. ...(9)

Applying the conditions (2) to the solution (9).

On applying $y(0) = 0$, *we get* $c_1 \cos 0 + c_2 \sin 0 = 0$,

and hence $c_1 = 0$.

On applying $y(\pi) = 0$, *we get* $c_1 \cos \alpha\pi + c_2 \sin \alpha\pi = 0$ ...(10)

But $c_1 = 0$, *equation (10) reduces to* $c_2 \sin \alpha\pi = 0$, ...(11)

which gives either $c_2 = 0$ *or* $\sin \alpha\pi = 0$.

But if $c_2 = 0$, *then solution (9) reduces to unwanted trivial solution*

Thus, we must set $\sin \alpha\pi = 0$, ...(12)

for obtaining non –trivial solutions.

Now, $\sin \alpha\pi = 0$ *gives* $\alpha\pi = n\pi, n \in \mathbb{Z}$ *[as* $\alpha \neq 0$*]*

which gives $\alpha = n, n \in \mathbb{Z}$.

Therefore, solution (9) takes the form $y = c_2 \sin nx$. *As* c_2 *is arbitrary, therefore*

$y = c_n \sin nx$, *where* $n \in \mathbb{Z}$ *and* $\lambda = \alpha^2 = n^2$; *which are the required non –trivial solutions of the given problem.*

3.3.1. Characteristic Values (or Eigen Values) and Characteristic Functions (or Eigen Functions)

Consider the Sturm-Liouville problem consisting of the second order homogenous linear differential equation $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda r(x)]\,y=0$, ...(1)

and the supplementary conditions

$$\left.\begin{aligned} A_1 y(a) + A_2 y'(a) = 0, \\ B_1 y(b) + B_2 y'(b) = 0, \end{aligned}\right\} \quad ...(2)$$

where A_1, A_2, B_1 *and* B_2 *are real constants such that* A_1 *and* A_2 *are not both zero and* B_1 *and* B_2 *are not both zero. Then the values of the parameter* λ *in equation (1) for which there exist* ***non-trivial*** *solutions of the problem are called the* ***characteristic values or eigen values*** *of the problem.*

The corresponding non – trivial solutions themselves are called the ***characteristic functions or eigen functions*** *of the problem.*

Example 3. *Eigen value for BVP* $X''(x) + \lambda X(x) = 0$; $x(0) = 0 \quad X(\pi) + X'(\pi) = 0$ satisfy

(A) $\lambda + \tan \lambda\pi = 0$ *(B)* $\sqrt{\lambda} + \tan\lambda\pi = 0$

(C) $\sqrt{\lambda} + \tan\sqrt{\lambda}\pi = 0$ *(D)* $\lambda + \tan\sqrt{\lambda}\pi = 0$

Solution :

Case (i) *When* $\lambda = 0$

$X''(x) = 0 \Rightarrow X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$

$X'(x) = A$

$X(\pi) + X'(\pi) = 0$

$\Rightarrow A\pi + A = 0 \Rightarrow A(1 + \pi) = 0 \quad \Rightarrow A = 0$ *(Trivial solution)*

Case (ii) *When* $\lambda > 0$, *let* $\lambda = \mu^2$ $(\mu \neq 0)$

$X(x) = A\cos\mu x + B\sin\mu x = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$X(0) = 0 \quad \Rightarrow A = 0$

$X'(x) = B\mu\cos\mu x = B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$X(\pi) + X'(\pi) = 0 \quad \Rightarrow B\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0 \quad \Rightarrow B[\tan\sqrt{\lambda}\pi + \sqrt{\lambda}] = 0, B \neq 0$

$\Rightarrow \sqrt{\lambda} + \tan\sqrt{\lambda}\pi = 0$, *Non–trivial solution.*

Example 4. *Find the characteristic values and the corresponding characteristic functions of the Sturm – Liouville problem.*

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad ...(1)$$

with $y(0) - y'(0) = 0$, *...(2)*

$y(\pi) - y'(\pi) = 0$. *...(3)*

Solution: *For finding characteristic values and characteristic functions, we consider separately the three cases, namely* $\lambda = 0$, $\lambda < 0$ *and* $\lambda > 0$.

Case I : *When* $\lambda = 0$

For the case when $\lambda = 0$, *the differential equation (1) reduces to* $\frac{d^2y}{dx^2} = 0$. *...(4)*

The general solution of equation (4) is $y(x) = c_1x + c_2$. *...(5)*

Now, relation (5) gives $y'(x) = c_1$. *...(6)*

We now apply the conditions (2) and (3) to the solution (5).

On applying (2) i.e. $y(0) - y'(0) = 0$, *we get* $c_2 - c_1 = 0$, *or* $c_1 = c_2$.

Therefore, (5) gives $y(x) = c_1x + c_1$.

On applying (4) i.e. $y(\pi) - y'(\pi) = 0$, *we get* $c_1\pi + c_1 - c_1 = 0$,
or $c_1 = 0$ $[\text{as } \pi \neq 0]$
But $c_1 = c_2$, *therefore* $c_1 = c_2 = 0$.
Thus, the solution (5) becomes $y(x) = 0$, *for all values of* x. *Therefore, for the case when the parameter* $\lambda = 0$, *we get only the trivial solution of the given problem.*

***Case II.** When* $\lambda < 0$

Let $\lambda = -\alpha^2$, $\alpha \neq 0$.

Then equation (1) gives $\frac{d^2y}{dx^2} - \alpha^2 y = 0$. ...(7)

The Auxiliary equation is $m^2 - \alpha^2 = 0$, *which gives* $m = \alpha, -\alpha$.

Notes on Sturm – Liouville Boundary Value Problem

1. *All the Eigen Values of SLP are real.*

2. *Eigen Values of SLP are countable*

3. *Eigen functions corresponding to different eigen values, are L.I.*

4. *If* $\int_a^b f(x)\, g(x) = 0$ *over (a, b), then f(x) and g(x) are orthogonal.*

5. *Every second order ODE can be written in S-L form.*

6. *In each SLP, there is a one – parameter family of characteristic functions corresponding to each characteristic value and two characteristic functions corresponding to same characteristic value are merely non – zero constant multiple of each other i.e. The SLP cannot have two L.I. eigen function corresponding to the same eigen values.*

7. *Each characteristic function* ϕ_n *corresponding to the characteristic value* λ_n $(n = 1, 2, 3, ...)$ *has exactly* $(n - 1)$ *zeroes in (a, b).*

8. *In SLP, each eigen function can be made real value by multiplying it by an appropriate non –zero constant.*

9. *The characteristic function corresponding to the different characteristic values of a SLP are orthogonal on the given interval.*

PRACTICE SET – 2

1. *Let $y:\mathbb{R}\to\mathbb{R}$ be differentiable and satisfy the ODE:*

$$\begin{cases} \dfrac{dy}{dx} = f(y), x\in\mathbb{R} \\ y(0)=y(1)=0 \end{cases}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function. Then **(CSIR UGC NET DEC-2014)**

(A) $f(x)=0$ if and only if $x\in\{0,1\}$ *(B) y is bounded*

(C) y is strictly increasing *(D) dy/dx is unbounded*

2. *Consider the boundary value problem (BVP)*

$$u''+\lambda u=0, u(0)=u'(\pi)=0, u'\equiv\frac{du}{dx},\ u''\equiv\frac{d^2u}{dx^2}, \lambda\in\mathbb{C}$$

Let k denote a non negative integer. Then, which of the following are correct?

(CSIR UGC NET JUNE-2013)

(A) There exist eigenvalues of the BVP and the corresponding eigen functions constitute an orthogonal set.

(B) The eigenvalues of the BVP are $\left(k+\frac{1}{2}\right)^2$ with the corresponding eigen functions $\left\{\sin\left(k+\frac{1}{2}\right)x\right\}$.

(C) The eigenvalues of the BVP are $(k+1)^2$ with the corresponding eigenfunctions $\{\sin(k+1)x\}$

(D) There exist no non-real eigenvalue for the BVP.

3. *The difference between the least two eigenvalues of the boundary value problem*

$y''+\lambda y=0,\quad 0<x<\pi$

$y(0)=0,\quad y'(\pi)=0$, *is equal to*_____ **(GATE-2016)**

4. *The problem*

$$\left.\begin{array}{l} -y''+(1+x)y=\lambda y, x\in(0,1) \\ y(0)=y(1)=0 \end{array}\right\}$$ *has a non zero solution* **(CSIR UGC NET JUNE-2016)**

(A) for all $\lambda<0$. *(B) for all $\lambda\in[0,1]$.*

(C) for some $\lambda\in(2,\infty)$. *(D) for a countable number of λ's.*

§ 3. 3. GREEN'S FUNCTION

Consider a linear homogeneous differential equation of order n: L(y)=0 ...(1)

where L is a differential operator $L\equiv P_o(x)\frac{d^n}{dx^n}+P_1(x)\frac{d^{n-1}}{dx^{n-1}}+...+P_n(x)$ ...(2)

where the functions $P_o(x), P_1(x),...,P_n(x)$ are continuous on [a,b] $P_0(x)\neq 0$ on [a,b] and the boundary conditions are

$V_k(y) = a_k y(a) + a_k^{(1)} y^1(a) + ... + a_k^{(n-1)} y^{(n-1)}(a) + \beta_k y(b) + \beta_k^{(1)} y^1(b) + ... + \beta_k^{(n-1)} y^{(n-1)}(b)$
$=0$; $(k=1,2,...,n)$ ...(3)

where the linear forms $V_1, ..., V_n$ in $y(a), y'(a), ..., y^{(n-1)}(a), y(b), y'(b), ..., y^{(n-1)}(b)$ are linearly independent.

If the homogeneous boundary value problem given by equation (1) to (3) has only the trivial solution y(x)=0 then the green's function G(x,t) constructed for any point a<t<b for BVP equations (1) to (3) which has the following four properties:

(a) *In each of the intervals [a,t) and [t,b] the function G(x,t) considered as a function of X is non-trivial solution of equation (1) i.e., L[G]=0.*

(b) *G(x,t) is continuous in X for fixed t and has continuous derivative with respective to x upto order (n-2) inclusive for* $a \le x \le b$.

(c) *(n-1)th derivative of G(x,t) with respect to x=t has the discontinuity of fixed kind and the jump being equal to* $-1/P_0(t)$ *i.e.* $\left[\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,t)\right]_{x=t^+} - \left[\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,t)\right]_{x=t^-} = \frac{-1}{P_0(t)}$

(d) *G(x,t) satisfies boundary conditions i.e.,* $V_k(G)=0$; $(k=1,2,...,n)$

Result: *If the boundary value problem given by (1) to (3) has only the trival solution y(x)=0 then the operator L has a unique green's function G(x,t)*

Self Adjoint Equation: *If the coefficients* $a_0(x), a_1(x), a_2(x)$ *in the differential equation*

$a_0(x)\ y'' + a_1(x)y' + a_2(x)y = 0$ *are continuous on* $a \le x \le b$ *and* $a_0(x) \ne 0$ *on* $a \le x \le b$. *Then the above differential equation can be transformed into the equivalent self adjoint equation* $[r(x)y']' + P(x)y = 0$ *where* $r(x) = e^{\int \frac{a_1(x)}{a_0(x)}dx}$; $P(x) = \frac{a_2 x}{a_0(x)} r(x)$

Note:

(1) $a_0(x)$ *has a continuous second order derivative*

(2) $a_1(x)$ *has continuous first order derivative*

(3) $a_2(x)$ *is continuous for all* $x \in [a,b]$

Result: *The necessary and sufficient condition that the above equation is self adjoint is* $a'_0(x) = a_1(x)$ *on* $x \in [a,b]$ *and in this case, the differential equation may be written in the form* $[a_0(x)y']' + a_2(x)y = 0$

e.g. Consider the equation $x^3y'' + 3x^2y' + y = 0$. *Here* $a_0(x) = x^3; a_1(x) = 3x^2$

$\Rightarrow a'_0(x) = a_1(x)$

here, the required self adjoint equation is $[a_0(x)y']' + a_2(x)y = 0 \Rightarrow (x^3y')' + y = 0$

Result: *If the boundary value problem is self adjoint then green's function is symmetric i.e., G(x,t)=G(t,x). The converse is also true.*

Example 1. *Find the green's function of the boundary value problem* $y''=0; y(0)=y(l)=0$

Solution: *Given boundary value problem is* $y''=0$ ...(1)

The general solution of (1) is $y(x)=Ax+B$ ...(2)

Given conditions are $y(0)=0=y(l)$

$y(0)=0 \Rightarrow B=0$

$y(l)=0 \Rightarrow Al=0 \Rightarrow A=0$

$\Rightarrow$ (2) *yields only the trivial solution for the given boundary value problem*

Hence, the green's function exists and is given by $G(x,t)=\begin{cases} a_1x+a_2; 0\le x<t \\ b_1x+b_2; t<x\le l \end{cases}$ ...(3)

Now the proposed green's function must satisfy the following properties :

(a) *G(x,t) is continuous at x=t i.e.,* $a_1t+a_2=b_1t+b_2$ *or* $(b_1-a_1)t+(b_2-a_2)=0$ ...(4)

(b) *The derivative of G has a discontinuity of magnitude* $-1/P_0(t)$ *at the point x=t, where* $P_0(x)$= *coefficient of highest order derivative in (1) which is 1*

$$\therefore \left(\frac{\partial G}{\partial x}\right)_{x=t^+}-\left(\frac{\partial G}{\partial x}\right)_{x=t^-}=-1$$

$\Rightarrow b_1-a_1=-1$ ...(5)

So (4) $\Rightarrow -t+(b_2-a_2)=0 \quad \Rightarrow b_2-a_2=t$ ...(6)

(c) *G(x,t) must satisfy the boundary conditions i.e..* $G(0,t)=0 \Rightarrow a_2=0$ ...(7)

and $G(l,t)=0 \Rightarrow b_1l+b_2=0$ ...(8)

Put the value of a_2 *in (6); we get* $b_2=t$ ...(9)

Now (8) $\Rightarrow b_1l+t=0 \quad \Rightarrow b_1=\frac{-t}{l}$ ...(10)

$\therefore$ (5) $\Rightarrow b_1=-1+a_1$

$\Rightarrow a_1=\frac{-t}{l}+1 \quad \Rightarrow a_1=\left(\frac{l-t}{l}\right)$ ...(11)

Substituting the above values in (3); the required green's function of the given boundary value problem is given by

$$G(x,t)=\begin{cases} \left(1-\frac{t}{l}\right)x; \; 0\le x<t \\ \frac{-tx}{l}+t; \quad t<x\le l \end{cases}$$

$$\Rightarrow G(x,t)=\begin{cases} \frac{x}{l}(l-t); \; 0\le x<t \\ \frac{t}{l}(l-x); \; t<x\le l \end{cases}$$

, is required Green's function satisfying differential equation and boundary conditions. ...(12)

Example 2. *Construct Green's function for the BVP* $u'' - u = 0; u(0) = u(1) = 0$

Solution: *Given equation is* $u'' - u = 0$

$\Rightarrow (D^2 - 1)u = 0$ ...(1)

$\Rightarrow m^2 - 1 = 0 \Rightarrow u = a\cosh x + b\sinh x$, *where a and b are constants* ...(2)

Given boundary conditions are u(0)=0=u(1)

$u(0)=0 \Rightarrow 0=a\cosh 0 \Rightarrow a=0$

$u(1)=0 \Rightarrow b\sinh 1 = 0 \Rightarrow b=0$ ($\because \sinh 1 \neq 0$)

$\Rightarrow$ *(2) yield only the trivial solution for the given boundary value problem*

Hence, the green's function exists and is given by

$$G(x,t) = \begin{cases} a_1 \cosh x + a_2 \sinh x; \ 0 \le x < t \\ b_1 \cosh x + b_2 \sinh x; \ t < x \le 1 \end{cases} \quad ...(3)$$

Now, the proposed green's function must satisfy the following properties:

(a) *G(x,t) is continuous at x=t i.e.,* $a_1 \cosh t + a_2 \sinh h = b_1 \cosh t + b_2 \sinh t$

$\Rightarrow (b_1 - a_1)\cosh t + (b_2 - a_2)\sinh t = 0$ ...(4)

(b) *The derivative of G has a discontinuity of magnitude* $-1/P_0(t)$

$$\Rightarrow \left[\frac{\partial G}{\partial x}\right]_{x=t^{+0}} - \left[\frac{\partial G}{\partial x}\right]_{x=t^{-0}} = \frac{-1}{P_o(t)}$$

$\Rightarrow [b_1 \sinh t + b_2 \cosh t] - [a_2 \cosh t + a_1 \sinh t] = -1$

$\Rightarrow (b_1 - a_1)\sinh t + (b_2 - a_2)\cosh t = -1$ ...(5)

Solve (4) and (5) we get $b_2 - a_2 = \dfrac{-\cosh t}{\cosh 2t}$ ...(6)

Now (4)

$\Rightarrow (b_1 - a_1)\cosh t - \dfrac{\cosh t \sinh t}{\cosh 2t} = 0 \quad \Rightarrow b_1 - a_1 = \dfrac{\sinh t}{\cosh 2t}$ ...(7)

(c) *G(x,t) must satisfy the boundary conditions i.e., G(0,t)=0* $\Rightarrow a_1 = 0$

$\Rightarrow b_1 = \dfrac{\sinh t}{\cosh 2t}$ *and G(1,t)=0* $\Rightarrow b_1 \cosh 1 + b_2 \sinh 1 = 0$

$\Rightarrow b_2 = \dfrac{-b_1 \cosh 1}{\sinh 1} \Rightarrow b_2 = \dfrac{-\sinh t \cosh 1}{\sinh 1 \cosh 2t}$ ...(8)

$\therefore$ (6) $\Rightarrow a_2 = b_2 + \dfrac{\cosh t}{\cosh 2t} = \dfrac{-\sinh t \cosh 1}{\sinh 1 \cosh 2t} + \dfrac{\cosh t}{\cosh 2t} = \dfrac{-\sinh t \cosh 1 + \cosh t \sinh 1}{\sinh 1 \cosh 2t}$

$\Rightarrow a_2 = \dfrac{\sinh(1-t)}{\sinh 1 \cosh 2t}$ ...(9)

Substituting the above value in (3) we get $a_1 \cosh x + a_2 \sinh x = \dfrac{\sinh(1-t)\sinh x}{\sinh 1 \cosh 2t}$

and $b_1 \cosh x + b_2 \sinh x = \frac{\sinh t \cosh x}{\cosh 2t} - \frac{\sinh t \cosh 1 \sinh x}{\cosh 2t \sinh 1}$

$= \frac{\sinh t[\cosh x \sinh 1 - \cosh 1 \sinh x]}{\cosh 2t \sinh 1} = \frac{\sinh t \sinh(1-x)}{\cosh 2t + \sinh 1}$

hence, $G(x,t) = \begin{cases} \frac{\sinh x \sinh(1-t)}{\sinh 1 \cosh 2t}; 0 \le x < t \\ \frac{\sinh t \sinh(1-x)}{\sinh 1 \cosh 2t}; t < x \le 1 \end{cases}$ *is required Green's function*

Exercise 1. *Find the green's function for the boundary value problem* $\frac{d^2y}{dx^2} + \mu^2 y = 0; y(0) = y(1) = 0$

Exercise 2. *Find green's function for the differential equation* $y'' + y' = 0;$ *y(0)=y(1);* $y'(0) = y'(1)$

PRACTICE SET – 3

1. *The Green's function* $G(x,\xi)$, $0 \le x, \xi \le 1$ *of the boundary value problem* $y'' + \lambda y = 0,\ y(0) = 0 = y(1)$ *is*

(CSIR UGC NET JUNE-2011)

(A) symmetric in x and ξ

(B) continuous at x= ξ

(C) $\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi^-} - \left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi^+} = -1$

(D) $\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi^-} - \left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi^+} = 1$

2. *The green's function G(x,t) of the boundary value problem* $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1,$ *y(0)=y(1)=0 is*

$G(x,t) = \begin{cases} f_1(x,t), & \text{if } x \le t \\ f_2(x,t), & \text{if } t \le x \end{cases}$, *where* ***(CSIR UGC NET DEC-2011)***

(A) $f_1(x,t) = -\frac{1}{2}t(1-x^2), f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$

(B) $f_1(x,t) = -\frac{1}{2x}t^2(1-x^2), f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$

(C) $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2), f_2(x,t) = -\frac{1}{2}t(1-x^2)$

(D) $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2), f_2(x,t) = -\frac{1}{2x}t^2(1-x^2)$

3. The solution of the differential equation $\frac{d^2y}{dx^2} = f(x)$; $x \in (0,1)$

$y(0) = y(1) = 0$ is given by $y(x) = \int_0^1 G(x,\xi) f(\xi) d\xi$ where **(CSIR UGC NET DEC-2012)**

(A) $G(x,\xi) = \begin{cases} x(\xi-1); x\le\xi \\ \xi(x-1); x>\xi \end{cases}$

(B) $G(x,\xi) = \begin{cases} x^2(\xi-1); x\le\xi \\ \xi^2(x-1); x>\xi \end{cases}$

(C) $G(x,\xi) = \begin{cases} x(\xi^2-1); x\le\xi \\ \xi(x^2-1); x>\xi \end{cases}$

(D) $G(x,\xi) = \begin{cases} \sin x(\xi-1); x\le\xi \\ \sin\xi(x-1); x>\xi \end{cases}$

4. Consider the boundary value problem (BVP) $u'' = -f, u(0) = u''(1) = 0$ on [0,1] where $u' \equiv \frac{du}{dx}$ and $u'' \equiv \frac{d^2u}{dx^2}$. Assume f(x) is a real-valued continuous function on [0,1]. Then, which of the following is/are correct? **(CSIR UGC NET JUNE-2013)**

(A) The Green's function $G(x,\xi), (x,\xi) \in [0,1]\times[0,1]$, for the above BVP is

$$G(x,\xi) = \begin{cases} x & for \quad 0 \le x \le \xi \\ \xi & for \quad \xi \le x \le 1 \end{cases}$$

(B) Both G and $\frac{\partial G}{\partial x}$ are continuous on $[0,1]\times[0,1]$ with $\frac{\partial^2 G}{\partial x^2}$ having a discontinuity along $x=\xi$

(C) $G(x, \xi)$ satisfies the homogeneous equation $u'' = 0$ and $0 \le x < \xi$ and $\xi < x \le 1$

(D) The solution of the given boundary problem is $u(x) = \int_0^x \xi f(\xi) d\xi + \int_x^1 x f(\xi) d\xi$

5. Let G(x,y) be the Green's function of the boundary value problem

$[(1+x)u']' + (\sin x)u = 0, x \in [0,1]$ $u(0)=u(1)=0$. Then the function g defined by

$g(x) = G\left(x, \frac{1}{2}\right), x \in [0,1]$ **(CSIR UGC NET DEC-2013)**

(A) is continuous
(B) is discontinuous at $x=1/2$
(C) is differentiable
(D) does not have the left derivative at $x=1/2$,

KEY POINTS

- *If in D.E.* $\frac{dy}{dx} = f(x),\ y(x_0) = y_0$, *if f(x) satisfies Lipschitz condition, it has unique solution otherwise it has infinitely many or no solutions.*

- *If* $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ *and* $|x - x_0| \le a,\ |y - y_0| \le b$, *f(x, y) is bounded over the given interval then IVP has atleast one solution in the interval* $|x - x_0| \le h$, *where* $h = \min(a, b/m)$ *and* $|f(x)| \le m$ *over the region given R.*

- *The function f is said to satisfy a Lipschitz condition (with respect to y) in D if* $\exists$ *a positive constant k such that* $|f(x, y_1) - f(x, y_2)| \le k|y_1 - y_2|$, *D is domain of f. The constant k is called the Lipschitz constant.*

- *If the boundary value second order differential equation of the form* $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$, *where p(x), q(x) and r(x) are real functions, p(x) has a continuous derivative on* $a \le x \le b$, $p(x) > 0$ *and* $r(x) > 0$ *for all values of x,* λ *is a parameter independent of x with two supplementary homogenous boundary conditions* $\left.\begin{array}{l} A_1 y(a) + A_2 y'(a) = 0, \\ B_1 y(b) + B_2 y'(b) = 0, \end{array}\right\}$. *Then this type of boundary value problem is called a Sturm – Liouville Problem.*

- *In the Sturm-Liouville problem the values of the parameter* λ *for which there exist non-trivial solutions of the problem are called the characteristic values or eigen values of the problem.*

- *The non-trivial solutions corresponding to eigen values are called the characteristic functions or eigen functions of the problem.*

- *For differential equation* $\frac{dy}{dx} = Ky^{\alpha}$, *where* $\alpha < 1$, *K is a constant and* $y(0) = 0$, *it has infinitely many solutions.*

- *When in boundary value problems, we get only trivial solution we find Green's function to get non-trivial solutions.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *Solve* $\frac{dy}{dx} = y^{\frac{1}{2}}$ *with* $y(0) = 0$

Solution: *The given differential equation is* $\frac{dy}{dx} = y^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{\sqrt{y}} = dx$$

$$\Rightarrow \text{Integrating } \frac{\sqrt{y}}{\frac{1}{2}} = x + c \quad \Rightarrow 2\sqrt{y} = x + c \quad \Rightarrow y = \frac{1}{4}(x+c)^2 \quad \text{as } y(0) = 0 \Rightarrow \; y = \frac{1}{4}x^2$$

Moreover, $y = \begin{cases} \frac{1}{4}(x-1)^2 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$ *, is also solution of differential equation*

In general, $y = \begin{cases} \frac{1}{4}(x-c)^2, & x \geq c \\ 0 & \text{, otherwise} \end{cases}$

hence, this differential equation has infinitely many solutions

Example 2. *Find eigen values and eigen functions of* $[ty'(t)]' + \left(\frac{\lambda}{t}\right)y(t) = 0,\; y'(1) = y'(e^{2\pi}) = 0$

Solution: *The given differential equation is* $[ty'(t)]' + \left(\frac{\lambda}{t}\right)y(t) = 0$

$\Rightarrow t^2y''(t) + ty'(t) + \lambda y = 0$

A.E. is $(t^2D^2 + tD + \lambda)\,y = 0$, *where* $D = \frac{d}{dt}$ ...(i)

It can be reducible to a homogenous equation with constant coefficients put $t = e^z \Rightarrow tD = \theta$, $t^2D^2 = \theta(\theta - 1)$

Equation (i) reduces to $\theta(\theta - 1) + \theta + \lambda = 0$

$\theta^2 + \lambda = 0$ ...(ii)

Here, it arises 3 cases:-

Case I, $\lambda = 0$

From (ii), $\theta^2 = 0 \Rightarrow y = Az + B \Rightarrow y = A\log t + B$ *as* $y'(1) = 0$ *and* $y'(e^{2\pi}) = 0$

$y'(t) = \frac{A}{t} \Rightarrow A = 0 \Rightarrow y(t) = B$, *B is arbitrary*

Taking $B = 1$ *(say)*

$\Rightarrow y(t) = 1$ *is eigen function corresponding to eigen value* $\lambda = 0$

Case II Let $\lambda = -\mu^2$, where $\mu \neq 0$

$\Rightarrow \theta^2 = \mu^2 \Rightarrow \theta = \pm\mu$

The solution is $y = Ae^{\mu z} + Be^{-\mu z} \Rightarrow y'(t) = A\mu\, t^{\mu-1} + B(-\mu)\, t^{-\mu-1}$

$y'(1) = A\mu - B\mu = 0$

$y'(e^{2\pi}) = A\mu\, e^{2\pi(\mu-1)} - B\mu e^{-2\pi(\mu+1)} = 0 \Rightarrow A - B = 0$ as $\mu \neq 0$ and $Ae^{2\pi\mu} - Be^{-2\pi\mu} = 0$

On solving gives $A = B = 0$

$\Rightarrow y(t) = 0$, which is trivial solution. Hence it is not an eigen function.

Case III: Let $\lambda = \mu^2$, where $\mu \neq 0 \Rightarrow \theta^2 = -\mu^2 \Rightarrow \theta = \pm i\mu$

The solution is $y = A\cos\mu z + B\sin\mu z$

$y' = A\cos\mu(\log t) + B\sin\mu(\log t)$

$$y'(t) = -\left(\frac{\mu A}{t}\right)\sin(\mu\log t) + \frac{B\mu}{t}\cos(\mu\log t)$$

Given $y'(1) = 0 \Rightarrow y'(1) = 0 + B\mu = 0 \Rightarrow B\mu = 0$

As, $\mu \neq 0 \quad \Rightarrow B = 0$

$y'(e^{2\pi}) = 0$

$$-\frac{\mu A}{2\pi}\sin(\mu\, 2\pi) = 0 \Rightarrow 2\pi\mu = n\pi,\ n = 1, 2, 3, \ldots$$

$\mu = \frac{n}{2}$ (n = 0 is not considered as it done in Case I)

$$\Rightarrow y(t) = A\cos\left(\frac{n}{2}\log t\right) \text{ with } \lambda = \mu^2 = \frac{n^2}{4}$$

So, the required eigen functions $y_n(t)$ with the corresponding eigenvalues λ_n are

$$y_n(t) = \cos\left(\frac{n}{2}\log t\right),\ n = 1, 2, 3, \ldots,\ \lambda_n = \frac{n^2}{4} \text{ and } y(t) = 1, \text{ with } \lambda = 0.$$

Example 3. For initial value problem $\frac{dy}{dx} = y^2 + \cos^2 x,\ y(0) = 0$ find the interval of existence of its solution given that $R = \left\{(x, y): 0 \le x \le a, |y| \le b, a > \frac{1}{2}, b > 0\right\}$ **(CSIR UGC NETJUNE 2015)**

Solution: Let $f(x, y) = y^2 + \cos^2 x$

$|f(x, y)| = |y^2 + \cos^2 x| \le |y|^2 + |\cos x|^2 \le b^2 + 1$

$$\left|\frac{\partial f}{\partial y}\right| = |2y| = 2b = K$$

$\Rightarrow f(x, y)$ satisfies Lipschitz condition

The y(x) exists for $0 \le x \le h = \min\left(a, \frac{b}{b^2+1}\right)$ ...(i)

Let $g(b) = \frac{b}{b^2+1} \Rightarrow g'(b) = \frac{1-b^2}{(b^2+1)^2}$

put $g'(b) = 0$ *gives* $b = \pm 1$

$g''(b) < 0$ *for* $b = 1, -1$

$\Rightarrow b = \pm 1$, *both are points of maxima.*

$g(1) = \frac{1}{2} \quad g(-1) = \frac{-1}{2}$

Maximum value of $\frac{b}{b^2+1}$ *is* $\frac{1}{2}$

$\Rightarrow h = \frac{1}{2}$ *[from (i)]*

So, solution exists in largest interval $0 \le x \le \frac{1}{2}$.

Example 4. *Derive the transcendental equation for determining* λ *for BVP*

$y'' + \lambda 2y = 0; \; y(0) = 0$

$y(1) = y'(1)$

Solution:

Case I: *If* $\lambda > 0$

$y = A\cos\lambda x + B\sin\lambda x$ *As,* $y(0) = 0 \Rightarrow A = 0 \Rightarrow y = B\sin\lambda x \Rightarrow y' = B\lambda\cos\lambda x$

$y(1) = y'(1)$

$\Rightarrow B\sin\lambda = B\lambda\cos\lambda \Rightarrow B(\sin\lambda - \lambda\cos\lambda) = 0$

For non–trivial solution, $B \ne 0$

$\Rightarrow \sin\lambda - \lambda\cos\lambda = 0$

Case II: *If* $\lambda = 0$

$y'' = 0 \Rightarrow y = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y(1) = y'(1)$

$\Rightarrow A = A$

$\therefore y(x) = Ax$ *is the eigen function corresponding to eigen value* $\lambda = 0$.

Example 5. *Let* P *be a polynomial of degree* N, *with* $N \ge 2$. *Then the initial value problem* $u'(t) = P(u(t)), u(0) = 1$, *has always* ***(CSIR UGC NET JUNE-2011)***

(A) A unique solution in $\mathbb{R}$

(B) N number of distinct solutions in $\mathbb{R}$

(C) No solution in any interval containing 0 for some P.

(D) A unique solution in an interval containing 0

Solution : (4) Given $u'(t) = P(u(t))$, $u(0) = 1$

Consider $P(x) = x^2$

$\therefore u'(t) = u^2(t) \Rightarrow \frac{du}{dt} = u^2 \Rightarrow \frac{du}{u^2} = dt$, Integrate on both sides

$\Rightarrow \frac{-1}{u} = t + C$

Also $u(0) = 1 \Rightarrow C = -1$

$\frac{-1}{u} = t - 1 \Rightarrow u = \frac{1}{1-t}$

which does not exist at $t = 1$ but exist in an interval not containing 0.

$\therefore$ Option 'D' is true.

Example 6. Let $\frac{d^2y}{dx^2} - q(x)y = 0$, $0 \le x < \infty$, $y(0)=1$, $\frac{dy}{dx}(0) = 1$, where $q(x)$ is a positive monotonically increasing continuous function. Then **(CSIR UGC NET DEC-2011)**

(A) $y(x) \to \infty$ as $x \to \infty$

(B) $\frac{dy}{dx} \to \infty$ as $x \to \infty$

(C) $y(x)$ has finitely many zeros in $[0, \infty)$

(D) $y(x)$ has infinitely many zeros in $[0, \infty)$

Solution: (A,B,C) Let $q(x)=1$ ($\because$ $q(x)$ is monotonically increasing and continuous function)

Then the given equation becomes

$$\frac{d^2y}{dx^2} - y = 0$$

$$(D^2 - 1)y = 0$$

$$\Rightarrow y = c_1 e^t + c_2 e^{-t} \Rightarrow y' = c_1 e^t - c_2 e^{-t}$$

As $y(0) = 1 \Rightarrow 1 = c_1 + c_2$

Also $\frac{dy}{dx}(0) = c_1 - c_2 = 1$

$\Rightarrow c_1 = 1, \; c_2 = 0 \Rightarrow y(x) = e^t$

$\Rightarrow y(x) \to \infty$ as $x \to \infty$ and $\frac{dy}{dx} \to \infty$ as $x \to \infty$

Also, $y(x)$ has finitely many zeroes in $[0, \infty)$

Example 7. *The solution to the initial value problem* $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=3e^{-t}\sin t,\ y(0)=0$ *and* $\frac{dy}{dt}(0)=3$, *is*

(GATE-2014)

(A) $y(t)=e^{t}(\sin t+\sin 2t)$ *(B)* $y(t)=e^{-t}(\sin t+\sin 2t)$

(C) $y(t)=3e^{t}\sin t$ *(D)* $y(t)=3e^{-t}\sin t$

Solution: (B) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=3e^{-t}\sin t$, *where* $y(0)=0, \frac{dy}{dt}(0)=3$

Auxiliary equation of $(D^2+2D+5)\,y=3e^{-t}\sin t$ *is* $(m^2+2m+5)=0 \Rightarrow m=-1\pm 2i$

$\Rightarrow y_c=e^{-t}(C_1\cos 2t+C_2\sin 2t)$

Particular integral is $3\frac{1}{D^2+2D+5}e^{-t}\sin t=3e^{-t}\frac{1}{(D-1)^2+2(D-1)+5}\sin t$

$=3e^{-t}\frac{1}{D^2+4}\sin t=3e^{-t}\frac{1}{-1+4}\sin t=e^{-t}\sin t$

$\Rightarrow y=e^{-t}(C_1\cos 2t+C_2\sin 2t)+e^{-t}\sin t$, *given* $y(0)=0 \Rightarrow C_1=0$

Therefore, $y=e^{-t}C_2\sin 2t+e^{-t}\sin t$

Given $y'(0)=3 \Rightarrow C_2=1 \Rightarrow y(t)=e^{-t}(\sin 2t+\sin t)$

ASSIGNMENT - 3.1

NOTE: CHOOSE THE BEST OPTION

1. The differential equation $\frac{dy}{dt} = \sqrt{|y|}$, for 0<y<10 and y(0)=0
   (A) has unique solution
   (B) has no solution
   (C) has two independent solutions
   (D) has infinite solutions

2. For the Sturm – Liouville problem $(1+x^2)y'' + 2xy' + \lambda x^2 y = 0$ with $y'(1) = 0$ and $y'(10) = 0$ the eigen values, λ satisfy
   (A) $\lambda \geq 0$
   (B) $\lambda < 0$
   (C) $\lambda \neq 0$
   (D) $\lambda \leq 0$

3. For IVP $y' = 2y^{1/2}, y(0) = 0$ which one is correct in a neighbourhood of 0?
   (A) It has unique solution
   (B) It has no solution
   (C) Solution exists but not uniquely
   (D) None of the above

4. The ordinary differential equation $x\frac{dy}{dx} - y = 2x^2$ with initial conditions y(0)=0, has
   (A) no solution
   (B) a unique solution
   (C) two distinct solutions
   (D) an infinite number of solutions

5. The initial value problem $x\frac{dy}{dx} = y, y(0) = 0, x \geq 0$ has
   (A) no solution
   (B) a unique solution
   (C) exactly two solutions
   (D) uncountably many solutions

6. The solution of $\frac{dy}{dx} = y^2, y(0) = 1$ exists for all
   (A) $x \in (-\infty, 1)$
   (B) $x \in [0,a]$ where $a>1$
   (C) $x \in (-\infty, \infty)$
   (D) $x \in [1,a]$ where $a>1$

7. For the Initial value problem (I.V.P) : $y' = f(x, y)$ with $y(0) = 0$, which of the following statements is true?
   (A) $f(x, y) = \sqrt{xy}$ satisfies Lipschitz's condition and so I.V.P has unique solution.
   (B) $f(x, y) = \sqrt{xy}$ does not satisfy Lipschitz's condition and so I.V.P. has no solution.
   (C) $f(x, y) = |y|$ satisfies Lipschitz's condition and so I.V.P. has unique solution.
   (D) $f(x, y) = |y|$ does not satisfy Lipschitz's condition still I.V.P. has unique solution.

8. The set of all eigenvalues of the Strum–Liouville problem $y'' + \lambda y = 0, y'(0) = 0, y'\left(\frac{\pi}{2}\right) = 0$, is given by
   (A) $\lambda = 2n$, n = 1, 2, 3, ...
   (B) $\lambda = 2n$, n = 0, 1, 2, 3, ...
   (C) $\lambda = 4n^2$, n = 1, 2, 3, ...
   (D) $\lambda = 4n^2$, n = 0, 1, 2, 3, ...

9. *If y(x) is the solution of the differential equation* $\frac{dy}{dx} = 2(1 + y)\sqrt{y}$ *satisfying* $y(0) = 0$; $y(\pi/2) = 1$, *then the largest interval (to the right of origin) on which the solution exists is*
(A) $[0, 3\pi/4)$ (B) $[0, \pi)$ (C) $[0, 2\pi)$ (D) $[0, 2\pi/3)$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

10. *For the initial value problem (IVP)* $\frac{dy}{dx} = f(x, y)$ *with* $y(0)=0$, *which of the following is/are true?*
(A) *If f(x,y) satisfies Lipschitz's condition, then IVP has unique solution*
(B) *IVP may not be unique although f(x,y) is continuous*
(C) *IVP does not satisfy Lipschitz's condition still IVP has unique solution*
(D) *IVP does not satisfy Lipschitz's condtion and so IVP has no solution*

11. *In a Sturm-Liouville Problem,* $[r(x)y']' + [q(x) + \lambda p(x)]y = 0$; $\alpha y(a) + \beta y'(a) = 0$ *and* $\gamma y(b) + \delta y'(b) = 0$, *which of the followings is/are true for* $x \in [a, b]$?
(A) $r'(x)$ *is continuous and* $p(x) > 0$
(B) *Atleast two from* $\alpha, \beta, \gamma, \delta$ *must be non zero*
(C) *Atleast one from* α, β *is non zero and atleast one from* γ, δ *is non zero*
(D) $\int_a^b p(x) y_m(x) y_n(x) dx = 0$, *where* $y_m(x)$ *and* $y_n(x)$ *are two eigen functions.*

12. *Which of the followings is/are true about Sturm-Liouville Problem (SLP)?*
(A) *All eigen values of SLP are real and non negative.*
(B) *Eigen functions corresponding to different eigen values are orthogonal with respect to weight function.*
(C) *SLP has always an eigen function.*
(D) *For each eigen values of a SLP there exists only one linearly independent eigen function.*

13. *Given, continuous function* $f(x,y) = y^{2/3}$ *on rectangle* $|x| \le 1, |y| \le 1$, *which of the following are true?*
(A) *This function satisfies Lipschitz condition on a rectangle.*
(B) *This function does not satisfy Lipschitz condition on a rectangle.*
(C) $\left|\frac{\partial f}{\partial y}\right| >$ *constant for* $y \ne 0$
(D) $\left|\frac{\partial f}{\partial y}\right| \le$ *constant for* $y = 0$

14. *Given, the initial value problem* $\frac{dy}{dx} = y^2$, $y(1) = -1$, *then*
(A) *there exist atleast one solution*
(B) *there does not exist any solution*
(C) *the initial value problem has no unique solution*
(D) *the given problem possesses a unique solution when* $|x - 1| < \frac{1}{4}$.

ASSIGNMENT - 3.2

NOTE: CHOOSE THE BEST OPTION

1. *If S is either a rectangle* $|x-x_0|\le a, |y-y_0|\le b (a,b>0)$ *or strip* $|x-x_0|\le a, |y|<\infty (a>0)$ *and if f is real valued continuous function defined on S and* $\frac{\partial f}{\partial y}$ *exist and also,* $\left|\frac{\partial}{\partial y} f(x,y)\right| \le k, (x,y)\in S$ *for a positive constant k, then*
   (A) *f satisfies Lipschitz condition on S with Lipschitz constant k.*
   (B) *f does not satisfy Lipschitz condition on S with Lipschitz constant k.*
   (C) *both (a) and (b) are true.*
   (D) *none of the above.*

2. *The largest value of c such that there exists a function h(x) for* $-c < x < c$ *that is solution of* $\frac{dy}{dx} = 1 + y^2$ *with h(0)=0, is given by*
   (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{4}$ (D) π

3. *In a sufficiently small neighbourhood around x = 2, the differential equation* $\frac{dy}{dx} = \frac{y}{\sqrt{x}}, y(2) = 4$ *has*
   (A) *no solution* (B) *a unique solution*
   (C) *exactly two solutions* (D) *infinitely many solutions*

4. *The Sturm – Liouville problem :* $y'' + \lambda^2 y = 0, y'(0) = 0, y'(\pi) = 0$ *has its eigenvectors given by*
   (A) $y = \sin\left(n+\frac{1}{2}\right)x$ (B) $y = \sin nx$
   (C) $y = \cos\left(n+\frac{1}{2}\right)x$ (D) $y = \cos nx$ *; where n = 0, 1, 2, ...*

5. *The eigenvalues of the Sturm Liouville system* $y'' + \lambda y = 0, 0 \le x \le \pi, y(0) = 0, y'(\pi) = 0$ *are*
   (A) $\frac{n^2}{4}$ (B) $\frac{(2n-1)^2\pi^2}{4}$ (C) $\frac{(2n-1)^2}{4}$ (D) $\frac{n^2\pi^2}{4}$

6. *Let n be a non-negative integer. The eigenvalues of the Sturm-Liouville problem* $\frac{d^2y}{dx^2} + \lambda y = 0$, *with boundary conditions* $y(0) = y(2\pi)$, $\frac{dy}{dx}(0) = \frac{dy}{dx}(2\pi)$, *are*
   (A) n (B) $n^2\pi^2$ (C) $n\pi$ (D) n^2

7. *For the Strum Liouville problems:* $(1+x^2)y'' + 2xy' + \lambda x^2 y = 0$ *with* $y'(1) = 0$ *and* $y'(10) = 0$, *the eigen-values* λ, *satisfy*

(A) $\lambda \geq 0$ (B) $\lambda < 0$ (C) $\lambda \neq 0$ (D) $\lambda \leq 0$

8. Consider the following statement for IVP $\frac{dy}{dx} = x^2 + y,\ y(0) = 1, D : |x| \leq 1, |y-1| \leq 1$

i. It has a solution which exists for all x

ii. The local interval for which the solution exists uniquely is $|x| \leq \frac{1}{3}$

iii. It has no solution in the interval $|x| \leq 3$ then select the correct code

(A) only (ii) is true
(B) (ii) and (iii) are true
(C) (i) and (ii) are true
(D) all are true

9. The Lipschitz constant of $f(x, y) = x^3 e^{-xy^2}$ in $D : 0 \leq x \leq p,\ |y| < \infty,\ (p > 0)$

(A) $2p^3$ (B) $2p^4$ (C) max $\{2p^3, 2p^4\}$ (D) none of these

10. The differential equation $y' = \frac{(y-1)}{x},\ y(0) = 1$ has

(A) no solution
(B) unique solution
(C) infinitely many solutions
(D) two solutions

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

11. For Sturm-Liouville Problem, $y'' + \lambda y = 0$ and $y(0) = 0 = y(\pi),\ 0 \leq x \leq \pi$.

(A) the eigen values of the problem are $\lambda = n^2, n = 1, 2, ...$
(B) the eigen values of the problem are $\lambda = n, n = 1, 2, ...$
(C) the eigen functions are $y(x) = \sin nx,\ n = 1, 2, ...$
(D) the eigen functions are $y(x) = \cos nx,\ n = -1, -2, ...$

12. $\frac{dy}{dx} = 1 + y^2,\ y(0) = 0$ in the domain $R = \{(x,y) ; |x| < 5, |y| < 3\}$ the equation has

(A) no solution
(B) unique solution for $|x| < 0.3$
(C) infinite number of solutions
(D) nothing about solution can be concluded

13. Which of the following satisfy Lipschitz condition?

(A) $f(t, x) = 4t^2 + x^2$ on $D : |t| < 1, |x| < 1$
(B) $f(t, x) = t^2 \cos^2 x + x \sin^2 t$ on $D : |t| \leq 1, |x| < \infty$
(C) Both (A) and (B)
(D) Neither (A) nor (B)

14. Which of the following is an eigen value of differential equation $x^2 y'' - \lambda(xy' - y) = 0,\ y(1) = 0,$ $y(2) - y'(2) = 0$?

(A) $n\pi, n \in \mathbb{N}$ (B) 1 (C) 0 (D) 2

ANSWERS TO EXERCISES

(PRACTICE SET – 1)

1. (D) 2. (B) 3. (A) 4. (A,B,C,D) 5. (C)

(PRACTICE SET – 2)

1. (B) 2. (A,B,D) 3. (2) 4. (C,D)

(PRACTICE SET – 3)

1. (A,B,D) 2. (A,C) 3. (A) 4. (A,C,D) 5. (A)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 3.1

1. (D) 2. (A) 3. (C) 4. (D) 5. (D) 6. (A) 7. (C)
8. (D) 9. (A)
10. (A, B) 11. (A, C, D) 12. (B, D) 13. (B, C) 14. (A, D)

ASSIGNMENT - 3.2

1. (A) 2. (B) 3. (B) 4. (D) 5. (C) 6. (D) 7. (A)
8. (C) 9. (C) 10. (C)
11. (A, C) 12. (B) 13. (A,B,C) 14. (C)

CHAPTER – 4
INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

INTRODUCTION

As same as ordinary differential equation, partial differential equations also arise in many fields e.g. geometry, mechanics, physics. It is an equation which arises when a dependent variable depends upon more than one independent varies. In this chapter, we will learn what are different types of PDE's, e.g. homogeneous, non-homogeneous, linear, non - linear and the various methods to solve them. By Charpit's method we can solve any type of non-linear P.D.E.

§ 4.1. PARTIAL DIFFERENTIAL EQUATION

If a dependent variable is a function of two or more independent variables, then an equation involving partial differential coefficients is called a partial differential equation. The order of a partial differential equation is the same as that of the highest order differential coefficient in it. If $z = f(x, y)$, where x and y are independent variables, then the partial differential coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are denoted by p and q respectively.

4.1.1. Formation Of Partial Differential Equations

The partial differential equations can be formed either by elimination of arbitrary constants from a relation between x, y, z or by the elimination of arbitrary functions of these variables.

4.1.2. Solutions Of Partial Differential Equations

*The solution $f(x,y,z,a,b) = 0$ of a first order partial differential equation, which contains two arbitrary constants is called a **complete solution or complete integral.***

*If in this solution, we put $b=\phi(a)$ and find the envelope of the family of surfaces $f(x,y,z,a,\phi(a)) = 0$, we get a solution involving an arbitrary function ϕ. This is called the **general solution or general integral.** A solution obtained from the complete integral by giving particular values to the arbitrary constant is called a **particular solution or particular integral.***

4.1.3. Types Of Partial Differential Equations

(i) ***Linear P.D.E. :*** *A first order p.d.e. is said to be a linear equation if it is linear in p,q and z, i.e., if it is of the form $P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$.*
e.g. $yp - xq = xyz + x$.

(ii) ***Semi- linear P.D.E. :*** *A first order p.d.e. is said to be a semi-linear equation if it is linear in p and q and the coefficients of p and q are the functions of x and y only, i.e., it is of the form*
$P(x,y)p + Q(x,y)q = R(x,y,z)$
e.g. $e^x p - yxq = xz^2$.

(iii) ***Quasi linear P.D.E.*** *: A first order p.d.e. is said to be a quasi linear equation if it is linear in p and q, i.e., if it is of the form* $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$.
e.g. $(x^2 + z^2)p - xyq = z^3x + y^2$.

(iv) ***Non-linear P.D.E.*** *: Partial differential equations of the form* $f(x,y,z,p,q) = 0$ *which do not come under the above three types are said to be non – linear equations.*
e.g. $pq = z$ *does not belong to any of the first three types. So, it is a non – linear first order p.d.e.*

Note: *We observed that by eliminating arbitrary functions, we always produce quasi- linear partial differential equations, only. However, we get both quasi-linear as well as non linear partial differential equations when we eliminate arbitrary constants.*

4.1.4. Classification of integrals

Let us consider a first order p.d.e. $f(x,y,z,p,q) = 0$ ...(1)
Essentially a solution of (1) in a region $D \subseteq R$ *is given by z as a continuously differentiable function of x and y and* $(x,y) \in D$
further if one computes p and q from it and substitutes them into (1), then the equation reduces to an identity in x and y. There are different types of solutions (integral surfaces) for the first order p.d.e. (1).

Note: *A solution* $z = z(x,y)$ *can be interpreted as a surface of the partial differential equation.*

(a) ***Complete integral :*** *A two – parameter family of solutions* $z(x,y,a,b)$, ...(2)
is called a 'complete integral' of (1) if in the region considered, the rank of the matrix

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}, \text{ is two.}$$

(b) ***General integral :*** *In (2), if we take* $b = \phi(a)$, *we get a one parameter family of solutions of (1) which is a sub-family of the two parameter family (2) as* $z = F(x,y,a,\phi(a))$. ...(3)
The envelope of (3) if it exists, is obtained by eliminating a between (3) and $F_a+F_b\,\phi'(a)=0$ ...(4)
In fact, if it can be solved for a , then $a = a(x,y)$.

Substituting value of a in (3), we obtain an integral surface as $Z=F\{x,y,a(x,y),\ \phi[a(x,y)]\}$ ...(5)
If the function ϕ *which defines this sub-family is arbitrary, then such a solution is called a general integral (general solution of (1)). When a particular function* ϕ *is used, we obtain a particular solution of the p.d.e. Different choices of* ϕ *may give different particular solutions of the p.d.e.*

Note: *A general integral hence involve an arbitrary function and the following lemma shows that it is indeed a solution of the given p.d.e.*

4.1.4. (i) Lemma. *Let* $z = F(x,y,a)$ *be a one parameter family of solutions of (1). Then the envelope of this one- parameter family, if it exists , is a solution of (1)*

Proof: *Note that the envelope is obtained by eliminating a between* $z = F(x,y,a)$, ...(6)
and $F_a(x,y,a) = 0$ ...(7)
Hence, the envelope will be given by $z = G(x,y) = F[x,y,a(x,y)]$, *where* $a(x, y)$ *is obtained from (7) by solving for a in terms of x and y;*
The envelope will satisfy the p.d.e. (1) for, $G_x = F_x + F_a a_x = F_x$ *and* $G_y = F_y + F_a a_y = F_y$ *Since* $F_a = 0$,
Thus the envelope will have the same partial derivatives as those of a member of the family. The partial differential equation at every point being only a relation to be satisfied between these derivatives, the envelope satisfies the p.d.e.(1).

(c) ***Singular integral :*** *In addition to be 'general integral', we can sometimes obtain yet another solution by finding the envelope of the two parameter family (2). This is obtained by eliminating a and b from the equations* $Z=F(x, y, a, b)$; $F_a=0$, $F_b=0$, ...(8).
And is called the singular integral of (1).

4.1.4. (ii) Lemma. *The singular integral is also a solution of the partial differential equation.*

Proof: *Let* $Z = F(x,y,a,b)$ *be a complete integral.*
We will show that the envelope of this two parameter family, if it exists, is also a solution.
Note that the envelope is obtained by eliminating a and b between
$Z = F(x,y,a,b)$ ...(9)
$F_a(x,y,a,b) = 0$ ...(10)
$F_b(x,y,a,b) = 0$ ...(11)
Hence, the envelope will be given by $Z = G(x,y) = F(x,y,a(x,y), b(x,y))$,
where $a(x,y)$ *and* $b(x,y)$ *are obtained from (10) and (11) by solving for a and b in terms of x and y.*
The envelope will satisfy the given partial differential equations. For,
$G_x = F_x + F_a a_x + F_b b_x = F_x$
$G_y = F_x + F_a a_y + F_b b_y = F_y$
since $F_a = 0$, $F_b = 0$.
That is, the envelope will have the same partial derivatives as a member of the family. This envelope, if it exists, is also a solution.
The singular integral can, however, be found by p.d.e. itself without knowing any complete Integral.

4.1.4. (iii) Lemma. *The singular solution is obtained by eliminating p and q from the equations*

$$\left.\begin{aligned} f(x,y,z,p,q)&=0\\ f_p(x,y,z,p,q)&=0\\ f_q(x,y,z,p,q)&=0 \end{aligned}\right\} \qquad ...(12)$$

Proof : *Since* $z = F(x,y,a,b)$ *is a complete integral of (1), the equation.*
$f(x, y, F(x,y,a,b), F_x(x,y,a,b), F_y(x,y,a,b)) = 0$ ...(13)
which holds identically for all a and b can be differentiated with respect to a and b, and hence leads to

$$\left.\begin{aligned} f_z F_a + f_p F_{xa} + f_q F_{ya} &= 0\\ f_z F_b + f_p F_{xb} + f_q F_{yb} &= 0 \end{aligned}\right\} \qquad ...(14)$$

On the singular integral, $F_a = 0$ *and* $F_b = 0$*. Therefore, the equations in (14) simplify to*

$$f_p F_{xa} + f_q F_{ya} = 0$$

$$f_p F_{xb} + f_q F_{yb} = 0$$

On this surface, $F_{xa}F_{yb} - F_{xb}F_{ya} \neq 0$ *(since* $F_a=0$*,* $F_b=0$*) and hence* $f_p=0, f_q=0$*. Otherwise the matrix.*

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix},$$

will not have rank two contradicting the fact that z=F(x,y,a,b) is a complete integral. Hence the lemma.

(d) ***Special integral :*** *Usually (but not always), the three classes (a), (b) and (c) given above include all the integrals of the first order p.d.e ...(1). However, there are some solutions of certain first order partial differential equations which do not fall under any of the three classes (a), (b) or (c). Such solutions are called '**Special Integrals**'.*

Example 1. $F(x + y, x-\sqrt{z}) = 0$ *is the general integral of the equation* $p-q=2\sqrt{z}$*. But* $z = 0$ *also satisfies this equation and it cannot be obtained from the general integral. It is a special integral of the equation. A complete integral of the p.d.e. is* $\sqrt{z} = \dfrac{(ax+y)}{(a-1)} + b$*.*

Example 2. *Consider* $F(x,y,z,p,q)=z-px-qy-p^2-q^2=0$ *...(15)*

The two parameter family of planes $z=F(x,y,a,b)=ax+by+a^2+b^2$*,* *...(16)*

is a complete integral of (15), since the matrix $\begin{pmatrix} x+2a & 1 & 0 \\ y+2b & 0 & 1 \end{pmatrix}$*, is of rank two and these planes satisfy the p.d.e. (15).*

Let us now take $b = \sqrt{(1-a^2)}$*. Then* $z = F(x,y,a,\sqrt{(1-a^2)}) = ax+\sqrt{(1-a^2)}\,y+1$*,*

$$\frac{\partial F}{\partial a} = x - \frac{ay}{\sqrt{(1-a^2)}} = 0$$

On eliminating a, we get $(z-1)^2 = (x^2+y^2)$*.*

This is a particular solution of the given p.d.e.

If we take $b = a$*, then* $z=ax+ay+2a^2$ $\dfrac{\partial F}{\partial a} = 0 \Rightarrow x+y = -4a$*.*

On eliminating a, the envelope is $8z = -(x+y)^2$*.*

This is another particular solution of the given p.d.e.

Now from equation (16), the conditions $F_a = 0$ *and* $F_b = 0$ *become* $F_a = x+2a = 0$ *...(17)*

$F_b = y+2b = 0$*, respectively* *...(18)*

On eliminating a and b between equations (16), (17) and (18), we obtain the singular integral as

$4z = -(x^2+y^2)$*.* *...(19)*

which is a paraboloid of revolution.

Note: *Using Lemma, the singular integral can also be obtained directly by eliminating p and q between (15) and* $F_p = -x-2p = 0$ *...(20)*

$F_q = -y-2q = 0$*,* *...(21)*

§ 4.2. LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A differential equation involving first order partial derivatives p and q only is called a partial differential equation of the first order.

If p and q both occur in the first degree only and are not multiplied together, then it is called a linear partial differential equation of the first order.

4.2.1. Lagrange's Linear Equation

The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is the standard form of a linear partial differential equation of the first order and is called Lagrange's Linear Equation. Working Procedure to solve $Pp + Qq = R$

(i) Form the Auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) Find two independent solutions of the auxiliary equations. Let $u=c_1$ and $v = c_2$ be two solutions of these equations.

(iii) Then $f(u, v) = 0$ or $u = f(v)$ is the solution of the given equation.

4.2.2. Integral surfaces passing through a given curve: *In the last section, we obtained general integral $Pp + Qq = R$. We shall now present method of using such a general solution for getting in the integral surface which passes through a given curve.*

Method : Let $Pp + Qq = R$ ...(1)

be the given equation, let its auxiliary equation gives the following two independent solutions.

$u(x, y,z)= c_1$ and $v(x,y,z)= c_2$ ...(2)

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by $x = x(t)$, $y = y(t)$ and $z = z(t)$ ...(3)

where t is a parameter. Then (2) may be expressed as $u[x(t),y(t),z(t)]= c_1$,

$v[x(t),y(t),z(t)]= c_2$ ...(4)

We eliminate single parameter t from the equation of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with the help of (2) and obtain the required integral surface.

Example 1. *Find the integral surface of the linear partial differential equation.*

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$, which contains the straight line $x+y=0$, $z = 1$.

Solution: Given $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ ...(1)

Lagrange's auxiliary equations of (1) are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad ...(2)$$

Solving, we get $xyz = c_1, x^2+y^2-2z = c_2$ ...(3)

Taking (t) as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form $x = t$, $y = -t$, $z = 1$ ...(4)

Using (3), (4) may be re-written as $-t^2 = c_1, 2t^2-2 = c_2$ These gives

$2(-c_1)-2 = c_2$ or $2c_1+c_2+2=0$. ...(5)

Putting values of c_1 and c_2 from (3) in (5), the required integral surface is

$2xyz + x^2 + y^2 - 2z + 2 = 0$

4.2.3. Surfaces orthogonal to a given system of surfaces

Let f(x,y,z)=C ...(1)

represents a system of surfaces, where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Let the surface $z = \phi(x, y)$ ...(2)

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, *-1 i.e. p, q, -1. Since normals at (x, y, z) to (1) and (2) are at right angles, we have*

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 0 \text{ or } p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \quad ...(3)$$

which is of the form Pp + Qq = R.

Conversely, we can easily verify that any solution of (3) is orthogonal to every surface of (1).

Example 2. *Find the surface which intersects the surfaces of the system z(x + y) = c(3z+1) orthogonally and which passes through the circle* $x^2 + y^2 = 1, z = 1$

Solution: *The given system of surfaces is given by f(x, y, z)* $= \frac{z(x+y)}{3z+1} = C$

$$f(x, y, z) \equiv \frac{z(x+y)}{3z+1} = C \quad ...(1)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \frac{\partial f}{\partial z} = (x+y)\frac{1(3z+1) - z.3}{(3z+1)^2} = \frac{x+y}{(3z+1)^2}$$

The required orthogonal surfaces is solution of $p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$

$$\text{or } \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2} \text{ or } z(3z+1)p + z(3z+1)q = x+y \quad ...(2)$$

Lagrange's auxiliary equations for (2) are $\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y}$ ...(3)

taking the first two fractions of (3), we get $dx - dy = 0$

Integrating both sides we get $x - y = C_1$ ...(4)

Choosing x, y, -z (3z+1) as multipliers, each fraction of (3)

$$= \frac{xdx + ydy - z(3z+1)dz}{0}$$

$$\Rightarrow xdx + ydy - 3z^2dz - zdz = 0$$

Integrating, $(1/2)x^2 + (1/2)y^2 - 3(z^3/3) - (1/2)z^2 = (1/2)C_2 \Rightarrow x^2 + y^2 - 2z^3 - z^2 = C_2$ ...(5)

Hence, any surface which is orthogonal to (1) has equation of the form

$x^2 + y^2 - 2z^3 - z^2 = \phi(x-y)$, *being an arbitrary function.*

In order to get the required surface passing through the circle $x^2 + y^2 = 1, z = 1$ *we must choose* $\phi(x-y) = -2$ *thus, the required particular surface is* $x^2 + y^2 - 2z^3 - z^2 = -2$

§ 4.3. NON–LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

These equations will contain p and q with powers higher than unity and the products of p and q. A solution of such an equation containing as many arbitrary constants as there are independent variables, is called the **complete integral.** *A* **particular integral** *is obtained by giving particular values to the constants.*

(a) Equations of the type that involves p and q only

These equations are of the form $f(p,q)=0$ ...(1)

Evidently $z=ax+by+c$, *where a and b are connected by the relation* $f(a, b) = 0$, *is a solution of the given equation. Differentiating* $z = ax + by + c$, *w.r.t x and y partially, we get* $p = \frac{\partial z}{\partial x} = a$ *and*

$q = \frac{\partial z}{\partial y} = b.$

substituting these in equations (1), we get, $f(a, b) = 0$. *From the relation* $f(a, b) = 0$, *we can find b in terms of a, say,* $b=F(a)$ *and then putting this value of b, the complete solution is given by* $z=ax+yF(a) + C$

Example 1. *Solve* $q=3p^2$ ...(i)

Solution: *The equation is of the form* $f(p,q)=0$.

$\therefore$ *The complete integral is given by* $z=ax + by + c$, *where* $b=3a^2$. *(By substituting in equation (i))*

$\therefore$ *The complete integral is* $z=ax + 3a^2y + c$

(b) Equations of the type $z = px + qy + f(p, q)$

This type of equation may be considered analogous to Clairaut's form $y = px + f(p)$, *where* $p = dy/dx$ *in ordinary differential equations.*

The complete integral is $z = ax + by + f(a, b)$, *obtained by putting* $p = a$ *and* $q = b$ *in the given equation.*

(c) Partial differential equations not containing x and y

These equations will be of the form $f(z, p, q) = 0$.

Put $u = x + cy$, *where c is an arbitrary constant and assume that z is a function of u,* $z = F(x + cy) = F(u)$. *Then* $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$; $\left[\because \frac{\partial u}{\partial x} = 1\right]$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = c\frac{dz}{du}. \quad \left[\because \frac{\partial u}{\partial y} = c\right]$$

The given equation then becomes, $f\left(z, \frac{dz}{du}, c\frac{dz}{du}\right) = 0$

which is an ordinary differential equation of the first order.

Rule to solve the partial differential equation of the type $f(z, p, q) = 0$.

Assume $u = x+cy$; *replace p and q by* $\frac{dz}{du}$ *and* $c\frac{dz}{du}$ *respectively in the given equation and then solve the resulting ordinary differential equation.*

Example 2. *Find the complete integral of* $16p^2z^2+9q^2z^2+4(z^2-1)=0$

Solution: *Given equation is of the form f(p,q,z)=0*

Let u=x + cy, c being an arbitrary constant. ...(1)

Now, replacing p and q by dz/du and c(dz/du) respectively in the given equation, we have

$$16z^2(dz/du)^2+9c^2z^2(dz/du)^2+4(z^2-1)=0 \text{ or } (16+9c^2)z^2\left(\frac{dz}{du}\right)^2=4(1-z^2)$$

$$\text{or } \frac{dz}{du}=\frac{2(1-z^2)^{1/2}}{z(16+9c^2)^{1/2}}$$

$$(1/2)\times(16+9c^2)^{1/2}(1-z^2)^{-1/2}z\,dz=du$$

Integrating, $-(16+9c^2)^{1/2}\dfrac{(1-z^2)^{1/2}}{2}=u+b=x+cy+b,$ *(by (1))*

or $(16+9c^2)(1-z^2)=2(x+cy+b)^2$ *is the complete integral, a,b being arbitrary constants*

(d) ***Equation of the type*** $f_1(x,p)=f_2(y,q)$

In this type of equation z is absent and the terms containing p and x can be separated from those containing q and y.

Put $f_1(x,p)=f_2(y,q)=c$, *(say).*

Then solving for p and q, we get $p=F_1(x)$ *and* $q=F_2(y)$.

Since $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore$ $dz=F_1(x)dx+F_2(y)dy$ *or integrating* $z=\int F_1(x)\,dx+\int F_2(y)\,dy+c_1$.

which is the complete integral containing two constants c and c_1.

§ 4.4. COMPATIBLE SYSTEMS OF FIRST ORDER EQUATIONS.

Two first order partial differential equations are said to be compatible, if they have a common solution. The compatibility condition for two partial differential equations f(x,y,z,p,q)=0 ...(1) *and g(x,y,z,p,q)=0* ...(2) *is* $\dfrac{\partial(f,g)}{\partial(x,p)}+p\dfrac{\partial(f,g)}{\partial(z,p)}+\dfrac{\partial(f,g)}{\partial(y,q)}+q\dfrac{\partial(f,g)}{\partial(z,q)}=0$

Example 1. *Show that partial differential equations* $p^2+q^2=1$ *and* $(p^2+q^2)x=pz$ *are compatible*

Solution: *Let* $f=p^2+q^2-1=0,\; g=p^2x+q^2x-pz=0$

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}0 & 2p\\ p^2+q^2 & -z+2px\end{vmatrix}=-2p(p^2+q^2)$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}0 & 2p\\ -p & -z+2px\end{vmatrix}=2p^2 \text{ and } \frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}0 & 2q\\ 0 & 2qx\end{vmatrix}=0$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}0 & 2q\\ -p & 2qx\end{vmatrix}=2pq$$

According to compatibility condition, $\dfrac{\partial(f,g)}{\partial(x,p)}+p\dfrac{\partial(f,g)}{\partial(z,p)}+\dfrac{\partial(f,g)}{\partial(y,q)}+q\dfrac{\partial(f,g)}{\partial(z,q)}=0$

$-2p^3-2pq^2+2p^3+2pq^2=0$

$0=0$, hence PDEs are compatible

§ 4.5. *CAUCHY'S PROBLEM FOR LINEAR PDE*

OR

INTEGRAL SURFACE PASSING THROUGH A GIVEN CURVE: *We obtained general integral of $Pp+Qq=R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.*

4.5.1. Method I: *Let $Pp+Qq=R$.* ...(1)

be the given equation. Let its auxiliary equations give the following two independent solutions $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$. ...(2)

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by $x=x(t),\ y=y(t),\ z=z(t)$, ...(3)

where t is a parameter. Then (2) may be expressed as

$u[x(t),y(t),z(t)]=c_1$ *and* $v[x(t),y(t),z(t)]=c_2$. ...(4)

We eliminate single parameter t from the equation of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

Example 1. *Find the integral surface of the linear partial differential equation $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$, which contains the straight line $x+y=0$, $z=1$.*

Solution: *Given $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$.* ...(1)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{x(y^2+z)}=\dfrac{dy}{-y(x^2+z)}=\dfrac{dz}{(x^2-y^2)z}$ ...(2)

Solving (2), we get $xyz=c_1$ and $x^2+y^2-2z=c_2$ ...(3)

taking t as parameter, the given equation of the straight line $x+y=0$, $z=1$ can be put in parameter form $x=t$, $y=-t$, $z=1$. ...(4)

Using (4), (3) may be re-written as $-t^2=c_1$ and $2t^2-2=c_2$. ...(5)

Eliminating t from the equations of (5), we have $2(-c_1)-2=c_2$ or $2c_1+c_2+2=0$ ...(6)

Putting values of c_1 and c_2 from (3) in (6), the required integral surface is $2xyz+x^2+y^2-2z+2=0$.

Example 2. *Find the equation of the integral surface of the differential equation $2y(z-3)p+(2x-z)q=y(2x-3)$, which passes through the circle $z=0$, $x^2+y^2=2x$.*

Solution: *Given equation is $2y(z-3)p+(2x-z)q=y(2x-3)$.* ...(1)

Given circle is $x^2+y^2=2x, z=0$ ...(2)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{2y(z-3)}=\dfrac{dy}{2x-z}=\dfrac{dz}{y(2x-3)}$. ...(3)

Taking the first and third fractions of (3) $(2x-3)dx-2(z-3)dz=0$.

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 *being an arbitrary constant.* ...(4)

Choosing 1/2, y,-1 as multipliers, each fraction of (3)

$$= \frac{(1/2)\,dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)\,dx + ydy - dz}{0},$$

hence, $(1/2)dx + ydy - dz = 0 \Rightarrow dx + 2ydy - 2dz = 0,$

Integrating, $x + y^2 - 2z = c_2, c_2$ *being an arbitrary constant.* ...(5)

Now, the parametric equations of given circle (2) are $x = t,\ y = (2t - t^2)^{1/2},\ z=0.$ ...(6)

Substituting these values in (4) and (5), we have $t^2 - 3t = c_1$ *and* $3t - t^2 = c_2$. ...(7)

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0$ ...(8)

Substituting the values of c_1, c_2 *from (4) and (5) in (8), the required integral surface is* $x^2 - y^2 - z^2 + 2x + 4z = 0.$

4.5.2. Method II: *Let* $Pp + Qq = R.$ ...(1)

be the given equation. Let its Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ *and* $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface passing through the curve which is determined by the following two equations $\phi(x, y, z) = 0$ *and* $\psi(x, y, z) = 0$ ...(3)

we eliminate x,y,z from four equations of (2) and (3) and obtain a relation between c_1 *and* c_2. *Finally, replace* c_1 *by* $u(x, y, z)$ *and* c_2 *by* $v(x, y, z)$ *in that relation and obtain the required integral surface.*

Example 3. *Find the integral surface of the partial differential equation*

$(x - y)p + (y - x - z)q = z$ *through the circle* $z = 1, x^2 + y^2 = 1.$

Solution: *Given* $(x - y)p + (y - x - z)q = z.$ ...(1)

Lagrange's auxiliary equations for (1) are $\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$ ...(2)

Choosing 1,1,1 as multipliers, each fraction on (2) $= \frac{(dx + dy + dz)}{0}$

$\therefore dx + dy + dz = 0$

integrating; $x + y + z = c_1$ ...(3)

taking the last two fractions of (2) and using (3), we get $\frac{dy}{y - (c_1 - y)} = \frac{dz}{z}$ *or* $\frac{2dy}{2y - c_1} - \frac{2dz}{z} = 0$

Integrating $\log(2y - c_1) - 2\log z = \log c_2$ *or* $(2y - c_1)/z^2 = c_2$ *or*

$(2y - x - y - z)/z^2 = c_2$ *or* [from (3)]

$(y - x - z)/z^2 = c_2.$ ...(4)

The given curve is $z=1$ *and* $x^2 + y^2 = 1$ ...(5)

Putting $z=1$ *in (3) and (4), we get* $x + y = c_1 - 1$ *and* $y - x = c_2 + 1$ ...(6)

But $2(x^2 + y^2) = (x+y)^2 + (y-x)^2.$ ...(7)

Using (5) and (6), (7) becomes

$2=(c_1-1)^2+(c_2+1)^2$ *or* $c_1^2+c_2^2-2c_1+2c_2=0$ ...(8)

Putting the values of c_1 *and* c_2 *from (3) and (4) in (8), required integral surface is*

$(x+y+z)^2+(y-x-z)^2/z^4-2(x+y+z)+2(y-x-z)/z^2=0$

$z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0$

Example 4. *Let u(x,y) be the solution of Cauchy's problem* $xu_x+u_y=1$

$u(x,0)=2\log x; x>1$ *then* $u(e,1)=?$

(a) -1 *(b) 0* *(c) 1* *(d) e*

Solution: $\dfrac{dx}{x}=\dfrac{dy}{1}=\dfrac{du}{1}$

$\log x = y + \log c_1$

$v_1=xe^{-y}=c_1$

Also, $y-u=c_2$

$v_2=y-u=c_2$

$v_2=\phi(v_1)$

$y-u=\phi(xe^{-y})$

use $u(x,0)=2\log x$

$0-2\log x=\phi(x) \Rightarrow -2\log x=\phi(x)$

Solution is $y-u=-2\log(xe^{-y})$

Put $x = e$, $y=1$

$1-u(e,1)=2\log(ee^{-1}) \Rightarrow u(e,1)=1-2\log(1) \Rightarrow u(e,1)=1 \Rightarrow$ *option (c) is correct*

§ 4.6. CHARPIT'S METHOD

This method is used for finding the complete integral of a non-linear partial differential equation.

Consider the equation $f(x, y, z, p, q) = 0$ ... (i)

Since z depends on x and y, we have $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = pdx + qdy$ ... (ii)

If we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ .... (iii)

then we can solve equations (i) and (iii) for p and q and substitute in equation (ii). This will give the solution provided (ii) is integrable.

ϕ *is determined by differentiating equation (i) and equation (iii) w.r.t. x and y and solving, we get*

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}=0$$

$$\text{or } \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}=0$$

This is Lagrange's linear equation with x, y, z, p, q as independent variables and ϕ *as the dependent variable. Its solution will depend on the solution of the subsidiary equations.*

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

An integral of these equations involving p or q or both, can be taken as the required relation (iii), which along with (i) will give the values of p and q to make (ii) integrable.

Example 1. *Find a complete integral of* $px + qy = pq$

Solution: *Here given equation is* $f(x,y,z,p,q) = px + qy - pq = 0$ ...(1)

Charpit's auxiliary equations are

$$\frac{dp}{(\partial f/\partial x)+p(\partial f/\partial z)}=\frac{dq}{(\partial f/\partial y)+q(\partial f/\partial z)}=\frac{dz}{-p(\partial f/\partial p)-q(\partial f/\partial q)}=\frac{dx}{-\partial f/\partial p}=\frac{dy}{-\partial f/\partial q}$$

or $$\frac{dx}{-(x-q)}=\frac{dy}{-(y-p)}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dp}{p+p.0}=\frac{dq}{q+q.0} \quad ..(2)$$

Taking the last two fractions of (2), $(1/p)\,dp = (1/q)\,dq$.

Integrating, $\log p=\log q+\log a$ *or* $p=aq$ ...(3)

Substituting this value of p in (1), we have $aqx + qy - aq^2 = 0$

or $aq = ax + y$, *as* $q \neq 0$ ...(4)

$\therefore$ *From (3) and (4),* $q = (ax + y)/a$ *and* $p = ax + y$

Putting these values of p and q in $dz = pdx + qdy$, *we get* $dz = (ax+y)dx + [(ax+y)/a]dy$

$adz = (ax+y)(adx+dy)$ *or* $adz = (ax+y)d(ax+y) = udu$, *where* $u = ax + y$

Integrating, $az = (1/2)u^2 + b = (1/2)(ax+y)^2 + b$,

which is a complete integral, a and b being arbitrary constants.

4.6.1. *Standard forms of Charpit's Method*

Type 1: *Equations containing p,q in this case, differential equation is given by* $f(p,q)=0$ ...(1)

Here $f_x=0$, $f_y=0$, $f_z=0$ *A.E. is given by* $\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(pf_z+f_x)}=\frac{dq}{-(qf_z+f_y)}$

which reduce to $\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}\Rightarrow dp=0\Rightarrow p=c_1$

Putting this value of p in (1), we get: $f(c_1,q) = 0$ *which gives* $q = Q(c_1)$

Now, equation $dz = pdx + qdy$ *gives* $= c_1dx + Q(c_1)dy$

On integrating both sides, we get $z = c_1x + Q(c_1)y + c_2$

which is the solution of differential equation

Example 2. *Find the Complete integral of* $p^2+q^2 = n^2$

Solution: $f(p,q) = p^2+q^2-n^2 = 0$ ...(1)

Auxilliary equations are:

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(pf_z+f_x)}=\frac{dq}{-(qf_z+f_y)}\Rightarrow\frac{dx}{2p}=\frac{dy}{2q}=\frac{dz}{2p^2+2q^2}=\frac{dp}{0}=\frac{dq}{0}\Rightarrow dp=0$$

On integrating, we get $p=a$ ... (2)

Using (2) in (1), we get $a^2+q^2=n^2 \Rightarrow q^2=n^2-a^2 \quad \Rightarrow q=\sqrt{n^2-a^2}$

$\therefore$ *The equation* $pdx+qdy=dz$ *reduces to* $adx+\sqrt{n^2-a^2}\ dy=dz$

$\Rightarrow dz=adx+\sqrt{n^2-a^2}\ dy$

Integrating both sides, we get $z=ax+\left(\sqrt{n^2-a^2}\right)y+C$

which is the required solution of the given differential equation.

Type II. *Equations Involving p, q and z*

In this case, differential equations are of the form $f(p,q,z)=0$ ... (1)

$\therefore$ *Here* $f_x=0$ *and* $f_y=0$

Charpit's Auxilliary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(pf_z+f_x)}=\frac{dq}{-(qf_z+f_y)}$$

$$\Rightarrow\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_q+qf_z}=\frac{dp}{-pf_z}=\frac{dq}{-qf_z} \qquad ...(2)$$

Taking last two fractions of (2) gives $\frac{dp}{p}=\frac{dq}{q}$

On integrating, we get $\log p=\log q+\log a$

$$\Rightarrow\frac{p}{q}=a \qquad ...(3)$$

Using (2) in (1), we can obtain p, q in terms of z and then using equation $p\,dx+q\,dy=dz$

We get the required solution of the given equation

Example 3. *Find the complete Integral of* $zpq=p+q$

Solution: *Here* $f(p,q,z)=zpq-p-q$ ... (1)

$$A.E.\ is\ \frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(pf_z+f_x)}=\frac{dq}{-(qf_z+f_y)}$$

$$\Rightarrow\frac{dx}{qz-1}=\frac{dy}{pz-1}=\frac{dz}{pqz-p+pqz-q}=\frac{dp}{-(p.pq)}=\frac{dq}{-(q.pq)}$$

$$=\frac{dp}{-(p.pq)}=\frac{dq}{q.pq}\Rightarrow\frac{dp}{-p^2q}=\frac{dq}{-pq^2}\Rightarrow\frac{dp}{p}=\frac{dq}{q}$$

On integrating, we get $\log p=\log q+\log a \quad \Rightarrow p=aq$. ... (2)

Using (2), (1) becomes $zaq^2-aq-q=0 \Rightarrow aqz-a-1=0\Rightarrow q=\frac{a+1}{az}$

$\therefore\ p=\frac{a+1}{z}$

Equation $p\ dx+q\,dy=dz$ *reduces to* $\frac{a+1}{z}dx+\frac{a+1}{az}dy=dz$

$$\Rightarrow (a+1)dx + \left(\frac{a+1}{a}\right)dy = z\ dx$$

On integrating, we get $(a+1)x + \left(\frac{a+1}{a}\right) y = \frac{z^2}{2} + C_1$

Type III. Separable Equations

A partial differential equation is said to be separable if it can be written in the form $f(x, p) = g(y, q)$
Auxillary equation becomes

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(p.0 + f_x)} = \frac{dq}{-(q.0 + f_y)} \Rightarrow \frac{dx}{f_p} = -\frac{dp}{f_x} \Rightarrow \frac{dx}{dp} = -\frac{f_p}{f_x}$$

which may be solved to give solution of p in terms of x, *we use this solution to get values of p and q in terms of* x *and* y *and then using equation* $pdx + q\,dy = dz$, *we get the required solution of the given equation.*

Example 4. *Find the complete integral of* $p^2 y(1+x^2) = qx^2$

Solution: *Given equation can be written as* $p^2\left(\frac{1+x^2}{x^2}\right) = \frac{q}{y}$

$$\Rightarrow p^2\left(1+\frac{1}{x^2}\right) = \frac{q}{y} \text{ which is the form } f(x,p) = g(y,q)$$

Auxilliary equation is $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(pf_z + f_x)} = \frac{dq}{-(qf_z + f_y)}$.

$$\Rightarrow \frac{dx}{f_p} = \frac{dp}{-f_x} \Rightarrow \frac{dp}{-p^2\left(\frac{-2}{x^3}\right)} = \frac{dx}{2p\left(\frac{1+x^2}{x^2}\right)}$$

$$\Rightarrow \frac{dp}{\frac{2p^2}{x^3}} = \frac{dx}{\frac{2p}{x^2}(1+x^2)} \Rightarrow \frac{xdp}{p} = \frac{dx}{1+x^2} \Rightarrow \frac{dp}{p} = \frac{dx}{x(1+x^2)} \Rightarrow \int\frac{dp}{p} = \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

On integrating, we get $\log p = \log x - \frac{1}{2}\log(1+x^2) + \log c$

$$\Rightarrow 2\log p = 2\log x - \log(1+x^2) + 2\log c \Rightarrow \log\ p^2 = \log\left(\frac{x^2}{1+x^2}\right) + \log\ c^2$$

$$\Rightarrow \log\ p^2 = \log\left(\frac{c^2x^2}{1+x^2}\right) \Rightarrow p^2 = \frac{c^2x^2}{1+x^2} \Rightarrow p = \frac{cx}{\sqrt{1+x^2}}$$

$\therefore$ (1) *becomes* $\frac{c^2x^2}{1+x^2}\, y(1+x^2) = qx^2 \Rightarrow q = c^2 y$

$\therefore$ *Equation* $pdx + qdy = dz$ *reduces to* $\frac{cx}{\sqrt{1+x^2}}dx + c^2 ydy = dz$

On integrating, we get $\frac{1}{2}c\frac{(1+x^2)^{1/2}}{\frac{1}{2}}+\frac{c^2y^2}{2}=z+c_1 \Rightarrow z+c_1=c(1+x^2)^{1/2}+\frac{c^2y^2}{2}$

Type-IV: Clairaut equation:

A first order partial differential equation is said to be of Clairaut form if it can be written in the form

$z = px + qy + f(p, q)$ ...(1)

Let $F(x, y, z, p, q) = px+qy+f(p,q)-z$

$\therefore$ *Charpit's auxiliary equation are* $\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}$

or $\frac{dp}{0}=\frac{dq}{0}=\frac{dz}{-px-qy-p(\partial f/\partial p)-q(\partial f/\partial q)}=\frac{dx}{-x-(\partial f/\partial p)}=\frac{dy}{-y-(\partial f/\partial q)}$

then, first and second fractions

$\Rightarrow dp=0$ *and* $dp=0 \Rightarrow p=a$ *and* $q=b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Example 5. *Solve* $z = px + qy + pq$

Solution: *Since, the given p.d.e. is* $z=px+qy+pq$*, which is the in Clairaut's form.*

The complete integral is $z = ax+ by + ab$ ...(1)

Singular integral: *Differentiating (1) partially w.r.t a and b.,* $0=x+b$ *and* $0=y+a$ ...(2)

Eliminating a and b between (1) and (2), we get $z=-xy-xy+xy$ *i.e.* $z = -xy$*, which is the required singular solution, which satisfies the given equation*

General integral: *Take* $b=\phi(a)$*, where* ϕ *denotes an arbitrary function. Then (1) becomes*

$z = ax+\phi(a)y+a\phi(a)$ ...(3)

Differentiating (3) partially w.r.t. a we have $0=x+\phi'(a)y+\phi(a)+a\phi'(a)$ ...(4)

the general integral is obtained by eliminating a between (3) and (4).

Example 6. *Find singular solution of* $z=px+qy-2\sqrt{(pq)}$

Solution: *The complete integral is* $z=ax+by-2\sqrt{(ab)}$ ...(1)

For Singular Integral, Differentiating (1) partially w.r.t a and b, we have

$0=x-\frac{2}{2\sqrt{(ab)}}b$ *i.e* $x=\sqrt{\left(\frac{b}{a}\right)}$ ...(2)

And $0=y-\frac{2}{2\sqrt{(ab)}}a$ *i.e.* $y=\sqrt{\left(\frac{a}{b}\right)}$ ...(3)

Eliminating a,b the singular solution is $xy=1$

Example 7. *Find complete and singular integrals of* $2xz - px^2 - 2qxy + pq = 0$.

Solution: *Here given equation is* $f(x,y,z,p,q)=2xz-px^2-2qxy+pq=0.$ ...(1)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}$, *by (1)*

The second fraction gives dq=0 so that q=a

Putting q=a in (1), we get $p=2x(z-ay)/(x^2-a)$

Putting values of p and q in dz = p dx+q dy, we get

$$dz=\frac{2x(z-ay)}{x^2-a}dx+ady \quad or \quad \frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}.$$

Integrating, $\log(z-ay)=\log(x^2-a)+\log b$ *or* $z-ay=b(x^2-a)$ *or* $z=ay+b(x^2-a).$ ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get $0=y-b$ *and* $0=x^2-a$ ...(3)

Solving (3) for a and b, $a=x^2$ *and* $b=y$ ...(4)

Substituting the values of a and b given by (4) in (2),

we get $z=x^2y$*, which is the required singular integral.*

PRACTICE SET – 1

1. *A general solution of the PDE* $uu_x+yu_y=x$ *is of the form* **(CSIR UGC NET JUNE-2011)**

(A) $f\left(u^2-x^2,\frac{y}{x+u}\right)=0$, *where* $f:\mathbb{R}^2\to\mathbb{R}$ *is* C^1 *and* $\nabla f\neq(0,0)$ *at every point*

(B) $u^2=g\left(\frac{y}{x+u}\right)+x^2, g\in C^1(\mathbb{R})$

(C) $f(u^2+x^2)=0, f\in C^1(\mathbb{R})$

(D) $f(x+y)=0, f\in C^1(\mathbb{R})$

2. *The Cauchy problem*

$$\left.\begin{array}{ll} u_x(x,y)+u_y(x,y)=0 & for\ (x,y)\in\mathbb{R}^2 \\ u(x,0)=0 & for\ all\ x\in\mathbb{R}\end{array}\right\}$$

has **(CSIR UGC NET JUNE-2011)**

(A) *a unique solution*

(B) *a family of straight lines as characteristics.*

(C) *solution which vanishes at (2, 1)*

(D) *infinitely many solutions.*

3. *The Cauchy problem*

$$\left.\begin{array}{l} xu_x + yu_y = 0 \\ u(x, y) = x, on\, x^2 + y^2 = 1 \end{array}\right\} has$$ **(CSIR UGC NET JUNE-2012)**

(A) a solution for all $x \in \mathbb{R}, y \in \mathbb{R}$ $x \in R, y \in R$

(B) a unique solution in $\{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$

(C) a bounded solution in $\{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$

(D) a unique solution in $\{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$, but the solution is unbounded.

4. Complete integral for the PDE $z = px + qy - \sin pq$ is not **(GATE-2003)**

(A) $z = ax + qy - \sin aq$ (B) $z = ax + by - \sin ab$

(C) $z = ax - by + \sin ab$ (D) $z = bx + ay + \sin ab$

5. If $f(x)$ and $g(y)$ are arbitrary functions, then the general solution of the partial differential equation $u\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0$, is given by **(GATE-2005)**

(A) $u(x, y) = f(x) + g(y)$ (B) $u(x, y) = f(x + y) + g(x - y)$

(C) $u(x, y) = f(x)\, g(y)$ (D) $u(x, y) = xg(y) + y f(x)$

§ 4.7. HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form $\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + ... + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$ ... (i)

in which k_i's are constants, is called a homogeneous linear partial differential equation of the nth order with constant coefficients. It is called homogeneous because all terms contain derivatives of the same order. This can be written as, $\phi(D, D')z = F(x, y)$ Its solution consists of two parts

(i) ***the Complementary Function*** *(C.F.) which is the complete solution of the equation $\phi(D, D')z = 0$. It must contain n arbitrary functions where n is the order of the differential equation.*

(ii) ***the Particular Integral*** *(P.I) which is a particular solution (free from arbitrary constants) of $\phi(D, D')z = F(x, y)$. The complete solution of above differential equation is z = C.F. + P.I.*

Rules to write Complementary Function

Consider the equation $\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0$ ... (i)

which in symbolic form is $(D^2 + k_1 DD' + k_2 D'^2)z = 0$ ... (ii)

Form the (A.E.) $m^2 + k_1 m + k_2 = 0$, by putting $D = m$ and $D' = 1$ in (ii). Solve the (A.E.) and find its roots. If

(a) *the roots of A.E. are different say m_1 and m_2 then $z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$ is the C.F.*

(b) the roots of the A.E. are equal each equal to say, m_1 then $z = \phi_1(y + m_1x) + x\phi_2(y + m_1x)$ is the C.F. In general if the A.E has r roots equal, then $z = \phi_1(y+mx) + x\phi_2(y+mx) + \ldots + x^{r-1}\phi_r(y + mx)$

Example 1. *Solve* $\left(D^4 - 2D^3D' + 2DD'^3 - D'^4\right)z = 0$

Solution: *The auxiliary equation of the given equation is* $m^4 - 2m^3 + 2m - 1 = 0$

or $(m+1)(m-1)^3 = 0$ so that $m = -1, 1, 1, 1$.

Hence, the general solution of the given equation is

$z = \phi_1(y-x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$ where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

Rules To Obtain Particular Integral

(i) **when $F(x, y) = e^{ax+by}$**

$$P.I. = \frac{1}{\phi(D, D')} e^{ax+by} = \frac{1}{\phi(a, b)} e^{ax+by}$$ (i.e. put $D = a$ and $D' = b$) provided $\phi(a,b) \neq 0$.

If $\phi(a, b) = 0$, we have the case of failure, in that case

$$P.I. = x.\frac{1}{\frac{\partial\phi}{\partial D}} e^{ax+by} \text{ or } y.\frac{1}{\frac{\partial\phi}{\partial D'}} e^{ax+by}$$

(ii) **When $F(x, y) = \sin(ax + by)$**

$$P.I. = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax + by)$$

(i.e. put $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$), provided $\phi(-a^2, -ab, -b^2) \neq 0$.

If $\phi(-a^2, -ab, -b^2) = 0$, then it is a case of failure and we can repeat the process of (i).

A similar rule holds when $F(x,y) = \cos(ax+by)$.

Example 2. *Solve* $\left(D^2 - 3DD' + 2D'^2\right)z = \cos(x + 2y)$

Solution: *Its auxiliary equation is given by* $m^2 - 3m + 2 = 0$

$\therefore$ Its C.F is $y = \phi_1(y+x) + \phi_2(y+2x)$

$$P.I = \frac{1}{(D^2 - 3DD' + 2D'^2)}\cos(x+2y) = \frac{1}{-1-3(-2)+2(-4)}\cos(x+2y).$$

$$= \frac{1}{-1+6-8}\cos(x+2y) = \frac{-1}{3}\cos(x+2y)$$

$\{\because D^2 = -a^2, DD' = -ab, D'^2 = -b^2\}$

$\Rightarrow$ solution is $z = \phi_1(y+x) + \phi_2(y+2x) - \frac{1}{3}\cos(x+2y)$

(iii) **When $F(x, y) = x^py^q$, where p, q are positive integers**

$$P.I. = \frac{1}{\phi(D, D')} x^py^q = [\phi(D, D')]^{-1} x^py^q$$

If p < q, expand $[\phi(D, D')]^{-1}$ *in powers of* $\frac{D}{D'}$

If q < p expand $[\phi(D, D')]^{-1}$ *in powers of* $\frac{D'}{D}$

Also, we have $\frac{1}{D}F(x,y) = \int_{y\,constant} F(x,y)\,dx$ *and* $\frac{1}{D'}F(x,y) = \int_{x\,constant} F(x,y)\,dy$

Example 3. *Solve* $(D^3 - D'^3)z = x^3y^3$

Solution: *Its auxiliary equation is* $m^3 - 1 = 0$

$\therefore$ *Its C.F is* $\phi_1(y+x)+\phi_2(y+\omega x)+\phi_3(y+\omega^2 x), \phi_1, \phi_2, \phi_3$ *being arbitrary functions, where* ω *and* ω^2 *are complex cube roots of unity.*

Now, $P.I = \frac{1}{(D^3 - D'^3)}x^3y^3 = \frac{1}{D^3[1-(D'^3/D^3)]}x^3y^3 = \frac{1}{D^3}\left(1-\frac{D'^3}{D^3}\right)^{-1}x^3y^3$

$= \frac{1}{D^3}\left(1+\frac{D'^3}{D^3}+...\right)x^3y^3$

$= \frac{1}{D^3}\left(x^3y^3+\frac{1}{D^3}6x^3\right) = \frac{1}{D^3}\left(x^3y^3+6\times\frac{x^6}{4\times5\times6}\right)$

$= (1/120)x^6y^3 + (1/10080)x^9.$

Hence, the required general solution is z=C.F+P.I

$\Rightarrow z = \phi_1(y+x)+\phi_2(y+\omega x)+\phi_3(y+\omega^2 x) + (1/120)x^6y^3 + (1/10080)x^9.$

(iv) ***If*** $f(x, y) = e^{ax+by}V(x, y)$***, where V(x, y) is a function of x and y***

$P.I. = \frac{1}{\phi(D,D')}e^{ax+by}.V = e^{ax+by}\frac{1}{\phi(D+a, D'+b)}.V$

(v) ***A short method when f(x, y) is a function of ax + by,***

We may apply a shorter method to find the particular integral.

Working rule. *To get the particular integral of an equation F(D, D') z =* $\phi(ax+by)$ *where F(D, D') is a homogeneous function of D, D' of degree n.*

Put ax + by = t; then integrate $\phi(t)$*, n times with respect to t. Put a for D and b for D' in F(D, D') we get F(a, b). Thus P.I. =* $\frac{1}{F(a,b)}\times$ *nth integral of* $\phi(t)$ *with respect to t, where t = ax + by.*

In case of failure

$$\frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{n!\,b^n}\phi(ax+by)$$

Example 4. *Solve* $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$

Solution: *Its auxiliary equation is* $m^3 - 2m^2 - m + 2 = 0$

$m^2(m-2)-(m-2)=0$

$(m^2-1)(m-2)=0 \Rightarrow m=1,-1,2$

$\therefore$ C.F is $\phi_1(y+2x)+\phi_2(y+x)+\phi_3(y-x)$

$$P.I=\frac{1}{(D^3-2D^2D'-DD'^2+2D'^3)}e^{x+y}=\frac{1}{(D-D')(D^2-DD'-2D'^2)}e^{x+y}=\frac{1}{D-D'}\frac{1}{1-1-2}e^{x+y}$$

$$=\frac{-1}{2}\frac{1}{D-D'}e^v \text{ (Put } x+y=v) = \frac{-1}{2}\frac{x}{1!}e^{x+y}=\frac{-x}{2}e^{x+y}$$

Hence, solution is $z=\phi_1(y+2x)+\phi_2(y+x)+\phi_3(y-x)-\frac{x}{2}e^{x+y}$

(vi) ***When F(x, y) = Any function***

Then $P.I. = \frac{1}{\phi(D,D')}F(x,y)$

Resolve $\frac{1}{\phi(D,D')}$ *into partial fractions.*

Considering ϕ *(D, D') as a function of D alone*

$P.I.=\frac{1}{D-mD'}F(x,y)=\int F(x,c-mx)\,dx$

where c is to be replaced by y+mx after integration.

§ 4.8. NON-HOMOGENEOUS LINEAR PDE

A linear PDE which is not homogeneous i.e., all the derivatives are not of the same order, is called a non-homogenous linear partial differential equation.

$F(D, D')z = f(x, y)$, where $F(D, D')$ is non-homogeneous in D and D'.

$F(D, D')$ is not always resolvable into linear factors as in homogeneous linear equations. Therefore we classify linear differential operators $F(D, D')$ into two following types.

(i) *$F(D, D')$ cannot be resolved into linear factors for example $D^2 - D'$.*

(ii) *$F(D, D')$ can be expressed as product of linear factors of the form $(\alpha D+\beta D'+\gamma)$ where α, β and γ are constants.*

Method of finding C.F. of non-homogeneous liner PDEs

Consider $F(D, D') = f(x, y)$

When $F(D, D')$ cannot be factorized into linear factors:

In such cases, we apply trial method.

Consider the equation $(D-D'^2)z=0$ ... (i)

Let a trial solution of (i) be $z=Ae^{hx+ky}$ ... (ii)

where, A, h and k are constants.

from (ii), $Dz=\frac{\partial z}{\partial x}=hAe^{hx+ky}$ *and* $D'^2z=\frac{\partial^2 z}{\partial y^2}=k^2Ae^{hx+ky}$

putting these in (i), we get $(h-k^2)Ae^{hx+ky}=0$ *or,* $h=k^2$ ... (iii)

Putting the value of h in (ii) a solution (which is also C.F.) of (i) is $z=Ae^{k^2x+ky}$ ... (iv)

Since all values of k satisfy the given equation (i), a more general solution (which is also C.F) is taken as

$z = \sum Ae^{k^2x+ky}$ ...(v)

where A and k are arbitrary constants.

Example 1. *Solve* $(2D^4 - 3D^2 D' + D'^2)z = 0$

Solution: *The given equation can be written as* $(2D^2 - D')(D^2 - D')z = 0$ ...(1)

Consider $(2D^2 - D')z = 0$ ...(2)

Let $z = Ae^{hx+ky}$ *be a trial solution of (1). Then, we have* $D^2z = Ah^2e^{hx+ky}$ *and* $D'z = Ake^{hx+ky}$.

Putting these values in (2), we get $A(2h^2-k)e^{hx+ky}=0$, *so that* $2h^2-k=0$ *or* $k=2h^2$

Hence, the most general solution of (2) is $z = \sum Ae^{hx+2h^2y}$ ...(3)

Next, consider $(D^2 - D')z = 0$ ...(4)

Let $z = A'e^{h'x+k'y}$ *be a trial solution of (1). Then, we have* $D^2z = A'h'^2e^{h'x+k'y}$ *and* $D'z = A'k'e^{h'x+k'y}$

putting these values in (4), we get $A'(h' - k')e^{h'x+k'y}=0$ *so that* $h' - k' = 0$ *or* $h' = k'$

Hence, the most general solution of (3) is $z = \sum A'e^{h'x+h'^2}$ ...(5)

From (3) and (5) the most general solution of (1) is $\sum Ae^{hx+2h^2y} + \sum A'e^{h'x+h'^2}$

A, A', h, h', k, k' *being arbitrary constants.*

Example 2. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = n^2 z$

Solution: *The given equation can be written as* $(D^2 + D'^2 - n^2)z = 0$ ...(1)

Let a trial solution of (1) be $z=Ae^{hx+ky}$ ...(2)

$\therefore D^2z=Ah^2e^{hx+ky}$ *and* $D'^2=Ak^2e^{hx+ky}$

Hence, (1) gives $A(h^2+k^2-n^2)e^{hx+ky} = 0$

or $h^2+k^2=n^2$ ...(3)

Taking α *as parameter, we see that (2) is satisfied if* $h = n\cos\alpha$ *and* $k = n\sin\alpha$ *putting these values in (2), the required general solution is* $z = \sum Ae^{n(x\cos\alpha + y\sin\alpha)}$,

where A and α *being arbitrary constants.*

Case II. *When F(D,D') can be expressed as product of linear factors:*

Let $(\alpha D + \beta D' + \gamma)$ *be a linear factor of F(D, D'). To find C.F. corresponding to this factor we consider the most simple non-homogeneous equation.*

$(\alpha D + \beta D' + \gamma)z = 0$ *or* $\alpha p + \beta q = -\gamma z$ ... (i)

which is of Lagrange's form

$$\therefore \frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{-\gamma z} \quad \text{... (ii)}$$

From first and second ratios of (ii)

$\alpha dy - \beta dx = 0$

Integrating; $\alpha y - \beta x = C$ ... (iii)

Again from first and third ratios of (ii), we get $\frac{dz}{z} = \frac{-\gamma}{\alpha} dx$

Integrating, $\log z = \frac{-\gamma}{\alpha} x + \log C_1$

$z = C_1 e^{\frac{-\gamma}{\alpha}x}, z = e^{\frac{-\gamma}{\alpha}x} \phi(C), z = e^{\frac{-\gamma}{\alpha}x} \phi(\alpha y - \beta x)$ {∵ from (iii)}

Thus, the part of C.F. corresponding to linear factor

$\alpha D + \beta D' + \gamma$ *is* $e^{-\gamma x/\alpha} \phi(\alpha y - \beta x)$ ... (iv)

where ϕ *is an arbitrary function.*

Similarly it can be shown that if F(D, D') has non-repeated linear factors of the type.

$F(D, D') = (\alpha_1 D + \beta_1 D' + \gamma_1)(\alpha_2 D + \beta_2 D' + \gamma_2) \dots (\alpha_n D + \beta_n D' + \gamma_n)$

then C.F. of equation $F(D, D')z = F(x, y) = e^{\frac{-\gamma_1}{\alpha_1}x} \phi_1(\alpha_1 y - \beta_1 x) + e^{\frac{-\gamma_2}{\alpha_2}x} \phi_2(\alpha_2 y - \beta_2 x) + \dots + e^{\frac{-\gamma_n}{\alpha_n}x} \phi_n(\alpha_n y - \beta_n x)$

Also, corresponding to a repeated factor

$(\alpha D + \beta D' + \gamma)^k$, *the part of C.F is* $e^{\frac{-\gamma}{\alpha}x} [\phi_1(\alpha y - \beta x) + x\phi_2(\alpha y - \beta x) + \dots + x^{k-1}\phi_k(\alpha y - \beta x)]$

Remark:

1. *Corresponding to each non repeated factor* $(D - mD' - \gamma)$ *the part of C.F. is* $e^{\gamma x}\phi(y + mx)$

2. *If the factor* $(D - mD' - \gamma)$ *repeats k times then the part of C.F. corresponding to it is* $e^{\gamma x}[\phi_1(y+mx) + x\phi_2(y+mx) + \dots + x^{k-1}\phi_k(y + mx)]$

3. *If a factor* $(\beta D' + \gamma)$ *occurs only once then C.F. corresponding to it is* $e^{\frac{-\gamma}{\beta}y} \phi(\beta x)$. *In case* $\beta D' + \gamma$ *repeats k times the part of C.F. is* $e^{\frac{-\gamma_1}{\beta}y} [\phi_1(\beta x) + x\phi_2(\beta x) + \dots + x^{k-1}\phi_k(\beta x)]$

Method *of finding P.I. of non-homogeneous linear partial differential equation :* $F(D, D')z = f(x, y)$

$$P.I = \frac{1}{F(D,D')} f(x, y)$$

Case I. *When* $F(x, y) = e^{ax+by}$ *and* $F(a, b) \neq 0$

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$$

Example 3. *Solve* $(D^2 - D'^2 + D - D')z = e^{2x+3y}$

Solution: *The given equation can be re-written as* $(D - D')(D + D' + 1)z = e^{2x+3y}$

∴ *C.F=* $\phi_1(y + x) + e^{-x}\phi_2(y - x)$, *where* ϕ_1, ϕ_2 *being arbitrary functions.*

And P.I is $\frac{1}{(D - D')(D + D' + 1)} e^{2x+3y} = \frac{1}{(2-3)(2+3+1)} e^{2x+3y} = -\frac{1}{6} e^{2x+3y}$

Hence, the required general solution is z=C.F.+P.I, i.e, $z = \phi_1(y+x) + e^{-x}\varphi_2(y-x) - \frac{1}{6}e^{2x+3y}$

Case II. *When f(x, y) = sin (ax+by) [or cos(ax+by)]*

$$P.I. = \frac{1}{F(D,D')} \sin(ax+by) = \frac{1}{F(D^2, DD', D'^2, D, D')} \sin(ax+by)$$

which can be evaluated further.

Example 4. *Solve* $(D-D'-1)(D-D'-2)z = \sin(2x+3y)$

Solution: *Here* $C.F = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x)$, where ϕ_1, ϕ_2 *being arbitrary functions and*

$$P.I = \frac{1}{(D-D'-1)(D-D'-2)} \sin(2x+3y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x+3y)$$

$$= \frac{1}{-2^2 + 2\times(2\times3) - 3^2 - 3D + 3D' + 2} \sin(2x+3y)$$

$$= \frac{1}{-3D+3D'+1} \sin(2x+3y) = D\frac{1}{-3D^2+3DD'+D} \sin(2x+3y)$$

$$= D\frac{1}{-3\times(-2^2) - 3\times(2\times3) + D} \sin(2x+3y) = D\frac{1}{D-6} \sin(2x+3y)$$

$$= D(D+6)\frac{1}{D^2-36} \sin(2x+3y)$$

$$= (D^2+6D)\frac{1}{-2^2-36} \sin(3x+2y)$$

$$= -(1/40)\times[D^2\sin(2x+3y) + 6D\sin(2x+3y)]$$

$$= -(1/40)\times[-4\sin(2x+3y) + 12\cos(2x+3y)]$$

Hence, solution is $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + (1/10)[\sin(2x+3y) - 3\cos(2x+3y)]$

Case III. *When f(x, y) =* $x^m y^n$, *m and n being positive integers, P.I. =* $\frac{1}{F(D,D')} x^m y^n = [F[D,D')]^{-1} x^m y^n$

Example 5. *Solve:* $r - s + 2q - z = x^2y^2$

Solution: $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) + 2(\partial z/\partial y) - z = x^2y^2$

or $(D^2 - DD' + 2D' - 1)z = x^2y^2$ ...(1)

Since $(D^2 - DD' + 2D' - 1)$ *cannot be resolved into linear factors in D and D′, hence C.F. of (1) is obtained by considering the equation* $(D^2 - DD' + 2D' - 1)z = 0$ ...(2)

Let a trial solution of (2) be $z = Ae^{hx+ky}$ ...(3)

$\therefore D^2z = Ah^2e^{hx+ky}, \quad DD'z = Ahke^{hx+ky}, D'z = Ake^{hx+ky}$ then (2) gives

$A(h^2 - hk + 2k - 1)e^{hx+ky} = 0$ or $h^2 - hk + 2k - 1 = 0$ so that $k = (1-h^2)/(2-h)$ ...(4)

$\therefore$ from (3), C.F.= $\sum Ae^{hx+ky}$, where A,h,k are arbitrary constants and h and k are related by (4).now,

$$P.I. = \frac{1}{D^2 - DD' + 2D' - 1}x^2y^2 = -\frac{1}{1-(D^2 - DD' + 2D')}x^2y^2$$

$$= -[1-(D^2 - D\ D' + 2D')]^{-1}x^2y^2$$

$$= -[1 + (D^2 - DD' + 2D') + (D^2 - DD' + 2D')^2 + \{D^2 + D'(2-D)\}^3 + ...]x^2y^2$$

$$= -[1 + (D^2 - DD' + 2D') + (D^2D'^2 + 4D^2D' - 4DD'^2 + ...) + 3D^2D'^2(2-D)^2 + ...]x^2y^2$$

$$= -[1 + (D^2 - DD' + 2D') + (D^2D'^2 + 4D^2D' - 4DD'^2 + 12D^2D'^2 + ...)x^2y^2$$

$$= -x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52$$

Hence, the required general solution is z=C.F.+P.I.

$$z = \sum Ae^{hx+ky} = x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$$

Case IV. *When f(x, y) =Ve^{ax+by} where V is a function of x and y, then*

$$P.I. = \frac{1}{F(D,D')}(Ve^{ax+by}) = e^{ax+by}\frac{1}{F(D+a, D'+b)}V,$$ which can be evaluated further.

Example 6. Solve $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y+3x)$

Solution: Hence, C.F.= $e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)]$

and $P.I. = \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\tan(y+3x) = 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\tan(y+3x)$

$$= 2e^{2x}\frac{1}{(D'-3D')^2}\tan(y+3x) = 2e^{2x}\frac{x^2}{1^2.2!}\tan(y+3x) = x^2e^{2x}\tan(y+3x)$$

$$\therefore z = e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)] + x^2e^{2x}\tan(y+3x)$$

Remark. *If f(x, y) = e^{ax+by} and F(a, b) = 0 then we have* $P.I. = \frac{1}{F(D,D')}e^{ax+by}$

$= e^{ax+by}.\frac{1}{F(D+a, D'+b)}.1$ *which can be evaluated further.*

PRACTICE SET - 2

1. The complete integral of the PDE $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = xe^{x+y}$ involving arbitrary functions ϕ_1 and ϕ_2 is **(CSIR UGC NET DEC-2011)**

(A) $\phi_1(y+x) + \phi_2(y+x) + \frac{1}{4}e^{x+y}$

(B) $\phi_1(y+x)+x\phi_2(y+x)+\dfrac{(x-1)}{4}e^{x+y}$

(C) $\phi_1(y-x)+\phi_2(y-x)+\dfrac{1}{4}e^{x+y}$

(D) $\phi_1(y-x)+x\phi_2(y-x)+\dfrac{(x-1)}{4}e^{x+y}$

2. *The general solution of the partial differential equation* $\dfrac{\partial^2 z}{\partial x \partial y}=x+y$ *is of the form* **(GATE-2010)**

(A) $\dfrac{1}{2}xy(x+y)+F(x)+G(y)$ (B) $\dfrac{1}{2}xy(x-y)+F(x)+G(y)$

(C) $\dfrac{1}{2}xy(x-y)+F(x)G(y)$ (D) $\dfrac{1}{2}xy(x+y)+F(x)G(y)$

3. *A general solution of the second order equation* $4u_{xx}-u_{yy}=0$ *is of the form* $u(x,y)=$ **(CSIR UGC NET JUNE-2011)**

(A) $f(x)+g(y)$ (B) $f(x+2y)+g(x-2y)$

(C) $f(x+4y)+g(x-4y)$ (D) $f(4x+y)+g(4x-y)$

where f and g are twice differentiable functions

4. *Let u(x,t) satisfy for* $x \in \mathbb{R}, t>0$

$\dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial u}{\partial t}+2\dfrac{\partial^2 u}{\partial x^2}=0$. *A solution of the form* $u=e^{ix}v(t)$ *with* $v(0)=0$ *and* $v'(0)=1$

(CSIR UGC NET DEC-2014)

(A) is necessarily bounded (B) satisfies $|u(x,t)|<e^t$

(C) is necessarily unbounded (D) is oscillatory in x.

5. *If u (x, t) satisfy the partial differential equation* $\dfrac{\partial^2 u}{\partial x^2}=4\dfrac{\partial^2 u}{\partial t^2}$ *then u (x,t) can be of the form*

(CSIR UGC NET DEC-2012)

(A) $u(x,t)=f(e^{x-2t})+g(x+2t)$. (B) $u(x,t)=f(x^2-4t^2)+g(x^2+4t^2)$

(C) $u(x,t)=f(2x-4t)+g(x+2t)$. (D) $u(x,t)=f(2x-4t)+g(2x+t)$

KEY POINTS

- *The solution $f(x,y,z,a,b) = 0$ of a first order partial differential equation, which contains two arbitrary constants is called a* ***complete solution***

- *The singular integral is obtained by eliminating a and b from the equations $Z=F(x, y, a, b)$, $F_a=0$, $F_b=0$.*

- *For the partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z and form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and find two independent solutions of the auxiliary equations. Let $u=c_1$ and $v = c_2$ be two solutions of these equations. Then $f(u, v) = 0$ or $u = f(v)$ is the solution of the given equation.*

- *The equations of the type $z = px + qy + f(p, q)$ is analogous to Clairaut's form $y = px + f(p)$, the complete integral is $z = ax + by + f(a, b)$, obtained by putting $p = a$ and $q = b$ in the given equation.*

- *In Charpit's method for the equation $f(x, y, z, p, q) = 0$*

  We have $$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

- *In homogeneous linear equations with constant coefficients, for the equation $\frac{\partial^2 z}{\partial x^2} + k_1\frac{\partial^2 z}{\partial x \partial y} + k_2\frac{\partial^2 z}{\partial y^2} = 0$ which in symbolic form is $(D^2 + k_1DD' + k_2D'^2)z = 0$. The A.E. is $m^2 + k_1m + k_2 = 0$, by putting $D = m$ and $D' = 1$. The roots of A.E. are say m_1 and m_2 then $z=\phi_1(y+m_1x)+\phi_2(y+m_2x)$ is the C.F. If the roots of the A.E. are equal, each equal to say, m_1 then $z = \phi_1(y + m_1x) + x\phi_2(y + m_1x)$ is the C.F.*

  For particular integral, when $F(x, y) = e^{ax+by}$ then P.I. $= \frac{1}{\phi(D,D')}e^{ax+by} = \frac{1}{\phi(a,b)}e^{ax+by}$

  In the case of failure P.I. $= x.\frac{1}{\frac{\partial \phi}{\partial D}}e^{ax+by}$ or $y.\frac{1}{\frac{\partial \phi}{\partial D'}}e^{ax+by}$

- *When $F(x, y) = \sin(ax + by)$, P.I. $= \frac{1}{\phi(D^2, DD', D'^2)}\sin(ax + by) = \frac{1}{\phi(-a^2, -ab, -b^2)}\sin(ax + by)$ (i.e. put $D^2=-a^2$, $DD'=-ab$, $D'^2 = -b^2$), provided $\phi(-a^2, -ab, -b^2) \neq 0$.*

- *When $F(x, y) = x^py^q$, where p, q are positive integers, P.I. $= \frac{1}{\phi(D,D')}x^py^q = [\phi(D, D')]^{-1}x^py^q$*

  If $p < q$, expand $[\phi(D, D')]^{-1}$ in powers of $\frac{D}{D'}$. If $q < p$ expand $[\phi(D, D')]^{-1}$ in powers of $\frac{D'}{D}$

- *If $F(D, D')z = \phi(ax + by)$, where $F(D, D')$ is a homogeneous function of D, D' of degree n. Put [$ax + by = t$,] then integrate $\phi(t)$, n times with respect to t. Put a for D and b for D' in F(D, D') we get [F(a, b).]*

  Thus P.I. = $\frac{1}{F(a,b)} \times$ *nth integral of $\phi(t)$ with respect to t, where $t = ax + by$. In case of failure,*

  $$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{n!\,b^n}\phi(ax + by)$$

- *For C.F. of non-homogeneous linear PDEs, when F(D, D') cannot be factorized into linear factors, then take trial solution of (i) be $z = Ae^{hx+ky}$ where, A, h and k are constants.*

- *When F(D,D') can be expressed as product of linear factors, let $(\alpha D + \beta D' + \gamma)$ be a linear factor of F(D, D'). Then, the part of C.F. corresponding to linear factor $\alpha D + \beta D' + \gamma$ is $e^{-\gamma x/\alpha}\phi(\alpha y - \beta x)$ where ϕ is an arbitrary function.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

1. *The integral surface of the partial differential equation $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ satisfying the condition $u(1, y) = y$, is given by* ***(GATE-2005)***

(A) $u(x, y) = \frac{y}{x}$ *(B)* $u(x, y) = \frac{2y}{x+1}$

(C) $u(x, y) = \frac{y}{2-x}$ *(D)* $u(x, y) = y + x - 1$

Solution: *The given differential equation is $xp + yq = 0$ and $u(1, y) = y$*

in option (D), $p = 1$ and $q = 1$

$x(p) + yq = x + y \neq 0$

Option (D) is not correct

in option (C), $p = \frac{y}{(2-x)^2}, q = \frac{1}{2-x}$

$$xp + yq = \frac{xy}{(2-x)^2} + \frac{y}{(2-x)} = \frac{xy + 2y - xy}{(2-x)^2} \neq 0$$

Option (C) is not correct

in option (B), $u(x, y) = \frac{2y}{x+1} \Rightarrow p = \frac{-2y}{(x+1)^2}$, $q = \frac{2}{x+1}$

$xp + yq = 0$

$$\Rightarrow \frac{-2xy}{(x+1)^2} + \frac{2y}{(x+1)} = \frac{-2xy + 2xy + 2y}{(x+1)^2} \neq 0$$

Option (B) is also incorrect

in option (A), $u(x,y)=\frac{y}{x}\Rightarrow p=\frac{-y}{x^2}$ *and* $q=\frac{1}{x}$

$$xp+yq=\frac{-y}{x}+\frac{y}{x}=0$$

Clearly, it satisfies the pde

$\Rightarrow$ *Option (A) is correct.*

2. *The complete integral of the partial differential equation* $x p^3 q^2 + y p^2 q^3 + (p^3+q^3) - z p^2 q^2 = 0$, *is not* ***(GATE-1997)***

(A) $ax + by + (ab^{-2} + ba^{-2})$ (B) $ax - by + (ab^{-2} - ba^{-2})$

(C) $-ax + by + (ba^{-2} - ab^{-2})$ (D) $ax + by - a(b^{-3} + ba^{-2})$

Solution: *The given PDE is* $xp^3q^2 + yp^2q^3 + (p^3 + q^3) - zp^2q^2 = 0 \Rightarrow z = xp + yq + pq^{-2} + qp^{-2}$

The general solution by Clairaut's form is $z = ax + by + ab^{-2} + ba^{-2}$ ...(1)

replace b by –b

$z = ax - by + ab^{-2} - ba^{-2}$ ...(2)

replace a by –a

$z = -ax + by - ab^{-2} + ba^{-2}$ ...(3)

By (1), (2) and (3)

Option (A), (B) and (C) are complete integrals

Clearly option (D) is not of above forms

So, answer is option (D).

3. *Solve* $(\partial^2 z/\partial x^2)-(\partial^2 z/\partial y^2)+(\partial z/\partial x)+3(\partial z/\partial y)-2z=e^{x-y}-x^2y$

Solution: *The given equation can be re-written as* $(D^2-D'^2+D+3D'-2)z=e^{x-y}-x^2y$

$or\{(D-D')(D+D')+2(D+D')-(D-D'+2)\}z=e^{x-y}-x^2y$

$or\{(D+D')(D-D'+2)\}-(D-D'+2)z=e^{x-y}-x^2y$

$or(D-D'+2)(D+D'-1)z=e^{x-y}-x^2y$

$\therefore C.F.=e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x),\phi_1,\phi_2$ *being arbitrary functions*

P.I. corresponding to e^{x-y} $=\frac{1}{(D-D'+2)(D+D'-1)}e^{x-y}$

$=\frac{1}{\{1-(-1)+2\}(1-1-1)}e^{x-y}=\frac{1}{4}e^{x-y}$ *and P.I. corresponding to* $(-x^2y)$

$$=\frac{1}{(D-D'+2)(D+D'-1)}(-x^2y)=\frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\{1-(D+D')\}^{-1}(x^2y)$$

$$=\frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\left(\frac{D-D'}{2}\right)^3+...\right\}\times\{1+(D+D')+(D+D')^2+(D+D')^3+...\}(x^2y)$$

$$=\frac{1}{2}\left(1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+...\right)(x^2y)$$

$$=(1/2)[1+(1/2)D+(3/2)D'+(3/4)D^2+(3/2)DD'+(21/8)D^8D'+...](x^2y)$$

$= (1/2)[x^2y + xy + (3/2)x^2 + (3/2)y + 3x + 21/4]$

Hence, the required general solution is [z=C.F.+P.I.,] i.e. $z = e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$
$+ (1/4)e^{(x-y)} + (1/2)x^2y + (1/2)xy + (3/4)x^2 + (3/4)y + (3/2)x + (21/8)$

4. *The partial differential equation of the family of surfaces z = (x + y) + A (xy), is* **(GATE-1998)**
(A) $xp - yq = 0$ (B) $xp - yq = x - y$ (C) $xp + yq = x + y$ (D) $xp + yq = 0$

Solution: *The given solution is* $z = x + y + A(xy)$
Let $A(xy) = xy$.
Then, the solution is $z = x + y + xy$
$p = 1 + y$ *and* $q = 1 + x$
In option (A), $xp - yq = x + xy - y - xy = x - y \neq 0$
$\Rightarrow$ *option (A) is incorrect*
In option (C), $xp + yq = x + y$
$x(1 + y) + y(1 + x) = x + y$
$x + xy + y + xy \neq x + y$
$\Rightarrow$ *Option (C) is incorrect.*
In option (D), $xp + yq$
$x(1 + y) + y(1 + x) \neq 0$
$\Rightarrow$ *options (A), (C) and (D) are incorrect*
$\Rightarrow$ *option (B) is correct answer.*

5. *The general solution of PDE* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = u$ *passing through the curve,* $x = t$, $y = 2t$, $u = 1$ *is*
(A) e^{x+2y} (B) e^{2x-y} (C) e^{2x+y} (D) e^{y-2x}

Trick: *Put value of x and y in given option, then either option (B) is correct or option (D). But option (D) do not satisfy given p.d.e.*
$\therefore$ *Option (B) is correct.*

Solution: $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{du}{u}$

$dx - dy = 0 \quad \Rightarrow x - y = c_1 = v_1$

$\dfrac{dx}{1} = \dfrac{du}{u} \quad \Rightarrow x + \log c_2 = \log u \quad \Rightarrow x = \log\dfrac{u}{c_2} \Rightarrow e^x = \dfrac{u}{c_2} \Rightarrow ue^{-x} = c_2 = v_2$

$v_2 = \phi(v_1) \quad \Rightarrow ue^{-x} = \phi(x - y)$ ...(1)

using initial condition, i.e., x=t, u=1, y=2t, we have

(1) $e^{-t} = \phi(t - 2t) \Rightarrow e^{-t} = \phi(-t) \Rightarrow \phi(t) = e^t \Rightarrow \phi(x) = e^x$

(1) $\Rightarrow ue^{-x} = e^{x-y} \quad \Rightarrow u = e^{2x-y}$

ASSIGNMENT 4.1

NOTE: CHOOSE THE BEST OPTION

1. *The complete solution of* $z = px + qy + p^2 + q^2$ *is*
   (A) $z = ax + by + a^2 + b^2$ (B) $z = ax + by$
   (C) $z = a^2x^2 + z^2 + by^2$ (D) $z = ax^2 + by^2 + 1$

2. *The relation* $z = (x + a)(y + b)$ *represent the partial differential equation*
   (A) $z = \frac{p}{q}$ (B) $z = pq$
   (C) $z = p - q$ (D) none of these

3. *Solution of* $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, *given* $z(x,0)=4e^{-3x}$ *is*
   (A) $z = 4e^{3x+t}$ (B) $z = 3e^{4x+t}$
   (C) $z = e^{-3x+t}$ (D) $z = 4e^{-3x+t}$

4. *The general solution of the given partial differential equation* $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$, *is*
   (A) $f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$ (B) $f_1(y + x) + f_2(y - x)$
   (C) $f_1(y + ix) + f_2(y - ix)$ (D) none of these

5. *The general solution of the partial differential equation* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$, *is*
   (A) $z = \phi(2y - x)$ (B) $z = \psi(y - 2x)$
   (C) $z = \phi(2y - x) + \psi(y - 2x)$ (D) $z = \phi(y - x) + \psi(y + x)$

6. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0$ *has the general solution*
   (A) $z = \phi(y) + f_1(y + 2x) + f_2(y - 2x)$ (B) $z = \phi(y) + 2f_1(y + 2x)$
   (C) $z = \phi(y) + f_1(y + 2x) + x f_2(y + 2x)$ (D) none of these

7. *The general solution of* $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = \sin(4x + y)$, *is*
   (A) $z = \frac{1}{3}x \cos(4x + y)$ (B) $z = f_1(y + x) + f_2(y + 4x)$
   (C) $z = f(y + x) - \frac{1}{3}x \cos(4x + y)$ (D) $z = f_1(y + x) + f_2(y + 4x) - \frac{1}{3}x \cos(4x + y)$

8. *The solution of the equation* $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}+2\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)+z=0$

(A) $z = 2e^{-x}f_1(y+x)$
(B) $z = e^{-x}f_1(y-x) + xe^{-x}f_2(y-x)$
(C) $z = f_1(y-x) + xf_2(y-x)$
(D) none of these

9. *The solution of* $(2D + D' - 1)(D + 2D' - 2)\,z = 0$ *is*

(A) $z = e^{x/2}\phi(x-2y) + e^{-2x}\phi(2x-y)$
(B) $z = e^{y}\phi(x-2y) + e^{y}\phi(2x-y)$
(C) $z = e^{x/2}\phi(2x-y) + e^{+2x}\phi(x-2y)$
(D) none of these

10. *The solution of* $\left(D^3-4D^2D'+4DD'^2\right)z=0$, *is*

(A) $z=\phi_1(x)+\phi_2(y+2x)+\phi_3(y+2x)$
(B) $z=\phi_1(y)+\phi_2(y+2x)+\phi_3(y+2x)$
(C) $z=\phi_1(x)+\phi_2(y+2x)+x\phi_3(y+2x)$
(D) $z=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x)$

11. *The solution of* $\left(D^3D'-4D^2D'^2+4DD'^3\right)z=0$, *is*

(A) $z=\phi_1(x)+\phi_2(y)+\phi_3(y+2x)+x\phi_4(y-2x)$
(B) $z=\phi_1(y)+\phi_2(x)+\phi_3(y+2x)+x\phi_4(y+2x)$
(C) $z=\phi_1(x)+\phi_2(y)+\phi_3(y-2x)+x\phi_4(y-2x)$
(D) $z=\phi_1(y)+\phi_2(x)+\phi_3(y-2x)+x\phi_4(y+2x)$

12. *The general integral of* $yzp + zxq = xy$, *is*

(A) $f(x+y,\, y+z) = 0$
(B) $f(x^2+y^2, x^2+z^2)=0$
(C) $f(x^2-y^2,\, x^2-z^2) = 0$
(D) none of these

13. *The relation* $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ *represents the partial differential equation*

(A) $z = p + q$
(B) $z = p - q$
(C) $2z = xp + yq$
(D) $2z = xp/yq$

14. *The partial differential equation formed by eliminating arbitrary functions from the equation* $z = f(x^2-y^2)$, *is*

(A) $xp + yq = 0$
(B) $xq + yp = 0$
(C) $\frac{x}{y} = p$
(D) $\frac{x}{y} = q$

15. *Solution of* $DD'(D-2D'-3)z=0$, *is*

(A) $z=\phi_1(y)+\phi_2(x)+e^{-3x}\phi_3(y+2x)$
(B) $z=\phi_1(y)+\phi_2(x)+e^{-3x}\phi_3(y-2x)$
(C) $z=\phi_1(y)+\phi_2(x)+e^{3x}\phi_3(y+2x)$
(D) None of these.

16. *The solution of* $\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right) = \frac{z}{a}$ *is*

(A) $z = e^{y/a}f(x-y)$
(B) $z = e^{y/a}f(x)$
(C) $z = e^{y}f(y)$
(D) $z = e^{a}f(x+y)$

17. *The general integral of* $y^2zp + x^2zq = xy^2$, *is*

(A) $\phi(x^2 - y^2, x^2 - z^2) = 0$
(B) $\phi(x^3 - y^3, x^3 - z^3) = 0$
(C) $\phi(x^3 - y^3, x^2 - z^2) = 0$
(D) $\phi(x^2 - y^2, x^3 - z^3) = 0$

18. *The general integral of* $xzp + yzq = xy$, *is*

(A) $\phi\left(\frac{x}{y}, xz - y^2\right)$
(B) $\phi\left(\frac{x}{z}, xz - z^2\right)$
(C) $\phi\left(\frac{x}{y}, xy - z^2\right)$
(D) *none of these.*

19. *The particular integral of* $r - 2s + t = \sin(2x+3y)$, *is*

(A) $2\cos(2x+3y)$
(B) $\sin(2x+3y)$
(C) $-\sin(2x+3y)$
(D) $-\frac{1}{2}(2x+3y)$

20. *The particular integral of* $(D^2 - D'^2 + D - D')z = e^{2x+3y}$, *is*

(A) $\frac{1}{6}e^{2x+3y}$
(B) $-\frac{1}{3}e^{2x+3y}$
(C) $-\frac{1}{6}e^{2x+3y}$
(D) $\frac{1}{3}e^{2x+3y}$

21. *The complete Integral of* $r + 2s + t + 2p + 2q + z = 0$, *is*

(A) $z = e^{-x}[\phi_1(y-x) + \phi_2(y-x)]$
(B) $z = e^{-x}[\phi(y-x) + \psi(y-x)]$
(C) $z = e^{-x}[\phi_1(y-x) + x\phi_2(y-x)]$
(D) *none of these.*

22. *The solution of* $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$ *is,*

(A) $z = e^{-x}\phi_1(y-2x) + \sum Ae^{(2k^2+1)x+ky}$
(B) $z = e^{x}\phi_1(y+2x) + \sum Ae^{(2k^2+1)x+ky}$
(C) $z = e^{x}\phi_1(y+2x) + \sum Ae^{-(2k^2+1)x+ky}$
(D) *none of these.*

23. *The solution of* $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = x + y$, *is*

(A) $z = \frac{2}{3}[(x+a)^{3/2} + (y-a)^{3/2}] + b$
(B) $z = \frac{2}{3}[(x+a) + (y-a)] + b$
(C) $z = \frac{2}{3}[(x+a)^2 + (y-a)^2] + b$
(D) $z = \frac{2}{3}[(x+a)^3 + (y-a)^3] + b$

24. *The complete integral of* $x(1+y)p = y(1+x)q$, *is*

(A) $z = a(\log xy + x + y) + b$
(B) $z = a(\log xy + x) + b$
(C) $z = a(1+x) + (1+y)$
(D) $z = ax + by + (a+b)xy$

25. *The complete integral of* $x^2p^2 + y^2q^2 - 4 = 0$, *is*

(A) $z = a\log x + \sqrt{4-a^2}\log y + b$
(B) $z^2 = ax^2 + \sqrt{4-a^2}\,y^2 + b$
(C) $z = ax^2 + by^2 + c$
(D) $z = a\log x^2 + \sqrt{4-a^2}\log y^2 + b$

26. *The complete integral of* $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$, *is*

(A) $f(x^2+y^2-2z, xyz) = 0$
(B) $f(x^2+y^2+2z, xyz) = 0$
(C) $f(x^2-y^2-2z, xyz) = 0$
(D) $f(x^2-y^2+2z^2, xyz) = 0$

27. *The complete integral of* $r+(a+b)s+abt = xy$, *is*

(A) $z = f_1(y+ax) + f_2(y-bx) + \frac{1}{6}x^3y - \frac{1}{24}(a+b)x^4$
(B) $z = f_1(y-ax) + f_2(y+bx) + \frac{1}{6}x^3y - \frac{1}{24}(a+b)x^4$
(C) $z = f_1(y-ax) + f_2(y-bx) + \frac{1}{6}x^3y - \frac{1}{24}(a+b)x^4$
(D) $z = f_1(y+ax) + f_2(y+bx) + \frac{1}{6}x^3y + \frac{1}{24}(a+b)x^4$

28. *The complete integral of* $q = px + p^2$, *is*

(A) $z = a(xe^y) + \frac{a^2}{2}e^{2y} + b$
(B) $z = a(xe^y) + \frac{a^2}{2}e^{2x} + b$
(C) $z = a(xe^y) - \frac{a^2}{2}e^{2y} + b$
(D) *none of these*

29. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial x} = 1$ *has the solution*

(A) $z = f_1(y) + e^{-x}f_2(y+x) + x$
(B) $z = f_1(y) + e^{-x}f_2(y+x)$
(C) $z = x$
(D) *none of these*

30. *The general integral of* $p + 3q = 5z + \tan(y-3x)$

(A) $\phi[y+3x, 5x - \log\{5z+\tan(y+3x)\}] = 0$
(B) $\phi[y+3x, 5x - \log\{5z-\tan(y+3x)\}] = 0$
(C) $\phi[y-3x, 5x - \log\{5z+\tan(y-3x)\}] = 0$
(D) $\phi[y-3x, 5x - \log\{5z-\tan(y-3x)\}] = 0$

31. *The characteristic curve of* $2yu_x + (2x+y^2)u_y = 0$ *passing through (0, 0) is*

(A) $y^2 = 2(e^x + x - 1)$
(B) $y^2 = 2(e^x - x + 1)$
(C) $y^2 = 2(e^x - x - 1)$
(D) $y^2 = 2(e^x + x + 1)$

32. Consider $4xyz = pq + 2px^2y + 2qxy^2$

(i) It can be reduced to Clairaut form by some suitable transformation

(ii) $z = ax + by + a \cdot b$ is complete integral

(iii) $z = -x^2y^2$ is singular solution

Choose correct code

(A) only I and II are correct

(B) only II and III are correct

(C) I and III are correct

(D) all are correct

33. Let u(x,y) be the solution to the Cauchy problem $xu_x + u_y = 1, u(x,0) = 2\ln(x), x > 1$. Then u (e,1) =

(A) -1 (B) 0 (C) 1 (D) e

34. The characteristic curve for the equation $xz_y - yz_x = z$ is

(A) straight line passing through origin

(B) circle with centre at origin

(C) parabola with vertex at origin

(D) rectangular hyperbola

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

35. General solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is not of the form

(A) $u = f(x+iy) + g(x-iy)$

(B) $u = f(x-iy) + g(x-iy)$

(C) $u = f(x+iy) - g(x-iy)$

(D) $u = f(x-iy) - g(x-iy)$

36. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$ has the solution

(A) $z = f_1(y+ix) + f_2(y-ix)$

(B) $z = f_1(y+ix) + f_2(y-ix) + 2x^3 + 6x^2y$

(C) $z = (x+y)^3$

(D) none of these

37. The general solution of the given partial differential equation $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$ cannot be

(A) $z = f_1(y+ax) + f_2(y-ax)$

(B) $z = 2f_1(y+ax)$

(C) $z = f_1(y+ax) + f_2(y)$

(D) $z = f_1(y+ax)$

38. The solution of the given partial differential equation $\frac{\partial^4 z}{\partial x^4} - 2\frac{\partial^4 z}{\partial x^3 \partial y} + 2\frac{\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0$ is not

(A) $z = f_1(y-x) + f_2(y+x) + xf_3(y+x) + x^2f_4(y+x)$

(B) $z = 2f_1(y-x) + 2f_2(y+x)$

(C) $z = f_1(y-x) + 3f_2(y+x)$

(D) $z = f_1(y-x) + f_2(y+x) + xf_3(y+x)$

39. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ *has not the general solution–*

(A) $z = f_1(y - x) + e^x f_2(y - x)$
(B) $z = f_1(y + x) + f_2(y - x)$
(C) $z = e^{-x} f(y - x)$
(D) $z = f_1(y + x) + e^{-x} f_2(y - x)$

40. *The partial differential equation formed by eliminating arbitrary functions from the relation* $z = f(x + at) + g(x - at)$ *is not*

(A) $\frac{\partial^2 z}{\partial t^2} - a^2 \frac{\partial^2 z}{\partial x^2} = 0$
(B) $a^2 \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} = 0$
(C) $a \frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = 0$
(D) $\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = 0$

41. *Elimination of a function f from* $z = f(y/x)$ *does not give a partial differential equation*

(A) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$
(B) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$
(C) $x + y = 0$
(D) $x - y = 0$

42. *Elimination of a and b from* $z = ae^{bt} \sin bx$ *does not give the partial differential equation*

(A) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
(B) $\frac{\partial z}{\partial x} + = 0$
(C) $\frac{\partial z}{\partial x} + z = 0$
(D) $x + y = 0$

43. *General solution of PDE* $(y + z)p + (z + x)q = x + y$*, is/are*

(A) $\phi\left[(x-y)^2(x+y+z), \left(\frac{x-y}{y-z}\right)\right] = 0$

(B) $\frac{x-y}{y-z} = f[(x-y)^2(x+y+z)]$

(C) $(x-y)^2(x+y+z) = F\left(\frac{x-y}{y-z}\right)$

(D) $\psi\left[(y-z)^2(x+y+z), \left(\frac{y-z}{z-x}\right)\right] = 0$

44. *Given,* $z = a(x+y) + b(x-y) + abt + c$*, then*

(A) $\frac{\partial z}{\partial x} = a + b$

(B) $\frac{\partial z}{\partial y} = a - b$

(C) $\frac{\partial z}{\partial t} = ab$

(D) elimination of arbitrary constant gives $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t}$

45. Given, $y = f(x - at) + F(x + at)$, then

(A) $\frac{\partial y}{\partial x} = f'(x-at) + F'(x+at)$

(B) $\frac{\partial y}{\partial t} = -af'(x-at) + aF'(x+at)$

(C) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}$

(D) elimination of arbitrary functions gives partial differential equation of second order.

ASSIGNMENT - 4.2

NOTE: CHOOSE THE BEST OPTION

1. The general integral of the partial differential equation $p_2 + p_3 = 1 + p_1$, is
(A) $f(x_1 + z, x_1 + x_2) = 0$ (B) $f(x_1 + x_2, x_1 + x_3) = 0$
(C) $f(x_1 + z, x_1 + x_2, x_1 + x_3) = 0$ (D) none of these

2. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = xyz$ gives the general solution
(A) $f\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right) = 0$ (B) $f\left(xyz - 3u, \frac{y}{z}\right) = 0$ (C) $f\left(xyz - 3u, \frac{x}{y}\right) = 0$ (D) $f(xyz - 3u) = 0$

3. The general integral of $\frac{\partial z}{\partial y} = 3\left(\frac{\partial z}{\partial x}\right)^2$, is
(A) $z = ax + 3ay + c$ (B) $z = ax + 3a^2y + c$ (C) $z = ax^2 + 3ay + c$ (D) none of these

4. The differential equation $\frac{\partial z}{\partial x} e^y = \frac{\partial z}{\partial y} e^x$ gives the general solution–
(A) $z = ae^x + be^x$ (B) $z = e^x + e^y$ (C) $z = a(e^x + e^y) + b$ (D) none of these

5. The general solution of the differential equation $p(y - z) + (x - y)q = z - x$, is given by
(A) $f(x^2 + 2yz, x + y + z) = 0$ (B) $f(y^2 + 2xz, x + y + z) = 0$
(C) $f(z^2 + 2xz, x + y + z) = 0$ (D) none of the these

6. The general solution of $px + qy = 3z$, is given by
(A) $f\left(\frac{z}{x^3}, \frac{y}{3}\right) = 0$ (B) $f\left(\frac{y}{x^3}, \frac{z}{x}\right) = 0$ (C) $f\left(\frac{x}{y^3}, \frac{z}{y}\right) = 0$ (D) $f\left(\frac{x}{y}, \frac{z}{x^3}\right) = 0$

7. Complete solution of $p^2 - q^2 = 1$ is given by
(A) $z = ax + (a^2 + 1)y + C$ (B) $z = aye^{-x}$
(C) $z = ax - (a^2 - 1)^{1/2} y + C$ (D) none of these

8. Complete Integral of $z^2(p^2+q^2) = x^2+y^2$, is
(A) $z^2 = x^2\sqrt{a^2 + x^2} + a^2 \log\left[x + \sqrt{x^2 + a^2} + y\sqrt{(y^2 - a^2)}\right] - a^2 \log\{y + \sqrt{(y^2 - a^2)}\} + b$
(B) $z^{3/2} = (x + a)^{3/2} + (y + b)^{3/2} + b$
(C) $z^2 = x^2\sqrt{a^2 - x^2} - a^2 \log\left[x + \sqrt{x^2 + a^2} - y\sqrt{(y^2 - a^2)}\right] - a^2 \log\{y + \sqrt{(y^2 - a^2)}\} + b$
(D) none of these

9. *The general solution of the partial differential equation* $(D^2 - D'^2 - 2D - 2D')z = 0$, *where* $D = \frac{\partial}{\partial x}$ *and* $D' = \frac{\partial}{\partial y}$ *is*

(A) $f(y + x) + e^{2x} g(y - x)$
(B) $e^{2x} f(y + x) + g(y - x)$
(C) $e^{-2x} f(y + x) + g(y - x)$
(D) $f(y + x) + e^{-2x} g(y - x)$

10. *The complete integral of the p.d.e.* $\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)z = \left(\frac{\partial z}{\partial x}\right)^2\left[x\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2\right] + \left(\frac{\partial z}{\partial y}\right)^2\left[y\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial y}\right)^2\right]$, *is*

(A) $z = ax + by + \frac{a^3 + b^3}{a + b}$
(B) $z = ax + by + \frac{a^4 + b^4}{a + b}$
(C) $z = -ax - by + \frac{a^4 + b^4}{ab}$
(D) $z = -ax - by - \frac{a^4 + b^4}{ab}$

11. *Integral surface of* $x^2p + y^2q = -z^2$ *which passes through hyperbola* $xy = x + y$, $z = 1$, *is given by*

(A) $xy + 2yz + xz = 3xyz$
(B) $yz + 2xy + xz = 3xyz$
(C) $xy + 2zx + xz = 3xyz$
(D) none of these

12. *Solution of partial differential equation* $r + s - 2t = (2x+y)^{1/2}$, *is*

(A) $\phi_1(x - y) + \phi_2(2x-y) + \frac{1}{10}(2x+y)^{3/2}$
(B) $\phi_1(x + y) + \phi_2(y-2x) + \frac{1}{15}(2x+y)^{3/2}$
(C) $\phi_1(x + y) + \phi_2(y-2x) + \frac{1}{15}(2x+y)^{5/2}$
(D) none of these

13. *The complete integral of* $(p + q)(z - xp - yq) = 1$, *is*

(A) $z = ax + by + a + b$
(B) $z = -ax + by + \frac{1}{a+b}$
(C) $z = ax - by + \frac{1}{a+b}$
(D) $z = ax + by + \frac{1}{a+b}$

14. *The complete integral of* $z = pq$, *is*

(A) $2\sqrt{y} = x\sqrt{a} + z\sqrt{b} + \sqrt{a}$
(B) $2\sqrt{z} = x\sqrt{a} + y\sqrt{b} + \sqrt{b}$
(C) $2\sqrt{z} = x\sqrt{a} + \left(\frac{1}{\sqrt{a}}\right)y + b$
(D) none of these

15. *The complete integral of the p.d.e.* $p^2x^2 + q^2y^2 = z^2$, *is*

(A) $z = x^{\cos\alpha}.y^{\sin\alpha}.k$
(B) $z = x^{\cos\alpha}.y^{\sin\beta}.k$
(C) $z = x^{\cos\alpha}.y^{\sin\alpha}$
(D) none of these

16. *The partial differential equation of the family of surface* $z = (x + y) + A(xy)$, *is*

(A) $xp - yq = 0$
(B) $xp - yq = x - y$
(C) $xp + yq = x + y$
(D) $xp + yq = 0$

17. *The complete integral of* $9(p^2z+q^2)=4$, *is*

(A) $(z+a^2)^{3/2}=y+ax+c$
(B) $(z+a^2)^{3/2}=x+ay+c$
(C) $(z-a^2)^{3/2}=y+ax+c$
(D) none of these

18. *The complete integral of* $q=2yp^2$, *is*

(A) $z=\lambda y+\lambda^2y^2+c$
(B) $z=\lambda x+\lambda^2y^2+c$
(C) $z=\lambda x+\lambda^2x^2+c$
(D) $z=\lambda x+\lambda y+c$

19. *The complete integral of* $q=xyp^2$, *is*

(A) $(2z-\lambda y^2-2c)^2=16\lambda x$
(B) $(2z-\lambda y-2c)^2=16\lambda x$
(C) $(2z-\lambda y^2-2c)=16\lambda x$
(D) $(2z-\lambda y^2-2c)^2=16\lambda x^2$

20. *The complete integral of* $yp=2xy+\log q$, *is*

(A) $z=\frac{(2x+\lambda)}{4}+\frac{e^{\lambda y}}{\lambda}+b$
(B) $z=\frac{(2x+\lambda)^2}{4}+\frac{e^{\lambda y}}{\lambda}+b$
(C) $z=\frac{(2x+\lambda)^2}{4}+\frac{e^{y}}{\lambda}+b$
(D) $z=\frac{(2x+\lambda)^2}{4}+e^{\lambda y}+b$

21. *The complete integral of* $2z+p^2+qy+2y^2=0$, *is*

(A) $2z=\frac{b}{y^2}-(a-x)^2-y^2$
(B) $2z=\frac{b}{y}-(a-x)^2-y^2$
(C) $2z=\frac{b}{y^2}-(a-x)-y^2$
(D) $2z=\frac{b}{y}-(a-x)^2-y$

22. *The complete integral of* $p(1+q^2)=q(z-a)$, *is*

(A) $4k(z-a)=(x+ky+b)+4$
(B) $4k(z-a)=(y+kx+b)^2+4$
(C) $4k(z-a)=(x+ky+b)^2+4$
(D) none of these

23. *The complete integral of* $p=(qy+z)^2$, *is*

(A) $yz=ay+2\sqrt{(ax)}+c$
(B) $z=ax+2\sqrt{(ay)}+c$
(C) $yz=ax+2\sqrt{(ay)}+c$
(D) $y=az+2\sqrt{(ay)+c}$

24. *The complete integral of* $(p^2+q^2)y=qz$, *is*

(A) $z^2-a^2y^2=(ax+b)^2$
(B) $z^2-a^2y^2=(ax+b)$
(C) $z^2-a^2x^2=(ay+b)^2$
(D) $z-a^2y^2=(ax+b)$

25. *The complete integral of* $z(p^2-q^2)=x-y$, *is*

(A) $z^{3/2}=(x+a)^{1/2}+(y+a)^{1/2}+b$
(B) $z^{1/2}=(x+a)^{1/2}+(y+a)^{1/2}+b$
(C) $z^{3/2}=(x+a)^{3/2}+(y+a)^{3/2}+b$
(D) none of these

26. *The complete integral of* $x^2y^3p^2q = z^3$, *is*

(A) $\log z = \sqrt{a}\log x - \frac{1}{2ay^2} + b$

(B) $\log z = \sqrt{a}\log x - \frac{1}{2ay} + b$

(C) $\log z = \sqrt{a}\log y - \frac{1}{2ax^2} + b$

(D) $\log z = \sqrt{a}\log y - \frac{1}{2ax} + b$

27. *The complete integral of* $r - s - 2t = (2x^2 + xy - y^2)\sin xy - \cos xy$, *is*

(A) $z = f_1(y+x) + f_2(y+x) + \sin xy$

(B) $z = f_1(y+2x) + f_2(y-x) + \sin xy$

(C) $z = f_1(y+2x) + f_2(y+x) + \cos xy$

(D) *none of these*

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

28. *Solution of the equation* $p\tan x + q\tan y = \tan z$ *is not*

(A) $\phi\left[\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right] = 0$

(B) $\phi[\sin x - \sin y, \sin y - \sin z] = 0$

(C) $\phi\left[\frac{\sin x}{\sin y}, \frac{\sin y}{\sin x}\right] = 0$

(D) $\phi[\sin x, \sin y] = 0$

29. *The particular Integral of the equation* $(D^2 + 3DD' + 2D'^2)z = x + y$ *cannot be*

(A) $z = \frac{1}{2}yx^2 - x^3$

(B) $z = \frac{1}{2}yx^2 + \frac{x^3}{3}$

(C) $z = yx^2 - \frac{x^2}{3}$

(D) $z = \frac{1}{2}yx^2 - \frac{x^3}{3}$

30. *The particular Integral of* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx.\cos ny$ *is not*

(A) $\frac{1}{(m^2+n^2)}\cos mx.\cos ny$

(B) $\frac{1}{(m^2+n^2)}\cos mx.\sin ny$

(C) $-\frac{1}{(m^2+n^2)}\cos mx.\cos ny$

(D) $\frac{-1}{(m^2+n^2)}\sin mx.\sin ny$

31. *For PDE* $(p^2+q^2)y = qz$

(A) *no singular solution exist*

(B) $z = x^2y$ *is the singular solution*

(C) $z = 0$ *is the singular solution*

(D) $z^2 = (ax+b)^2 + a^2y^2$ *is the complete solution*

32. *The surface passing through the parabola* $u = 0$, $y^2 = 4ax$ *and* $u = 1$, $y^2 = -4ax$ *and satisfying the equation* $x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = 0$

(A) $u = -\frac{y^2}{8ax} + \frac{1}{2}$

(B) *no such surface exist*

(C) $u = -y^2 + \cos x$

(D) $u = y^2 \sin x$

33. The surface satisfying $\frac{\partial^2 u}{\partial y^2} = \sigma x^3 y$ containing two lines $y = 0 = u$ and $y=1=u$ is

(A) $u = x^3y^3 + y(1-x^3)$ (B) no such surface exist

(C) $u = -\frac{y^2}{8ax} + 1$ (D) $u = \sin x \cos y$

34. C.F. of the P.D.E. $(bD - aD' - c)(bD - aD' - c)z = 0$ is not

(A) $z = e^{\frac{c}{a}x}[\phi_1(by+ax) + y\phi_2(by+ax)]$ if $b \neq 0$ (B) $z = e^{\frac{c}{b}x}[\phi_1(by+ax)+x\phi_2(by+ax)]$ if $b \neq 0$

(C) $z = e^{\frac{c}{b}y}[\psi_1(by+ax)+x\psi_2(by+ax)]$ if $a \neq 0$ (D) none of these.

35. The complete integral of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x^2 \sin(x+y)$ cannot be

(A) $z = f_1(y+2x) + f_2(y-3x) - \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y)$

(B) $z = f_1(y+2x) + f_2(y-3x) + \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y)$

(C) $z = f_1(y-2x) + f_2(y-3x) + \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y)$

(D) $z = f_1(y-2x) + f_2(y+3x) - \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y)$

36. The complete integral of $(x^2+y^2)(p^2+q^2) = 1$ cannot be

(A) $z = a\log\sqrt{(x^2+y^2)} + \sqrt{(1-a^2)}\tan^{-1}\frac{y}{x} + b$ (B) $z = a\log\sqrt{(x^2+y^2)} + \sqrt{(1-a^2)}\tan^{-1}\frac{x}{y} + b$

(C) $z = a\log\sqrt{(x^2-y^2)} + \sqrt{(1-a^2)}\tan^{-1}\frac{y}{x} + b$ (D) $z = a\log\sqrt{(x^2-y^2)} + \sqrt{(1-a^2)}\tan^{-1}\frac{x}{y} + b$

37. The complete integral of $(p+q)(px+qy) - 1 = 0$, is

(A) $\sqrt{z(1+a)} = 2\sqrt{(ax+y)} + b$ (B) $z\sqrt{(1+a)} = 2\sqrt{(ax+y)} + b$

(C) $z\sqrt{(1+a)} = 2\sqrt{(ay+x)} + b$ (D) none of these

ANSWERS TO EXERCISES

(PRACTICE SET – 1)

1. (A,B)	2. (A,B,C)	3. (B,C)	4. (A,D)	5. (C)

(PRACTICE SET – 2)

1. (D)	2. (A)	3. (B)	4. (B,C,D)	5. (A,C)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT 4.1

1. (A)	2. (B)	3. (D)	4. (A)	5. (C)	6. (C)	7. (D)
8. (B)	9. (B)	10. (D)	11. (B)	12. (C)	13. (C)	14. (B)
15. (C)	16. (A)	17. (C)	18. (C)	19. (C)	20. (C)	21. (C)
22. (B)	23. (A)	24. (A)	25. (A)	26. (A)	27. (C)	28. (A)
29. (A)	30. (C)	31. (C)	32. (C)	33. (C)	34. (B)	
35. (B,D)	36. (B,C)	37. (B,C,D)	38. (B,C,D)	39. (A,B,C)	40. (B,C,D)	41. (B,C,D)
42. (B,C,D)	43. (A,B,C,D)	44. (A,B,C,D)	45. (A,B,C,D)			

ASSIGNMENT 4.2

1. (C)	2. (A)	3. (B)	4. (C)	5. (A)	6. (D)	7. (C)
8. (D)	9. (B)	10. (C)	11. (B)	12. (C)	13. (D)	14. (C)
15. (A)	16. (B)	17. (B)	18. (B)	19. (A)	20. (B)	21. (A)
22. (C)	23. (C)	24. (A)	25. (C)	26. (A)	27. (B)	
28. (B,C,D)	29. (A,B,C)	30. (A,B,D)	31. (C,D)	32. (A)	33. (A)	34. (A,C,D)
35. (A,C,D)	36. (C,D)	37. (B,C)				

CHAPTER – 5
PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER AND SOME BOUNDARY VALUE PROBLEMS

INTRODUCTION

In this chapter, we will study the second order partial differential equations and their behaviours. We will learn to reduce them to the canonical forms. Moreover, here we will study some particular partial differential equations which arises from actual problems in nature e.g. by diffusion of heat from one point to other, the behaviour of waves and their governing sequence.

§ 5.1. CLASSIFICATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER IN TWO INDEPENDENT VARIABLES

Let us consider the equation of 2^{nd} order in two independent variables x and y.

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad ...(1)$$

where $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$

and R, S, T are functions of x and y then Equation (1) is

Hyperbolic *if* $S^2 - 4RT > 0$

Parabolic *if* $S^2 - 4RT = 0$

Elliptic *if* $S^2 - 4RT < 0$

Note: (1) *When equation (1) is hyperbolic then the characteristic equation of (1) is*

$$R\lambda^2 + S\lambda + T = 0 \quad ...(2)$$

Here roots are real and distinct [$\because S^2 - 4RT > 0$] *say* λ_1 *and* λ_2 *are roots of (2)*

Now, the characteristic curves are given by $\dfrac{dy}{dx} + \lambda_1 = 0$ *and* $\dfrac{dy}{dx} + \lambda_2 = 0$

Note: (2) *When equation (1) is elliptic then roots are complex.*

Here characteristic curves cannot be calculated.

Note: (3) *When equation (1) is parabolic then the characteristic equation of (1) is* $R\lambda^2 + S\lambda + T = 0$,

Here, the roots are real and equal i.e. $\lambda_1 = \lambda_2 = \lambda$ *(say)*

So, the characteristic curve is given by $\dfrac{dy}{dx} + \lambda = 0$.

Note: (4) *The number of real characteristic curves*

In hyperbolic = 2

In parabolic = 1

In elliptic = 0 (No real characteristic curve).

Example 1. *Consider the PDE* $x\,u_{xx} + 2xy\,u_{xy} + y\,u_{yy} + x\,u_y + y\,u_x = 0$, *then*

(A) elliptic in the region $x < 0, y < 0, xy > 1$

(B) elliptic in the region $x > 0, y > 0, xy > 1$

(C) parabolic in the region $x < 0, y > 0, xy > 1$

(D) hyperbolic in the region $x < 0, y < 0, xy > 1$

Solution: Here $R = x,\ S = 2xy,\ T = y$

Now, $S^2 - 4RT = (2xy)^2 - 4xy = 4x^2y^2 - 4xy$

If $x < 0, y < 0, xy > 1$, then $S^2 - 4RT = 4xy(xy - 1) > 0$

$\Rightarrow S^2 - 4RT > 0$

$\Rightarrow$ The given P.D.E. is hyperbolic in the region $x < 0, y < 0, xy > 1$.

$\therefore$ Option (D) is correct.

Example 2. *The* 2^{nd} *order PDE*

$$\frac{(x-y)^2}{4}u_{xx} + (x-y)\sin(x^2+y^2)u_{xy} + \cos^2(x^2+y^2)$$

$$u_{yy} + (x-y)u_x + \sin^2(x^2+y^2) + 4y + 4 = 0 \text{ is}$$

(A) elliptic in the region $\{(x, y) : x \neq y, x^2 + y^2 < \frac{\pi}{6}\}$

(B) hyperbolic in the region $\{(x, y) : x \neq y, \frac{\pi}{4} < x^2 + y^2 < \frac{3\pi}{4}\}$

(C) both true

(D) none of the above.

Solution: Here $R = \frac{(x-y)^2}{4}, S = (x-y)\sin(x^2+y^2)\ T = \cos^2(x^2+y^2)$

Thus, $S^2 - 4RT = (x-y)^2\sin^2(x^2+y^2) - (x-y)^2\cos^2(x^2+y^2)$

$= (x-y)^2\{\sin^2(x^2+y^2) - \cos^2(x^2+y^2)\} > 0$

in the region in option (B).

Example 3. *Classify the operators*

(i) $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x\partial t} + \frac{\partial^2 u}{\partial x^2}$ (ii) $\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial^2 u}{\partial x^2}$ (iii) $\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x\partial t} + 4\frac{\partial^2 u}{\partial x^2}$.

Solution:

(i) Here $A = 1, B = 1, C = 1$ and so $B^2 - 4AC = 1 - 4 = -3 < 0$.
Therefore, the given operator is elliptic.

(ii) Here $A = 1, B = -4, C = 1$ and so $B^2 - 4AC = 16 - 4 = 12 > 0$.
Therefore, the given operator is hyperbolic.

(iii) Here $A=1, B=4, C=4$ and so $B^2 - 4AC = 16 - 16 = 0$. Therefore, the given operator is parabolic.

Example 4. *Classify the following equations:*

(i) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$ *(Laplace equation)*

(ii) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$ *(Wave equation)*

(iii) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{C^2}\frac{\partial u}{\partial t}$ *(Heat equation)*

Solution:

(i) *Here the operator*

$\phi=\delta_1^2+\delta_2^2+\delta_3^2$, $a_{11}=a_{22}=a_{33}=1$, $a_{13}=a_{23}=a_{31}=0$

ϕ *is +ve for all real values of* δ_1, δ_2, δ_3 *and it reduces to zero only when* $\delta_1=\delta_2=\delta_3=0$.

hence, the given Laplace's equation is elliptic.

(ii) *Here the operator* $\phi=\delta_1^2+\delta_2^2+\delta_3^2-\frac{1}{C^2}\delta_4^2$.

This can be both positive or negative. Hence, the equation is hyperbolic.

(iii) *Here* $a_{11}=a_{22}=a_{33}=1$, $a_{44}=0$ *and* $a_{12}=a_{13}=a_{14}=a_{21}=a_{23}=a_{24}=a_{31}=a_{32}=a_{34}=a_{41}=a_{42}=a_{43}=0$.

$$\therefore \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

hence, the equation is parabolic.

Example 5. *Classify the equation.*

$$(1-x^2)\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+(1-y^2)\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+3x^2y\frac{\partial z}{\partial y}-2z=0.$$

Solution: *Consider the operator* $\phi=A\delta_1^2+B\delta_1\delta_2+C\delta_2^2$, *where* $\delta_1\equiv\frac{\partial}{\partial x}$, $\delta_2\equiv\frac{\partial}{\partial y}$.

Here $A=1-x^2$, $B=-2xy$, $C=1-y^2$, *and so,*

$B^2-4AC=4x^2y^2-4(1-x^2)(1-y^2)=4(-1+x^2+y^2)$.

Since A, B, C are functions of x and y, the given differential equation is hyperbolic in the region where $B^2-4AC>0$ *i.e.,* $x^2+y^2>1$, *parabolic in the region where* $B^2-4AC=0$ *i.e., at points on the circle* $x^2+y^2=1$, *and elliptic in the region where* $B^2-4AC<0$ *i.e.,* $x^2+y^2<1$.

Example 6. *Find the regions where the following operator is hyperbolic, parabolic and elliptic*

(i) $\frac{\partial^2 u}{\partial t^2}+t\frac{\partial^2 u}{\partial x\partial t}+x\frac{\partial^2 u}{\partial x^2}$ (ii) $x^2\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u$ (iii) $t\frac{\partial^2 u}{\partial t^2}+2\frac{\partial^2 u}{\partial x\,\partial t}+x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}$

Solution:

(i) *Here* $A=1$, $B=t$, $C=x$.

$\therefore B^2-4AC=t^2-4x$. *Thus, the operator is hyperbolic if* $t^2-4x>0$ *i.e., if* $t^2>4x$, *parabolic if* $t^2=4x$ *and elliptic if* $t^2<4x$.

(ii) Here $A = x^2$, $B = 0$, $C = -1$.

$\therefore B^2 - 4AC = 4x^2$.

Thus, the operator is hyperbolic if $4x^2 > 0$ i.e., if $x^2 > 0$ i.e., if $x>0$ or $x<0$ parabolic if $4x^2 = 0$ i.e., if $x = 0$.

Since $4x^2$ cannot be negative so the operator cannot be elliptic.

(iii) Here $A = t$, $B = 2$, $C = x$.

$\therefore B^2 - 4AC = 4 - 4tx$. Thus the operator is hyperbolic if $4 - 4tx > 0$ i.e., if $tx < 1$, parabolic if $tx = 1$ and elliptic if $tx > 1$.

§ 5.2. CANONICAL FORMS (METHOD OF TRANSFORMATION)

Now, we shall consider the equation of the type $Rr + Ss + Tt + F(x, y, z, p, q) = 0$, ... (1)

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. We shall show that any equation of the type (1) can be reduced to one of the three canonical forms by a suitable change of the independent variables. Suppose we change the independent variables from x,y to u,v were $u = u(x, y)$, $v = v(x, y)$. ...(2)

Then, we have $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$, $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$.

$$\therefore \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial}{\partial v},\quad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial}{\partial v}.$$

Now, $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}.\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}\right)$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y\partial x}$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}.\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

Substituting these values of p, q, r, s and t in (1), it takes the form

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u\partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \quad ...(3)$$

where $A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2$ ...(4)

$B = R\frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$, ...(5)

$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}.\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2$ ...(6)

and the function F is the transformed form of the function f.
Now, the problem is to determine u and v so that the equation (3) takes the simplest possible form. The procedure is simple when the discriminant $S^2 - 4RT$ *of the quadratic equation.*

$$R\lambda^2 + S\lambda + T = 0 \qquad \dots (7)$$

is everywhere either positive, negative or zero, and we shall discuss these three cases separately.

Case I. $S^2 - 4RT > 0$. *If this condition is satisfied then the roots* λ_1, λ_2 *of the equation (7) are real and distinct. The coefficient of* $\frac{\partial^2 z}{\partial u^2}$ *and* $\frac{\partial^2 z}{\partial v^2}$ *in the equation (3) will vanish if we choose u and v such that* $\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y}$, ... (8) *and* $\frac{\partial v}{\partial x} = \lambda_2 \frac{\partial v}{\partial y}$, ... (9)

The differential equations (8) and (9) will determine the form of u and v as functions of x and y. For this, from (8), Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$. *The last member gives* $du = 0$ *i.e., u = constant. The first two members gives* $\frac{dy}{dx} + \lambda_1 = 0$. ... (10)

Let $f_1(x, y)$ = *constant be the solution of the equation (10).*
Then the solution of the equation (8) can be taken as $u = f_1(x, y)$. ... (11)

Similarly, if $f_2(x, y)$ = *constant is a solution of* $\frac{dy}{dx} + \lambda_2 = 0$, *then the solution of the equation (9) can be taken as* $v = f_2(x, y)$ ... (12)

Also, it can be easily seen that, in general,

$$AC - B^2 = (4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2,$$

So that when A and C are zero

$$B^2 = (S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \qquad \dots (13)$$

It follows that $B^2 > 0$ *since* $S^2 - 4RT > 0$ *and hence we can divide both sides of the equation by it.*
Thus making the substitutions defined by the equations (11) and (12), the equation (1) transforms to the form.

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right), \qquad \dots (14)$$

which is the canonical form in this case.

Case II. $S^2 - 4RT = 0$. *In this case, the roots of the equation (7) are equal. We define the function u as in Case I and take v to be any function of x and y, which is independent of u. Then, we have, as before,* $A = 0$.
Since $S^2 - 4RT = 0$, *hence from (13),* $B^2 = 0$ *i.e.,* $B = 0$. *On the other hand, in this case,* $C \neq 0$, *otherwise v would be a function of u.*
Putting $A = 0$, $B = 0$ *and dividing by C, we see that in this case the canonical form of the equation (1) is,*

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right). \qquad \dots (15)$$

Case III. $S^2 - 4RT < 0$. *Formally, it is the same as Case I except that now the roots of the equation (7) are complex.*

Proceeding as in Case I, we find that the equation (1) reduces to the form (14) but that the variables u, v are not real but are in fact complex conjugates. To find a real canonical form let

$u = \alpha + i\beta,\ v = \alpha - i\beta$

So that $\alpha = \frac{1}{2}(u + v),\ \beta = \frac{1}{2}i\,(v - u)$.

Now $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$

Similarly $\frac{\partial z}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$

$\therefore\ \frac{\partial^2 z}{\partial u\,\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right)$.

Thus, transforming the independent variables u, v to α, β *the desired canonical form is*

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \quad \ldots(16)$$

Second order partial differential equations of the type (1) are classified by their canonical forms; we say that an equation of this type is:

(i) Hyperbolic if $S^2 - 4RT > 0$ *(ii) Parabolic if* $S^2 - 4RT = 0$. *(iii) Elliptic if* $S^2 - 4RT < 0$.

Example 1. *Reduce the equation* $(y-1)\,r - (y^2-1)s + y\,(y-1)\,t + p - q = 2ye^{2x}\,(1-y)^3$ ... (1) *to canonical form and hence solve it.*

Solution: *Comparing the equation (1) with* $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, *we have*

$R = (y-1),\ S = -(y^2-1),\ T = y(y-1)$.

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ *therefore becomes*

$(y-1)\lambda^2 - (y^2-1)\lambda + y(y-1) = 0$ or $\lambda^2 - (y+1)\lambda + y = 0$ or $(\lambda - 1)(\lambda - y) = 0$

$\Rightarrow \lambda = 1, y$ *(real and distinct roots).*

The equations $\frac{dy}{dx} + 1 = 0$ *and* $\frac{dy}{dx} + y = 0$.

These on integration give $x + y$ = *constant and* ye^x = *constant, so that to change the independent variables from x, y to u, v, we take* $u = x + y$ *and* $v = ye^x$.

$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + ye^x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$,

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}$

$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2v\frac{\partial^2 z}{\partial u\,\partial v} + v^2\frac{\partial^2 z}{\partial v^2} + v\frac{\partial z}{\partial v}$

$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + e^x\frac{\partial z}{\partial v}$

$= \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right) + e^x\left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right) + e^x\frac{\partial z}{\partial v} = \frac{\partial^2 z}{\partial u^2} + (e^x + v)\frac{\partial^2 z}{\partial u\,\partial v} + ve^x\frac{\partial^2 z}{\partial v^2} + e^x\frac{\partial z}{\partial v}$

and $t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right)$

$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + e^x \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} + e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$

$= \frac{\partial^2 z}{\partial u^2} + 2e^x \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2}.$

Substituting these values in (1), it reduces to $(1-y)^3 e^x \frac{\partial^2 z}{\partial u \partial v} = 2y\, e^{2x} (1-y)^3$

or $\frac{\partial^2 z}{\partial u \partial v} = 2ye^x$ *or* $\frac{\partial^2 z}{\partial u \partial v} = 2v,$ ... (2)

which is the canonical form of the equation (1).

Integrating (2) w.r.t. v, we get $\frac{\partial z}{\partial u} = v^2 + \phi_1(u),$ ... (3)

where $\phi_1(u)$ *is an arbitrary function of u. Again integrating (3) w.r.t. u, we get*
$z = uv^2 + \psi_1(u) + \psi_2(v)$, *where* ψ_1 *is an integral of* ϕ_1 *and* ψ_2 *is an arbitrary function.*
or $z = (x+y)y^2e^{2x} + \psi_1(x+y) + \psi_2(ye^x)$.

Example 2. *Reduce the equation* $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ *to canonical form.*

Solution: *The given equation can be written as* $r - x^2 t = 0$. ... (1)

Comparing the equation (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, *we have* $R = 1, S = 0, T = -x^2$.

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ *therefore becomes* $\lambda^2 - x^2 = 0$

$\Rightarrow \lambda = x, -x$ *(real and distinct roots).*

The equations $\frac{dy}{dx} + \lambda_1 = 0$ *and* $\frac{dy}{dx} + \lambda_2 = 0$ *become* $\frac{dy}{dx} + x = 0$ *and* $\frac{dy}{dx} - x = 0$.

On integrating, we get $y + \frac{1}{2}x^2$ = *constant and* $y - \frac{1}{2}x^2$ = *constant, so that to change the independent variables from x, y to u, v, we take* $u = y + 1/2\, x^2$ *and* $v = y - 1/2\, x^2$.

$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v} = x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$

$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

$= x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$ *and* $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}.$

Substituting these values in (1), it reduces to $\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$

or $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, *which is the required canonical form of the given equation.*

Example 3. *Classify the type of following p.d.e's*

(a) $u_{xx} + 2yu_{xy} + xu_{yy} - u_x + u = 0$

(b) $2xyu_{xy} + xu_y + yu_x = 0$

Solution:

(a) *Here* $R = 1$, $S = y$ *and* $T = x$ $\therefore S^2 - 4RT = 4(y^2 - x)$

$\therefore$ *The equation is hyperbolic in the region* $y^2 > x$, *parabolic on the curve* $y^2 = x$ *and elliptic in the region* $y^2 < x$.

(b) *Here* $R = 0$, $S = 2xy$ *and* $T = 0$ $\therefore S^2 - 4RT = x^2y^2$

(c) *which is positive except on the coordinate axis. The equation is hyperbolic for all x, y except x = 0 or y = 0. Along the coordinate axes the equation degenerates to first – order and the second-order categories do not apply.*

Example 4. *Use the transformation* $\xi = \phi(x, y), \eta = \psi(x, y)$ *to express all the x-and y-derivatives in* $u_{xx} + 2bu_{xy} + du_x + eu_y + fu = g$ *in terms of* ξ *and* η

Solution: *By the chain rule* $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x}$ *or* $u_x = u_\xi \phi_x + u_\eta \psi_x$

By the product rule, $u_{xx} = u_\xi \phi_{xx} + (u_\xi)_x \phi_x + u_\eta \psi_{xx} + (u_\eta)_x \psi_x$

which, after using the chain rule to find $(u_\xi)_x$ *and* $(u_\eta)_x$ *yields*

$$u_{xx} = u_\xi \phi_{xx} + (u_{\xi\xi}\phi_x + u_{\xi\eta}\psi_x)\phi_x + u_\eta \psi_{xx} + (u_{\eta\xi}\phi_x + u_{\eta\eta}\psi_x)\psi_x$$

$$= u_{\xi\xi}\phi_x^{\,2} + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x^{\,2} + u_\xi\phi_{xx} + u_\eta\psi_{xx}$$

Similarly, $u_{yy} = u_\xi \phi_{yy} + (u_\xi)_y \phi_y + u_\eta \psi_{yy} + (u_\eta)_y \psi_y$

$$u_{yy} = u_\xi \phi_{yy} + (u_{\xi\xi}\phi_y + u_{\xi\eta}\psi_y)\phi_y + u_\eta \psi_{yy} + (u_{\eta\xi}\phi_y + u_{\eta\eta}\psi_y)\psi_y$$

$$= u_{\xi\xi}\phi_y^2 + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y^2 + u_\xi\phi_{yy} + u_\eta\psi_{yy}$$

Finally, $u_{xy} = u_\xi \phi_{xy} + (u_\xi)_y \phi_x + u_\eta \psi_{xy} + (u_\eta)_y \psi_x$

$$u_{xy} = u_\xi \phi_{yy} + (u_{\xi\xi}\phi_y + u_{\xi\eta}\psi_y)\phi_x + u_\eta \psi_{xy} + (u_{\eta\xi}\phi_y + u_{\eta\eta}\psi_y)\psi_x$$

$$= u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + u_\xi\phi_{xy} + u_\eta\psi_{xy}$$

Example 5. *Classify according to type and determine the characteristics of*

(a) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$ (b) $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0$

(c) $u_{xx} - x^2yu_{yy} = 0$ $(y > 0)$ (d) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$

Solution:

(a) $a = 2,\ b = -2,\ c = -6 \therefore b^2 - ac = 16$ and the equation is hyperbolic. The characteristics are determined by $\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = -1 \pm 2$

Thus, the lines $x - y =$ constant and $3x + y =$ constant are the characteristics of the equation.

(b) In this case $a = 4,\ b = 6,\ c = 9,\ b^2 - ac = 0$ and the equation is parabolic

If follows that there is a single family of characteristics given by $\frac{dy}{dx} = \frac{b}{a} = \frac{3}{2}$

or $2y - 3x =$ constant

(c) In the region $y > 0$, $b^2 - ac = x^2 y$ is positive, so that the equation is hyperbolic type. The characteristics are determined by $\frac{dy}{dx} = \pm x\sqrt{y}$ or $\frac{dy}{\sqrt{y}} \pm xdx = 0$

From which it follows that the characteristics are $x^2 \pm 4\sqrt{y} =$ constant

(d) $b^2 - ac = \left(e^{x+y}\right)^2 - e^{2x}e^{2y} = 0$ and the equation is parabolic. The characteristics are given by

$\frac{dy}{dx} = \frac{e^{x+y}}{e^{2x}}$ or $e^{-x}dx - e^{-y}dy = 0$

$e^{-x} - e^{-y} =$ constant.

Example 6. *Transform the hyperbolic equations.*

(a) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

(b) $u_{xx} - x^2 y u_{yy} = 0\ (y > 0)$ to a canonical form with principal part $u_{\xi\eta}$.

Solution:

(a) The characteristics of the given equation are $x - y =$ constant and $3x + y =$ constant

$\therefore$ We take $\xi = \phi(x, y) = x - y$ and $\eta = \psi(x, y) = 3x + y$

Transforming, we get $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 16u_{\xi\eta} + u_\xi + 3u_\eta$

$\therefore$ the required canonical form is $u_{\xi\eta} + \frac{1}{16}u_\xi + \frac{3}{16}u_\eta = 0$

(b) The characteristics are $x^2 \pm 4\sqrt{y} =$ constant.

$\therefore$ We take $\xi = \phi(x, y) = x^2 + 4\sqrt{y}$ $\eta = \psi(x, y) = x^2 - 4\sqrt{y}$

We get $u_{xx} - x^2 y u_{yy} = 16x^2 u_{\xi\eta} + (2 + x^2 y^{-1/2})u_\xi + (2 - x^2 y^{-1/2})u_\eta$

$= 8(\xi + \eta)u_{\xi\eta} + \left(\frac{6\xi + 2\eta}{\xi - \eta}\right)u_\xi - \left(\frac{2\xi + 6\eta}{\xi - \eta}\right)u_\eta$

where the last equality follows from $x^2 = (\xi + \eta)/2$ and $y^{1/2} = (\xi - \eta)/8$

the desired canonical form is $u_{\xi\eta} + \frac{3\xi + \eta}{4(\xi^2 - \eta^2)}u_\xi - \frac{\xi + 3\eta}{4(\xi^2 - \eta^2)}u_\eta = 0$, $(\xi > \eta)$

PRACTICE SET - 1

1. The number of characteristic curves of the PDE $(x^2 + 2y)u_{xx} + (y^3 - y + x)u_{yy} + x^2(y - 1)u_{xy} + 3u_x + u = 0$ passing through the point $x = 1, y = 1$ is

(CSIR UGC NET JUNE-2011)

(A) 0 (B) 1 (C) 2 (D) 3

2. Let a, b, c be continuous functions defined on $\mathbb{R}^2$. Let V_1, V_2, V_3 be non empty subsets of $\mathbb{R}^2$ and the PDE:

$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = 0$ is elliptic in V_1, parabolic in V_2 and hyperbolic in V_3, then **(CSIR UGC NET DEC-2013)**

(A) V_1, V_2 and V_3 are open sets in $\mathbb{R}^2$ (B) V_1 and V_3 are open sets in $\mathbb{R}^2$
(C) V_1 and V_2 are open sets in $\mathbb{R}^2$ (D) V_2 and V_3 are open sets in $\mathbb{R}^2$

3. In the region $x > 0, y > 0$, the partial differential equation

$(x^2 - y^2)\frac{\partial^2 u}{\partial x^2} + 2(x^2 + y^2)\frac{\partial^2 u}{\partial x \partial y} + (x^2 - y^2)\frac{\partial^2 u}{\partial y^2} = 0$, is not **(GATE-2006)**

(A) changes type (B) elliptic (C) parabolic (D) hyperbolic

4. The second order PDE $u_{yy} - yu_{xx} + x^3u = 0$, is **(CSIR UGC NET JUNE-2012)**

(A) elliptic for all $x \in \mathbb{R}, y \in \mathbb{R}$. (B) parabolic for all $x \in \mathbb{R}, y \in \mathbb{R}$.

(C) elliptic for all $x \in \mathbb{R}, y < 0$. (D) hyperbolic for all $x \in \mathbb{R}, y < 0$.

5. The second order partial differential equation

$$\frac{(x-y)^2}{4}\frac{\partial^2 u}{\partial x^2} + (x-y)\sin(x^2+y^2)\frac{\partial^2 u}{\partial x \partial y} + \cos^2(x^2+y^2)\frac{\partial^2 u}{\partial y^2} + (x-y)\frac{\partial u}{\partial x} + \sin^2(x^2+y^2)\frac{\partial u}{\partial y} + u = 0,$$

is **(CSIR UGC NET DEC-2011)**

(A) Elliptic in the region $\{(x,y) : x \neq y, x^2 + y^2 < \pi/6\}$

(B) Hyperbolic in the region $\left\{(x,y) : x \neq y, -\frac{\pi}{4} < x^2 + y^2 < \frac{3\pi}{4}\right\}$

(C) Elliptic in the region $\left\{(x,y) : x \neq y, \frac{\pi}{4} < x^2 + y^2 < \frac{3\pi}{4}\right\}$

(D) Hyperbolic in the region $\left\{(x,y) : x \neq y, x^2 + y^2 < \frac{\pi}{4}\right\}$

6. The partial differential equation $y\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0$, is hyperbolic in **(CSIR UGC NET DEC-2012)**

(A) the second and fourth quadrants. (B) the first and second quadrants.
(C) the second and third quadrants. (D) the first and third quadrants.

§ 5.3. BOUNDARY VALUE PROBLEM

1. *Heat Equation:* $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$

   In two dimensional $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$

2. *Wave Equation:* $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

3. *Laplace Equation:* $\nabla^2 u = 0$

5.3.1. Solution of one dimensional wave equation with initial value problem

Let problem be $u_{tt} - c^2 u_{xx} = 0,\ -\infty < x < \infty, t \geq 0$ ...(1)

InitialConditions $u(x,0) = \eta(x), u_t(x,0) = v(x)$ ...(2)

Let characteristic lines be $\xi = x - ct, \eta = x + ct$

$\therefore$ *we have* $u_x = \mu_\xi \xi_x + u_\eta \eta_x = \mu_\xi + \mu_n$

$u_t = \mu_\xi \xi_t + u_\eta \eta_t = c(\mu_\eta - \mu_\xi)$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(u_\xi + u_n) = \mu_{\xi\xi} + 2u_{\xi\eta} + \mu_{\eta\eta}$ ...(3)

Similarly, $\frac{\partial^2 u}{\partial t^2} = c^2(\mu_{\xi\xi} - 2u_{\xi\eta} + \mu_{\eta\eta})$ ...(4)

Substituting (3) and (4) in equation (1), we get $4\mu_{\xi\eta} = 0$

Integrating, $\mu(\xi,\eta) = \phi(\xi) + \psi(\eta)$, *where* ϕ *and* ψ *are arbitrary functions*

$\therefore$ *the general solution is given by*

$\mu(x,t) = \phi(x - ct) + \psi(x + ct)$ ...(5)

Substituting initial conditions in (5), we get

$\phi(x) + \psi(x) = \eta(x)$ ...(6)

$c[\phi'(x) - \psi'(x)] = v(x)$

Integrating, $\phi(x) - \psi(x) = \frac{1}{c}\int_0^x v(\xi)d\xi$ ...(7)

From equation (6) and (7), $\phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(\xi)d\xi$ *and* $\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(\xi)d\xi$

$\therefore$ *equation (5) gives* $u(x,t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi$,

which is known as D'Alembert's solution of one dimensional wave equation.

If v=0 then $u(x,t)=\frac{1}{2}[\eta(x+ct)-\eta(x-ct)]$

5.3.2. Solution of wave equation with initial and boundary conditions.

Consider one dimensional wave equation

$u_{tt}=c^2u_{xx},\ 0\le x\le L, t>0$ ...(1)

Boundary Conditions: $u(0,t)=0=(L,t), t>0$

Initial Conditions: $u(x,0)=f(x), u_t(x,0)=g(x)$

Let u(x,t) be solution of (1)

u (x, t) = X(x) T(t) and substituting into equation (1) we obtain

$$X\frac{d^2T}{dt^2}=c^2T\frac{d^2X}{dx^2}$$

i.e., $\frac{d^2X/dx^2}{X}=\frac{d^2T/dt^2}{c^2T}=k$ ...(2)

Case- I *When k>0, we have* $k=\lambda^2$. *Then* $\frac{d^2X}{dx^2}-\lambda^2X=0$ *and* $\frac{d^2T}{dt^2}-c^2\lambda^2T=0$

Their solution can be put in the form

$X=c_1e^{\lambda x}+c_2e^{-\lambda x}$

$T=c_3e^{c\lambda t}+c_4e^{-c\lambda t}$

Therefore, $u(x,t)=(c_1e^{\lambda x}+c_2e^{-\lambda x})(c_3e^{c\lambda t}+c_4e^{-c\lambda t})$ ...(3)

Now, use the BCs:

$u(0,t)=0=(c_1+c_2)(c_3e^{c\lambda t}+c_4e^{-c\lambda t})$ ...(4)

which imply that $c_1+c_2=0$. *Also,* $u(L,t)=0$ *gives* $c_1e^{-\lambda L}+c_2e^{-\lambda L}=0$ ...(5)

equations (4) and (5) possess a non trivial solution iff

$\begin{vmatrix}1 & 1\\ e^{\lambda L} & e^{-\lambda L}\end{vmatrix}=e^{-\lambda L}-e^{\lambda L}=0$ *or* $1-e^{2\lambda L}=0$ *implying* $e^{2\lambda L}=1$ *or* $\lambda L=0$

This implies that $\lambda=0$, *since L cannot be zero, which is against the assumption hence, this solution is not acceptable.*

Case II: *Let k=0. Then we have* $\frac{d^2X}{dx^2}=0, \frac{d^2T}{dt^2}=0$

Their solutions are found to be X=Ax + B, T= Ct + D

therefore, the required solution of the PDE is u(x, t) = (Ax + B)(Ct + D) using the boundary conditions, we have

u(0,t) = 0 = B(Ct + D), implying B = 0

u(L, t) = 0 = AL (Ct + D), implying A = 0

hence, only a trivial solution is possible. Since we are looking for a non trivial solution, consider the following case.

Case III: *When k<0, say* $k=-\lambda^2$, *the differential equations are*

$$\frac{d^2X}{dx^2}+\lambda^2X=0,\ \frac{d^2T}{dt^2}+c^2\lambda^2T=0$$

Their general solutions give

$$u(x,t)=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos c\lambda t+c_4\sin c\lambda t) \qquad ...(6)$$

using the boundary conditions C : u (0, t) = 0 we obtain $c_1=0$.

Also, using the Boundary Conditions: u(L, t) = 0, we get sin $\lambda L=0$ *implying that* $\lambda=n\pi/L, n=1,2,....,$ *which are the eigenvalues. Hence the possible solution is*

$$u_n(x,t)=\sin\frac{n\pi x}{L}\left(A\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right), n=1,2... \qquad ...(7)$$

Using the superposition principle, we have

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right) \qquad ...(8)$$

The initial conditions gives

$$u(x,o)=f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}$$

which is a half range Fourier sine series, where $A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$ ...(9)

$$Also,\ u_t(x,0)=g(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}\left(\frac{n\pi}{L}C\right)$$

which is also a half range sine series, where

$$B_n=\frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx \qquad ...(10)$$

Hence, the required physically meaningful solution is obtained from equation (8) where A_n *and* B_n *are given by equations (9) and (10).* $u_n(x,t)$ *given by equation (7) are called normal modes of vibration and* $n\pi c/L=\omega_n, n=1,2,...$ *are called normal frequencies.*

Working Rule: *Suppose problem is* $u_{tt}=c^2u_{xx}$ ...(1)

Subject to boundary condition u(0,t) = u (L,t) = 0 $\forall$ *t and initial conditions u(x,0)=f(x),* $u_t(x,0)=g(x)$, $0\le x\le L$.

$$\therefore u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right)\frac{\sin n\pi x}{L},\ where$$

$$A_n=\frac{2}{L}\int_0^L f(x)\frac{\sin n\pi x}{L}dx\ \ \&\ \ B_n=\frac{2}{n\pi c}\int_0^L g(x)\frac{\sin n\pi x}{L}dx \qquad ....(2)$$

(i) *Find particular values of C,L,f(x) and g(x).*

(ii) *Substitute values of L, f(x) and g(x) in (1) and find* A_n *and* B_n

(iii) *Put values of* A_n *and* B_n *in u(x,t) i.e., solution of problem.*

Case-I: *If f(x) = 0, then*

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L} \quad \text{where } B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

Case-II: *If g(x)=0, then* $u(x,t)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}+\cos\frac{n\pi ct}{L}$, *where* $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

Example 1. *Let u(x,t) be solution of initial boundary value problem*

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\ 0<x<\infty, t>0$$

$$u(x,0) = \cos\left(\frac{\pi x}{2}\right)\ 0 \le x < \infty$$

$$\frac{\partial u}{\partial t}(x,0)=0, 0\le x<\infty$$

$$\frac{\partial u}{\partial x}(0,t)=0,\ t\ge 0$$

Solution: $u_{xx}=u_{tt}$,

$$u(x,0)=\eta(x)=\cos\frac{\pi x}{2}$$

$$u_t(x,0)=0=v(\xi)$$

$\therefore$ *By D' Alembert principle solution is given by*

$$u(x,t)=\frac{1}{2}[\eta(x+ct)-\eta(x-ct)]$$

$$u(x,t)=\frac{1}{2}\left[\cos\frac{\pi(x+ct)}{2}-\cos\frac{\pi(x-ct)}{2}\right]=-\sin\frac{\pi x}{2}\sin\frac{\pi ct}{2}$$

Example 2. $u_{tt}=c^2u_{xx}, 0\le x\le l, t\ge 0$ *subject to* $u(0,t)=0, u(l,t)=0\ \forall t$

$$u(x,0)=0, u_t(x,0)=b\sin^3\left(\frac{\pi x}{l}\right)$$

Solution: *By D' Alembert principle solution is*

$$u(x,t)=\frac{1}{2}[\eta(x+ct)-\eta(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}v(\xi)\,d\xi \ \therefore u(x,t)=\frac{b}{2c}\int_{x-ct}^{x+ct}\frac{\sin^3 \pi\xi}{l}d\xi$$

$$=\frac{b}{2c}\int_{x-ct}^{x+ct}\left(\frac{3}{4}\frac{\sin\pi\xi}{l}-\frac{1}{4}\frac{\sin 3\pi\xi}{l}\right)d\xi$$

$$=\frac{3bl}{8c\pi}\left[\cos\pi\frac{(x+ct)}{l}-\cos\frac{\pi}{l}(x-ct)\right]-\frac{lb}{24c\pi}\left[\cos\frac{3\pi}{l}(x+ct)-\frac{\cos 3\pi}{l}(x-ct)\right]$$

$$=\frac{3bl}{4c\pi}\sin\frac{\pi x}{l}\sin\frac{\pi}{l}ct-\frac{bl}{12c\pi}\sin\frac{3\pi x}{l}\frac{\sin 3\pi ct}{l}=\frac{bl}{12c\pi}\left[9\frac{\sin\pi x}{l}\frac{\sin\pi}{l}ct-\sin\frac{3\pi x}{l}\frac{\sin 3\pi ct}{l}\right]$$

Uniqueness Theorem. *The solution to the wave equation* $u_{tt} = c^2 u_{xx}, 0 < x < L, t > 0$
satisfying the initial conditions
$u(x,0) = f(x), 0 \le x \le L$
$u_t(x,0) = g(x), 0 \le x \le L$
and boundary conditions u(0, t) = u(L, t) = 0 where u(x, t) is twice differentiable function w.r.t to x and t. is always unique

5.3.3. Solution of Heat Equation

Let the problem be $C\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial u}{\partial t}$ ...(1)

subject to boundary conditions $u(0,t)=0$ *and* $u(a,t)=0 \ \forall t$. *and* ...(2)
initial condition is $u(x,0)=f(x)$, $0<x<a$. ...(3)
Suppose that solution is of the form $u(x,t)=X(x)T(t)$ ...(4)
where X and T are respectively the functions of x and t alone. Using the values of (4) in (1), we get

$\frac{X''}{X} = \frac{T'}{CT} = \mu$ *(say)* ...(5)

where μ *is a separation constant.*
From equation (5), we can deduce that $X'' - \mu X = 0$ ...(6)
and $T' = \mu CT$ ...(7)
using (2) and (4), we get $X(0) = 0$ *and* $X(a) = 0$ ...(8)
Now, we want to solve (6) subject to the boundary condition. Hence, we have the following cases:

Case I: *Let* $\mu=0$. *Then solution of (6) is given by* $X(x) = Ax + B$
Using (8), we get $A = B = 0 \Rightarrow X(x) = 0 \ \Rightarrow u = 0$, *which does not satisfy (3)*

Case II: *Let* $\mu = \lambda^2, \lambda \ne 0$. *In this case, solution of (6) is given by* $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$
Using (8), we get A+B=0 and $0 = Ae^{a\lambda} + De^{-a\lambda} \Rightarrow A = B = 0 \quad \Rightarrow X = 0 \quad \Rightarrow u = 0$
Thus, we reject this case also.

Case III: *Let* $\mu = -\lambda^2, \lambda \ne 0$. *In this case, solution of (6) is given by* $X(x) = A\cos\lambda x + B\sin\lambda x$
Using (8), we get
$A=0$ *and* $A\cos\lambda a + B\sin\lambda a = 0$
Let $B \ne 0$, *then* $\sin\lambda a = 0$
$\Rightarrow \lambda a = n\pi, n = 1,2,...$
$\Rightarrow \lambda = \frac{n\pi}{a}, n = 1,2,...$
Therefore, non zero solution of (6) is given
by $X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$

putting, $\lambda = \frac{n\pi}{a}$ *in (7), we get* $\frac{dT}{T} = -\frac{n^2\pi^2 C}{a^2} dt \Rightarrow \frac{dT}{T} = -C_n^2 dt$

whose solution is given by $T_n(t) = D_n e^{-C_n^2 t}$

Thus, we have $u_n(x,t) = X_n(x)T_n(t) = E_n \sin\left(\frac{n\pi x}{a}\right)e^{-C_n^2 t}$ ...(9)

$\therefore$ *The more general solution of (1) is given by*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)e^{-C_n^2 t} \quad ...(10)$$

Putting t = 0 in (10) and using (3) we get $f(x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)$

which is a fourier sine series, thus the constants E_n *are given by*

$$E_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx, n = 1,2,3,...$$

***Example** 3. Solve the equation in region* $0 \le x \le \pi, t \ge 0$, *subject to conditions*

(1) *T remains finite as* $t \to \infty$

(2) *T =0 if x=0 and* π *for all t.*

(3) $At\ t = 0, T = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$

***Solution:** The solution of equation is* $T(x,t) = \sum_{n=1}^{\infty} E_n \sin nx e^{-\alpha n^2 t}$.

$$T(x,0) = \sum_{n=1}^{\infty} E_n \sin nx. \text{ where}$$

$$E_n = \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-x\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left((-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right] = \frac{4\sin(n\pi/2)}{n^2\pi}$$

$$\therefore T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha n^2 t}\sin(n\pi/2)}{n^2}\sin nx$$

PRACTICE SET - 2

1. *Let* $u(x,t)$ *be the bounded solution of* $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ *with* $u(x,0) = \frac{e^{2x}-1}{e^{2x}+1}$. *Then,* $\lim_{t\to\infty} u(1,t)$ *equals* **(GATE-2006)**

   (A) $-1/2$ (B) $1/2$ (C) -1 (D) 1

2. *Consider the Neumann problem* $u_{xx} + u_{yy} = 0$, $0 < x < \pi, -1 < y < 1$, $u_x(0,y) = u_x(\pi,y) = 0$, $u_y(x,-1) = 0, u_y(x,1) = \alpha + \beta\sin(x)$. *The problem admits solution for* **(GATE-2007)**

(A) $\alpha = 0, \beta = 1$ (B) $\alpha = -1, \beta = \frac{\pi}{2}$ (C) $\alpha = 1, \beta = \frac{\pi}{2}$ (D) $\alpha = 1, \beta =$ [illegible]

3. *Consider the boundary value problem* $u_{xx} + u_{yy} = 0, x \in (0, \pi), y \in (0, \pi), u(x, 0) = u(x, \pi) = u(0, y) = 0$. *Any solution of this boundary value problem is of the form* **(GATE-2008)**

(A) $\sum_{n=1}^{\infty} a_n \sinh nx \sin ny$ (B) $\sum_{n=1}^{\infty} a_n \cosh nx \sin ny$

(C) $\sum_{n=1}^{\infty} a_n \sinh nx \cos ny$ (D) $\sum_{n=1}^{\infty} a_n \cosh nx \cos ny$

4. *Consider the wave equation* $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, 0 < x < \pi, t > 0$, *with* $u(0,t) = u(\pi, t) = 0$, $u(x,0) = \sin x$ *and* $\frac{\partial u}{\partial t} = 0$ *at* $t = 0$. *Then* $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is* **(GATE-2010)**

(A) 2 (B) 1 (C) 0 (D) -1

5. *If* $u(x,t)$ *is the D'Alembert's solution to the wave equation* $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, x \in \mathbb{R}, t > 0$, *with the condition* $u(x,0) = 0$ *and* $\frac{\partial u}{\partial t}(x,0) = \cos x$, *then* $u\left(0, \frac{\pi}{4}\right)$ *is* ________ **(GATE-2014)**

KEY POINTS

- For second order pde, $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$ and R,S, T are functions of x and y. Then the equation is
  Hyperbolic if $S^2 - 4RT > 0$
  Parabolic if $S^2 - 4RT = 0$
  Elliptic if $S^2 - 4RT < 0$

- The number of real characteristic curves in second order p.d.e.,
  Hyperbolic = 2
  Parabolic = 1
  Elliptic = 0 (No real characteristic curve).

- For the canonical transformations we change the independent variables from x, y to u, v where $u = u(x, y)$, $v = v(x, y)$. To determine u and v, solve $R\lambda^2 + S\lambda + T = 0$, we get roots of this equation. Say λ_1 and λ_2, then solve $\frac{dy}{dx} + \lambda_1 = 0$ ...(1) and $\frac{dy}{dx} + \lambda_2 = 0$ ...(2). We get $u = f_1(x, y)$ and $v = f_2(x, y)$, the solution of (1) and (2) respectively.

- The canonical form (1) For hyperbolic is $\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$

  (2) For parabolic $\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$

  (3) For elliptic $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$

- Heat Equation: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$

- Wave Equation: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

- Laplace Equation: $\nabla^2 u = 0$

- The solution of one dimensional wave equation with initial value problem $u_{tt} - c^2 u_{xx} = 0, -\infty < x < \infty, t \geq 0$, $u(x,0) = \eta(x), u_t(x,0) = v(x)$, by D'Alembert's formula is

$$u(x,t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi$$

SOLVED QUESTIONS FROM PREVIOUS PAPERS

1. *The partial differential equation $y^3u_{xx} - (x^2+1)u_{yy} = 0$, is not* *(GATE-1998)*

(A) parabolic in $\{(x,y): x<0\}$ *(B) hyperbolic in $\{(x,y): y>0\}$*

(C) elliptic in $\mathbb{R}^2$. *(D) parabolic in $\{(x,y): x>0\}$*

Solution:- *The given PDE is $y^3u_{xx} - (x^2 + 1)\, u_{yy} = 0$*

Here $S = 0$, $R = y^3$, $T = -(x^2 + 1)$

$S^2 - 4RT = 0 + 4y^3 (x^2 + 1) = 4y^3 (x^2 + 1)$

when $y > 0 \Rightarrow S^2 - 4RT > 0$

$\Rightarrow$ hyperbolic in $\{(x, y) : y > 0\}$

Option (B) is clearly true

In option (A),

parabolic in $\{(x, y); x < 0\}$

If $x < 0$, $S^2 - 4RT > 0$ in $y > 0$ and $S^2 - 4RT < 0$ in $y < 0$

$\Rightarrow$ not true

In Option (C), it is clearly not true

Option (D) is similarly with option (A) not true.

Ans. is (A), (C) and (D).

2. *Transform the parabolic equations*

(a) $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0$

(b) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$

to a canonical form with principal part $u_{\xi\xi}$

Solution:

(a) $\eta = \psi(x,y) = 3x - 2y$

Any $\phi(x,y)$ satisfying $\phi_x\psi_y - \phi_y\psi_x \neq 0$ can be chosen as the second variable

$\therefore$ *Choose* $\xi = \phi(x,y) = y$

$\therefore \quad 4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 9u_{\xi\xi} - 3u_\eta + u$

hence the canonical form is $u_{\xi\xi} - \frac{1}{3}u_\eta + \frac{1}{9}u = 0$

(b) the characteristics is $e^{-x} - e^{-y} =$ constant

$\therefore \eta = \psi(x,y) = e^{-x} - e^{-y}$

Choose the other new variable as

$\xi = \phi(x,y) = x \quad \therefore \quad e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy}$

$= e^{2x}u_{\xi\xi} + 2u_\eta = e^{2\xi}u_{\xi\xi} + 2u_\eta$.

hence the canonical form is $u_{\xi\xi} + 2e^{-2\xi}u_\eta = 0$

3. *Transform the elliptic equations*

(a) $u_{xx} + 2u_{xy} + 17u_{yy} = 0$

(b) $x^2u_{xx}+y^2u_{yy}=0(x>0,y>0)$

to canonical form with principal part $u_{\xi\xi}+u_{\eta\eta}$

Solution:

(a) $a=1, b=1, c=17$

$$\therefore \frac{dy}{dx}=\frac{b+i\sqrt{|b^2-ac|}}{a}$$

Gives $\frac{dy}{dx}=1+i\,4$ which has the solution $z=(x-y)+i4x=$ constant

Thus setting $\xi=\phi(x,y)=x-y$, $\eta=\psi(x,y)=4x$

We have $u_{xx}+2u_{xy}+17u_{yy}=16u_{\xi\xi}+16u_{\eta\eta}$

hence the canonical form $u_{\xi\xi}+u_{\eta\eta}=0$ (Laplace's equation)

(b) $\frac{dy}{dx}=i\frac{y}{x}$ with solution $z=\log x+i\log y=$ constant

Getting $\xi=\phi(x)=\log x$ and $\eta=\psi(y)=\log y$

We have, $x^2u_{xx}+y^2u_{yy}=u_{\xi\xi}+u_{\eta\eta}-u_{\xi}-u_{\eta}=0$ as the required canonical form.

4. The diffusion equation $\frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial t}, u=u(x,t), u(0,t)=u(\pi,t)=0, u(x,0)=\cos x\sin 5x$ admits the solution (GATE-2012)

(A) $\frac{e^{-36t}}{2}[\sin 6x+e^{20t}\sin 4x]$ (B) $\frac{e^{-36t}}{2}[\sin 4x+e^{20t}\sin 6x]$

(C) $\frac{e^{-20t}}{2}[\sin 3x+e^{15t}\sin 5x]$ (D) $\frac{e^{-36t}}{2}[\sin 5x+e^{20t}\sin x]$

Solution: (A) By putting the conditions in the options given. (A) is the only option that satisfies. Therefore

$$u(x,t)=\frac{e^{-36t}}{2}[\sin 6x+e^{20t}\sin 4x]$$

ASSIGNMENT - 5.1

NOTE: CHOOSE THE BEST OPTION

1. PDE of second order in canonical form is $Rr+Ss+Tt+f(x,y,z,p,q)=0$, then $S^2-4RT>0$ represent
   (A) hyperbolic (B) parabolic
   (C) elliptic (D) straight line

2. In PDE, $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$, then
   (A) $r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x \partial y}$ (B) $r=\frac{\partial^2 z}{\partial y^2}, s=\frac{\partial^2 z}{\partial x \partial y}$
   (C) $r=\frac{\partial^2 z}{\partial y^2}, s=\frac{\partial^2 z}{\partial x^2}$ (D) $r=\frac{\partial^2 z}{\partial x \partial y}, s=\frac{\partial^2 z}{\partial x^2}$

3. Classify the equation $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$
   (A) hyperbolic (B) parabolic (C) elliptic (D) none of these

4. Classify the equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$
   (A) hyperbolic (B) parabolic (C) elliptic (D) none of these

5. Classify the equation $\frac{\partial^2 u}{\partial t^2}-4\frac{\partial^2 u}{\partial x \partial t}+4\frac{\partial^2 u}{\partial x^2}$
   (A) elliptic (B) hyperbolic (C) parabolic (D) none of these

6. Let u(x, t) be the solution of $u_{tt}=u_{xx}$; $0<x<1$, $t>0$, $u(x,0)=x(1-x)$, $u_t(x,0)=0$. Then $u\left(\frac{1}{2},\frac{1}{4}\right)$ is
   (A) $\frac{3}{16}$ (B) $\frac{1}{4}$ (C) $\frac{3}{4}$ (D) $\frac{1}{16}$

7. The solution of the Cauchy's problem $u_{yy}(x,y)-u_{xx}(x,y)=0$; $u(x,0)=0$, $u_y(x,0)=x$ is $u(x,y)=$
   (A) $\frac{x}{y}$ (B) xy (C) $xy+\frac{x}{y}$ (D) 0

8. The solution of the initial value problem $u_{tt}=4u_{xx}$, $t>0$, $-\infty<x<\infty$. Satisfying the conditions $u(x,0)=x$, $u_t(x,0)=0$, is
   (A) x (B) $\frac{x^2}{2}$ (C) $2x$ (D) $2t$

9. *Let u be a solution of the initial value problem* $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$; $u(x, 0) = x^2$, $\frac{\partial u}{\partial t}(x, 0) = 0$. *Then* $u(0,1)$ *equals to*

(A) 1 (B) -2 (C) 2 (D) $\frac{1}{2}$

10. *Let* $u = \psi(x, t)$ *be the solution to the initial value problem* $u_{tt} = u_{xx}$ *for* $-\infty < x < \infty$, $t > 0$ *with* $u(x, 0) = \sin(x)$, $u_t(x, 0) = \cos(x)$ *then the value of* $\psi(\pi/2, \pi/6)$ *is*

(A) $\sqrt{3}/2$ (B) 1/2 (C) $1/\sqrt{2}$ (D) 1

11. *The variables* ξ *and* η *which reduce the differential equation* $\frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0$ *to the canonical form are*

(A) $\xi = y^2 + \frac{1}{2}x, \eta = y^2 - \frac{1}{2}x$ (B) $\xi = y + \frac{1}{2}x^2, \eta = y - \frac{1}{2}x^2$

(C) $\xi = y + x^2, \eta = y - x^2$ (D) $\xi = y^2 + x, \eta = y^2 - x$

12. $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x + 2y)$ *has the solution*

(A) $z = f_1(2y + x) + x f_2(2y + x) + 2x^2 \log(x + 2y)$
(B) $z = 2x^2 \log(x + 2y)$
(C) $z = 2f_1(2y + x) + 2x^2 \log(x + 2y)$
(D) *none of these*

13. *Classify the equation* $\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2}$

(A) elliptic (B) hyperbolic (C) parabolic (D) none of these

14. *Let* $u(x,t)$ *be the solution of the initial value problem* $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$; $u(x,0) = \sin x$; $\frac{\partial u}{\partial t}(x,0) = 1$

Then, $u(\pi, \pi/2)$ *equals*

(A) $\pi/2$ (B) $1-(\pi/2)$ (C) 1 (D) $1+\pi/2$

15. *Consider the wave equation*

$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi, t > 0$, *with* $u(0,t) = u(\pi,t) = 0$ $u(x,0) = \sin x$ *and* $\frac{\partial u}{\partial t} = 0$ *at* $t = 0$.

Then $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right) =$

(A) 2 (B) 1 (C) 0 (D) -1

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

16. *Which of the following represent hyperbolic?*

(A) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial y}$ (B) $\frac{\partial^2 z}{\partial x^2}=c^2\frac{\partial^2 z}{\partial y^2}$

(C) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}$ (D) $\frac{\partial^2 z}{\partial x^2}=c\frac{\partial z}{\partial y}$

17. *For the PDE* $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$

(A) general solution is $z=\phi_1(y-x)+x\phi_2(y-x)$

(B) general solution is $z=(x+y)\phi_1(x-y)+\phi_2(x-y)$

(C) characteristic curves are orthogonal

(D) canonical form is $\frac{\partial^2 z}{\partial v^2}=0$

18. *Consider PDE* $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$ $(c>0)$ *such that* $u(x,0)=f(x)$ *and* $u_t(x,0)=g(x)$, *then*

(A) if $g(x)=0$, then $u(x,t)=\frac{1}{2}[f(x-ct)+f(x+ct)]$

(B) if $f(x)=\sin x$, $g(x)=0$, then $u(x,t)=\sin x\cos(ct)$

(C) if $g(x)=1$, then $u(x,t)=f(x,t)+t$

(D) canonical form of PDE is $\frac{\partial^2 z}{\partial u\partial v}=0$

19. *Consider a general PDE of second order of the form* $Rr+Ss+Tt+f=0$, *where R,S,T are continuous functions of x and y, then PDE at a point (x,y) is*

(A) parabolic, if $S^2-4RT=0$ (B) hyperbolic, if $S^2-4RT<0$

(C) elliptic, if $S^2-4RT>0$ (D) hyperbolic or elliptic, when $S^2-4RT\neq 0$

20. *Consider PDE* $\frac{\partial^2 u}{\partial x^2}=x^2\frac{\partial^2 u}{\partial y^2}$, *then select the correct option/s*

(A) PDE is elliptic for $x\neq 0$ (B) PDE is parabolic for $x=0$

(C) characteristic equations are $dy\pm x^2dx=0$ (D) characteristic curves are $y=c\pm\frac{x^2}{2}$

21. *Consider the PDE* $x^2r-y^2t+px-qy=x^2$, *select the correct options.*

(A) PDE is hyperbolic for $x\neq 0\ \&\ y\neq 0$

(B) $xy = C_1$ and $\frac{x}{y} = C_2$ are the characteristic curves

(C) $x^2 y = C_1$ and $\frac{x}{y^2} = C_2$ are the characteristic curves

(D) canonical form of PDE is $4\frac{\partial^2 z}{\partial u \partial v} = 1$ where, $u=xy$, $v=x/y$

22. *Which of the followings is/are true?*
    (A) two dimensional Laplace's equation is elliptic
    (B) one dimensional wave equation is hyperbolic
    (C) one dimensional heat equation is parabolic
    (D) canonical form of one-dimensional wave equation is of the form $\frac{\partial^2 z}{\partial u \partial v} = 0$

23. *Which of the following is/are true?*
    (A) canonical form of hyperbolic PDE is of the form $\frac{\partial^2 z}{\partial u \partial v} = \phi(u, v, z, z_u, z_v)$
    (B) canonical form of parabolic PDE is of the form $\frac{\partial^2 z}{\partial v^2} = \phi(u, v, z, z_u, z_v)$
    (C) canonical form of elliptic PDE does not exist .
    (D) canonical form of elliptic PDE is $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi(\alpha, \beta, z, z_\alpha, z_\beta)$

24. Given, the equation $u_{xx} + \frac{2N}{x} u_x = \frac{1}{a^2} u_{tt}$ where N and a are constant, then
    (A) the given partial differential equation is hyperbolic
    (B) the given partial differential equation is parabolic
    (C) the canonical form is $\mu_{\xi\eta} + \frac{N}{\xi + \eta}(u_\xi + u_\eta) = 0$
    (D) $\mu_{\xi\eta} = 0$ is the canonical form

25. Given, the equation $(\sin^2 x)u_{xx} + (\sin 2x)u_{xy} + (\cos^2 x)u_{yy} = x$, then
    (A) the equation is parabolic
    (B) the equation is hyperbolic
    (C) the canonical equation is $(\cos^2 x)u_{\eta\eta} + u_\xi = x$
    (D) the canonical equation is $u_{\eta\eta} = \frac{\sin^{-1}(e^{\eta-\xi}) - u_\xi}{1 - 2^{2(\eta-\xi)}}$

26. *Given,* $y^2u_{xx} - 2xyu_{xy} + x^2u_y = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$, *then*

(A) *the canonical equation is* $4x^2y^2u_{\eta\eta} = 0$

(B) *the canonical equation is* $u_{\eta\eta} = 0$

(C) *the solution is* $u = y^2 f(x^2 + y^2) + g(x^2 + y^2)$

(D) *no solution exist*

27. *Given, the equation* $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$, *then*

(A) *the discriminant of the given partial differential equation is* $B^2 - 4AC = -4(1+x^2)(1+y^2) < 0$

(B) *the given partial differential equation is elliptic.*

(C) *the characteristic equation are* $\frac{dy}{dx} = \pm\sqrt{\frac{1+y^2}{1+x^2}}$

(D) *canonical equation for the given partial differential equation is* $u_{\alpha\alpha} + u_{\beta\beta} = 0$

28. *Particular solution of differential equation* $r + s - 6t = y \cos x$ *cannot be*

(A) $y \sin x + \cos x$

(B) $\sin x + y \cos x$

(C) $y \sin x - \cos x$

(D) $\sin x - y \cos x$

29. *Given, the wave equation* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, *then*

(A) *the general solution of the wave equation is* $y(x,t) = \phi(x + ct) + \psi(x - ct)$

(B) *no general solution exist*

(C) *the D'Alembert's solution of the given equation is* $y(x,t) = f(x + ct) + f(x - ct)$

(D) *no D'Alembert's solution exist*

ANSWERS TO EXERCISES

(PRACTICE SET – 1)

1. (C)   2. (B)   3. (A,B,C)   4. (C)   5. (A,B)   6. (A)

(PRACTICE SET – 2)

1. (A)   2. (B)   3. (A)   4. (D)   5. (Ans. 0.71)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT – 5.1

1. (A)   2. (A)   3. (A)   4. (C)   5. (C)   6. (A)   7. (B)
8. (A)   9. (A)   10. (A)   11. (B)   12. (A)   13. (C)   14. (A)
15. (D)
16. (B, C)   17. (A,B,C,D)   18. (A,B,C,D)   19. (A,D)   20. (B,D)   21. (A,B,D)   22. (A,B,C,D)
23. (A,B,D)   24. (A,C)   25. (A,D)   26. (A,B)   27. (A,B,D)   28. (A,B,C)   29. (A,C)