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QARSHI DAVLAT UNIVERSITETI

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ODDIY DIFFERENSIAL TENGLAMALAR

5130100 – ”Matematika” ta’lim yo‘nalishi talabalari uchun darslik

«Sano-standart» nashriyoti
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Mazkur kitob oddiy differensial tenglamalar kursi bo‘yicha 5130100 – ”matematika” ta’lim yo‘nalishi talabalari uchun darslik. Unda amaldagi o‘quv dasturidagi barcha mavzular to‘la yoritilgan. Darslikda nazariyaning matematik aniqligi va qat’iyligi bilan bir qatorda amaliy masalalarni yechishga ham katta e’tibor berilgan. Bundan tashqari, bilimlarni chuqrlashtirish va mustahkamlash maqsadida mustaqil yechish uchun nazariy masalalar ham tavsiya etilgan. Bu masalalarning ko‘pini yechish uchun ko‘rsatmalar va/yoki javoblar keltirilgan.

Kitobdan differensial tenglamalarni mustaqil o‘rganmoqchi bo‘lgan barcha xohlovchilar unumli foydalanishlari mumkin.

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MUNDARIJA

SO‘ZBOSHI

Mazkur kitob 5130100 − “Matematika” yo‘nalishi bo‘yicha tahsil oluvchi bakalavriat talabalari uchun oddiy differensial tenglamalar o‘quv kursi dasturining barcha mavzularini o‘z ichiga qamrab olgan bo‘lib, u shu kursni o‘rganish uchun darslik sifatida tayyorlangan. Mustaqil o‘zlashtirish mavzulari ham to‘la yoritilgan. Kitobni yozishda bu boradagi chet el tajribalaridan keng foydalanish bilan bir qatorda muallifning QarshiDU da uzoq yillar davomida mazkur o‘quv kursdan ma’ruzalar o‘qish va amaliy mashg‘ulotlar olib borish tajribalariga ham asoslangan.

Darslik materiallari shartli ravishda 17 modulga (bo‘lakka) bo‘lingan (ajratilgan). Bunday bo‘linish o‘zlashtirishni nazorat qilishni optimallashtirish uchun maqsadga muvofiq deb hisobladik.

Matematika olamini o‘rganishga bel bog‘lagan har bir o‘quvchi chidamli va matonatli bo‘lmog‘i lozim. Uchragan tushunchalar, fikrlarning mantiqiy tuzilishi, teorema va formulalarning ma’nosi va o‘rinlilik shartlari to‘g‘risida, albatta, fikrlash hamda matematik masalalar yechish kerak bo‘ladi. Masalalarni yechish jarayonida Siz oddiy qoida va metodlardan foydalanish bilan bir qatorda o‘zingiz ham elementar ilmiy ijodlar qilishingizga to‘g‘ri keladi. Sizda ijodiy fikrlar ham paydo bo‘lib, keyinchalik ular mustahkamlanadi va matematik ijodingizda ajralmas yo‘ldoshingizga aylanadi.

Darslikda nazariyani tushuntiruvchi ko‘pdan-ko‘p misollar to‘la yechimlari bilan birgalikda keltirilgan. Bundan tashqari, har bir paragraf oxirida mustaqil yechish uchun nazariy masalalar taklif etilgan. Bu masalalarning yechimlari va javoblari ham berilgan. Ularni avval mustaqil yechishga urinib ko‘ring. Lekin, agar siz uni uzoq vaqt hal qila olmasangiz, yechilishiga qarab oling, so‘ngra esa, albatta, o‘zingiz mustaqil uning yechilishini tiklang.

Ba’zi nazariy masalalarni ilg‘or (kuchli) talabalarga taqiqot topshiriqlari sifatida taklif etish mumkin.

Qo‘shimcha misol va masalalarni muallifning ”Differensial tenglamalardan misollar, masalalar va mustaqil ishlar” (Qarshi:

Nasaf, 2014) o‘quv qo‘llanmasidan topish mumkin. Bundan tashqari, kitob oxirida keltirilgan adabiyotlardan ham foydlanish maqsadga muvofiq bo‘ladi.

Muallif qo‘llanmadagi o‘quv materiallarini aprobatsiyadan o‘tkazishda yordam bergan barcha shogirdlari va talabalaridan hamda kasbdoshlaridan, bundan tashqari, taqrizchilardan ham, minnatdor ekanligini mamnunlik bilan e’tirof etadi.

Kitob haqidagi fikr va mulohazalaringizni nosir_d@mail.ru elektron manzilga yozsangiz, muallif sizdan minnatdor bo‘ladi.

Asosiy belgilashlar ro‘yxati

$\forall$ — har qanday, ixtiyoriy, har bir (umumiylik kvantori).

$\exists$ — mavjud, kamida bitta mavjud (mavjudlik kvantori).

$\Rightarrow$ — kelib chiqadi (implikatsiya belgisi).

$\Leftrightarrow$ — teng kuchli (ekvivalent).

$\stackrel{def}{\Leftrightarrow}$ — ta’rifga ko‘ra ekvivalent (teng kuchli).

$\stackrel{def}{=}$ — ta’rifga ko‘ra teng.

$\{x\in E\,|P(x)\}$ — E to‘plamning $P(x)$ xossaga ega bo‘lgan barcha x elementlari to‘plami.

$\mathbb{N}$ — natural sonlar to‘plami; n - natural son, $n\in\mathbb{N}$.

$\mathbb{R}$ — haqiqiy sonlar to‘plami.

$\mathbb{C}$ — kompleks sonlar to‘plami.

$\mathbb{R}^n$ — n o‘lchamli haqiqiy Evklid fazosi.

$c, c_1, c_2, \ldots$ — ixtiyoriy o‘zgarmaslar (doimiylar).

const — o‘zgzrmas (doimiy).

$(a,b)\stackrel{def}{=}\{x\in\mathbb{R}\,|\,a<x<b\}$ $(a<b)$ — interval.

$[a,b]\stackrel{def}{=}\{x\in\mathbb{R}\,|\,a\le x\le b\}$ $(a<b)$ — segment.

$(a,b]\stackrel{def}{=}\{x\in\mathbb{R}\,|\,a<x\le b\}$ $(a<b)$ — yarim segment.

$[a,b)\stackrel{def}{=}\{x\in\mathbb{R}\,|\,a\le x<b\}$ $(a<b)$ — yarim segment.

$\mathbb{R}_+\stackrel{def}{=}[0,+\infty)$.

I — sonli oraliq (ichi bo‘sh bo‘lmagan bog‘lanishli sonli to‘plam).

D (yoki G) — soha ($\mathbb{R}^n$ dagi) , ya’ni bo‘shmas, ochiq va bog‘lanishli to‘plam.

max$E$ — E sonli to‘plamning maksimumi (eng katta elementi).

min$E$ — E sonli to‘plamning minimumi (eng kichik elementi).

sup$E$ — E sonli to‘plamning supremumi (yuqori crgaralarning eng kichigi, aniq yuqori chegara).

inf$E$ — E sonli to‘plamning infimumi (quyi crgaralarning eng kattasi, aniq quyi chegara).

$\|\ \|$ — norma (yoki matritsa) belgisi.

∂E — E to‘plamning chegarasi.

E^C — E to‘plamning (qaralayotgan fazogacha) to‘ldiruvchisi.

$\bar{E}$ — E to‘plamning yopig‘i (yop[ilmasi).

$B_\delta(a)$ — δ radiusli a markazli (ochiq) shar, $B_\delta = B_\delta(o)$.

$X\times Y$ — to‘plamlarning to‘g‘ri (Dekart) ko‘paytmasi.

$\cup$, $\cap$, $\setminus$ — mos ravishda to‘plamlar birlashmasi, kesishmasi, ayirmasi.

$f{:}X\to Y$ — X to‘plamda aniqlangan, qiymatlari Y to‘plamda joylashgan f funksiya (akslantirish).

$D(f)$ — f funksiyaning aniqlanish to‘plami (sohasi).

$f|_E$ — f funksiyaning E to‘plamga torayishi.

$f|_a = f(a)$

$g\circ f$ — f va g funksiyalar kompozitsiyasi (ketma-ket bajarilishi).

$f(x)=o(g(x)),\ x\to a,$ — asimptotik tenglik (kichik o);

u $f(x)=\varepsilon(x)\cdot g(x),\ \lim\limits_{x\to a}\varepsilon(x)=0$, ekanligini anglatadi.

$f(x)=O(g(x)),\ x\to a,$— (katta o); u $f(x)$ funksiya $g(x)$ ni a nuqtaning biror atrofida chegaralangan $h(x)$ funksiyaga ko‘paytirishdan hosil bo‘lishini ($f(x)= h(x)\cdot g(x)$) anglatadi.

$C(X,Y)$ — barcha uzluksiz $f: X\to Y$ funksiyalar sinfi (to‘plami).

$C(X)=C(X,\mathbb{R})$.

$C^k(X,Y)$) — k- tartibli barcha hosilalari (demak, undan past tartiblilari ham) uzluksiz bo‘lgan $f: X\to Y$ funksiyalar sinfi.

$C^k(X)=C^k(X,\mathbb{R})$

dist(X,Y) — to‘plamlar orasidagi masofa (distance – masofa).

dim$X$ — X fazoning o‘lchami (dimension – o‘lcham).

deg$P$ — P ko‘phadning darajasi (degree – daraja).

$\mathrm{M}_{n\times n}(\mathbb{R})$ ($\mathrm{M}_{n\times n}(\mathbb{C})$) — haqiqiy (kompleks) sonlardan tuzilgan $n\times n$ o‘lchamli matritsalar to‘plami.

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}, \boldsymbol{h}, \boldsymbol{f}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q},\ldots$ (qalin harflar) — vektorlar.

Silliq funksiya — qralayotgan masalada barcha kerak bo‘lgan tartibli uzluksiz hosilalarga ega bo‘lgan funksiya.

MYaT — mavjudlik va jagonalik teoremasi.

DT — differensial tenglama.

ODT (=DT) — oddiy differensial tenglama.

🗝 — masala (misol) yechilishining, isbotning boshlanishi belgisi.

👍 — masala (misol) yechilishining, isbotning tugallanganligi belgisi.

MODUL 1. ODDIY DIFFERENSIAL TENGLAMALAR FANIGA KIRISH

§ 1.1. Oddiy differensial tenglama va uning yechimi tushunchalari

Umumiy ma’lumotlar (tushunchalar). Bizga o‘rta maktabdan yaxshi tanish bo‘lgan kvadrat tenglamada noma’lum sondan iborat bo‘ladi. Differensial tenglamada esa noma’lum matematikaning songa qaraganda murakkabroq ob’yekti bo‘lgan funksiyadan iborat. Differensial tenglama deb noma’lun funksiya hosilalari va argument(lar) qatnashgan tenglamaga aytiladi (bu ta’rif aniq va qat’iy emas, aniq va qat’iy ta’rifni keyinroq keltiramiz). Differensial tenglamalar ikki turga bo‘linadi: oddiy differensial tenglamalar (qisqacha differensial tenglamalar) va xususiy hosilali differensial tenglamalar. Oddiy differensial tenglamada noma’lum funksiya bir dona (odatda haqiqiy) erkli o‘zgaruvchiga bog‘liq, xususiy hosilali tenglamada esa noma’lum funksiya ikki yoki undan ortiq argumentlarga bog‘liq bo‘ladi. Differensial tenglamaning tartibi deb shu tenglamada qatnashgan noma’lum funksiya hosilasining eng yuqori tartibiga aytiladi.

Masalan, ushbu

$$y'^3 - y^2\ln(2-x) + 1 + \sin x = 0$$

($y = y(x)$ – bir o‘zgaruvchining noma’lum funksiyasi),

$$x''' + x''x'^2 - 2\sin t = 0$$

($x = x(t)$ – bir o‘zgaruvchining noma’lum funksiyasi),

tenglamalar mos ravishda birinchi va uchinchi tartibli oddiy differensial tenglamalar,

$$u'_x + uu'_y = 0$$

($u = u(x, y)$ – ikki o‘zgaruvchining noma’lum funksiyasi),

$$u''_{xx} + u''_{yy} + u''_{zz} = 0$$

($u = u(x, y, z)$ – uch o‘zgaruvchining noma’lum funksiyasi),

tenglamalar esa mos ravishda birinchi va ikkinchi tartibli xususiy hosilali differensial tenglamalardir.

Tenglamaning yechimi deb noma’lumning tenglamani qanoatlantiruvchi “qiymatiga” aytiladi. Differensial tenglamada noma’lum – funksiya, uning “qiymati” esa konkret funksiya. Funksiyaning differensial tenglamani qanoatlantirishi uni shu tenglamani ayniyatga aylantirishini anglatadi. Yechimning mavjud yoki mavjud emasligi yechimning qaysi to‘plamda (sinfda) izlanishiga bog‘liq. Masalan, agar $x^2+1=0$ tenglamaning yechimi haqiqiy sonlar to‘plamida izlansa, yechim mavjud emas, yechim kompleks sonlar to‘plamida izlansa esa, ikkita yechim mavjud: $x=\pm i$ ($i-$mavhum birlik).

Ko‘pdan-ko‘p amaliy, fizik, ekologik va h.k. masalalarni yechish differensial tenglamalarni yechishga keltiriladi. Bu yerda biz ularning ba’zilarini keltiramiz. Shunga o‘xshash boshqa masalalar bilan o‘quv kursi davomida tanishamiz.

1. ***Radioaktiv yemirilish.*** $m(t)$ bilan radioaktiv moddaning t paytdagi massasini belgilaylik. Bizga $m(t)$funksiyani topish kerak bo‘lsin. Fizikadan ma’lumki, radioaktiv moddaning yemirilish tezligi $-\dfrac{dm(t)}{dt}$ ($\dfrac{dm(t)}{dt}$ hosila o‘sish tezligini ifodalaydi) mavjud modda miqdoriga to‘g‘ri proporsional, ya’ni

$$-\frac{dm(t)}{dt}=km(t) \text{ yoki qisqaroq } m'=-km; \qquad (1.1.1)$$

bu yerda o‘zgarmas $k, k>0,$ – proporsionallik koeffitsienti. Demak, m=$m(t)$ noma’lum funksiya (1.1.1) differensial tenglamani qanoatlantiradi.

2. ***Nyutonning sovish qonuni.*** Temperaturasining o‘zgarish qonuniyati $T(t)$ ni topish kerak bo‘lgan jism (masalan, bir piyola issiq choy) temperaturasi T_m ma’lum bo‘lgan muhitda (masalan, xonada) joylashgan bo‘lsin. Nyutonning sovish qonuniga ko‘ra jism temperaturasining kamayish tezligi $-T'(t)$ jism va muhit temperaturalarining ayirmasi $T(t)-T_m$ga to‘g‘ri proporsional, ya’ni

$$T'(t)=-\lambda\left(T(t)-T_m\right), \qquad (1.1.2)$$

bu yerda $\lambda=\text{const}>0-$ proporsionallik koeffitsienti (jism temperaturasi muhit temperaturasidan katta bo‘lganda jismning temperaturasi kamayadi, ya’ni u soviydi, $T'(t)<0$ bo‘ladi). Demak, jism temperaturasi $T(t)$ (1.1.2) differensial tenglamadan topiladi.

3. ***To‘g‘ri chiziqli harakat.*** Inersial sanoq sistemasida Ox oqi bo‘ylab m massali moddiy nuqta $f(t,x,x')$ kuch ta’sirida harakat qilsin, bunda $x = x(t)$ – nuqtaning t paytdagi koordinatasi, $x' = x'(t)$ – tezligi. Nyutonning ikkinchi qonuniga ko‘ra massaning tezlanishga ko‘paytmasi ta’sir etuvchi kuchga teng, ya’ni

$$mx'' = f(t,x,x') \tag{1.1.3}$$

Harakat ($x = x(t)$ funksiya)ni topish ana shu ikkinchi tartibli differensial tenglamani yechish demakdir.

Agar moddiy nuqta elastiklik kuchi $f(t,x,x') = -kx,\ k = \text{const} > 0,$ ta’sirida harakat qilsa, garmonik ossilyator tenglamasi deb ataluvchi

$$x'' + \omega^2 x = 0,\ \omega = \sqrt{k/m}\ , \tag{1.1.4}$$

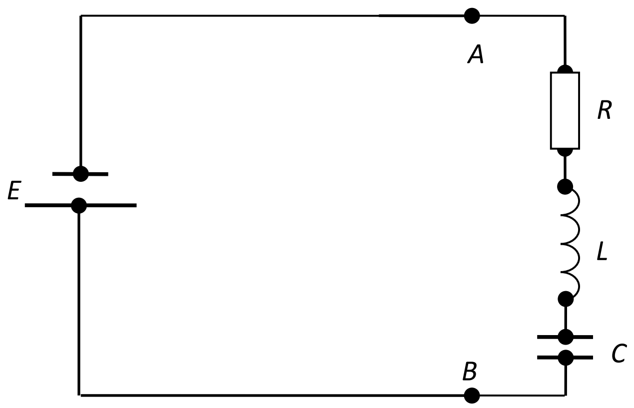
differensial tenglama hosil bo‘ladi.

Agar $f(t,x,x')$ funksiya oshkor ko‘rinishda x ga bog‘liq bo‘lmasa, ya’ni $f(t,x,x') \equiv f(t,x')$ bo‘lsa, (1.1.3) harakat tenglamasidan $v = v(t) = x'(t)$ tezlikka nisbatan quyidagi birinchi tartibli differensial tenglama hosil bo‘ladi:

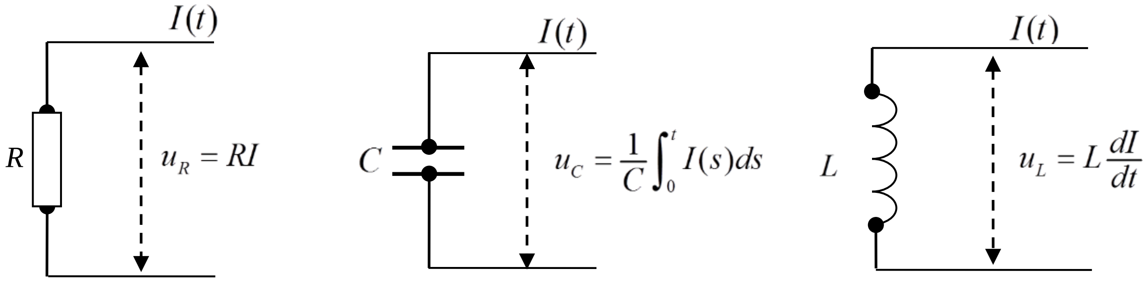
$$mv' = f(t,v)$$

Endi qat’iy ta’riflarga o‘taylik.

4. RLCE ***elektr zanjir.*** Quyidagi rasmda ko‘rsatilgan kuchlanish manbai E, R qarshilik, L g‘altak va C kondensatorli iste’molchilardan iborat elektr zanjirida $I(t)\ (t - \text{vaqt})$ elektr tokining o‘zgarish qonunini topish masalasini qaraylik:



Fizika kursidan ma’lumki, R aktiv qarshilikdagi u_R, L g‘altakdagi u_L va C kondesatordagi u_C kuchlanishlar (potensiallar ayirmasi) mos ravishda quyidagilarga teng bo‘ladi:



Iste'molchining A va B nuqtalaridagi potentsiallar ayirmasi t paytda $E=E(t)$ bo'ladi. Butun zanjir uchun Om (yoki zanjir uchun Kirxgoffning 2-) qonuniniga ko'ra $E=E(t)$ kuchlanish A va B nuqtalar orasidagi kuchlanishlar yig'indisiga teng:

$$L\frac{dI(t)}{dt}+RI(t)+\frac{1}{C}\int_0^t I(\tau)d\tau=E(t).$$

Bu tenglamadan differensial tenglamaga o'tish uchun uning har ikkala tomonidan hosila olish kerak. Bu yerdagi funksiyalarni silliq deb hisoblab, tenglamaning har ikkala tomonini differensiallaymiz. Natijada $I(t)$ elektr toki uchun quyidagi ikkinchi tartibli differensial tenglama hosil bo'ladi:

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE(t)}{dt}.$$

Agar elektr zanjiridan kuchlanish manbasi olib tashlansa ($E(t)=0$ bo'lsa), passiv elektr zanjiri hosil bo'ladi, ya'ni

$$L\frac{dI(t)}{dt}+RI(t)+\frac{1}{C}\int_0^t I(\tau)d\tau=0 \text{ yoki } \frac{d^2I}{dt^2}+\frac{R}{L}\frac{dI}{dt}+\frac{1}{LC}I=0$$

differensial tenglamaga ega bo'lamiz.

Endi qat'iy ta'riflarga o'taylik. $n\in\mathbb{N}$ bo'lsin. n**- tartibli oddiy differensial tenglama** deb

$$F(x,y,y',y'',...,y^{(n)})=0 \qquad (1.1.7)$$

ko'rinishdagi tenglamaga aytiladi; bu yerda $x-$ argument (erkli haqiqiy o'zgaruvchi), $y=y(x)-$ noma'lum haqiqiy funksiya, $y'=y'(x),\ y''=y''(x),\ ...,y^{(n)}=y^{(n)}(x)-$ noma'lum funksiya hosilalari, $F(x,y,p_1,p_2,\dots,p_n)-$biror $G\subset\mathbb{R}^{2+n}$ sohada (soha bu − bo'shmas, ochiq va bog'lanishli to'plam) aniqlangan $2+n$ ta haqiqiy o'zgaruvchining uzluksiz haqiqiy funksiyasi, ya'ni $F\in C(G,\mathbb{R})$ (yoki qisqaroq: $F \in C(G)$), bu funksiya p_n o‘zgaruvchiga tom ma’noda bog‘liq, ya’ni u p_n argumentning funksiyasi sifatida (boshqa argumentlar tayinlanganda) o‘zgarmasga aylanmaydi deb faraz qilinadi; oxirgi bu shart tenglamaning n- tartibli ekanligini ta’minlaydi.

$n=1$ holida birinchi tartibli oddiy differensial tenglama hosil bo‘ladi. Birinchi tartibli oddiy differensial tenglama

$F(x, y, y')=0$

ko‘rinishga ega.

Qisqalik uchun “oddiy differensial tenglama” atamasi o‘rniga “differensial tenglama” atamasini ishlatamiz.

Differensial tenglamaning yechimi oraliqda aniqlangan bo‘ladi. I bilan sonlar o‘qidagi biror **oraliq**ni (ya’ni bog‘lanishli va kamida bitta ichki nuqtaga ega bo‘lgan sonli to‘plamni) belgilaymiz. Matematik analizdan ma’lumki, oraliq ushbu

$$(-\infty,+\infty),\ (-\infty,b),\ (-\infty,b],\ [a,b),\ (a,b],\ (a,b),\ [a,b],\ (a,+\infty),\ [a,+\infty)$$

sonli to‘plamlarning biridir (bu terda $a<b$).

$n-$ tartibli differensial tenglama (1.1.7) ning I **oraliqda (aniqlangan) yechimi** deb quyidagi shartlarni qanoatlantiruvchi $y=\varphi(x)$ funksiyaga aytiladi:

1^0. $\varphi(x) \in C^n(I)$, ya’ni $\varphi^{(n)}(x)$ hosila I oraliqda uzluksiz (yechim $C^n(I)$ sinfda izlanadi),

2^0. $\forall x \in I\ \ F(x, \varphi(x), \varphi'(x), \varphi''(x), \ldots, \varphi^{(n)}(x))=0$, ya’ni $y=\varphi(x)$ funksiya (1.1.7) tenglamani I oraliqda ayniyatga aylantiradi.

Eslatma 1. Ta’rifda $y=\varphi(x)$ yechim **oraliqda** aniqlangan ekanligi muhim, ya’ni yechim faqat oraliqda aniqlangan bo‘lishi kerak.

Eslatma 2. Ta’rifdagi 2^0 shartning bajarilishi uchun

$\forall x \in I\ \ (x, \varphi(x), \varphi'(x), \varphi''(x), \ldots, \varphi^{(n)}(x)) \in G$

bo‘lishi, ya’ni $(x, \varphi(x), \varphi'(x), \varphi''(x), \ldots, \varphi^{(n)}(x))$, $x \in I$, nuqtalarning (1.1.7) tenglama aniqlangan sohaga tegishli bo‘lishi zarurdir. Demak, (1.1.7) tenglama aniqlangan G soha $y=\varphi(x)$ yechimning aniqlanish oralig‘i I ni, $y=\varphi(x)$ va $y'=\varphi'(x), \ldots, y^{(n)}=\varphi^{(n)}(x)$ hosilalarning o‘zgarish to‘plamlarini ma’lum ma’noda chegaralaydi.

Eslatma 3. Agar I oraliqning chegaraviy nuqtasi Iga tegishli bo‘lsa, bu nuqtadagi hosila sifatida mos bir tomonli hosila tushuniladi. Masalan, $I=[a,b)$ oraliqda aniqlangan (qaralayotgan) $y=\varphi(x)$ funksiyaning $x=a$ nuqtadagi $\varphi'(a)$ hosilasi sifatida shu nuqtadagi o‘ng hosila $\varphi'(a+0)$ qabul qilinadi.

Yechim oshkormas ko‘rinishda ham berilishi mumkin. Faraz qilaylik, $\Phi(x,y)=0$ tenglama biror I oraliqda $y=\varphi(x)\in C^n(I)\ (=C^n(I,\mathbb{R}))$ funksiyani oshkormas ko‘rinishda aniqlasin va bu $y=\varphi(x)$ funksiya I da (1.1.7) tenglamani qanoatlantirsin. U holda $\Phi(x,y)=0$ munosabat (1.1.7) tenglamaning (I oraliqda) oshkormas ko‘rinishdagi yechimi deyiladi.

Tushunarliki, yechim parametrik ko‘rinishda ham berilishi mumkin.

Differensial tenglama yechimining grafigi **integral chiziq** deb ataladi. Ravshanki, integral chiziq silliq chiziqdir. Biz keyinroq birinchi tartibli differensial tenglama uchun integral chiziq tushunchasining ma’nosini kengaytiramiz.

Misol 1. Ushbu

$$xy'^2-2yy'+x=0$$

birinchi tartibli differensial tenglamada $F(x,y,p)=xp^2-2yp+x$. Ravshanki, bu F funksiya $G=\mathbb{R}^3$ da aniqlangan va uzluksiz (aslida cheksiz differensiallanuvchi).

a) $y=x$ funksiya berilgan differensial tenglamaning $(-\infty,+\infty)$ intervalda yechimi. Haqiqatan ham,

1^0. $\varphi(x)=x$ funksiya $(-\infty,+\infty)$ intervalda uzluksiz differensiallanuvchi, chunki $y'=\varphi'(x)=1\in C(\mathbb{R})$;

2^0. ixtiyoriy $x\in(-\infty,+\infty)$ nuqtada

$$xy'^2-2yy'+x=x\cdot 1^2-2\cdot x\cdot 1+x=0.$$

b) Lekin $y=2x-1$ funksiya hech qanday $I\subset\mathbb{R}$ oraliqda berilgan differensial tenglamaning yechimi bo‘la olmaydi. Chunki, bu holda $y'=\varphi'(x)=2\in C(I)$ bo‘lsa-da,

$$xy'^2-2yy'+x=x\cdot 4-2(2x-1)\cdot 2+x=-3x+4$$

tenglik hech qanday I oraliqda ayniyat emas: u I ning ko‘pi bilan bitta nuqtasida qanoatlanishi mumkin ($x=4/3$ bo‘lganda) xolos, I da esa cheksiz ko‘p nuqtalar mavjud.

Misol 2. Ushbu

$$(1+x^2)y'-xy+x=0 \qquad (1.1.8)$$

birinchi tartibli differensial tenglama uchun $F(x,y,p)=(1+x^2)p-xy+x$ funksiya $G=\mathbb{R}^3$ da aniqlangan va silliq. Ushbu

$$y=1+c\sqrt{1+x^2} \qquad (1.1.9)$$

funksiya (1.1.8) tenglamaning $(-\infty,+\infty)$ oraliqda yechimi; bu yerda $c-$ ixtiyoriy o‘zgarmas son. Haqiqatan ham, c tayinlanganda bu funksiya $x\in(-\infty,+\infty)$ oraliqda uzluksiz differensiallanuvchi,

$$y'(x)=\frac{cx}{\sqrt{1+x^2}}\in C\big((-\infty,+\infty)\big),$$

va u berilgan tenglamani $(-\infty,+\infty)$ oraliqda qanoatlantiradi:

$$(1+x^2)y'-xy+x=(1+x^2)\frac{cx}{\sqrt{1+x^2}}-x\left(1+c\sqrt{1+x^2}\right)+x=0.$$

Misol 3. Ushbu $y'-f(x)=0$ ($f\in C(I)$ – berilgan), sodda differensial tenglama matematik analiz kursida o‘rganilgan. Uning barcha yechimlari $y=\int f(x)dx+c$ formula bilan berilishi isbotlangan, $c-$ ixtiyoriy o‘zgarmas son. Bu yerda va bundan buyon $\int f(x)dx$ aniqmas integral $f(x)$ ning biror tayin boshlang‘ich funksiyasini (barcha boshlang‘ich funksiyalarni emas!) belgilaydi.

Yuqori tartibli hosilaga nisbatan yechilgan tenglama. Faraz qilaylik, (1.1.7) tenglama $y^{(n)}$ hosilaga nisbatan yechilgan bo‘lsin:

$$y^{(n)}=f(x,y,y',\ldots,y^{(n-1)}). \qquad (1.1.10)$$

Bu yerda berilgan $f(x,y,p_1,\ldots,p_{n-1})$ funksiya biror $D\subset\mathbb{R}^{1+n}$ sohada uzluksiz (haqiqiy) funksiya deb hisoblanadi, ya’ni $f\in C(D)$. (1.1.10) tenglama **yuqori tartibli hosilaga nisbatan yechilgan differensial tenglama** deyiladi. $n=1$ holida hosilaga nisbatan yechilgan $y'=f(x,y)$ tenglama **normal ko‘rinishdagi** birinchi tartibli differensial tenglama deb ataladi.

Yuqorida keltirilgan misoldagi $(1+x^2)y'-xy+x=0$ (1.1.8) tenglamani osongina hosilaga nisbatan yechilgan ko‘rinishga keltirish mumkin:

$$y' = \frac{xy - x}{1 + x^2}.$$

Umumiy holda (1.1.7) tenglamani (1.1.10) ko‘rinishga keltirish murakkab masala. Bunday masalalar matematik analizda o‘rganiladi (oshkormas funksiya haqidagi teoremani eslang). Biz (1.1.7) tenglamadan y' ni topishda to‘xtalmasdan birdaniga (1.1.10) tenglama berilgan va $f \in C(D)$ deb faraz qilamiz.

Agar $y = \varphi(x)$ funksiya I oraliqda aniqlangan bo‘lib, u

1^0. $\varphi(x) \in C^n(I)$,

2^0. $\forall x \in I \quad \varphi^{(n)}(x) = f\left(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n-1)}(x)\right)$

shartlarni qanoatlantirsa, $y = \varphi(x)$ funksiya (1.1.4) **tenglamaning** I **oraliqda (aniqlangan) yechimi** deyiladi.

Eslatma. Bu yerda yechimdan 1^0 shart o‘rniga $\varphi(x)$ning I da n marta differensiallanuvchi bo‘lishini talab etsak , $\varphi(x) \in C^n(I)$ ham bo‘ladi, chunki bu holda $\varphi(x) \in C^{n-1}(I)$ bo‘lib, $f(x, y, p_1, \ldots, p_{n-1})$ funksiyaning uzluksizligi va 2^0 shartga ko‘ra $\varphi^{(n)}(x) = f\left(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n-1)}(x)\right)$ murakkab funksiya ham I da uzluksizdir.

Tenglamani yechish, har doimgidek, uning barcha yechimlarini topish demakdir.

Yuqoridagi misollardan ko‘rinadiki, differensial tenglama cheksiz ko‘p yechimlarga ega bo‘lishi mumkin.

Ba’zan barcha yechimlarni bitta formula bilan berish mumkin bo‘ladi. Agar $y = \varphi(x, c_1, c_2, \ldots, c_n)$ funksiya $c_1, c_2, \ldots, c_n$ o‘zgarmaslarning ixtiyoriy joiz (yo‘l qo‘yilgan) qiymatlarida (1.1.10) differensial tenglamaning yechimini bersa hamda differensial tenglamaning har qanday yechimi shu $y = \varphi(x, c_1, c_2, \ldots, c_n)$ formuladan $c_1, c_2, \ldots, c_n$ larning biror joiz qiymatida hosil bo‘lsa, u holda $y = \varphi(x, c_1, c_2, \ldots, c_n)$ funksiya berilgan (1.1.10) tenglamaning **umumiy yechimi** deyiladi. Umumiy yechim oshkormas ko‘rinishda $\Phi(x, y, c_1, c_2, \ldots, c_n) = 0$ tenglama bilan yoki parametrik $x = x(p, c_1, c_2, \ldots, c_n)$, $y = y(p, c_1, c_2, \ldots, c_n)$ (p – yechimdagi nuqtalarni belgilovchi parametr, $c_1, c_2, \ldots, c_n$ esa yechimlarni belgilaydi) ko‘rinishda ham berilishi mumkin.

Yuqoridagi misol 2 da biz ixtiyoriy tayin c da $y=1+c\sqrt{1+x^2}$ (1.1.9) funksiya $(1+x^2)y'-xy+x=0$ (1.1.8) tenglamaning yechimi ekanligini tekshirgan edik. Endi bu tenglamaning har qanday yechimi (1.1.9) ko‘rinishda bo‘lishini isbotlaylik. (1.1.8) tenglamaning ixtiyoriy $y=y(x)$ yechimi berilgan bo‘lsin:

$$(1+x^2)y'(x)-xy(x)+x=0\,,\ \ x\in(-\infty,+\infty).$$

Bu yechimga ko‘ra ushbu

$$\psi(x)=\frac{y(x)-1}{\sqrt{1+x^2}}$$

funksiyani tuzaylik. Uning hosilasi $(-\infty,+\infty)$ intervalda aynan nolga teng:

$$\psi'(x)=\frac{y'(x)\sqrt{1+x^2}-\dfrac{x}{\sqrt{1+x^2}}(y(x)-1)}{1+x^2}=\frac{(1+x^2)y'(x)-xy(x)+x}{(1+x^2)\sqrt{1+x^2}}=0\,.$$

Analizdan ma’lumki, oraliqda hosilasi nolga teng funksiya o‘zgarmasdir. Demak, tuzilgan funksiya o‘zgarmasdan iborat:

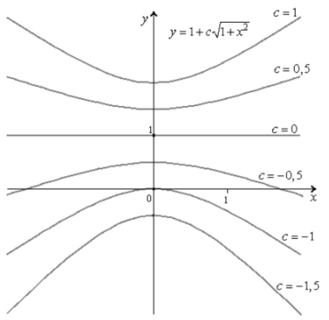
$$\psi(x)=c_0,\ c_0=\text{const}\,,$$

ya’ni

$$\frac{y(x)-1}{\sqrt{1+x^2}}=c_0\,,\ \ x\in(-\infty,+\infty)\,.$$

Bundan $y(x)=1+c_0\sqrt{1+x^2}$ ekanligini topamiz.

Biz (1.1.9) formula (1.1.8) tenglamaning umumiy yechimini ifodalashini isbotladik. Demak, (1.1.8) differensial tenglamaning integral chiziqlari ushbu $y=1+c\sqrt{1+x^2}$ chiziqlar oilasidan iborat ($c-$ ixtiyoriy o‘zgarmas) (1.1-rasm).



1.1-rasm. $(1+x^2)y'-xy+x=0$ differensial tenglamaning integral chiziqlari.

Izoh. (1.1.8) tenglamaning har qanday yechimi (1.1.9) ko‘rinishda bo‘lishini quyidagicha ham isbotlash mumkin. Aytaylik, $y = y(x),\ x \in (-\infty, +\infty)$, yechim bo‘lsin:

$$(1+x^2)y'(x) - xy(x) + x = 0\ ,\ x \in (-\infty, +\infty).$$

Bu ayniyatni $(1+x^2)\sqrt{1+x^2}$ ga bo‘lib topamiz:

$$\frac{y'(x)}{\sqrt{1+x^2}} - \frac{x}{(1+x^2)^{3/2}}\, y(x) + \frac{x}{(1+x^2)^{3/2}} = 0\ ,\ x \in (-\infty, +\infty).$$

Bundan

$$\left(\frac{y(x)}{\sqrt{1+x^2}}\right)' = -\frac{x}{(1+x^2)^{3/2}}\ ,\ x \in (-\infty, +\infty).$$

Endi bu tenglikni integrallab, umumiy yechim ko‘rinishini hosil qilamiz:

$$\frac{y(x)}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} + c,\ y(x) = 1 + c\sqrt{1+x^2}\ .$$

Misol 4. Ushbu

$$\frac{d^2x}{dt^2} = 6t$$

differensial tenglamaning umumiy yechimi ikki marta ketma-ket integrallash natijasida hosil bo‘ladi:

$$\frac{dx}{dt} = 3t^2 + c_1\ ,\ x = t^3 + c_1 t + c_2\ (c_1, c_2 - \text{ixtiyoriy o‘zgarmaslar}).$$

Masalalar

1. Differensial tengalamani yeching $y'(x) = \sin(ax)\cdot\cos(bx)$.

2. Differensial tenglamani yeching $y' = e^x / x$,

3. Differensial tenglamani yeching $y' = 2\,|x|$.

4. $y = |2x-1| + 1$ funksiya hech qanday differensial tenglamaning $(0;1)$ intervalda yechimi bo‘1a olmaydi. Nega?

5. Ushbu $y = \sqrt{x^2+1}$ funksiya hech qanday oraliqda $y' = xy^2 - 1$ tenglamaning yechimi bo‘la olmaydi. Shuni isbotlang.

6. Aytaylik, $f(x,y) \in C(\mathbb{R}^2)$ va $y = \varphi(x), x \in \mathbb{R},$ funksiya $y' = f(x,y)$ tenglamaning yechimi bo‘lsin. Quyidagilarni isbotlang:

a) agar $f(-x,y) \equiv -f(x,y)$ bo‘lsa, $y = \varphi(-x)$ ham tenglamaning yechimi;

b) agar $f(x,-y) \equiv -f(x,y)$ bo‘lsa, $y = -\varphi(x)$ ham tenglamaning yechimi;

v) agar $f(-x,-y) \equiv f(x,y)$ bo‘lsa, $y = -\varphi(-x)$ ham tenglamaning yechimi.

7. Ushbu

$$f(x)=\left(\int_0^x e^{-s^2}ds\right)^2,\ g(x)=\int_0^1 \frac{e^{-x^2(s^2+1)}}{s^2+1}ds$$

funksiyalar uchun $f'(x)+g'(x)=0,\ x\in\mathbb{R},$ ayniyatni isbotlang. Undan foydalanib, $\int_0^{+\infty} e^{-s^2}ds=\frac{\sqrt{\pi}}{2}$ tenglikni hosil qiling.

§ 1.2. $y'=f(x,y)$ tenglama uchun Koshi masalasi va uning geometrik talqini

Hosilaga nisbatan yechilgan ushbu

$$y'=f(x,y) \tag{1.2.1}$$

birinchi tartibli differensial tenglamani qaraylik; bu yerda $f\in C(D)$, D – tekislikdagi soha. (1.2.1) tenglamaning

$$y(x_0)=y_0,\ (x_0,y_0)\in D, \tag{1.2.2}$$

shartni qanoatlantiruvchi (berilgan x_0 nuqtada berilgan y_0 qiymatni qabul qiluvchi) va biror I, $x_0\in I$, oraliqda aniqlangan $y=\varphi(x)$ yechimini topish **Koshi masalasi** yoki **boshlang‘ich masala** deyiladi va u

$$\begin{cases} y'=f(x,y) \\ y\big|_{x_0}=y_0 \end{cases} \quad \text{yoki} \quad y'=f(x,y),\ y(x_0)=y_0\ (K)$$

ko‘rinishda yoziladi. Bu yerda va bundan keyin $y\big|_{x_0}$ yozuv $y=y(x)$ funksiyaning $x=x_0$ nuqtadagi qiymatini anglatadi, ya’ni $y\big|_{x_0}=y(x_0)$.

(1.2.2) shart **boshlang‘ich shart** yoki **Koshi sharti** deb yuritiladi.

Agar shunday I, $x_0\in I$, oraliq topilsaki, bu oraliqda biror $y=y(x)$ funksiya (1.2.1) ning yechimi bo‘lib, (1.2.2) shartni ham qanoatlantirsa, u holda (K) masalaning yechimi mavjud deyiladi, bu $y=y(x)$ funksiya esa (K) masalaning I oraliqda aniqlangan yechimi deb yuritiladi.

(K) masalaga nisbatan tabiiy ravishda quyidagi savollar tug‘iladi:

1. (K) masala x_0 ga yetarlicha yaqin x larda aniqlangan biror yechimga egami?
2. Agar yechim mavjud bo‘lsa, u yagonami?
3. (K) masalaning yechimi qaysi eng katta I, $I\ni x_0$, oraliqda mavjud?

Misol 1. *a*) Ushbu

$$y' = 3y^{2/3},\ y(0) = 0,$$

Koshi masalasi $y = 0$ yechim bilan birgalikda $y = x^3$ yechimga ham ega (tekshirib ko‘ring). Shunday qilib, *a*) masalaning yechimi yagona emas.

b) Ushbu

$$y' = y^2,\ y|_0 = 1,$$

masalaning yechimini topaylik. Faraz qilaylik, $y = y(x)$ yechim mavjud bo‘lsin. Bu yechim $(\in C^1)$ $x = 0$ nuqtaning kichik atrofida noldan farqli, chunki $y(0) = 1$. Berilgan tenglamadan $\frac{dy}{y^2} = dx$ tenglikni topib, uni 0 dan x gacha integrallab, quyidagini olamiz: $\frac{1}{y(0)} - \frac{1}{y(x)} = x$. Bundan berilgan $y|_0 = 1$ boshlang‘ich shartga ko‘ra $y(x) = \frac{1}{1-x}$ ekanligini hosil qilamiz. Topilgan bu funksiya qaralayotgan masalaning $(-\infty,1)$ oraliqda aniqlangan yechimidir (tekshirib ko‘ring). Demak, qaralayotgan masalaning yechimi yagona. Ravshanki, berilgan masala yechimi $(-\infty,1)$dan kattaroq (kengroq) oraliqda mavjud bo‘la olmaydi. Chunki, $x \to 1-0$ da yechim $+\infty$ ga ketib qoladi, ya’ni yechim “portlaydi”.

(*K*) masala yechimining mavjudligi va yagonaligi to‘grisidagi teoremani § 3.1 da keltiramiz. Koshi masalasining yechimi aniqlangan eng katta oraliq masalasi (davomsiz yechim) § 3.4 da o‘rganiladi.

Agar $D \subset \mathbb{R}^2$ sohaning ha bir nuqtasidan $y' = f(x, y)$ tenglamaning yagona, ya’ni bir dona integral chizig‘i (yechimi) o‘tsa, D soha shu tenglama uchun **yagonalik sohasi** deb ataladi.

Masalan, $D = \left\{(x, y) \in \mathbb{R}^2 \middle| y > 0\right\}$ yarim tekislik $y' = 3y^{2/3}$ tenglamaning yagonalik sohasi, lekin $D = \mathbb{R}^2$ tekislik bu tenglama uchun yagonalik sohasi emas.

Endi $y' = f(x, y)$ (1.2.1) differensial tenglama va integral chizig‘ining geometrik talqini bilan tanishamiz.

Har bir $(x_0, y_0) \in D$ nuqtaga shu nuqtadan o‘tuvchi va absissalar o‘qining musbat yo‘nalishi bilan hosil qilgan (og‘ish) burchagining tangensi $k = f(x_0, y_0)$ ga teng bo‘lgan vektorni (yo‘nalishni) mos qo‘yib, uni “kichik uzunlikli vektor” (yo‘nalish) bilan tasvirlaylik. Natijada D sohada (1.3.1) differensial tenglamaga mos keluvchi yo‘nalishlar maydoni hosil bo‘ladi. har bir “kichik uzunlikli vektor” mos nuqtadagi **maydon yo‘nalishi** deyiladi.

Yechimning ta’rifiga ko‘ra D sohadagi $y = y(x)$ silliq egri chiziq (1.3.1) tenglamaning integral chizig‘i (yechim grafigi) bo‘lishi uchun u o‘zining ixtiyoriy (x_0, y_0) nuqtasida maydonning shu nuqtadagi yo‘nalishiga urinishi yetarli va zarurdir ($y'(x_0) = f(x_0, y_0) = k$ hosila $y = y(x)$ chiziqning (x_0, y_0) nuqtasidagi urinmasining burchak koeffitsientidir; maydon yo‘nalishining Ox o‘qiga og‘ish burchagi α uchun $\mathrm{tg}\alpha = k = f(x_0, y_0)$ bo‘ladi).

Maydon yo‘nalishlarini ko‘rsatuvchi vektorlarni D sohada «yetarlicha zich» tasvirlab, va bu vektorlarga urinuvchi egri chiziqlarni chizib, berilgan differensial tenglamaning yechimlarini taqriban qurish mumkin.

Yechimlarni aniqroq qurish maqsadida ularning ekstremum va bukilish nuqtalarini topish mumkin. Yechimlarning statsionar (kritik) va, demak, ekstremum nuqtalari ham $f(x, y) = 0$ tenglamani qanoatlantiradi. Bukilish nuqtalarida $y'' = 0$ bo‘lishi kerak. $y = y(x)$ yechim uchun $y'(x) = f(x, y(x))$ ayniyat bajariladi. Bu tenglikni differensiallab, y'' ni topamiz ($f \in C^1$ deb hisoblaymiz):

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot y' \text{ , ya'ni } y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot f \text{ .}$$

Demak, bukilish nuqtalari

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot f = 0$$

tenglamani qanoatlantirdi.

Differensial tenglamaning yo‘nalishlar maydoni va integral chiziqlarini qurish uchun Maple, Mathematika kabi kompyuter amaliy dasturlaridan foydalanish maqsadga muvofiq.

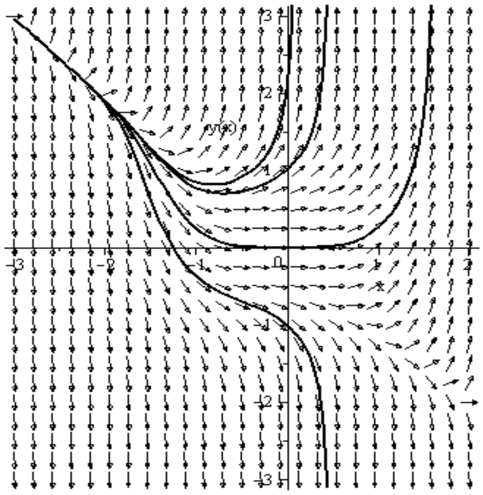
Misol. Ushbu $y' = x^3 + y^3$ differensial tenglama uchun yo‘nalishlar maydoni va 4 dona integral chiziq 1.2-rasmda ko‘rsatilgan. Bu rasmni hosil qilish uchun Maple da quyidagi buyruqlar berilgan.

> with(DEtools):deq:=D(y)(x)=x^3+y(x)^3;

$$deq := D(y)(x) = x^3 + y(x)^3$$

> P1:=DEplot(deq,y(x),x=-3..2,[[y(0)=- 1],[y(0)=0],[y(0)=1], [y(0)=2]],y=-3..3,stepsize=.05,arrows=‘SLIM’, scaling=constrained,color=black,linecolour=black,thickness=2, dirfield=[25,25]):

> display(P1);



1.2-rasm. $y' = x^3 + y^3$ tenglamaning yo‘nalishlar maydoni va integral chiziqlari.

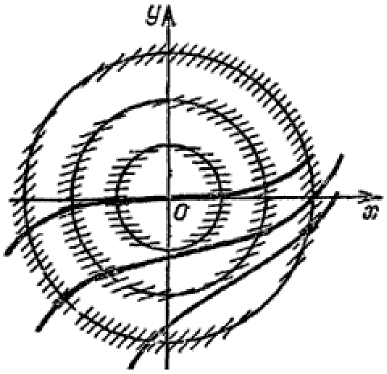
Yo‘nalishlar maydonini qurishda (tasvirlashda) ba’zan izoklinalardan foydalanish maqsadga muvofiq.

Barcha nuqtalarida maydon yo‘nalishining og‘ish burchagi bir xil bo‘lgan to‘plam **izoklina**[1] deyiladi. Izoklinalar $f(x,y)=k$ (k – const) tenglama bilan beriladi. $f(x,y)=k$ izoklinaning har bir nuqtasida maydon yo‘nalishi x lar o‘qining musbat yo‘nalishi bilan bir xil $\alpha = \operatorname{arctg} k$ burchak tashkil etadi

Misol. Ushbu $y' = x^2 + y^2$ differensial tenglama uchun izohlinalar $x^2 + y^2 = k$, k – const, tenglikdan topiladi. Izoklinalar markazi koordinatalar boshida joylashgan konsentrik aylanalardan ($k > 0$) va (0;0) nuqtadan ($k = 0$) iborat. Masalan, $x^2 + y^2 = 1$ izohlina nuqtalarida yo‘nalishlar maydoni x lar o‘qi bilan bir xil $\alpha = \operatorname{arctg} k = \operatorname{arctg} 1 = 45^\circ$li burchak tashkil etadi. Bir nechta izoklina va ulardagi maydon yo‘nalishlarini chizamiz hamda bu

[1] Grekchadan: *isos* – bir xil, teng; *klino* – og‘dirish.

yo‘nalishlarga urintirib egri chiziqlar (yechimlar grafigi) ni o‘tkazamiz (1.3-rasm). Bu egri chiziqlarning ko‘rinishi orqali qaralayotgan differensial tenglamaning yechimlari haqida tasavvur hosil qilamiz. 1.3-rasm.



1.3-rasm. $y' = x^2 + y^2$ tenglama izokhlinalari va integral chiziqlari.

Masalalar

1. Ushbu

$$y' = 3y^{2/3},\ y(0) = 0,$$

masalaning turli yechimlarini toping va mos integral chiziqlarini quring.

2. Ushbu

$$y' = y^2,\ y|_0 = 1,$$

masalaning integral chizig‘ini quring.

3. Ushbu $y' = |x^2 - y^2| / (x^2 - y^2)$ differensial tenglamaning yo‘nalishlar maydonini quring va bir necha yechimlari grafigini tasvirlang.

4. Ushbu $y' = x - y^3$ differensial tenglamaning yo‘nalishlar maydonini va bir necha integral chiziqlarini quring.

§ 1.3. Differensiallarda yozilgan tenglamalar

Yuqorida qaralgan

$$y' = f(x, y),\ f \in C(D), \tag{1.3.1}$$

tenglamada x va y o‘zgaruvchilar teng huquqli emas: x – erkli o‘zaruvchi, $y = y(x)$ esa – uning funksiyasi, ya’ni erksiz o‘zgaruvchi. Buning natijasida, masalan, integral chiziq Oy o‘qiga parallel urinmaga ega bo‘la olmaydi. (1.3.1) tenglamani

$$dy = f(x, y)dx$$

ko‘rinishda differensiallarda yozish mumkin. D sohaning $f(x, y)$ funksiya nolga aylanmagan qism sohasida oxirgi tenglamani

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \quad \text{yoki} \quad dx = \frac{1}{f(x, y)}dy \tag{1.3.2}$$

ko‘rinishga keltirish mumkin. Bu (1.3.2) tenglamadagi $x = x(y)$ noma’lum funksiya va (1.3.1) tenglamadagi $y = y(x)$ noma’lum funksiya o‘zaro teskari funksiyalardir.

Endi differensiallarda yozilgan umumiy ko‘rinishdagi tenglamani kiritaylik:

$$M(x,y)dx + N(x,y)dy = 0, \tag{1.3.3}$$

bu yerda $\{M,N\} \subset C(D)$ deb hisoblanadi. (1.3.3) differensial tenglama o‘zgaruvchilari teng huquqli qatnashgan yoki simmetrik ko‘rinishdagi tenglama deb ham yuritiladi. (1.3.3) differensiallarda yozilgan tenglama D sohaning $N(x,y)$ funksiya nolga aylanmagan qismsohasida $y = y(x)$ noma’lum funksiyaga nisbatan

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \tag{1.3.4}$$

differensial tenglama kabi; D sohaning $M(x,y)$ funksiya nolga aylanmagan qismsohasida esa $x = x(y)$ noma’lum funksiyaga nisbatan ushbu

$$\frac{dx}{dy} = -\frac{N(x,y)}{M(x,y)} \tag{1.3.5}$$

differensial tenglama kabi tushuniladi.

D sohaning $N(x,y)$ va $M(x,y)$ funksiyalar birgalikda nolga aylanmagan qism sohasida (1.3.4) va (1.3.5) tenglamalar ekvivalent va bu tenglamalardagi noma’lum funksiyalar lokal (“kichik sohada”) o‘zaro teskari. Shuning uchun bunday sohada (1.3.3) tenglama o‘rnida (1.3.4) va (1.3.5) tenglamalarning ixtiyoriy birini ishlatish mumkin.

Ushbu $x = x(t)$, $y = y(t)$ funksiyalarni qaraylik. Agar

1^0. $x'(t), y'(t)$ hosilalar $\tilde{I} \subset \mathbb{R}$ oraliqda uzluksiz va bir vaqtda nolga aylanmaydi $\left(|x'(t)| + |y'(t)| \neq 0\right)$ va

2^0. $\forall t \in \tilde{I} \quad M(x(t), y(t))x'(t)dt + N(x(t), y(t))y'(t)dt = 0$

shartlar bajarilsa, u holda $x = x(t)$, $y = y(t)$ funksiyalar (1.3.3) tenglamaning parametrik ko‘rinishda berilgan yechimi deyiladi.

Eslatma. 1^0 shart integral chiziqning maxsus nuqtaga ega bo‘lmagan silliq chiziq ekanligini anglatadi. $|x'(t)| + |y'(t)| \neq 0$

bo‘lgani uchun integral chiziq o‘zining ixtiyoriy nuqtasi atrofida (lokal)

$$y=\varphi(x) \quad \text{yoki} \quad x=\psi(y)$$

oshkor ko‘rinishda beriladi.

Integral chiziqni oshkormas ko‘rinishda, ya’ni

$$u(x,y)=0$$

tenglama bilan ham berish mumkin; bu yerda $u\in C^1(D)$ va

$$\left|\frac{\partial u(x,y)}{\partial x}\right|+\left|\frac{\partial u(x,y)}{\partial y}\right|\neq 0$$

deb hisoblanadi.

(1.3.3) tenglama D sohada $\boldsymbol{v}=\boldsymbol{v}(x,y)=\left(N(x,y),-M(x,y)\right)$ uzluksiz vektor maydonni aniqlaydi. Agar $(x_0,y_0)\in D$ nuqtada

$$M(x_0,y_0)=N(x_0,y_0)=0$$

bo‘lsa, (x_0,y_0) nuqta (1.3.3) tenglamaning maxsus nuqtasi deyiladi. Maxsus bo‘lmagan nuqta regular nuqta deb ataladi. Regular nuqtada M va N funksiyalarning birortasi, aytaylik, N (yoki M) noldan farqli bo‘lgani uchun bu nuqtaning biror atrofida ham $N\neq 0$ (yoki $M\neq 0$). Demak, ixtiyoriy regular nuqtaning yetarlicha kichik atrofida (1.3.3) tenglama (1.3.4) (yoki (1.3.5)) ko‘rinishga keladi. Shuning uchun regular nuqtalarda $\boldsymbol{v}$ vektor noldan farqli, ya’ni (1.3.3) tenglama regular nuqtalarda yo‘nalishni aniqlaydi (yo‘nalish ordinatalar o‘qiga parallel bo‘lishi mumkin). Regular nuqtada yechim (integral chiziq) shu nuqtadagi yo‘nalishga ($\boldsymbol{v}$ vektorga) urinadi.

Misol. Ushbu

$$xdx+ydy=0$$

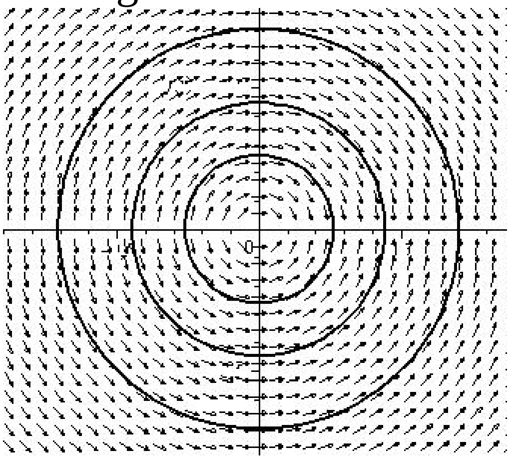
differensial tenglama $\boldsymbol{v}=\left(-y,x\right)$ vektor maydonni aniqlaydi (1.4-rasm). $(0;0)$ maxsus nuqtadan boshqa barcha nuqtalarda, ya’ni regular nuqtalarda $\boldsymbol{v}$ vektor noldan farqli. Qaralayotgan tenglamani quyidagicha yechish mumkin

$$2xdx+2ydy=0,\ d(x^2)+d(y^2)=0, d(x^2+y^2)=0.$$

Demak, $x^2+y^2=c$. Berilgan $(x_0,y_0)\neq(0;0)$ nuqtadan o‘tuvchi integral chiziq aylanadan iborat:

$$x^2+y^2=x_0^2+y_0^2.$$

Bu aylana (integral chiziq) ixtiyoriy nuqtasining biror atrofida $y=\varphi(x)$ yoki $x=\psi(y)$ formula bilan oshkor ko‘rinishda berilishi mumkin. Lekin uni butunligicha oshkor ko‘rinishda berib bo‘lmaydi.



1.4-rasm. $xdx+ydy=0$ tenglama aniqlovchi vektor maydon va integral chiziqlar.

Masalalar

1. Ushbu $xdx+ydy=0$ tenglamaning parametrik ko‘rinishdagi yechimi $x=\cos t, y=\sin t$ ekanligini isbotlang.

2. Ushbu $ydx-xdy=0$ tenglama uchun yo‘nalishlar maydonini tasvirlang. Tenglamani yeching.

3. Ushbu $(x+|x|)dx+(y+|y|)dy=0$ differensial tenglama yechimlari grafigklarini quring.

MODUL 2. KVADRATURALARDA YECHILADIGAN BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR

§ 2.1. O‘zgaruvchilari ajraladigan tenglamalar

1. Ushbu

$$y' = g(y),\ g \in C((c,d)), \tag{2.1.1}$$

differensial tenglamani qaraylik. Bu (2.1.1) tenglamada erkli o‘zgaruvchi x bevosita qatnashmagan. U avtonom tenglama deb ataladi.

Faraz qilaylik, g funksiya $I = (c,d)$ intervalda nolga aylanmasin. $g \in C(I)$ (uzluksiz) bo‘lganligi uchun u I da o‘z ishorasini saqlaydi. Aniqlik uchun $y \in I$ bo‘lganda $g(y) > 0$ deylik.

Agar $y = y(x)$ funksiya (2.1.1) tenglamaning $y(x_0) = y_0$ $(y_0 \in I)$ shartni qanoatlantiruvchi biror yechimi bo‘lsa, u holda

$$\frac{dy(x)}{dx} = g(y(x)),\ y(x_0) = y_0$$

bo‘ladi. Demak,

$$\frac{dy(s)}{g(y(s))} = ds\,.$$

Bu tenglikning har ikkala tomonini $s = x_0$ dan $s = x$ gacha integral-laymiz:

$$\int_{x_0}^{x} \frac{dy(s)}{g(y(s))} = x - x_0 \text{ yoki } \int_{y_0}^{y(x)} \frac{dz}{g(z)} = x - x_0\,. \tag{2.1.2}$$

Ushbu

$$\Phi_{y_0}(y) = \int_{y_0}^{y} \frac{dz}{g(z)}\,,\quad (\{y_0, y\} \subset (c,d)) \tag{2.1.3}$$

belgilashni kiritib, (2.1.2) tenglikni

$$\Phi_{y_0}(y(x)) = x - x_0 \tag{2.1.4}$$

ko‘rinishga keltiramiz. Demak, qaralayotgan $y = y(x)$ yechim ushbu

$$\Phi_{y_0}(y) = x - x_0 \qquad (y_0 = y(x_0)) \tag{2.1.5}$$

tenglamani qanoatlantiradi. Bu (2.1.5) tenglamada

$$\frac{\partial \Phi_{y_0}(y)}{\partial y} = \frac{1}{g(y)} > 0$$

bo‘lgani uchun teskari funksiya haqidagi teoremaga ko‘ra (2.1.5) munosabat biror $y = y(x)$ funksiyani (oshkormas ko‘rinishda) aniqlaydi va bu funksiya uchun

$$\frac{\partial \Phi_{y_0}(y)}{\partial y} y' = 1, \text{ ya'ni } y' = g(y) \text{ bo‘ladi.}$$

Demak, bu $y = y(x)$ funksiya (2.1.1) differensial tenglamanining yechimi. $u = \Phi_{y_0}(y)$ funksiyaga teskari funksiyani $y = \Phi_{y_0}^{-1}(u)$ bilan belgilaylik:

$$u = \Phi_{y_0}(\Phi_{y_0}^{-1}(u)), \qquad y = \Phi_{y_0}^{-1}(\Phi_{y_0}(y)).$$

Tushunarliki, teskari $y = \Phi_{y_0}^{-1}(u)$ funksiya to‘g‘ri funksiyaning qiymatlar to‘plami bo‘lmish

$$\Phi_{y_0}(c) < u < \Phi_{y_0}(d)$$

intervalda aniqlangan bo‘ladi. Demak, $y = y(x)$ yechim

$$\Phi_{y_0}(c) < x - x_0 < \Phi_{y_0}(d)$$

bo‘lganda mavjud. Shunday qilib, $y = y(x)$yechimning aniqlanish sohasi (2.1.3) ga ko‘ra

$$x_0 + \int_{y_0}^{c} \frac{dz}{g(z)} < x < x_0 + \int_{y_0}^{d} \frac{dz}{g(z)}$$

intervaldan iborat bo‘lib, u

$$\int_{y_0}^{c} \frac{dz}{g(z)} \text{ va } \int_{y_0}^{d} \frac{dz}{g(z)} \tag{2.1.6}$$

integrallar uzoqlashuvchi bo‘lgan holdagina $(-\infty, +\infty)$ oraliqda aniqlangan bo‘ladi.

Biz quyidagi teoremani isbotladik.

Teorema. *Aytaylik, (2.1.1) differensial tenglamada* $g \in C((c,d))$ *va* $g(y) \neq 0$ $(y \in (c,d))$ *bo‘lsin. U holda ushbu* $\{(x,y) \mid -\infty < x < +\infty,\ c < y < d\}$ *polosaning ixtiyoriy* (x_0, y_0) *nuqtasidan (2.1.1) tenglamaning yagona* $y = y(x)$ *integral chizig‘i o‘tadi va bu yechim (2.1.5) formula bilan oshkormas ko‘rinishda beriladi.*

Bunda yechim $(-\infty,+\infty)$ *oraliqda aniqlangan bo'lishi uchun (2.1.6) integrallarning uzoqlashuvchi bo'lishi yetarli va zarurdir.*

Demak, keltirilgan teoremaning shartlarida $\{(x,y)\,|-\infty < x < +\infty,\, c < y < d\}$ polosa qaralayotgan (2.1.1) tenglamaning yagonalik sohasi bo'ladi.

Endi (2.1.1) tenglamadagi g funksiyaning (c,d) intervalda nolga aylangan holida to'xtalaylik. Faraz qilaylik, $g(y)$ funksiya (c,d) ning bitta $y=\tilde{y}\in(c,d)$ nuqtasida nolga aylansin. Bu holda (2.1.1) tenglamaning $y(x)=\tilde{y}$ o'zgarmas yechimi mavjud. Undan farqli $y=y(x)$ yechim uchun (2.1.1) tenglamadan

$$\frac{dy(x)}{g(y(x))}=x$$

yoki

$$\Phi_{y_0}(y(x))=x-x_0,\ \tilde{y}\neq y(x_0)=y_0.$$

(x_0,y_0) nuqtadan o'tgan (integral chiziqning) yechimning chekli x da $\tilde{y}$ ga aylanishi ushbu

$$\int_{\tilde{y}-0}^{c}\frac{dz}{g(z)}\,,\quad \int_{\tilde{y}+0}^{d}\frac{dz}{g(z)} \qquad (2.1.7)$$

xosmas integrallarning yaqinlashuvchiligi bilan aniqlanadi.

Agar (2.1.7) xosmas integrallarning birortasi yaqinlashuvchi bo'lsa, $y=\tilde{y}$ to'g'ri chiziqning ixtiyoriy nuqtasidan kamida ikkita integral chiziq o'tadi (yechimning yagonalik xossasi buziladi).

$y=\tilde{y}$ to'g'ri chiziq atrofida integral chiziqlarning turli hollardagi tabiatini ko'rsatuvchi grafiklarni mustaqil quring.

2. Ushbu

$$y'=f(x)g(y)$$

tenglama **o'zgaruvchilari ajraladigan differensial tenglama** deb ataladi. Bu yerda $f\in C((a,b))$ va $g\in C((c,d))-$ berilgan funksiyalar.

Agar g funksiya nolga aylanmasa, u holda ixtiyoriy $(x_0,y_0),\, a<x_0<b,\, c<y_0<d$, nuqtadan bu tenglamaning yagona integral chizig'i o'tadi. Bu $y=y(x)$ yechim

$$\int_{y_0}^{y} \frac{dz}{g(z)} = \int_{x_0}^{x} f(x)dx$$

tenglama bilan oshkormas ko‘rinishda beriladi.

Agar biror $\tilde{y}$ nuqtada $g(\tilde{y}) = 0$, lekin $g(y) \neq 0, y \neq \tilde{y},$ va (2.1.7) xosmas integrallarning ikkalasi ham uzoqlashuvchi bo‘lsa, bu holda ham yechimning yagonaligi saqlanadi; (2.1.7) xosmas integrallarning kamida bittasi yaqinlashuvchi bo‘lgan holda esa $y = \tilde{y}$ to‘g‘ri chiziqning har bir nuqtasidan kamida 2 ta (va, demak, cheksiz ko‘p) integral chiziq o‘tadi (bu to‘g‘ri chiziqda yotmagan ixtiyoriy nuqtadan bitta va faqat bitta integral chiziq o‘tadi). Bu tasdiqlar 1- banddagi tekshirishlardan bevosita kelib chiqadi.

Differensiallarda yozilgan ushbu

$$M(x)N(y)dx + P(x)Q(y)dy = 0 \tag{2.1.8}$$

$$(\{M(x), P(x)\} \subset C((a,b)), \{N(y), Q(y)\} \subset C((c,d))$$

tenglama ham o‘zgaruvchilari ajraladigan tenglama deb ataladi.

Agar $P(x_0)N(y_0) \neq 0$ bo‘lsa, (x_0, y_0) nuqtaning yetarlicha kichik atrofida tenglamaning har ikkala tomonini $P(x)N(y) \neq 0$ ga bo‘lib, o‘zgaruvchilarni ajratamiz:

$$\frac{M(x)}{P(x)}dx + \frac{Q(y)}{N(y)}dy = 0$$

Bu tenglikning har ikkala tomonini integrallab, yechimni oshkormas ko‘rinishda topamiz:

$$\int \frac{M(x)}{P(x)}dx + \int \frac{Q(y)}{N(y)}dy = c \ \ (c - \text{const}). \tag{2.1.9}$$

Agar $N(y_0) = 0$ $(P(x_0) = 0)$ bo‘lsa, $y = y_0$ $(x = x_0)$ o‘zgarmas yechimlar ham mavjud. Topilgan (2.1.9) yechimlar orasida bu yechimlar bo‘lmasligi, ya’ni ular yo‘qolgan bo‘lishi mumkin. Masalalar yechish jarayonida ana shuni esda tutish lozim.

Misol 1. Ushbu

$$ye^x dx + (e^x + 1)dy = 0 \tag{2.1.10}$$

tenglamani yeching.

🗝 Tenglamada o‘zgaruvchilar ajraladi. Tenglamani (ya’ni uning har ikkala tomonini) $y(e^x + 1)$ $(y \neq 0)$ ga bo‘lib, integrallashlarni bajaramiz:

$$\frac{e^x}{e^x+1}dx+\frac{dy}{y}=0\ ,\ \int\frac{e^x}{e^x+1}dx+\int\frac{dy}{y}=c_1\ (c_1-const),$$

$$\ln(e^x+1)+\ln|y|=c_1,\ y=\frac{\pm e^{c_1}}{e^x+1}.$$

Bu yerdagi $\pm e^{c_1}$ ni c $(c\neq 0)$ bilan belgilab, $y=\dfrac{c}{e^x+1}(c\neq 0)$ yechimlarni hosil qilamiz. Bu formuladan $c=0$ da yo‘qolgan $y=0$ yechim hosil bo‘ladi. Demak, berilgan (2.1.10) tenglamaning umumiy yechimi $y=\dfrac{c}{e^x+1}$ ($c-$ixtiyoriy o‘zgarmas) formula bilan aniqlanadi.

Radioaktiv yemirilish. $m(t)$ bilan radioaktiv moddaning t paytdagi massasini belgilaylik. Bizga $m(t)$funksiyani topish kerak bo‘lsin. Fizikadan ma’lumki, radioaktiv moddaning yemirilish tezligi $-\dfrac{dm(t)}{dt}$ ($\dfrac{dm(t)}{dt}$ hosila o‘sish tezligini ifodalaydi) mavjud modda miqdoriga to‘g‘ri proporsional, ya’ni

$$-\frac{dm(t)}{dt}=km(t)\ \text{ yoki qisqaroq }\ m'=-km; \qquad (2.1.11)$$

bu yerda o‘zgarmas $k=\text{const}>0-$ proporsionallik koeffitsienti. Demak, $m=m(t)$ noma’lum funksiya (2.1.11) o‘zgaruvchilari ajraladigan tenglamani qanoatlantiradi. Uni yechib, $m=ce^{-kt}$ umumiy yechimni topamiz. Agar boshlang‘ich, ya’ni $t=0$ paytdagi massa $m_0>0$ bo‘lsa, $c=m_0$ bo‘ladi. Shunday qilib, radioaktiv moddaning massasi ushbu

$$m=m_0e^{-kt} \qquad (2.1.12)$$

qonuniyatga ko‘ra o‘zgaradi. Massa miqdori vaqt o‘tishi blan eksponensial tezlik bilan kamayib 0 ga intiladi.

Yarim yemirilish davri T deb dastlabki radioaktiv moddaning yarmi yemirilishi uchun ketgan vaqt oralig‘iga aytiladi. Demak,

$$\frac{m_0}{2}=m_0e^{-kT}\ ,\ \text{ya'ni }\ T=\frac{\ln 2}{k}\ \text{ yoki }\ k=\frac{\ln 2}{T}.$$

Oxirgi formuladan T ga ko‘ra (uni o‘lchash nisbatan oson) k ni topish uchun foydalanish mumkin.

Yuqoridagi (2.1.12) formulaning yana bir tatbig‘ini e’tirof etaylik. Ma’lumki, tirik organizmlarda C^{12} turg‘un uglerod bilan birgalikda oz miqdorda C^{14} radioaktiv izotop ham bo‘ladi. Atmosferaning yuqori qatlamlarida γ-nurlar hosil qiluvchi C^{14} izotoplar tirik organizmlarda yutiladi va biologik o‘zgarish (almashinuv) jarayonlari natijasida C^{14} miqdori o‘zgarmas va biror m_0 ga teng bo‘ladi. Organizm o‘lishi bilanoq unda radioaktiv izotopning yutilishi to‘xtaydi va C^{14} ning miqdori kamaya boshlaydi. C^{14} ning yarim yemirilish davri $T \approx 5570$(yil). Bundan, $k = \frac{\ln 2}{T} \approx \frac{0,6931}{5570} \approx 1,24 \cdot 10^{-4} \approx \frac{1}{8000}$(1/yil). Demak, agar $m(t)$ bilan C^{14} izotopning organizm o‘lgan paytdan boshlab hisoblangan t-yildagi massasini belgilasak, u holda

$$m'(t) = -\frac{1}{8000} m(t), \text{ ya'ni } m(t) = m_0 e^{-t/8000}$$

bo‘ladi. Bundan, agar $m(t)$ aniqlangan (uni C^{14} chiqaradigan β-zarrachalar soni orqali topish mumkin) bo‘lsa, u holda organizmning o‘lganidan keyin o‘tgan t vaqtni yillarda ushbu

$$t = 8000 \ln \frac{m_0}{m(t)}$$

formula orqali topish mumkinligi kelib chiqadi. Hosil bo‘lgan bu formula o‘lgan organizmlarning yoshini aniqlashga imkon beradi.

Bakteriyalarning ko‘payishi. t vaqtdagi bakteriyalar sonini $n(t)$ bilan belgilatlik. $n(t)$ ning qiymatlari juda katta bo‘lgani uchun bu funksiyani silliq deb faraz qilish mumkin. Bakteriyalar sonining o‘sish tezligi $\frac{dn(t)}{dt}$ bakeriyalarning tabiiy tug‘ilishi, tabiiy o‘lishi va yashash uchun kurash tufayli sodir bo‘luvchi yo‘qotilishlar (o‘limlar) natijasida sodir bo‘ladi. Tabiiy tug‘ilish va tabiiy o‘lish tezliklarini mavjud bakteriyalar soniga proporsional deb hisoblab, ularni mos ravishda $k_1 n(t)$, va $k_2 n(t)$ga teng deymiz, $k_1, k_2 = \text{const} > 0$. Yashash uchun kurash jarayonida ikki bakteriya uchrashganda ular orasida raqobat paydo bo‘ladi (jang ketadi) va bunda o‘lish tezligi mavjud bakteriyalar sonining kvadratiga proporsional, ya’ni $an^2(t)$

ga teng deb faraz qilamiz, $a = \text{const} > 0$. Natijada quyidagi differensial tenglama hosil bo‘ladi:

$$\frac{dn(t)}{dt} = k_1 n(t) - k_2 n(t) - an^2(t).$$

$k = k_1 - k_2 > 0$ deb hisoblaymiz. Bu $k > 0$ soni raqobat bo‘lmaganda ($a = 0$) bakteriyalar sonining o‘sishini anglatadi, ya’ni

$$\frac{dn(t)}{dt} = kn(t) > 0.$$

Shunday qilib, $n(t)$ noma’lum funksiya uchun

$$\frac{dn(t)}{dt} = kn(t) - an^2(t) \quad \left(k, a = \text{const} > 0\right) \qquad (2.1.13)$$

differensial tenglama hosil bo‘ldi. U logistik tenglama deb ataladi. Vaqt boshidagi bakteriyalar soni $n_0 > 0$ ma’lum bo‘lsin:

$$n(0) = n_0. \qquad (2.1.14)$$

Bakteriyalar soni $n(t)$ ni aniqlash uchun (2.1.13) tenglamaning (2.1.14) boshlang‘ich shartni qanoatlantiruvchi yechimini topish kerak, ya’ni (2.1.13),(2.1.14) Koshi masalasini yechish kerak. (2.1.13) tenglamani o‘zgaruvchilarini ajratib yechamiz $\left(n(t) \neq k/a\right)$

$$\frac{dn(t)}{kn(t) - an^2(t)} = dt; \quad \int \frac{dn}{kn - an^2} = \int dt + c_1, \quad n = n(t);$$

$$\frac{1}{k}\int\left(\frac{1}{n} + \frac{a}{k - an}\right)dn = t + c_1; \quad \frac{1}{k}\ln\left|\frac{n}{k - an}\right| = t + c_1;$$

$$\frac{k - an}{n} = ce^{-kt}, c \neq 0.$$

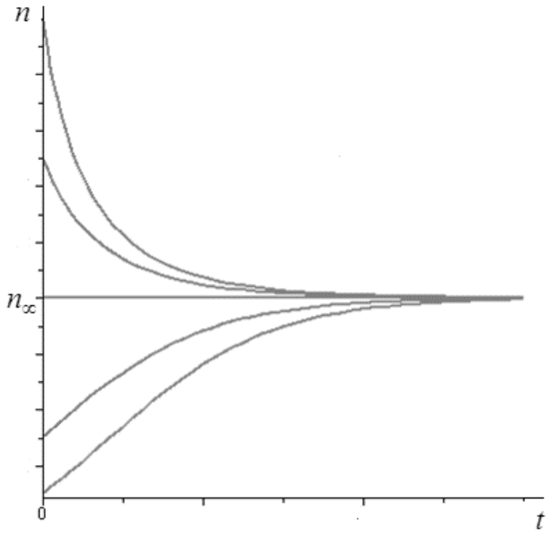
Oxirgi tenglikdan

$$n = \frac{k}{a + ce^{-kt}} \qquad (2.1.15)$$

umumiy yechim formulasini topamiz, bu yerda $c = 0$ da yo‘qolgan $n = k/a$ yechim hosil bo‘ladi. Yechim $n(0) = n_0$ boshlang‘ich shartni qanoatlantirishi, ya’ni $n_0 = \dfrac{k}{a + c}$ bo‘lishi kerak. Bu yerdan c ning qiymatini topib, uni (2.1.15) ga qo‘yamiz va (2.1.13)-(2.1.14) boshlang‘ich masala yechimini hosil qilamiz:

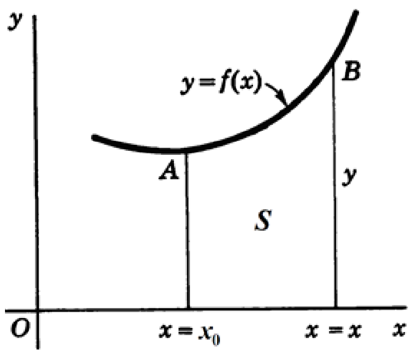
$$n = \frac{n_\infty n_0}{n_0 + (n_\infty - n_0)ce^{-kt}}, \; n_\infty = \frac{k}{a}. \qquad (2.1.16)$$

Yechimni tahlil qilaylik. Agar $n_0 = \frac{k}{a} = n_\infty$ bo‘lsa, yechim o‘zgarmas funksiyadan iborat: $n = n_0$ (vaqt o‘tishi bilan bakteriyalar soni o‘zgarmaydi). Agar boshlang‘ich qiymat $n_0 < n_\infty$ bo‘lsa, vaqt o‘tishi bilan bakteriyalar soni ortadi va n_∞ ga intiladi. Agar vaqt boshida bakteriyalar soni $n_0 > n_\infty$ bo‘lsa, keyingi paytlarda bakteriyalar soni kamayadi va vaqt ortishi bilan n_∞ ga intiladi (2.1-rasm).



2.1-rasm. $\frac{dn(t)}{dt} = kn(t) - an^2(t)$ tenglamaning integral chiziqlari.

Bir geometrik masala. Shunday $y = f(x)$ silliq funksiyani topingki, tekislikda yuqoridan $y = f(x)$, quyidan $y = 0$, chapdan $x = x_0$ va o‘ngdan "$x = x$" chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzi ordinatalar ayirmasi $f(x) - f(x_0)$ ning kvadratiga proportsional bo‘lsin (2.2-rasm).



2.2-rasm.

Shartda aytilgan egri chiziqli trapetsiya yuzi $S=\int_{x_0}^{x} f(x)dx$ ga teng. Masala shartiga ko‘ra $S=k\left(f(x)-f(x_0)\right)^2$, $0<k=\text{const}-$ proporsionallik koeffitsienti. Demak,

$$\int_{x_0}^{x} f(x)dx=k\left(f(x)-f(x_0)\right)^2.$$

Bu tenglikni hadma-had differensiallab, $y=f(x)$ noma’lum funksiyaga nisbatan differensial tenglama hosil qilamiz:

$$f(x)=2k\left(f(x)-f(x_0)\right)f'(x)$$

yoki qisqaroq

$$2k\left(y-y_0\right)y'=y\quad \left(y_0=f(x_0)>0\right).$$

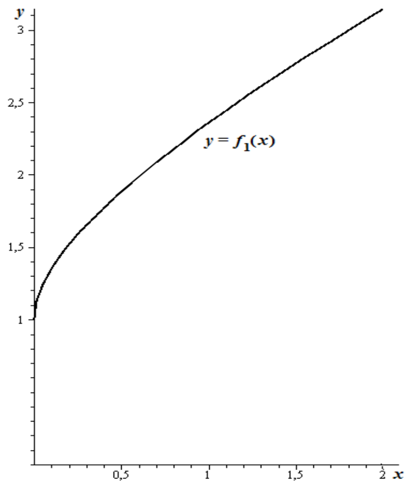
Bu o‘zgaruvchilari ajraladigan tenglamani $y(x_0)=y_0$ shartda yechib, yechimni oshkormas ko‘rinishda topamiz:

$$x-2k(y-y_0\ln y)=x_0-2ky_0(1-\ln y_0)$$

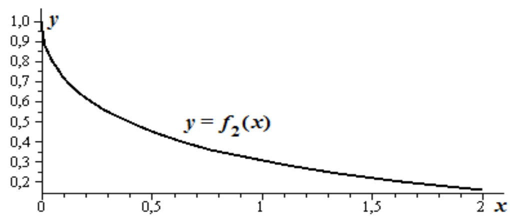
Oxirgi tenglamadan $x=x(y)$ bog‘lanish osongina topiladi:

$$x=x(y)=2k(y-y_0\ln y)x_0-2ky_0(1-\ln y_0).$$

U ikkita teskari funksiyani aniqlaydi; ular $y=f_1(x)$ va $y=f_2(x)$. Bu funksiyalar aniqlovchi egri chiziqlar (grafiklar) $k=1,\ x_0=0, y_0=1$ holda 2.3- va 2.4-rasmlarda ko‘rsatilgan.



2.3- rasm.



2.4- rasm.

Masalalar

1. Ushbu $F(x,y)=x+y$ ikki erkli o‘zgaruvchining funksiyasi $f(x)$ va $g(y)$ bir o‘zgaruvchining funksiyalari ko‘paytmasi sifatida ifodalanmaydi. Shu tasdiqni isbotlang.

2. Differensial tenglamani yeching : $y' = |x| y$.

3. Ushbu $y' = \max(x, y),\ y(0) = 0,$ boshlang‘ich masalaning $[0, +\infty)$ oraliqda aniqlangan notrivial ($y \neq 0$) yechimini toping.

4. Agar biror $(-a; a)$ $(a > 0)$ intervalda aniqlangan $y = f(x)$ haqiqiy funksiya

$$f(u+v) = \frac{f(u) + f(v)}{1 - f(u) \cdot f(v)} \quad (\{u, v, u+v\} \subset (-a; a)) \qquad (*)$$

(funksional) tenglamani qanoatlantirsa va 0 nuqtada $f'(0)$ hosilaga ega bo‘lsa, bu $y = f(x)$ funksiyani toping.

5. Logistik tenglamada $n = Nx$, $t = T\tau$ deb o‘lchash masshtablarini o‘zgartiring. N va T masshtab birliklarini qanday tanlasak, logistik tenglama x, τ o‘lchovsiz miqdorlarda ushbu

$$\frac{dx}{d\tau} = x(1-x)$$

ko‘rinishga keladi (tenglamada parametrlar yo‘qoladi)?

6. $y' = f(x, y)$ tenglamaning o‘ng tomonidagi $f(x, y)$ funksiya x bo‘yicha davriy bo‘lsin: $f(x+T, y) = f(x, y), T \neq 0$. Bu tenglamaning davriy yechimlari munosabati bilan quyidagi tenglamalarni yeching va yechimlarni davriylikka tekshiring:

$y' = y^2 \cos x$; $y' = (y^2 + 1)(2 + \cos x)$; $y' = (y^2 - 1)(2 + \cos x)$.

§ 2.2. O‘zgaruvchilariga nisbatan bir jinsli differensial tenglamalar

Agar $y' = f(x, y)$ differensial tenglamadagi $f(x, y)$ funksiya x, y o‘zgaruvchilarni mos ravishda tx, ty $(t > 0)$ bilan almashtirilganda o‘zgarmasa, ya’ni

$$f(tx, ty) = f(x, y),\ (x, y) \in D(f),\ t > 0 \qquad (2.2.1)$$

bo‘lsa, u holda bu $f(x, y)$ funksiya (0- tartibli) bir jinsli, mos

$$y' = f(x, y)$$

differensial tenglama esa o‘zgaruvchilariga nisbatan bir jinsli tenglama deyiladi.

Agar $f(x, y)$ bir jinsli funksiya bo‘lsa, $x \neq 0$ bo‘lganda

$$f(x, y) = f\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right), \text{ bunda } g(t) \stackrel{def}{=} f(1, t).$$

Aksincha, ravshanki, bir o‘zgaruvchining $g(t)$ funksiyasi orqali ifodalangan $f(x,y)=g\left(\frac{y}{x}\right)$ ($x\neq 0$ bo‘lgan sohada) (yoki $f(x,y)=g\left(\frac{x}{y}\right)$ ($y\neq 0$ bo‘lgan sohada)) funksiya bir jinslidir.

Bir jinsli tenglamada

$$y=xu \tag{2.2.2}$$

deb, yangi $u=u(x)$ noma’lum funksiyaga o‘tamiz. U holda

$$y'=u+xu'$$

va berilgan tenglama

$$u+xu'=f(x,xu)$$

yoki, $f(x,xu)=f(1,u)=g(u)$ bo‘lgani uchun,

$$xu'=g(u)-u$$

ko‘rinishni oladi. Bu o‘zgaruvchilari ajraladigan differensial tenglamadir. Oxirgi tenglamaning $u=u(x)$ yechimi topilgach (oshkor yoki oshkormas ko‘rinishda), (2.2.2) formulaga ko‘ra berilgan tenglamaning $y=xu(x)$ yechimini hosil qilamiz.

Misol. Ushbu

$$xy'=y\ln y-y\ln x,\ y|_{x=1}=e^2,$$

boshlang‘ich masalani yeching.

🔑 Dastlab berilgan tenglamaning barcha yechimlarini topamiz. So‘ngra ular orasidan ko‘rsatilgan boshlang‘ich shartni qanoat-lantiradiganini ajratamiz. Berilgan tenglama – o‘zgaruvchilarga nisbatan bir jinsli. Yangi $u=u(x)$ noma’lum funksiyani $y=xu$, $u=y/x>0$, formula bilan kiritamiz. Zarur hisoblashlarni va shakl almashtirishlarni bajaramiz:

$$y'=u+xu',\ y'=\frac{y}{x}\ln\frac{y}{x},\ u+xu'=u\ln u,\ xu'=u(\ln u-1).$$

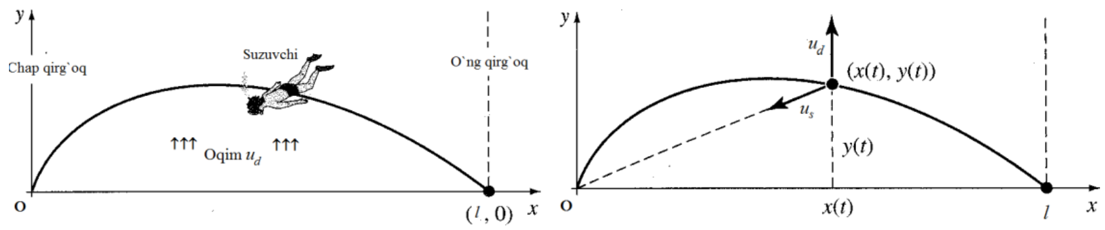
Oxirgi tenglamada o‘zgaruvchilarni ajratamiz va integrallashlarni bajaramiz:

$$\frac{du}{u(\ln u-1)}=\frac{dx}{x},\ u=e^{1+cx}.$$

Tenglamani $u(\ln u - 1)$ ga bo‘lishda $u = e$, yechim yo‘qolishi mumkin (bizda $u > 0$). Lekin bu yechim $u = e^{1+cx}$ formuladan $c = 0$ da hosil bo‘ladi. Demak, berilgan tenglamaning barcha yechimlari $y = xu = xe^{1+cx}$ formula bilan beriladi. $y|_{x=1} = e^2$ boshlang‘ich shart qanoatlanishi uchun $e^2 = e^{1+c}$, ya’ni $c = 1$ bo‘lishi kerak. Shunday qilib, berilgan Koshi masalasining yechimi umumiy yechim formulasidan $c = 1$ da hosil bo‘ladi: $y = xe^{1+x}$.

Daryoning o‘ng qirg‘og‘idan chap qirg‘og‘iga suzib o‘tish masalasi.

Daryoning o‘ng qirgog‘i $x = l$, chap qirg‘og‘i esa $x = 0$ to‘g‘ri chiziqlar bilan tasvirlansin (2.5-rasm). Daryo Oy o‘qi bo‘ylab $\vec{u}_d = (0, u_d)$ o‘zgarmas tezlik bilan oqadi. Suzuvchi $(l, 0)$ nuqtada suvga tushib, chap qirg‘oqdagi $(0, 0)$ nuqtaga suzib o‘tmoqchi. Suzuvchining daryoga nisbatan tezligi $\vec{u}_s$ bo‘lib, uning moduli o‘zgarmas: $|\vec{u}_s| = u_s = \text{const}$. Suzuvchi $(0, 0)$ nuqtaga o‘tish maqsadida shu nuqtaga qarab suzadi, ya’ni $\vec{u}_s$ tezlik vektori $(0, 0)$ nuqtaga yo‘nalgan. Suzuvchining traektoriyasini toping. U $(l, 0)$ nuqtadan $(0, 0)$ nuqtaga suzib o‘tishi mumkinmi?



2.5-rasm.

Suzuvchi traektoriyasining tenglamasi $x = x(t), y = y(t)$ bo‘lsa, uning tezlik vektori $\vec{u}(t) = (x'(t), y'(t))$ daryo oqimining tezligi $\vec{u}_d = (0, u_d)$ ga suzuvchining daryoga nisbatan tezligi $\vec{u}_s$ ni qo‘shishdan hosil bo‘ladi: $\vec{u}(t) = (x', y') = \vec{u}_d + \vec{u}_s$. Berilganga ko‘ra $\vec{u}_s(t)$ vektor $-\big(x(t), y(t)\big)$ vektor bo‘ylab yo‘nalgan, ya’ni $\vec{u}_s = \vec{u}_s(t) = -\lambda \cdot \big(x(t), y(t)\big) = -\lambda \cdot (x, y)$ $(\lambda = \text{const} > 0)$. Demak,

$$|\vec{u}_s| = u_s = \lambda\sqrt{x^2 + y^2}, \text{ ya’ni } \lambda = u_s / \sqrt{x^2 + y^2}.$$

Bundan tashqari,

$$\vec{u}(t) = (x', y') = (0, u_d) - \lambda(x, y) = (-\lambda x, u_d - \lambda y) \Rightarrow \begin{cases} x' = -\lambda x, \\ y' = u_d - \lambda y. \end{cases}$$

Bundan

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{u_d - \lambda y}{-\lambda x} = \frac{u_s y - u_d \sqrt{x^2 + y^2}}{u_s x}.$$

Demak, suzuvchining traektoriyasi uchun quyidagi differensial tenglama hosil bo'ldi:

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \text{ bu yerda } k = \frac{u_d}{u_s} = \text{const} > 0.$$

Boshlang'ich shart:

$y(0) = l$ (suzuvchi $(l, 0)$ nuqtada daryoga tushgan).

Agar x ning o'rniga x/l, y ning o'rniga y/l o'lchovsiz miqdorlarni kiritsak, quyidagi Koshi masalasi yosil bo'ladi:

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \; y(0) = 1.$$

Hosil bo'lgan tenglama o'zgaruvchilariga nisbatan bir jinsli. Uni $y = xu$ almashtirish yordamida yechamiz:

$$u + x\frac{du}{dx} = u - k\sqrt{1 + u^2}, \; \frac{du}{\sqrt{1 + u^2}} = -\frac{k}{x}dx,$$

$$\ln\left(u + \sqrt{1 + u^2}\right) = -k \ln x + c,$$

$$\ln\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = -k \ln x + c, \; y + \sqrt{x^2 + y^2} = xe^c x^{-k}.$$

Bu yerdagi e^c konstantani $y(0) = 1$ boshlang'ich shart yordamida topamiz, y ni x orqali ifodalaymiz va quyidagi natijaga kelamiz:

$$y = \frac{1 - x^{2k}}{2x^{k-1}}.$$

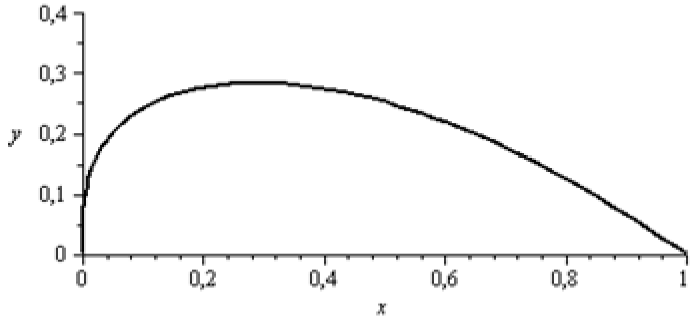
Endi suzuvchi $(0, 0)$ nuqtaga kela oladimi yoki yo'qligini aniqlaymiz. Buning uchun y ning $x \to +0$ dagi limitini hisoblash kerak.

$0 < k < 1$ (daryoning tezligi suzuvchining tezligidan kichik) bo‘lsin. Bu holda $\lim\limits_{x\to 0+} y = \lim\limits_{x\to 0+} \dfrac{x^{1-k}(1-x^{2k})}{2} = 0$, ya’ni suzuvchi $(0,0)$ nuqtaga suzib keladi (2.6-rasm).

$k > 1$(daryoning tezligi suzuvchining tezligidan katta) bo‘lsin. Bu holda

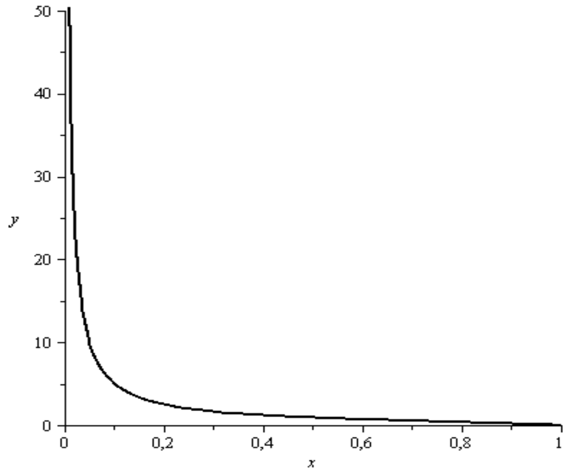
$\lim\limits_{x\to 0+} y = \lim\limits_{x\to 0+} \dfrac{1-x^{2k}}{2x^{k-1}} = +\infty$, ya’ni daryo oqimi suzuvchini chap qirg‘oqning $+\infty$ nuqtasiga olib ketadi (2.7-rasm).

$k = 1$ (daryoning tezligi suzuvchining tezligiga teng) bo‘lsin. Bu holda $\lim\limits_{x\to 0+} y = \lim\limits_{x\to 0+} \dfrac{1-x^2}{2} = \dfrac{1}{2}$, ya’ni suzuvchi chap qirg‘oqdagi $(0,0)$ nuqtadan yuqoridagi $(0, l/2)$ nuqtaga suzib keladi (2.8-rasm).



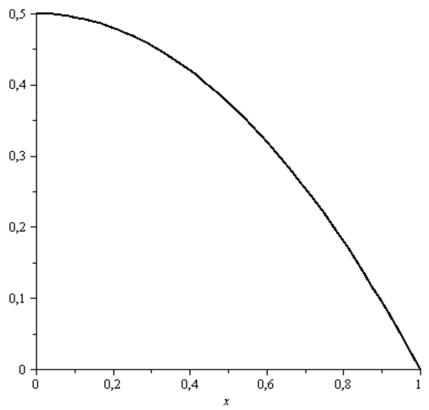
$k = 0,7$

2.6-rasm. Daryoning tezligi suzuvchining tezligidan kichik.



$k = 2$.

2.7-rasm. Daryoning tezligi suzuvchining tezligidan katta.



$k = 1$

2.8-rasm. Daryoning tezligi suzuvchining tezligiga teng.

Ba’zi differensial tenglamalar (x, y) o‘zgaruvchilarga nisbatan bir jinsli bo‘lmasa-da, ularni $y = z^k$ almashtirish yordamida, $k-$ biror tayin son, $z = z(x)-$ yangi noma’lum funksiya, (x, z) o‘zgaruvchilarga nisbatan bir jinsli tenglamaga keltirib yechish mumkin. Bunday tenglamalar o‘zgaruvchilariga nisbatan umumlashgan bir jinsli differensial tenglamalar deb ataladi.

Ushbu

$$y' = \frac{y}{x} + h(x)g(\frac{y}{x})$$

ko‘rinishdagi tenglamani ham (u bir jinsli bo‘lmasa-da) $y = xu$ almashtirish yordamida yechish mumkin.

Masalalar

1. Differensial tenglamani yeching $yy' + 2y + x = 0$

2. Ushbu

$$y' = \frac{y}{x} + h(x)g(\frac{y}{x})$$

differensial tenglamani umumiy holda yeching.

3. Differensial tenglamani yeching $x^3 y' = y^2 + x^4$.

§ 2.3. Chiziqli tenglamalar

Ushbu

$$y' + p(x)y = q(x), \ \{p, q\} \subset C(I), \qquad (2.3.1)$$

tenglama birinchi tartibli **chiziqli differensial tenglama** deyiladi. Bu yerdagi $q(x)$ ozod had deb ataladi. Ozod had nolga teng bo‘lganda hosil bo‘luvchi

$$y' + p(x)y = 0 \qquad (2.3.2)$$

tenglama (2.3.1) ga mos bir jinsli tenglama deb ataladi. (2.3.2) o‘zgaruvchilari ajraladigan tenglamaning umumiy yechimini topish oson. Biz bu yerda umumiy yechimni boshqa usulda topamiz.

Lemma. Ushbu

$$y' + p(x)y = 0 \qquad (2.3.2)$$

differensial tenglamaning umumiy yechimi

$$y = c\exp\left(-\int_{x_0}^{x} p(s)ds\right) \qquad (2.3.3)$$

($x_0, x_0 \in I, -$tayinlangan nuqta , $c -$ixtiyoriy o‘zgarmas) ko‘rinishda bo‘ladi.

Osongina tekshirib ko‘rish mumkinki, (2.3.3) formula bilan berilgan funksiya (2.3.2) differensial tenglamaning I oraliqda aniqlangan yechimi. Endi ixtiyoriy yechimning (2.3.3) ko‘rinishda ekanligini isbotlaymiz. Ixtiyoriy $y(x)$ yechimni qaraylik:

$$y'(x) + p(x)y(x) = 0.$$

Bu tenglikni $\exp\left(\int_{x_0}^{x} p(s)ds\right) > 0$ ga ko‘paytirib,

$$\left(y(x)\cdot\exp\left(\int_{x_0}^{x} p(s)ds\right)\right)' = 0$$

munosabatni hosil qilamiz. Demak, matematik analizdan ma’lum teoremaga ko‘ra

$$y(x)\cdot\exp\left(\int_{x_0}^{x} p(s)ds\right) = c \;\; (c - \text{const}) \;\; \Rightarrow \;\; y(x) = c\exp\left(-\int_{x_0}^{x} p(s)ds\right).$$

Tushunarliki, (2.3.3) formuladagi c o‘zgarmas son noma’lum funksiyaning x_0 nuqtadagi qiymatiga teng, ya’ni $c = y(x_0)$. Endi ravshanki, $I \times (-\infty, +\infty)$ soha (polosa)ning har bir (x_0, y_0) nuqtasidan (2) tenglamaning yagona integral chizig‘i o‘tadi. U ushbu

$$y = y_0\exp\left(-\int_{x_0}^{x} p(s)ds\right)$$

formula bilan beriladi, ya’ni $I \times (-\infty, +\infty)$ polosa (2.3.2) tenglamaning yagonalik sohasi.

Shunday qilib, (2.3.3) formula chiziqli bir jinsli tenglama (2) ning $I\times(-\infty,+\infty)$ sohadagi umumiy yechimini beradi, ya'ni (2.3.2) tenglamaning $I\times(-\infty,+\infty)$ sohadagi barcha yechimlari va ulargina (2.3.3) formula bilan aniqlanadi..

Endi bir jinsli bo'lmagan (2.3.1) tenglamani yechimiz. Yechimi

$$y = v(x)\cdot\exp\left(-\int_{x_0}^{x} p(s)ds\right) \tag{2.3.4}$$

ko'rinishda izlaylik (mos bir jinsli tenglamaning umumiy yechimi (2.3.3) dagi ixtiyoriy o'zgarmasni «variatsiyalab», ya'ni o'zgartirib, (2.3.1) ning yechimini quramiz); bu – **Lagranj metodi**. (2.3.4) ni (2.3.1) ga qo'yib,

$$v'(x)\cdot\exp\left(-\int_{x_0}^{x} p(s)ds\right) = q(x)$$

tenglikni hosil qilamiz. Bundan

$$v'(x) = q(x)\exp\left(\int_{x_0}^{x} p(s)ds\right) \Rightarrow v(x) = c + \int_{x_0}^{x} q(\tau)\exp\left(\int_{x_0}^{\tau} p(s)ds\right)d\tau \tag{2.3.5}$$

Endi (2.3.5) ni (2.3.4) ga qo'yib, (2.3.1) ning umumiy yechimini hosil qilamiz:

$$y = c\exp\left(-\int_{x_0}^{x} p(s)ds\right) + \exp\left(-\int_{x_0}^{x} p(s)ds\right)\int_{x_0}^{x} q(\tau)\exp\left(-\int_{x_0}^{\tau} p(s)ds\right)d\tau \tag{2.3.6}$$

(2.3.6) formuladagi birinchi qo'shiluvchi bir jinsli tenglama (2.3.2) ning umumiy yechimini, ikkinchi qo'shiluvchi esa (2.3.1) ning xususiy (biror tayin) yechimini ifodalaydi. Shunday qilib, bir jinsli bo'lmagan (2.3.1) chiziqli tenglamaning umumiy yechimi uning xususiy yechimiga mos bir jinsli tenglama (2.3.2) ning umumiy yechimini qo'shishdan hosil bo'ladi.

Tenglamaning yagona (birorta) xususiy yechimini ajratish uchun qo'shimcha shart qo'yish kerak. (2.3.1) tenglamaning $x_0 \in I$ nuqtada berilgan y_0 qiymatni qabul qiluvchi, ya'ni

$$y(x_0) = y_0 \tag{2.3.7}$$

shartni qanoatlantiruvchi yechimi (2.3.6) formuladan osongina topiladi. Bu yechim bitta va u

$$y = y_0 \exp\left(-\int_{x_0}^{x} p(s)ds\right) + \exp\left(-\int_{x_0}^{x} p(s)ds\right)\int_{x_0}^{x} q(\tau)\exp\left(-\int_{x_0}^{\tau} p(s)ds\right)d\tau \tag{2.3.8}$$

formula bilan ifodalanadi. Demak, $I\times(-\infty,+\infty)$ polosa (2.3.1) tenglamaning yagonalik sohasidir. Ravshanki, (2.3.1),(2.3.7) boshlang‘ich masalaning (2.3.8) yechimi I oraliqda, ya’ni (2.3.1) tenglamada berilgan funksiyalarning uzluksizlik oralig‘ida aniqlangan.

Bino temperaturasining o‘zgarishini modellashtirish. Binoning t paytdagi temperaturasini $T = T(t)$ bilan belgilaymiz. Binodagi temperatura tashqi muhit va ichki issiqlik manbalari ta’sirida o‘zgaradi. Bino tashqi muhit bilan Nyuton qonuniga ko‘ra issiqlik almashinadi deb faraz qilamiz, ya’ni agar tashqi muhit temperaturasinining t paytdagi qiymatini $T_{tash}(t)$ bilan belgilasak, bino temperaturasining o‘zgarish tezligi $T'(t)$ temperaturalar ayirmasi $T_{tash}(t)-T(t)$ ga proporsional bo‘ladi. Binoning ichida esa issiqlik manbalari bor: binoda yashovchi odamlar, issiqlik chiqaruvchi turli mashinalar va h.k. Bu issiqlik manbalari ta’sirida $T'(t)$ tezlik o‘zgarmas v_0 miqdorga ortadi deb faraz qilamiz. Bu farazlarga ko‘ra bino temperaturasining o‘zgarish qonuni uchun quyidagi differensial tenglama hosil bo‘ladi:

$$T'(t) = k\left(T_{tash}(t) - T(t)\right) + v_0,$$

bu yerda k, v_0 o‘zgarmas sonlar va $k > 0$. Bu tenglamani quyidagi ko‘rinishda yozib olamiz:

$$T'(t) + p(t)T(t) = q(t), \tag{2.3.9}$$

bu yerda

$$p(t) = k,\ q(t) = kT_{tash}(t) + v_0. \tag{2.3.10}$$

Binoning noma’lum temperaturasi $T(t)$ uchun chiziqli differensial tenglama (2.3.9) hosil bo‘ldi. Bu tenglamani $e^{p_0 t}$ ga ko‘paytitib va integrallashni bajarib, topamiz:

$$T(t) = ce^{-kt} + \frac{v_0}{k} + e^{-kt}k\int e^{kt}T_{tash}(t)dt. \tag{2.3.11}$$

Ba'zi xususiy hollarni qarab chiqaylik.

1- hol. Faraz qilaylik, ish kunining oxirida t_0 paytda ishchilar binoni tark etsin; tashqi temperatura o'zgarmasin, $T_{tash}(t)=\tilde{T}=\text{const}$; ichki issiqlik manbalari ochirilsin, $v_0=0$. $T(t_0)=T_0$ ma'lum bo'lsin. Bino temperaturasining o'zgarish qonunini topaylik. Bu holda bino temperaturasi uchun

$$T'(t)+kT(t)=k\tilde{T},$$

tenglama hosil bo'ladi. Uning $T(t_0)=T_0$ boshlang'ich shartni qanoatlantiruvchi yechimi

$$T(t)=\left(T_0-\tilde{T}\right)e^{-k(t-t_0)}+\tilde{T}.$$

Vaqt o'tishi bilan boshlang'ich temperaturaning qiymatidan qat'iy nazar bino temperaturasi eksponensial tezlik bilan $\tilde{T}$ tashqi o'zgarmas temperaturaga yaqinlashadi. Bunda $T(t)-\tilde{T}$ ayirmaning qiymati $1/k$ vaqtda $T_0-\tilde{T}$ dan $\left(T_0-\tilde{T}\right)/e$ gacha o'zgaradi.

2- hol. Faraz qilaylik, tashqi temperatura davri 24 soat bo'lgan sinusoida kabi o'zgarsin, ya'ni

$$T_{tash}(t)=\bar{T}+A\sin\omega t \qquad (2.3.12)$$

bo'lsin, bu yerda $\bar{T}-$ tashqi temperaturaning o'rtacha qiymati , $0<A-$ tashqi temperaturaning o'zgarish (tebranish) amplitudasi, $\omega=2\pi/24=\pi/12$ $(1/\text{soat})$; $t=0$ da (yarim kecha) $T_{tash}(t)$ eng kichik, tush payti $t=12$ (soat) da $T_{tash}(t)$ eng katta qiymatga ega. Boshlang'ich temperatura $T(0)=T_0$ berilgan. Bino temperaturasi uchun tenglama

$$T'(t)+p(t)T(t)=q(t),\ \ p(t)=k,\ q(t)=k\left(\bar{T}+A\sin\omega t\right)+v_0. \qquad (2.3.13)$$

Bu tenglamaning yechimi (2.3.11) ga ko'ra topiladi. Ikki marta bo'laklab integrallash yordamida $T(t)$ ni topamiz:

$$\begin{aligned} T(t)&=ce^{-kt}+\frac{v_0}{k}+e^{-kt}k\int e^{kt}(\tilde{T}+A\sin\omega t)dt= \\ &=ce^{-kt}+\frac{v_0}{k}+\tilde{T}+A\frac{k}{k^2+\omega^2}(k\sin\omega t-\omega\cos\omega t). \end{aligned} \qquad (2.3.14)$$

Bu yerdagi c konstanta $T(0)=T_0$ boshlang'ich shartdan aniqlanadi:

$$c=T_0-\bar{T}-\frac{v_0}{k}+A\frac{k\omega}{k^2+\omega^2}.$$

c ning qiymatidan qat’iy nazar $t \to +\infty$ (katta t lar) da binoda

$$T(t) \approx \frac{v_0}{k} + \tilde{T} + A\frac{k}{k^2+\omega^2}(k\sin\omega t - \omega\cos\omega t) \qquad (2.3.15)$$

davriy o‘zgaruvchi temperatura o‘rnatiladi. Bu temperaturaning o‘zgarish fazasi tashqi temperaturaning o‘zgarish fazasidan farq qiladi, chunki

$$A\frac{k}{k^2+\omega^2}(k\sin\omega t - \omega\cos\omega t) = Ak\sin(\omega t - \varphi),\ \varphi = \mathrm{arctg}(\omega/k);$$

fazalar siljishi (farqi) φ/ω ga teng.

3- hol (umumiy hol). Qo‘shimcha holda binoda termorele o‘rnatilgan bo‘lib, u $T(t) > T_*$ ($T_* = \mathrm{const}$) bo‘lganda sovutgichni, $T(t) < T_*$ bo‘lganda esa qizitgichni ishga tushirsin; natijada bino temperaturasining o‘zgarish tezligi $T_* - T(t)$ ga proporsional miqdorga ortadi va bino temperaturasi uchun ushbu

$$T'(t) = k\left(T_{tash}(t) - T(t)\right) + v_0 + \lambda\left(T_* - T(t)\right)$$

differensial tenglama hosil bo‘ladi. Bu tenglamani quyidagi ko‘rinishda yozaylik:

$$T'(t) + p(t)T(t) = q(t), \qquad (2.3.16)$$

bu yerda

$$p(t) = p_0 = k + \lambda\ (= \mathrm{const}),\ q(t) = kT_{tash}(t) + v_0 + \lambda T_*. \qquad (2.3.17)$$

Bu oxirgi (2.3.16) va (17.17) munosabatlar termorele bo‘lmagan holdagi (2.3.6) va (2.3.16) tenglamalardan faqat koeffitsientlari bilan farq qiladi. Termorelening qo‘shilishi bino temperaturasiga ”qo‘shimcha” tashqi temperatura kabi ta’sir etadi.

Pechga qo‘yilgan metall parchasi temperaturasining o‘zgarish qonunini topish. Temperaturasi T_0 (gradus) bo‘lgan metall parchasi pechga qo‘yildi. Pechning temperaturasi T_1 (gradus) dan boshlab minutiga τ (gradus) tezlik bilan tekis ravishda ko‘tarila boshlaydi. Agar metall parchasi bilan pech orasidagi issiqlik almashinish Nyuton qonuniga bo‘yso‘nsa, metall parchasi temperaturasining o‘zgarish qonunini toping.

Masalaning shartiga ko‘ra pechning temperaturasi t vaqt bo‘yicha $T_{pech}(t) = T_1 + \tau t$ qonunga ko‘ra o‘zgaradi. Issiqlik almashinish to‘g‘risidagi Nyuton qonunuga ko‘ra metall parchasi temperaturasi temperaturasi $T = T(t)$ uchun ushbu

$$T' = k(T_{pech}(t) - T), \text{ ya'ni } T' = k(T_1 + \tau t - T)$$

chiziqli differensial tenglamani olamiz ($k = \text{const} > 0 -$ proporsionallik koeffitsienti). Bu tenglamani

$$T' + kT = k(T_1 + \tau t) \qquad (2.3.18)$$

ko‘rinishda yozaylik. Mos bir jinsli tenglama $T' + kT = 0$ ning umumiy yechimi $T = ce^{-kt}$, bir jinsli bo‘lmagan (2.3.18) tenglamaning xususiy yechimi esa $T = (kT_1 - \tau)/k + \tau t$. Demak, (2.3.18) tenglamaning umumiy yechimi

$$T = ce^{-kt} + (kT_1 - \tau)/k + \tau t$$

formula bilan beriladi. Boshlang‘ich shart: $t = 0$ da $T = T_0$. Bunga ko‘ra $c = T_1 - T_0 + \tau/k$. Demak, metall parchasi temperaturasining o‘zgarish qonuni

$$T = \left(T_1 - T_0 + \frac{\tau}{k}\right)e^{-kt} + \frac{kT_1 - \tau}{k} + \tau t$$

formula bilan aniqlanadi (vaqt minutlarda temperatura graduslarda o‘lchanadi).

Masalalar

1. $xy' = -2y + e^x, y(1) = 0$ Koshi masalasini yeching. Yechim qaysi oraliqda aniqlangan?

2. Agar

$y' + p(x)y = q(x),\ \{p,q\} \subset C(I),$

chiziqli tenglamaning ikki dona $y = y_1(x)$ va $y = y_2(x)$ yechimlari ma’lum bo‘lsa, uning umumiy yechimi

$y = y_1(x) + c\big(y_2(x) - y_1(x)\big),\ c = \text{const},$

formula bilan berilishini ko‘rsating.

3. $p(t), q(t) \in C\big([0, +\infty)\big)$ bo‘lsin. Agar $x = x(t)$ va $u = u(t)$ funksiyalar uchun

$$x' + p(t)x = q(t), x(0) = x_0,$$
$$u' + p(t)u \ge q(t), u(0) \ge x_0,$$

bo‘lsa, u holda $t \in [0, +\infty)$ oraliqda $u(t) \ge x(t)$ tengsizlik o‘rinli. Shuni isbotlang.

4. Tenglamani yeching $x(e^y - y') = a\ (a = \text{const} \ne 1)$.

5. Tenglamani yeching $(x^2 - 1)y' \sin y - 2x \cos y = x^2$.

6. $y' = y - \exp(-x^2)$ tenglamaning $\lim\limits_{x\to-\infty} y(x) = \lim\limits_{x\to+\infty} y(x) = 0$ shartlarni qanoatlantiruvchi yechimlarini toping.

7. Ushbu $y' + 2|y| = 1,\ y(0) = 1/4$, Koshi masalasini yeching.

§ 2.4. Bernulli va Rikkati tenglamalari

Bernulli tenglamasi deb ushbu

$$y' = p(x)y + q(x)y^{\alpha} \quad (\alpha \neq 1, \alpha \neq 0, \{p(x), q(x)\} \subset C(I)) \qquad (2.4.1)$$

ko‘rinishdagi tenglamaga aytiladi. Bernulli tenglamasi $u(x) = y^{1-\alpha}$ almashtirish yordamida

$$u' = (1-\alpha)p(x)u + (1-\alpha)q(x)$$

chiziqli tenglamaga keltiriladi.

Bernulli tenglamasini **Eyler-Bernulli usuli** deb ataluvchi usul bilan ham yechish mumkin. Bu usulga ko‘ra yechim $y = uv$ ko‘rinishda izlanadi, bunda $u, v -$hozircha noma’lum funksiyalar. $y = uv$ ni berilgan tenglamaga qo‘yamiz:

$$u'v + uv' - p(x)uv = q(x)y^{\alpha}, \;\; \left(u' - p(x)u\right)v + uv' = q(x)u^{\alpha}v^{\alpha}.$$

Endi $u' - p(x)u = 0$, ya’ni $u = \exp\left(\int p(x)dx\right)$ deb, v uchun $uv' = q(x)u^{\alpha}v^{\alpha}$, ya’ni $v' = q(x)\exp\left((\alpha-1)\int p(x)dx\right)v^{\alpha}$ o‘zgaruvchilari ajraladigan tenglamaga kelamiz. Oxirgi tenglamani yechib, topilgan u, v larga ko‘ra Bernulli tenglamasining $y = uv$ yechimini yozamiz.

Rikkati tenglamasi deb

$$y' = a(x)y^2 + b(x)y + c(x) \qquad (2.4.2)$$

ko‘rinishdagi tenglamaga aytiladi; bunda $\{a(x), b(x), c(x)\} \subset C(I), a(x) \neq 0, c(x) \neq 0$ deb hisoblanadi ($a(x) \equiv 0$ bo‘lganda chiziqli tenglama hosil bo‘ladi, $c(x) \equiv 0$ bo‘lganda esa – Bernulli tenglamasi).

Rikkati tenglamasi differensial tenglamalar nazariyasida alohida o‘rin tutadi. Bu tenglama amaliy masalalarni yechishda ham ko‘p uchraydi.

Umumiy holda Rikkati tenglamasining yechimi kvadraturalarda ifodalanmaydi, ya’ni u $a(x), b(x), c(x)$ va elementar funksiyalar orqali funksiyalarni qo‘shis, ayirish, ko‘paytirish, bo‘lish, integrallash va kompositsiya olish amallarini chekli marta ishlatilgan holda topilmaydi. Ba’zi maxsus hollardagina u kvadraturalarda yechiladi. Shu hollarning ba’zilarini keltiraylik:

1) $y' = f(x)(ay^2 + by + c)$, 2) $y' = a\dfrac{y^2}{x^2} + b\dfrac{y}{x} + c$, 3) $y' = ay^2 + \dfrac{b}{x}y + \dfrac{c}{x^2}$

(uchala holda ham a,b,c – o‘zgarmas sonlar). 1) tipdagi tenglamalarda o‘zgaruvchilar ajraladi, 2) tipdagilari – o‘zgaruvchilariga nisbatan bir jinsli tenglamalar, 3) tipdagilari esa $y = z/x$ almashtirish yordamida o‘zgaruvchilari ajraladigan tenglamalarga keltiriladi.

Agar Rikkati tenglamasining biror $y = \varphi(x)$ xususiy yechimi ma’lum bo‘lsa, uning umumiy yechimini topish mumkin. Buning uchun tenglamada $y = \varphi(x) + u$ almashtirishni bajarish kerak, bunda u –yangi noma’lum funksiya. U holda ushbu

$$u' = (2a(x)\varphi(x) + b(x))u + a(x)u^2$$

Bernulli tenglamasini hosil qilamiz. Oxirgi tenglamani yechish uchun $u = 1/z$ deb yangi $z = z(x)$ noma’lum funksiyani kiritib, qaralayotgan Rikkati tenglamasidan chiziqli tenglama hosil qilamiz. Demak, yangi noma’lum funksiya $z = z(x)$ ni

$y = \varphi(x) + 1/z$

formula bilan kiritib, z ga nisbatan chiziqli tenglamaga kelamiz.

Quyidagi ikki holda xususiy yechim osongina topiladi:

$c(x) = -a(x)d^2 - b(x)d$ bo‘lganda $y = \varphi(x) = d$ xususiy yechim;

$c(x) = -a(x)x^2 - b(x)x + 1$ bo‘lganda esa $y = \varphi(x) = x$ xususiy yechim.

Noma’lum funksiyani almashtirish yordamida Rikkati tenglamasining ko‘rinishini soddalashtirish mumkin. y noma’lum funksiya o‘rniga yangi $v = \alpha(x)y$ noma’lum funksiyani kiritib, ushbu

$$v' = \frac{a(x)}{\alpha(x)}v^2 + \left(b(x) + \frac{\alpha'(x)}{\alpha(x)}\right)v + c(x)\alpha(x)$$

tenglamani hosil qilamiz. Endi $\alpha(x)$ sifatida $a(x)$ ni tanlab ($\alpha(x) = a(x)$), tenglamada noma’lum funksiya kvadrati oldidagi koeffitsientni birga tenglashtiramiz:

$$v' = v^2 + \tilde{b}(x)v + \tilde{c}(x),$$

bu yerda $\tilde{b}(x) = b(x) + \dfrac{a'(x)}{a(x)}$, $\tilde{c}(x) = c(x)a(x)$.

Agar v noma’lum funksiya o‘rniga yangi $w = \beta(x) + v$ noma’lum funksiyani kiritsak, $\beta(x)$ ni tanlash evaziga tenglamada noma’lum funksiya qatnashgan hadni yo‘qotish mumkin.

Rikkati tenglamasining yana bir xususiy holida to‘xtalaylik. Agar $a(x)=a=\text{const}\neq 0, b(x)\equiv 0,\quad c(x)=cx^m\quad (c,m-$o‘zgarmaslar) bo‘lsa, ushbu

$$y'-ay^2=cx^m \tag{2.4.3}$$

maxsus Rikkati tenglamasi deb ataluvchi tenglamaga kelamiz. Bu tenglama yechimlarining kvadraturalarda ifodalanishi yoki ifodalanmasligi to‘la o‘rganilgan. $m=0$ bo‘lganda (2.4.3) − o‘zgaruvchilari ajraladigan tenglama, $m=-2$ bo‘lganda esa u $y=z/x$ almashtirish bilan o‘zgaruvchilari ajraladigan tenglamaga keltiriladi. Shunday qilib, bu hollarda (2.4.3) maxsus Rikkati tenglamasi kvadraturalarda yechiladi (aslida elementar funksiyalarda). Bundan tashqari $m/(2m+4)$ butun son bo‘lgan hollarda ham (ya’ni $m=4k/(1-2k), k\in\mathbb{Z}$, holida) (2.4.3) tenglama elementar funksiyalarda yechiladi. m ning qolgan boshqa qiymatlarida esa bu tenglamaning yechimi elementar funksiyalarning chekli sondagi integrallari orqali ifodalanmasligini Liuvill 1841-yil isbotlagan.

Masalalar

1. Ushbu $(x^2y^3+xy)y'=1$ tenglamani yeching .

2. Ushbu

$$y'=c(y+a)(y+b)\ , (a,b,c-\text{const})$$

Rikkati tenglamasida $z=1/(y+a)$ almashtirishni bajaring.

3. Agar $y(x),\ y_1(x),\ y_2(x),\ y_3(x)-$ funksiyalar

$$y'=p(x)y^2+q(x)y+r(x)\ \ (p(x),q(x),r(x)-\text{uzluksiz})$$

Rikkati tenglamasining yechimlari bo‘lsa, ushbu

$$\frac{y_2(x)-y(x)}{y_2(x)-y_1(x)}:\frac{y_3(x)-y(x)}{y_3(x)-y_1(x)}$$

nisbatning o‘zgarmas ekanligini ko‘rsating.

4. Rikkati tenglamasining har qanday yechimi uning uchta yechimi orqali ifodalanishini isbotlang.

§ 2.5. To‘la differensialli tenglamalar va integrallovchi ko‘paytuvchi

To‘la differensialli tenglama. x va y o‘zgaruvchilar teng huquqli qatnashgan differensial tenglamani qaraylik:

$$M(x,y)dx+N(x,y)dy=0\ \ (\{M,N\}\subset C(D)). \tag{2.5.1}$$

Agar D sohada (2.5.1) tenglamaning chap tomonidagi differensial ifoda biror $u = u(x, y)$ funksiyaning to‘la differensialidan iborat, ya’ni

$$du = M(x, y)dx + N(x, y)dy, \ (x, y) \in D, \qquad (2.5.2)$$

bo‘lsa, u holda (2.5.1) tenglama D sohada **to‘la (to‘liq) differensialli tenglama** deyiladi. Bu yerdagi u funksiya **potensial** deb ataladi.

To‘la differensialli tenglama uchun differensial ta’rifi $du = u'_x dx + u'_y dy$ va (2.5.2) tenglikka ko‘ra

$$u'_x = M, \ \ u'_y = N \qquad (2.5.3)$$

bo‘ladi hamda u $du(x, y) = 0$ ko‘rinishga keladi. Bu holda, ravshanki, $y = \varphi(x) \in C^1(I)$ (yoki $x = \psi(y) \in C^1(I)$)funksiya (2.5.1) ning yechimi bo‘lishi uchun I oraliqda $u(x, \varphi(x)) \equiv \text{const}$ (yoki $u(\psi(y), y) \equiv \text{const}$) ayniyat bajarilishi kerak. Demak, to‘la differensialli (2.5.1) tenglamaning har qanday regular nuqtasi atrofida integral chiziqlar ushbu

$$u(x, y) = c \ \ (c - \ \text{const}) \qquad (2.5.4)$$

tenglama bilan oshkormas ko‘rinishda beriladi.

Bu yerda shuni e’tirof etaylikki, u potensial bir qiymatli topilmayadi, chunki ixtiyoriy o‘zgarmas c uchun u bilan birgalikda $u + c$ ham potensial bo‘ladi.

Shunday qilib, to‘la differensialli tenglamani yechish $u(x, y)$ potensialni topishga keltirildi, bu tenglamaning yechimi (2.5.4) formula bilan oshkormas ko‘rinishda beriladi.

$Mdx + Ndy$ differensial ifoda biror funksiyaning to‘la differensialidan iborat bo‘lishi uchun yetarli shartlar matematik analiz kursida o‘rganiladi. Biz bu yerda quyidagi lokal teoremani keltiramiz.

Teorema. Aytaylik, M, N funksiyalar hamda M'_y va N'_x hosilalar (x_0, y_0) nuqtaning biror doiraviy $B_\delta = B_\delta(x_0, y_0)$ atrofida uzluksiz bo‘lsin. U holda (2.5.1) tenglamaning B_δ da to‘la differensialli bo‘lishi uchun $B_\delta$ning har bir nuqtasida

$$M'_y = N'_x \qquad (2.5.5)$$

shartning bajarilishi yetarli va zarurdir. Bu shart bajarilganda

$$u(x,y)=\int\limits_{x_0}^{x} M(s,y)ds+\int\limits_{y_0}^{y} N(x_0,t)dt\,,\ (x,y)\in B_\delta\,, \qquad (2.5.6)$$

funksiyaning to‘la differensiali $Mdx+Ndy$ ifodadan iborat bo‘ladi.

Zarurligi. (2.5.1) tenglama B_δ da to‘la differensialli bo‘lsin. Demak, B_δ da $u=u(x,y)$ potensial uchun (2.5.3) shartlar bajariladi:

$$u'_x=M\,,\ \ u'_y=N\,.$$

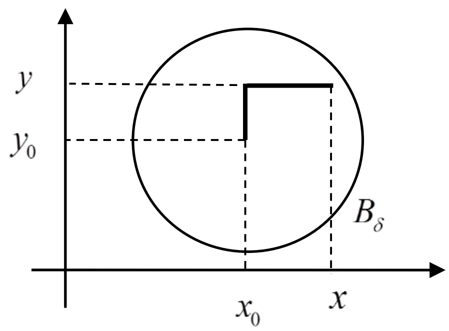
Bundan hosila olib, topamiz

$$u''_{yx}=M'_y,\ \ u'_{xy}=N'_x.$$

M'_y va N'_x hosilalar uzluksiz bo‘lgani uchun, analizdan ma’lum teoremaga ko‘ra, $u''_{yx}=M'_y,\ u'_{xy}=N'_x$ uzluksiz aralash hosilalar o‘zaro teng, ya’ni (2.5.5) shart o‘rinli.

Yetarliligi. Faraz qilaylik, (2.5.5) shart bajarilsin. $u=u(x,y)$, $(x,y)\in B_\delta$ potensialni topish uchun (2.5.3) shartdagi birinchi tenglikni (y ni tayinlab) x_0 dan x gacha integrallaylik (2.9-rasm):

$$u(x,y)=\int\limits_{x_0}^{x} M(s,y)ds+u(x_0,y)\ \ .$$



2.9-rasm.

Bu yerdagi $u(x_0,y)$ni topish maqsadida (2.5.3) shartdagi ikkinchi tenglikni yozaylik:

$$u'_y(x_0,y)=N(x_0,y).$$

Bu tenglikni y_0 dan y gacha integrallab, $u(x_0,y)$ni topamiz

$$u(x_0,y)=\int\limits_{y_0}^{y} N(x_0,t)dt+u(x_0,y_0)\,.$$

Oxirgi tenglik va yuqoridagi $u(x,y)=\int\limits_{x_0}^{x} M(s,y)ds+u(x_0,y)$ tenglikka ko‘ra (2.5.6) ni hosil qilamiz. (2.5.6) formula bilan aniqlangan $u=u(x,y)$ funksiya uchun B_δ da

$$du(x,y)=M(x,y)dx+N(x,y)dy$$

bo‘lishi bevosita tekshiriladi. ☝

Misol 1. Ushbu

$$(x^2+2y)dx+(2x+y^2)dy=0 \qquad (2.5.7)$$

tenglama $D=\mathbb{R}^2$ tekislikda to‘la differensiallidir, chunki bu tenglama uchun

$$M=x^2+2y,\ N=2x+y^2 \text{ va } M'_y=2=N'_x$$

$u=u(x,y)$ potensialni (2.5.3) shartlardan foydalanib topamiz;

$$u'_x=x^2+2y,\ u'_y=2x+y^2;$$

$$u'_x=x^2+2y \Rightarrow u=\frac{x^3}{3}+2xy+\varphi(y),\ u'_y=2x+\varphi'(y);$$

$$u'_y=2x+y^2=2x+\varphi'(y)\Rightarrow \varphi'(y)=y^2 \Rightarrow \varphi(y)=y^3/3;$$

$$u=\frac{x^3}{3}+2xy+\varphi(y)=x^3/3+2xy+y^3/3 .$$

Qaralayotgan misolda u potensialni differensial xossalaridan foydalanib, bevosita topsa ham bo‘ladi:

$$(x^2+2y)dx+(2x+y^2)dy=x^2dx+y^2dy+2ydx+2xdy=$$
$$=d\left(x^3/3\right)+d\left(y^3/3\right)+2d(xy)=$$
$$=d\left(x^3/3+y^3/3+2xy\right),\ u=x^3/3+y^3/3+2xy.$$

Demak, (2.5.7) tenglamaning yechimi

$$x^3/3+y^3/3+2xy=c/3,\ c=\text{const},$$

ya’ni

$$x^3+y^3+6xy=c$$

tenglik bilan oshkormas ko‘rinishda beriladi.

Integrallovchi ko‘paytuvchi. «Agar berilgan tenglama uchun (2.5.5) shart bajarilmasa, tenglamani biror «integrallovchi» ko‘paytuvchiga ko‘paytirib, uni to‘la differensialli ko‘rinishga keltirish mumkinmi?» – degan savol tug‘iladi.

Agar D sohada nolga aylanmaydigan $\mu=\mu(x,y)\in C^1(D)$ funksiya uchun

$$\mu Mdx+\mu Ndy=0 \tag{2.5.8}$$

tenglama to‘la differensialli bo‘lsa, u holda μ funksiya (2.5.1) tenglamaning **integrallovchi ko‘paytuvchisi** deyiladi.

Agar $\mu=\mu(x,y)$ funksiya (2.5.1) tenglamaning integrallovchi ko‘paytuvchisi va

$$\mu\cdot(Mdx+Ndy)=du,\ u=u(x,y),$$

bo‘lsa, bir o‘zgaruvchining ixtiyoriy $\varphi(u)$ uzluksiz funksiyasi uchun $\varphi(u(x,y))\cdot\mu(x,y)$ ham shu tenglamaning integrallovchi ko‘paytuvchisi bo‘ladi, chunki

$$\varphi(u)\mu\cdot(Mdx+Ndy)=\varphi(u)d(u)=d\left(\int^{u}\varphi(s)ds\right).$$

$D-$ bir bog‘lamli soha bo‘lsin. (2.5.8) dan $\mu\in C^1(D)$ integrallovchi ko‘paytuvchi uchun

$$(\mu M)'_y=(\mu N)'_x \tag{ 2.5.9}$$

shartni hosil qilamiz. (2.5.9) dan

$$N\mu'_x-M\mu'_y=(M'_y-N'_x)\mu\,. \tag{2.5.10}$$

Shunday qilib, μ integrallovchi ko‘paytuvchi xususiy hosilali differensial tenglama (2.5.10) ning yechimi sifatida aniqlanishi kerak. Umumiy holda bu tenglamani yechish dastlabki tenglama (2.5.1) ni yechishdan oson emas. Lekin ba’zi hollarda (2.5.10) dan integrallovchi ko‘paytuvchi μ ni topish uchun foydalanish mumkin.

Integrallovchi ko‘paytuvchini biror $\omega=\omega(x,y)$ funksiyaning

$$\mu=\mu(\omega) \tag{2.5.11}$$

funksiyasi sifatida izlab ko‘raylik. (2.5.11)ni (2.5.10) ga qo‘yamiz:

$$\left(N\omega'_x-M\omega'_y\right)\frac{d\mu}{d\omega}=(M'_y-N'_x)\mu \tag{2.5.12}$$

Bu tenglik qanoatlanishi uchun

$$\frac{M'_y-N'_x}{N\omega'_x-M\omega'_y}\equiv f(\omega) \tag{2.5.13}$$

bo‘lishi kerak, ya’ni (2.5.13) ning chap tomonidagi x va y ga bog‘liq bo‘lgan funksiya $\omega=\omega(x,y)$ ning funksiyasi sifatida ifodalanishi lozim. Bu holda (2.5.13) ni (2.5.12) ga qo‘yib,

$$\mu = e^{\int f(\omega)d\omega} \qquad (2.5.14)$$

integrallovchi ko‘paytuvchini topamiz. (2.5.13) tenglik $\mu=\mu(\omega)$ shakldagi integrallovchi ko‘paytuvchining mavjudlik shartini beradi. Mashqlar bajarganda ω funksiyani $\omega(x,y)\equiv x$, $\omega(x,y)\equiv y$, $\omega(x,y)\equiv x+y$, $\omega(x,y)\equiv x^2+y^2$ va hokazo ko‘rinishlarda tanlashga harakat qilib ko‘rish mumkin.

Masalan, berlgan (2.5.1) tenglamaning $\mu=\mu(x+y)$ ko‘rinishdagi integrallovchi ko‘paytuvchisini aniqlaylik. $\mu=\mu(\omega)$, $\omega=x+y$, deb, (2.5.12) tenglamadan $\mu=\mu(\omega)$ funksiyaga nisbatan ushbu

$$\frac{d\mu}{d\omega}=\frac{M'_y-N'_x}{N-M}\mu$$

shartni hosil qilamiz. Bu shartdan ravshanki, agar $(M'_y-N'_x)/(N-M)$ ifoda $x+y=\omega$ ning funksiyasi, ya’ni $(M'_y-N'_x)/(N-M)=f(x+y)=f(\omega)$ bo‘lsa, u holda faqat $x+y$ ga bog‘liq bo‘lgan $\mu=\exp\left(\int^{x+y} f(s)ds\right)$ ko‘rinishdagi integrallovchi ko‘paytuvchi mavjud bo‘ladi.

Umumiy holda integrallovchi ko‘paytuvchining mavjud bo‘lishi uchun yetarli va zaruriy shartlar Li gruppalari nazariyasida o‘rganiladi.

Misol 2. Ushbu

$$3ydx+x(\ln y+2\ln x+1)dy=0 \ (x>0, y>0)$$

differensial tenglamani yeching.

🗝 Bu tenglama to‘la differensialli emas. $\mu=\mu(\omega)$, $\omega=\omega(x,y)$, ko‘rinishdagi integrallovchi ko‘paytuvchini topishga harakat qilamiz. Integrallovchi ko‘paytuvchini

$$3y\mu dx+x(\ln y+2\ln x+1)\mu dy=0$$

tenglamaning to‘la differensialli bo‘lishi shartidan, ya’ni

$$(3y\mu)'_y=(x(\ln y+2\ln x+1)\mu)'_x$$

tenglamadan topamiz. Zarur hisoblashlarni bajarib, oxirgi tenglamani quyidagi ko‘rinishga keltiramiz:

$$(3y\omega'_y - x(\ln y + 2\ln x + 1)\omega'_x)\mu' = (\ln y + 2\ln x)\mu .$$

Bu tenglamaning ko‘rinishidan kelib chiqib, $\omega = \omega(x, y)$ funksiya uchun

$$\omega'_x = -\frac{1}{x} \text{ va } 3y\omega'_y + 1 = 0$$

shartlarni qo‘yamiz va $\mu = \mu(\omega)$ funksiya uchun $\mu' = \mu$, ya’ni $\mu = e^{\omega}$ ekanligini topamiz. Endi zarur integrallashlarni va ixchamlashtirishlarni bajarib, integrallovchi ko‘paytuvchini topamiz:

$$\omega'_x = -\frac{1}{x} \Rightarrow \omega = -\ln x + \psi(y),\ \omega'_y = \psi'(y);$$

$$3y\omega'_y + 1 = 0 \Rightarrow \omega'_y = -\frac{1}{3y} = \psi'(y) \Rightarrow \psi(y) = -\frac{1}{3}\ln y;$$

$$\omega = -\ln x + \psi(y) = -\ln x - \frac{1}{3}\ln y = \ln\frac{1}{xy^{1/3}};$$

$$\mu = e^{\omega} = \exp(\ln\frac{1}{xy^{1/3}}) = \frac{1}{xy^{1/3}}.$$

Endi berilgan differensial tenglamani topilgan $\mu = \dfrac{1}{xy^{1/3}}$ integrallovchi ko‘paytuvchiga ko‘paytirib, ushbu

$$\frac{3}{x}y^{2/3}dx + \frac{1}{y^{1/3}}(\ln y + 2\ln x + 1)dy = 0$$

to‘la differensialli tenglamani hosil qilamiz. Bu tenglamaning $u = u(x, y)$ potensiali uchun

$$u'_x = \frac{3}{x}y^{2/3},\ u'_y = \frac{1}{y^{1/3}}(\ln y + 2\ln x + 1)$$

bo‘lishi kerak. Oxirgi ikki shartdan $u = u(x, y)$ funksiyani topamiz:

$$u = 3y^{2/3}\ln x + \frac{3}{2}y^{2/3}\ln y - \frac{3}{4}y^{2/3},\ u = 12y^{2/3}(4\ln x + 2\ln x - 1) .$$

Demak, berilgan tenglamaning yechimi

$$y^{2/3}(4\ln x + 2\ln y - 1) = c$$

tenglama bilan oshkormas ko‘rinishda beriladi.

Masalalar

1. Matematik analizda quyidagi teorema isbotlanadi.

Teorema. Aytaylik, D soha bir bog‘lamli (ya’ni ”teshiksiz”), M, N funksiyalar hamda M'_y va N'_x hosilalar D da uzluksiz bo‘lsin. U holda $Mdx+Ndy$ differensial ifodaning D sohada to‘la differensialdan iborat bo‘lishi uchun D da

$$M'_y = N'_x \qquad (*)$$

bo‘lishi yetarli va zarurdir.

Bu teoremada D sohaning bir bog‘lamli bo‘lishi muhim. Ushbu

$$\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$$

differensial ifoda bir bog‘lamli bo‘lmagan $D=\mathbb{R}^2\setminus(0;0)$ sohada aniqlangan, shu sohada silliq koeffitsientli va $\frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2}$ shart bajarilishini tekshiring. D sohada $du = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ shartni qanoatlantiruvchi $u=u(x,y)$ funksiyaning mavjud emasiligini isbotlang.

2. Aytaylik, D soha biror $A_0 \in D$ nuqtaga nisbatan yulduzsimon bo‘lsin, ya’ni ixtiyoriy $A\in D$ nuqta bilan birgalikda A_0A kesma ham D da yotsin. Bundan tashqari, M va N funksiyalar D da C^1 sinfga tegishli ham bo‘lsin. U holda, agar D da $M'_y = N'_x$ bo‘lsa, $A_0=(x_0,y_0)$ deb,

$$u(x,y)=\int_0^1\left[M(t)\cdot(x-x_0)+N(t)\cdot(y-y_0)\right]dt,$$

$$M(t)=M(x_0+t(x-x_0),\, y_0+t(y-y_0)),\, N(t)=N(x_0+t(x-x_0),\, y_0+t(y-y_0)),$$

funksiyani tuzsak, uning uchun D da $du(x,y)=M(x,y)dx+N(x,y)dy$ bo‘lishini isbotlang.

Differensial tenglamalarni yeching

3. $(1+x)dy+(1+y)dx=0$. **4.** $(2x^2-y^2+y)dx+x(2y-y)dy=0$.

5. $x(xy-3)dy+y(xy-1)dx=0$.

6. Ushbu $yf(xy)dx+xg(xy)dy=0$ $\left(f(t), g(t) - \text{silliq funksiyalar}\right)$ tenglama uchun $\mu=\mu(\omega)=1/\left(\omega\cdot(f(\omega))-g(\omega)\right)$, $\omega=xy$, integrallovchi ko‘paytuvchisi bo‘lishini isbotlang.

MODUL 3. NORMAL KO‘RINISHDAGI BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR UCHUN KOSHI MASALASI

§ 3.1. Hosilaga nisbatan yechilgan tenglama uchun yechimining mavjudligi va yagonaligi to‘g‘risidagi teorema. Ketma-ket yaqinlashishlar usuli

Bu paragrafda hosilaga nisbatan yechilgan birinchi tartibli tenglama $y' = f(x,y)$ uchun ushbu

$$\text{(K)}\quad \begin{cases} y' = f(x,y) & (3.1.1) \\ y\big|_{x_0} = y_0 & (3.1.2) \end{cases}$$

Koshi masalasi yechimining mavjudligi va yagonaligi to‘g‘risidagi teoremani ketma-ket yaqinlashishlar usuli yordamida isbotlaymiz. Teoremani ifodalash uchun bizga Lipshits sharti kerak bo‘ladi.

Lipshits sharti. Aytaylik, $f(x,y)$ funksiya va $E \subset D(f) \subset \mathbb{R}^2$ to‘plam berilgan bo‘lsin. Agar shunday musbat L son mavjud bo‘lib, E to‘plamdagi ixtiyoriy (x,y_1) va (x,y_2) nuqtalar uchun $\left|f(x,y_1) - f(x,y_2)\right| \le L\left|y_1 - y_2\right|$ tengsizlik bajarilsa, ya’ni

$$\exists L > 0 \ \ \forall\left\{(x,y_1),(x,y_2)\right\} \subset E \quad \left|f(x,y_1) - f(x,y_2)\right| \le L\left|y_1 - y_2\right|$$

bo‘lsa, $f(x,y)$ funksiya E to‘plamda y ga nisbatan (yoki y bo‘yicha) ***Lipshits shartini qanoatlantiradi*** deyiladi.

Jumla. *Aytaylik,* D *ochiq yoki yopiq soha* Oy *o‘qiga nisbatan qavariq, ya’ni*

$$\{(x,y_1),(x,y_2)\} \subset D \Rightarrow \left\{\left(x, y_1 + \theta(y_2 - y_1)\right) \big| 0 < \theta < 1\right\} \subset D \ \textit{bo‘lsin.}$$

Agar $f(x,y)$ *funksiya* D *da* y *bo‘yicha uzluksiz va uning ichki nuqtalarida chegaralangan* f'_y *xususiy hosilaga ega bo‘lsa, u holda* $f(x,y)$ *funksiya* D *da* y *ga nisbatan Lipshits shartini qanoatlantiradi.*

Chekli orttirmalar haqidagi Lagranj teoremasidan foydalanib, mustaqil isbotlang.

(K) Koshi masalasini yechish mos integral tenglamani yechishga keltiriladi.

Ekvivalentlik lemmasi. $f(x,y)$ funksiya $(x_0, y_0) \in \mathbb{R}^2$ nuqtaning biror U atrofida uzluksiz bo‘lsin. U holda (K) masalaning I, $I \ni x_0$, oraliqda yechimini topish ushbu

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds \tag{i}$$

integral tenglamaning shu I oraliqda uzluksiz bo‘lgan yechimini topishga ekvivalent.

Aytaylik, $y = y(x) \in C^1(I)$, $I \ni x_0$, funksiya (K) masalaning yechimi bo‘lsin:

$$\begin{cases} y'(s) = f(s, y(s)), \; s \in I, \\ y(x_0) = y_0 \end{cases}$$

Bu yerdagi uzluksiz funksiyalarning aynan tengligini ifodalovchi birinchi ayniyatni $s = x_0$ dan $s = x \in I$ gacha integrallab va boshlang‘ich shartni hisobga olib, $y = y(x) \in C(I) \subset C^1(I)$ funksiya (i) integral tenglamaning yechimi ekanligini ko‘ramiz.

Endi faraz qilaylik, $y(x) \in C(I)$, $I \ni x_0$, funksiya (i) integral tenglamaning yechimi bo‘lsin:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds, \; x \in I.$$

Bu ayniyatdan ravshanki, $y(x) \in C^1(I)$. Ayniyatda $x = x_0$ deb, $y(x_0) = y_0$ boshlang‘ich shartning qanoatlanganligini, hosila olib esa, $y = y(x)$ ning (3.1.1) differensial tenglamani qanoatlantirishini ham topamiz. Demak, $y = y(x)$ funksiya (K) masalaning yechimi.

Yechimlarni biriktirish lemmasi. Aytaylik, $y = \varphi_+(x)$ funksiya (K) masalaning $[x_0, b>$ oraliqda , $y = \varphi_-(x)$ funksiya esa uning $<a, x_0]$ oraliqdagi yechimi hamda $\varphi_+(x_0) = \varphi_-(x_0)$ bo‘lsin. U holda ushbu biriktirilgan

$$y = \varphi(x) = \begin{cases} \varphi_+(x), \text{ agar } x \in [x_0, b> \text{ bo`lsa,} \\ \varphi_-(x), \text{ agar } x \in <a, x_0] \text{ bo`lsa,} \end{cases}$$

funksiya $I = <a, x_0] \cup [x_0, b> = <a, b>$ oraliqda (K) masalaning yechimi bo‘ladi.

Bu yerda $<$ va $>$ belgilar (va] belgilarning ixtiyoriy birini anglatadi.

🗝 Ekvivalentlik lemmasiga ko‘ra

$$\varphi_+(x)\in C\big([x_0,b>\big) \text{ va } \varphi_+(x)=\varphi_+(x_0)+\int_{x_0}^{x} f(s,\varphi_+(s))ds,\ x\in[x_0,b>;$$

$$\varphi_-(x)\in C\big(<a,x_0]\big) \text{ va } \varphi_-(x)=\varphi_-(x_0)+\int_{x_0}^{x} f(s,\varphi_-(s))ds,\ x\in<a,x_0].$$

Berilgan $\varphi_+(x_0)=\varphi_-(x_0)=y_0$ shartga ko‘ra bu tengliklarni birlashtirib,

$$\varphi(x)\in C\big(<a,b>\big) \text{va } \varphi(x)=y_0+\int_{x_0}^{x} f(s,\varphi(s))ds,\ x\in<a,b>$$

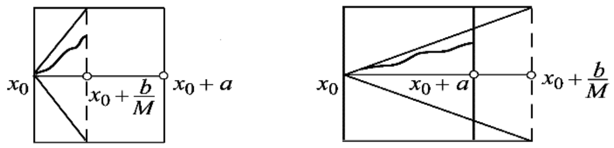
ekanligini topamiz. Bu, yana ekvivalentlik lemmasiga ko‘ra, isbotni tugatadi. 👍

Mavjudlik va yagonalik teoremasi (MYaT).

Teorema (Koshi-Pikar-Lindelyof teoremasi yoki to‘rtburchakda MYaT). *Aytaylik,* $f(x,y)$ *funksiya ushbu* $T=\{(x,y)\in\mathbb{R}^2\mid |x-x_0|\le a, |y-y_0|\le b\}$ $(a>0, b>0)$ *to‘rtburchakda uzluksiz va* y *argumentga nisbatan Lipshits shatini qanoatlantirsin.* $f(x,y)$ *funksiya chegaralangan va yopiq* $T\subset\mathbb{R}^2$ *to‘plamda uzluksiz bo‘lgani uchun u shu* T *da chegaralangan; demak, shunday* $M>0$ *son mavjudki, barcha* $(x,y)\in T$ *nuqtalar uchun* $|f(x,y)|\le M$ *bo‘ladi.* $h=\min\{a,\frac{b}{M}\}$ $(h>0)$ *deylik. U holda* (K) *Koshi masalasining* $I=[x_0-h,x_0+h]$ *oraliqda aniqlangan* $y=\varphi(x)$ *yechimi mavjud va bu yechim yagonadir (3.1-rasm). Bu* $y=\varphi(x)$ *yechim ushbu*

$$y_0(x)=y_0,\ y_n(x)=y_0+\int_{x_0}^{x} f(s,y_{n-1}(s))ds,\ n\in\mathbb{N},$$

formulalar bilan rekurrent aniqlangan $y_n(x)$ *funksional ketma-ketlikning* $|x-x_0|\le h$ *oraliqdagi tekis limitidan iborat.*



3.1-rasm.

🗝 Isbotni $T_+ = \{(x,y)\in\mathbb{R}^2 \mid x_0 \le x \le x_0 + a, |y - y_0| \le b\}$ o‘ng to‘rtburchakda bajaramiz. Chap

$$T_- = \{(x,y)\in\mathbb{R}^2 \mid x_0 - a \le x \le x_0, |y - y_0| \le b\}$$

to‘rtburchakda teorema x ning o‘rniga $2x_0 - x$ o‘zgaruvchini kiritishdan hosil bo‘ladi va teorema $T = T_+ \bigcup T_-$ to‘la to‘rtburchakda yechimlarni biriktirish lemmasidan bevosita kelib chiqadi.

1. Integral tenglamaga o‘tish. Ekvivalentlik lemmasiga ko‘ra T_+ da (K) masalani yechish o‘rniga unga teng kuchli bo‘lgan quyidagi integral tenglamani yechamiz:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds, \; y(x) \in C\big([x_0, x_0 + h]\big) - ?. \qquad (3.1.3)$$

2. Yechimga ketma-ket yaqinlashishlarni qurish. (3.1.3) integral tenglamani ketma-ket yaqinlashishlar metodi yordamida yechamiz. Boshlang‘ich (nolinchi) yaqinlashish sifatida ushbu

$$y_0(x) = y_0 \qquad (3.1.4_0)$$

o‘zgarmas funksiyani tanlaymiz. Endi ketma-ket (rekurrent usulda) quyidagi funksiyalarni kiritamiz:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds, \qquad (3.1.4_1)$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds, \qquad (3.1.4_2)$$

…………………………..

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds, \qquad (3.1.4_n)$$

……………………………...

Bu formulalardagi integrallarning mavjud bo‘lishini, ya’ni $y_1(x), y_2(x), ..., y_n(x), ...$ ketma-ket yaqinlashishlarning aniqlangan bo‘lishini ta’minlashimiz kerak.

Agar

$$x \in [x_0, x_0 + a] \qquad (3.1.5)$$

bo‘lsa, (3.1.4$_1$)dagi integral mavjud va $y_1(x)$ uzluksiz funksiyadan iborat (integral ostidagi funksiya uzluksiz). (3.1.4$_2$)dagi integral aniqlangan bo‘lishi uchun $(s, y_1(s))$ o‘zgaruvchi nuqta T_+ to‘rtburchakdan chiqib ketmasligi kerak. (3.1.4$_1$)ga ko‘ra

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f(s, y_0(s))ds \right| \le \int_{x_0}^{x} |f(s, y_0)|\, ds \le \int_{x_0}^{x} Mds = M \cdot (x - x_0).$$

Bundan ravshanki, $|y_1(x) - y_0| \le b$ bo‘lishi uchun $0 \le M \cdot (x - x_0) \le b$, ya’ni

$$0 \le x - x_0 \le \frac{b}{M} \qquad (3.1.6)$$

shartning bajarilishi yetarli. Demak, (3.1.5) va (3.1.6) shartlar birgalikda o‘rinli, ya’ni $x_0 \le x \le x_0 + h$, $h = \min\{a, \frac{b}{M}\}$ $(h > 0)$ bo‘lganda $(x, y_1(x)) \in T_+$ bo‘ladi. Bundan keyin x o‘zgaruvchi uchun ana shu $x \in [x_0, x_0 + h]$ shart bajarilgan deb hisoblaymiz. Endi tushunarliki, agar $y_k(x)$ funksiya $x \in [x_0, x_0 + h]$ oraliqda aniqlangan, grafigi T_+ da joylashgan va uzluksiz bo‘lsa, $y_{k+1}(x)$ funksiya ham shu xususiyatlarga ega bo‘ladi, chunki

$$|y_{k+1}(x) - y_0| = \left| \int_{x_0}^{x} f(s, y_k(s))ds \right| \le$$

$$\le \int_{x_0}^{x} |f(s, y_k)|\, ds \le \int_{x_0}^{x} Mds = M \cdot (x - x_0) \le Mh \le b.$$

Demak, matematik induksiya prinsipiga ko‘ra $y_n(x)$ $(n \in \mathbb{N})$ funksiyalarning barchasi $[x_0, x_0 + h]$ oraliqda aniqlangan, grafiklari T_+ da joylashgan va uzluksiz (aslida $y_n'(x)$ hosilalar ham uzluksiz, chunki (3.1.4$_n$) formulada integral ostidagi funksiya uzluksiz).

3. Ketma-ket yaqinlashishlarning tekis yaqinlashuvchiligi. Yuqorida aniqlangan $y_0(x), y_1(x), y_2(x),..., y_n(x),...$ uzluksiz funksiyalardan tuzilgan ketma-ketlikning $[x_0, x_0+h]$ oraliqda tekis yaqinlashuvchi ekanligini ko‘rsatamiz. Buning uchun ushbu

$$y_0(x)+\big(y_1(x)-y_0(x)\big)+\big(y_2(x)-y_1(x)\big)+... \qquad (3.1.7)$$

funksional qatorning tekis yaqinlashishini asoslash kerak. Tekis yaqinlashish haqidagi Veyershtrass alomatidan foydalanamiz:

agar $|u_n(x)|\le c_n,\ x\in I,\ n\in\mathbb{N}$ va $\sum_n c_n$ sonli qator yaqinlashuvchi bo‘lsa, u holda $\sum_n u_n(x)$ qator I oraliqda tekis yaqinlashuvchi bo‘ladi.

Teoremaning shartiga ko‘ra $f(x,y)$ funksiya y o‘zgaruvchiga nisbatan Lipshits shartini qanoatlantiradi:

$$\exists\, L>0\ |\, f(x,u)-f(x,v)|\le L|u-v|\ \big((x,u)\in T_+,(x,v)\in T_+\big). \quad (3.1.8)$$

Quyidagi baholashlarni bajaramiz.

$$|y_0(x)|=|y_0|;$$

$$|y_1(x)-y_0(x)| \overset{(3.1.4_1)}{\le} \int_{x_0}^{x}|f(s,y_0)|\,ds\le M\cdot(x-x_0) \qquad (3.1.9_1)$$

$$(\,T_+ \text{ da } |f(x,y)|\le M);$$

$$|y_2(x)-y_1(x)| = \left|\int_{x_0}^{x}(f(s,y_1(s))-f(s,y_0(s)))ds\right|\le$$

$$\le\int_{x_0}^{x}|f(s,y_1(s))-f(s,y_0(s))|\,ds \overset{(3.1.8)}{\le} \int_{x_0}^{x}L|y_1(s)-y_0(s)|\,ds \overset{(3.1.9_1)}{\le}$$

$$\overset{(3.1.9_1)}{\le}\int_{x_0}^{x}LM(s-x_0)ds=ML\frac{(x-x_0)^2}{2}; \qquad (3.1.9_2)$$

$$|y_3(x)-y_2(x)|\le\int_{x_0}^{x}|f(s,y_2(s))-f(s,y_1(s))|\,ds \overset{(3.1.8)}{\le}\int_{x_0}^{x}L|y_2(s)-y_1(s)|\,ds \overset{(3.1.9_2)}{\le}$$

$$\overset{(3.1.9_2)}{\le}\int_{x_0}^{x}LML\frac{(s-x_0)^2}{2}ds=ML^2\frac{(x-x_0)^3}{3!}; \qquad (3.1.9_3)$$

Matematik induksiya metodiga ko‘ra ravshanki, $\forall n\in\mathbb{N}$ uchun

$$\forall x\in I=[x_0,x_0+h]\quad |y_n(x)-y_{n-1}(x)|\ \le ML^{n-1}\frac{(x-x_0)^n}{n!}. \quad (3.1.9_n)$$

Demak,

$$0\le x-x_0\le h \text{ bo‘lganda } |y_n(x)-y_{n-1}(x)|\ \le ML^{n-1}\frac{h^n}{n!},\ n\in\mathbb{N}$$

va ushbu $\sum_n ML^{n-1}\frac{h^n}{n!}$ sonli qator yaqinlashuvchi bo‘lgani uchun Veyershtrass alomatiga ko‘ra (3.1.7) funksional qator I oraliqda tekis yaqinlashuvchi.

4. Ketma-ket yaqinlashishlar limitining yechim ekanligi. Biz uzluksiz funksiyalardan tuzilgan $y_0(x), y_1(x), y_2(x),..., y_n(x),...$ ketma-ketlikning $[x_0,x_0+h]$ oraliqda tekis yaqinlashuvchi ekanligini ko‘rsatdik. Demak, analizdan ma’lum teoremaga ko‘ra (uzluksiz funksyalarning tekis limiti uzluksiz)

$$\varphi(x)=\lim_{n\to\infty} y_n(x),\ x\in[x_0,x_0+h], \quad (3.1.10)$$

limit funksiya qaralayotgan oraliqda uzluksiz. Uning (3.1.3) integral tenglamani qanoatlantirishini ko‘rsatamiz. ($3.1.4_n$) formulaga qaytaylik:

$$y_n(x)=y_0+\int_{x_0}^{x} f(s,y_{n-1}(s))ds. \quad (3.1.4_n)$$

Agar

$$\lim_{n\to\infty}\int_{x_0}^{x} f(s,y_{n-1}(s))ds=\int_{x_0}^{x} f(s,\varphi(s))ds,\ x\in[x_0,x_0+h], \quad (3.1.11)$$

ekanligini ko‘rsatsak, u holda ($3.1.4_n$) tenglikda limitga o‘tib, (3.1.10) funksiyaning (3.1.3) integral tenglama yechimi ekanligini asoslagan bo‘lamiz:

$$\varphi(x)=y_0+\int_{x_0}^{x} f(s,\varphi(s))ds,\ x\in[x_0,x_0+h]. \quad (3.1.12)$$

(3.1.11) dagi tasdiq (3.1.8) Lipshits sharti va (3.1.10) dagi yaqinlashishning tekis ekanligidan kelib chiqadi: yetarlicha katta n nomerlar uchun barcha $x\in[x_0,x_0+h]$ nuqtalarda $|y_{n-1}(x))-\phi(x)|<\varepsilon$ bo‘lganligiga ko‘ra ana shu n har uchun

$$\left|\int_{x_0}^{x} f(s,y_{n-1}(s))ds-\int_{x_0}^{x} f(s,\varphi(s))ds\right|\le\int_{x_0}^{x}|f(s,y_{n-1}(s))-f(s,\varphi(s)|ds\le$$

$$\le\int_{x_0}^{x} L|y_{n-1}(s))-\varphi(s)|ds<L\varepsilon\cdot(x-x_0)\le Lh\varepsilon.$$

5. Yechimning yagonaligi. Biz (3.1.3) integral tenglamaning $y=\varphi(x),\ x\in[x_0,x_0+h],$ yechimini topdik. Uning har qanday yechimi ana shu yechim bilan ustma-ust tushishini ko‘rsatamiz.

(3.1.3) tenglamaning ixtiyoriy $y=\psi(x),\ x\in[x_0,x_0+h],$ yechimini qaraylik. Demak,

$$\psi(x)\in C(I)\ \text{ va }\ \psi(x)=y_0+\int_{x_0}^{x} f(s,\psi(s))ds\ ,\ x\in I. \qquad (3.1.13)$$

$y=\varphi(x)$ yechimga intiluvchi $y_n(x)$ ketma-ket yaqinlashishlar bilan $y=\psi(x)$ yechim orasidagi farqning modulini baholaymiz. Quyidagilarga egamiz:

$$|y_0(x)-\psi(x)|\overset{(4_0),(13)}{=}\left|\int_{x_0}^{x} f(s,\psi(s))ds\right|\le\int_{x_0}^{x}|f(s,\psi(s))|ds\le M\cdot(x-x_0)\qquad (3.1.14_1)$$

$$|y_1(x)-\psi(x)|\overset{(4_1),(13)}{=}\left|\int_{x_0}^{x}(f(s,y_0(s))-f(s,\psi(s)))ds\right|\le$$

$$\le\int_{x_0}^{x}|f(s,y_0(s))-f(s,\psi(s))|ds\overset{(8)}{\le}\int_{x_0}^{x} L|y_0(s)-\psi(s)|ds\overset{(14_1)}{\le}$$

$$\overset{(14_1)}{\le}\int_{x_0}^{x} LM(s-x_0)ds=ML\frac{(x-x_0)^2}{2},\qquad (3.1.14_2)$$

Matematik induksiya metodiga ko‘ra $\forall n\in\mathbb{N}$ uchun

$$|y_n(x)-\psi(x)|\le ML^n\frac{(x-x_0)^{n+1}}{(n+1)!},\ x\in[x_0,x_0+h].\qquad (3.1.14_n)$$

Oxirgi tengsizlikda $x\in[x_0,x_0+h]$ nuqtani tayinlab, $n\to\infty$da limitga o‘tamiz. U holda $\frac{a^n}{n!}\to 0$ bo‘lganligi uchun quyidagilarga ega bo‘lamiz:

$$x\in[x_0,x_0+h]\ \text{uchun}\ |\varphi(x)-\psi(x)|\le 0\ \Rightarrow\ \varphi(x)\equiv\psi(x).$$ ☝

Eslatma. ($3.1.14_n$) tengsizlikdan (K) Koshi masalasining yechimi $y=\varphi(x)$ va unga ketma-ket yaqinlashishlar orasidagi farqning baholanishi kelib chiqadi:

$$| y_n(x)-\varphi(x)|\le ML^n \frac{|x-x_0|^{n+1}}{(n+1)!}, \ |x-x_0|\le h.$$

Misol. Noma'lum $y=y(x)$ funksiyaga nisbatan qo'yilgan ushbu

$$\begin{cases} y'=y \\ y(0)=1 \end{cases}$$

Koshi masalasini ketma-ket yaqinlashishlar metodi yordamida yeching.

🗝 Berilgan tenglamaning o'ng tomonidagi $f(x,y)=y$ funksiya MYaT shartlarini ixtiyoriy $T=\{(x,y)\in\mathbb{R}^2 \, \| \, |x|\le a, |y-1|\le b\}$ ($a>0, b>0$) to'rtburchakda qanoatlantiradi. Ravshanki, qaralayotgan misolda $M=\sup\limits_T |f(x,y)|=\sup\limits_T |y|=1+b$ va $h=\min\left\{a,\frac{b}{1+b}\right\}<1.$ Berilgan Koshi masalasi ushbu

$$y=1+\int_0^x y(s)ds$$

integral tenglamaga ekvivalent. Bu tenglamani ketma-ket yaqinlashishlar metodi yordamida yechamiz:

$$y_0(x)=1,$$

$$y_1(x)=1+\int_0^x y_0(s)ds=1+\int_0^x 1ds=1+x,$$

$$y_2(x)=1+\int_0^x y_1(s)ds=1+\int_0^x (1+s)ds=1+x+\frac{x^2}{2!},$$

$$y_3(x)=1+\int_0^x y_2(s)ds=1+\int_0^x \left(1+s+\frac{s^2}{2!}\right)ds=1+x+\frac{x^2}{2!}+\frac{x^3}{3!},$$

....

$$y_n(x)=1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!},$$

....

Teoremada isbotlandiki, bu $y_n(x)$ ketma-ket yaqinlashishlar berilgan masalaning yagona yechimi bo‘lmish $y = e^x$ funksiyaga $|x| \le h$, $h = \min\left\{a, \frac{b}{1+b}\right\} < 1$, oraliqda tekis yaqinlashadi:

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$$

Oxirgi formula, analizdan ma’lumki, aslida ixtiyoriy $x \in \mathbb{R}$ uchun o‘rinli (bu yerdagi qator ixtiyoriy chegaralangan oraliqda tekis yaqinlashuvchi ham bo‘ladi). ✍

Eslataylikki, agar $D \subset \mathbb{R}^2$ sohaning har bir nuqtasidan $y' = f(x, y)$ differensial tenglamaning yagona integral chizig‘i (yechimi) o‘tsa, D soha shu tenglama uchun yagonalik sohasi deyiladi.

Quyidagi teorema isbotlangan teoremaning bevosita natijasidir.

Teorema (sohada MYaT). *Agar $f(x, y)$ funksiya $D \subset \mathbb{R}^2$ sohada uzluksiz va ixtiyoriy $(x_0, y_0) \in D$ nuqtaning yetarlicha kichik atrofida y bo‘yicha Lipshits shartini qanoatlantirsa, u holda D soha $y' = f(x, y)$ differensial tenglamaning yagonalik sohasi bo‘ladi. Xususan, agar D sohada $f(x, y)$ funksiya va uning $f'_y(x, y)$ hosilasi uzluksiz bo‘lsa, D soha $y' = f(x, y)$ differensial tenglama uchun yagonalik sohasidir.*

Misol. Ushbu $y' = x^2 y^3 - \ln y + 1$ differensial tenglama uchun biror yagonalik sohasini ko‘rsating.

🔑 Ravshanki, $f(x, y) = x^2 y^3 - \ln y + 1$ funksiya va uning $f'_y(x, y) = 3x^2 y^2 - \frac{1}{y}$ hosilasi $y > 0$ yarim tekislikda uzluksiz. Demak, shu $y > 0$ yarim tekislik berilgan tenglama uchun yagonalik sohasidir, ya’ni $y > 0$ yarim tekislikning har bir nuqtasidan $y' = x^2 y^3 - \ln y + 1$ differensial tenglamaning yagona integral chizig‘i o‘tadi. ✍

Masalalar

1. Ushbu

$$f(x) = \begin{cases} x \ln x, \text{ agar } x > 0 \text{ bo'lsa}, \\ 0, \text{ agar } x = 0 \text{ bo'lsa}, \end{cases}$$

funksiyaning ixtiyoriy $[0,b]$ $(b>0)$ yoki $[a,+\infty)$ $(a>0)$ ko‘rinishdagi oraliqda (x bo‘yicha) Lipshits shartini qanoatlantirmasligini ko‘rsating.

2. Faraz qilaylik, $f(x,y)\in C^1(\mathbb{R}^2)$ va $f(x,0)\equiv 0$ bo‘lsin. $y=\sin x$ funksiya $y'=f(x,y)$ tenglamaning biror $(-a;a)$ $(a>0)$ oraliqda yechimi bo‘lishi mumkinmi?

3. $y_0:[-1,1]\to\mathbb{R}$ uzluksiz funksiya bo‘lsin.

Ushbu $y_n(x)=\int\limits_0^x \sqrt[3]{y_{n-1}(s)}ds$, $n\in\mathbb{N}$, ketma-ket yaqinlashishlar ushbu $y'=\sqrt[3]{y}$, $y(0)=0$, Koshi masalasining yechimiga intilishini isbotlang ($f(y)=\sqrt[3]{y}$ funksiya ixtiyoriy $[-a,a]$ $(a>0)$ segmentda Lipshits shartini qanoatlantirmaydi, MYaTni qo‘llab bo‘lmaydi).

4. Ushbu $xy'=2y$, $y(0)=0$ masala

$$y_1(x)\equiv 0 \text{ va } y_2(x)=\begin{cases}0, \text{ agar } 0<x \text{ bo'lsa},\\ x^2, \text{agar } x\geq 0 \text{ bo'lsa},\end{cases}$$

yechimlarga ega ekanligini ko‘rsating. Bu MYaTga zid emasmi? Nega?

5. $y'=3y^{2/3}$ differensial tenglama uchun $y>0$ yuqori yarim tekislik va $y<0$ quyi yarim tekisliklar yagonalik sohalari, lekin $\mathbb{R}^2$ to‘la tekislik yagonalik sohasi bo‘lmasligini isbotlang.

6. $f(x,y)\in C(T)$ bo‘lsin (Lipshits sharti talab qilinmagan). Agar $\varphi_n(I)$, $n\in\mathbb{N}$, funksiyalarning gragiklari T to‘rtburchakda joylashsa va ular biror $\varphi(x), x\in I$, funksiyaga I da tekis intilsa ($n\to\infty$ da $\varphi_n(x)\overset{x\in I}{\rightrightarrows}\varphi(x)$), ushbu

$\lim\limits_{n\to\infty}\int\limits_{x_0}^{x} f(x,\varphi_n(s))ds=\int\limits_{x_0}^{x} f(x,\varphi(s))ds, x\in I$, munosabatning to‘g‘riligini isbotlang.

7. Quyidagi funksiyalar uchun y bo‘yicha Lipshits sharti qaysi $E\subset\mathbb{R}^2$ to‘plamlarda bajariladi:

$$x+y,\ \frac{y}{1+x^2},\ 1+y^2,\ e^x+xy^2,\ \frac{1}{1+y^2},\ \frac{x}{1+y^2},\ xe^{x+y},\ |xy| \ ?$$

8. Agar $y(x)\in C^1(I)$ funksiya va $M\geq 0$ son uchun $|y'(x)|\leq M$, $x\in I$, bo‘lsa, ushbu

$|y(x)|\leq|y(x_0)|+M|x-x_0|$, $x,x_0\in I$, baholashning o‘rinli ekanligini isbotlang.

§ 3.2. Gronuoll-Bellman tipidagi tengsizliklar va ularning ba’zi tatbiqlari

Teorema (Gronuoll-Bellman tipidagi tengsizlik). Aytaylik, biror α, β va $\gamma>0$ sonlar hamda uzluksiz $u:[x_0,b)\to\mathbb{R}$, $u\in C([x_0,b),\mathbb{R})$, funksiya uchun ushbu

$$u(x) \le \alpha + \beta(x - x_0) + \gamma \int_{x_0}^{x} u(s)ds,\ x \in [x_0, b), \tag{3.2.1}$$

tengsizlik bajarilsin. U holda

$$u(x) \le \alpha e^{\gamma(x - x_0)} + \frac{\beta}{\gamma}\left(e^{\gamma(x - x_0)} - 1\right),\ x \in [x_0, b), \tag{3.2.2}$$

baholash o‘rinlidir.

🗝 Ushbu

$$v(x) = \alpha + \beta(x - x_0) + \gamma \int_{x_0}^{x} u(s)ds,\ x \in [x_0, b),$$

yordamchi funksiyani qaraylik. U holda $u(x) \le v(x)$. Endi $v(x)$ ni yuqoridan baholaymiz. Ravshanki, $v'(x) = \beta + \gamma u(x) \le \beta + \gamma v(x)$, chunki $\gamma > 0$. Demak, $v'(x) - \gamma v(x) \le \beta$. Bu tengsizliknig har ikkala tomonini $e^{-\gamma x}$ ga ko‘paytiramiz: $\left(v(x)e^{-\gamma x}\right)' \le \beta e^{-\gamma x}$. Oxirgi tengsizlikni x_0 dan $x \in [x_0, b)$ gacha integrallaymiz:

$$v(x)e^{-\gamma x} - v(x_0)e^{-\gamma x_0} \le \frac{\beta}{\gamma}\left(e^{-\gamma x_0} - e^{-\gamma x}\right).$$

Bundan $v(x_0) = \alpha$ ekanligini hisobga olib, (3.2.2) tengsizlikni hosil qilamiz. 👍

Teoremadan $\beta = 0$ holida Gronuoll tengsizligi hosil bo‘ladi.

Natija (Gronuoll tengsizligi). Aytaylik, biror α va $\gamma > 0$ sonlar hamda uzluksiz $u : [x_0, b) \to \mathbb{R}$, $u \in C([x_0, b), \mathbb{R})$, funksiya uchun ushbu

$$u(x) \le \alpha + \gamma \int_{x_0}^{x} u(s)ds,\ x \in [x_0, b), \tag{3.2.3}$$

tengsizlik bajarilsin. U holda

$$u(x) \le \alpha e^{\gamma(x - x_0)},\ x \in [x_0, b), \tag{3.2.4}$$

baholash o‘rinlidir.

Natija (Gronuoll-Bellman tipidagi tengsizlik). Agar $u \in C((a, b), \mathbb{R}_+)$ funksiya va biror $\gamma > 0,\ \alpha, \beta$ hamda $x_0 \in (a, b)$ sonlar uchun

$$u(x) \le \alpha + \beta |x - x_0| + \gamma \left| \int_{x_0}^{x} u(s)ds \right|, \quad x \in (a,b), \tag{3.2.5}$$

tengsizlik o‘rinli bo‘lsa, u holda

$$u(x) \le \alpha e^{\gamma|x-x_0|} + \frac{\beta}{\gamma}\left(e^{\gamma|x-x_0|} - 1\right), \; x \in (a,b), \tag{3.2.6}$$

baholash ham o‘rinli.

🗝 $x \ge x_0$ holi yuqorida isbotlandi. $x \le x_0$ bo‘lganda $z - x_0 = x_0 - x$ deb, $v(z) = u(2x_0 - z) = u(x)$ funksiyani kiritamiz. U holda $z \ge x_0$ va

$$\left| \int_{x_0}^{x} u(s)ds \right| = \int_{x_0}^{z} v(t)dt, \;\; |x - x_0| = z - x_0,$$

bo‘ladi va isbotlanadigan tengsizlik yana yuqorida isbotlangandan kelib chiqadi. 👍

Gronuoll-Bellman tipidagi tengsizliklar ko‘pdan-ko‘p tatbiqlarga ega. Biz hozircha bunday tengsizliklarning ikkita tatbig‘ini keltiramiz.

Teorema (yechimning yagonalik xossasi). $f(x,y)$ funksiya D sohada uzluksiz va D da joylashgan ixtiyoriy kompaktda y bo‘yicha Lipshits shartini qanoatlantirsin. $y = \varphi(x)$ funksiya $y' = f(x,y)$ tenglamaning I_1 oraliqda, $y = \psi(x)$ shu tenglamaning I_2 oraliqda aniqlangan yechimlari bo‘lsin. Agar $I_1 \cap I_2 \ne \varnothing$ va $x_0 \in I_1 \cap I_2$ nuqtada $\varphi(x_0) = \psi(x_0) = y_0$ bo‘lsa, $y = \varphi(x)$ va $y = \psi(x)$ yechimlar $I_1 \cap I_2$ oraliqda ustma-ust tushadi.

🗝 Ekvivalentlik lemmasi va teorema shartlariga ko‘ra

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(s,\varphi(s))ds, \;\; x \in I_1; \tag{3.2.7}$$

$$\psi(x) = y_0 + \int_{x_0}^{x} f(s,\psi(s))ds, \;\; x \in I_2. \tag{3.2.8}$$

Ixtiyoriy $x \in I_1 \cap I_2$ nuqtani tayinlaylik. Aniqlik uchun $x > x_0$ deylik. $(x,\varphi(x))$ va $(x,\psi(x))$ nuqtalar D sohaga tegishli bo‘lgani uchun $\{(s,\varphi(s)) \in D | \; s \in [x_0, x]\}$ va $\{(s,\psi(s)) \in D | \; s \in [x_0, x]\}$ integral chiziq qismlari biror $K \subset D$ kompaktda joylashadi. Teorema shartlariga ko'ra shu kompaktda $f(x,y)$ funksiya y bo'yicha Lipshits shartini qanoatlantiradi:

$$\exists L>0 \ \forall (x,y),(x,\tilde{y})\in K \ \left| f(x,y)-f(x,\tilde{y})\right| \le L\left| y-\tilde{y}\right| \qquad (3.2.9)$$

(3.2.7) va (3.2.8) tengliklardan

$$\left|\varphi(x)-\psi(x)\right| = \left|\int_{x_0}^{x}\left(f(s,\varphi(s))-f(s,\psi(s))\right)ds\right| \le \int_{x_0}^{x}\left| f(s,\varphi(s))-f(s,\psi(s))\right|ds.$$

Bu tengsizlikdan (3.2.9) Lipshishits shartiga ko'ra

$$\left|\varphi(x)-\psi(x)\right| \le L\int_{x_0}^{x}\left|\varphi(s)-\psi(s)\right|ds .$$

Endi Gronuoll tengsizligidan ravshanki,

$$\left|\varphi(x)-\psi(x)\right| \le 0\cdot e^{L(x-x_0)} = 0 .$$

Demak, barcha $x\in I_1\cap I_2$ nuqtalarda $\varphi(x)=\psi(x)$. 👍

Teorema (yechimning boshlang'ich qiymatga uzluksiz bog'liqligi). $f(x,y)$ funksiya $C(D)$ sinfga tegishli va D sohada joylashgan har qanday kompaktda y bo'yicha Lipshits shartini qanoatlantirsin. Ushbu

$$\begin{cases} y' = f(x,y) \\ y(x_0)=y_0 \end{cases} \quad \left(\ (x_0,y_0)\in D \ \right)$$

Koshi masalasi bilan birgalikda boshlang'ich qiymati o'zgargan

$$\begin{cases} y' = f(x,y) \\ y(x_0)=\tilde{y}_0 \end{cases} \quad \left(\ (x_0,\tilde{y}_0)\in D \ \right)$$

masalani ham qaraylik. Ularning yechimlarini mos ravishda $y=\varphi(x,y_0)$ va $y=\varphi(x,\tilde{y}_0)$ bilan belgilaylik. Bu yechimlar farqining modulini yuqoridan baholaylik (x argumentni x_0 nuqtaning yetarlicha kichik yopiq atrofida o'zgaradi deb faraz qilamiz). Ekvivalentlik lemmasiga ko'ra

$$\varphi(x,y_0) = y_0 + \int_{x_0}^{x} f(s,\varphi(s,y_0))ds ,$$

$$\varphi(x,\tilde{y}_0) = \tilde{y}_0 + \int_{x_0}^{x} f(s,\varphi(s,\tilde{y}))ds .$$

Bundan

$$|\varphi(x,y_0)-\varphi(x,\tilde{y}_0)| = \left| y_0 - \tilde{y}_0 + \int_{x_0}^{x} \left(f(s,\varphi(s,y_0)) - f(s,\varphi(s,\tilde{y}_0)) \right) ds \right| \le$$

$$\le |y_0 - \tilde{y}_0| + \left| \int_{x_0}^{x} | f(s,\varphi(s,y_0)) - f(s,\varphi(s,\tilde{y}_0)) | ds \right| .$$

Lipshits shartidan foydalanib, baholashni davom ettiramiz:

$$|\varphi(x,y_0)-\varphi(x,\tilde{y}_0)| \le |y_0 - \tilde{y}_0| + L \left| \int_{x_0}^{x} |\varphi(s,y_0)-\varphi(s,\tilde{y}_0)| ds \right| .$$

Gronuoll-Bellman tipidagi tengsizlikka ko‘ra

$$|\varphi(x,y_0)-\varphi(x,\tilde{y}_0)| \le |y_0 - \tilde{y}_0| e^{L|x-x_0|}.$$

Bu tengsizlikdan yechimning boshlang‘ich qiymatga uzluksiz bog‘liq ekanligini ko‘ramiz.

Masalalar

1. $\varphi_1(x) \in C([x_0,b),\mathbb{R}_+)$, $\varphi_2(x) \in C([x_0,b),\mathbb{R})$ funksiyalar va $v_0 \in \mathbb{R}$ son berilgan bo‘lsin. Agar $u \in C([x_0,b),\mathbb{R})$, funksiya uchun

$$u(x) \le v_0 + \int_{x_0}^{x} \left(\varphi_1(s)u(s) + \varphi_2(s) \right) ds, \; x \in [x_0,b),$$

integral tengsizlik bajarilsa, ushbu

$$u(x) \le v_0 \exp\left(\int_{x_0}^{x} \varphi_1(s)ds \right) + \int_{x_0}^{x} \varphi_2(s) \exp\left(\int_{s}^{x} \varphi_1(t)dt \right) ds, \; x \in [x_0,b),$$

baholash ham o‘rinlidir. Shu tasdiqni isbotlang.

2. $F(x,s,y) \in C\left([x_0,b)\times[x_0,b)\times\mathbb{R}\right)$ $(x_0 < b)$ funksiya y o‘zgaruvchi bo‘yicha (keng ma’noda) o‘suvchi va $v_0(x) \in C\left([x_0,b)\right)$ bo‘lsin. Agar $u(x) \in C\left([x_0,b)\right)$ funksiya ushbu

$$u(x) < v_0(x) + \int_{x_0}^{x} F(x,s,u(s))ds, \; x \in [x_0,b),$$

integral tengsizlikni qanoatlantirsa va

$$v(x) \overset{def}{=} v_0(x) + \int_{x_0}^{x} F(x,s,v(s))ds, \; x \in [x_0,b), \; v(x) \in C\left([x_0,b)\right)$$

desak, u holda ushbu

$$u(x) < v(x), \; x \in [x_0,b),$$

baholash o‘rinli bo‘ladi. Shuni isbotlang.

3. $f(x,y)$ va $g(x,y)$ funksiyalar D sohada uzluksiz va $f(x,y)$ funksiya y bo‘yicha Lipshits shartini $L>0$ konstanta bilan qanoatlantirsin. D sohada $|f(x,y)-g(x,y)| < \varepsilon$ $(\varepsilon > 0)$ ham bo‘lsin. Ushbu

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \big((x_0, y_0) \in D\big) \text{ va } \begin{cases} u' = g(x,u) \\ u(x_0) = u_0 \end{cases} \big((x_0, u_0) \in D\big)$$

boshlang‘ich masalalarning $y = y(x)$ va $u = u(x)$ yechimlari uchun

$$|y(x) - u(x)| \le |y_0 - u_0| e^{L|x-x_0|} + \frac{\varepsilon}{L}\left(e^{L|x-x_0|} - 1\right)$$

tengsizlikni isbotlang.

4. $f(x,y)$ funksiya $C(D)$ sinfga tegishli va D da joylashgan har qanday kompaktda y bo‘yicha Lipshits shartini qanoatlantirsin. Ushbu

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \big((x_0, y_0) \in D \big)$$

Koshi masalasi bilan birgalikda boshlang‘ich ma’lumotlar o‘zgargan

$$\begin{cases} y' = f(x,y) \\ y(\tilde{x}_0) = \tilde{y}_0 \end{cases} \big((\tilde{x}_0, \tilde{y}_0) \in D \big)$$

masalani qarang. Ularning yechimlarini mos ravishda $y = \varphi(x; x_0, y_0)$ va $y = \varphi(x; \tilde{x}_0, \tilde{y}_0)$ bilan belgilab, $|\varphi(x; x_0, y_0) - \varphi(x; \tilde{x}_0, \tilde{y}_0)|$ miqdorni yetarli kichik $|x_0 - \tilde{x}_0|, |y_0 - \tilde{y}_0|$ uchun, $K \subset D$ kompaktda f funksiyaning Lipshits doimiysi va $\sup\limits_K |f(x,y)|$ orqali baholang.

5. Agar $u, f, p \in C([x_0, b), \mathbb{R})$ va $p(x) \ge 0$ funksiyalar uchun

$$u(x) \le f(x) + \int_{x_0}^{x} p(s) u(s) ds, \; x \in [x_0, b),$$

bo‘lsa, ushbu

$$u(x) \le f(x) + \int_{x_0}^{x} p(s) f(s) \exp\left(\int_{s}^{x} p(t) dt \right) ds, \; x \in [x_0, b),$$

tengsizlikning o‘rinli bo‘lishini isbotlang.

6. Agar $y(x) \in C^1(I)$ funksiya va $\beta \ge 0, \gamma > 0$ sonlar uchun

$$|y'(x)| \le \beta + \gamma |y(x)|, \; x \in I,$$

bo‘lsa, ushbu

$$|y(x)| \le |y(x_0)| e^{\gamma|x-x_0|} + \frac{\beta}{\gamma}\left(e^{\gamma|x-x_0|} - 1\right); \; x, x_0 \in I,$$

tengsizlikning o‘rinli ekanligini isbotlang.

§ 3.3. Eyler siniq chiziqlari va ularning xossalari

Eyler siniq chiziqlari. Faraz qilaylik, $f(x,y)$ funksiya ushbu $T_+ = \{(x,y) \in \mathbb{R}^2 \; | x \in [x_0, x_0 + a], y \in [y_0 - b, y_0 + b]\}$ $(a > 0, b > 0)$ (o‘ng) to‘rtburchakda uzluksiz bo‘lsin. $T_+ -$ kompakt (chegaralangan va yopiq) va $f(x,y)\in C(T_+)$ bo‘lgani uchun $f(x,y)$ funksiya shu T_+ da chegaralangan; demak, shunday $M>0$ son mavjudki, barcha $(x,y)\in T_+$ nuqtalar uchun $|f(x,y)|\le M$ bo‘ladi. $d=\min\{a,\frac{b}{M}\}$ $(d>0)$ deb belgilaylik.

Ushbu

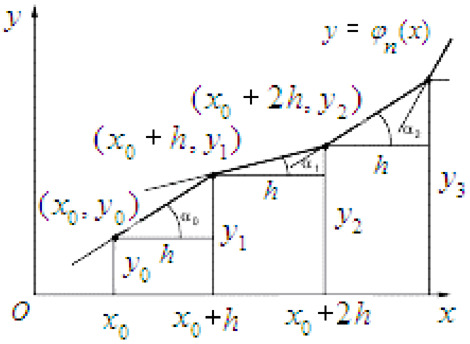
$$\text{(K)}\ \begin{cases} y'=f(x,y) \\ y\big|_{x_0}=y_0, \end{cases}$$

Koshi masalasini qaraylik. Ixtiyoriy $n\in\mathbb{N}$ uchun $h=h_n=\frac{d}{n}$ sonni aniqlab,

$x_0^{(n)}=x_0, x_1=x_1^{(n)}=x_0+h, x_2=x_2^{(n)}=x_0+2h,\dots,x_n=x_n^{(n)}=x_0+nh=x_0+d$ nuqtalarni (tugunlarni) kiritaylik. $[x_0,x_1]$ segmentda (K) masalaning $y=\varphi_n(x)$ taqribiy yechimini $f(x_0,y_0)=\mathrm{tg}\,\alpha_0$ burchak koeffitsientli chiziqli funksiya kabi aniqlaylik va $y_1=y_0+hf(x_0,y_0)$ deb belgilaylik (integral chiziqni unga urinma bilan almashtirdik). Endi $[x_{k-1},x_k], k=2,3,...,n,$ segmentlarda $y=\varphi_n(x)$ taqribiy yechimni ketma-ket $f(x_{k-1},y_{k-1})=\mathrm{tg}\,\alpha_{k-1}$ burchak koeffitsientli chiziqli funksiya kabi aniqlab, $y_k=y_{k-1}+f(x_{k-1},y_{k-1})h$ deb belgilaymiz. Bunda $y=\varphi_n(x), x\in[x_0,x_0+d],$ bo‘lakli-chiziqli funksiya hosil bo‘ladi. Uning grafigi

$$\Delta=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x_0\le x\le x_0+d, |y-y_0|\le M(x-x_0)\right\}$$

uchburchakda joylashadi $\left(\Delta\subset T_+\right)$, chunki ketma-ket $k=0,1,2,...,n-1$ larda $(x_k,y_k)\in\Delta$ nuqtadan chiqarilgan kesmaning $f(x_k,y_k)=\mathrm{tg}\,\alpha_k$ burchak koeffitsienti uchun $\left|f(x_k,y_k)\right|\le M$ bo‘ladi. $y=\varphi_n(x)$ funksiyalar grafiklari Eyler siniq chiziqlari deb ataladi (3.2- rasm).



3.2- rasm. Eyler siniq chizig‘i.

Ko‘rish qiyin emaski, $x\in[x_{k-1},x_k],\ k=1,2,...,n,$ bo‘lganda

$$\varphi_n(x)=y_{k-1}+f(x_{k-1},y_{k-1})\cdot(x-x_{k-1})=y_{k-1}+\int_{x_{k-1}}^{x}f(x_{k-1},y_{k-1})ds,$$

$$y_{k-1}=y_0+\int_{x_0}^{x_1}f(x_0,y_0)ds+\int_{x_1}^{x_2}f(x_1,y_1)ds+ +\ldots+\int_{x_{k-2}}^{x_{k-1}}f(x_{k-2},y_{k-2})ds, \qquad (3.3.1)$$

bo‘ladi. Quyidagi teorema Eyler siniq chiziqlarining (K) masala yechimiga ”yaqinligini” anglatadi.

Teorema 1. *$f(x,y)\in C(T_+)$ va $\varphi_n(x)$ – Eyler siniq chiziqlarini beruvchi yuqorida aniqlangan bo‘lakli-chiziqli funksiya bo‘lsin. U holda har qanday $\varepsilon>0$ son uchun shunday $\nu\in\mathbb{N}$ son topiladiki, barcha $n>\nu$ nomerlar va barcha $x\in[x_0,x_0+d]$ lar uchun*

$$\left|\varphi_n(x)-\left(y_0+\int_{x_0}^{x}f(s,\varphi_n(s))ds\right)\right|<\varepsilon \qquad (3.3.2)$$

tengsizlik bajariladi.

🗝 $\forall\varepsilon>0$ son berilgan bo‘lsin. $f(x,y)\in C(T_+)$ va T_+ kompakt bo‘lgani uchun $f(x,y)$ funksiya T_+ da tekis uzluksiz hamdir (Kantor teoremasi). Demak, $\varepsilon/d>0$ soniga ko‘ra shunday $\delta>0$ son topiladiki, barcha $(x,y)\in T_+$ va $(\tilde{x},\tilde{y})\in T_+$ nuqtalar uchun

$$|x-\tilde{x}|<\delta,\ |y-\tilde{y}|<\delta\ \Rightarrow\ |f(x,y)-f(\tilde{x},\tilde{y})|<\frac{\varepsilon}{d}. \qquad (3.3.3)$$

$n\in\mathbb{N}$ sonni $\dfrac{d}{n}<\delta$ tengsizlikdan tanlaylik. Barcha $x\in(x_{k-1},x_k]$, $x_k-x_{k-1}=\dfrac{d}{n}$, nuqtalar uchun T_+ da $|f(x,y)|\le M$ ekanligiga ko‘ra

$$\left|y_{k-1}-\varphi_n(x)\right| = \left|f(x_{k-1},y_{k-1})\cdot(x-x_{k-1})\right| \le M\left|x-x_{k-1}\right| \le M\frac{d}{n}.$$

Endi $n\in\mathbb{N}$ son uchun $\frac{d}{n}<\delta$ shart bilan birgalikda $M\frac{d}{n}<\delta$ shartni ham talab qilaylik, ya'ni $n>\nu$, $\nu \overset{def}{=} \max\left\{\frac{d}{\delta},\frac{Md}{\delta}\right\}$, deylik. $x=x_0$ nuqtada (3.3.2) tengsizlik, ravshanki, bajariladi. $\forall x\in(x_0,x_0+d]$ nuqta biror $(x_{k-1},x_k]$ oraliqqa tegishli bo'ladi. Demak, (3.3.1) formulaga ko'ra (3.3.3) dan foydalanib, ixtiyoriy $x\in(x_{k-1},x_k]$ nuqta uchun quyidagilarni topamiz:

$$\left|\varphi_n(x)-\left(y_0+\int_{x_0}^{x} f(s,\varphi_n(s))ds\right)\right| = \left|\left(\int_{x_0}^{x_1} f(x_0,y_0)ds-\int_{x_0}^{x_1} f(s,\varphi_n(s)ds\right)+\right.$$

$$+\ldots+\left(\int_{x_{k-2}}^{x_{k-1}} f(x_{k-2},y_{k-2})ds-\int_{x_{k-2}}^{x_{k-1}} f(s,\varphi_n(s))ds\right)+$$

$$\left.+\left(\int_{x_{k-1}}^{x} f(x_{k-1},y_{k-1})ds-\int_{x_{k-1}}^{x} f(s,\varphi_n(s))ds\right)\right| \le$$

$$\le \int_{x_0}^{x_1}\left|f(x_0,y_0)-f(s,\varphi_n(s))\right|ds+$$

$$+\ldots+\int_{x_{k-2}}^{x_{k-1}}\left|f(x_{k-2},y_{k-2})-f(s,\varphi_n(s))\right|ds+$$

$$+\int_{x_{k-1}}^{x}\left|f(x_{k-1},y_{k-1})-f(s,\varphi_n(s))\right|ds \le$$

$$\le \frac{\varepsilon}{d}(x_1-x_0)+\ldots+\frac{\varepsilon}{d}(x_{k-1}-x_{k-2})+\frac{\varepsilon}{d}(x-x_{k-1}) =$$

$$= \frac{\varepsilon}{d}(x-x_0) \le \frac{\varepsilon}{d}\cdot d = \varepsilon.$$ ☝

Umumiy holda Eyler siniq chiziqlarini aniqlovchi $y=\varphi_n(x)$, $x\in[x_0,x_0+d]$, funksiyalar ketma-ketligi yaqinlashuvchi bo'lmasiligi mumkin. Lekin quyidagi teorema o'rinli.

Teorema 2. *Aytaylik, $f(x,y)$ funksiya T_+ to'rtburchakda uzluksizlik sharti bilan birgalikda y argumentga nisbatan Lipshits shatini ham qanoatlantirsin. U holda $\varphi_n(x)$ funksiyalar ketma-ketligi $n\to\infty$ da $[x_0,x_0+d]$ oraliqda tekis yaqinlashuvchi va limit funksiya (K) masalaning yechimi bo'ladi.*

🗝 Dastlab $\varphi_n(x)$ ketma-ketlik $[x_0, x_0+d]$ segmentda quyidagi tekis fundamentallik shartini qanoatlantirishini ko‘rsatamiz:

$\forall\varepsilon>0\ \exists\nu\in\mathbb{N}\ \forall n>\nu\ \forall p\in\mathbb{N}\ \forall x\in[x_0,x_0+d]\ \left|\varphi_n(x)-\varphi_{n+p}(x)\right|<\varepsilon$.(3.3.4)

$\forall\varepsilon>0$ son berilgan bo‘lsin. Teorema 1 ga ko‘ra, agar

$$\Delta_n(x)\overset{def}{=}\varphi_n(x)-\left(y_0+\int_{x_0}^{x} f(s,\varphi_n(s))ds\right)$$

desak, $\Delta_n(x)$ funksiya $n\to\infty$ da $[x_0,x_0+d]$ segmentda tekis nolga intiladi:

$$\varphi_n(x)=y_0+\int_{x_0}^{x} f(s,\varphi_n(s))ds+\Delta_n(x)\,;\ \Delta_n(x)\ \overset{x\in[x_0,x_0+d]}{\rightrightarrows}\ 0\,,\ n\to\infty\,.$$

Shunday $\nu\in\mathbb{N}$ sonni topaylikki, $\forall n>\nu$ nomerlar va $\forall x\in[x_0,x_0+d]$ sonlar uchun $\left|\Delta_n(x)\right|<\varepsilon$ bo‘lsin. Bunda o‘sha n va $\forall p\in\mathbb{N}$, $\forall x\in[x_0,x_0+d]$ uchun $\left|\Delta_{n+p}(x)\right|<\varepsilon$ ham o‘rinli. $f(x,y)$ funksiya T_+ da y bo‘yicha Lipshits shartini qanoatlantirgani uchun

$$\exists L>0\ \ \forall(x,y),(x,\tilde{y})\in T_+\ \ \left|f(x,y)-f(x,\tilde{y})\right|\le L\left|y-\tilde{y}\right|\ .$$

Endi ravshanki, $\forall n>\nu$, $\forall p\in\mathbb{N}$ va $\forall x\in[x_0,x_0+d]$ uchun

$$\left|\varphi_n(x)-\varphi_{n+p}(x)\right|=\left|\int_{x_0}^{x} f(s,\varphi_n(s))ds-\int_{x_0}^{x} f(s,\varphi_{n+p}(s))ds+\Delta_n(x)-\Delta_{n+p}(x)\right|\le$$

$$\le\int_{x_0}^{x}\left|f(s,\varphi_n(s))-f(s,\varphi_{n+p}(s))\right|ds+\left|\Delta_n(x)\right|+\left|\Delta_{n+p}(x)\right|\le$$

$$\le L\int_{x_0}^{x}\left|\varphi_n(s)-\varphi_{n+p}(s)\right|ds+2\varepsilon\,,$$

ya’ni

$$\left|\varphi_n(x)-\varphi_{n+p}(x)\right|\le L\int_{x_0}^{x}\left|\varphi_n(s)-\varphi_{n+p}(s)\right|ds+2\varepsilon$$

tengsizlik o‘rinli. Bu tengsizikdan Gronuoll tengsizligiga ko‘ra barcha $n>\nu$, $p\in\mathbb{N}$ va $x\in[x_0,x_0+d]$ lar uchun ushbu

$$\left|\varphi_n(x)-\varphi_{n+p}(x)\right|\le 2\varepsilon e^{L(x-x_0)}\le 2e^{Ld}\varepsilon=\text{const}\cdot\varepsilon$$

baholashni topamiz. Bu - (3.3.4) shartga ekvivalent. Demak, $\varphi_n(x)$ funksiyalar ketma-ketligi $n\to\infty$ da $[x_0,x_0+d]$ oraliqda tekis yaqinlashuvchi. Uning limitini $\varphi(x)$ bilan belgilaylik: $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$, $x\in[x_0,x_0+d]$. Analizdan ma’lumki, uzluksiz funksiyalarning tekis limiti uzluksiz. Demak, $\varphi(x) \in C([x_0, x_0 + d])$. Endi

$$\varphi_n(x) = y_0 + \int_{x_0}^{x} f(s, \varphi_n(s))ds + \Delta_n(x)$$

tenglikda $n \to \infty$ deb limitga o‘tamiz. Bunda $\varphi_n(x) \underset{\to}{\overset{x\in[x_0,x_0+d]}{\rightarrow}} \varphi(x)$ ekanligidan $\int_{x_0}^{x} f(s, \varphi_n(s))ds \to \int_{x_0}^{x} f(s, \varphi(s))ds$, $x \in [x_0, x_0 + d]$ kelib chiqishini hisobga olib (to‘rtburchakda MYaTning isbotidagi 4-qadamga qarang), limitda

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))ds, \; x \in [x_0, x_0 + d],$$

tenglikni hosil qilamiz. $\varphi(x) \in C([x_0, x_0 + d])$ bo‘lganligi uchun $y = \varphi(x)$ funksiya ekvivalentlik lemmasiga ko‘ra (K) masalaning $x \in [x_0, x_0 + d]$ oraliqda aniqlangan yechimi. ☝

MYaT ga ko‘ra teorema 2 ning shartlarida (K) masalaning yechimi yagona. $y = \varphi_n(x)$ funksiyalar ana shu yagona yechimga intiladi.

Misol. Ushbu

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Koshi masalasinining yagona yechimi, ma’lumki, $y = e^x$. Yechimni [0,1] oraliqda taqriban qurish uchun Eyler siniq chiziqlaridan foydalanaylik. Bizda $f(x, y) = y$.

[0,1] segmentni $n = 3$ ta teng bo‘lakchalarga bo‘luvchi nuqtalar $x_0 = 0;\ x_1 = 1/3;\ x_2 = 2/3;\ x_3 = 1$(bunda qadam $h = 1/3$). Bu nuqtalarda mos ravishda

$y_0 = 1;\ y_1 = y_0 + f(x_0, y_0)h = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1{,}3333$ (aniq qiymat $e^{1/3} \approx 1{,}3956$);

$y_2 = y_1 + f(x_1, y_1)h = \frac{4}{3} + \frac{4}{3} \cdot \frac{1}{3} = \frac{16}{9} \approx 1{,}7778$ (aniq qiymat $e^{2/3} \approx 1{,}9477$);

$y_3 = y_2 + f(x_2, y_2)h = \frac{16}{9} + \frac{16}{9} \cdot \frac{1}{3} = \frac{64}{27} \approx 2,3704$ (aniq qiymat $e^1 \approx 2,7185$).

Endi segmentni $n = 4$ ta teng bo‘lakchalarga bo‘luvchi nuqtalarni qaraylik: $x_0 = 0;\ x_1 = 1/4;\ x_2 = 2/4 = 1/2;\ x_3 = 3/4;\ x_4 = 1$ (bunda qadam $h = 1/4$). Bu nuqtalarda mos ravishda

$y_0 = 1;\ y_1 = y_0 + f(x_0, y_0)h = 1 + 1 \cdot \frac{1}{4} = \frac{5}{4} = 1,25$ (aniq qiymat $e^{1/4} \approx 1,2840$);

$y_2 = y_1 + f(x_1, y_1)h = \frac{5}{4} + \frac{5}{4} \cdot \frac{1}{4} = \frac{25}{16} = 1,5625$ (aniq qiymat $e^{1/2} \approx 1,6487$);

$y_3 = y_2 + f(x_2, y_2)h = \frac{25}{16} + \frac{25}{16} \cdot \frac{1}{4} = \frac{125}{64} \approx 1,9531$ (aniq qiymat $e^{3/4} \approx 2,1170$);

$y_4 = y_3 + f(x_3, y_3)h = \frac{125}{64} + \frac{125}{64} \cdot \frac{1}{4} = \frac{625}{256} \approx 2,4414$ (aniq qiymat $e \approx 2,7185$). ***Mavjudlik teoremasi.*** Dastlab haqiqiy funksiyalar nazariyasidan ba’zi ma’lumotlarni keltiraylik.

$[a,b]$ segmentda aniqlangan $\psi_n(x)$, $n \in \mathbb{N}$, funksiyalar ketma-ketligi berilgan bo‘lsin. Agar bu funksiyalarning barchasi moduli bo‘yicha bitta umumiy son bilan yuqoridan chegaralangan bo‘lsa, berilgan funksiyalar ketma-ketligi tekis chegaralangan deyiladi. Demak, $\psi_n(x), n \in \mathbb{N}$, funksiyalarning $[a,b]$ segmentda tekis chegaralanganlik sharti quyidagicha:

$$\exists M > 0\ \ \forall n \in \mathbb{N}\ \ \forall x \in [a,b]\ \ |\psi_n(x)| \le M .$$

Masalan, x^n funksiyalar ketma-ketligi $[0;1]$ segmentda $M = 1$ soni bilan tekis chegaralangan: $\forall n \in \mathbb{N}\ \forall x \in [0;1]\ |x^n| = |x|^n \le 1.$ nx^n funksiyalarning har biri $[0;1]$ segmentda (n soni bilan) yuqoridan chegaralangan bo‘lsa-da, ular tekis chegaralangan emas.

Agar ixtiyoriy $\varepsilon > 0$ songa ko‘ra shunday $\delta = \delta(\varepsilon) > 0$ topilsaki, $|x - \tilde{x}| < \delta$ shartni qanoatlantiruvchi har qanday $x, \tilde{x} \in [a,b]$ nuqtalar va barcha $\psi_n(x)$, $n \in \mathbb{N}$, funksiyalar uchun $|\psi_n(x) - \psi_n(\tilde{x})| < \varepsilon$

tengsizlik bajarilsa, qaralayotgan funksiyalar [a,b] da tekis darajali uzluksiz deyiladi.

Tushunarliki, tekis darajali uzluksiz funksiyalar ketma-ketligidagi har bir funksiya tekis uzluksiz; bundan tashqari, tekis uzluksizlik ta’rifidagi $\varepsilon > 0$ ga ko‘ra topiladigan $\delta > 0$ soni barcha funksiyalar uchun bitta (umumiy) tanlanishi mumkin.

Masalan, agar $\psi_n(x), n \in \mathbb{N}$, funksiyalar [$a,b$] segmentda uzluksiz, ($a,b$) intervalda differensiallanuvchi va $\psi_n'(x)$ hosilalar (a,b) da tekis chegaralangan, ya’ni $|\psi_n'(x)|$ larning barchasi (a,b) da bitta $L > 0$ son bilan yuqoridan chegaralangan bo‘lsa ($\exists L > 0\ \forall x \in (a,b)\ \forall n \in \mathbb{N}\ |\psi_n'(x)| \le L$), qaralayotgan funksiyalar ketma-ketligi [a,b] segmentda tekis darajali uzluksiz bo‘ladi, chunki $|x - \tilde{x}| < \delta \Rightarrow |\psi_n(x) - \psi_n(\tilde{x})| < \varepsilon$ bo‘lishi uchun chekli orttirmalar haqidagi Lagranj

teoremasiga ko‘ra $\delta = \dfrac{\varepsilon}{L}$ tanlash kifoya ($0 < \theta < 1$):

$$|\psi_n(x) - \psi_n(\tilde{x})| = |\psi_n'(x + \theta(\tilde{x} - x))| \cdot |x - \tilde{x}| \le L|x - \tilde{x}| < L\delta = L\frac{\varepsilon}{L} = \varepsilon .$$

$x^n, n \in \mathbb{N}$, funksiyalarning har biri [0;1] segmentda tekis uzluksiz, lekin ular [0;1] segmentda tekis darajali uzluksiz emas (tekshirib ko‘ring).

Teorema (Arsela-Askoli). [a,b] *segmentda tekis chegaralangan va tekis darajali uzluksiz bo‘lgan* $\psi_n(x), n \in \mathbb{N}$, *funksiyalar ketma-ketligidan* [a,b] *da tekis yaqinlashuvchi qismiy ketma-ketlik* $\psi_{k_n}(x)$ *ajratish mumkin* (k_n *– natural sonlardan tuzilgan qat’iy o‘suvchi ketma-ketlik*).

Biz bu teoremani isbotlamaymiz. Nisbatan sodda isbot Петровский И.Г. [12] kitobida bor. Arsela-Askoli teoremasidan foydalanib (K) masala yechimining mavjudligi haqidagi Peano teoremasini isbotlaymiz.

Teorema 3 (Peano). *Agar* $f(x, y) \in C(T_+)$ *bo‘lsa, (K) masala* $[x_0, x_0 + d]$ *oraliqda aniqlangan kamida bitta yechimga ega*.

🗝 (K) masala uchun Eyler siniq chiziqlarini, ya’ni $\varphi_n(x)$, $x \in [x_0, x_0 + d]$, funksiyalar ketma-ketligini qaraylik. Ular $[x_0, x_0 + d]$

da tekis chegaralangan, chunki, bilamizki, $x\in[x_0,x_0+d]$ lar uchun $|\varphi_n(x)-y_0|\le M|x-x_0|\le Md$ va oxirgi tengsizlikdan $|\varphi_n(x)|=|\varphi_n(x)-y_0+y_0|\le|\varphi_n(x)-y_0|+|y_0|\le Md+|y_0|$, ya'ni $|\varphi_n(x)|$ larning barchasi $[x_0,x_0+d]$ segmentda bitta umumiy $Md+|y_0|$ son bilan yuqoridan chegaralangan. Bundan tashqari, $\varphi_n(x)$ funksiyalar $[x_0,x_0+d]$ da tekis darajali uzluksiz ham; bu tasdiq ushbu $|\varphi_n(x)-\varphi_n(\tilde{x})|\le M|x-\tilde{x}|$ tengsizlikdan bevosita kelib chiqadi. Demak, Arsela-Askoli teoremasiga ko‘ra $\varphi_n(x)$ funksiyalar ketma-ketligidan $[x_0,x_0+d]$ da tekis yaqinlashuvchi qismiy ketma-ketlik $\varphi_{k_n}(x)$ ajratish mumkin. $n\to\infty$ da $\varphi_{k_n}(x)\overset{x\in[x_0,x_0+d]}{\rightrightarrows}\varphi(x)$ deylik. Endi teorema 2 ning isbotidagi ikkinchi qismga o‘xshash ish tutamiz. Ravshanki, $\varphi(x)\in C\big([x_0,x_0+d]\big)$. Bundan tashqari, teorema 1 dan kelib chiquvchi ushbu

$$\varphi_{k_n}(x)=y_0+\int_{x_0}^{x}f(s,\varphi_{k_n}(s))ds+\Delta_{k_n}(x)$$

tenglikda $n\to\infty$ da limitga o‘tib, $\Delta_{k_n}(x)\xrightarrow[n\to\infty]{}0$ va $\int_{x_0}^{x}f(s,\varphi_{k_n}(s))ds\xrightarrow[n\to\infty]{}\int_{x_0}^{x}f(s,\varphi(s))ds$ munosabatlarga ko‘ra

$$\varphi(x)=y_0+\int_{x_0}^{x}f(s,\varphi(s))ds,\ x\in[x_0,x_0+d],$$

tenglikni hosil qilamiz. ☝

Biz yuqorida Eyler siniq chiziqlarini T_+ o‘ng to‘rtburchakda qurdik va xossalarini o‘rgandik. x ning o‘rniga yangi $2x_0-x$ o‘zgaruvchi kiritish yordamida barcha qilingan ishlarni $T_-=\{(x,y)\in\mathbb{R}^2 \mid x_0-a\le x\le x_0, |y-y_0|\le b\}$ chap to‘rtburchakka o‘tkazish mumkin. So‘ngra birlashtirish yordamida $T=T_+\bigcup T_-$ to‘rtburchakda Eyler siniq chiziqlari va uning xossalarini topamiz. Buning o‘rniga T_+ qilingan ishlarni birdaniga T to‘la to‘rtburchakda ham bajarish mumkin.

Agar $f(x,y)$ funksiyadan uzluksizlikdan boshqa shart talab qilmasak, nuqtadan o‘tuvchi yechimning yagonaligini umumiy holda tasdiqlab bo‘lmaydi.

M. A. Lavrent'ev tekislikda uzluksiz bo'lgan shunday $f(x,y)$ funksiya qurganki, u orqali tuzilgan $y' = f(x,y)$ differensial tenglamaning $\forall (x_0, y_0) \in \mathbb{R}^2$ nuqta orqali cheksiz ko'p integral chizig'i o'tadi.

Masalalar

1. $f(x,y)$ funksiya ushbu
$T = \{(x,y) \in \mathbb{R}^2 \mid x \in [x_0 - a, x_0 + a], y \in [y_0 - b, y_0 + b]\}$ $(a > 0, b > 0)$ to'rtburchakda uzluksiz bo'lsin. Ushbu

$$\text{(K)} \qquad \begin{cases} y' = f(x,y) \\ y\big|_{x_0} = y_0, \end{cases}$$

Koshi masalasini qaraylik. $T-$kompakt va $f(x,y) \in C(T)$ bo'lgani uchun $|f(x,y)|$ funksiya T da chegaralangan: demak, shunday $M > 0$ son mavjudki, barcha $(x,y) \in T$ nuqtalar uchun $|f(x,y)| \le M$ bo'ladi. $d = \min\left\{a, \frac{b}{M}\right\}$ $(d > 0)$ deylik. Ixtiyoriy $n \in \mathbb{N}$ uchun $h = h_n = \frac{d}{n}$ sonni aniqlab, $x_j = x_j^{(n)} = x_0 + jh,\ j = \overline{-n,n}$ $\left(x_{-n}^{(n)} = x_0 - nh = x_0 - d,\ x_n^{(n)} = x_0 + nh = x_0 + d\right)$ nuqtalarni (tugunlarni) kiritaylik. $[x_{k-1}, x_k]$ $(k = \overline{1,n})$ segmentda (K) masalaning $y = \varphi_n(x)$ taqribiy yechimini yuqoridagidek aniqlaymiz. x_0 dan chap tomondagi segmentlarda ham taqribiy yechimni shunga o'xshash aniqlaymiz. Bunda $y = \varphi_n(x)$, $x \in [x_0 - d, x_0 + d]$, bo'lakli-chiziqli funksiya hosil bo'ladi. Uning grafigi

$$\Delta = \left\{(x,y) \in \mathbb{R}^2 \,\big|\, |x - x_0| \le d, |y - y_0| \le M|x - x_0|\right\}$$

qo'sh uchburchakda joylashadi, $\Delta \subset D$ ham bo'ladi. Qurilgan $y = \varphi_n(x)$, $x \in [x_0 - d, x_0 + d]$ fnksiyalar ketma-ketligi uchun yuqorida keltirilgan teoremalarni isbotlang.

2. Ushbu $\sin nx,\ n \in \mathbb{N}$, funksiyalar ketma-ketligi $x \in [0, \pi]$ segmentda tekis darajali uzluksiz emas. Shuni isbotlang.

3. Masala 1 dagi boshlang'ich masala (K)ni qaraylik. Bu masala uchun quyidagi ketma-ket yaqinlashishlarni quraylik (to'rtburchakda MYaTga qarang):

$$y_0(x) = y_0\ ,\ y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds,\ n = 1, 2, 3, \ldots$$

$y_n(x)$ funksiyalar ketma-ketligining $|x - x_0| \le h$ oraliqda tekis chegaralangan va tekis darajali uzluksiz ekanligini, Arsela-Askoli teoremasiga ko'ra ajratilgan qismiy ketma-ketlikni esa (K) masala yechimi ekanligini isbotlang.

§ 3.4. Davomsiz yechimlar

Yechimni davom ettirish. Ushbu

$$y' = f(x,y) \qquad (3.4.1)$$

tenglamani qaraylik; bu yerda $f \in C(D)$, $f'_y \in C(D)$ deb faraz qilamiz. Bu farazga ko'ra $\forall (x_0, y_0) \in D$ uchun ushbu

$$(K) \qquad \begin{cases} y' = f(x,y) \\ y|_{x_0} = x_0 \end{cases}$$

Koshi masalasi biror $[x_0, x_0 + h_0]$ $(h_0 > 0)$ segmentda aniqlangan yagona $y = \varphi(x)$ yechimga ega.

Agar $y = \varphi(x)$ funksiya (3.4.1) differensial tenglamaning $I = [a;b]$ oraliqda, $y = \psi(x)$ funksiya esa uning $J = [a;b^*], b < b^*$, yoki $J = [a;b^*), b < b^*$, oraliqda aniqlangan yechimi bo'lib, ular I da ustma-ust ham tushsa, u holda $y = \psi(x)$ yechim $y = \varphi(x)$ **yechimning** I dan J gacha **o'ngga davomi** (**davom ettirilishi**) deb ataladi.

Yechimning boshqa tur oraliqlardan o'ngga hamda chapga davomi shunga o'xshash aniqlanadi.

Yechimning o'ng uchi bo'lmish $(x_1; y_1) \overset{def}{=} (x_0 + h_0, \varphi(x_0 + h_0)) \in D$ nuqtaga ko'ra

$$\begin{cases} y' = f(x,y) \\ y\big|_{x_1} = y_1 \end{cases}$$

Koshi masalasini yechib, $[x_1, x_1 + h_1]$ $(h_1 > 0)$ segmentda aniqlangan yagona $y = \varphi_1(x)$ yechimni topamiz. Yuqoridagi ikki yechimni biriktirib (birlashtirib) ushbu

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), \text{ agar } x \in [x_0, x_1] \text{ bo'lsa,} \\ \varphi_1(x), \text{ agar } x \in [x_1, x_1 + h_1] \text{ bo'lsa,} \end{cases} \quad (x_1 = x_0 + h_0, \varphi(x_1) = \varphi_1(x_1) = y_1)$$

funksiyani quramiz. Yechimlarni biriktirish lemmasiga ko'ra bu $\tilde{\varphi}(x)$ funksiya (K) masalaning $[x_0, x_1] \cup [x_1, x_1 + h_1] = [x_0, x_1 + h_1]$ segmentda aniqlangan yechimi. U $[x_0, x_0 + h_0]$ da aniqlangan $y = \varphi(x)$ yechimning $[x_0, x_1 + h_1]$ segmentgacha (o'ngga) davomidir.

Bu yechimni (funksiyani) yana $y = \varphi(x)$ bilan belgilaymiz; endi bu $y = \varphi(x)$ funksiya (*K*) masalaning $[x_0, x_1 + h_1]$ segmentda aniqlangan yechimi bo‘ladi. Yechimning bu davomi bir qiymatli aniqlanadi. Endi bu yechimni yana o‘ngga davom ettiramiz va h.k. Natijada $[x_0, b)$ oraliqda aniqlangan yechimni hosil qilamiz. Yechimning chapga davomi yuqoridagiga o‘xshash amalga oshiriladi. Yechimni grafigi D da joylashgan har qanday K kompaktdan tashqariga chiqib ketgunga qadar davom ettirish mumkin. Bu quyidagi teoremadan kelib chiqadi.

Teorema. $f \in C(D)$, $f'_y \in C(D)$ bo‘lsin. U holda D sohada yotgan ixtiyoriy K kompakt uchun shunday $h_K > 0$ mavjudki, har qanday $(x_0, y_0) \in K$ nuqtadan o‘tgan yechim $\left[x_0 - h_K; x_0 + h_K\right]$ oraliqda aniqlangan bo‘ladi.

⊶ K kompakt D sohaning qismi bo‘lganligi uchun K va Dning chegarasi ∂D orasidagi masofa qat’iy musbat: $\operatorname{dist}(K, \partial D) = d > 0$ (§ 8.2 dagi jumlaga qarang). $K^{d/2}$ bilan Kdan ko‘pi bilan $d/2$ masofaga uzoqlashgan nuqtalar to‘plamini belgilaylk. Ravshanki, $K^{d/2}$ –kompakt va $K^{d/2} \subset D$. $f \in C(K^{d/2})$ bo‘lgani uchun $\exists M = M(d) > 0$ $\forall (x, y) \in K^{d/2}$ $\left|f(x, y)\right| \le M$. $\forall (x_0, y_0) \in K$ nuqta uchun $\left|x - x_0\right| \le \dfrac{d}{2\sqrt{2}}$, $\left|y - y_0\right| \le \dfrac{d}{2\sqrt{2}}$ to‘rtburchakni qaraylik. Ravshanki, bu to‘rtburchak $K^{d/2}$ da yotadi. MYaTni (K) Koshi masalasiga shu to‘rtburchakda tatbiq etaylik. $h_K = \min\left\{\dfrac{d}{2\sqrt{2}}, \dfrac{d}{2\sqrt{2}M}\right\}$ deb (K) masala $\left|x - x_0\right| \le h_K$ oraliqda aniqlangan yagona yechimga ega bo‘lishini topamiz. Bu yerda aniqlangan h_K son $(x_0, y_0) \in K$ nuqtaga bog‘liq emas, K ga bog‘liq. Barcha $(x_0, y_0) \in K$ nuqtalar uchun umumiy h_K topilgan. ☝

Demak, $K \subset D$ kompaktning ixtiyoriy nuqtasidan chiqqan integral chiziq o‘ngga $h_K > 0$ masofaga davom etadi. Chekli marta yechimni K ning nuqtalaridan o‘ngga hamda chapga davom ettirib, K dan tashqariga chiqib ketamiz, ya’ni davom ettirilgan yechim D sohaning chegarasidan chegarasigacha “boradi”.

Shunday qilib, (ξ,η) intervalda aniqlangan yechim hosil bo‘ladi ($\xi=-\infty$ yoki/va $\eta=+\infty$ bo‘lishi ham mumkin). Bu yechim (K) masalaning (D sohadagi) **davomsiz yechimi** deyiladi. Davomsiz yechimning aniqlanish sohasi eng katta bo‘ladi va bu yechim chegaradan (∂D) chegaragacha (∂D) “davom” etadi.

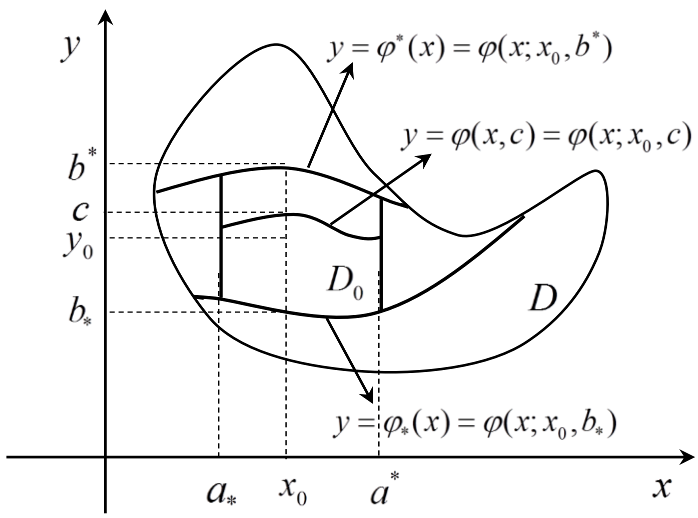
Umumiy yechim. Qulaylik uchun $y=\varphi(x;\xi,\eta)$ bilan $y'=f(x,y)$ tenglamaning $x=\xi$ da η ga aylanuvchi davomsiz yechimini belgilaylik, $\varphi(\xi;\xi,\eta)=\eta$, $(\xi,\eta)\in D$. $\forall(x_0,y_0)\in D$ nuqtani tayinlaylik. D ochiq bo‘lgani uchun shunday $b_*<y_0,\, b^*>y_0$ va $a_*<x_0,\, a^*>x_0$ sonlar topiladiki, yuqoridan $y=\varphi^*(x)\equiv\varphi(x;x_0,b^*)$ quyidan $y=\varphi_*(x)\equiv\varphi(x;x_0,b_*)$ yechimlar bilan, chapdan $x=a_*$ o‘ngdan $x=a^*$ to‘g‘ri chiziqlar bilan chegaralangan ushbu

$$D_0=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a_*<x<a^*,\ \varphi_*(x)<y<\varphi^*(x)\right\}$$

ochiq to‘plam o‘zining yopilmasi

$$\overline{D_0}=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a_*\le x\le a^*,\ \varphi_*(x)\le y\le\varphi^*(x)\right\}$$

bilan birgalikda D ning qismi bo‘ladi, $D_0\subset\overline{D_0}\subset D$ (3.3- rasm).



3.3- rasm.

Bu yerda ravshanki, $\varphi_*(x)<\varphi^*(x)$, chunki $\varphi_*(x_0)=b_*<b^*=\varphi^*(x_0)$ va biror $x=s$ nuqtada $\varphi_*(s)=\varphi^*(s)$ bo‘lganda edi, $\left(s,\varphi_*(s)\right)=\left(s,\varphi^*(s)\right)$ nuqtada yechimning yagonalik xossasi buzilardi. Endi x_0 ni tayinlab, barcha $c\in(b_*,b^*)$ o‘zgarmaslar uchun $a_*<x<a^*$ oraliqda aniqlangan (davom etgan) ushbu

$$y=\varphi(x,c)\equiv\varphi(x;x_0,c)$$

yechimlar oilasini qaraylik. Ularning grafiklari D_0 sohada joylashadi, turli $c\in(b_*,b^*)$ larda grafiklar kesishmaydi (yechimning yagonalik xossasiga ko‘ra). Bundan tashqari, D_0 soha shu grafiklar bilan to‘ladi ham. Haqiqatan ham, $\forall(\xi,\eta)\in D$ uchun $y=\varphi(x;\xi,\eta)$, $a_*<x<a^*$, yechimni qaraylik. Bu yechimning $x=x_0$ dagi qiymati $c=\varphi(x_0;\xi,\eta)\in(b_*,b^*)$ bo‘ladi va yechimning yagonalik xossasiga ko‘ra $y=\varphi(x;\xi,\eta)=\varphi(x;x_0,c)=\varphi(x,c)=\varphi(x,\varphi(x_0;\xi,\eta))$. Demak, grafigi D_0 sohada joylashgan har qanday yechim $y=\varphi(x,c)$ formuladan biror $c\in(b_*,b^*)$ qiymatda hosil bo‘1adi. Shunday qilib, $y=\varphi(x,c)$ formula (1) tenglamaning D_0 sohada umumiy yechimini ifodalaydi. Biz quyidagi teoremani isbotladik.

Teorema. Ushbu $y'=f(x,y),\ f,f'_y\in C(D),$ differensial tenglama ixtiyoriy $(x_0,y_0)\in D$ nuqtaning biror kichik atrofida $y=\varphi(x,c)$ umumiy yechimga ega.

Masalalar

1. $y'=f(x,y)$ tenglamada $f(x,y)\in C^1(\mathbb{R}^2)$ bo‘lsin. Bu tenglamaning $\mathbb{R}$ da chegaralangan har qanday yechimi $(-\infty,+\infty)$ gacha davom etishini ko‘rsating.

2. $(1+x^2+y^{2018})y'=1$ tenglamaning barcha yechimlari $(-\infty,\infty)$ da chegaralangan. Shuni isbotlang.

3. $f(x,y)$ funksiya $C^1(D)$ sinfga tegishli, D da chegaralangan va D da joylashgan har qanday kompaktda y bo‘yicha Lipshits shartini qanoatlantirsin. Bundan tashqari, ushbu

$$\begin{cases} y'=f(x,y) \\ y(x_0)=y_0 \end{cases} \quad \left(\ (x_0,y_0)\in D\ \right)$$

Koshi masalasining $y=\varphi(x)$ yechimi $x\in(\alpha,\beta)$ intervalda aniqlangan bo‘lsin. $\lim\limits_{x\to\alpha+}\varphi(x)=\delta$ $\left(\lim\limits_{x\to\beta-}\varphi(x)=\gamma\right)$ bir tomonli limit mavjud bo‘lishini ko‘rsating. Agar $(\alpha,\delta)\in D$ $\left((\beta,\gamma)\in D\right)$bo‘lsa, $y=\varphi(x)$ yechimni chapga (o‘ngga) davom ettirish mumkinligini isbotlang.

4. $D=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,x>0\right\}$ yarim tekislikda ushbu

$$\begin{cases} y'=-\dfrac{1}{x^2}\cos\dfrac{1}{x}, \\ y(x_0)=y_0\ \ (x_0>0) \end{cases}$$

Koshi masalasining yagona davomsiz yechimi $x \in (0,+\infty)$ da aniqlangan va

$$y(x) = \sin\frac{1}{x} + y_0 - \sin\frac{1}{x_0}$$

ko‘rinishga ega. $\lim\limits_{x\to 0+} y(x)$ mavjudmi?

5. $f(x,y)$ funksiya $C^1(D)$ sinfga tegishli, D da chegaralangan va D da joylashgan har qanday kompaktda y bo‘yicha Lipshits shartini qanoatlantirsin. Ushbu

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad \left((x_0, y_0) \in D \right)$$

Koshi masalasining $y = \varphi(x)$ davomsiz yechimi $x \in (\alpha, \beta)$ intervalda aniqlangan bo‘lsin. U holda yo $\alpha = -\infty$ yoki $(\alpha, \lim\limits_{x\to\alpha+} \varphi(x)) \in \partial D$ bo‘ladi. β uchun ham shunga o‘xshash tasdiq (alternativa) o‘rinli. Shu tasdiqlarni isbotlang.

6. Ushbu

$$y' = \frac{xy}{1+y^4} + 2\cos x$$

tenglama uchun quyidagi tasdiqlarni isbotlang.

1^0. Tenglamaning har qanday $y = y(x)$ yechimi uchun ushbu

$$|y(x)| \le c_1 \exp(c_2 x^2) \quad (\ c_1, c_2 - \text{ biror musbat sonlar})$$

baholash o‘rinli.

2^0. Tenglamaning har qanday yechimi $(-\infty,+\infty)$ oraliqqacha davom etadi.

MODUL 4. HOSILAGA NISBATAN YECHILMAGAN BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR

§ 4.1. Hosilaga nisbatan yechilmagan tenglama uchun yechimning mavjudlik va yagonalik teoremasi

Quyidagi differensial tenglamani qaraylik:

$$F(x,y,y')=0; \tag{4.1.1}$$

bu yerda $F(x,y,p)$ funksiya $G\subset\mathbb{R}^3$ sohada aniqlangan, uzluksiz va p ga tom ma’noda bog‘liq deb hisoblanadi.

Berilgan $(x_0,y_0)\in D\subset\mathbb{R}^2$ (bu yerda D – G sohaning $\mathbb{R}^2=\{(x,y)\}$ dagi ortogonal proyeksiyasi) nuqta orqali bir nechta (hatto, cheksiz ko‘p) integral chiziqlar o‘tishi mumkin. Chunki $F(x,y,y')=0$ tenglamani y' ga nisbatan yechib, umumiy holda bir nechta $y'=f_i(x,y)\,(i=1,2,...)$ qiymatlarni hosil qilamiz. Endi, agar har bir $f_i(x,y)$ funksiya (x_0,y_0) nuqta atrofida yechimning mavjudlik va yagonaligi haqidagi teorema shartlarini qanoatlantirsa, har bir $y'=f_i(x,y)$ differensial tenglama uchun $y(x_0)=y_0$ shartni qanoatlantiruvchi yagona $y=y(x)$ yechim mavjud va bu yechim uchun $y'(x_0)=f_i(x_0,y_0)$ bo‘ladi. Agar $y'(x_0)=p_0$ bo‘lsa, $y=y(x)$ yechim (integral chiziq) (x_0,y_0) nuqtadan p_0 yo‘nalishda o‘tadi (chiqadi) deb aytamiz.

Ushbu

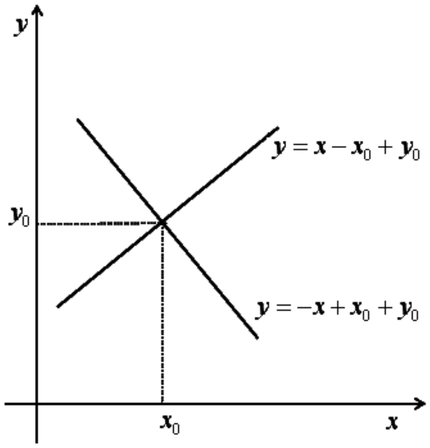
$$\begin{cases} F(x,y,y')=0 \\ y(x_0)=y_0 \end{cases}$$

masala yechimining yagonaligi yechimning mavjudligi va berilgan (x_0,y_0) nuqtadan har qanday yo‘nalish bo‘ylab ko‘pi bilan bitta integral chiziqning o‘tishini anglatadi.

Agar berilgan (x_0,y_0)nuqtadan biror tayin yo‘nalish bo‘ylab kamida ikkita yechim (integral chiziq) o‘tsa, bu nuqta yagonalik (yechimning yagonaligi) buzilgan nuqta deb ataladi.

Misol 1. Ushbu $(y')^2-1=0$ tenglama uchun yagonalik xossasi ixtiyoriy (x_0,y_0) nuqtada o‘rinli, chunki (x_0,y_0) nuqtadan ikkita integral chiziq ikki xil (turli) yo‘nalish bo‘ylab o‘tadi (4.1-rasm):

$$\frac{dy}{dx}=\pm 1,\ y=x-x_0+y_0,\ y=-x+x_0+y_0$$

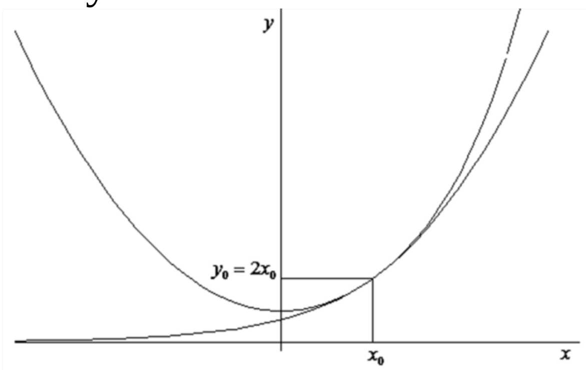


4.1-rasm

Misol 2. Ushbu $(y')^2-(2x+y)y'+2xy=0$ tenglamani y' ga nisbatan yechib, $y'=y,\ y'=2x$ differensial tenglamalarni hosil qilamiz. Ularni yechib, $y=c_1e^x$ va $y=x^2+c_2$ yechimlar oilasini topamiz. $y=2x$ to‘g‘ri chiziqning nuqtalarida yechimning yagonalik xossasi buzilgan, chunki bu to‘g‘ri chiziqning ixtiyoriy $(x_0,2x_0)$ nuqtasi orqali bir xil $y'(x_0)=2x_0$ yo‘nalish bo‘ylab ikkita

$$y=x_0e^{x-x_0}\ ,\ y=x^2+2x_0-x_0^2$$

yechim o‘tadi (4.2-rasm). Bu yechimlar qismlarini $(x_0,2x_0)$ nuqtada tutashtirib, yana ikkita yechim tuzish mumkin.



4.2- rasm

Bizga matematuk analiz kursida isbotlangan oshkormas funksiya to‘g‘risidagi teorema kerak bo‘ladi. Shu teoremani biz uchun qulay ko‘rinishda keltiraylik.

Teorema. *Faraz qilaylik, uch haqiqiy o‘zgaruvchining haqiqiy funksiyasi* $F(x,y,p)$ *uchun quyidagi uch shart o‘rinli bo‘lsin:*

1. $F(x_0,y_0,p_0)=0$,

2. $(x_0,y_0,p_0)\in\mathbb{R}^3$ *nuqtaning biror atrofida* $F(x,y,p)\in C^1$, *3.* $\dfrac{\partial F(x_0, y_0, p_0)}{\partial p} \neq 0.$

U holda $(x_0, y_0) \in \mathbb{R}^2$ *nuqtaning shunday* V *va* $p_0 \in \mathbb{R}$ *nuqtaning shunday* W *atroflari (simmetrik bo'lishi shart emas) topiladiki, ixtiyoriy* $(x,y) \in V$ *uchun* $F(x,y,p) = 0$ *tenglamaning* W *ga tegishli bo'lgan yagona* $p = f(x,y)$ *yechimi mavjud* $\big(F(x,y,f(x,y)) \equiv 0,$ $(x,y) \in V,\ f(x,y) \in W,\ p_0 = f(x_0,y_0)\big)$. *Bundan tashqari,* $f \in C^1(V,W)$ *va*

$$\frac{\partial f(x,y)}{\partial x} = -\frac{F'_x(x,y,f(x,y))}{F'_p(x,y,f(x,y))},\ \frac{\partial f(x,y)}{\partial y} = -\frac{F'_y(x,y,f(x,y))}{F'_p(x,y,f(x,y))}$$

formulalar o'rinli bo'ladi.

Quyidagi teorema (4.1.1) tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligini ta'minlovchi yetarli shartlarni beradi.

Teorema 1. *Aytaylik,* $p = p_0$ *son* $F(x_0, y_0, p) = 0$ *tenglamaning biror haqiqiy ildizi bo'lsin, ya'ni* $F(x_0, y_0, p_0) = 0$. *Agar* $(x_0, y_0, p_0) \in G \subset \mathbb{R}^3$ *nuqtaning biror atrofida* $F(x,y,p)$ *funksiya* C^1 *sinfga tegishli va*

$$\frac{\partial F(x_0, y_0, p_0)}{\partial p} \neq 0$$

bo'lsa, u holda shunday $h > 0$ *mavjudki,* $x_0 - h \le x \le x_0 + h$ *oralig'ida (4.1.1) diffrensial tenglamaning* $y(x_0) = y_0, y'(x_0) = p_0$ *shartlarni qanoatlantiruvchi yagona* $y = y(x)$ *yechimi mavjud.*

🗝 Oshkormas funksiya haqidagi teoremaga ko'ra (x_0, y_0) nuqtaning biror V atrofida (4.1.1) tenglama $y' = f(x,y)$ ko'rinishga keltiriladi; bunda $p_0 = f(x_0, y_0)$ hamda (x_0, y_0) nuqtaning yetarli kichik atrofida $f \in C^1$ va $F(x,y,f(x,y)) = 0$ bo'ladi. Teoremaning shartlaridan $\dfrac{\partial F}{\partial y}$ va $\dfrac{\partial F}{\partial p}$ funksiyalar (x_0, y_0, p_0) nuqtaning yetarlicha kichik atrofida uzluksiz va $\dfrac{\partial F}{\partial p}$ shu atrofda noldan farqli ekanligi kelib chiqadi. Shuning uchun $(x_0, y_0) \in \mathbb{R}^2$ nuqtaning yetarlicha

kichik atrofida $\left|\dfrac{\partial F(x,y,f(x,y))}{\partial p}\right| \geq \text{const} > 0$, $\left|\dfrac{\partial f(x,y)}{\partial y}\right| =$
$= \left|\dfrac{F'_y(x,y,f(x,y))}{F'_p(x,y,f(x,y))}\right| \leq \text{const}$ va, demak, shu atrofda $f(x,y)$ funksiya y bo‘yicha Lipshits shartini qanoatlantiradi. Mavjudlik va yagonalik teoremasiga ko‘ra $y' = f(x,y)$ tenglamaning (x_0, y_0) nuqtadan p_0 yo‘nalish bo‘ylab o‘tuvchi (shu nuqtadagi urinmaning burchak koeffitsiyenti p_0 ga teng, $y'(x_0) = f(x_0, y_0) = p_0$) yagona $y = \varphi(x)$ integral chizig‘i (yechimi) biror yetarlicha kichik $\left[x_0 - h,\ x_0 + h\right]$ $(h > 0)$ oraliqda mavjud, $\varphi'(x) = f(x, \varphi(x))$. Ravshanki, bu $y = \varphi(x)$ funksiya uchun $0 = F\left(x, \varphi(x), f(x, \varphi(x))\right) = F\left(x, \varphi(x), \varphi'(x)\right)$. Demak, $y = \varphi(x)$ funksiya (1) tenglamaning $y(x_0) = y_0$, $y'(x_0) = p_0$ shartlarni qanoatlantiruvchi yechimi. Teoremaning mavjudlik qismi isbot bo‘ldi.

Endi yagonalik qismini isbotlaymiz. Faraz qilaylik, $y = y(x)$ ixtiyoriy yechim bo‘lsin:

$$F\left(x, y(x), y'(x)\right) = 0,\ x \in [x_0 - h, x_0 + h], y(x_0) = y_0, y'(x_0) = p_0.$$

Yetarlicha kichik $h > 0$ uchun $\left(x, y(x)\right) \in V,\ x \in [x_0 - h, x_0 + h]$, bo‘ladi. Demak, oshkormas funksiya haqidagi teoremaning yagonalik qismiga ko‘ra

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

bo‘lishi kerak. Shunday qilib, yuqorida topilgan $y = \varphi(x)$ va berilgan $y = y(x)$ funksiyalar ushbu

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Koshi masalasining yechimi. Bu masala yechimining yagonalik xossasidan $\varphi(x) \equiv y(x), x \in [x_0 - h, x_0 + h]$. ✍

Isbotlangan bu teoremadan quyidagi teorema bevosita kelib chiqadi.

Teorema 2. *Faraz qilaylik,* $G \subset \mathbb{R}^3$ *sohada* $F(x,y,p)$ *funksiya* C^1 *sinfga tegishli va* G *sohaning* $F(x,y,p)$ *funksiya nolga aylangan, ya'ni* $F(x,y,p)=0$ *bo'lgan nuqtalarida* $\frac{\partial F(x,y,p)}{\partial p}$ *hosila nolga aylanmasin, ya'ni* $\frac{\partial F(x,y,p)}{\partial p} \neq 0$. *U holda* G *sohaning* x, y *tekisligidagi ortogonal proeksiyasida joylashgan ixtiyoriy* (x_0, y_0) *nuqta orqali* $F(x,y,y')=0$ *differensial tenglamaning ushbu* $F(x_0, y_0, \tilde{p})=0$ *tenglamani qanoatlantiruvchi har bir* $\tilde{p}$ *yo'nalish bo'ylab bittadan yechimi (integral chizig'i) o'tadi.*

Teoremani yuqorida qaralgan

$$(y')^2-(y+2x)y'+2xy=0$$

tenglamaga tatbiq etaylik. Tenglamada $F(x,y,p)=p^2-(y+2x)p+2xy \in C^1(\mathbb{R}^2)$ va $F(x,y,p)$ funksiya nolga aylangan nuqtalar uchun $F(x,y,p)=p^2-(y+2x)p+2xy=0$, ya'ni $p=2x$ yoki $p=y$ bo'ladi. Demak, bunday nuqtalar uchun mos ravishda

$$\frac{\partial F}{\partial p}=2p-(y+2x)=2x-y \text{ yoki } \frac{\partial F}{\partial p}=2p-(y+2x)=y-2x.$$

Agar $y \neq 2x$ bo'lsa, $\frac{\partial F}{\partial p} \neq 0$ va bunday $(x,y) \in \mathbb{R}^2$ nuqtadan qaralayotgan tenglamaning $p=2x$ va $p=y$ yo'nalishlar bo'ylab bittadan yechimi o'tadi, ya'ni $2x<y$ va $2x>y$ sohalar (yarim tekisliklar) berilgan tenglama uchun yagonalik sohalaridir. Yuqorida $y=2x$ to'g'ri chiziq nuqtalarida yagonalik buzilishi e'tirof etilgan edi. ☝

Masalalar

1. Ushbu

$$(y')^2-(3x^2+y)y'+3x^2y=0$$

tenglamani yeching. Yagonalik sohalarini aniqlang.

2. Ushbu

$$(y')^2-(2x+1)yy'+2xy^2=0$$

tenglamani yeching. Yechimlar oilasining tabiatini tekshiring.

§ 4.2. Hosilaga nisbatan yechilmagan differensial tenglamani yechish usullari

Ushbu $F(x,y,y')=0$ (4.1.1) differensial tenglama yechimini topishning ikki usulini keltiraylik.

1. *Tenglamani* y' *ga nisbatan yechish usuli.* Agar (4.1.1) tenglama y' ga nisbatan oshkor yechilishi mumkin bo‘lsa, u holda bitta yoki bir nechta (hatto cheksiz ko‘p)

$$y'=f(x,y) \tag{4.2.1}$$

ko‘rinishdagi differensial tenglamaga kelamiz. Biz $f(x,y)$ funksiya kamida uzluksiz bo‘lishi kerak deb talab qilamiz. Normal ko‘rinishdagi (4.2.1) tenglama(lar)ni yechish uchun yuqoridan bizga ma’lum metodlarni qo‘llaymiz (agar mumkin bo‘lsa).

Misol 1. Ushbu $y'^2-(y+x)^2=0$ tenglamadan y'ni x va $y$ning ***silliq funksiyasi*** sifatida topsak, ikki dona

$$y'=y+x,\ y'=-y-x,$$

yechimni hosil qilamiz. Tenglamadan y' ni x va $y$ning ***uzluksiz funksiyasi*** sifatida izlasak,

$$y'=y+x,\ y'=-y-x, y'=|y+x|,\ y'=-|y+x|\,,$$

to‘rtta yechim hosil bo‘ladi. 👍

Misol 2. Quyidagi differensial tenglamani yeching:

$$y'^2-4x^2=0.$$

🗝 Berilgan tenglamani ushbu

$$y'=2x,\ y'=-2x,\ y'=2|x| \text{ va } y'=-2|x|$$

tenglamalarga ajrataylik. Bu tenglamalarni yechib, berilgan differensial tenglamaning yechimlar oilasini hosil qilamiz:

$$y=x^2+c,\ y=-x^2+c,\ y=x|x|+c\,,\ y=-x|x|+c$$

$$(c-\text{ixtiyoriy o‘zgarmas}).$$ 👍

2. *Parametr kiritish metodi.*

a) Aytaylik, (4.2.1) tenglama y o‘zgaruvchiga nisbatan oshkor ko‘rinishda yechilsin:

$$y=f(x,y') \quad (f\in C^1,\ f'_{y'}\neq 0 \text{ deb faraz qilinadi }). \tag{4.2.3}$$

Oshkormas funksiya haqidagi teoremadan ko‘rsatilgan shartlarda (4.2.3) differensial tenglama yechimlari C^2 sinfga tegishli ekanligi kelib chiqadi: (4.2.3) dan $y' = g(x, y)$, $g \in C^1 \Rightarrow y \in C^2$. Qaralayotgan (4.2.3) tenglamada $y' = p$ $(dy = pdx)$ deb p parametrni kiritamiz va

$$y = f(x, p) \tag{4.2.4}$$

munosabatni hosil qilamiz. Agar x ni p ning funksiyasi sifatida topa olsak, uni (4.2.4) ga qo‘yib, y ni p ning funksiyasi sifatida ifodalaymiz va yechimni parametrik ko‘rinishda topgan bo‘lamiz. Bu ishni bajarish uchun (4.2.4) ning har ikkala tomonini differensiallaymiz:

$$dy = f_x{}' dx + f_p{}' dp; \quad pdx = f_x{}' dx + f_p{}' dp.$$

Oxirgi tenglikdan

$$\left(p - f_x'\right) dx = f_p' dp \tag{4.2.5}$$

munosabatni topamiz. Endi yana qo‘shimcha ravishda

$$f_x' \neq p \tag{4.2.6}$$

deb faraz qilamiz. (4.2.6) shartda (4.2.5) tenglamani

$$\frac{dx}{dp} = \frac{f_p'}{p - f_x'} \tag{4.2.7}$$

ko‘rinishga keltirib, uni yechamiz va $x = \varphi(p, c)$ bog‘lanishni hosil qilamiz. Bu bog‘lanishni (4.2.4) ga qo‘yib, (4.2.3) differensial tenglamaning yechimi ushbu

$$\begin{cases} x = \varphi(p, c) \\ y = f(\varphi(p, c), p) \end{cases} \tag{4.2.8}$$

parametrik (p – parametr, c – ixtiyoriy o‘zgarmas yechimni belgilaydi) ko‘rinishda bo‘lishi kerakligini topamiz. Endi ixtiyoriy o‘zgarmas c uchun (4.2.8) formulalar bilan parametrik ko‘rinishda berilgan funksiya (4.2.3) differensial tenglamaning yechimi ekanligini ko‘rsatamiz. Buning uchun ixtiyoriy tayinlangan c da (4.2.8) dagi birinchi tenglamadan $p = p(x)$ bog‘lanishni topib ($\varphi_p' \neq 0$), uni ikkinchisiga qo‘yishdan hosil bo‘lgan $y = y(x)$ funksiyaning hosilasi uchun $\frac{dy}{dx} = p$ bo‘lishini ko‘rsatish kifoya. $\frac{dy}{dx} = p$ ekanligi esa ravshan: (4.2.8) dagi ikkinchi tenglikdan (4.2.5) formulaga ko‘ra

$$\frac{dy}{dx} = f_x' + f_p' \cdot \frac{dp}{dx} = f_x' + f_p' \cdot \frac{p - f_x'}{f_p'} = p\,.$$

Shunday qilib, qo‘yilgan $f_p' \neq 0$ va $f_x' \neq p$ shartlarda (4.2.3) differensial tenglamaning umumiy yechimi (4.2.8) formulalar bilan parametrik ko‘rinishda ifodalanadi.

Agar $f_x'(x, p) = p$ tenglik $x = x(p) \in C^1$ funksiyani aniqlasa va $f_p'(x(p), p) \equiv 0$ ham bo‘lsa, u holda (4.2.5) tenglama, ravshanki, $x = x(p)$ yechimga ega. Buni (4.2.4) formulaga qo‘yib, qaralayotgan (4.2.3) differensial tenglamaning $\{\, x = x(p),\quad y = f(x(p), p)\}$ parametrik yechimini hosil qilamiz. Endi (4.2.5) tenglamani $x - x(p)$ ga “qisqartirib”, so‘ngra yuqoridagidek ish ko‘rib, (4.2.3) tenglamaning bir parametrli yechimlar oilasini (agar mavjud bo‘lsa) topamiz.

$f_x'(x, p) = p$ tenglama biror $p = p^*$ ($-$o‘zgarmas son) yechimga ega bo‘lgan holi $f(x, p) = x\psi(p) + \chi(p)$ bo‘lganda, ya’ni $y = x\psi(y') + \chi(y') -$ Lagranj tenglamasi holida uchrashi mumkin; bu tenglamani biz keyinroq o‘rganamiz. $f_x' \equiv p$ holi esa Klero tenglamasi bandida o‘rganiladi.

Misol 3. Ushbu

$$x^3 y'^2 + x^2 y y' - 1 = 0$$

differensial tenglamani yechaylik.

⊶Berilgan tenglamadan y osongina topiladi:

$$y = \frac{1}{x^2 y'} - xy'\,. \tag{4.2.6}$$

Yuqorida aytilganidek, $y' = p$, ya’ni $dy = pdx$ deb, p parametrni kiritamiz. U holda

$$y = \frac{1}{x^2 p} - xp\,. \tag{4.2.7}$$

Endi x ni p ning funksiyasi sifatida topish maqsadida, bu tenglikni differensiallaymiz va kerakli shakl almashtirishlarni bajaramiz:

$$dy = d\left(\frac{1}{x^2 p}\right) - d(xp),\;\; pdx = -\frac{2}{x^3 p}dx - \frac{1}{x^2 p^2}dp - xdp - pdx,$$

$$\left(\frac{1}{x^2p^2}+x\right)\left(dp+\frac{2p}{x}dx\right)=0.$$

Oxirgi tenglamadan

$$\frac{1}{x^2p^2}+x=0,\ dp+\frac{2p}{x}dx=0\ .$$

ya'ni

$$x=-\sqrt[3]{\frac{1}{p^2}},\ x=\frac{c}{\sqrt{|p|}}$$

bog‘lanishlarni topamiz. Ularni (4.2.7) ga qo‘yib, y ni p ning funksiyasi sifatida topamiz:

$$y=2\sqrt[3]{p},\ y=\frac{1}{c^2}\frac{|p|}{p}-c\sqrt{p}\,.$$

Yechimlarni parametrik ko‘rinishda hosil qildik:

$$\begin{cases} x=-\sqrt[3]{\dfrac{1}{p^2}} \\ y=2\sqrt[3]{p} \end{cases},\quad \begin{cases} x=\dfrac{c}{\sqrt{|p|}} \\ y=\dfrac{|p|}{p}\left(\dfrac{1}{c^2}-c\sqrt{|p|}\right) \end{cases}.$$

Bu yerda p ni yo‘qotib, yechimlarni oshkor ko‘rinishda ham ifodalash mumkin:

$$x=-\frac{4}{y^2},\quad y=\frac{c}{x}-\frac{1}{c}\ .$$

b) (4.2.1) tenglama x ga nisbatan yechilgan

$$x=f(y,y')\quad (f\in C^1,\ f'_{y'}\neq 0 \text{ deb faraz qilnadi})$$

ko‘rinishga bo‘lsa ham, uning parametrik yechimlari yuqoridagi usul bilan topiladi.

c) Umumiy holda parametr kiritishning nazariyasida to‘xtalmaymiz.

Masalalar

l. Quyidagi differensial tenglamani $y'=p$ parametr kiritish usuli bilan yeching:

$$yy'^2+x^3y'-x^2y=0.$$

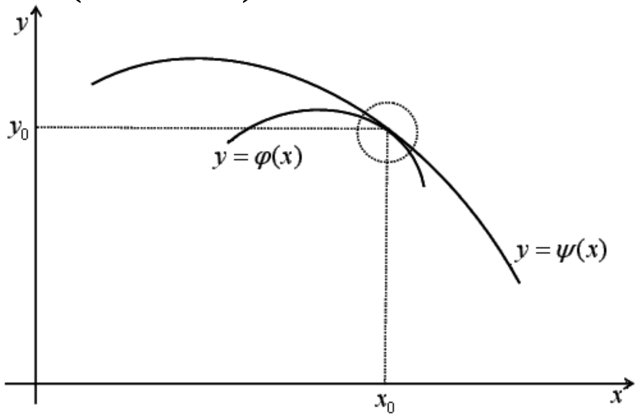
2. Ushbu $y'^2+y^2=1$ $(y=y(x))$ differensial tenglamani, u parametrni $y'=\cos u,\ y=\sin u$ formulalar bilan kiritib, yeching.

§ 4.3. Maxsus yechimlar

Ushbu

$$F(x,y,y')=0 \qquad (4.3.1)$$

differensial tenglamani qaraylik; bu yerda ham avvalgi paragrafdagidek $F\in C^1$ deb hisoblanadi. (4.3.1) ***tenglamaning maxsus yechimi*** deb uning shunday $y=\psi(x)$ yechimiga aytiladiki, bu yechim grafigining har qanday nuqtasidan shu integral chiziqqa (grafikka) urinib boshqa bir $y=\varphi(x)$ yechim grafigi ham oʻtadi; bunda $y=\psi(x)$ va $y=\varphi(x)$ yechimlar urinish nuqtasining ixtiyoriy atrofida ustma-ust tushmasliklari kerak. Shunday qilib, maxsus yechim grafigi yechimning yagonalik xossasi buziladigan nuqtalardan tuziladi (4.3-rasm).



4.3-rasm.

Eslatamizki, $y=\varphi(x)$ va $y=\psi(x)$ funksiyalar grafiklarining (x_0,y_0) nuqtada urinishi ularning shu nuqtada umumiy urinmaga egaligini, ya’ni

$$\varphi(x_0)=\psi(x_0)=y_0\,,\ \varphi'(x_0)=\psi'(x_0)$$

shartlarning bajarilishini anglatadi.

$F\in C^1$ boʻlgani uchun § 4.3 dagi teorema 2 dan ravshanki, (4.3.1) differensial tenglamaning maxsus yechimi grafigidagi ixtiyoriy (x,y) nuqta p ning biror qiymatida ushbu

$$\begin{cases} F(x,y,p)=0 \\ \dfrac{\partial F(x,y,p)}{\partial p}=0 \end{cases} \qquad (4.3.2)$$

sistemani qanoalantiradi.

Bu (4.3.2) sistemani qanoatlantiruvchi barcha (x, y)juftliklar to‘plami (4.3.1) **differensial tenglamaning** $p-$**diskriminant chizig‘i** deyiladi. Demak, maxsus yechim $p-$diskriminant chiziqning tarkibida bo‘ladi. $p-$diskriminant chiziqda yechimning yagonalik xossasi buziladigan nuqtalardan tashqari yana boshqa nuqtalar ham bo‘lishi mumkin.

$p-$diskriminant chiziq mos differensial tenglamaning yechimi bo‘lmasligi mumkin. Quyidagi misol bu tasdiqni asoslaydi.

Misol 1. Ushbu

$$yy'^2-2y'+1=0$$

tenglama uchun $F(x,y,p)=yp^2-2p+1$ va $F'_p(x,y,p)=2yp-2$. Demak, $p-$diskriminant chiziq

$$\begin{cases} yp^2-2p+1=0 \\ 2yp-2=0 \end{cases}$$

sistemadan $p$ni yo‘qotish yordamida topiladi. Bu sistemaning ikkinchi tenglamasidan $p=1/y$ ni topib, birinchisiga qo‘yamiz va $y=1$ $p-$diskriminant chiziqqa ega bo‘lamiz. Lekin, ravshanki, bu $y=1$ funksiya, ya’ni $p-$diskriminant chiziq, qaralayotgan differensial tenglamaning yechimi emas.

$p-$diskriminant chiziq mos differensial tenglamaning yechimi bo‘lganda ham u maxsus yechim bo‘lmasligi mumkin. Bu fikrni asoslash uchun quyidagi misolni keltiramiz.

Misol 2. Ushbu

$$y'^2+4y^3(y-1)=0$$

differensial tenglamaning $p-$diskriminant chizig‘i va maxsus yechim(lar)ini topaylik. Berilgan tenglama uchun $F(x,y,p)=p^2+4y^3(y-1)$, $F'_p(x,y,p)=2p$. Demak, $p-$diskriminant chiziq

$$\begin{cases} p^2+4y^3(y-1)=0 \\ 2p=0 \end{cases}$$

sistemadan topiladi. Bu sistemadagi ikkinchi tenglamadan $p=0$ ni topib, birinchisiga qo‘ysak, $y^3(y-1)=0$ tenglamaga kelamiz. Demak, $p-$diskriminant chiziq $y=0$ va $y=1$ shoxlardan iborat.

Ravshanki, ularning har biri berilgan differensial tenglamaning yechimidir. Endi bu yechimlarni maxsus yechim bo‘lish yoki bo‘lmaslikka tekshiraylik. Buning uchun tenglamaning boshqa yechimlarini topamiz. Berilgan tenglamani y' ga nisbatan yechib, ikki dona o‘zgaruvchilari ajraladigan tenglama hosil qilamiz: $y' = \pm 2y\sqrt{y(1-y)}$. Bu tenglamalardan ushbu

$$y = \frac{1}{(x+c)^2+1},\ y=0,\ y=1$$

yechimlarni topamiz (ularning grafiklarini chizing). Ravshanki, $y=1$ yechim maxsus yechim, chunki bu yechim grafigining barcha nuqtalarida yechimning yagonalik xossasi buziladi: $y=1$ yechimning ixtiyoriy $(x_0,1)$ nuqtasidan bu yechimga urinib $y = \dfrac{1}{(x-x_0)^2+1}$ boshqa yechim ham o‘tadi. $y=0$ yechim grafigining nuqtalarida esa yechimning yagonalik xossasi saqlanadi, ya’ni $y=0$ yechim maxsus yechim emas.

Shunday qilib, (4.3.1) differensial tenglamaning maxsus yechim(lar)ini topishni quyidagi qadamlarni ketma-ket bajarib amalga oshirish mumkin:

1. $p-$diskriminant chiziqni aniqlash, ya’ni (4.3.2) sistemani tuzib, undan p ni yo‘qotish;
2. $p-$diskriminant chiziq shoxlari orasidan differensial tenglama yechimlarini ajratish;
3. topilgan yechimlar orasidan maxsus yechimlarni aniqlash.

Maxsus yechimni differensial tenglamaning bir parametrli yechimlar oilasining o‘rama chizig‘i (mavjud bo‘lganda) sifatida ham topish mumkin (4.5. paragrafga qarang).

§ 4.4. Lagranj va Klero tenglamalari

Lagranj tenglamasi deb ushbu

$$y = x\psi(y') + \chi(y') \tag{4.4.1}$$

ko‘rinishdagi differensial tenglamaga aytiladi; bu yerda $\{\psi,\chi\}\subset C^1(I,\mathbb{R}),\ \ I\subset\mathbb{R}-$biror oraliq, va $\psi(p)\not\equiv p$ deb faraz qilinadi.

Lagranj tenglamasini § 4.2 da aytilgan parametr kiritish usuli bilan yechamiz. Bu usuldan kelib chiqib, $y' = p$ deb parametr kiritamiz va (4.4.1) tenglamadan

$$y = x\psi(p) + \chi(p)$$

bog‘lanishni topamiz. Endi, agar $x$ni $p$ning funksiyasi sifatida topsak, u holda yechimni parametrik ko‘rinishda topgan bo‘lamiz. Buning uchun oxirgi tenglikni differensiallaymiz va

$$p = \psi(p) + x\psi'(p)\frac{dp}{dx} + \chi'(p)\frac{dp}{dx} \quad \text{yoki} \quad p - \psi(p) = \left(x\psi'(p) + \chi'(p)\right)\frac{dp}{dx}$$

tenglamani hosil qilamiz. Bu yerdan $\forall\, p \in I$ uchun $p - \psi(p) \neq 0$ deb, quyidagini topamiz:

$$\frac{dx}{dp} = \frac{\psi'(p)}{p - \psi(p)}x + \frac{\chi'(p)}{p - \psi(p)}.$$

Bu tenglama $x = x(p)$ funksiyaga nisbatan chiziqli birinchi tartibli differensial tenglamadir. Uning yechimi kvadraturalarda topiladi. Umumiy yechimini

$$x = a(p)c + b(p) \ \ (a(p), b(p) - \text{aniq funksiyalar})$$

ko‘rinishda topamiz. Demak, berilgan (4.4.1) Lagranj tenglamasining parametrik ko‘rinishdagi umumiy yechimi

$$\begin{cases} x = a(p)c + b(p) \\ y = \left(a(p)c + b(p)\right)\psi(p) + \chi(p) \end{cases} \quad (p - \text{parametr})$$

formulalar bilan beriladi.

Agar $p - \psi(p) = 0$ tenglama yechimga ega bo‘lsa, uning ixtiyoriy $p = p^*$ yechimidan Lagranj tenglamasining $y = x\psi(p^*) + \chi(p^*)$ yechimini ham topamiz.

Misol 1. Ushbu

$$y = \frac{1}{2}xy' + \ln y'$$

Lagranj tenglamasini yechaylik.

$y' = p$ parametr kiritib,

$$y = \frac{1}{2}xp + \ln p$$

bog‘lanishni hosil qilamiz. Bu tenglikning har ikkala tomonidan differensial olib, $dy = pdx$ ekanligini hisobga olib, $x = x(p)$

noma’lum funksiyaga nisbatan chiziqli differensial tenglama hosil qilamiz (bizda $p>0$):

$$\frac{dx}{dp}=\frac{1}{p}x+\frac{2}{p^2}.$$

Bu chiziqli tenglamaning umumiy yechimi osongina topiladi:

$$x=cp-\frac{1}{p}.$$

Endi berilgan tenglamaning parametrik ko‘rinishdagi umumiy yechimini yozamiz:

$$\begin{cases} x=cp-\dfrac{1}{p} \\ y=\dfrac{1}{2}(cp-\dfrac{1}{p})p+\ln p \end{cases}.$$

Klero tenglamasi deb

$$y=xy'+\chi(y') \qquad (4.4.2)$$

ko‘rinishdagi differensial tenglamaga aytiladi; bu yerda $\chi\in C^1(I,\mathbb{R})$ deb faraz qilinadi. Klero tenglamasini yechish uchun $y'=p$ parametrni kiritamiz. U holda $y$ning p parametrga bog‘lanish qonunini hosil qilamiz:

$$y=xp+\chi(p)$$

x ning p parametrga bog‘lanish qonuniyatini topish uchun Klero tenglamasi (4.4.2) ning har ikkala tomonini x bo‘yicha differensiallaymiz:

$$p=p+x\frac{dp}{dx}+\chi'(p)\frac{dp}{dx} \text{ yoki } (x+\chi'(p))\frac{dp}{dx}=0.$$

Bundan $\dfrac{dp}{dx}=0$ yoki $x+\chi'(p)=0$ kelib chiqadi. 1-holda $p=c$ va

$$y=cx+\chi(c) \text{ (to‘g‘ri chiziqlar oilasi)}$$

ko‘rinishdagi yechimlarni topsak, 2-holda esa

$$\begin{cases} x=-\chi'(p) \\ y=-\chi'(p)\cdot p+\chi(p) \end{cases} (p - \text{parametr}) \qquad (4.4.3)$$

yechimni topamiz. Ko‘rsatish mumkinki, agar $\chi(\cdot)$ funksiya nochiziqli bo‘lsa, $y=cx+\chi(c)$ to‘g‘ri chiziqlar oilasining o‘ramasi (4.4.3) chiziqdan iborat bo‘ladi; agar $\chi(\cdot)$ funksiya chiziqli (

$\chi(c)=\alpha c+\beta$) bo‘lsa, u holda $y=cx+\chi(c)$ to‘g‘ri chiziqlarning barchasi bitta tayin (ya'ni $(-\alpha,\beta)$) nuqtadan o‘tadi va ular o‘rama chiziqqa ega bo‘lmaydi.

Masalalar

1. Tenglamalarni yeching. Maxsus yechimlarni (agar bor bo‘lsa) toping.

1. $y=y'+\sqrt{1-y'^2}$. **2.** $y=x(1+y')+y'^2$. **3.** $y=xy'+\sqrt{1+y'^2}$.

§ 4.5. Maxsus yechimni yechimlar o‘ramasi sifatida topish

Maxsus yechim bir parametrli chiziqlar oilasining o‘ramasi (yopishmasi) tushunchasi bilan bevosita bog‘liq. Dastlab zarur ma’lumotlarni eslaylik.

Bizga bir parametrli silliq chiziqlar oilasi ushbu

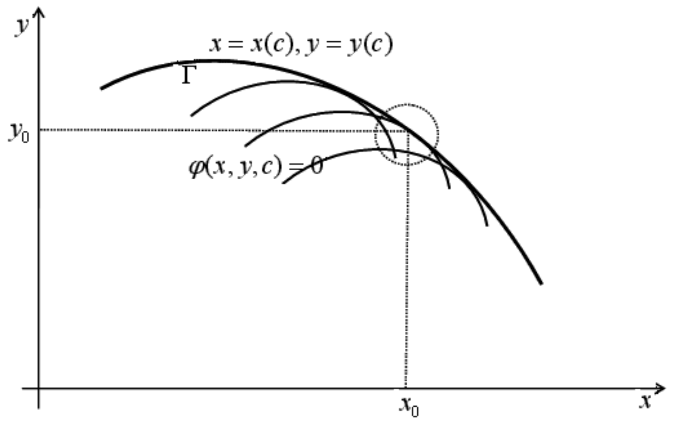
$$\varphi(x,y,c)=0 \tag{4.5.1}$$

tenglama bilan berilgan bo‘lsin; bu yerda $\varphi(x,y,c)$ funksiya $(x,y)\in D$ ($D-\mathbb{R}^2$ tekislikdagi soha) va $c\in(c_1,c_2)$ bo‘lganda C^1 sinfga tegishli, $\varphi\in C^1\left(D\times(c_1,c_2),\mathbb{R}\right)$, hamda $D\times(c_1,c_2)$ sohada $\varphi_x'^2+\varphi_y'^2\neq 0$.

Ushbu

$$\begin{cases} x=x(c) \\ y=y(c) \end{cases} \left(c\in(c_1,c_2)\right) \tag{4.5.2 }$$

parametrik ko‘rinishda berilgan Γ silliq chiziqni qaraylik, bunda $\{x(c),y(c)\}\subset C^1\left((c_1,c_2)\right)$ va $|x'(c)|+|y'(c)|\neq 0$. Agar har qanday $(x_0,y_0)\in\Gamma$, $x_0=x(c_0), y_0=y(c_0), c_0\in(c_1,c_2)$, nuqtadan shu nuqtada Γ chiziqqa urinib (4.5.1) oilaning biror chizig‘i o‘tsa va bu chiziq (x_0,y_0) nuqtaning ixtiyoriy atrofida Γ bilan ustma-ust tushmasa, u holda Γ chiziq (4.5.1) oilaning o‘ramasi (o‘rama chizig‘i) deyiladi (4.4-rasm). Bu yerda Γ o‘ramaning nuqtalari berilgan oilaning Γga urinuvchi mos chizig‘ini belgilovchi parametr qiymatlari bilan patametrlangan.



4.4-rasm.

(4.5.1) oilaning c–diskriminant chizig‘i deb

$$\begin{cases} \varphi(x,y,c)=0 \\ \dfrac{\partial\varphi(x,y,c)}{\partial c}=0 \end{cases}$$

sistemani biror c da qanoatlantiruvchi (x,y) nuqtalar to‘plamiga aytiladi.

Teorema (o‘rama mavjudligining zaruriy sharti). *Berilgan (4.5.1) chiziqlar oilasining Γ o‘ramasi shu oilaning c–diskriminant chizig‘ida joylashadi.*

🗝 Γ o‘ramadan ixtiyoriy $(x(c),y(c))\in\Gamma$ $(c\in(c_1,c_2))$ nuqtani olaylik. Bu nuqta berilgan oilaning $\varphi(x,y,c)=0$ chizig‘ida yotadi va shu nuqtada bu chiziqlar urinadi, ya’ni

$$\varphi(x(c),y(c),c)=0 \ \ (c\in(c_1,c_2)) \qquad (4.5.3)$$

va Γ ning $\left(x'(c),y'(c)\right)$ urinma vektori mos chiziqning $\left(\varphi'_x(x,y,c),\varphi'_y(x,y,c)\right)$normal vektoriga ortogonal (ularning skalyar ko‘paytmasi nolga teng):

$$\varphi'_x(x,y,c)\cdot x'(c)+\varphi'_y(x,y,c)\cdot y'(c)=0\,. \qquad (4.5.4)$$

(4.5.3) ayniyatni c bo‘yicha differensiallaymiz:

$$\varphi'_x(x,y,c)\cdot x'(c)+\varphi'_y(x,y,c)\cdot y'(c)+\varphi'_c(x(c),y(c),c)=0\,.$$

Endi (4.5.3) va (4.5.4) ayniyatlardan

$$\varphi'_c(x(c),y(c),c)=0 \qquad (4.5.5)$$

ekanligini topamiz. Yuqoridagi (4.5.3) va (4.5.5) tengliklar teoremani isbotlaydi. 👍

Qo‘shimcha tekshirishlar o‘tkazib, c–diskriminant chiziqdan o‘rama chiziq ajratib olinadi (agar u mavjud bo‘lsa).

Faraz qilaylik, (4.1.1) differensial tenglamaning bir paramertli yechimlar oilasi

$$\Phi(x, y, c) = 0 \qquad (4.5.6)$$

ma’lum bo‘lsin. Agar bu oila o‘rama chiziqqa (o‘ramaga) ega bo‘lsa, u holda bu o‘rama

1) differensial tenglamaning yechimidan iborat bo‘ladi, chunki o‘ramaning har bir (x, y) nuqtasida (x, y, y') uchlik (4.5.6) oilaning biror bir egri chizig‘ining mos nuqtasidagi mos uchlikka teng bo‘ladi.

2) (4.1.1) differensial tenglamaning maxsus yechimini berada, chunki o‘ramaning har bir nuqtasidan o‘ramaning o‘zi va unga urinib (4.5.6) oilaning biror chizig‘i o‘tadi; bu chiziqlar turli yechimlarni ifodalaydi.

(4.5.6) oilaning o‘ramasini topish uchun, ma’lumki, c diskriminant chiziq deb ataluvchi chiziqlarni topib ular orasidan o‘rama egri chiziqni ajratish kerak. c –diskriminant chiziqlar ushbu

$$\begin{cases} \Phi(x, y, c) = 0 \\ \dfrac{\partial \Phi(x, y, c)}{\partial c} = 0 \end{cases} \qquad (4.5.7)$$

sistemadan aniqlanadi. Umumiy holda bu sistema nafaqat o‘ramani, balki u karrali nuqtalar to‘plamini ham aniqlashi mumkin. (4.5.7) sistemadan c ni yo‘qotib, ushbu

$$\varphi(x, y) = 0 \qquad (4.5.8)$$

c –diskriminant chiziqni topamiz. Uning o‘ramadan iborat bo‘lgan qismini ajratib, (4.1.1) tenglamaning maxsus yechimini aniqlaymiz.

MODUL 5. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR

§ 5.1. Umumiy ko‘rinishdagi $n-$tartibli differensial tenglama va uning yechimi

Bilamizki (§ 1.1), $n-$**tartibli differensial tenglama** ushbu

$$F(x,y,y',...,y^{(n)})=0. \tag{5.1.1}$$

ko‘rivishga ega. Bu yerda $F(x,y,p_1,...,p_n)-\mathbb{R}^{2+n}$ $(n\in\mathbb{N}, n>1)$ fazoning biror G sohasida berilgsn uzluksiz funksiya, u p_n o‘zgaruvchiga tom ma’noda bog‘liq; $y=y(x)-$noma’lum fuksiya.

Ba’zi shartlarda (masalan, oshkormas funksiya haqidagi teorema shartlari bajarilganda) (5.1.1) tenglamani yuqori tartibli hosila $y^{(n)}$ga nisbatan yechilgan ko‘rinishga keltirish mumkin:

$$y^{(n)}=f(x,y,y',....,y^{(n-1)}); \tag{5.1.2}$$

bu yerda $f\in C(D), D\in\mathbb{R}^{1+n}-$soha, deb hisoblanadi.

Tenglama yechimining ta’rifini eslaylik. $y=\phi(x)$ funksiya berilgan bo‘lsin. Agar

1^0. $\varphi(x)\in C^n(I)$;

2^0. $\forall x\in I \quad F\left(x,\varphi(x),\varphi'(x),....,\varphi^{(n)}(x)\right)=0$;

bo‘lsa, $y=\varphi(x)$ funksiya (5.1.1) differensial tenglamaning ***I*** **oraliqda aniqlangan yechimi** deyiladi.

Shunday qilib, (5.1.1) differensial tenglamaning I oraliqda yechimi deb shunday funksiyaga aytiladiki, uning tenglamada qatnashgan barcha hosilalari shu oraliqda uzluksiz va uni (5.1.1) tenglamaga qo‘yilganda, $x\in I$ ga nisbatan ayniyat hosil bo‘ladi.

Agar

$$y=\varphi(x,c_1,c_2,...,c_n) \tag{5.1.3}$$

formula $c_1,c_2,...,c_n$ parametrlarning tayinlangan joiz qiymatlarida (5.1.1) tenglamaning ($\tilde{D}\subset\mathbb{R}^2$ sohada joylashgan) yechimini bersa, shu bilan birgalikda (5.1.1) tenglamaning (shu sohadagi) har qanday yechimi (5.1.3) formuladan $c_1,c_2,...,c_n$ larning biror joiz qiymatlarida hosil bo‘lsa, u holda $y=\varphi(x,c_1,c_2,...,c_n)$ oila (5.1.1) tenglamaning ($\tilde{D}$ sohadagi) **umumiy yechimi** deb ataladi.

Misol 1. Ushbu

$$\frac{d^2x}{dt^2}+\omega^2 x=0 \ \ (\omega=\text{const}>0) \qquad (5.1.4)$$

garmonik ossilyator tenglamasida $F(t,x,p_1,p_2)=\omega^2 x+p_2,\ G=\mathbb{R}^4$. Bu tenglamaning ($\mathbb{R}^2$dagi) umumiy yechimi

$$x=c_1\cos\omega t+c_2\sin\omega t \ \ (c_1,c_2 - \text{ixtiyoriy o‘zgarmaslar})$$

formula bilan beriladi. Bu funksiyaning ixtiyoriy o‘zgarmas c_1,c_2 larda yechim bo‘lishi osongina tekshiriladi ($x(t)\in C^2(\mathbb{R})$):

$$x''=c_1(\cos\omega t)''+c_2(\sin\omega t)''=-c_1\omega^2\cos\omega t-c_2\omega^2\sin\omega t=$$

$$=-\omega^2(c_1\cos\omega t+c_2\sin\omega t)=-\omega^2 x\ ;\ x''+\omega^2 x=0.$$

Qaralayotgan tenglamasining har qanday yechimi keltirilgan ko‘rinishda bo‘lishini biz keyinroq § 6.3 da isbotlaymiz (elementar isboti to‘g‘risida masalalar bo‘limiga qarang). ✍

(5.1.1) tenglama uchun ***Koshi masalasi*** (***boshlang‘ich masala***) quyidagicha qo‘yiladi:

n- tartibli (5.1.1) tenglamaning ushbu

$$y\Big|_{x_0}=y_0,\ y'\Big|_{x_0}=y_0',\ \dots,\ y^{(n-1)}\Big|_{x_0}=y_0^{(n-1)} \qquad (5.1.5)$$

n ta shartni qanoatlantiruvchi yechimini biror $I\ni x_0$ oraliqda toping; bunda $x_0,y_0,y_0',\ \dots,y_0^{(n-1)}-$ berilgan sonlar va $(x_0,y_0,y_0',\ \dots,y_0^{(n-1)})\in G$.

(5.1.5) shartlar **Koshi shartlari** yoki **boshlang‘ich shartlar** deb ataladi. Shunga e’tibor berish kerakki, bu shartlarning hammasi bitta $x=x_0$ boshlang‘ich nuqtada qo‘yilgan.

Misol 2. To‘g‘ri chiziq bo‘ylab $m=1$ massali moddiy nuqta $f(t,x,x')$ kuch ta’sirida harakat qilsin, bunda $x=x(t)-$ nuqtaning t paytdagi koordinatasi, $x'=x'(t)-$ tezligi. Nyutonning ikkinchi qonuniga ko‘ra harakatni boshqaruvchi tenglama

$$mx''=f(t,x,x')$$

ko‘rinishga ega.

Bu ikkinchi taribli differensial tenglama uchun Koshi masalasi quyidagicha yoziladi:

$$\begin{cases} x''=f(t,x,x'), \\ x|_{t_0}=x_0\ ,\ x'|_{t_0}=v_0\ . \end{cases}$$

Bu masala nuqtaning $t=t_0$ paytdagi x_0 koordinatasi (o‘rni) va v_0 (boshlang‘ich) tezligiga ko‘ra uning $x=x(t)$ harakat qonunini aniqlashni anglatadi. Bu – Koshi masalasining mexanik talqini.

Geometrik nuqtai nazaridan ushbu

$$\begin{cases} y'' = f(x,y,y') \\ y\big|_{x_0} = y_0, \ y'\big|_{x_0} = y'_0 \end{cases}$$

boshlang‘ich masala (x_0, y_0) nuqta orqali berilgan y'_0 yo‘nalishda o‘tgan yechim grafigini topishni anglatadi.

Masalalar

1. Ushbu

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \ \ (\omega = \text{const} > 0)$$

garmonik ossilyator tenglamasining har qanday yechimi $x = c_1 \cos\omega t + c_2 \sin\omega t$ formuladan c_1, c_2 o‘zgarmaslarning biror tayin qiymatida hosil bo‘lishini ko‘rsating.

2. h balandlikdan v_0 boshlang‘ich tezlik bilan yuqoriga tik otilgan jismning harakat tenglamasini yozing. Mos boshlang‘ich masalani qo‘ying va uni yeching.

§ 5.2. Koshi masalasi yechimining mavjudligi va yagonaligi

Lipshits sharti. $f(x,y,p_1,...,p_{n-1})$ funksiya va $E \subset D(f) \subset \mathbb{R}^{1+n}$ to‘plam berilgan bo‘lsin. Agar shunday $L>0$ mavjud bo‘lib, ixtiyoriy $(x,y,p_1,...,p_{n-1}) \in E$ va $(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1}) \in E$ nuqtalar uchun

$$\left| f(x,y,p_1,...,p_{n-1}) - f(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1}) \right| \le$$
$$\le L\left(|y-\tilde{y}| + |p_1-\tilde{p}_1| + ... + |p_{n-1}-\tilde{p}_{n-1}| \right) \quad (5.2.1)$$

tengsizlik bajarilsa, $f(x,y,p_1,...,p_{n-1})$ funksiya E to‘plamda $y,p_1,...,p_{n-1}$ o‘zgaruvchilarga nisbatan **Lipshits shartini qanoatlantiradi** deyiladi.

Teorema (Lipshits shartining bajarilishi uchun yetarli shart). *Agar*

1. $G \subset \mathbb{R}^{1+n}$ *ochiq yoki yopiq sohaga ixtiyoriy* $(x,y,p_1,...,p_{n-1}) \in G$ *va* $(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1}) \in G$ *nuqtalar bilan birgalikda ularni tutashtiruvchi kesma*

$$\left\{ (x, y+\theta(\tilde{y}-y), p_1+\theta(\tilde{p}_1-p_1), ..., p_{n-1}+\theta(\tilde{p}_{n-1}-p_{n-1})) \,\middle|\, 0<\theta<1 \right\}$$

ham tegishli,

2. G sohada $f(x,y,p_1,...,p_{n-1})$ funksiya uzluksiz va G ning ichki nuqtalarida

$$\frac{\partial f}{\partial y},\frac{\partial f}{\partial p_1},...,\frac{\partial f}{\partial p_{n-1}}$$

xususiy hosilalar mavjud va chegaralangan bo'lsa, u holda f funksiya G sohada $y,p_1,...,p_{n-1}$ o'zgaruvchilarga nisbatan Lipshits shartini qanoatlantiradi.

🗝 Teoremaning 2- shartiga ko'ra shunday $L>0$ son topiladiki, uning uchun G ning ichki nuqtalarida

$$\left|\frac{\partial f}{\partial y}\right|\le L,\ \left|\frac{\partial f}{\partial p_1}\right|\le L,\ ...,\ \left|\frac{\partial f}{\partial p_{n-1}}\right|\le L \qquad (5.2.2)$$

tengsizliklar o'rinli bo'ladi. Ixtiyoriy $(x,y,p_1,...,p_{n-1})\in G$ va $(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1})\in G$ nuqtalar uchun teoremaning 1-sharti va chekli orttirmalar haqidagi Lagranj teoremasiga ko'ra

$$f(x,y,p_1,...,p_{n-1})-f(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1})=$$
$$=\frac{\partial f}{\partial y}(y-\tilde{y})+\frac{\partial f}{\partial p_1}(p_1-\tilde{p}_1)+...+\frac{\partial f}{\partial p_{n-1}}(p_{n-1}-\tilde{p}_{n-1});$$

bu yerda xususiy hosilalar biror

$$(x,y+\theta^*(\tilde{y}-y),p_1+\theta^*(\tilde{p}_1-p_1),...,p_{n-1}+\theta^*(\tilde{p}_{n-1}-p_{n-1}))\in G$$
$$(0<\theta^*<1)$$

nuqtada hisoblangan. Demak, (5.2.2) baholashlarga ko'ra

$$|f(x,y,p_1,...,p_{n-1})-f(x,\tilde{y},\tilde{p}_1,...,\tilde{p}_{n-1})|\le$$
$$\le\left|\frac{\partial f}{\partial y}\right||y-\tilde{y}|+\left|\frac{\partial f}{\partial p_1}\right||p_1-\tilde{p}_1|+...+\left|\frac{\partial f}{\partial p_{n-1}}\right||p_{n-1}-\tilde{p}_{n-1}|\le$$
$$\le L\big(|y-\tilde{y}|+|p_1-\tilde{p}_1|+...+|p_{n-1}-\tilde{p}_{n-1}|\big).$$ 👍

***Mavjudlik va yagonalik teoremasi* (MyaT).** Endi ushbu

$$\begin{cases} y^{(n)}=f(x,y,y',....,y^{(n-1)}),\\ y\big|_{x_0}=y_0,\ y'\big|_{x_0}=y_0',\ ...,\ y^{(n-1)}\big|_{x_0}=y_0^{(n-1)}. \end{cases} \qquad (5.2.3)$$

Koshi masalasini qaraylik. Bu yerda $x_0,y_0,y_0',...,y_0^{(n-1)}$ – berilgan sonlar va $(x_0,y_0,y_0',...,y_0^{(n-1)})\in D(f)$.

Teorema (MYaT). *Ushbu*

$$\Pi=\left\{(x,y,p_1,...,p_{n-1})\in\mathbb{R}^{1+n}\,\middle|\,|x-x_0|\le a,|y-y_0|\le b,|p_1-y_0'|\le b,...,\left|p_{n-1}-y_0^{(n-1)}\right|\le b\right\}$$

$(a>0,b>0)$ *parallelipipedda* $f(x,y,p_1,...,p_{n-1})$ *funksiya birato'la barcha argumetnlari bo'yicha uzluksiz hamda* $(y,p_1,...,p_{n-1})$ *o'zgaruvchilarga nisbatan Lipshits shartini ham qanoatlantirsin. U holda (5.2.3) Koshi masalasi biror* $|x-x_0|\le h$ $(h>0)$ *oraliqda* $y=y(x)$ *yagona yechimga ega bo'ladi.*

Bu teoremani keyinroq § 8.3 da isbotlaymiz.

Agar $f(x,y,p_1,...,p_{n-1})$ funksiya $G\subset\mathbb{R}^{1+n}$ sohada C^1 sinfga tegishli bo'lsa,

$$y^{(n)}=f(x,y,y',....,y^{(n-1)})$$

tenglamaning lokal umumiy yechimi $y=\varphi(x,c_1,c_2,\ldots,c_n)$ formula bilan beriladi, ya'ni u n dona ixtiyoriy o'zgarmasga bog'liq bo'ladi. Bu tasdiq MyaTdan $x_0$ni tayinlab, $y\big|_{x_0}=c_1,\ y'\big|_{x_0}=c_2,\ ...,\ y^{(n-1)}\big|_{x_0}=c_n$ qiymatlarni o'zgartirish natijasida hosil bo'ladi (Birinchi tartibli ODT uchun umumiy yechim haqida § 3.4 ga va sistemalar uchun § 8.3 ga qarang).

$n-$ tartibli **chiziqli differensial tenglama** deb

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x)$$

ko'rinishdagi tenglamaga aytiladi. Bu tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi teorema quyida keltirilgan.

Teorema. *Aytaylik, chiziqli tenglama uchun quyidagi boshlang'ich masala qo'yilgan bo'lsin:*

$$\begin{cases} y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x), \\ y\big|_{x_0}=y_0,\ y'\big|_{x_0}=y_0',\ ...,\ y^{(n-1)}\big|_{x_0}=y_0^{(n-1)}; \end{cases}$$

bu yerda $x_0\in I;\ y_0,y_0',...,y_0^{(n-1)}-$ *berilgan sonlar. Agar* $\{a_{n-1}(x),...a_1(x),a_0(x),g(x)\}\subset C(I)$ *bo'lsa, u holda qo'yilgan bu masalaning birato'la* I *oraliqda aniqlangan yechimi mavjud va yagonadir.*

Bu teoremani ham keyinroq § 8.3 da isbotlaymiz.

Masalalar

1. Lipshits shartidagi (5.2.1) tengsizlikni quyidagi tengsizlik bilan almashtirish mumkinligini ko‘rsating:

$$|f(x,y,p_1,...,p_{n-1})-f(x,\tilde y,\tilde p_1,...,\tilde p_{n-1})|\le$$
$$\le \tilde L\sqrt{|y-\tilde y|^2+|p_1-\tilde p_1|^2+...+|p_{n-1}-\tilde p_{n-1}|^2}\,.$$

2. Faraz qilaylik, $T=\{(t,x)\in\mathbb{R}^2\,|\,|t-t_0|\le a,\ |x-x_0|\le b\}$ $(a>0,b>0)$ va $f\in C(T)$ uchun

$$\begin{cases} x''=f(t,x), \\ x(t_0)=x_0,\ x'(t_0)=y_0 \end{cases} \quad (y_0-\text{ixtiyoriy son})$$

Koshi masalasi berilgan bo‘lsin. $x=\varphi(t)$ noma’lum funksiyaga nisbatan ushbu

$$\varphi(t)=x_0+(t-t_0)y_0+\int_{t_0}^{t}(t-s)f(s,\varphi(s))ds \qquad (*)$$

integral tenglamani tuzaylik.

1^0. Bu integral tenglamaning $x=\varphi(t)\in C(I)$, $t_0\in I$, yechimi berilgan Koshi masalasining yechimi ekanligini isbotlang.

2^0. Quyidagi funksional ketma-ketlikni qarang:

$$x_0(t)=x_0,$$
$$x_k(t)=x_0+(t-t_0)y_0+\int_{t_0}^{t}(t-s)f(s,x_{k-1}(s))ds,\ k=1,2,\dots .$$

$m=|y_0|+\frac{a}{2}\sup\limits_T|f(t,x)|$, $h=\min\{a,\frac{b}{m}\}$ deylik. Barcha $x_k(t)$ lar $|t-t_0|\le h$ oraliqda aniqlangan va uzluksiz ekanligini ko‘rsating.

3^0. Agar $x_k(t)$ yoki uning biror qismiy ketma-ketligi $\varphi(t)$ ga $[t_0-h,t_0+h]$ da tekis yaqinlashsa, $x=\varphi(t)$ funksiya (*) integral tenglamaning, demak, berilgan Koshi masalasining ham yechimi ekanligini isbotlang ($x_k(t)$ dan tekis yaqinlashuvchi qismiy ketma-ketlik ajratish mumkinligi Askoli-Arsela teoremasi yordamda isbotlanishi mumkin).

§ 5.3. Yuqori tartibli tenglamaning tartibini pasaytirish

1. Ushbu

$$y^{(n)}=f(x),\ f\in C(I), \qquad (5.3.1)$$

tenglamani n marta integrallash orqali uning tartibini pasaytirish va yechish mumkin. Bunda yechim n dona ixtiyoriy o‘zgarmasga bog‘liq bo‘lib chiqadi: $y=y(x,c_1,c_2,\dots,c_n)$.

2. Tenglamada noma’lum funksiya $y=y(x)$ va uning $y',y'',\dots,y^{(k-1)}$ hosilalari oshkor ko‘rinishda qatnashmagan:

$$F(x,y^{(k)},y^{(k+1)},\dots,y^{(n)})=0. \qquad (5.3.2)$$

Bu tenglamada

$$y^{(k)} = z \tag{5.3.3}$$

deb, yangi $z = z(x)$ noma'lum funksiyani kiritamiz. U holda z ga nisbatan $(n-k)$- tartibli ushbu

$$F(x, z, z', \ldots, z^{(n-k)}) = 0$$

tenglamani hosil qilamiz. Bu tenglamadan

$$z = z(x, c_1, c_2, \ldots, c_{n-k})$$

yechimni topamiz. Buni (5.3.3) belgilashga qo'yamiz va k marta integrallashni bajarib, y noma'lumni topamiz. Bunda yana k ta ixtiyoriy o'zgarmas paydo bo'ladi va y yechim n dona ixtiyoriy o'zgarmasga bog'liq bo'lib topiladi:

$$y = y(x, c_1, c_2, \ldots, c_{n-k}, \ldots, c_n).$$

Misol 1. Ushbu $y'' = y'^2$ tenglamani yeching.

🗝 Tenglamada y oshkor ko'rinishda qatnashmagan. $y' = z$ yangi o'zgaruvchini kiritamiz. Demak, $z' = z^2$. Bu tenglamaning $z = 0$ yechimi mavjudligi ravshan. Qolgan yechimlarni o'zgaruvchilarni ajratish yordamida topamiz:

$$\frac{dz}{z^2} = dx, \quad \int \frac{dz}{z^2} = \int dx, \quad -\frac{1}{z} = x + c_1, \quad z = -\frac{1}{x + c_1}, \quad c_1 \in \mathbb{R}.$$

Endi $y' = z$ belgilashga ko'ra dastlabki noma'lum y ga qaytamiz.

1) $y' = 0$ va 2) $y' = -\dfrac{1}{x + c_1}$ tenglamalardan

$$y = c_2 \text{ va } y = c_2 - \ln(x + c_1)$$

yechimlar majmuasini hosil qilamiz. 👍

3. Erkli o'zgaruvchi bevosita qatnashmagan tenglama ushbu

$$F(y, y', y'', \ldots, y^{(n)}) = 0 \tag{5.3.4}$$

ko'rinishda bo'ladi. U **avtonom tenglama** deb yuritiladi.

$y' = p$ yangi noma'lum funksiyani kiritamiz va erkli o'zgaruvchi sifatida y ni qabul qilamiz. $y'', y''' \ldots, y^{(n)}$ hosilalarni $p = p(y)$ va uning y bo'yicha hosilalari orqali ifodalab chiqamiz:

$$y'' = \frac{dy'}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p$$

$$y''' = \frac{dy''}{dx} = \frac{d}{dx}\left(\frac{dp}{dy}\cdot p\right) = \frac{d}{dy}\left(\frac{dp}{dy}\cdot p\right)\cdot\frac{dy}{dx} = \left(\frac{d^2p}{dy^2}p + \left(\frac{dp}{dy}\right)^2\right)p$$

$$y^{(n)} = \omega\left(p, \frac{dp}{dy}, ..., \frac{d^{n-1}p}{dy^{n-1}}\right)$$

Shuning uchun (5.3.4) tenglama ushbu

$$F\left(y, p, \frac{dp}{dy}\cdot p, ..., \omega\left(p, \frac{dp}{dy}, ..., \frac{d^{n-1}p}{dy^{n-1}}\right)\right) = 0 \tag{5.3.5}$$

ko‘rinishni oladi. Bu (5.3.5) tenglama esa $n-1$ tartiblidir. Agar (5.3.5) ni yechib,

$$p = \varphi(y, c_1, ..., c_{n-1})$$

yechimni topsak, dastlabki nomalum y ga qaytib, ushbu

$$y' = \varphi(y, c_1, ..., c_{n-1})$$

birinchi tartibli (o‘zgaruvchilari ajraladigan) tenglamani hosil qilamiz.

Biz yuqorida y noma’lum funksiyani erkli o‘zgaruvchi sifatida qaradik, bunda $y = \text{const}$ o‘zgarmas yechimlarni yo‘qotishimiz mumkin. Shuning uchun (5.3.4) tenglamaga $y = b$ qo‘yib, hosil bo‘lgan

$$F(b, 0, ..., 0) = 0$$

tenglamani yechib, $b = b_i$ ildizlarni topsak, u holda (5.3.4) tenglamaning $y = b_i$ ko‘rinishdagi o‘zgarmas yechimlariga ega bo‘lamiz.

Misol 2. Ushbu

$$yy'' + y'^2 + 1 = 0$$

tenglamada x erkli o‘zgaruvchi oshkor ko‘rinishda qatnashmagan. U – avtonom tenglama. Uni yechish uchun $y' = p$, $p = p(y)$ noma’lum funksiyani kiritamiz. U holda $y'' = \frac{dy'}{dx} = \frac{dp}{dy}\cdot p$ va berilgan tenglama $yp\frac{dp}{dy} + p^2 + 1 = 0$ ko‘rinishga keladi. Bu birinchi tartbli tenglama osongina yechiladi: $(p^2+1)y^2 = c_1$. Bundan $y' = p$ belgilashga ko‘ra $y^2y'^2 + y^2 = c_1$ tenglamaga kelamiz va uni yechib topamiz:

$y^2+(x-c_2)^2=c_1$. Dastlabki tenglamaning, ravshanki, $y=\text{const}$ ko'rinishdagi yechimi mavjud emas. ☝

4. Noma'lum funksiya va uning hosilalariga nisbatan bir jinsli tenglama. Agar ushbu

$$F(x,y,y'...,y^{(n)})=0 \qquad (5.3.6)$$

tenglamaning chap tomonidagi funksiya $y,y',...,y^{(n)}$ o'zgaruvchilarni λ ga ko'paytirganda u λ ning biror $m-$ darajasiga ko'paysa, ya'ni

$$F(x,\lambda y,\lambda y',...,\lambda y^{(n)})=\lambda^m F(x,y,y',...,y^{(n)}) \qquad (5.3.7)$$

bo'lsa, u holda (5.3.6) tenglama $y,y',...,y^{(n)}$ larga nisbatan **bir jinsli tenglama** deyiladi. Bunday tenglamaning tartibini kamaytirish uchun $y'=zy$ deb, yangi $z=z(x)$ noma'lum funksiyani kiritamiz.

Quyidagilarga egamiz:

$$y'=yz, \quad y''=y'z+yz'=(yz)z+yz'=y(z^2+z'),$$

$$y'''=y(z^3+3z'+z''), \; ..., \; y^{(n)}=y\cdot\omega(z,z',...,z^{(n-1)}).$$

Endi bu munosabatlarni (5.3.6) ga qo'yib, (5.3.7) shartni e'tiborga olsak, quyidagi tenglikka kelamiz:

$$y^m F(x,1,z,z^2+z',...,\omega(z,z',...,z^{(n-1)}))=0$$

Oxirgi tenglikni, y^m ga qisqartirib, z ga nisbatan $(n-1)$- tartibli tenglamani hosil qilamiz. Agar hosil bo'lgan tenglamaning ushbu

$$z=\varphi(x,c_1,c_2,...,c_{n-1})$$

yechimini topa olsak, bu yerda z ni $\frac{y'}{y}$ bilan almashtirib,

$$\frac{y'}{y}=\varphi(x,c_1,c_2,...,c_{n-1})$$

tenglamaga ega bo'lamiz. Nihoyat, oxirgi tenglamadan

$$y=c_n\exp\left(\int\varphi(x,c_1,c_2,...,c_{n-1})dx\right)$$

yechimni hosil qilamiz. Bu yechimlar oilasi ($c_n=0$ bo'lganda) $y=0$ yechimni o'z ichiga oladi, ya'ni yuqorida $m>0$ bo'lganda y^m ga qisqartirishda yechim yo'qolmaydi.

Misol 3. Ushbu

$$yy''-xy'^2+yy'=0$$

tenglamani yeching.

🗝 Berilgan tenglama uchun

$$F(x,y,y',y'') = yy'' - xy'^2 + yy' \text{ va}$$
$$F(x,\lambda y,\lambda y',\lambda y'') = \lambda^2(yy'' - xy'^2 + yy') = \lambda^2 F(x,y,y',y'').$$
Demak, berilgan tenglama y, y', y'' larga nisbatan bir jinsli ($m = 2$). Tenglama tartibini pasaytirish uchun unda $y' = zy$ va $y'' = y(z' + z^2)$ almashtirish bajaramiz. U holda berilgan tenglama quyidagi ko‘rinishni oladi:
$$y^2(z' + z^2) - xz^2y^2 + y^2z = 0 \Rightarrow z' + z^2 - xz^2 + z = 0.$$
Oxirgi tenglamaning har ikkala tomonini z^2ga bo‘lamiz va $1/z = u$ deymiz. Natijada ushbu $u' - u = 1 - x$ tenglamani hosil qilamiz. Bu chiziqli tenglama osongina yechiladi: $u = x + c_1e^x$. Bu yerda $u = 1/z$, $z = y'/y$ deb, dastlabki noma’lumga qaytamiz va
$$\frac{y'}{y} = \frac{1}{x + c_1e^x} \Rightarrow y = c_2\exp\left(\int \frac{dx}{x + c_1e^x}\right)$$
yechimni topamiz (yo‘qolgan $y = 0$ yechim $c_2 = 0$ da hosil bo‘ladi). ☝

5. Umumlashgan bir jinsli tenglama.

Ushbu
$$F(x,y,y'...,y^{(n)}) = 0 \qquad (5.3.8)$$
tenglama berilgan bo‘lib, $x, y, y'..., y^{(n)}$ larni mos ravishda $\lambda, \lambda^k, \lambda^{(k-1)}, ..., \lambda^{k-n}$larga ko‘paytirilganda tenglamaning chap tomoni uchun ushbu
$$F(\lambda x, \lambda^k y, \lambda^{(k-1)}y', ..., \lambda^{k-n}y^{(n)}) = \lambda^m F(x,y,y',...,y^{(n)}) \qquad (5.3.9)$$
shart bajarilsa, u holda bunday (5.3.8) tenglama **umumlashgan bir jinsli tenglama** deyiladi. Bu turdagi tenglamalarni yechish uchun x va y o‘rniga
$$x = e^t,\ \ y = ze^{kt}\ \ (x > 0) \qquad (5.3.10_0)$$
deb yangi t va $z = z(t)$ o‘zgaruvchilarni kiritamiz (bu almashtirish $x > 0$ bo‘lganda qo‘llaniladi, $x < 0$ bo‘lganda esa $x = -e^t$ $y = ze^{kt}$ almashtirishdan foydalanish kerak). $y$ning x bo‘yicha hosilalarini yangi nomalum funksiya z ning t argument bo‘yicha hosilasi orqali ifodalaymiz. Murakkab funksiyaning hosilasini hisoblash qoidasiga ko‘ra
$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}} = \frac{dy}{dt}\cdot\frac{1}{e^t}, \text{ yoki } y' = \frac{dy}{dt}e^{-t}. \qquad (5.3.11)$$

(5.3.10$_0$) tengliklardagi ikkinchi formulani t bo‘yicha differensiallab,

$$\frac{dy}{dt}=\left(\frac{dz}{dt}+kz\right)e^{kt}$$

tenglikka ega bo‘lamiz. Buni (5.3.11) ga qo‘yib,

$$y'=\left(\frac{dz}{dt}+kz\right)e^{(k-1)t} \qquad (5.3.10_1)$$

munosabatni hosil qilamiz. Biz y ning x bo‘yicha hosilasini z ning t bo‘yicha hosilasi orqali ifodalovchi munosabatga ega bo‘ldik.

Xuddi shu yo‘sinda ishni davom ettirib, y ning x bo‘yicha yuqori tartibli hosilalarini ham z ning t bo‘yicha hosilalari orqali ifodalab chiqamiz:

$$y''=\frac{dy'}{dx}=\frac{dy'}{dt}e^{-t}=\left(\frac{d^2z}{dt^2}+(2k-1)\frac{dz}{dt}+k(k-1)z\right)e^{(k-2)t}, \qquad (5.3.10_2)$$

$$y'''=\frac{dy''}{dx}=\frac{dy''}{dt}e^{-t}=\left(\frac{d^3z}{dt^3}+(3k-3)\frac{d^2t}{dt^2}+\right.$$

$$\left.+\big(k(k-1)+(k-2)(2k-1)\big)\frac{dz}{dt}+k(k-1)(k-2)z\right)e^{(k-3)t}, \qquad 5.3.10_3)$$

va, nihoyat,

$$y^{(n)}=\omega\left(z,\frac{dz}{dt},...,\frac{d^nz}{dt^n}\right)e^{(k-n)t} \; . \qquad (5.3.10_n)$$

Endi (5.3.8) tenglamada (5.3.10$_0$), (5.3.10$_1$), (5.3.10$_2$), ..., (5.3.10$_n$) almashtirishlarni bajarib, ushbu

$$F\left(e^t,ze^{kt},\left(\frac{dz}{dt}+kz\right)e^{(k-1)t},...,\omega\left(z,\frac{dz}{dt},...,\frac{d^nz}{dt^n}\right)e^{(k-n)t}\right)=0$$

tenglamaga kelamiz. (5.3.9) tenglikdagi t o‘rniga e^t ni qo‘yib, e^{mt} ni F funksiya ishorasi oldiga chiqarib, va unga qisqartirib,

$$F\left(1,z,\frac{dz}{dt}+kz,...,\omega\left(z,\frac{dz}{dt},...,\frac{d^nz}{dt^n}\right)\right)=0$$

n-tartibli tenglamani hosil qilamiz. Bu tenglama erkli o‘zgaruvchi t ni oshkor ko‘rinishda o‘z ichiga olmagan (avtonom tenglama); shuning uchun 3- bandga ko‘ra uning tartibi bittaga kamayadi.

Misol 4. Ushbu

$$xyy''-yy'-x^3=0$$

tenglamaning tartibini pasaytiring.

🗝 Qaralayotgan tenglama uchun $F(x, y, y', y'') = xyy'' - yy' - x^3\, x$. Demak,

$$F(\lambda x, \lambda^k y, \lambda^{k-1} y', \lambda^{k-2} y'') = \lambda^{1+k+k-2} xyy'' - \lambda^{k+k-1} yy' - \lambda^3 x^3 =$$
$$= \lambda^{2k-1} xyy'' - \lambda^{2k-1} yy' - \lambda^3 x^3.$$

va ushbu $2k-1 = 2k-1 = 3$ shartlar bajarilganda, ya'ni $k=2$ bo'lganda (5.3.9) umumlashgan bir jinslilik sharti bajariladi ($m=3$). Shuning uchun berilgan umumlashgan bir jinsli tenglamada (5.3.10$_0$) $x = e^t$, $y = ze^{2t}$ $(x>0)$ almashtirishni bajarib, uni erkli o'zgaruvchi bevosita qatnashmagan ko'rinishga olib kelish mumkin. Buning uchun y', y'' larni z yangi noma'lum funksiya, t erkli o'zgaruvchi va z ning t bo'yicha hosilalari orqali ifodalaymiz:

$$y' = \frac{dy}{dx} = \frac{dy}{dt} e^{-t} = \frac{d(ze^{2t})}{dt} e^{-t} = (z' + 2z)e^t,$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} e^{-t} = \frac{d\big((z'+2z)e^t\big)}{dt} e^{-t} = \big((z''+2z')e^t + (z'+2z)e^t\big)e^{-t} =$$
$$= z'' + 3z' + 2z.$$

Endi berilgan tenglamada zarur almashtirishlarni bajaramiz:

$$e^t ze^{2t}(z'' + 3z' + 2z) - ze^{2t}(z' + 2z)e^t - e^{3t} = 0,$$
$$zz'' + 2zz' - 1 = 0.$$

Oxirgi tenglamada erkli o'zgaruvchi t oshkor ko'rinishda qatnashmagan. Uning tartibini pasaytirish uchun yangi noma'lum funksiya sifatida $p = z'$ni olib, z ni erkli o'zgaruvchi sifatida qaraymiz, $p = p(z)$. U holda $z'' = \dfrac{dz'}{dt} = \dfrac{dz'}{dz} \cdot \dfrac{dz}{dt} = \dfrac{dp}{dz} \cdot p$ bo'lgani uchun ushbu

$$z\frac{dp}{dz} \cdot p + 2zp - 1 = 0$$

birinchi tartibli tenglamaga kelamiz. 👍

6. Chap tomoni to'la hosiladan iborat bo'lgan tenglama.

Faraz qilaylik,

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{5.3.17}$$

tenglamaning **chap tomoni** $x, y, y', ..., y^{(n-1)}$ o‘zgaruvchilarning funksiyasi bo‘lgan biror $\Phi(x, y, y', ..., y^{(n-1)})$ ning x bo‘yicha **to‘la hosilasidan iborat** bo‘lsin, ya’ni

$$F(x, y, y', ..., y^{(n)}) = \frac{d}{dx}\Phi(x, y, y', ..., y^{(n-1)}) \qquad (5.3.18)$$

yoki

$$F(x, y, y', ..., y^{(n)}) = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}y' + \frac{\partial\Phi}{\partial y'}y'' + ... + \frac{\partial\Phi}{\partial y^{(n-1)}}y^{(n)} \qquad (5.3.19)$$

tenglik $x, y, y', ..., y^{(n)}$ o‘zgaruvchilarning barcha joiz qiymatlarida aynan bajarilsin. U holda (5.3.17) tenglamaning tartibi bittaga kamayadi va quyidagi ko‘rinishni oladi:

$$\Phi(x, y, y', ..., y^{(n-1)}) = c_1$$

Agar (5.3.17) tenglamaning chap tomoni to‘la hosiladan iborat bo‘lmasa, ba’zi hollarda shunday $\mu = \mu(x, y, y', ..., y^{(n-1)})$ funksiyani topish mumkin bo‘ladiki, (5.3.17) tenglamaning har ikkala tomonini shu μ ga ko‘paytirib, to‘la hosilali tenglama hosil qilinadi. Bu μ funksiya **integrallovchi ko‘paytuvchi** deb ataladi.

Masalan, agar berilgan tenglama

$$y'' = \frac{\dfrac{\partial G(x,y)}{\partial x} + \dfrac{\partial G(x,y)}{\partial y}y'}{\dfrac{dF(y')}{dy'}} \qquad (G(x,y), F(y') - \text{berilgan funksiyalar})$$

ko‘rinishda bo‘lsa, $\mu = \dfrac{dF(y')}{dy'}$ integrallovchi ko‘paytuvchi yordamida u quyidagi birinchi tartibili differensial tenglamaga keltiriladi: $F(y') = G(x, y) + c_1$.

Misol 5. Ushbu

$$xy'' - y' - x^2yy' = 0$$

tenglamani yechaylik.

🗝 Berilgan tenglamaning har ikkala tomonini $\mu = \dfrac{1}{x^2}$ integrallovchi ko‘paytuvchiga ko‘paytirib, quyidagilarni hosil qilamiz:

$$\frac{xy''-y'}{x^2}-yy'=0,\ \left(\frac{y'}{x}-\frac{y^2}{2}\right)'=0,\ \frac{y'}{x}-\frac{y^2}{2}=c_1.$$

Oxirgi tenglama o‘zgaruvchilarini ajratib yechiladi. 👍

Masalalar

Tenglamalarni yeching:

1. $2yy''-y'^2-y'^3=0$. **2.** $x^2yy''=(y+xy')^2$.

3. $y''(1-y')=e^{y'}$. **4.** $y''=xy'+y+2x$.

MODUL 6. YUQORI TARTIBLI CHIZIQLI DIFFERENSIAL TENGLAMALAR

§ 6.1. $n-$tartibli chiziqli differensial tenglamaning umumiy xossalari

Biz bu paragrafda ushbu

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x) \qquad (6.1.1)$$

$n-$tartibli chiziqli differensial tenglama yechimlarining umumiy xossalarini o‘rganamiz; bunda $y=y(x)-$ noma’lum funksiya, berilgan $a_{n-1}(x),...,a_1(x),a_0(x)$ koeffitsientlar va $g(x)$ ozod had (o‘ng tomon) biror I oraliqda aniqlangan va uzluksiz deb hisoblanadi, ya’ni $\{a_{n-1}(x),...a_1(x),a_0(x),g(x)\}\subset C(I)$.

Quyidagi Koshi shartlarini (boshlang‘ich shartlarni) qaraylik:

$$y\Big|_{x_0}=y_0,\ y'\Big|_{x_0}=y'_0,...,y^{(n-1)}\Big|_{x_0}=y_0^{(n-1)}; \qquad (6.1.2)$$

bunda $x_0\in I$ va $y_0,\ y'_0,...,y_0^{(n-1)}$ berilgan ixtiyoriy sonlar.

Yuqorida e’tirof etilganidek, aytilgan shartlarda (6.1.1),(6.1.2) Koshi masalasi birato‘la I oraliqda aniqlangan yechimga ega va bunday yechim yagona.

(6.1.1) tenglamaning ozod hadi o‘rniga 0 qo‘yib, (6.1.1) ga **mos bir jinsli tenglama**ni hosil qilamiz:

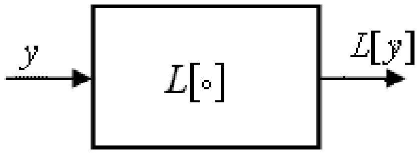
$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=0. \qquad (6.1.1_0)$$

Ravshanki, bu bir jinsli tenglama $y=0$ trivial yechimga ega.

Qulaylik uchun ushbu

$$L_n[y]\overset{def}{=}y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y \qquad (6.1.3)$$

$n-$tartibli differensial operatorni kiritaylik. Qisqalik uchun ba’zan $L_n[y]$ o‘rniga $L[y]$ yozamiz. Biz bu operatorni I oraliqda n marta uzluksiz differensiallanuvchi funksiyalarga ta’sir ettiramiz, ya’ni $L[y]$ (6.1.3) operatorni $y=y(x)\in C^n(I)$ funksiyalarda aniqlangan deb hisoblaymiz: $L[\circ]:C^n(I)\to C(I)$. $L[\circ]$ operator kirish funksiyasi y ga ko‘ra chiqish funksiyasi $L[y]$ ni hisoblaydi (aniqlaydi):



Jumla. *(6.1.3) differensial operator chiziqlidir, ya'ni*

a) $\forall\{y_1, y_2\} \subset C^n(I) \quad L[y_1 + y_2] = L[y_1] + L[y_2]$ *(additivlik)*

b) $\forall\lambda \in \mathbb{R} \quad \forall y \in C^n(I) \quad L[\lambda \cdot y] = \lambda \cdot L[y]$ *(bir jinslilik)*

xossalar o'rinli.

🗝 Isboti hosilaning chiziqlilik xossasidan bevosita kelib chiqadi. 👍

Natija. *Agar* $\{y_1, y_2, ..., y_k\} \subset C^n(I)$ va $\{c_1, ..., c_k\} \subset \mathbb{R}$ *bo'lsa, ushbu*

$$L\left[\sum_{j=1}^{k} c_j y_j\right] = \sum_{j=1}^{k} c_j L\left[y_j\right]$$

tenglik o'rinli.

$y_1, y_2, ..., y_k$ **funksiyalarning chiziqli kombinatsiyasi** deb ushbu $c_1 y_1 + c_2 y_2 + ... + c_k y_k$ yig'indiga aytiladi, bunda $\{c_1, ..., c_k\} \subset \mathbb{R}$.

Teorema. *(6.1.1$_0$) chiziqli bir jinsli differensial tenglama yechimlarining chiziqli kombinatsiyasi yana shu tenglamaning yechimidir.*

🗝 $y_1 y_2, ... y_k$ – (6.1.1$_0$) ning yechimlari bo'lsin: $L[y_j] = 0, \ j = \overline{1, k}.$ Biz ularning $\sum_{j=1}^{k} c_j y_j$ chiziqli kombinatsiya ham (6.1.1$_0$) tenglamani qanoatlantirishini asoslashimiz kerak. Bu esa $L[\circ]$ operatorning chiziqliligi va berilganga ko'ra ravshan:

$$L\left[\sum_{j=1}^{k} c_j y_j\right] = \sum_{j=1}^{k} c_j L[y_j] = \sum_{j=0}^{k} c_j \cdot 0 = 0.$$ 👍

(6.1.1$_0$) bir jinsli tenglamaning barcha yechimlari to'plamini V_n bilan belgilaylik:

$$V_n = \left\{ y \in C^n(I) \,\middle|\, L_n[y] = 0 \right\}. \tag{6.1.4}$$

V_n da funksiyalarni qo'shish va funksiyani songa ko'paytirish amallari odatdagidek (nuqtama-nuqta) kiritilgan deb hisoblaymiz.

Agar $\{y_1, y_2\} \subset V_n$ bo'lsa, $y_1 + y_2 \in V_n$, chunki (6.1.1$_0$) ning yechimlari yig'indisi yana yechim. Shunga o'xshash $y \in V_n \Rightarrow \lambda y \in V_n \ (\lambda \in \mathbb{R})$, chunki (6.1.1$_0$) ning yechimini o'zgarmasga ko'paytirishdan yana yechim hosil bo'ladi. Bular V_n to'plam kiritilgan amallarga nisbatan yopiqligini anglatadi.

Endi ravshanki, (1_0) tenglamaning V_n yechimlari to‘plami chiziqli (vektor) fazoni tashkil etadi. Bunda nol-vektor tenglamaning $y=0$ trivial yechimidan iborat, $y \in V_n$ vektorga qarama-qarshi vektor $(-1)y = -y \in V_n$.

Masalalar

1. Ushbu $y = x^2$ funksiya biror $(-a,a)\ (a>0)$ oraliqda

$$y'' + p(x)y' + q(x)y = 0, \{p,q\} \subset C\left((-a,a)\right),$$

ko‘rinishdagi tenglamaning yechimi bo‘la oladimi? $y = 1-\cos x$ funksiyachi ?

2. $x_0 \in I$ nuqtani tayinlaylik.

$$V_n{}^0 = \left\{ y \in C^n(I) \mid L[y]=0, y(x_0)=0 \right\} = \left\{ y \in V_n \middle| y(x_0)=0 \right\} \subset V_n$$

to‘plam V_n ning qismfazosimi?

3. $y'' + p(x)y' + q(x)y = 0$ tenglamaning p va q koeffitsientlariga qanday yetarli va zaruriy shart qo‘yilsa u

a) $y_1(x)$ va $xy_1(x)$,

b) $y_1(x)$ va $1/y_1(x)$

yechimlarga ega bo‘ladi?

§ 6.2. Chiziqli erkli va chiziqli bog‘langan funksiyalar. Vronskian

Biror I oraliqda aniqlangan $y_1 = y_1(x),\ y_2 = y_2(x),\ ...,\ y_n = y_n(x)$ funksiyalar berilgan bo‘lsin. Bu funksialarning ushbu

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x)$$

chiziqli kombinatsiyasini qaraylik; bu yerda $\lambda_1, \lambda_2, ..., \lambda_n$ – o‘zgarmas sonlar, ular **chiziqli kombinatsiya koeffitsientlari** deb ataladi. Barcha koeffitsientlari nolga teng bo‘lgan $(\lambda_1 = \lambda_2 = = ... = \lambda_n = 0)$ chiziqli kombinatsiya **trivial chiziqli kombinatsiya** deyiladi. Ravshanki, trivial chiziqli kombinatsiya I oraliqda aynan nolga teng. Agar berilgan funksiyalarning biror ***notrivial*** chiziqli kombinatsiyasi I oraliqda aynan nolga teng, ya’ni

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x) = 0, x \in I,$$

bunda $|\lambda_1| + |\lambda_2| + ... + |\lambda_n| \neq 0,$ bo‘lsa,

bu $y_1(x), y_2(x), ..., y_n(x)$ funksiyalar (funksiyalar sistemasi) I oraliqda **chiziqli bog‘langan funksiyalar** deb ataladi. Aks holda esa, ya'ni $y_1(x), y_2(x), ..., y_n(x)$ funksiyalarning faqat trivial chiziqli kombinatsiyasigina I oraliqda aynan nolga teng bo‘lsa, bu funksiyalar I oraliqda **chiziqli erkli funksiyalar** deb yuritiladi. Shunday qilib, $y_1(x), y_2(x), ..., y_n(x)$ funksiyalarning I oraliqda chiziqli erkliligi quyidagi implikatsiyaning rostligini anglatadi:

$$\sum_{j=1}^{n} \lambda_j y_j(x) = 0, x \in I \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0 .$$

Tushunarliki, $y_1(x), y_2(x), ..., y_n(x)$ funksiyalarning chiziqli erkliligi bu funksiyalarning yozilish tartibiga bog‘liq emas.

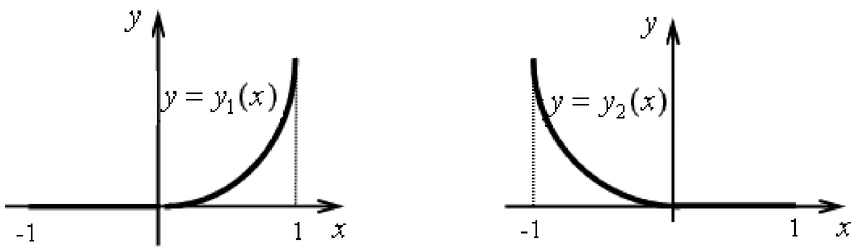
Misol 1. Ushbu $1, x, x^2, x^3$ funksiyalarni chiziqli erklilikka tekshiring.

🗝 Faraz qilaylik, bu funksiyalar biror I oraliqda chiziqli bog‘langan bo‘lsin. U holda ularning biror notrivial chiziqli kombinatsiyasi I oraliqning har bir nuqtasida nolga aylanadi, ya'ni hammasi bir vaqtda nolga teng bo‘lmagan biror $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ $(|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4| \neq 0)$ sonlar uchun $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 = 0, \ x \in I,$ bo‘ladi. Lekin bu holda $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ ko‘phad ko‘pi bilan 3-darajali va, demak, u ko‘pi bilan 3ta nuqtada nolga aylanishi mumkin. har qanday I oraliqda esa nuqtalar cheksiz ko‘p. Shuning uchun ham $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ (bunda $|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4| \neq 0$) ko‘phad hech qanday oraliqda aynan nolga teng bo‘la olmaydi. Hosil bo‘lgan ziddiyat farazimizning noto‘g‘ri ekanligini ko‘rsatadi. Demak, berilgan funksiyalar har qanday oraliqda chiziqli erkli. 👍

Misol 2. Ushbu

$$y_1(x) = \begin{cases} x^2, \text{ agar } x \geq 0 \text{ bo'lsa,} \\ 0, \text{ agar } x < 0 \text{ bo'lsa} \end{cases} \quad \text{va} \quad y_2(x) = \begin{cases} 0, \text{ agar } x \geq 0 \text{ bo'lsa,} \\ x^2, \text{ agar } x < 0 \text{ bo'lsa} \end{cases}$$

funksiyalar $I = [-1,1]$ oraliqda chiziqli erkli (2.1-rasm.).



2.1-rasm.

🗝 Haqiqatan ham, $\lambda_1 y_1(x)+\lambda_2 y_2(x)=0$, $x\in[-1,1]$, deylik. U holda oxirgi tenglikda $x=-1$ va $x=1$deb topamiz:

$\lambda_1 y_1(-1)+\lambda_2 y_2(-1)=\lambda_1\cdot 0+\lambda_2\cdot 1=\lambda_2=0$, $\lambda_1 y_1(1)+\lambda_2 y_2(1)=\lambda_1=0$, ya'ni $\lambda_1=\lambda_2=0$. Demak, qaralayotgan funksiyalarning faqat trivial chiziqli kombinatsiyasigina $I=[-1,1]$ oraliqda nolga teng. 👍

Misol 3. Ushbu $1, \sin^2 x, \cos^2 x$ funksiyalar $(-\infty,+\infty)$ oraliqda chiziqli bog'langan, chunki ularning ushbu $(-1)\cdot 1+1\cdot\sin^2 x+1\cdot\cos^2 x$ notrivial chiziqli kombinatsiyasi, ma'lumki, aynan nolga teng: $-1+\sin^2 x+\cos^2 x=0$.

Jumla 1. Berilgan funksiyalar chiziqli bog'langan bo'lishi uchun ularning birortasi qolganlarining chiziqli kombinatsiyasidan iborat bo'lishi yetarli va zarurdir.

Mustaqil isbotlang (chiziqli algebrani eslang).

Faraz qilaylik, berilgan ushbu $y_1=y_1(x), y_2=y_2(x), ..., y_n=y_n(x)$ funksiyalar I oraliqda $n-1$ marta differensiallanuvchi bo'lsin. Ularning **vronskiani (Vronskiy determinanti**) deb ushbu

$$W[y_1,y_2,...,y_n]=\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

determinantga aytiladi.

Teorema 1. *Agar* $y_1(x), y_2(x), ..., y_n(x)$ *funksiyalar* I *oraliqda* $n-1$ *marta differensiallanuvchi va chiziqli bog'langan bo'lsa, ularning vronskiani shu* I *oraliqda aynan nolga teng*.

🗝 Teoremaning shartiga ko'ra hammasi bir vaqtda nolga teng bo'lmagan $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ sonlar uchun I oraliqda ushbu

$$\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0, x\in I,$$

ayniyat o'rinli. Bu ayniyatni ketma-ket $n-1$ marta differensiallab, quyidagi ayniyatlar sistemasini hosil qilamiz:

$$\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0$$
$$\lambda_1 y_1'(x)+\lambda_2 y_2'(x)+...+\lambda_n y_n'(x)=0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\lambda_1 y_1^{(n-1)}(x)+\lambda_2 y_2^{(n-1)}(x)+...+\lambda_n y_n^{(n-1)}(x)=0$$

yoki vektor ko‘rinishda

$$\lambda_1\begin{pmatrix} y_1(x) \\ y_1'(x) \\ \vdots \\ y_1^{(n-1)}(x) \end{pmatrix}+\lambda_2\begin{pmatrix} y_2(x) \\ y_2'(x) \\ \vdots \\ y_2^{(n-1)}(x) \end{pmatrix}+\ldots+\lambda_n\begin{pmatrix} y_n(x) \\ y_n'(x) \\ \vdots \\ y_n^{(n-1)}(x) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Oxirgi vektor tenglik berilgan funksiyalar Vronskiy determinantining ustunlari orasida ixtiyoriy $x\in I$ nuqtada chiziqli bog‘lanish mavjudligini anglatadi $(|\lambda_1|+|\lambda_2|+\ldots+|\lambda_n|\neq 0)$. Demak, algebradan ma’lum teoremaga ko‘ra berilgan funksiyalarning vronskiani har bir $x\in I$ nuqtada nolga teng. ☝

Natija. *Agar I oraliqda $n-1$ marta differensiallanuvchi $y_1(x),\, y_2(x),\ldots, y_n(x)$ funksiyalarning $W(x)=W[y_1, y_2,\ldots, y_n]$ vronskiani I ning biror nuqtasida noldan farqli bo‘lsa, u holda bu funksiyalar shu I oraliqda chiziqli erkli bo‘ladi.*

Bu yerda shuni e’tirof etaylikki, teorema 1 va uning natijasining teskarisi o‘rinli emas, ya’ni I oraliqda $W(x)=W[y_1, y_2,\ldots, y_n]=0$ ekanligidan $y_1, y_2,\ldots, y_n$ funksiyalarning I oraliqda chiziqli bog‘langanligi kelib chiqmaydi. Bu tasdiqni quyidagi misol asoslaydi.

Misol 3. Ushbu

$$y_1(x)=\begin{cases} x^2, \text{ agar } x\geq 0 \text{ bo'lsa,} \\ 0, \text{ agar } x<0 \text{ bo'lsa} \end{cases} \quad \text{va} \quad y_2(x)=\begin{cases} 0, \text{ agar } x\geq 0 \text{ bo'lsa,} \\ x^2, \text{ agar } x<0 \text{ bo'lsa} \end{cases}$$

funksiyalarni qaraylik. Ravshanki, ular $C^1([-1,1])$ sinfga tegishli. Osongina tekshirib ko‘rish mumkinki, $[-1,1]$ oraliqda $W(x)=W[y_1, y_2]=0$. Lekin biz bu funksiyalarning $[-1,1]$ oraliqda chiziqli erkli ekanligini yuqorida misol 2 da ko‘rsatgan edik. ☝

Agar qaralayotgan funksiyalar biror chiziqli (uzluksiz koeffitsientli) bir jinsli differensial tenglamaning yechimlari bo‘lsa, u holda bu funksiyalar vronskianining nolga tengligidan ularning chiziqli bog‘langan ekanligi kelib chiqadi. Bu tasdiq quyidagi teoremaning bir qismidir.

Teorema 2. *Aytaylik, $y_1(x),\, y_2(x),\ldots, y_n(x)$ funksiyalar $n-$tartibli chiziqli (I oraliqda uzluksiz koeffitsientli) bir jinsli differensial tenglama $L_n[y]=0$ ning yechimlari bo‘lsin. U holda bu*

$y_1(x), y_2(x), ..., y_n(x)$ yechimlarning (I da) chiziqli bog‘langan bo‘lishi uchun $W(x)=W[y_1, y_2, ..., y_n]$ vronskianning I ning biror nuqtasida nolga teng bo‘lishi yetarli va zarurdir.

🗝 **Yetarliligi.** Faraz qilaylik, $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning vronskiani $x_0 \in I$ nuqtada nolga teng bo‘lsin, $W(x_0)=0$. Demak, algebradan ma’lum teoremaga ko‘ra $\lambda_1, \lambda_2, ..., \lambda_n$ noma’lumlarga nisbatan

$$\lambda_1 y_1(x_0)+\lambda_2 y_2(x_0)+...+\lambda_n y_n(x_0)=0$$
$$\lambda_1 y_1'(x_0)+\lambda_2 y_2'(x_0)+...+\lambda_n y_n'(x_0)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_1 y_1^{(n-1)}(x_0)+\lambda_2 y_2^{(n-1)}(x_0)+...+\lambda_n y_n^{(n-1)}(x_0)=0$$

chiziqli bir jinsli algebraik sistema $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ notrivial yechimga ega. Endi ana shu notrivial yechimlarning birortasi $\lambda_1, \lambda_2, ..., \lambda_n$ ni olib, bu sonlarga ko‘ra $y(x)=\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)$ funksiyani tuzaylik. Bu funksiya yechimlarning chiziqli kombinatsiyasi sifatida $L_n[y]=0$ tenglamaning yechimi va $\lambda_1, \lambda_2, ..., \lambda_n$ larning tanlab olinishiga ko‘ra $y(x_0)=0, y'(x_0)=0, ..., y^{(n-1)}(x_0)=0$. Yechimning yagonalik xossasiga ko‘ra $y(x)\equiv 0$, ya’ni tanlangan $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ lar uchun I oraliqda $\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0$. Bu $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning chiziqli bog‘langan ekanligini anglatadi.

Zarurligi. $y_1(x), y_2(x), ..., y_n(x)$ yechimlar chiziqli bog‘langan bo‘lsin. Bu holda ularning vronskiani teorema 1ga ko‘ra barcha nuqtalarda nolga teng. 👍

Isbotlangan bu teoremadan quyidagi teorema bevosita kelib chiqadi.

Teorema 3. *Aytaylik, $y_1(x), y_2(x), ..., y_n(x)$ funksiyalar n–tartibli chiziqli bir jinsli $L_n[y]=0$ differensial tenglamaning I oraliqda aniqlangan yechimlari bo‘lsin. U holda $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning vronskiani yo I oraliqda nolga aylanmaydi va bu yechimlar chiziqli erkli, yo yechimlarning vronskiani I oraliqda aynan nolga teng va bu yechimlar chiziqli bog‘liq bo‘ladi.*

🗝 Faraz qilaylik, $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning $W(x) = W[y_1, y_2, ..., y_n]$ vronskiani I oraliqda nolga aylanmasin. U holda bu yechimlar chiziqli erkli bo‘ladi, chunki, agar ular chiziqli bog‘langan bo‘lganda edi, teorema 2 ga ko‘ra $W(x)$ vronskian farazimizga zid ravishda aynan nolga teng bo‘lardi.

Endi teskarisini faraz qilaylik, ya’ni $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning $W(x)$ vronskiani I oraliqning biror nuqtasida nolga aylangan bo‘lsin. U holda yana teorema 2 ga ko‘ra bu yechimlar chiziqli bog‘langan va, demak, $W(x)$ vronskian I oraliqda aynan nolga teng. 👍

Masalalar

1. $y_1 \equiv 0, y_2(x), \dots, y_k(x), x \in I$, funksiyalar chiziqli erkli bo‘lishi mumkinmi?

2. Ikkita funksiya biri ikkinchisiga proporsional bo‘lgan taqdirdagina chiziqli bog‘liq bo‘ladi. Shuni isbotlang.

3. Jumla 1 ni isbotlang.

4. Berilgan $y_1(x), y_2(x), \dots, y_n(x), x \in I$, funksiyalar orqali ularning chiziqli kombinatsiyalaridan ushbu

$$z_i(x) = \sum_{j=1}^{n} a_{ij} y_j(x) \, , \; i = \overline{1,n} \, ,$$

funksiaylarni quraylik. Agar a_{ij} sonlardan tuzilgan matritsaning determinanti $\det[a_{ij}] \neq 0$ bo‘lsa, $\{y_i(x)\}_{i=1}^n$ va $\{z_i(x)\}_{i=1}^n$ funksiyalar bir vaqtda chiziqli erkli (chiziqli bog‘langan) bo‘ladi. Shu tasdiqni isbotlang.

§ 6.3. Chiziqli bir jinsli tenglama umumiy yechimining tuzilishi

Bu paragrafda quyidagi n–tartibli chiziqli uzluksiz koeffitsientli bir jinsli differensial tenglama yechimining tuzilishini o‘rganamiz:

$$L[y] = 0, \qquad (6.3.1)$$

bu yerda $L[y] = L_n[y] = y^{(n)} + a_{n-1}(x) y^{(n-1)} + ... + a_1(x) y' + a_0(x) y$ va $\{a_{n-1}(x), ..., a_1(x), a_0(x)\} \subset C(I)$.

n–tartibli chiziqli bir jinsli differensial tenglama (6.3.1) ning n dona chiziqli erkli yechimlari bu tenglama **yechimlarining fundamental sistemasi** (qisqaroq **fundamental sistema**) yoki **bazis yechimlari** deb ataladi. Yuqoridagi § 6.3 da asoslanganiga ko‘ra (6.3.1) tenglamaning n dona yechimlari bazis yechimlarni tashkil etishini ularning vronskiani orqali aniqlash mumkin: agar bu yechimlarning vronskiani nolga aylanmasa, ular bazis yechimlarni tashkil etadi. Quyidagi teorema (6.3.1) tenglama umumiy yechimining tuzilishini ochadi.

Teorema. *Chiziqli bir jinsli (6.3.1) differensial tenglamaning bazis yechimlari mavjud va uning har qanday yechimi biror bazis yechimlarining chiziqli kombinatsiyasidan iborat, ya'ni* $\dim V_n = n$.

🗝 Quyidagi n guruh boshlang'ich shartlarni qaraylik ($x_0 \in I$):

$$y(x_0)=1, y'(x_0)=0, y''(x_0)=0, ..., y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=0; \quad (6.3.2_1)$$

$$y(x_0)=0, y'(x_0)=1, y''(x_0)=0,\ ..., y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=0; \quad (6.3.2_2)$$

. . . .

$$y(x_0)=0, y'(x_0)=0, y''(x_0)=0, ..., y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=1; \quad (6.3.2_n)$$

Ushbu (6.3.1),(6.3.2$_1$); (6.3.1),(6.3.2$_2$); …. ; (6.3.1),(6.3.2$_n$) n dona Koshi masalalarining har biri I oraliqda aniqlangan yagona yechimga ega. Bu yechimlarni mos ravishda $y=\varphi_1(x)$, $y=\varphi_2(x), ..., y=\varphi_n(x)$ bilan belgilaylik. Shunday qilib,

$$L[\varphi_i(x)]=0,\ \varphi_i^{(i-1)}(x_0)=1, \varphi_i^{(j)}(x_0)=0;\ i,j=\overline{1,n},\ j \neq i-1 \qquad (6.3.3)$$

Bu $y=\varphi_1(x), y=\varphi_2(x), ..., y=\varphi_n(x)$ yechimlar chiziqli erkli, chunki ularning vronskiani $x=x_0$ nuqtada, (6.3.3) ga ko'ra, birga teng. Demak, ular bazis yechimlarni tashkil etadi.

Endi ixtiyoriy $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)$ bazis yechimlar berilgan bo'lsin. Ular yuqorida qurilgan bazis yechimlardan farqli bo'lishi ham mumkin. Uxtiyoriy $y=y(x)$ yechim shu bazis yechimlarning chiziqli kombinatsiyasidan iborat ekanligini ko'rsatishimiz kerak. $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)$ chiziqli erkli yechimlar bo'lgani uchun ularning vronskiani ixtiyoriy $x_0 \in I$ nuqtada noldan farqli. Demak, ushbu

$$c_1\varphi_1(x_0)+c_2\varphi_2(x_0)+...+c_n\varphi_n(x_0)=y(x_0),$$
$$c_1\varphi_1'(x_0)+c_2\varphi_2'(x_0)+...+c_n\varphi_n'(x_0)=y'(x_0),$$

. . . .

$$c_1\varphi_1^{(n-1)}(x_0)+c_2\varphi_2^{(n-1)}(x_0)+...+c_n\varphi_n^{(n-1)}(x_0)=y^{(n-1)}(x_0)$$

algebraik sistemani qanoatlantiruvchi yagona $c_1, c_2, ..., c_n$ yechim mavjud (Kramer qoidasiga ko'ra). Ana shu $c_1, c_2, ..., c_n$ larga ko'ra

$\tilde{y}(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + ... + c_n\varphi_n(x)$ funksiyani tuzaylik. Yechimlarning chiziqli kombinatsiyasi sifatida $\tilde{y}(x)$ ham yechim. Bundan tashqari, $c_1, c_2, ..., c_n$ larning tanlanishiga ko‘ra

$$\tilde{y}(x_0) = y(x_0), \tilde{y}'(x_0) = y'(x_0), ..., \tilde{y}^{(n-1)}(x_0) = y^{(n-1)}(x_0) .$$

Demak, $\tilde{y}(x)$ va $y(x)$ yechimlar bir xil boshlang‘ich shartlarni qanoatlantiradi. Yechimning yagonalik xossasiga ko‘ra $\tilde{y}(x) \equiv y(x)$, ya’ni $y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + ... + c_n\varphi_n(x)$. ✍

Isbotlangan teoremada $L[\circ]$ operatorning koeffitsientlari uzluksiz ekanligi ahamiyatli. Agar $L[\circ]$ operatorning koeffitsientlari uzluksiz bo‘lmasa, teoremaning xulosasi noto‘g‘ri bo‘lishi mumkin. Buni quyidagi misol asoslaydi.

Misol. Ushbu

$$y'' - a(x)y' = 0 \,,\; a(x) = \begin{cases} \dfrac{2}{x}, \text{ agar } x \neq 0 \text{ bo'lsa,} \\ 0, \text{ agar } x = 0 \text{ bo'lsa,} \end{cases} ,$$

ikkinchi tartibli (uzluksiz bo‘lmagan koeffitsientli) chiziqli differensial tenglamani qaraylik. Bu tenglama $(-\infty; +\infty)$ oraliqda aniqlangan $y_1 = 1, y_2 = x^3, y_3 = |x|^3$ yechimlarga ega (tekshirib ko‘ring). Bu yechimlar $\mathbb{R}$ da chiziqli erkli. Haqiqatan ham, faraz qilaylik,

$$\lambda_1 + \lambda_2 x^3 + \lambda_3 |x|^3 = 0, \; x \in \mathbb{R},$$

bo‘lsin. U holda

$$\lambda_1 + \lambda_2 x^3 + \lambda_3 x^3 = 0, \; x \geq 0,$$

$$\lambda_1 + \lambda_2 x^3 - \lambda_3 x^3 = 0, \; x \leq 0,$$

bo‘ladi. Bu tengliklardan $\lambda_1 = 0, \lambda_2 + \lambda_3 = 0, \lambda_2 - \lambda_3 = 0,$ ya’ni $\lambda_1 = \lambda_2 = \lambda_3 = 0$ ekanligini topamiz. Demak, qaralayotgan $y_1 = 1, y_2 = x^3, y_3 = |x|^3$ funksiyalarning faqat trivial chiziqli kombinatsiyasigina aynan nolga teng, ya’ni ular chiziqli erkli. Shunday qilib, berilgan ikkinchi tartibli chiziqli differensial tenglama uch dona chiziqli erkli yechimga ega ekan. Agar $a(x)$ koeffitsient $(-\infty; +\infty)$ oraliqda uzluksiz bo‘lganda edi, bu differensial tenglama ikkita chiziqli erkli yechimga ega bo‘lib, ixtiyoriy uchinchi yechim shu ikki yechimning chiziqli kombinatsiyasidan iborat bo‘lar edi. ☝

Masalalar

1. $y'' + \omega^2 y = 0,\ \omega = \mathrm{const} > 0,$ tenglamaning umumiy yechimi $y = c_1 \cos\omega x + c_2 \sin\omega x$ formula bilan berilishini ko‘rsating.

2. Ushbu $x^2 y'' + pxy' + qy = 0,\ p, q - \mathrm{const},\ x > 0$, Eyler tenglamasini qaraylik. $\lambda^2 + (p-1)\lambda + q = 0$ kvadrat tenglamaning diskriminantini $D = (p-1)^2 - 4q$, ildizlarini esa λ_1, λ_2 bilan belgilaylik. Eyler tenglamasining umumiy yechimi

$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$, agar $D > 0$ bo‘lsa;

$y = (c_1 + c_2 \ln x) x^{\lambda_1}$, agar $D = 0$ bo‘lsa;

$y = (c_1 \cos(\mathrm{Im}\lambda_1 \ln x) + c_2 \sin(\mathrm{Im}\lambda_1 \ln x)) x^{\mathrm{Re}\lambda_1}$, agar $D < 0$ bo‘lsa,

ko‘rinishda tasvirlanadi. Shuni isbotlang.

§ 6.4. Bazis yechimlariga ko‘ra chiziqli bir jinsli differensial tenglamani tiklash. Ostrogradskiy-Liuvill formulasi

Bazis yechimlariga ko‘ra mos chiziqli bir jinsli differensial tenglamani tiklash. Biz yuqorida n–tartibli bosh koeffitsienti birga teng bo‘lgan chiziqli bir jinsli differensial tenglama $L[y] = 0$ ning bazis yechimlari mavjud ekanligini asoslagan edik. Endi teskari masala bilan shug‘ullanamiz, ya’ni bazis yechimlariga ko‘ra mos differensial tenglamani tiklash masalasini o‘rganamiz.

Teorema. *Aytaylik,* $\{\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)\} \subset C^n(I)$ *funksiyalarning* $W(x) = W[\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)]$ *vronskiani* I *oraliqda nolga aylanmasin. U holda bazis yechimlari shu funksiyalardan iborat bo‘lgan* $L[y] = 0$ *ko‘rinishdagi* n*- tartibli chiziqli uzluksiz koeffitsientli bir jinsli differensial tenglama mavjud va yagonadir.*

🗝 $y = y(x)$ noma’lum funksiyaga nisbatan ushbu

$$\frac{1}{W(x)} \cdot \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) & y \\ \varphi_1'(x) & \varphi_2'(x) & \cdots & \varphi_n'(x) & y' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) & y^{(n-1)} \\ \varphi_1^{(n)}(x) & \varphi_2^{(n)}(x) & \cdots & \varphi_n^{(n)}(x) & y^{(n)} \end{vmatrix} = 0 \qquad (6.4.1)$$

differensial tenglamani tuzaylik. Ravshanki, bu tenglama n dona $y=\varphi_1(x), y=\varphi_2(x), \ldots, y=\varphi_n(x)$ yechimlarga ega (ikkita ustuni bir xil bo‘lgan determinantning qiymati nolga teng). (6.4.1) dagi determinantni oxirgi ustuni bo‘ylab Laplas formulasiga ko‘ra yoyib, tuzilgan tenglamani quyidagi ko‘rinishda yozamiz:

$$y^{(n)}+\tilde{a}_{n-1}(x)y^{(n-1)}+\cdots+\tilde{a}_1(x)y'+\tilde{a}_0(x)y=0;$$

bu yerda $\tilde{a}_{n-1}(x),\ldots,\tilde{a}_1(x),\tilde{a}_0(x)$ koeffitsientlar berilgan funksiyalar va ularning n- tartibligacha hosilalari orqali ko‘paytirish, qo‘shish va $W(x)\neq 0$ vronskianga bo‘lish amallari yordamida ifodalanadi va shuning uchun ular I oraliqda uzluksiz. Tushunarliki, berilgan funksiyalar qurilgan shu n- tartibli chiziqli bir jinsli differensial tenglamaning bazis yechimlaridir ($W(x)\neq 0$).

Endi bunday ko‘rinishdagi tenglamaning yagonaligini ko‘rsatamiz. Faraz qilaylik, berilgan $y=\varphi_1(x), y=\varphi_2(x), \ldots, y=\varphi_n(x)$ funksiyalar ushbu

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=0$$

uzluksiz koeffitsientli tenglamaning yechimlari bo‘lsin, ya’ni I oraliqda

$$\varphi_i^{(n)}(x)+a_{n-1}(x)\varphi_i^{(n-1)}(x)+\cdots+a_1(x)\varphi_i'(x)+a_0(x)\varphi_i(x)y=0, i=\overline{1,n}.$$

Bu ayniyatlarni $a_0(x),a_1(x),\ldots,a_{n-1}(x)$ noma’lumlarga nisbatan chiziqli algebraik tenglamalar sistemasi deb qaraylik. Sistemaning determinanti

$$\begin{vmatrix} \varphi_1(x) & \varphi_1'(x) & \ldots & \varphi_1^{(n-1)}(x) \\ \varphi_2(x) & \varphi_2'(x) & \ldots & \varphi_2^{(n-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n(x) & \varphi_n'(x) & \ldots & \varphi_n^{(n-1)}(x) \end{vmatrix} = W(x)\neq 0, x\in I.$$

Demak, $a_0(x),a_1(x),\ldots,a_{n-1}(x)$ lar bu sistemadan Kramer formulalari yordamida $\varphi_i^{(j)}(x)$ $\left(i,j=\overline{1,n}\right)$ uzluksiz funksiyalar orqali bir qiymatli topiladi. 👍

Ostrogradskiy-Liuvill formulasi. $L[y]=0$ chiziqli bir jinsli differensial tenglama yechimlarining vronskiani uchun Ostrogradskiy-Liuvill formulasi deb ataluvchi formulani hosil qilamiz.

Dastlab determinantni differensiallash qoidasini keltiramiz.

Lemma. *Differensiallanuvchi* $d_{ij} = d_{ij}(x)$ *funksiyalardan tuzilgan* n*- tartibli*

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{vmatrix}$$

determinantning hosilasi uchun quyidagi formula o'rinli:

$$\Delta' = \Delta_1 + \Delta_2 + \cdots + \Delta_n, \qquad (6.4.2)$$

bunda Δ_i *determinant* Δ *ning* i*- satridagi elementlarning o'rniga ularning hosilasini yozishdan hosil bo'lgan.*

🗝 Determinant ta'rifiga ko'ra

$$\Delta = \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} d_{1j_1} d_{2j_2} \ldots d_{nj_n}, \qquad (6.4.3)$$

bunda yig'indi $1, 2, \ldots, n$ sonlarining barcha $j_1, j_2, \ldots, j_n$ o'rin almashtirishlari bo'yicha hisoblangan, $\sigma(j_1, j_2, \ldots, j_n)$ bilan $j_1, j_2, \ldots, j_n$ o'rinalmashtirishning juftliligi belgilangan. Quyidagilarga egamiz:

$$\Delta' = \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} (d_{1j_1} d_{2j_2} \ldots d_{nj_n})' =$$

$$= \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} \left(d'_{1j_1} d_{2j_2} \ldots d_{nj_n} + d_{1j_1} d'_{2j_2} \ldots d_{nj_n} + \cdots + d_{1j_1} d_{2j_2} \ldots d'_{nj_n} \right) =$$

$$= \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} d'_{1j_1} d_{2j_2} \ldots d_{nj_n} + \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} d_{1j_1} d'_{2j_2} \ldots d_{nj_n} +$$

$$+ \cdots + \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} d_{1j_1} d_{2j_2} \ldots d'_{nj_n} = \Delta_1 + \Delta_2 + \cdots + \Delta_n;$$

bu yerda

$$\Delta_i = \sum (-1)^{\sigma(j_1, j_2, \ldots, j_n)} d_{1j_1} d_{2j_2} \ldots d'_{ij_i} \ldots d_{nj_n} \ (i = \overline{1, n})$$

determinant Δ determinantdagi i- satrni uning hosilasi bilan almashtirishdan hosil bo'lgan. 👍

Teorema (Ostrogradskiy-Liuvill formulasi). n*- tartibli chiziqli bir jinsli* $y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = 0$ *differensial tenglamaning* n *dona* $y = y_1(x),\ y = y_2(x),\ \ldots,\ y = y_n(x)$ *yechimlarining* $W(x)$ *vronskiani uchun ushbu*

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} a_{n-1}(s)ds\right) \ \left(x_0, x \in I\right) \qquad (6.4.4)$$

***Ostrogradskiy-Liuvill formulasi** deb ataluvchi formula o'rinli.*

🗝 Vronskian ta'rifiga ko'ra

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Bu determinantni (6.4.2) formuladan foydalanib differensiallaymiz. Bunda hosil bo'luvchi determinantlarning dastlabki $n-1$ tasi nolga teng bo'ladi, chunki ularning ikkita satri bir xil. Natijada

$$W'(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix} \qquad (6.4.5)$$

tenglik hosil bo'ladi. Endi $y_i(x)$ larning yechim ekanligidan foydalanamiz:

$$y_i^{(n)}(x) + a_{n-1}(x)y_i^{(n-1)}(x) + ... + a_1(x)y_i'(x) + a_0(x)y_i(x) = 0,$$

ya'ni

$$y_i^{(n)}(x) = -a_{n-1}(x)y_i^{(n-1)}(x) - ... - a_1(x)y_i'(x) - a_0(x)y_i(x) \ \ (i = \overline{1,n}). \qquad (6.4.6)$$

(6.4.5) da o'ng tomondagi determinantning birinchi satrini $a_0(x)$ ga, ikkinchi satrini $a_1(x)$ ga va h.k. $n-1$ satrini $a_{n-2}(x)$ ga ko'paytirib, ularni oxirgi n- satrga qo'shamiz. (6.4.6) tengliklarga ko'ra oxirgi satrda $-a_{n-1}(x)$ umumiy ko'paytuvchi hosil bo'ladi. Uni determinant oldiga chiqarib, ushbu

$$W'(x) = -a_{n-1}(x)W(x)$$

tenglikni topamiz. Bu tenglikdan endi (6.4.4) formula ravshan. 👍

Chiziqli tenglama tartibini pasaytirish. Agar $L[y] = 0$ chiziqli tenglamaning $y = \varphi_1(x), \varphi_1(x) \neq 0,$ yechimi ma'lum bo'lsa, bu tenglamaning tartibini bittaga kamaytirish mumkin. Buning uchun tenglamada $y=\varphi_1(x)u$ almashtirishni bajaramiz; bunda $u=u(x)$ yangi noma’lum funksiya. Kerakli hosilalalarni hisoblaymiz:

$$y'=\varphi_1(x)u'+\varphi_1'(x)u,$$
$$y''=\varphi_1(x)u''+2\varphi_1'(x)u'+\varphi_1''(x)u,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y^{(n)}=\varphi_1(x)u^{(n)}+n\varphi_1'(x)u^{(n-1)}+\frac{n(n-1)}{2}\varphi_1''(x)u^{(n-2)}+\cdots+\varphi_1^{(n)}(x)u\,.$$

Bularni $L[y]=0$ tenglamaga qo‘yib, quyidagi tenglamani hosil qilamiz:

$$L[\varphi_1(x)u]\equiv\varphi_1(x)u^{(n)}+\tilde{a}_{n-1}(x)u^{(n-1)}+\cdots+\tilde{a}_1(x)u'+\tilde{a}_0(x)u=0\,.$$

Ravshanki, $u=1$ bu tenglamaning yechimi, chunki $L[\varphi_1(x)]=0$. Demak, $\tilde{a}_0(x)\equiv 0$ va $u=u(x)$ noma’lum funksiya uchun ushbu

$$\varphi_1(x)u^{(n)}+\tilde{a}_{n-1}(x)u^{(n-1)}+\cdots+\tilde{a}_1(x)u'=0$$

tenglama hosil bo‘ladi. Bu tenglamada $u=u(x)$ noma’lum oshkor ko‘rinishda qatnashmaganligi uchun $v=u'$ deb, ($\varphi_1(x)$ nolga aylanmagan oraliqda) v ga nisbatan $(n-1)$- tartibli differensial tenglamaga kelamiz.

Agar ikkinchi tartibli chiziqli bir jinsli tenglama $y''+a_1(x)y'+a_0(x)y=0$ ning biror $y=\varphi_1(x),\ \varphi_1(x)\neq 0,$ yechimi ma’lum bo‘lsa, tenglamaning $y=y(x)$ yechimini Ostrogradskiy-Liuvill formulasidan foydalanib topish ham mumkin:

$$\begin{vmatrix}\varphi_1(x) & y\\ \varphi_1'(x) & y'\end{vmatrix}=c_2\exp\left(-\int_{x_0}^{x}a_1(s)ds\right),\ \varphi_1(x)y'-\varphi_1'(x)y=c_2\exp\left(-\int_{x_0}^{x}a_1(s)ds\right),$$

$$\left(\frac{y}{\varphi_1(x)}\right)'=\frac{c_2}{\varphi_1^2(x)}\exp\left(-\int_{x_0}^{x}a_1(s)ds\right),$$

$$y=c_1\varphi_1(x)+c_2\varphi_1(x)\int_{x_0}^{x}\frac{1}{\varphi_1^2(t)}\exp\left(-\int_{x_0}^{t}a_1(s)ds\right)dt\,.$$

Masalalar

1. $\varphi(x)$ funksiya $C^2(I)$ sinfga tegishli va I oraliqda nolga aylanmasin. Yechimlari

a) $\varphi(x)$ va $x\varphi(x)$;

b) $\varphi(x)$va $1/\varphi(x)$ ($\varphi'(x)\neq 0$)

bo‘lgan chiziqli ikkinchi tartibli differensial tenglama tuzing.

2. Agar ikkinchi tartibli chiziqli bir jinsli tenglama $y''+a_1(x)y'+a_0(x)y=0$ ning biror $y=\varphi_1(x),\ \varphi_1(x)\neq 0,$ yechimi ma’lum bo‘lsa, tenglamaning bu yechimga chiziqli bog‘liq bo‘lmagan ikkinchi yechimini toping.

3. Agar $y''+a_1(x)y'+a_0(x)y=0$, $\{a_1(x),a_0(x)\}\subset C([a,+\infty))$, tenglamaning har qanday yechimi o‘zining birinchi tartibli hosilasi bilan birgalikda $x\to+\infty$ da nolga intilsa, $a_1(x)$ funksiya to‘g‘risida nima deyish mumkin?

§ 6.5. n-tartibli chiziqli bir jinsli bo‘lmagan tenglama. Ixtiyoriy o‘zgarmaslarni variatsiyalash usuli

Bir jinsli bo‘lmagan $L[y]=g(x)$ tenglamani qaraylik. Bu tenglama umumiy yechimining tuzilishini quyidagi teorema ochadi.

Teorpema. *Aytaylik, $y=\psi(x)$ funksiya bir jinsli bo‘lmagan $L[y]=g(x)$ tenglamaning biror xususiy yechimi ($L[\psi]=g(x)$), $\phi_1,...,\phi_n$ funksiyalar esa mos bir jinsli $L[y]=0$ tenglamaning bazis yechimlari bo‘lsin. U holda $L[y]=g(x)$ tenglamaning umumiy yechimi $y=\psi+\sum_{i=1}^{n}c_i\varphi_i$ $(c_1,\dots,c_n-const)$ formula bilan beriladi, ya’ni bir jinsli bo‘lmagan tenglamaning umumiy yechimi shu tenglamaning biror xususiy yechimiga mos bir jinsli tenglamaning umumiy yechimini qo‘shishdan hosil bo‘ladi.*

🗝 Ravshanki, $y=\sum_{i=1}^{n}c_i\varphi_i+\psi$ funksiya $c_1,\dots,c_n$ larning ixtiyoriy tayinlangan qiymatlarida bir jinsli bo‘lmagan $L[y]=g(x)$ tenglamaning yechimi:

$$L[y]=L\left[\sum_{i=1}^{n}c_i\varphi_i+\psi\right]=\sum_{i=1}^{n}c_iL[\varphi_i]+L[\psi]=\sum_{i=1}^{n}c_i\cdot 0+L[\psi]=g(x).$$

Endi $L[y]=g(x)$ tenglamaning ixtiyoriy y yechimini qaraylik: $L[y]=g(x)$. $L[\psi]=g(x)$ ham bo‘lgani uchun $L[\circ]$ operatorning chiziqliligidan $L[y-\psi]=0$ ekanligi kelib chiqadi, ya’ni $y-\psi$ funksiya mos bir jinsli tenglamaning yechimi. U fundamental sistemaning biror chiziqli kombinatsiyasidan iborat bo‘lishi kerak, ya’ni biror $c_1,\dots,c_n$ larda $y-\psi=\sum_{i=1}^{n}c_i\varphi_i$, bo‘ladi.

Demak, $y=\psi+\sum_{i=1}^{n}c_i\varphi_i$. ☝

Bir jinsli bo‘lmagan tenglamaning xususiy yechimini topishda ba’zan quyidagi superpozitsiya prinsipi qo‘l keladi.

Jumla (superpozitsiya prinsipi). *Agar* $y=\psi_1(x)$ *funksiya* $L[y]=g_1(x)$ *tenglamaning* $y=\psi_2(x)$ *funksiya esa* $L[y]=g_2(x)$ *tenglamaning yechimi bo‘lsa, u holda* $y=\psi_1(x)+\psi_2(x)$ *funksiya ushbu* $L[y]=g_1(x)+g_2(x)$ *tenglamaning yechimi bo‘ladi.*

⊷ Isboti ravshan.

Misol 1. Ushbu

$$y''-3y'+2y=2+4e^{3x}$$

tenglamaning xususiy yechimini toping.

⊷ $y''-3y'+2y=2$ va $y''-3y'+2y=4e^{3x}$ tenglamalarni qaraylik. Birinchi tenglama, ravshanki, $y=1$ yechimga ega. Ikkinchi tenglamaning yechimini $y=ke^{3x}$ ko‘rinishda izlab ko‘raylik. Uni tenglamaga qo‘yib, noma’lum k soni uchun $2k=4$ munosabatni hosil qilamiz. Demak, $k=2$, ya’ni $y=2e^{3x}$ funksiya $y''-3y'+2y=4e^{3x}$ tenglamaning xususiy yechimi. Superpozitsiya prinsipiga ko‘ra $y=1+2e^{3x}$ funksiya berilgan $y''-3y'+2y=2+4e^{3x}$ tenglamaning (xususiy) yechimidir. ☝

Lagranjning ixtiyoriy o‘zgarmaslarni variatsiyalash usuli. Bir jinsli tenglama $L_n[y]=0$ нинг $\varphi_1(x),\varphi_2(x),...,\varphi_n(x)$ bazis yechimlari ma’lum bo‘lsin. U holda bu tenglamaning umumiy yechimi, ma’lumki, $y=\sum_{i=1}^{n}c_i\varphi_i(x)$ ko‘rinishda ifodalanadi; bu yerda $c_1,c_2,...,c_n$ – ixtiyoriy o‘zgarmaslar. Lagranjning ixtiyoriy o‘zgarmaslarni variatsiyalash usuliga ko‘ra bir jinsli bo‘lmagan $L[y]=g(x)$ tenglamaning yechimi ushbu

$$y=\sum_{i=1}^{n}c_i(x)\varphi_i(x) \qquad (6.5.1_0)$$

ko‘rinishda izlanadi; bu yerda $c_1(x), c_2(x), ..., c_n(x)-$ hozircha noma’lum uzluksiz differensiallanuvchi funksiyalar. Ularning soni n ta. Bu noma’lum funksiyalarni (6.5.1$_0$) ifoda $L[y]=g(x)$ tenglamaning yechimi bo‘lishi kerakligidan aniqlaymiz; bu bitta shart. Umuman olganda, biz yana $n-1$ ta shartni o‘zimizdan qo‘yishimiz mumkin. Hosilani hisoblaymiz:

$$y' = \sum_{i=1}^{n} c_i'(x)\varphi_i(x) + \sum_{i=1}^{n} c_i(x)\varphi_i'(x).$$

Hosilaning ko‘rinishi sodda bo‘lishi uchun noma’lum funksiyalarga nisbatan ushbu

$$\sum_{i=1}^{n} c_i'(x)\varphi_i(x) = 0 \qquad (6.5.2_0)$$

shartni qo‘yib,

$$y' = \sum_{i=1}^{n} c_i(x)\varphi_i'(x) \qquad (6.5.1_1)$$

tenglikni topamiz. Endi y'' ni hisoblaymiz:

$$y'' = \sum_{i=1}^{n} c_i'(x)\varphi_i'(x) + \sum_{i=1}^{n} c_i(x)\varphi_i''(x).$$

Ushbu

$$\sum_{i=1}^{n} c_i'(x)\varphi_i'(x) = 0 \qquad (6.5.2_1)$$

shartni qo‘yib, ikkinchi tartibli hosila uchun

$$y'' = \sum_{i=1}^{n} c_i(x)\varphi_i''(x) \qquad (6.5.1_2)$$

sodda ko‘rinishli formulani hosil qilamiz. Shunga o‘xshash fikr yuritib, $c_i'(x)$ larga nisbatan mos shartlarni qo‘yib, topamiz:

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-2)}(x) = 0, \qquad (6.5.2_{n-2})$$

$$y^{(n-1)} = \sum_{i=1}^{n} c_i(x)\varphi_i^{(n-1)}(x). \qquad (6.5.1_{n-1})$$

Endi

$$y^{(n)} = \sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) + \sum_{i=1}^{n} c_i(x)\varphi_i^{(n)}(x) \qquad (6.5.1_n)$$

hosilani hisoblab, va $y, y', y'', ..., y^{(n)}$ larning qiymatlarini (6.5.1$_0$)-(6.5.1$_n$) dan yechilayotgan $L[y] = g(x)$ tenglamaga qo‘yib,

$$L[y] = \sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) + \sum_{i=1}^{n} c_i(x)L[\varphi_i] = g(x),$$

ya’ni

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) = g(x) \qquad (6.5.2_{n-1})$$

tenglikni topamiz. Shunday qilib, $c_1'(x), c_2'(x), ..., c_n'(x)$ larga nisbatan quyidagi chiziqli algebraik sistemani (ya’ni (6.5.2$_0$)-(6.5.2$_{n-1}$) shartlarni) hosil qildik:

$$\sum_{i=1}^{n} c_i'(x)\varphi_i(x) = 0$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i'(x) = 0$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-2)}(x) = 0$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) = g(x).$$

Bu sistema $c_1'(x), c_2'(x), ..., c_n'(x)$ larga nisbatan yagona yechimga ega, chunki sistemaning determinanti $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)$ bazis yechimlarning $W(x)$ vronskianidan iborat bo‘lgani uchun u I oraliqda nolga aylanmaydi. Kramer formulalariga ko‘ra

$$c_1'(x) = \frac{1}{W(x)} \begin{vmatrix} 0 & \varphi_2(x) & \dots & \varphi_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_2^{(n-2)}(x) & \dots & \varphi_n^{(n-2)}(x) \\ g(x) & \varphi_2^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{vmatrix} = g(x)\omega_1(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_n'(x) = \frac{1}{W(x)} \begin{vmatrix} \varphi_1(x) & \dots & \varphi_{n-1}(x) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1^{(n-2)}(x) & \dots & \varphi_{n-1}^{(n-2)}(x) & 0 \\ \varphi_1^{(n-1)}(x) & \dots & \varphi_{n-1}^{(n-1)}(x) & g(x) \end{vmatrix} = g(x)\omega_n(x);$$

bu yerdagi $\omega_1(x), \omega_2(x), ..., \omega_n(x)$ funksiyalar I oraliqda C^1 sinfga tegishli.

Endi $c_1(x), c_2(x), ..., c_n(x)$ noma'lum funksiyalarni ushbu

$$c_i(x) = \int_{x_0}^{x} g(s)\omega_i(s)\,ds,\ i = \overline{1,n}\ \ (x_0 \in I - \text{tayinlangan nuqta})$$

ko'rinishda tanlaymiz va (6.5.1$_0$) formulaga ko'ra $L[y] = g(x)$ tenglamaning

$$y = \sum_{i=1}^{n} c_i(x)\varphi_i(x) = \sum_{i=1}^{n} \varphi_i(x)\int_{x_0}^{x} g(s)\omega_i(s)\,ds$$

yoki

$$y = \int_{x_0}^{x} K(x,s)g(s)\,ds \ \text{(bunda } K(x,s) = \sum_{i=1}^{n} \varphi_i(x)\omega_i(s)\text{)} \qquad (6.5.4)$$

yechimini topamiz. Bu (6.5.4) formula **Koshi formulasi**, $K(x,s)$ funksiya esa **Koshi funksiyasi** deyiladi.

Ko'rsatish mumkinki, $K(x,s)$ Koshi funksiyasini quyidagi masala yechimi sifatida tanlash mumkin (hosilalar x bo'yicha hisoblangan, $s-$ tayinlangan):

$$L[K] = 0, K\big|_{x=s} = 0, \ldots, K^{(n-2)}\big|_{x=s} = 0, K^{(n-1)}\big|_{x=s} = 1.$$

Misol 2. Ushbu

$$y'' + \omega^2 y = f(x) \ \ (\omega > 0,\ f \in C(I))$$

tenglamani yeching (garmonik ossilyatorning majburiy harakati, $x-$ vaqt). Mos bir jinsli tenglama $y'' + \omega^2 y = 0$ ning bazis yechimlari $y_1 = \cos\omega x,\ y_2 = \sin\omega x$ umumiy yechimi $y = c_1\cos\omega x + c_2\sin\omega x$ (bu garmonik tebranishlarni ifodalaydi). Berilgan bir jinsli bo'lmagan tenglamaning xususiy yechimini Lagranj usulidan foydalanib topamiz, ya'ni bu tenglamaning yechimini

$$y = c_1(x)\cos\omega x + c_2(x)\sin\omega x$$

ko'rinishda izlaymiz. Ushbu

$$c_1'(x)\cos\omega x + c_2'(x)\sin\omega x = 0$$

shartni qo'yib, $y' = -c_1(x)\omega\sin\omega x + c_2(x)\omega\cos\omega x$ formulani hosil qilamiz. Bundan y'' ni hisoblab, berilgan tenglamadan

$$-c_1'(x)\omega\sin\omega x + c_2'(x)\omega\cos\omega x = f(x)$$

shartni hosil qilamiz. Shunday qilib, $c_1(x), c_2(x)$ noma'lum funksiyalar uchun

$$\begin{cases} c_1'(x)\cos\omega x + c_2'(x)\sin\omega x = 0 \\ -c_1'(x)\omega\sin\omega x + c_2'(x)\omega\cos\omega x = f(x) \end{cases}$$

sistemani topdik. Bu sistemadan $c_1'(x), c_2'(x)$ lar bir qiymatli aniqlanadi:

$$c_1'(x) = -\frac{1}{\omega}\sin\omega x \cdot f(x), \ \ c_2'(x) = \frac{1}{\omega}\cos\omega x \cdot f(x).$$

Bundan $c_1(x) = -\frac{1}{\omega}\int \sin\omega x \cdot f(x)dx, \ \ c_2(x) = \frac{1}{\omega}\int \cos\omega x \cdot f(x)dx.$

Demak, berilgan bir jinsli bo‘lmagan tenglamaning

$$y = c_1(x)\cos\omega x + c_2(x)\sin\omega x =$$

$$= -\frac{1}{\omega}\cos\omega x\int \sin\omega x \cdot f(x)dx + \frac{1}{\omega}\sin\omega x\int \cos\omega x \cdot f(x)dx.$$

xususiy yechimini topdik. Bu yechimni aniq integral yordamida quyidagicha yozish mumkin:

$$y = -\frac{1}{\omega}\cos\omega x\int_{x_0}^{x} \sin\omega s \cdot f(s)ds + \frac{1}{\omega}\sin\omega x\int_{x_0}^{x} \cos\omega s \cdot f(s)ds$$

yoki

$$y = \frac{1}{\omega}\int_{x_0}^{x} \sin\omega(x-s)\cdot f(s)ds \quad .$$

Berilgan bir jinsli bo‘lmagan tenglamaning umumiy yechimi

$$y = \frac{1}{\omega}\int_{x_0}^{x} \sin\omega(x-s)\cdot f(s)ds + c_1\cos\omega x + c_2\sin\omega x$$

ko‘rinishda ifodalanadi.

Masalalar

1. Ushbu $x^2y'' + 3xy' + y = 1 + \cos x$, . ., tenglamaning umumiy yechimini quring (mos bir jinsli tenglamaning yechimlari $y_1 = \frac{1}{x}, \ y_2 = \frac{\ln x}{x}$) .

2. Ushbu $x'' + (1+\varphi(t))x = 0, \ |\varphi(t,x,x')| \le \frac{c}{t^2}, \ t \ge t_0 > 0 \ (c = \text{const} > 0)$

ikkinchi tartibli chiziqli differensial tenglamaning har qanday yechimi $t \to +\infty$da chegaralangan bo‘lishini isbotlang. $\varphi(t,x,x') \in C\big((0,+\infty)\times\mathbb{R}\times\mathbb{R}\big)$ deb hisoblanadi.

3. Quyidagi Dyuamel prinsipini isbotlang.

Dyuamel prinsipi. Ixtiyoriy tayinlangan $s \in I$ nuqta uchun

$$\begin{cases} L[\Gamma]=0, \\ \Gamma|_{x=s}=0,\ \Gamma'|_{x=s}=0,\ ...,\ \Gamma^{(n-2)}|_{x=s}=0,\ \Gamma^{(n-1)}|_{x=s}=g(s) \end{cases}$$

Koshi masalasining yechimini $\Gamma=\Gamma(x,s)$ bilan belgilaylik. U holda

$$y=\int_{x_0}^{x}\Gamma(x,s)ds$$

funksiya ushbu

$$\begin{cases} L[y]=g(x), \\ y|_{x=x_0}=0,\ y'|_{x=x_0}=0,...,\ y^{(n-1)}|_{x=x_0}=0 \end{cases}$$

Koshi masalasining yechimi bo‘ladi.

MODUL 7. CHIZIQLI O‘ZGARMAS KOEFFITSIENTLI DIFFERENSIAL TENGLAMALAR

§ 7.1. Tenglamani komplekslashtirish

Biz yuqorida $L[y]=g(x)$ chiziqli tenglamaning $y=\varphi(x),\ \varphi: I\to\mathbb{R},$ haqiqiy yechimlarini o‘rgandik (tenglamada qatnashgan funksiyalar haqiqiy qiymatli edi). Ba’zan bu tenglamaning kompleks yechimlarini topishga to‘g‘ri keladi.

Dastlab $I\subset\mathbb{R}$ oraliqda aniqlangan kompleks funksiya, uning uzluksizligi, hosilasi va integrali tushunchalarini kiritaylik.

Kompleks sonlar maydonini odatdagidek $\mathbb{C}$ bilan belgilaymiz. $w: I\to\mathbb{C}$ akslantirish I oraliqda aniqlangan **kompleks funksiya** deyiladi. U har qanday $x\in I$ haqiqiy songa $w(x)\in\mathbb{C}$ kompleks sonni mos keltiradi. Bu $w(x)$ kompleks sonning haqiqiy va mavhum qismlarini ajratib, uni $w(x)=u(x)+iv(x)$ ko‘rinishda yozish mumkin; bu yerda $i-$mavhum birlik $(i^2=-1),$ $u: I\to\mathbb{R},\ u(x)=\mathrm{Re}\,w(x)-$haqiqiy qism, $v: I\to\mathbb{R},\ v(x)=\mathrm{Im}\,w(x)-$ mavhum qism. Demak, bitta kompleks funksiyani berish ikkita haqiqiy funksiyani (haqiqiy va mavhum qismlarni) berish demakdir. $w(x)=u(x)+iv(x)$ kompleks funksiyani $\left(u(x),v(x)\right)$ vektor-funksiya kabi tushunish ham mumkin. Vektor-funksiya uchun koordinatalar bo‘ylab kiritilgan analiz tushunchalari (limit, uzluksizlik, hosila, integral, …) bevosita kompleks funksiya holiga ko‘chiriladi. Masalan, $x_0\in I$ nuqtadagi limit uchun

$$\lim_{x\to x_0} w(x)=\lim_{x\to x_0}(u(x)+iv(x))=\lim_{x\to x_0}u(x)+i\lim_{x\to x_0}v(x)$$

deb hisoblanadi. Agar berilgan $x_0\in I$ nuqtada $u(x)$ va $v(x)$ haqiqiy funksiyalar uzluksiz bo‘lsa, u holda $w(x)=u(x)+iv(x)$ kompleks funksiya shu x_0 **nuqtada uzluksiz** deyiladi. Xuddi shunga o‘xshash hosila va integral tushunchalari kiritiladi:

$$w'(x)=\left(u(x)+iv(x)\right)'=u'(x)+iv'(x),$$

$$\int_a^b w(x)dx=\int_a^b u(x)dx+i\int_a^b v(x)dx,\ \{a,b\}\subset I.$$

Ko‘rsatish mumkinki, ushbu

$$w'(x)=\lim_{h\to 0}\frac{w(x+h)-w(x)}{h}$$

formula ham o‘rinli bo‘ladi. Haqiqiy sohada hosila hisoblash uchun asosiy qoidalar kompleks funksiyalar uchun ham saqlanadi: z_1,z_2 kompleks sonlar va differensiallanuvchi (ya’ni hosilaga ega bo‘lgan) $w_1(x),w_2(x),w(x)$ kompleks va $x(t)$ haqiqiy funksiyalar uchun ushbu

$$\left(z_1w_1(x)+z_2w_2(x)\right)'=z_1w_1'(x)+z_2w_2'(x),$$

$$\left(w_1(x)\cdot w_2(x)\right)'=w_1'(x)\cdot w_2(x)+w_1(x)\cdot w_2'(x)\ ,$$

$$\left(\frac{w_1(x)}{w_2(x)}\right)'=\frac{w_1'(x)\cdot w_2(x)-w_1(x)\cdot w_2'(x)}{w_2^2(x)}\ \ (w_2(x)\neq 0)\ \ (*)$$

$$\frac{dw(x(t))}{dt}=\left.\frac{dw(x)}{dx}\right|_{x=x(t)}\cdot\frac{dx(t)}{dt}$$

formulalar o‘rinli. Agar $w(x)=u(x)+iv(x)$ kompleks funksiyaning $u(x)$haqiqiy va $v(x)$ mavhum qismlari I oraliqda uzluksiz, ya’ni $\{u(x),v(x)\}\subset C(I,\mathbb{R})$ bo‘lsa, u holda $w(x)$ kompleks funksiya I **oraliqda uzluksiz** deyiladi va bu $w(x)\in C(I,\mathbb{C})$ kabi ifodalanadi.

Agar $\{u(x),v(x)\}\subset C^1(I,\mathbb{R})$ bo‘lsa, u holda $w(x)=u(x)+iv(x)$ kompleks funksiya I oraliqda uzluksiz differensiallanuvchi deyiladi va bu $w(x)\in C^1(I,\mathbb{C})$ kabi yoziladi. I oraliqda n- tartibli hosilasi bilan birgalikda uzluksiz (n marta uzluksiz differensiallanuvchi) bo‘lgan kompleks funksiyalar sinfi ham shunga o‘xshash kiritiladi va bu sinf $C^n(I,\mathbb{C})$ bilan belgilanadi.

z_1,z_2 kompleks sonlar va $w_1(x),w_2(x)$ integrallanuvchi kompleks funksiyalar uchun

$$\int_a^b\left(z_1w_1(x)+z_2w_2(x)\right)dx=z_1\int_a^b w_1(x)dx+z_2\int_a^b w_2(x)dx,$$

$$\left|\int_a^b w_1(x)dx\right|\leq\int_a^b\left|w_1(x)\right|dx\ \ (a<b)$$

munosabatlar o‘rinlidir. Agar $w(x)\in C^1(I,\mathbb{C})$ bo‘lsa, har qanday $\{a,b\}\subset I$ uchun ushbu

$$\int_a^b w'(x)dx = w(b) - w(a)$$

Nyuton-Leybnits formulasi ham o‘rinli bo‘ladi.

Ikki t va x haqiqiy o‘zgaruvchining $w(t,x)$ kompleks funksiyasi yuqoridagiga o‘xshash kiritiladi va o‘rganiladi. Bizga quyidagi tasdiq kerak bo‘ladi. Agar $\frac{\partial^2 w}{\partial t \partial x}$, $\frac{\partial^2 w}{\partial x \partial t}$ aralash xususiy hosilalarning biri uzluksiz bo‘lsa, ular teng bo‘ladi: $\frac{\partial^2 w}{\partial t \partial x} = \frac{\partial^2 w}{\partial x \partial t}$. Bu tasdiq haqiqiy funksiyalar uchun mos teoremadan bevosita kelib chiqadi. Demak, haqiqiy funksiyalar holidagidek silliq kompleks funksiyaning yuqori tartibli xususiy hosilasi hosilani hisoblash tartibiga bog‘liq bo‘lmaydi.

Endi ixtiyoriy $z = \alpha + i\beta \in \mathbb{C}$ $(\{\alpha, \beta\} \subset \mathbb{R})$ kompleks son uchun

$$e^z \overset{def}{=} e^\alpha (\cos\beta + i\sin\beta) \tag{7.1.1}$$

kompleks sonni aniqlaylik. Agar bu formulada $z = ix,\ x \in \mathbb{R}$, desak,

$$e^{ix} = \cos x + i\sin x \tag{7.1.2}$$

tenglik hosil bo‘ladi. Ravshanki, ushbu

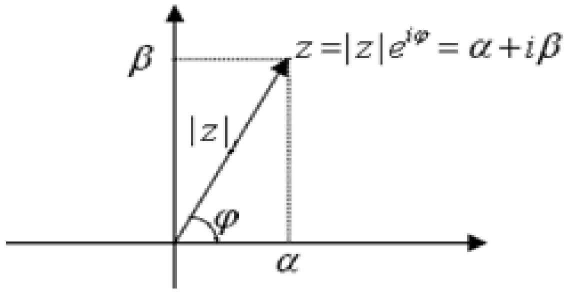
$$e^{-ix} = \cos x - i\sin x, \tag{7.1.3}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},\ \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{7.1.4}$$

formulalar ham o‘rinli. Bu (7.1.2) – (7.1.4) formulalar **Eyler formulalari** deb ataladi. Agar $z = \alpha + i\beta \neq 0$ kompleks sonning modulini (absolyut qiymatini) $|z|$ $\left(|z| = \sqrt{\alpha^2 + \beta^2}\right)$ bilan, argumentini esa $\arg z = \varphi$ $(|z|\cos\varphi = \alpha, |z|\sin\varphi = \beta;\ 0 \le \varphi < 2\pi)$ bilan belgilasak, ravshanki,

$$z = |z|e^{i\varphi}\ \left(z = |z|(\cos\varphi + i\sin\varphi)\right)$$

bo‘ladi (7.1 – rasm). $z = 0$ bo‘lganda $e^z = e^0 = 1$ deb qabul qilamiz.



7.1- rasm. $z = \alpha + i\beta = |z|e^{i\varphi}$ kompleks sonning moduli $|z|$ va argumenti φ

Osongina tekshirib ko‘rish mumkinki, ixtiyoriy z_1 va z_2 kompleks sonlar uchun

$e^{z_1+z_2}=e^{z_1}e^{z_2}$ bo‘ladi. Ixtiyoriy $a>0$ haqiqiy va ixtiyoriy z kompleks son uchun $a^z=e^{z\ln a}$ deymiz.

Endi ixtiyoriy $z=\alpha+i\beta$ ($\{\alpha,\beta\}\subset\mathbb{R}$) kompleks sonni tayinlab, $x\in\mathbb{R}$ haqiqiy o‘zgaruvchining ushbu

$$w(x)=e^{zx}=e^{\alpha x}(\cos\beta x+i\sin\beta x)$$

kompleks funksiyasini kiritamiz. Uning hosilasini hisoblaylik. Ravshanki,

$$\begin{aligned}(e^{zx})'&=(e^{\alpha x})'(\cos\beta x+i\sin\beta x)+e^{\alpha x}(\cos\beta x+i\sin\beta x)'=\\&=\alpha e^{\alpha x}(\cos\beta x+i\sin\beta x)+e^{\alpha x}(-\beta\sin\beta x+i\beta\cos\beta x)=\\&=\alpha e^{\alpha x}(\cos\beta x+i\sin\beta x)+e^{\alpha x}i\beta(i\sin\beta x+\cos\beta x)=\\&=(\alpha+i\beta)e^{\alpha x}(\cos\beta x+i\sin\beta x)=ze^{zx},\end{aligned}$$

ya’ni

$$(e^{zx})'=ze^{zx}. \qquad (7.1.5)$$

Demak, haqiqiy z uchun bizga ma’lum bo‘lgan bu formula kompleks z uchun ham o‘z kuchini saqlaydi.

Endi noma’lumi haqiqiy o‘zgaruvchining kompleks funksiyasidan iborat bo‘lgan differensial tenglamalarni o‘rganish mumkin. (7.1.5) formuladan ravshanki, $w'=zw$ ($z\in\mathbb{C}$ berilgan o‘zgarmas son) tenglama $w(x)=e^{zx}$ yechimga ega. Bu tenglamaning barcha yechimlari $w=ce^{zx}$ formula bilan berilishini ko‘rsatish mumkin; bu yerda c –ixtiyoriy o‘zgarmas kompleks son.

Faraz qilaylik,

$$L[y]\equiv y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x) \qquad (7.1.6)$$

tenglamadagi berilgan funksiyalar I oraliqda uzluksiz kompleks funksiyalar bo‘lsin, ya’ni $\{a_{n-1}(x),...,a_1(x),a_0(x),g(x)\}\subset C(I,\mathbb{C})$. U holda bu tenglamaning berilgan

$$y\Big|_{x_0}=y_0,\ y'\Big|_{x_0}=y_0',...,y^{(n-1)}\Big|_{x_0}=y_0^{(n-1)} \qquad (7.1.7)$$

boshlang‘ich shartlarni $(x_0\in I,\{y_0,y_0',\dots,y_0^{(n-1)}\}\subset\mathbb{C})$ qanoat-lantiruvchi $y=y(x)$ kompleks yechimi bira to‘la I oraliqda aniqlangan, $C^n(I,\mathbb{C})$ sinfga tegishli va yagona bo‘ladi (bu tasdiqni keyinroq isbotlaymiz).

Berilgan $y_1=y_1(x),\ y_2=y_2(x),\ ...,\ y_n=y_n(x)$ kompleks funksiyalarning ($\mathbb{C}$ maydon ustida) chiziqli kombinatsiyasi ushbu

$\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)$ yig‘indidan iborat; bu yerda endi $\lambda_1, \lambda_2, ..., \lambda_n$ koeffitsientlar kompleks sonlardir. Kompleks funksiyalarning ($\mathbb{C}$ maydon ustida) chiziqli erkliligi va chiziqli bog‘langanligi tushunchalari haqiqiy funksiyalar holidagidek kiritiladi. Quyidagi tasdiqlarni keltiraylik.

Agar berilgan **haqiqiy** funksiyalar $\mathbb{C}$ maydon ustida chizuqli erkli (chiziqli bog‘langan) bo‘lsa, ular $\mathbb{R}$ maydon ustida ham chiziqli erkli (mos ravishda chiziqli bog‘langan) bo‘ladi. Agar $2n$ ta $u_1(x), v_1(x), u_2(x), v_2(x), ..., u_n(x), v_n(x)$ haqiqiy funksiyalar chiziqli erkli (chiziqli bog‘langan) bo‘lsa, u holda

$$y_1(x)=u_1(x)+iv_1(x),\ y_2(x)=u_2(x)+iv_2(x),\ ,\ ...,\ y_n(x)=u_n(x)+iv_n(x)$$

$$(u_j(x)=\operatorname{Re} y_j(x)\,,\ v_j(x)=\operatorname{Im} y_j(x)\,;\ j=\overline{1,n})$$

kompleks funksiyalar ham chiziqli erkli (mos ravishda chiziqli bog‘langan) bo‘ladi. Haqiqiy funksiyalar holida § 6.1-§ 6.5 larda isbotlangan teoremalar kompleks funksiyalar uchun ham o‘z kuchini saqlaydi. Faqat endi kompleks koeffitsientli $L[y]=0$ tenglamaning (kompleks) yechimlari ($\mathbb{C}$ maydon ustida qurilgan) n o‘lchamli kompleks chiziqli fazoni tashkil etadi.

Ravshanki, agar $\mu \neq 0$ bo‘lsa, $y_1(x), y_2(x), ..., y_n(x)$ va $\mu y_1(x), y_2(x), ..., y_n(x)$ funksiyalar bir vaqtda chiziqli erkli (chiziqli bog‘langan) bo‘ladi.

$$y_1(x), y_2(x), y_3(x), ...,\ y_n(x) \quad \text{va}$$

$$y_1(x)+y_2(x), y_1(x)-y_2(x), y_3(x), ...,\ y_n(x)$$

funksiyalar bir vaqtda chiziqli erkli (yoki chiziqli bog‘langan). Bu tasdiq chiziqi erkllik ta’rifidan bevosita kelib chiqadi.

Bu yerda biz funksiyalar bir muhim sinfining chiziqli erkliligini isbotlaymiz. Dastlab yordamchi lemmani keltiraylik.

Lemma. *Kompleks koeffitsientli* $P(x)$ *ko‘phad va* $\theta \neq 0$ *kompleks son uchun*

$$\theta P(x)+P'(x)=0 \Leftrightarrow P(x)=0 \qquad (7.1.8)$$

ekvivakentlik o‘rinli..

⊶ Aytaylik,

$$P(x)=a_k x^k + a_{k-1}x^{k-1}+\cdots+a_1 x+a_0 \qquad (7.1.9)$$

bo‘lsin. Eslaylikki, agar $P(x)$ ko‘phaq nolga teng bo‘lsa, uning barcha koeffitsientlari nolga teng (va aksincha) bo‘ladi.

Ravshanki, agar $P(x)=0$ bo‘lsa, $\theta P(x)+P'(x)$ ko‘phad ham nolga teng. Aksincha: agar

$$\theta P(x)+P'(x)=0 \qquad (7.1.10)$$

bo‘lsa, bu tenglikni k marta differensiallab, $\theta k!a_k=0$ va $\theta\neq 0$ bo‘lgani uchun $a_k=0$ ekanligini topamiz. Bu ishni takrorlab, qolgan koeffitsientlarning ham nolga tengligini hosil qilamiz. Demak, $P(x)=0$. ✍

Teorema. *Berilgan* $\lambda_1,\lambda_2,\ldots,\lambda_s$ *turli kompleks sonlar va* $k_1,k_2,\ldots,k_s$ *nomanfiy butun sonlar uchun ushbu*

$$e^{\lambda_1 x},\ xe^{\lambda_1 x},\ \ldots,\ x^{k_1}e^{\lambda_1 x},\ e^{\lambda_2 x},\ xe^{\lambda_2 x},\ \ldots,\ x^{k_2}e^{\lambda_2 x},\ \ldots,\ e^{\lambda_s x},\ xe^{\lambda_s x},\ \ldots,\ x^{k_s}e^{\lambda_s x}$$

(kompleks) funksiyalar chiziqli erkli.

🗝 s bo‘yicha matematik induksiya metodini qo‘llaymiz. $s=1$ holida teorema ravshan. Teoremani s o‘rnida $s-1$ bo‘lganda o‘rinli deb faraz qilamiz va teoremani keltirib chiqaramiz. Berilgan funksiyalarning biror chiziqli kombinatsiyasi nolga teng bo‘lsin:

$$c_0e^{\lambda_1 x}+c_1xe^{\lambda_1 x}+\cdots+c_{k_1}x^{k_1}e^{\lambda_1 x}+d_0e^{\lambda_2 x}+d_1xe^{\lambda_2 x}+\cdots d_{k_2}x^{k_2}e^{\lambda_2 x}+$$

$$+\cdots+u_0e^{\lambda_s x}+u_1xe^{\lambda_s x}+\cdots+u_{k_s}x^{k_s}e^{\lambda_s x}=0;$$

bu yerdagi koeffitsientlar – kompleks sonlar. Bu ayniyatni qisqaroq ko‘rinishda yozaylik:

$$P_1(x)e^{\lambda_1 x}+P_2(x)e^{\lambda_2 x}+\cdots+P_s(x)e^{\lambda_s x}=0; \qquad (7.1.11)$$

bu yerda $P_j(x)$–ko‘phadlar $(\deg P_j(x)\le k_j\ ,\ j=\overline{1,n})$, biz ularning nolga tengligini ko‘rsatishimiz kerak. Oxirgi ayniyatning har ikkala tomonini $e^{-\lambda_1 x}$ ga ko‘paytiramiz va differensiallaymiz:

$$P_1'(x)+\big((\lambda_2-\lambda_1)P_2(x)+P_2'(x)\big)e^{(\lambda_2-\lambda_1)x}+$$
$$+\cdots+\big((\lambda_s-\lambda_1)P_s(x)+P_s'(x)\big)e^{(\lambda_s-\lambda_1)x}=0; \qquad (7.1.12)$$

bunda $\lambda_2-\lambda_1\neq 0,\ \ldots,\ \lambda_s-\lambda_1\neq 0$ va yuqoridagi lemmaga ko‘ra

$$(\lambda_2-\lambda_1)P_2(x)+P_2'(x)=0 \Leftrightarrow P_2(x)=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.,$$

$$(\lambda_s-\lambda_1)P_s(x)+P_s'(x)=0 \Leftrightarrow P_s(x)=0$$

ekvivalentliklar o‘rinli. Endi (7.1.12) ayniyatni ketma-ket k_1 marta differensiallab topamiz:

$$\tilde{P}_2(x)e^{(\lambda_2-\lambda_1)x} + \cdots + \tilde{P}_s(x)e^{(\lambda_s-\lambda_1)x} = 0;; \qquad (7.1.13)$$

bunda yana yuqoridagi lemmaga ko‘ra

$$\tilde{P}_2(x)=0 \Leftrightarrow P_2(x)=0, \ldots, \tilde{P}_s(x)=0 \Leftrightarrow P_s(x)=0.$$

Oxirgi (7.1.13) ayniyatdagi qo‘shiluvchilar soni $s-1$ ta. Induksiya faraziga ko‘ra $\tilde{P}_2(x)=...=\tilde{P}_{s-1}(x)=0$. Demak, $P_2(x)=...=P_{s-1}(x)=0$. Bu tengliklarni (7.1.11) ga qo‘yib, undan $P_1(x)=0$ ekanligini ham topamiz. ☝

Endi

$$L[y] \equiv y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0$$

tenglamaning koeffitsientlari yana haqiqiy bo‘lsin. Haqiqiy funksiyalar kompleks funksiyalarning xususiy holi bo‘lgani uchun biz bu tenglamani kompleks sohada o‘rganishimiz, ya’ni uning kompleks yechimlarini topishimiz mumkin. Bu – berilgan haqiqiy koeffitsientli **tenglamaning komplekslashtirilishi** deyiladi.

Jumla. *Faraz qilaylik, $L[y]=0$ haqiqiy koeffitsientli tenglamaning $y=u(x)+iv(x)$ kompleks yechimi ma’lum bo‘lsin. U holda bu yechimning $u(x)$ haqiqiy va $v(x)$ mavhum qismlari ham shu $L[y]=0$ tenglamaning yechimlari bo‘ladi.*

⊶ Haqiqatan ham, $L[\circ]$ operatorning chiziqlilik xossasi va koeffitsientlarining haqiqiyligiga ko‘ra

$$L[u(x)+iv(x)] = L[u(x)] + iL[v(x)] = 0 \Rightarrow L[u(x)] = L[v(x)] = 0.$$ ☝

Masalalar

1. Yuqorida keltirilgan (*) munosabatlarni isbotlang.

2. $w(x)=(x+z)^k$ $(z\in\mathbb{C}, k\in\mathbb{N})$ funksiyaning hosilasi $w'(x)=k(x+z)^{k-1}$ ga teng bo‘lishini isbotlang.

3. Ixtiyoriy $z\in\mathbb{C}$ uchun $e^z\neq 0$ ekanligini ko‘rsating.

4. Agar $I\subset\mathbb{R}$ oraliqda aniqlangan $y(x)$ kompleks funksiya uchun $y'(x)=0, x\in I$, bo‘lsa, u holda bu funksiya I oraliqda o‘zgarmas, $y(x)=\text{const}$, ekanligini ko‘rsating.

5. Ushbu $y'(x)=zy(x)$ $(z\in\mathbb{C})$ tenglamaning barcha kompleks yechimlari $y=ce^{zx}$ formula bilan berilishini isbotlang; bunda c – ixtiyoriy o‘zgarmas kompleks son.

§ 7.2. n- tartibli chiziqli o‘zgarmas koeffitsientli bir jinsli tenglamalar

Quyidagi n- tartibli chiziqli o‘zgarmas koeffitsientli bir jinsli differensial tenglamani qaraylik:

$$L[y] \overset{def}{=} y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \qquad (7.2.1)$$

Bu yerdagi koeffitsientlar kompleks bo‘lishi mumkin. Qaralayotgan (7.2.1) tenglamaning umumiy yechimini topish uchun uning n dona chiziqli erkli yechimlarini, ya’ni bazis yechimlarni qurish kerak. Umumiy yechim bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi ko‘rinishida ifodalanadi.

Berilgan (7.2.1) differensial tenglamaning bazis yechimlarini Eyler usuli yordamida topamiz. Bu usulga ko‘ra tenglamaning yechimi

$$y = e^{\lambda x} \qquad (7.2.2)$$

ko‘rinishda izlanadi; bu yerda $\lambda -$hozircha noma’lum kompleks son.

Ravshanki,

$$(e^{\lambda x})' = \lambda e^{\lambda x}, \ (e^{\lambda x})'' = \lambda^2 e^{\lambda x}, \ \ldots, \ (e^{\lambda x})^{(n)} = \lambda^n e^{\lambda x},$$

$$L[e^{\lambda x}] = L(\lambda) e^{\lambda x}; \qquad (7.2.3)$$

bu yerda

$$L(\lambda) \overset{def}{=} \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \qquad (7.2.4)$$

Demak, (7.2.2) funksiya (7.2.1) tenglamaning yechimi bo‘lishi uchun λ soni ushbu

$$L(\lambda) = 0, \text{ ya’ni}$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \qquad (7.2.5)$$

n- darajali algebraik tenglamaning ildizi bo‘lishi kerak. (7.2.4) ko‘phad va (7.2.5) algebraik tenglama (7.2.1) differensial tenglamaning mos ravishda **xarakteristik ko‘phadi** va **xarakteristik tenglamasi** deb ataladi. Xarakteristik tenglamaning ildizlari mos **differensial tenglamaning xarakteristik sonlari** deyiladi.

Masalan, $y'' - 3y' + 2y = 0$ differensial tenglamaning xarakteristik ko‘phadi $\lambda^2 - 3\lambda + 2$, xarakteristik tenglamasi $\lambda^2 - 3\lambda + 2 = 0$, xarakteristik sonlari esa $\lambda_1 = 1$ va $\lambda_2 = 2$ dan iborat bo‘ladi.

Algebradan ma'lumki, (7.2.5) n- darajali algebraik tenglama (xarakteristik tenglama) kompleks sohada karraliligi bilan birgalikda hisoblanganda n ta ildizga ega. Aniqrog'i, k_1 ($k_1 \ge 1$) karrali λ_1, k_2($k_2 \ge 1$) karrali λ_2, . . . , k_s ($k_s \ge 1$) karrali λ_s har xil ildizlar mavjud va $k_1 + k_2 + \cdots + k_s = n$ ($s \le n$) bo'ladi, ya'ni

$$L(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_s)^{k_s} . \quad (7.2.6)$$

Dastlab bir kerakli jumlani isbotlaylik.

Jumla 1. *Agar* $\lambda = \mu + i\nu$, $\{\mu, \nu\} \subset \mathbb{R}$, *bo'lsa, ixtiyoriy natural* m *uchun*

$$L[x^m e^{\lambda x}] == \sum_{j=0}^{m} C_m^j L^{(j)}(\mu + i\nu)\, x^{m-j} e^{\lambda x} \quad (7.2.7)$$

formula o'rinli bo'ladi; bu yerdagi barcha $L^{(j)}(\mu + i\nu)$ *hosilalar* μ *haqiqiy o'zgaruvchi bo'yicha hisoblangan,* $C_m^j = \binom{m}{j} = \dfrac{m!}{j!(m-j)!}$ – *binomial koeffitsientlar.*

🗝 (7.2.3) formulaga ko'ra $\lambda = \mu + i\nu$, $\{\mu, \nu\} \subset \mathbb{R}$, uchun

$$L[e^{\mu x} e^{i\nu x}] = L(\mu + i\nu) e^{\mu x} e^{i\nu x} .$$

Bu ayniyatni μ haqiqiy o'zgaruvchi bo'yicha m marta differensiallaymiz. Chap tomonda x va μ bo'yicha differensiallash tartibini almashtirib, topamiz:

$$\frac{\partial}{\partial \mu} L[e^{\mu x} e^{i\nu x}] = L[\frac{\partial}{\partial \mu} e^{\mu x} e^{i\nu x}] = L[x e^{\mu x} e^{i\nu x}] = L[x e^{\lambda x}],$$

ya'ni

$$L[x e^{\lambda x}] = \frac{\partial}{\partial \mu}\Big(L(\mu + i\nu) e^{(\mu + i\nu)x} \Big).$$

Shu ishni takrorlab,

$$L[x^m e^{\lambda x}] = \frac{\partial^m}{\partial \mu^m}\Big(L(\mu + i\nu) e^{(\mu + i\nu)x} \Big)$$

formulaga ega bo'lamiz. Endi quyidagi hisoblashlarni, ko'paytmaning hosilasi uchun (kompleks funkiyalar holida ham o'rinli bo'lgan) Leybnits formulasidan foydalanib, bajaramiz:

$$L[x^m e^{\lambda x}] = \frac{\partial^m}{\partial \mu^m}\Big(L(\mu + i\nu) e^{\mu x} \Big) e^{i\nu x} = \sum_{j=0}^{m} C_m^j L^{(j)}(\mu + i\nu)(e^{\mu x})^{(m-j)} e^{i\nu x} =$$

$$=\sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)x^{m-j}e^{\mu x}e^{i\nu x}=\sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)x^{m-j}e^{\lambda x}.$$ ✍

Jumla 2. *Agar λ_j xarakteristik son k_j karrali bo'lsa, u holda (7.2.1) differensial tenglama k_j dona*

$$y_1=e^{\lambda_j x},\ y_2=xe^{\lambda_j x},\ \ldots,\ y_{k_j}=x^{k_j-1}e^{\lambda_j x} \qquad (7.2.8)$$

yechimlarga ega.

🗝 λ_j ning k_j karrali xarakteristik son ekanligi ushbu

$$L(\lambda)=(\lambda-\lambda_j)^{k_j}M(\lambda)$$

tenglikning o'rinliligini anglatadi; bu yerda $M(\lambda)$ ko'phadning darajasi $(n-k_j)$ga teng va $M(\lambda_j)\neq 0$. Demak, $\lambda=\mu+i\nu$, $\{\mu,\nu\}\subset\mathbb{R}$, uchun

$$L(\mu+i\nu)=(\mu+i\nu-\lambda_j)^{k_j}M(\mu+i\nu),\ L(\lambda_j)=L'(\lambda_j)=\ldots=L^{(k_j-1)}(\lambda_j)=0,$$

$$L^{(k_j)}(\lambda_j)=k_j!\cdot M(\lambda_j)\neq 0\ . \qquad (7.2.9)$$

Yuqoridagi (7.2.3) ayniyatdan $L[e^{\lambda_j x}]=0$, ya'ni $y_1=e^{\lambda_j x}$ funksiya (7.2.1) tenglamaning yechimi ekanligi ravshan. Endi (7.2.8) dagi qolgan funksiyalarning ham yechim bo'lishini ko'rsatamiz. Buning uchun (7.2.7) formuladan $m=1,2,\ldots,k_j-1$ uchun $\lambda=\mu+i\nu=\lambda_j$ deb va (7.2.9) ni hisobga olib, topamiz:

$$L[xe^{\lambda_j x}]=\left(L(\lambda_j)x+L'(\lambda_j)\right)e^{\lambda_j x}=0,$$

$$L[x^2e^{\lambda_j x}]=\left(L(\lambda_j)x^2+2L'(\lambda_j)x+L''(\lambda_j)\right)e^{\lambda_j x}=0,$$

....

$$L[x^{k_j-1}e^{\lambda_j x}]=\Big(L(\lambda_j)x^{k_j-1}+C_{k_j-1}^1L'(\lambda_j)x^{k_j-2}+$$

$$+C_{k_j-1}^2L''(\lambda_j)x^{k_j-3}+\cdots+L^{(k_j-1)}(\lambda_j)\Big)e^{\lambda_j x}=0.$$

Shunday qilib, (7.2.8) da keltirilgan barcha funksiyalar (7.2.1) tenglamaning yechimi bo'lishi isbotlandi. ✍

Izoh. Biz kompleks funksiyadan kompleks o'zgaruvchi bo'yicha hosila tushunchasi va uning xossalari bilan tanish bo'lmaganligimiz sababli $\lambda\in\mathbb{C}$ ning haqiqiy μ va mavhum ν qismlarini ajratib, μ haqiqiy o'zgaruvchi bo'yicha differensiallashdan foydalandik. Kompleks o'zgaruvchiga nisbatan hosila tushunchasi bilan tanish o‘quvchilar to‘g‘ridan-to‘g‘ri $\lambda \in \mathbb{C}$ o‘zgaruvchi bo‘yicha hosiladan foydalanishi mumkin.

Isbotlangan jumla 2 dan foydalanib, (7.2.1) differensial tenglamaning k_1 karrali λ_1, k_2 karrali λ_2, . . . , k_s karrali λ_s turli xarakteristik sonlariga ko‘ra uning $k_1 + k_2 + \cdots + k_s = n$ dona yechimini tuzamiz:

$$e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^{k_1 -1}e^{\lambda_1 x} \ \ (k_1 \text{ ta})$$
$$e^{\lambda_2 x}, xe^{\lambda_2 x}, \ldots, x^{k_2 -1}e^{\lambda_2 x} \ \ (k_2 \text{ ta})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e^{\lambda_s x}, xe^{\lambda_s x}, \ldots, x^{k_s -1}e^{\lambda_s x} \ \ (k_s \text{ ta})$$

Bu yechimlarning chiziqli erkliligi § 7.1 da isbotlangan edi. Demak, biz (7.2.1) tenglamaning bazis yechimlarini topdik.

Misol 1. Ushbu

$$y''' - 4y'' + 5y' - 2y = 0$$

diefferensial tenglamani yeching.

🗝 Mos xarakteristik tenglamani tuzamiz:

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0.$$

Xarakteristik tenglamaning ildizlarini topamiz:

$$\begin{aligned} &\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda^3 - 4\lambda^2 + 4\lambda) + \lambda - 2 = \\ &= \lambda(\lambda - 2)^2 + \lambda - 2 = (\lambda - 1)^2(\lambda - 2) = 0. \end{aligned};$$

Demak, $\lambda = 1-$ ikki karrali, $\lambda = 2-$oddiy (bir karrali) ildiz. Bu xarakteristik sonlarga ko‘ra qaralayotgan differensial tenglamaning bazis yechimlarini tuzamiz:

$$e^x, xe^x, e^{2x}.$$

Berilgan differensial tenglamaning umumiy yechimi $y = c_1 e^x + c_2 xe^x + c_3 e^{2x}$ formula bilan ifodalanadi. Bu yerdagi c_1, c_2, c_3 lar ixtiyoriy kompleks qiymatlar qabul qilsa, kompleks sohadagi umumiy yechim, ular ixtiyoriy haqiqiy qiymatlar qabul qilganda esa, haqiqiy sohadagi umumiy yechim hosil bo‘ladi. 👍

Endi (7.2.1) tenglamaning koeffitsientlari haqiqiy bo‘lganda uning haqiqiy yechimlarini qurishda to‘xtalamiz. Xarakteristik tenglama (7.2.5) ning koeffitsientlari haqiqiy bo‘lgani uchun u k karrali $\lambda = \mu + i\nu \ (\nu \neq 0)$ kompleks ildiz bilan birgalikda unga qo‘shma bo‘lgan k karrali $\overline{\lambda}=\mu-i\nu$ $(\nu\neq 0)$ kompleks ildizga ham ega. Eyler formulalariga ko‘ra $2k$ ta

$$e^{(\mu+i\nu)x},\ xe^{(\mu+i\nu)x},\ \ldots,\ x^{k-1}e^{(\mu+i\nu)x},\ e^{(\mu-i\nu)x},\ xe^{(\mu-i\nu)x},\ \ldots,\ x^{k-1}e^{(\mu-i\nu)x}$$

kompleks yechimlardan yana $2k$ ta quyidagi haqiqiy yechimlarni quramiz:

$$e^{\mu x}\cos\nu x=\frac{1}{2}(e^{(\mu+i\nu)x}+e^{(\mu-i\nu)x}),\ e^{\mu x}\sin\nu x=\frac{1}{2i}(e^{(\mu+i\nu)x}-e^{(\mu-i\nu)x}),$$

$$xe^{\mu x}\cos\nu x=\frac{1}{2}(xe^{(\mu+i\nu)x}+xe^{(\mu-i\nu)x}),$$

$$xe^{\mu x}\sin\nu x=\frac{1}{2i}(xe^{(\mu+i\nu)x}-xe^{(\mu-i\nu)x}),$$

....

$$x^{k-1}e^{\mu x}\cos\nu x=\frac{1}{2}(x^{k-1}e^{(\mu+i\nu)x}+x^{k-1}e^{(\mu-i\nu)x}),$$

$$x^{k-1}e^{\mu x}\sin\nu x=\frac{1}{2i}(x^{k-1}e^{(\mu+i\nu)x}-x^{k-1}e^{(\mu-i\nu)x}).$$

Boshqa kompleks xarakteristik sonlarga ko‘ra ham mos haqiqiy yechimlarni tuzamiz va berilgan haqiqiy koeffitsientli differensial tenglamaning haqiqiy bazis yechimlarini topamiz.

Misol 2. Ushbu

$$y''-2y'+2y=0$$

differensial tenglamaning umumiy yechimini toping.

🗝Bu differensial tenglamaning xarakteristik sonlari kompleks: $\lambda^2-2\lambda+2=0\Rightarrow\lambda_1=1+i$ va $\lambda_2=1-i$. Demak, (kompleks) bazis yechimlar $e^{(1+i)x}, e^{(1-i)x}$; umumiy yechim esa kompleks sohada $y=c_1e^{(1+i)x}+c_2e^{(1-i)x}$ ko‘rinishga ega (c_1,c_2-ixtiyoroy kompleks o‘zgarmaslar). Biz yechimni kompleks sohada topdik. Differensial tenglama esa haqiqiy sohada berilgan edi. Bunday holda odarda haqiqiy yechimlarni topish talab etiladi. Haqiqiy bazis yechimlarni qurish uchun kompleks yechimning haqiqiy va mavhum qismilarini ajratamiz yoki Eyler formulalaridan foydalanamiz. Tushunarliki, $e^x\cos x$, $e^x\sin x$ $-$haqiqiy bazis yechimlar. Qaralayotgan differensial tenglamaning haqiqiy sohadagi umumiy yechimi $y=c_1e^x\cos x+c_2e^x\sin x$ (c_1,c_2-ixtiyoroy haqiqiy o‘zgarmaslar). 👍

Misol 3. Ushbu

$$y^{V}-y^{IV}+18y'''-18y''+81y'-81y=0$$

differensial tenglamani yeching.

Mos xarakteristik tenglamani tuzamiz va uni yechamiz:

$$\lambda^5-\lambda^4+18\lambda^3-18\lambda^2+81\lambda-81=0, (\lambda-1)(\lambda^2+9)^2=0.$$

Demak, xarakteristik sonlar : $\lambda=1$ (oddiy), $\lambda=\pm 3i$ (ikki karrali). Haqiqiy bazis yechimlar: $e^x, \cos 3x, \sin 3x, x\cos 3x, x\sin 3x$. Umumiy yechim (haqiqiy sohada):

$$y=c_1e^x+c_2\cos 3x+c_3\sin 3x+c_4x\cos 3x+c_5x\sin 3x.$$

Eyler tenglamasi (bir jinsli) deb ushbu

$$x^n y^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\cdots+xa_1y'+a_0y=0 \qquad (7.2.10)$$

ko‘rinishdagi chiziqli differensial tenglamaga aytiladi; bunda $a_{n-1},\dots,a_1,a_0$ – o‘zgarmas sonlar. Bu tenglama erkli o‘zgaruvchini $x>0$ bo‘lganda $x=e^t$ ($x<0$ bo‘lganda esa $x=-e^t$) almashtirish yordamida o‘zgarmas koeffitsientli chiziqli tenglamaga keltiriladi. Bunga ishonch hosil qilish uchun y noma’lum funksiyaning x bo‘yicha hosilalarini $y$ning t bo‘yicha hosilalari orqali ifodalash kerak. Quyidagi hisoblashlarni bajaramiz ($x>0, x=e^t$):

$$x'_t=e^t=x,\ t'_x=x^{-1}=e^{-t},\quad y'_x=y'_t\cdot t'_x=y'_te^{-t}\ (=x^{-1}y_t),$$

$$y''_{xx}=(y'_te^{-t})'_t\cdot t'_x=(y''_{tt}-y'_t)e^{-2t}\ \left(=x^{-2}(y''_{tt}-y'_t)\right),$$

Endi ixtiyoriy $k\in\mathbb{N}$ uchun

$$y_x^{(k)}=(b_{k,k}y_t^{(k)}+b_{k,k-1}y_t^{(k-1)}+\cdots+b_{k,1}y'_t)e^{-kt},$$

bunda $b_{k,j}$ lar – o‘zgarmas sonlar, bo‘lishini ko‘rish qiyin emas. Buni matematik induksiya metodi yordamida qat’iy isbotlashni o‘quvchiga havola etamiz. Hosilalar uchun tayyorlangan ifodalarni (7.2.10) tenglamaga qo‘yib, o‘zgarmas koeffitsientli chiziqli tenglamaga kelamiz. Uning xarakteristik tenglamasini tuzish uchun tenglamaga $y=e^{\lambda t}$ qo‘yib, $e^{\lambda t}$ ga qisqartirish kerak. $x=e^t$ bo‘lgani uchun buning o‘rniga $y=x^\lambda$ funksiyani to‘g‘ridan-to‘g‘ri (7.2.10) tenglamaga qo‘yib, x^λga qisqartirish mumkin. Natijada λ noma’lumga nisbatan ushbu

$$\begin{aligned}&\lambda(\lambda-1)\dots(\lambda-n+1)+a_{n-1}\lambda(\lambda-1)\dots(\lambda-n+2)+\\&+\cdots+a_1\lambda+a_0=0\end{aligned} \qquad (7.2.11)$$

n- darajali algebraik tenglama hosil bo‘ladi. Bu (7.2.11) tenglama va uning ildizlari (7.2.10) ning mos ravishda **xarakteristik tenglamasi** va **xarakteristik sonlari** deb ataladi. (7.2.10) Eyler tenglamasining umumiy yechimi uning xarakteristik sonlari bilan aniqlanadi. Har bir k karrali λ xarakteristik songa (7.2.10) tenglamaning k dona ushbu

$$x^{\lambda},\ x^{\lambda}\ln x,\ \ldots,\ x^{\lambda}(\ln x)^{k-1}$$

chiziqli erkli yechimlari mos keladi. Agar (7.2.10) tenglamaning barcha koeffitsientlari haqiqiy bo‘lsa, (7.2.11) xarakteristik tenglama k karrali $\lambda=\mu+i\nu$ $(\mu\neq 0)$ xarakteristik son bilan birgalikda k karrali $\bar{\lambda}=\mu-i\nu$ xarakteristik songa ham ega bo‘ladi. Bu xarakteristik sonlarga $2k$ dona ushbu

$$x^{\mu}\cos(\nu\ln x),\ x^{\mu}\sin(\nu\ln x),\ x^{\mu}\ln x\cos(\nu\ln x),$$

$$x^{\mu}\ln x\sin(\nu\ln x),\ \ldots,\ x^{\mu}(\ln x)^{k-1}\cos(\nu\ln x),\ x^{\mu}(\ln x)^{k-1}\sin(\nu\ln x)$$

chiziqli erkli haqiqiy yechimlar mos keladi. Barcha xarakteristik sonlarga mos keluvchi yechimlarni to‘plab, Eyler tenglamasining bazis yechimlarini hosil qilamiz.

Misol 7. Tenglamaning umumiy yechimini toping

$$x^3y''' + 2x^2y'' - xy' + y = 0.$$

🗝 Xarakteristik tenglamani tuzamiz:

$$\lambda(\lambda-1)(\lambda-2)+2\lambda(\lambda-1)-\lambda+1=0.$$

Xarakteristik sonlar: $\lambda_1=-1$, $\lambda_2=\lambda_3=1$. Oddiy xarakteristik son ($\lambda_1=-1$) ga mos keluvchi yechim: $y_1=\dfrac{1}{x}$. Ikki karrali xarakteristik son ($\lambda_2=\lambda_3=1$) ga mos keluvchi yechimlar: $y_2=x$, $y_3=x\ln x$ $(x>0$). Demak, $x>0$ sohadagi umumiy yechim:

$$y=\frac{c_1}{x}+c_2x+c_3x\ln x\ .$$

$x<0$ sohadagi umumiy yechim esa $y=\dfrac{c_1}{x}+c_2x+c_3x\ln(-x)$ formula bilan beriladi. 👍

Masalalar

Tenglamalarni yeching:

1. $y'''-3y''+3y'-y=0$. **2.** $y^{V}-6y^{IV}+16y'''-24y''+20y'-8y=0$.

Xarakteristik sonlariga ko‘ra mos differensial tenglamalarni tuzing:

3. $\lambda_1=\lambda_2=\lambda_3=1$. **4.** $\lambda_1=\lambda_2=1-i,\lambda_3=\lambda_4=1+i$.

Umumiy yechimiga ko‘ra mos differensial tenglamani tuzing:

5. $y = c_1 e^{2x} + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$.

6. Eyler tenglamasini yeching $x^3 y''' - x^2 y'' + 2xy' - 2y = 0 \ (x > 0)$

§ 7.3. Bir jinsli bo‘lmagan tenglamalar: ozod had – kvaziko‘phad

Chiziqli o‘zgarmas koeffitsientli differensial operator $L[\circ]$ (7.2.1) formula bilan aniqlangan bo‘lsin. Bir jinsli bo‘lmagan

$$L[y] = g(x) \tag{7.3.1}$$

tenglamaning umumiy yechimini topish uchun, ma’lumki, bu tenglamaning xususiy yechimiga mos bir jinsli $L[y]=0$ tenglamaning umumiy yechimini qo‘shish kerak. $L[y]=0$ tenglamaning bazis yechimlarini qurishni o‘rgandik. Bu yechimlardan foydalanib $L[y]=g(x)$ tenglamaning xususiy yechimini Lagranjning ixtiyoriy o‘zgarmaslarni variatsiyalash usuli bilan yoki § 6.5 da keltirilgan Koshi formulasiga ko‘ra qurish mumkin. Bunda integrallash amali ishlatiladi. Lekin o‘ng tomondagi $g(x)$ funksiya $g(x) = P(x)e^{\sigma x}$ ($P(x)$ – ko‘phad, σ – son) maxsus ko‘rinishga ega bo‘lganda xususiy yechimni quyida bayon etilgan **noma’lum koeffitsientlar metodi** deb ataluvchi metod yordamida integrallash amalini ishlatmasdan turib topish mumkin. $P(x)e^{\sigma x}$ funksiya kvaziko‘phad deb ataladi.

Faraz qilaylik, $P(x)$ ko‘phad

$$P(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 , \ b_m \neq 0, \tag{7.3.2}$$

ko‘rinishda bo‘lsin. Ushbu

$$L[y] = P(x)e^{\sigma x} \tag{7.3.3}$$

tenglamaning xususiy yechimini topish uchun (7.2.3) va (7.2.7) formulalardan kelib chiqib, quyidagicha ish tutish mumkin. σ son $L[y]=0$ tenglamaning xarakteristik soni bo‘lgan holni ($L(\sigma) \neq 0$) norezonans holi, aks holni esa ($L(\sigma) = 0$) rezonans holi deb ataymiz.

Teorema 1. *Agar (7.3.3) tenglamadagi* σ *son uchun* $L(\sigma) \neq 0$ *bo‘lsa, ya’ni norezonans holida*

$$L[y] = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x}$$

(7.3.3) tenglama

$$y = (r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0)e^{\sigma x} \qquad (7.3.4)$$

ko'rinishdagi xususiy yechimga ega.

🗝 $P(x)$ ko'phadning darajasi m bo'yicha matematik induksiya metodini qo'llaymiz. Dastlab $m = \deg P(x) = 0$ deymiz va

$$L[y] = b_0 e^{\sigma x}$$

tenglama $y = r_0 e^{\sigma x}$ ko'rinishdagi xususiy yechimga egaligini ko'rsatamiz. (7.3.4) formulaga ko'ra

$$L[r_0 e^{\sigma x}] = r_0 e^{\sigma x} L(\sigma) = b_0 e^{\sigma x}.$$

Bundan $L[y] = b_0 e^{\sigma x}$ tenglama qanoatlanishi uchun $r_0 = \dfrac{b_0}{L(\sigma)}$ deyish kifoya ekanligi kelib chiqadi. Demak, $m = 0$ holida teorema isbotlandi.

Endi induksiya farazini qilib, ya'ni $L[y] = \tilde{P}(x)e^{\sigma x}$, $\deg\tilde{P}(x) \le m-1$, tenglamaning $y = \tilde{Q}(x)e^{\sigma x}$, $\deg\tilde{Q}(x) \le m-1$, ko'rinishdagi yechimga ega ekanligini ma'lum deb, $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m$, tenglamaning $y = Q(x)e^{\sigma x}$, $\deg Q(x) = m$, ko'rinishdagi yechimga egaligini isbotlashimiz kerak. Berilgan

$$L[y] = P(x)e^{\sigma x}, P(x) = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0),$$

tenglamaning yechimini

$$y = r_m x^m e^{\sigma x} + u \qquad (7.3.5)$$

ko'rinishda izlaymiz; bu yerda r_m – hozircha noma'lum son, u – yangi noma'lum funksiya. Demak,

$$r_m L[x^m e^{\sigma x}] + L[u] = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x} \qquad (7.3.6)$$

tenglik qanoatlanishi kerak. Endi r_m ni tanlash evaziga (7.3.6) tenglamadan x^m qatnashgan hadni yo'qotishga harakat qilamiz. (7.2.3) ayniyatga ko'ra

$$\begin{aligned} L[x^m e^{\sigma x}] &= \big(L(\sigma)x^m + C_m^1 L'(\sigma)x^{m-1} + C_m^2 L''(\sigma)x^{m-2} + \\ &+ \cdots + L^{(m)}(\sigma)\big)e^{\sigma x} = L(\sigma)x^m e^{\sigma x} + R(x)e^{\sigma x} \end{aligned}; \qquad (7.3.7)$$

bu yerdagi $R(x) = C_m^1 L'(\sigma)x^{m-1} + C_m^2 L''(\sigma)x^{m-2} + \cdots + L^{(m)}(\sigma)$ ko'phadning darajasi $\deg R(x) \le m-1$. Endi (7.3.7) ni (7.3.6) ga qo‘yamiz , $r_m = \dfrac{b_m}{L(\sigma)}$ deb tanlaymiz va u noma’lum funksiya uchun ushbu

$L[u]= \tilde{P}(x)e^{\sigma x}$, $\tilde{P}(x) \overset{def}{=} b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 - R(x), \deg\tilde{P}(x) \le m-1,$ x^m qatnashmagan tenglamani hosil qilamiz.

Induksiya faraziga ko‘ra bu tenglama $u = \tilde{Q}(x)e^{\sigma x}$, $\deg\tilde{Q}(x) \le m-1,$ ko‘rinishdagi yechimga ega. Demak, (7.3.5) formulaga ko‘ra $L[y] = P(x)e^{\sigma x}, \deg P(x) = m,$ tenglama

$y = r_m x^m e^{\sigma x} + \tilde{Q}(x)e^{\sigma x}$, ya’ni

$y = Q(x)e^{\sigma x}, Q(x) \overset{def}{=} r_m x^m + \tilde{Q}(x), \deg Q(x) = m,$

ko‘rinishdagi yechimga ega. ✍

Teorema 2. *Rezonans holida, agar (7.3.3) tenglamadagi* σ *son* $L(\lambda) = 0$ *xarakteristik tenglamaning* k *karrali ildizi bo‘lsa, u holda*

$$L[y] = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x}$$

(7.3.3) tenglama

$$y = x^k(r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0)e^{\sigma x} \qquad (7.3.8)$$

ko‘rinishdagi yechimga ega.

🗝Teorema 1 ning isbotiga o‘xshash fikr yuritamiz, ya’ni $\deg P(x) = m$ bo‘yicha matematik induksiya metodini qo‘llaymiz, faqat endi sal nozikroq fikrlash kerak bo‘ladi. Dastlab teoremaning shartiga ko‘ra $L(\lambda) = (\lambda - \sigma)^k M(\lambda),\ M(\sigma) \ne 0\ (k \in \mathbb{N}),$ ekanligini e’tirof etaylik. Demak,

$$L(\sigma) = L'(\sigma) = \ldots = L^{(k-1)}(\sigma) = 0,\ L^{(k)}(\sigma) = k!\cdot M(\sigma) \ne 0. \qquad (7.3.9)$$

$m = 0$ bo‘lsin. Ushbu

$$L[y] = b_0 e^{\sigma x} \qquad (7.3.10)$$

tenglama $y = r_0 x^k e^{\sigma x}$ ko‘rinishdagi yechimga ega bo‘lishini ko‘rsatishimiz kerak. Buning uchun $L[r_0 x^k e^{\sigma x}]$ ni hisoblaymiz. (7.2.7) formuladan (7.3.9) ga ko‘ra quyidagini topamiz:

$$L[r_0 x^k e^{\sigma x}] = r_0\big(L(\sigma)x^k + C_k^1 L'(\sigma)x^{k-1} +$$
$$+ C_k^2 L''(\sigma)x^{k-2} + \cdots + L^{(k)}(\sigma)\big)e^{\sigma x} = r_0 k!\cdot M(\sigma)e^{\sigma x}.$$

Demak, agar $r_0 = \dfrac{b_0}{k!\cdot M(\sigma)}$ deb tanlasak, $y = r_0 x^k e^{\sigma x}$ funksiya (7.3.10) tenglamaning yechimi bo‘ladi. Teorema $m = 0$ holida isbot bo‘ldi.

Endi induksiya farazini qilamiz, ya’ni $L[y] = \tilde{P}(x)e^{\sigma x}$, $\deg\tilde{P}(x) \le m-1$, tenglama $y = x^k\tilde{Q}(x)e^{\sigma x}$, $\deg\tilde{Q}(x) \le m-1$, ko‘rinishdagi yechimga ega ekanligini ma’lum deymiz va $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m,$ tenglamaning $y = x^k Q(x)e^{\sigma x}$, $\deg Q(x) = m,$ ko‘rinishdagi yechimga egaligini isbotlaymiz. Berilgan tenglamaning yechimini

$$y = r_m x^{k+m} e^{\sigma x} + u \qquad (7.3.11)$$

ko‘rinishda izlaymiz; bu yerda $u = u(x) -$yangi noma’lum funksiya, $r_m -$noma’lum son. Biz ularni berilgan tenglamaning qanoatlanishi, ya’ni

$$L[r_m x^{k+m} e^{\sigma x} + u] = P(x)e^{\sigma x}$$

shartidan topishimiz kerak. Bu tenglamani

$$r_m L[x^{k+m} e^{\sigma x}] + L[u] = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x} \qquad (7.3.12)$$

ko‘rinishda yozib olamiz va $r_m$ni shunday tanlaymizki, natijada (7.3.12) tenglamada x^m qatnashmasin. Buning uchun $L[x^{k+m}e^{\sigma x}]$ni hisoblaymiz. (7.2.7) formuladan (7.3.9) munosabatlarni hisobga olib topamiz:

$$L[x^{k+m}e^{\sigma x}] = \Big(L(\sigma)x^{k+m} + C^1_{k+m}L'(\sigma)x^{k+m-1} + \cdots + C^{k-1}_{k+m}L^{(k-1)}(\sigma)x^{m-1} +$$

$$+ C^k_{k+m}L^{(k)}(\sigma)x^m + C^{k+1}_{k+m}L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma)\Big)e^{\sigma x} =$$

$$= \Big(C^k_{k+m}L^{(k)}(\sigma)x^m + C^{k+1}_{k+m}L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma)\Big)e^{\sigma x} =$$

$$= \Big(C^k_{k+m}k!\cdot M(\sigma)x^m + S(x)\Big)e^{\sigma x}, \qquad (7.3.13)$$

bunda $S(x) = C^{k+1}_{k+m}L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma)$ ko‘phadning darajasi $\deg S(x) \le m-1.$ Endi (7.3.13) ni (7.3.12) ga qo‘yib, $r_m = \dfrac{b_m}{C^k_{k+m}k!\cdot M(\sigma)}$ deb tanlab, u noma’lum funksiya uchun ushbu

$$L[u] = \tilde{P}(x)e^{\sigma x},$$
$$\tilde{P}(x) \overset{def}{=} (b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 - S(x)),\ \deg\tilde{P}(x) \le m-1, \tag{7.3.14}$$
tenglamani hosil qilamiz. Induksiya faraziga ko‘ra (7.3.14) tenglama $u = x^k\tilde{Q}(x)e^{\sigma x}$, $\deg\tilde{Q}(x) \le m-1$, ko‘rinishdagi yechimga ega. (7.3.11) formulaga ko‘ra $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m$, tenglama esa
$$y = r_m x^{k+m}e^{\sigma x} + u = r_m x^{k+m}e^{\sigma x} + x^k\tilde{Q}(x)e^{\sigma x} =$$
$$= x^k\left(r_m x^m + \tilde{Q}(x)\right)e^{\sigma x} = x^k P(x)e^{\sigma x}$$
ko‘rinishdagi yechimga ega bo‘ladi; bunda $P(x) = r_m x^m + \tilde{Q}(x)$, $\deg P(x) = m$. ✍

Isbotlangan teoremalardan
$$L[y] = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x}$$
tenglamaning xususiy yechimini topish uchun quyidagi **noma’lum koeffitsientlar metodidan** foydalanish mumkinligi kelib chiqadi:

1^0. $L(\sigma)$ ni hisoblaymiz. Agar $L(\sigma) \ne 0$ bo‘lsa (norezonans holi), $k = 0$ deymiz, aks holda esa (rezonans holi), ya’ni $L(\sigma) = 0$ bo‘lganda k bilan σ ildizning karralilik darajasini belgilaymiz.

2^0. Tenglamaning yechimini
$$y = x^k(r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0)e^{\sigma x}$$
ko‘rinishda izlaymiz; bunda $r_m, r_{m-1}, \ldots, r_1, r_0 -$ hozircha noma’lum koeffitsientlar.

3^0. Bu y ni berilgan tenglamaga qo‘yib, chap tomonni soddalashtirib, tenglamani $e^{\sigma x}$ ga qisqartirib, ko‘phadlarning tengligini hosil qilamiz.

4^0. Chap va o‘ng tomondagi kophadlardagi $x$ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, $r_m, r_{m-1}, \ldots, r_1, r_0 -$ noma’lum koeffitsientlarga nisbatan chiziqli algebraik tenglamalar sistemasini topamiz.

5^0. Bu sistemadan noma’lum koeffitsientlarni aniqlaymiz (sistemaning yechimga egaligini yuqorida isbotlangan teoremalar ta’minlaydi).

6^0. Koeffitsientlarning topilgan qiymatlarini
$$y = x^k(r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0)e^{\sigma x}$$

ga qo‘yib, izlangan xususiy yechimni hosil qilamiz.

Misol 4. Tenglamani yeching

$$y'' - (1+2i)y' + (-1+i)y = 2xe^{ix}.$$

🗝Berilgan tenglamaning umumiy yechimi mos bir jinsli tenglamaning umumiy yechimiga berilgan bir jinsli bo‘lmagan tenglamaning xususiy yechimini qo‘shishdan hosil bo‘ladi. Bir jinsli tenglamaning xarakteristik ko‘phadi: $L(\lambda) = \lambda^2 - (1+2i)\lambda - 1 + i$. Uning ildizlari $\lambda_1 = i, \lambda_2 = 1+i$. Bir jinsli tenglamaning umumiy yechimi

$$y = c_1 e^{ix} + c_2 e^{(1+i)x} \quad (\{c_1, c_2\} \subset \mathbb{C}).$$

Endi berilgan bir jinsli bo‘lmagan tenglamaning xususiy yechimini topamiz. Tenglamaning o‘ng tomoni $P(x)e^{\sigma x}$ uchun $\sigma = i,\ P(x) = 2x,\ \deg P(x) = 1$. Ravshanki i soni $L(\lambda) = 0$ tenglamaning $k = 1$ karrali (oddiy) ildizi. Demak, xususiy yechimni $y = x(r_1 x + r_0)e^{ix}$ ko‘rinishda izlash mumkin. Bu y ni va uning $y' = \left(ir_1 x^2 + (2r_1 + ir_0)x + r_0\right)e^{ix}, \quad y'' = \left(-r_1 x^2 + (-r_0 + i4r_1)x + 2r_1 + r_0\right)e^{ix}$ hosilalarini berilgan tenglamaga qo‘yib, e^{ix}ga qisqartirib, chap tomondagi o‘xshash hadlarni ixchamlab, topamiz:

$$-2r_1 x + 2r_1 - r_0 = 2x.$$

Bundan $r_1 = -1,\ r_0 = 2r_1 = -2$ bo‘lishi kelib chiqadi. Bu qiymatlarni $y = x(r_1 x + r_0)e^{ix}$ ga qo‘yib, berilgan tenglamaning $y = -x(x+2)e^{ix}$ xususiy yechimini hosil qilamiz. Uning umumiy yechimi esa

$$y = c_1 e^{ix} + c_2 e^{(1+i)x} - x(x+2)e^{ix} \quad (\{c_1, c_2\} \subset \mathbb{C})$$

formula bilan beriladi. 👍

Endi haqiqiy sohada berilgan va o‘ng tomoni maxsus ko‘rinishga ega bo‘lgan ushbu

$$L[y] = e^{\alpha x}(P(x)\cos\beta x + Q(x)\sin\beta x) \qquad (7.3.15)$$

tenglamaning xususiy yechimini topishda to‘xtalamiz. Bunda $L[\circ]$ operatorning koeffitsientlari va $\alpha, \beta -$haqiqiy sonlar; $P(x), Q(x) -$ haqiqiy koeffitsientli ko‘phadlar va $\deg P(x) = m_1,\ \deg Q(x) = m_2$. $e^{\alpha x}(P(x)\cos\beta x + Q(x)\sin\beta x)$ korinishdagi funksiya ham (haqiqiy sohadagi) kvaziko‘phad deb ataladi. Kompleks sohaga o‘tib Eyler formulalariga ko‘ra quyidagini yozamiz:

$$e^{\alpha x}(P(x)\cos\beta x+Q(x)\sin\beta x)=e^{\alpha x}(P(x)\operatorname{Re}e^{i\beta x}+$$
$$+Q(x)\operatorname{Re}(-ie^{i\beta x}))=\operatorname{Re}\Big(\big(P(x)-iQ(x)\big)e^{(\alpha+i\beta)x}\Big).$$

Demak,

$$L[y]=\big(P(x)-iQ(x)\big)e^{(\alpha+i\beta)x}$$

tenglamaning $y=x^k(r_m x^m+r_{m-1}x^{m-1}+\cdots+r_1x+r_0)e^{(\alpha+i\beta)x}$ (bunda $m=\max\{m_1,m_2\}$, agar $L(\alpha+i\beta)\neq 0$ bo‘lsa, $k=0$, $L(\alpha+i\beta)=0$ bo‘lganda esa k soni $\sigma=\alpha+i\beta$ ildizning karralilik darajasi) ko‘rinishdagi kompleks yechimini topib, unung haqiqiy qismini hisoblasak, berilgan (7.3.15) tenglamaning xususiy yechimini topgan bo‘lamiz.

Misol 5. Ushbu

$$y''-4y'+5y=e^x(3\cos x+\sin x) \qquad (7.3.16)$$

tenglamaning xususiy yechimini toping.

🗝 Tenglamaning o‘ng tomonini kompleks funksiyaning haqiqiy qismi ko‘rinishida ifodalaymiz:

$$e^x(3\cos x+\sin x)=e^x\Big(3\operatorname{Re}e^{ix}-\operatorname{Re}(ie^{ix})\Big)=\operatorname{Re}\Big((3-i)e^{(1+i)x}\Big).$$

Endi ushbu

$$y''-4y'+5y=(3-i)e^{(1+i)x} \qquad (7.3.17)$$

tenglamaning xususiy yechimini topamiz. Bu tenglamaga mos bir jinsli tenglamaning xarakteristik ko‘phadi $L(\lambda)=\lambda^2-4\lambda+5$ bo‘lgani uchun $\sigma=1+i$ uning ildizi emas. Demak, (7.3.17) differensial tenglamaning xususiy yechimini $y=re^{(1+i)x}$ ko‘rinishda izlash mumkin. Bu $y=re^{(1+i)x}$ni tenglamaga qo‘yib, $e^{(1+i)x}$ ga qisqartirib, topamiz:

$$(1+i)^2r-4(1+i)r+5r=3-i.$$

Bundan $r=1+i$. Demak, $y=(1+i)e^{(1+i)x}$ funksiya (7.3.17) tenglamaning xususiy yechimi. Endi oxirgi funksiyaning haqiqiy qismini topamiz:

$$\operatorname{Re}\Big((1+i)e^{(1+i)x}\Big)=e^x\operatorname{Re}\big((1+i)(\cos x+i\sin x)\big)=e^x(\cos x-\sin x).$$

Topilgan $y=e^x(\cos x-\sin x)$ haqiqiy funksiya berilgan (7.3.16) tenglamaning xususiy yechimidir. 👍

Biz yuqorida haqiqiy sohada berilgan (7.3.15) tenglamaning xususiy yechimini kompleks sohaga o‘tib topdik. Xususiy yechimni kompleks sohaga chiqmasdan, ya’ni haqiqiy sohada turib ham topish mumkin. Buning uchun (7.3.15) ning yechimini

$$y = x^k e^{\alpha x}\left(M(x)\cos\beta x + N(x)\sin\beta x\right) \tag{7.3.18}$$

ko‘rinishda izlash kerak; bunda $k-$ yuqorida aytilgan ma’noga ega, $M(x)$ va $N(x)$ ko‘phadlarning darajalari esa $\le m = \max\{m_1, m_2\}$. Bu ko‘phadlarlarning noma’lum koeffitsientlarini topish uchun (7.3.18) funksiyani (7.3.15) tenglamaga qo‘yib, ixchamlashlarni bajarib, chap va o‘ng tomondagi o‘xshash hadlardagi koefftsientlarni tenglashtirib, hosil bo‘lgan chiziqli algebraik tenglamalar sistemasini yechish kerak.

Misol 6. Ushbu

$$y'' - 4y' + 5y = 4e^{2x}(x\cos x + \sin x)$$

tenglamaning xususiy yechimini toping.

🗝Ravshanki, $\sigma = 2 + i$ son $\lambda^2 - 4\lambda + 5 = 0$ xarakteristik tenglamaning $k = 1$ karrali ildizi,

$$m_1 = 1,\ m_2 = 0,\ m = \max\{m_1, m_2\} = 1.$$

Demak, berilgan tenglamaning yechimini $y = xe^{2x}\left((ax+b)\cos x + (cx+d)\sin x\right)$ ko‘rinishda izlash mumkin; bunda $a, b, c, d-$hozircha noma’lum sonlar. Bu $y = e^{2x}\left((ax^2+bx)\cos x + (cx^2+dx)\sin x\right)$ ni tenglamaga qo‘yib, o‘xshash hadlarni ixchamlab va $2e^{2x}$ga qisqartirib, topamiz:

$$2cx\cos x - 2ax\sin x + (a+d)\cos x + (c-b)\sin x = 2x\cos x + 2\sin x.$$

Bundan, chap va o‘ng tomondagi o‘xshash hadlardagi koefftsientlarni tenglashtirib, $2c = 2,\ -2a = 0,\ a + d = 0,\ c - b = 2$ shartlarga ega bo‘lamiz. Oxirgi sistemadan $a = 0,\ b = -1,\ c = 1,\ d = 0$ qiymatlarni topamiz. Bularni $y = xe^{2x}\left((ax+b)\cos x + (cx+d)\sin x\right)$ ga qo‘yib, berilgan tenglamaning $y = xe^{2x}\left(x\sin x - \cos x\right)$ xususiy yechimini topamiz. 👍

Chiziqli o‘zgarmas koeffitsitntli diffeferensial tenglamalarni yechishda operatorli metoddan ham foydalanish mumkin [13].

Masalalar

1. Tenglamaning umumiy yechimini toping:

$$y''' - 4y'' + 9y' - 10y = 5e^{2x}.$$

2. Ushbu $L[y] = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0) e^{\sigma x}$ tenglamada $y = e^{\sigma x} u$ almashtirishni bajaring ($u = u(x)$ – yangi noma'lum funksiya). Bunda hosil bo'luvchi

$$\tilde{a}_n u^{(n)} + \tilde{a}_{n-1} u^{(n-1)} + \cdots + \tilde{a}_1 u' + \tilde{a}_0 u = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

tenglamaning koeffitsientlarini hisoblang.

3. Agar uzluksiz koeffitsientli

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0$$

tenglama barcha $\tilde{x} = x + \lambda$ almashtirishlarga nisbatan invariant (ko'rinishi o'zgarmas) bo'lsa, tenglamaning koeffitsientlari o'zgarmas bo'lishini isbotlang.

4. Agar uzluksiz koeffitsientli

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0$$

tenglama barcha $\tilde{x} = \lambda x$ almashtirishlarga nisbatan invariant bo'lsa, $a_j(x) = a_j x^j$ (a_j – const, $j = \overline{1, n}$) ekanligini, ya'ni bu tenglama Eyler tenglamasidan iborat bo'lishini isbotlang.

MODUL 8. NORMAL KO‘RINISHDAGI DIFFERENSIAL TENGLAMALAR SISTEMASI

§ 8.1. Umumiy ko‘rinishdagi differensial tenglamalar sistemasini birinchi tartibli tenglamalar sistemasiga keltirish

Soddalik uchun t skalyar argumentning ikkita $x = x(t)$ va $y = y(t)$ noma’lum haqiqiy funksiyalariga nisbatan ushbu $(m, n \in \mathbb{N})$

$$\begin{cases} F_1(t, x, x', ..., x^{(m)}, y, y', ..., y^{(n)}) = 0 \\ F_2(t, x, x', ..., x^{(m)}, y, y', ..., y^{(n)} = 0 \end{cases} \qquad (8.1.1)$$

differensial tenglamalar sistemasini qaraylik. Bu yerda F_1 va F_2 funksiyalari $m+n+3$ dona haqiqiy o‘zgaruvchilarning haqiqiy funksiyalari; ular $\mathbb{R}^{m+n+3}$ fazoning biror G sohasida aniqlangan va uzluksiz deb faraz qilinadi; $t-$ erkli o‘zgaruvchi, x va y lar t ning noma’lum funksiyalari. Agar x funksiyaning (8.1.1) sistemada qatnashgan hosilalarining maksimal tartibi m bo‘lsa, u holda (8.1.1) sistema x ga nisbatan m- tartibli differensial tenglamalar sistemasi deyiladi. Differensial tenglamalar sistemasining y ga nisbatan tartibi shunga o‘xshash aniqlanadi. Agar (8.1.1) sistema x ga nisbatan m- tartibli, y ga nisbatan esa n- tartibli bo‘lsa, u holda $m+n$ soni (8.1.1) sistemaning tartibi deyiladi.

Agar $I \subset \mathbb{R}$ oraliq va $x = \varphi(t)$, $y = \psi(t)$ funksiyalar uchun

1) $\varphi(t) \in C^m(I), \psi(t) \in C^n(I)$

2) $$\begin{cases} F_1(t, \varphi(t), \varphi'(t), ..., \varphi^{(m)}(t), \psi(t), \psi'(t), ..., \psi^{(n)}(t)) \equiv 0, \quad \forall t \in I \\ F_2(t, \varphi(t), \varphi'(t), ..., \varphi^{(m)}(t), \psi(t), \psi'(t), ..., \psi^{(n)}(t)) \equiv 0, \quad \forall t \in I \end{cases}$$

shartlar bajarilsa, bu $x = \phi(t)$, $y = \psi(t)$ funksiyalar (8.1.1) sistemaning I oraliqda aniqlangan yechimi deyiladi.

Agar (8.1.1) sistemani $x^{(m)}$, va $y^{(n)}$ hosilalarga nisbatan yechish mumkin bo‘lsa (oshkormas funksiya to‘g‘risdagi teoremani eslang), u holda uni

$$\begin{cases} x^{(m)} = f(t, x, x', ..., x^{(m-1)}, y, y', ..., y^{(n-1)}) \\ y^{(n)} = g(t, x, x', ..., x^{(n-1)}, y, y', ..., y^{(m-1)}) \end{cases} \qquad (8.1.2)$$

ko‘rinishda yozish mumkin. (8.1.2) sistema yuqori hosilalarga nisbatan yechilgan deb ataladi.

$(n+m)-$tartibli (8.1.1) sistema $(n+m)$ dona har bir noma'lumga nisbatan birinchi tartibli differensial tenglamalar sistemasiga keltiriladi. Buning uchun quyidagi belgilashlarni kiritaylik:

$$\begin{cases} x = x_1, x' = x_2, ..., x^{(m-1)} = x_m, \\ y = x_{m+1}, \ y' = x_{m+2}, ..., y^{(n-1)} = x_{m+n}. \end{cases} \qquad (8.1.3)$$

Bu belgilashlar natijasida (8.1.2) sistemadan ushbu

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ .... \\ x_{m-1}' = x_m \\ x_m' = f(t, x_1, ..., x_m, x_{m+1}, ...x_{m+n}) \\ x_{m+1}' = x_{m+2} \\ .... \\ x_{m+n-1}' = x_{m+n} \\ x_{m+n}' = g(t, x_1, ..., x_m, x_{m+1}, ...x_{m+n}) \end{cases} \qquad (8.1.4)$$

sistemani hosil qilamiz.

Shunday qilib, agar $x = \varphi(t)$ va $y = \psi(t)$ funksiyalar (8.1.2) sistemaning yechimi bo'lsa, u holda $x_1 = \varphi(t), \ x_2 = \varphi'(t),$ $...., x_m = \varphi^{(m-1)}(t),$ $x_{m+1} = \psi(t), x_{m+2} = \psi'(t), ..., x_{m+n} = \psi^{(n-1)}(t)$ funksiyalar (8.1.4) sistemaning yechimi bo'ladi.

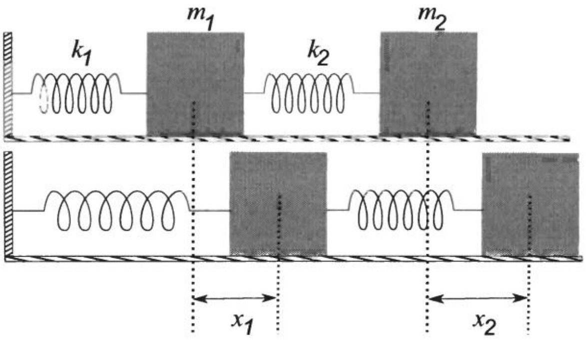
Aksincha, agar

$$x_1 = x_1(t), \ x_2 = x_2(t), ..., \ x_m = x_m(t), x_{m+1} = x_{m+1}(t), ..., \ x_{m+n} = x_{m+n}(t)$$

funksiyalar (8.1.4) sistemaning yechimi bo'lsa, sistemadagi $1-, ..., (m-1)-$ tenglamalardan $x_2 = x_1', ..., x_m = x_1^{(m-1)}$ tengliklarni, $(m+1)-, ..., (m+n-1)-$ tenglamalardan esa $x_{m+2} = x_{m+1}', ..., x_{m+n} = x_{m+1}^{(n-1)}$ tengliklarni topamiz va ularni m- va $(m+n)$- tenglamalarga qo'yib, $x = x_1(t)$ va $y = x_{m+1}(t)$ funksiyalar (8.1.2) sistemaning yechimi bo'lishini ko'ramiz.

Misol 1. Quyidagi rasmda tasvirlangan ikki massa va ikki elastik prujinanadan tashkil topgan mexanik sistemani qaraylik. Massalarning miqdorlarini m_1 va m_2, elastik prujinanilarning bikirlik koeffitsientlarini k_1 va k_2 hamda massalarning koordinatalarini x_1 va x_2 bilan belgilaylik;

bunda $x_1 = 0$ va $x_2 = 0$ muvozanat holatiga mos keladi deb hisoblaymiz (muvozanat holatga nisbatan ko‘chish).



Guk qonuniga ko‘ra m_1 massali jismga $-k_1x_1 + k_2(x_2 - x_1)$ kuch, m_2 massali jismga esa $-k_2(x_2 - x_1)$ kuch ta’sir etadi. Nyuton qonuniga ko‘ra harakat tenglamalari

$$\begin{cases} m_1 \dfrac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1), \\ m_2 \dfrac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) \end{cases}$$

kabi bo‘ladi. Bu sistema quyidagi normal sistema ko‘rinishiga keladi:

$$\begin{cases} \dfrac{dx_1}{dt} = x_3, \\ \dfrac{dx_2}{dt} = x_4, \\ \dfrac{dx_3}{dt} = -\dfrac{k_1}{m_1}x_1 + \dfrac{k_2}{m_1}(x_2 - x_1), \\ \dfrac{dx_4}{dt} = -\dfrac{k_2}{m_2}(x_2 - x_1). \end{cases}$$

Shunday qilib, (8.1.2) differensial tenglamalar sistemasini yechish quyidagi **normal ko‘rinishdagi sistema** (yoki qisqaroq **normal sistema)** deb ataluvchi sistemani yechishga keltiriladi:

$$\begin{cases} x_1' = f_1(t, x_1, ..., x_k) \\ x_2' = f_2(t, x_1, ..., x_k) \\ \dots\dots\dots\dots\dots\dots\dots \\ x_k' = f_k(t, x_1, ..., x_k) \end{cases} \qquad (8.1.5)$$

Shu munosabat bilan biz, asosan, (8.1.5) ko‘rinishdagi normal sistemalarni o‘rganamiz.

Ba'zi shartlar bajarilganda (8.1.5) normal sistemani yechish bitta k-tartibli differensial tenglamani yechishga keltiriladi. Aytaylik, x_1 ga nisbatan bitta n-tartibli differensial tenglama hosil qilish kerak bo'lsin. Buning uchun differensiallash va yo'qotish usulidan foydalanamiz. Bu usulga ko'ra (8.1.5) sistemadagi 1- tenglikni $k-1$ marta ketma-ket differensiallaymiz (buning mumkinligi faraz qilinadi) va bunda har bir qadamda (II.2.10) dagi tengliklardan foydalaninib noma'lum funksiyalar hosilalarini yo'qotib boramiz.

$$x_1'' = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} \cdot x_1' + ... + \frac{\partial f_1}{\partial x_k} \cdot x_k' = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + ... + \frac{\partial f_1}{\partial x_k} f_k \equiv g_2(t, x_1, ...x_k)$$

$$x_1''' = \frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1} f_1 + ... + \frac{\partial g_2}{\partial x_k} f_k \equiv g_3(t, x_1, ..., x_k)$$

....

$$x_1^{(k)} = \frac{\partial g_{k-1}}{\partial t} + \frac{\partial g_{k-1}}{\partial x_1} f_1 + ... + \frac{\partial g_{k-1}}{\partial x_k} f_k \equiv g_k(t, x_1, ..., x_k)$$

Endi $x_2, x_3, ..., x_n$ o'zgaruvchilarni yo'qotish uchun quyidagi sistemani tuzamiz:

$$\begin{cases} x_1' = g_1(t, x_1, ..., x_k) \qquad\qquad (f_1 = g_1) \\ x_1'' = g_2(t, x_1, ..., x_k) \\ .... \\ x_1^{(k)} = g_k(t, x_1, ..., x_k) \end{cases} \tag{8.1.6}$$

Bu sistemaning dastlabki $k-1$ ta tenglamasidan

$$x_2 = x_2(t,\ x_1,\ x_1', ..., x_1^{(k-1)}), ...,\ x_k = x_k(t,\ x_1,\ x_1', ..., x_1^{(k-1)}) \tag{8.1.7}$$

larni topib (buning mumkinligi faraz qilinadi), oxirgi tenglamaning o'ng tomoniga qo'yamiz. Natijada k- tartibli yuqori hosilaga nisbatan yechilgan tenglamaga kelamiz:

$$x_1^{(k)} = g_k\big(t, x_1,\ x_2(t,\ x_1,\ x_1', ..., x_1^{(k-1)}), ..., x_k(t,\ x_1,\ x_1', ..., x_1^{(k-1)})\big) \equiv$$
$$\equiv g(t,\ x_1,\ x_1', ..., x_1^{(k-1)}). \tag{8.1.13}$$

Ba'zi shartlar bajarilganda (8.1.13) tenglamadan topilgan $x_1 = x_1(t)$ yechim va unga ko'ra (8.1.7) tengliklar yordamida aniqlangan $x_2 = x_2(t), ..., x_k = x_k(t)$ funksiyalar (8.1.5) normal sistemaning yechimini tashkil etishini ko'rsatish mumkin. Biz bunda to'xtalmaymiz.

Biz yuqorida ikki noma'lum funksiya qatnashgan differensial tenglamalar sistemasi (8.1.1) uchun yechim tushunchasini kiritdik, uni birinchi tartibli differensial tenglamalar sistemasiga keltirish bilan shug'ullandik. Ixtiyoriy chekli sondagi $x_1, x_2, ..., x_n$ noma'lum funksiyalarga nisbatan ushbu

$$\begin{cases} F_1(t, x_1, x_1', ..., x_1^{(m_1)}, x_2, x_2', ..., x_2^{(m_2)}, ..., x_n, x_n', ..., x_n^{(m_n)}) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(t, x_1, x_1', ..., x_1^{(m_1)}, x_2, x_2', ..., x_2^{(m_2)}, ..., x_n, x_n', ..., x_n^{(m_n)}) = 0 \end{cases}$$

sistema uchun ham yechim tushunchasi yuqoridagiga o'xshash kiritiladi. Bu sistemani ham normal sistemani yechishga keltirish mumkin (ba'zi shartlar bajarilganda).

Izoh. Ba'zi hollarda k- tartibli normal sistema bir dona k-tartibli tenglamaga keltirilmaydi. Masalan, ushbu

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

ikkinchi tartibli tenglamalari ajralgan normal sistema, tushunarliki, bir dona ikkinchi tartibli differensial tenglamaga kelmaydi. Bu sistemaning yechimi $x = c_1 e^t$, $y = c_2 e^t$ ($c_1, c_2 -$const).

Yuqori tartibli differensial tenglamalar sistemasidan umumiy holda bitta noma'lum funksiyaga nisbatan bitta yuqori tartibli differensial tenglama hosil qilish mumkin. Bunda ham differensiallash va yo'qotish usulidan foydalaniladi. Ba'zi hollarda esa yo'qotish jarayonida chiziqli algebra metodlarini ishlatish mumkin.

Misol. Ushbu

$$\begin{cases} x' - xy - y = 0 \\ y' - x^2 + 2y^2 + x = 0 \end{cases}$$

sistemadagi $y = y(t)$ noma'lum funksiya qanoatlantiruvchi bitta differensial tenglamani topaylik. Berilgan sistemadagi ikkinchi tenglamani differesiallaymiz va bunda hosil bo'luvchi x' hosilani birinchi tenglamadan $x' = xy + y$ ekanligini topib, yo'qotamiz:

$$y'' - 2xx' + 4yy' + x' = 0\,,\; y'' - 2x(xy + y) + 4yy' + xy + y = 0,$$

$$y'' + 4yy' - yx - 2yx^2 = 0\,.$$

Endi berilgan sistemaning ikkinchi tenglamasi va hosil qilingan tenglamadan tuzilgan

$$\begin{cases} y' + 2y^2 + x - x^2 = 0 \\ y'' + 4yy' + y - yx - 2yx^2 = 0 \end{cases}$$

sistemadan x noma'lumni yo'qotish kerak. Bu ishni radikallarsiz bajarish mumkin. Buning uchun oxirgi sistemaning tenglamalarini x ga ko'paytirib, $1, x, x^2, x^3$ noma'lumlarga nisbatan quyidagi chiziqli bir jinsli algebraik sistemani tuzaylik:

$$\begin{pmatrix} y'+2y^2 & 1 & -1 & 0 \\ 0 & y'+2y^2 & 1 & -1 \\ y''+4yy'+y & -y & -2y & 0 \\ 0 & y''+4yy'+y & -y & -2y \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Bu sistema notrivial yechimga ega bo'lganligi sababli (masalan, yechimning birinchi komponentasi noldan farqli: $1 \neq 0$) uning determinanti nolga teng bo'lishi kerak:

$$\begin{vmatrix} y'+2y^2 & 1 & -1 & 0 \\ 0 & y'+2y^2 & 1 & -1 \\ y''+4yy'+y & -y & -2y & 0 \\ 0 & y''+4yy'+y & -y & -2y \end{vmatrix} = 0.$$

Bu yerdagi determinantni hisoblab, $y = y(t)$ noma'lum funksiyaga nisbatan quyidagi ikkinchi tartibli differensial tenglamani hosil qilamiz:

$$y''^2 + y(4y' - 8y^2 - 1)y'' + 4y^2y'^2 - 11y^2y' + 2(8y^2 - 7)y^4 = 0.$$

Masalalar

1. Ushbu

$$\begin{cases} x' - xy - y = 0 \\ y' - x^2 + y^2 + 2x = 0 \end{cases}$$

sistemadan $y = y(t)$ noma'lum funksiya qanoatlantiruvchi bitta oddiy differensial tenglama hosil qiling.

2. Ushbu

$$\begin{cases} x' - xy - y^2 + x^3 = 0 & (1) \\ y' + y^2 - xy - x^2 = 0 & (2) \end{cases}$$

sistemadan $x = x(t)$ noma'lum funksiya uchun bir dona differensial tenglama tuzing.

§ 8.2. Yordamchi ma'lumotlar. $\mathbb{R}^n$ fazoda matematik analiz elementlari

1⁰. n o'lchamli Evklid fazosi $\mathbb{R}^n$ ning elementlari – vektorlar – $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$, $x_j\in\mathbb{R}, j=\overline{1,n}$, n dona tartiblangan haqiqiy sonlardan, ya'ni koordinatalardan tuzilgan. Ba'zi hollarda vektornining koordinatalarini ustun bo'ylab yozamiz: $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)^T$ (T – transpozitsiya belgisi). Vektorlar odatdagicha (koordinatalar bo'ylab) qo'shiladi va songa ko'paytiriladi; $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ va $\boldsymbol{y}=(y_1,y_2,\ldots,y_n)$ vektorlarning skalyar ko'paytmasi

$$(\boldsymbol{x},\boldsymbol{y})=\sum_{j=1}^{n}x_jy_j \tag{8.2.1}$$

formula bilan, $\boldsymbol{x}$ vektorning normasi (uzunligi) esa

$$\|\boldsymbol{x}\|=\sqrt{\sum_{j=1}^{n}x_j^2}=\sqrt{(\boldsymbol{x},\boldsymbol{x})} \tag{8.2.2}$$

formula bilan kiritilgan. $\boldsymbol{x}$ va $\boldsymbol{y}$ orasidagi masofa

$$\|\boldsymbol{x}-\boldsymbol{y}\|=\sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}=\sqrt{(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y})} \tag{8.2.3}$$

ga teng.

Biz Koshi-Bunyakovskiy tengsizligidan ko'p foydalanamiz. U quyidagi ko'rinishga ega:

$$\left|\sum_{j=1}^{n}x_jy_j\right|\le\sqrt{\sum_{j=1}^{n}x_j^2}\sqrt{\sum_{j=1}^{n}y_j^2}\text{ , ya'ni }\left|(\boldsymbol{x},\boldsymbol{y})\right|\le\|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|. \tag{8.2.4}$$

Norma (vektor uzunligi) quyidagi xossalarga ega:

1. $\|\boldsymbol{x}\|\ge 0$; $\|\boldsymbol{x}\|=0\Leftrightarrow\boldsymbol{x}=(0,0,\ldots,0)\in\mathbb{R}^n$.
2. $\|\lambda\boldsymbol{x}\|=|\lambda|\cdot\|\boldsymbol{x}\|$, $\lambda\in\mathbb{R}$.
3. $\|\boldsymbol{x}-\boldsymbol{y}\|\le\|\boldsymbol{x}\|+\|\boldsymbol{y}\|$ (uchburchak tengsizligi).

Norma fazoda limit (yaqinlashish) va uzluksizlik tushunchalarini kiritishga imkon beradi. Agar $\{\boldsymbol{x}^k\}\subset\mathbb{R}^n$ ketma-ketlik va $\boldsymbol{x}\in\mathbb{R}^n$ uchun $\lim\limits_{k\to\infty}\left\|\boldsymbol{x}^k-\boldsymbol{x}\right\|=0$ bo'lsa, $\lim\limits_{k\to\infty}\boldsymbol{x}^k=\boldsymbol{x}$ deb yozamiz. Bu yaqinlashish koordinatalar bo'yicha yaqinlashishga ekvivalent, ya`ni, agar

$\boldsymbol{x}^k=(x_1^k,x_1^k,\ldots,x_n^k)$ va $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ bo'lsa,

$$\lim_{k\to\infty}\boldsymbol{x}^k=\boldsymbol{x}\Leftrightarrow\lim_{k\to\infty}x_j^k=x_j,\ j=\overline{1,n},$$

bo'ladi. Bu tasdiq quyidagi tengsizliklardan bevosita ravshan:

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left|x_j-y_j\right|\le\|\boldsymbol{x}-\boldsymbol{y}\|\le\sum_{j=1}^{n}\left|x_j-y_j\right| \qquad (8.2.5)$$

Bu yerdagi chap tengsizlik Koshi-Bunyakovskiy tengsizligidan kelib chiqadi, o‘ng tengsizlik esa ravshan. Funksiyaning uzluksizligi endi ketma-ketliklar tilida (Geyne bo‘yicha) odatdagidek kiritiladi.

$\mathbb{R}^n$ dagi chegaralangan va yopiq to‘plam kompaktdir. Kompaktdan olingan har qanday ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin. Kompaktning uzluksiz aksi kompakt, ya’ni agar $K\subset\mathbb{R}^m$ kompakt va $\boldsymbol{f}:K\to\mathbb{R}^n$ uzluksiz funksiya bo‘lsa, u holda K ning $\boldsymbol{f}(K)$ aksi ham kompaktdir.

Jumla. Agar $K\subset\mathbb{R}^n$ kompakt, $F\subset\mathbb{R}^n$ yopiq to‘plam va $K\bigcap F=\varnothing$ bo‘lsa, u holda ular orasidagi masofa $\operatorname{dist}(K,F)$ qat’iy musbat bo‘ladi, ya’ni

$$\operatorname{dist}(K,F)\overset{def}{=}\inf\left\{\|\boldsymbol{x}-\boldsymbol{y}\|\mid\boldsymbol{x}\in K,\boldsymbol{y}\in F\right\}>0.$$

🗝 Teskarisini faraz qilamiz, ya’ni teoremaning shartlari o‘rinli, lekin $\operatorname{dist}(K,F)=0$ bo‘lsin. Aniq quyi chegara ($\inf$) ta’rifiga ko‘ra shunday $\{\boldsymbol{x}^j\}\subset K$ va $\{\boldsymbol{y}^j\}\subset F$ ketma-ketliklar mavjudki, ular uchun $\lim\limits_{j\to\infty}\left\|\boldsymbol{x}^j-\boldsymbol{y}^j\right\|=0$ bo‘ladi. K kompakt bo‘lganligi uchun $\{\boldsymbol{x}^j\}\subset K$ ketma-ketlikdan K ning biror $\boldsymbol{x}$ elementiga yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin. Shuning uchun umumiylikni buzmasdan $\{\boldsymbol{x}^j\}\subset K$ ketma-ketlikning o‘zi yaqinlashuvchi deb hisoblaymiz: $\lim\limits_{j\to\infty}\boldsymbol{x}^j=\boldsymbol{x}\in K$. U holda $\lim\limits_{j\to\infty}\boldsymbol{y}^j=\boldsymbol{x}\in F$ ham bo‘ladi, chunki $\lim\limits_{j\to\infty}\left\|\boldsymbol{x}^j-\boldsymbol{y}^j\right\|=0$ va $\left\|\boldsymbol{y}^j-\boldsymbol{x}\right\|\le\left\|\boldsymbol{y}^j-\boldsymbol{x}^j\right\|+\left\|\boldsymbol{x}^j-\boldsymbol{x}\right\|$. Shunday qilib, $\boldsymbol{x}\in K\cap F$. Bu esa $K\cap F=\varnothing$ ekanligiga zid. Demak farazimiz noto‘g‘ri. 👍

$A=[a_{ij}]\in\mathbb{M}_{n\times m}(\mathbb{R})$ matritsani $\mathbb{R}^{n\cdot m}$ Evklid fazosining elementi deb tushunamiz va bu matritsning normasi sifatida

$$\|A\|=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2} \qquad (8.2.6)$$

miqdorni ishlatamiz. Agar $A\in\mathbb{M}_{n\times m}(\mathbb{R})$ va $B\in\mathbb{M}_{m\times k}(\mathbb{R})$ bo‘lsa, $AB\in\mathbb{M}_{n\times k}(\mathbb{R})$ ko‘paytma matrisaning normasi uchun

$$\|AB\|\le\|A\|\cdot\|B\|$$

tengsizlik o‘rinli bo‘ladi (bu Koshi-Bunyakovskiy tengsizligidan kelib chiqadi).

2⁰. $\boldsymbol{f}: I \to \mathbb{R}^n$ vektor-funksiyaning $t_0 \in I$ nuqtadagi luimiti va uzluksizligi tabiiy ravishda (koordinatlar bo`ylab) kiritiladi. Uning $t_0 \in I$ nuqtadagi hosilasi $\boldsymbol{f}'(t_0) = \dfrac{d\boldsymbol{f}(t_0)}{dt}$ tabiiy ravishda

$$\boldsymbol{f}'(t_0) = \lim_{\substack{h\to 0\\ t_0+h\in I}} \frac{\boldsymbol{f}(t_0+h) - \boldsymbol{f}(t_0)}{h}$$

formula bilan aniqlanadi. Oxirgi tenglik quyidagini anglatadi:

$$\left\| \frac{\boldsymbol{f}(t_0+h) - \boldsymbol{f}(t_0)}{h} - \boldsymbol{f}'(t_0) \right\| \xrightarrow[h\to 0]{} 0.$$

Agar $\boldsymbol{f}(t) = \left(f_1(t), f_2(t), ..., f_n(t) \right)$, $f_j : I \to \mathbb{R}, j = \overline{1,n}$, bo‘lsa, ko‘rish qiyin emaski,

$$\boldsymbol{f}'(t_0) = \left(f_1'(t_0), f_2'(t_0), ..., f_n'(t_0) \right)$$

bo‘ladi (hosila korninatalar bo‘ylab hisoblanadi).

$\boldsymbol{f}: [a,b] \to \mathbb{R}^n$ vektor-funksiyaning $[a,b]$ segment bo‘yicha (Riman) integrali skalyar funksiyaning integraliga o‘xshash kiritiladi. $[a,b]$ segmentni $a = t_0 < t_1 < t_2 < ... < t_k = b$ nuqtalar bilan k ta $[t_0,t_1], [t_1,t_2], \dots, [t_{k-1},t_k]$ bo‘lakchalarga ajratamiz va $[t_{i-1},t_i]$ bo‘lakchadan ixtiyoriy α_i nuqta tanlab, quyidagi integral yig‘indini tuzamiz: $\boldsymbol{\sigma} = \sum_{i=1}^{k} \boldsymbol{f}(\alpha_i)\Delta t_i$, $\Delta t_i = t_i - t_{i-1}$.

Ravshanki, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n) \in \mathbb{R}^n$; $\sigma_j = \sum_{i=1}^{k} f_j(\alpha_i)\Delta t_i$, $j = \overline{1,n}$. Agar $d \overset{def}{=} \max_{1\le i\le n} \Delta t_i \to 0$ bo‘lganda $\boldsymbol{\sigma}$ integral yig‘indining limiti $\alpha_i \in [t_{i-1},t_i]$ nuqtalarning tanlanishiga bog‘liqsiz holda mavjud bo‘lsa, u holda $\boldsymbol{f}(t)$ vektor-funksiya $[a,b]$ segmentda integrallanuvchi deyiladi va uning shu segment bo‘yicha integralini odatdagidek $\int_a^b \boldsymbol{f}(t)dt$ bilan belgilanadi:

$$\int_a^b \boldsymbol{f}(t)dt = \lim_{d\to 0} \boldsymbol{\sigma} \quad \left(\left\| \boldsymbol{\sigma} - \int_a^b \boldsymbol{f}(t)dt \right\| \xrightarrow[d\to 0]{} 0 \right). \qquad (8.2.7)$$

Ko‘rsatish mumkinki, $\boldsymbol{f}(t)$ vektor-funksiyaning $[a,b]$ segmentda integrallanuvchiligi uning $(f_1(t), f_2(t),..., f_n(t))$ koordinata funksiyalarining shu segmentda integrallanuvchanligiga ekvivalent va

$$\int_a^b \boldsymbol{f}(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, ..., \int_a^b f_n(t)dt\right). \qquad (8.2.8)$$

Agar $\boldsymbol{f}(t)$ vektor-funksiya $[a,b]$ da integrallanuvchi bo‘lsa, $\|\boldsymbol{f}(t)\|$ haqiqiy funksiya ham $[a,b]$ da integrallanuvchi va quyidagi tengsizlik o‘rinlidir:

$$\left\|\int_a^b \boldsymbol{f}(t)dt\right\| \le \int_a^b \|\boldsymbol{f}(t)\| dt. \qquad (8.2.9)$$

Haqiqatan ham, $\left\|\boldsymbol{\sigma} - \int_a^b \boldsymbol{f}(t)dt\right\| \xrightarrow[d\to 0]{} 0$ bo‘lgani uchun $\left| \|\boldsymbol{\sigma}\| - \left\|\int_a^b \boldsymbol{f}(t)dt\right\| \right| \le \left\|\boldsymbol{\sigma} - \int_a^b \boldsymbol{f}(t)dt\right\|$ tengsizlkdan $\|\boldsymbol{\sigma}\| \xrightarrow[d\to 0]{} \left\|\int_a^b \boldsymbol{f}(t)dt\right\|$ ekanligini ko‘ramiz. Endi ushbu

$$\|\boldsymbol{\sigma}\| = \left\|\sum_{i=1}^k \boldsymbol{f}(\alpha_i)\Delta t_i\right\| \le \sum_{i=1}^k \|\boldsymbol{f}(\alpha_i)\| \Delta t_i .$$

tengsizlikda limitga o‘tsak, (8.2.9) hosil bo‘ladi.

Agar $\boldsymbol{f}:[a,b]\to \mathbb{R}^n$ vektor-funksiya $[a,b]$ segmentda uzluksiz bo‘lsa, ushbu

$$\frac{d}{dt}\int_a^t \boldsymbol{f}(s)ds = \boldsymbol{f}(t) \ \ (t \in [a,b])$$

formula o‘rinli bo‘ladi (chunki u har bir koordinata funksiyasi f_j uchun o‘rinli).

3^0. Matritsaviy *funksiyalar.* $\Phi: I \to \mathbb{M}_{n\times m}(\mathbb{R})$ akslantirish $t \in I \subset \mathbb{R}$ skalyar argumentning (haqiqiy) matritsaviy funksiyasi deyiladi. U har bir $t\in I$ songa $\Phi(t) \in \mathbb{M}_{n\times m}(\mathbb{R})$ matritsani mos keltiradi. Ushbu

$$\frac{d\Phi(t)}{dt} = \Phi'(t) = \lim_{h\to 0}\frac{1}{h}[\Phi(t+h) - \Phi(t)]$$

formula bilan Φ ning hosilasi kiritiladi. $\mathbb{R}^{nm}$ fazodagi yaqinlashish kordinatalar bo‘yicha bo‘lgani uchun quyidagi formula ham o‘rinli:

$$\frac{d}{dt}\begin{pmatrix} \varphi_{11}(t) & \dots & \varphi_{1m}(t) \\ \dots & \dots & \dots \\ \varphi_{n1}(t) & \dots & \varphi_{nm}(t) \end{pmatrix} = \begin{pmatrix} \frac{d\varphi_{11}(t)}{dt} & \dots & \frac{d\varphi_{1m}(t)}{dt} \\ \dots & \dots & \dots \\ \frac{d\varphi_{n1}(t)}{dt} & \dots & \frac{d\varphi_{nm}(t)}{dt} \end{pmatrix}.$$

Osongina tekshirib ko‘rish mumkinki, agar $A: I \to \mathbb{M}_{n\times m}(\mathbb{R})$ va $B: I \to \mathbb{M}_{m\times l}(\mathbb{R})$ matritsaviy funksiyalar hosilaga ega bo‘lsa, u holda $(AB)' = A'B + AB'$ bo‘ladi.

$\Phi(t)$ matritsaviy funksiyaning integrali vektor-funksiyaning integrali kabi kiritiladi. Bunda, agar $\Phi(t)$ matritsaviy funksiya $[a;b]$ segmentda integrallanuvchi bo‘lsa, baholashlarda ishlatiladigan ushbu

$$\left\| \int_a^b \Phi(t)dt \right\| \le \int_a^b \|\Phi(t)\| dt$$

tengsizlik ham o‘rinli bo‘ladi ((8.2.9) tengsizlikka qarang).

4^0. *Ko‘p o‘zgaruvchining vektor-funksiyalari.* $E \subset \mathbb{R}^m$– ixtiyoriy to‘plam bo‘lsin. $\boldsymbol{f}: E \to \mathbb{R}^n$ akslantirish E da aniqlangan m ta $x_1, x_2, \dots, x_m$ haqiqiy o‘zgaruvchining $\big((x_1, x_2, \dots, x_m) = \boldsymbol{x} \in E\big)$ n o‘lchamli vektor-funksiyasi deyiladi. $\boldsymbol{f}(\boldsymbol{x}) = \big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots,, f_n(\boldsymbol{x})\big)$, $\boldsymbol{x} \in E$ vektor-funksiyaning koordinata funksiyalari $f_i: E \to \mathbb{R}$, $i = \overline{1,n}$, m ta haqiqiy o‘zgaruvchining skalyar funksiyalaridan iborat bo‘ladi.

Agar berilgan vektor-funksiyaning barcha koordinata funksiyalari E da chegaralangan bo‘lsa, bu vektor-funksiya chegaralangan deyiladi. Vektor-funksiyaning chegaralanganligi $\|\boldsymbol{f}(\boldsymbol{x})\| = \sqrt{\sum_{j=1}^n f_j^2(x)}$ haqiqiy funksiyaning (normaning) E da chegaralanganligiga ekvivalent.

$\boldsymbol{f}: E \to \mathbb{R}^n$ funksiyaning uzluksizligi, ta’rifga ko‘ra, uning barcha koordinata funksiyalarining uzluksizligiga ekvivalent.

$G \subset \mathbb{R}^m$ –bo‘shmas ochiq to‘plam, $\boldsymbol{f}: G \to \mathbb{R}^n$ vektor-funksiya va $\boldsymbol{x}^0 \in G$ berilgan bo‘lsin. Agar biror $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ matritsa uchun ushbu

$$\|\boldsymbol{f}(\boldsymbol{x}^0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}^0) - A\boldsymbol{h}\| = o(\|\boldsymbol{h}\|),\ \boldsymbol{h} \to 0, \qquad (8.2.10)$$

asimptotik tenglik o‘rinli bo‘lsa, $\boldsymbol{f}: G \to \mathbb{R}^n$ funksiya $\boldsymbol{x}^0$ nuqtada **hosilaga ega (differensiallanuvchi)** va bu hosila A matritsaga teng deyiladi hamda $\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x}^0)=A$ ko‘rinishda yoziladi. $\boldsymbol{f}:G\to\mathbb{R}^n$ vektor-funksiyaning $\boldsymbol{x}^0$ nuqtadagi hosilasi ushbu

$$\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x}^0)=\begin{pmatrix} \frac{\partial f_1(\boldsymbol{x}^0)}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x}^0)}{\partial x_2} & \cdots & \frac{\partial f_1(\boldsymbol{x}^0)}{\partial x_m} \\ \frac{\partial f_2(\boldsymbol{x}^0)}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x}^0)}{\partial x_2} & \cdots & \frac{\partial f_2(\boldsymbol{x}^0)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\boldsymbol{x}^0)}{\partial x_1} & \frac{\partial f_n(\boldsymbol{x}^0)}{\partial x_2} & \cdots & \frac{\partial f_n(\boldsymbol{x}^0)}{\partial x_m} \end{pmatrix} \qquad (8.2.11)$$

xususiy hosilalardan tuzilgan matritsadan iborat. U **Yakobi matritsasi** deb ataladi. G ochiq to‘plamning har bir nuqtasida differensiallanuvchi (hosilaga ega) funksiya G da differensiallanuvchi (hosilaga ega) deyiladi.

Agar $\boldsymbol{f}:G\to\mathbb{R}^n$ funksiyaning barcha koordinata funksiyalari $\boldsymbol{x}^0\in G$ nuqtaning biror atrofida barcha birinchi tartibli xususiy hosilalarga ega va bu hosilalar shu $\boldsymbol{x}^0$ nuqtada uzluksiz ham bo‘lsa, $\boldsymbol{f}$ funksiya $\boldsymbol{x}^0$ nuqtada differensiallanuvchi (va, demak, uzlukisiz ham) bo‘ladi.

Agar $\boldsymbol{f}:G\to\mathbb{R}^n$ funksiyaning barcha koordinata funksiyalari ochiq to‘plam G da barcha birinchi tartibli uzluksiz $\frac{\partial f_k}{\partial x_i}$, $k=\overline{1,n}$, $i=\overline{1,m}$, xususiy hosilalarga ega bo‘lsa, $\boldsymbol{f}$ funksiya G da uzluksiz differensiallanuvchi funksiya deyiladi va bu $\boldsymbol{f}\in C^1(G,\mathbb{R}^n)$ kabi yoziladi. Sohada uzluksiz differensiallanuvchi funksiya shu sohada differensiallanuvchi hamdir.

Differensiallanuvchi $\boldsymbol{f}$ va $\boldsymbol{u}$ funksiyalar kompozitsiyasi $\boldsymbol{f}\circ\boldsymbol{u}$ ham differensiallanuvchi hamda $(\boldsymbol{f}\circ\boldsymbol{u})'=\boldsymbol{f}'\boldsymbol{u}'$ matritsaviy tenglik o‘rinli. Bu tasdiqdan foydalanmagan holda matritsalarni ko‘paytirish qoidasiga ko‘ra quyidagi tasdiqni osongina tekshirib ko‘rish mumkin:

agar $G\subset\mathbb{R}^m$, $D\subset\mathbb{R}^l$, $\boldsymbol{u}\in C^1(G,D)$, $\boldsymbol{f}\in C^1(D,\mathbb{R}^n)$ bo‘lsa, u holda $\boldsymbol{g}=\boldsymbol{f}\circ\boldsymbol{u}\in C^1(G,\mathbb{R}^n)$ va $\boldsymbol{g}'_{\boldsymbol{x}}(\boldsymbol{x})=\boldsymbol{f}'_{\boldsymbol{u}}(\boldsymbol{u}(\boldsymbol{x}))\cdot\boldsymbol{u}'_{\boldsymbol{x}}(\boldsymbol{x})$ matritsaviy tenglik o‘rinli bo‘ladi.

$G\subset\mathbb{R}^m$ – qavariq soha $(\{\boldsymbol{x},\boldsymbol{y}\}\subset G\Rightarrow\{\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x})\,|\,0\le s\le 1\}\subset G)$ va $\boldsymbol{f}\in C^1(G,\mathbb{R}^n)$ bo‘lsin. $\{\boldsymbol{x},\boldsymbol{y}\}\subset G$ uchun

$\boldsymbol{u}(s)=\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x}),\ 0\le s\le 1,$ funksiyani qaraylik. G qavariq bo‘lgani uchun $s\in[0;1]\Rightarrow \boldsymbol{u}(s)\in G$. Ravshanki,

$$\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})=\int_0^1 \frac{d\boldsymbol{f}(\boldsymbol{u}(s))}{ds}ds.$$

Lekin $\frac{d\boldsymbol{f}(\boldsymbol{u}(s))}{ds}=\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{u}(s))\cdot(\boldsymbol{y}-\boldsymbol{x})$. Demak,

$$\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})=\int_0^1 \boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x}))ds\cdot(\boldsymbol{y}-\boldsymbol{x}) \qquad (8.2.12)$$

Bu chekli orttirmalar formulasi deyiladi. U $\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})$ chekli orttirmani hisoblashga hamda baholashga imkon beradi:

$$\|\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})\|\le \sup_{0\le s\le 1}\|\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x}))\|\cdot\|\boldsymbol{y}-\boldsymbol{x}\|. \qquad (8.2.13)$$

Agar $\boldsymbol{f}'_{\boldsymbol{x}}$ chegaralangan, ya’ni $\|\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x})\|\le c,\quad c>0$, bo‘lsa, ravshanki,

$$\|\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})\|\le \sup_{0\le s\le 1}\|\boldsymbol{f}'_{\boldsymbol{x}}(\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x}))\|\cdot\|\boldsymbol{y}-\boldsymbol{x}\|\le c\|\boldsymbol{y}-\boldsymbol{x}\| \quad (8.2.14)$$

baholash o‘rinli bo‘ladi.

Matematik analiz kursida isbotlanadigan oshkormas funksiyalar to‘g‘risidagi teoremani keltiramiz. Bu teoremada vektorli ko‘rinishda yozilgan ushbu

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}$$

sistemadan $\boldsymbol{y}=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$ o‘zgaruvchini $\boldsymbol{x}=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m$ o‘zgaruvchining silliq funksiyasi sifatida topib olish (ifodalash) uchun yetarli shartlar keltiriladi.

***Teorema* (*oshkormas funksiyalar to‘g‘risida*).** Faraz qilaylik, ushbu

1^0. $\boldsymbol{F}(\boldsymbol{x}^0,\boldsymbol{y}^0)=\boldsymbol{0}$,

2^0. $\boldsymbol{F}$ vektor-funksiya $(\boldsymbol{x}^0,\boldsymbol{y}^0)=(x_1^0,x_2^0,\dots,x_m^0,y_1^0,y_2^0,\dots,y_n^0)\in\mathbb{R}^{m+n}$ nuqtaning biror atrofida C^1 sinfga tegishli,

3^0. $(\boldsymbol{x}^0,\boldsymbol{y}^0)\in\mathbb{R}^{m+n}$ nuqtada

$$\det\boldsymbol{F}'_y=\det\left\|\frac{\partial F_i}{\partial y_j}\right\|\ne 0$$

shartlar bajarilsin. U holda $\boldsymbol{x}^0\in\mathbb{R}^m$ nuqtaning shunday $U\subset\mathbb{R}^m$ va $\boldsymbol{y}^0\in\mathbb{R}^n$ nuqtaning shunday $V\subset\mathbb{R}^n$ atroflari mavjudki, har qanday $\boldsymbol{x}\in U$ uchun $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}$ tenglama V da yagona

$$\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})\in V$$

yechimga ega va bu yerdagi $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya $C^1(U,V)$ sinfga tegishli bo‘ladi.

5⁰. *Lipshits sharti.* $E\subset\mathbb{R}^{1+m}$ – ixtiyoriy to‘plam bo‘lsin. $\mathbb{R}^{1+m}$ fazoning nuqtalarini (vektorlarini) $(t,\boldsymbol{x})$, $t\in\mathbb{R}$, $\boldsymbol{x}\in\mathbb{R}^m$, ko‘rinishda belgilaylik. $\boldsymbol{f}:E\to\mathbb{R}^n$, $(t,\boldsymbol{x})\to\boldsymbol{f}(t,\boldsymbol{x})$, vektor-funksiyani qaraylik. Agar shunday $L>0$ son mavjud bo‘lib, $\forall\{(t,\boldsymbol{x}^1),(t,\boldsymbol{x}^2)\}\subset E$, nuqtalar uchun,

$$\|\boldsymbol{f}(t,\boldsymbol{x}^1)-\boldsymbol{f}(t,\boldsymbol{x}^2)\|\le L\|\boldsymbol{x}^1-\boldsymbol{x}^2\| \qquad (8.2.15)$$

tengsizlik o‘rinli bo‘lsa, $\boldsymbol{f}$ funksiya E to‘plamda “$\boldsymbol{x}$” – o‘zgaruvchi bo‘yicha (global) **Lipshits shartini qanoatlantiradi** deyiladi. Bu tengsizlikda yozish mumkin bo‘lgan eng kichik L son **Lipshits doimiysi** deyiladi. U ushbu

$$L=\sup\frac{\|\boldsymbol{f}(t,\boldsymbol{x}^1)-\boldsymbol{f}(t,\boldsymbol{x}^2)\|}{\|\boldsymbol{x}^1-\boldsymbol{x}^2\|},\ \boldsymbol{x}^1\ne\boldsymbol{x}^2,\ \{(t,\boldsymbol{x}^1),(t,\boldsymbol{x}^2)\}\subset E,$$

formula bilan hisoblanishi mumkin.

Yuqoridagi (8.2.15) Lipshits shartini

$$\|\boldsymbol{f}(t,\boldsymbol{x}^1)-\boldsymbol{f}(t,\boldsymbol{x}^2)\|\le\tilde{L}\sum_{j=1}^{m}\left|x_j^1-x_j^2\right|$$

ko‘rinishda ham yozish mumkin; bu (8.2.5) qo‘sh tengsizlikdan ravshan.

Endi $G\subset\mathbb{R}^{1+m}$ ochiq to‘plamda berilgan $\boldsymbol{f}:G\to\mathbb{R}^n$, $(t,\boldsymbol{x})\to\boldsymbol{f}(t,\boldsymbol{x})$, funksiyani qaraylik. Agar har bir $(t_0,\boldsymbol{x}^0)\in G$ nuqtaning biror atrofida bu funksiya Lipshits shartini “$\boldsymbol{x}$” bo‘yicha qanoatlantirsa, $\boldsymbol{f}$ funksiya G da “$\boldsymbol{x}$” bo‘yicha lokal Lipshits shartini qanoatlantiradi deyiladi. Ravshanki, agar $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya G da (global) Lipshits shartini (“$\boldsymbol{x}$” bo‘yicha) qanoatlantirsa, u G da lokal Lipshits shartini ham (“$\boldsymbol{x}$” bo‘yicha) qanoatlantiradi.

Normaning ta’rifidan ravshanki, $\boldsymbol{f}(t,\boldsymbol{x})$ funksiyaning Lipshits shartini qanoatlantirishi uning barcha koordinata funksiyalari $f_i(t,\boldsymbol{x}),i=\overline{1,n}$, ning Lipshits shartini qanoatlantirishiga teng kuchli.

Misol. Chegaralangan matritsaviy funksiya $A:I\to\mathbb{M}_{n\times m}(\mathbb{R})$ orqali tuzilgan $\boldsymbol{f}:I\times\mathbb{R}^m\to\mathbb{R}^n$, $\boldsymbol{f}(t,\boldsymbol{x})=A(t)\boldsymbol{x}$ vektor-funksiya,

$\boldsymbol{x}=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^m$, $I\times\mathbb{R}^m$ to‘plamda $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantiradi. Haqiqatan ham, A chegaralangan bo‘lgani uchun $\exists L>0\ \forall t\in I\ \|A(t)\|\le L$. Endi ravshanki,

$$\|\boldsymbol{f}(t,\boldsymbol{x}^1)-\boldsymbol{f}(t,\boldsymbol{x}^2)\|=\|A(t)(\boldsymbol{x}^1-\boldsymbol{x}^2)\|\le\|A(t)\|\cdot\|\boldsymbol{x}^1-\boldsymbol{x}^2\|\le L\|\boldsymbol{x}^1-\boldsymbol{x}^2\|.$$

Lemma 1. Agar $\boldsymbol{f}:G\to\mathbb{R}^n$, $(t,\boldsymbol{x})\to\boldsymbol{f}(t,\boldsymbol{x})=(f_1(t,\boldsymbol{x}),f_2(t,\boldsymbol{x}),\ldots,f_n(t,\boldsymbol{x}))$, funksiya uchun $\dfrac{\partial f_i(t,x_1,\ldots,x_m)}{\partial x_j}$ $(i=\overline{1,n},\ j=\overline{1,m})$ xususiy hosilalar G da uzluksiz bo‘lsa, bu funksiya G da $\boldsymbol{x}=(x_1,x_2,\ldots,x_m)$ bo‘yicha lokal Lipshits shartini qanoatlantiradi.

🗝 Berilgan funksiya ixtiyoriy $(t_0,\boldsymbol{x}^0)\in G$ nuqtaning biror atrofida $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantirishini ko‘rsatish kerak. G ochiq bo‘lgani uchun $(t_0,\boldsymbol{x}^0)\in G$ nuqta o‘zining biror sharsimon yopiq atrofi F bilan birgalikda G da joylashadi. $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya F da $\boldsymbol{x}$ bo‘yicha uzluksiz differensiallanuvchi bo‘lgani uchun uning $\boldsymbol{f}'_x$ hosilaviy matritsasi $\dfrac{\partial f_i}{\partial x_j}$ chegaralangan xususiy hosilalardan tuzilgan. Demak, $\exists M>0\ \forall(t,\boldsymbol{x})\in F\ \|\boldsymbol{f}'_x(t,\boldsymbol{x})\|\le M$. Endi chekli orttirmalar formulasidan $\forall\left\{(t,\boldsymbol{x}^1)\in E,(t,\boldsymbol{x}^2)\right\}\subset F$ uchun

$$\left\|\boldsymbol{f}(t,\boldsymbol{x}^1)-\boldsymbol{f}(t,\boldsymbol{x}^2)\right\|=\left\|\int_0^1\boldsymbol{f}'_x(t,\boldsymbol{x}^1+s(\boldsymbol{x}^1-\boldsymbol{x}^2))ds\cdot(\boldsymbol{x}^1-\boldsymbol{x}^2)\right\|\le$$

$$\le\int_0^1\left\|\boldsymbol{f}'_x(t,\boldsymbol{x}^1+s(\boldsymbol{x}^1-\boldsymbol{x}^2))\right\|ds\cdot\left\|\boldsymbol{x}^1-\boldsymbol{x}^2\right\|\le M\|\boldsymbol{x}^1-\boldsymbol{x}^2\|.$$

ekanligini topamiz. 👍

Lemma 2. Agar $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya ($\boldsymbol{f}:G\to\mathbb{R}^n$) G da $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantirsa, u G ning ixtiyoriy kompakt qismida $\boldsymbol{x}$ bo‘yicha global Lipshits shartini ham qanoatlantiradi.

🗝 Teskarisini faraz qilaylik. U holda biror $K\subset G$ kompaktda $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantirmaydi, ya’ni

$$\forall k\in\mathbb{N}\ \ \exists\{(t_k,\boldsymbol{x}^k),(t_k,\boldsymbol{y}^k)\}\subset K\ \ \|\boldsymbol{f}(t,\boldsymbol{x}^k)-\boldsymbol{f}(t,\boldsymbol{y}^k)\|\ge k\|\boldsymbol{x}^k-\boldsymbol{y}^k\|. \tag{8.2.16}$$

K – kompakt bo‘lgani uchun $\{(t_k, \boldsymbol{x}^k)\} \subset K$ va $\{(t_k, \boldsymbol{y}^k)\} \subset K$ ketma-ketliklardan ushbu $\{(t_{k_i}, \boldsymbol{x}^{k_i})\}$ va $\{(t_{k_i}, \boldsymbol{y}^{k_i})\}$ yaqinlashuvchi qismiy ketma-ketliklarni ajratishimiz mumkin. Aytaylik, $k_i \to \infty$ da $t_{k_i} \to t_0$, $\boldsymbol{x}^{k_i} \to \boldsymbol{x}^0$, $\boldsymbol{y}^{k_i} \to \boldsymbol{y}^0$ bo‘lsin. K yopiq bo‘lganligi uchun $(t_0, \boldsymbol{x}^0) \in K$ va $(t_0, \boldsymbol{y}^0) \in K$. Shunday qilib, $t_{k_i} \to t_0$, $\boldsymbol{x}^{k_i} \to \boldsymbol{x}^0$, $\boldsymbol{y}^{k_i} \to \boldsymbol{y}^0$ ketma-ketliklar uchun

$$\left\| \boldsymbol{f}(t_{k_i}, \boldsymbol{x}^{k_i}) - \boldsymbol{f}(t_{k_i}, \boldsymbol{y}^{k_i}) \right\| \ge k_i \left\| \boldsymbol{x}^{k_i} - \boldsymbol{y}^{k_i} \right\| \qquad (8.2.17)$$

tengsizlik o‘rinli.

Agar $\boldsymbol{x}^0 = \boldsymbol{y}^0$ bo‘lsa, bu tengsizlik $\boldsymbol{f}(t, \boldsymbol{x})$ funksiya $(t_0, \boldsymbol{x}^0) = (t_0, \boldsymbol{y}^0) \in G$ nuqta atrofida $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantirmasligini anglatadi. Bu berilganga zid.

Endi $\boldsymbol{x}^0 \ne \boldsymbol{y}^0$ bo‘lsin. $(t_0, \boldsymbol{x}^0)$ va $(t_0, \boldsymbol{y}^0)$ nuqtalarning yetarlicha kichik atrofida $\boldsymbol{f}$ ning normasi 1 soni bilan chegaralangan (Bu $\boldsymbol{f}$ ning shu nuqtalar atrofida Lipshits shartini qanoatlantirishidan ravshan). (8.2.16) tengsizlikdan yetarli katta k_i lar uchun ushbu

$$2 \ge k_i \left\| \boldsymbol{x}^{k_i} - \boldsymbol{y}^{k_i} \right\|$$

tengsizlikni hosil qilamiz. $\left\| \boldsymbol{x}^{k_i} - \boldsymbol{y}^{k_i} \right\| \to \left\| \boldsymbol{x}^0 - \boldsymbol{y}^0 \right\| > 0$ bo‘lgani uchun oxirgi tengsizlikdan $k_i \to \infty$ da yana ziddiyatga kelamiz. ✍

Masalalar

1. Aytaylik, $G \subset \mathbb{R}^m$ – qavariq soha, $f: G \to \mathbb{R}^n$ differensiallanuvchi funksiya bo‘lsin. $\boldsymbol{x} \in G$ va $\boldsymbol{y} \in G$ nuqtalar uchun

$$\| \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) \| \le \sup_{0<s<1} \| \boldsymbol{f}'(\boldsymbol{x} + s(\boldsymbol{y} - \boldsymbol{x})) \| \cdot \| \boldsymbol{y} - \boldsymbol{x} \|$$

tengsizlikning o‘rinli ekanligini isbotlang.

2. $G \subset \mathbb{R}^m$ – qavariq soha, $\boldsymbol{f}: G \to \mathbb{R}^n$ differensiallanuvchi funksiya va $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ bo‘lsin. Ushbu

$$\| \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - A\boldsymbol{h} \| \le \sup_{0<s<1} \| \boldsymbol{f}_{\boldsymbol{x}}'(\boldsymbol{x} + s\boldsymbol{h}) - A \| \cdot \| \boldsymbol{h} \| \ \left(\boldsymbol{x} \in G,\ \boldsymbol{x} + \boldsymbol{h} \in G \right)$$

tengsizlikni isbotlang.

3. (Lagranj formulasi). $G \subset \mathbb{R}^m$ – ochiq to‘plam va $f: G \to \mathbb{R}$ differensiallanuvchi haqiqiy qiymatli funksiya berilgan bo‘lsin. Agar $\boldsymbol{x} \in G$ va $\boldsymbol{x} + \boldsymbol{h} \in G$ nuqtalarni tutashtiruvchi $\{\boldsymbol{x} + s\boldsymbol{h} \mid 0 \le s \le 1\}$ kesma G da joylashsa, shunday $\theta \in (0;1)$ son mavjudki, uning uchun

$$f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})=f'(\boldsymbol{x}+\theta\boldsymbol{h})\cdot\boldsymbol{h},$$

ya’ni

$$f(x_1+h_1,\dots,x_n+h_n)-f(x_1,\dots,x_n)=\sum_{i=1}^{n} f'_{x_i}(\boldsymbol{x}+\theta\boldsymbol{h})h_i\,.$$

Lagranj formulasi o‘rinli bo‘lishini ko‘rsating.

4. $A_k, B_k (k\in\mathbb{N})$ va A,B,C,D matritsalar uchun $A_k\to A, B_k\to B$ bo‘lsin. Quyidagilarni isbotlang (matritsa o‘lchamlari moslangan deb hisoblanadi):

a) $CA_kD\to CAD,\ A_kB_k\to AB$;

b) agar λ_k sonlar λ ga intilsa ($\lambda_k\to\lambda$), $\lambda_kA_k\to\lambda A$ ham bo‘ladi;

c) kvadrat matritsa holida $\det A_k\to\det A$, $\det A\neq 0$ bo‘lganda $A_k^{-1}\to A^{-1}$.

5. $f(t,\boldsymbol{x})=\|\boldsymbol{x}\|,\ \boldsymbol{x}\in\mathbb{R}^n,$ funksiya $\boldsymbol{x}$ bo‘yicha $\mathbb{R}^n$ da Lipshits shartini qanoatlantirishini ko‘rsating. Bu funksiya differensiallanuvchimi?

6. Agar $\boldsymbol{f}:E\to\mathbb{R}^n\quad(E\subset\mathbb{R}^{1+m})$ vektor-funksiyaning $f_i:E\to\mathbb{R}\,, i=\overline{1,n}$, komponentalari E da Lipshits shartini qanoatlantirsa, u holda **f** ning o‘zi ham E da Lipshits shartini qanoatlantirishini isbotlang. Teskari tasdiqning ham o‘rinliligini ko‘rsating.

7. $f(\boldsymbol{x})=\|\boldsymbol{x}\|^2,\ \boldsymbol{x}\in\mathbb{R}^n,$ funksiya har qanday $K\subset\mathbb{R}^n$ kompaktda Lipshits shartini qanoatlantirishini $\mathbb{R}^n$ da esa qanoatlantirmasligini ko‘rsating.

8. $\boldsymbol{f}(t,x_1,x_2)=\left(\sqrt{|x_1x_2|};\dfrac{x_1+x_2}{1+t^2}\right)^T$ funksiya $|t|<1$, $|x_1|<1$, $|x_2|<1$ to‘plamda x_1,x_2 bo‘yicha Lipshits shartini qanoatlantiradimi?

$|t|<1$, $\varepsilon<|x_1|<1$, $\varepsilon<|x_2|<1\quad(0<\varepsilon<1)$ to‘plamda-chi?

9. Ixtiyoriy $E\subset\mathbb{R}^n$ to‘plamda $f:E\to\mathbb{R}$ funksiya Lipshits shartini qanoatlantirsin:

$$\exists L>0\ \forall\{\boldsymbol{x}^1,\boldsymbol{x}^2\}\subset E\ \ |f(\boldsymbol{x}^1)-f(\boldsymbol{x}^2)|\le L\ |\boldsymbol{x}^1-\boldsymbol{x}^2|.$$

Har qanday $\boldsymbol{x}\in\mathbb{R}^n$ uchun

$$\tilde f(\boldsymbol{x})=\inf_{\boldsymbol{y}\in E}\{f(\boldsymbol{y})+L\,|\,\boldsymbol{x}-\boldsymbol{y}\,|\}$$

deb, yangi funksiyani aniqlaylik. E to‘plamda $\tilde f=f$ va $\tilde f$ funksiya $\mathbb{R}^n$ fazoda Lipshits shartini qanoatlantirishini ko‘rsating ($\tilde f$ funksiya f ning E dan $\mathbb{R}^n$ gacha Lipshits sharti saqlangan holda davom ettirilishidir).

§ 8.3. Mavjudlik va yagonalik teoremasi

Vektor ko‘rinishda yozilgan defferentsial tenglamalar sistemasi uchun quyidagi Koshi masalasini qaraylik:

$$(\mathrm{K})\ \begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}), & (8.3.1)\\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0. & (8.3.2)\end{cases}$$

Bu yerda $\boldsymbol{x} = \boldsymbol{x}(t) - n\times 1$ o‘lchamli noma’lum vektor-funksiya, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya $D \subset \mathbb{R}^{1+n}$ sohada aniqlangan va uzluksiz, $\boldsymbol{f}(t,\boldsymbol{x}) \in C(D,\mathbb{R}^n)$, va $(t_0,\boldsymbol{x}^0) \in D$. Bu Koshi masalasini yechish (8.3.1) sistemaning biror I, $t_0 \in I$, oraliqda aniqlangan va (8.3.2) boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish demakdir.

(K) yoki (8.3.1), (8.3.2) Koshi masalasining skalyar ko‘rinishi

$$\begin{cases} x_1' = f_1(t,x_1,x_2,...,x_n) \\ x_2' = f_2(t,x_1,x_2,...,x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = f_n(t,x_1,x_2,...,x_n) \\ x_1|_{t_0} = x_1^0,\ x_2|_{t_0} = x_2^0,\ ...,\ x_n|_{t_0} = x_n^0. \end{cases}$$

Bu paragrafdagi ma’lumotlar $n = 1$ holidagi mos ma’lumotlarga juda ham o‘xshash.

(K) Koshi masalasiga quyidagi vektor ko‘rinishda yozilgan integral tenglamalar sistemasini mos qo‘yaylik :

$$\boldsymbol{x}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{x}(s))ds \qquad (8.3.3)$$

Agar I oraliqda aniqlangan $\boldsymbol{\varphi}: I \to \mathbb{R}^n$ vektor-funksiya uchun

1) $\boldsymbol{\varphi} \in C(I,\mathbb{R}^n) - \boldsymbol{\varphi}$ vektor-funksiya I da uzluksiz

2) $\forall t \in I$ uchun $\boldsymbol{\varphi}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{\varphi}(s))ds$, ya’ni $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ vektor-funksiya I da (8.3.3) ni qanoatlantiradi shartlar bajarilsa, u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ vektor-funksiya (8.3.3) integral tenglamalar sistemasining I oraliqda yechimi deyiladi.

Ekvivalentlik lemmasi. $\boldsymbol{f}(t,\boldsymbol{x}) \in C(D,\mathbb{R}^n), t_0 \in I, (t_0,\boldsymbol{x}^0) \in D$ bo‘lsin. $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ vektor-funksiya (8.3.1), (8.3.2) Koshi masalasining yechimi bo‘lishi uchun uning (8.3.3) integral tenglama yechimi bo‘lishi yetarli va zarurdir .

🗝 Isboti bevosita tekshirish yo‘li bilan amalga oshiriladi. 👍

Endi normal sistema uchun (8.3.1), (8.3.2) Koshi masalasi lokal (t_0 nuqtaning biror atrofida aniqlangan) yechimining mavjudligi va yagonaligi haqidagi teoreama (MYaT)ni keltiramiz. U Koshi-Pikar-Lindelyof teoremasi deb ham yuritiladi.

Teorema 1 (Koshi – Pikar – Lindelyof, MYaT). Aytaylik, $S=\{(t,\boldsymbol{x})\in\mathbb{R}^{1+n}\,\big|\,|t-t_0|\le a\,,\,\|\boldsymbol{x}-\boldsymbol{x}^0\|\le b\}$ $(a>0,b>0)$ silindr, $\boldsymbol{f}(t,\boldsymbol{x})\in C(S,\mathbb{R}^n)$ va u S da $\boldsymbol{x}=(x_1,x_2,...,x_n)$ bo‘yicha Lipshits shartini qanoatlantirsin. S kompakt bo‘lgani uchun S da uzluksiz $\|\boldsymbol{f}\|$ funksiya chegaralangan:

$$\exists M>0\ \forall(t,\boldsymbol{x})^T\in S\ \ \|\boldsymbol{f}(t,\boldsymbol{x})\|\le M \tag{8.3.4}$$

$h=\min\left\{a,\dfrac{b}{M}\right\}$ deylik. U holda (8.3.1), (8.3.2) Koshi masalasining $t\in[t_0-h;t_0+h]$ segmentda aniqlangan yechimi mavjud va yagonadir.

🗝 Isboti $n=1$ holidagiga o‘xshash. (8.3.1), (8.3.2) Koshi masalasining o‘rniga unga ekvivalent bo‘lgan (8.3.3) integral tenglamani yechamiz. Yechimni ketma-ket yaqinlashishlar metodi yordamida quramiz. $|t-t_0|\le h$ segmentda ketma-ket yaqinlashishlar deb ataluvchi $\boldsymbol{x}^0(t),\boldsymbol{x}^1(t),\ldots,\,\boldsymbol{x}^k(t),\ldots$ vektor-funksiyalar ketma-ketligini quyidagicha (rekurrent usulda) kiritaylik:

$$\boldsymbol{x}^0(t)=\boldsymbol{x}^0,\ \ \boldsymbol{x}^k(t)=\boldsymbol{x}^0+\int_{t_0}^{t}\boldsymbol{f}(s,\boldsymbol{x}^{k-1}(s))ds,\ k\in\mathbb{N}\ . \tag{8.3.5$_k$}$$

Bu yerdagi barcha integrallar mavjud bo‘lishi uchun $|t-t_0|\le h$ bo‘lganda har qanday $k=0,1,2,...$ uchun $(t,\boldsymbol{x}^k(t))^T\in S$ bo‘lishini ko‘rsatish kerak. $k=0$ bo‘lganda bu tushunarli. Endi matematik induksiyani qo‘llaymiz. Faraz qilaylik, $|t-t_0|\le h$ bo‘lganda $(t,\boldsymbol{x}^k(t))^T\in S$ bo‘lsin. Biz $|t-t_0|\le h$ bo‘lganda $(t,\boldsymbol{x}^{k+1}(t))^T\in S$ ekanligini ko‘rsatamiz. Farazimizga ko‘ra

$$\boldsymbol{x}^{k+1}(t)=\boldsymbol{x}^0+\int_{t_0}^{t}\boldsymbol{f}(s,\boldsymbol{x}^k(s))ds$$

funksiya $|t-t_0|\le h$ oraliqda aniqlangan. Demak,

$$\|\boldsymbol{x}^{k+1}(t)-\boldsymbol{x}^0\|=\left\|\int_{t_0}^{t}\boldsymbol{f}(s,\boldsymbol{x}^k(s))ds\right\|\le\left|\int_{t_0}^{t}\|\boldsymbol{f}(s,\boldsymbol{x}^k(s))\|ds\right|\le M\,|t-t_0|\le Mh\le b\,,$$

ya'ni $|t-t_0|\le h$ bo'lganda $(t,\boldsymbol{x}^{k+1}(t))^T\in S$. Shunday qilib, o'sha $t$lar uchun (8.3.5$_k$) dagi barcha integrallar mavjud hamda $x^k(t)$ larning hammasi uzluksiz funksiyalardan iborat bo'ladi.

Qurilgan $\boldsymbol{x}^k(t)$ yaqinlashishlar $[t_0-h;t_0+h]$ segmentda tekis yaqinlashuvchi bo'ladi. Buni isbotlash uchun ushbu

$$\boldsymbol{x}^0(t)+(\boldsymbol{x}^1(t)-\boldsymbol{x}^0(t))+(\boldsymbol{x}^2(t)-\boldsymbol{x}^1(t))+\ldots+(\boldsymbol{x}^{k+1}(t)-\boldsymbol{x}^k(t))+\ldots \quad (8.3.6)$$

vektor-funksiyalardan tuzilgan funksional qatorning tekis yaqinlashuvchi ekanligini ko'rsatish kerak. Buni ko'rsatish uchun esa funksional qatorning tekis yaqinlashishi haqidagi Veyrshtrass alomatidan foydalanamiz. Buning uchun (8.3.6) qator hadlarini normasi bo'yicha yuqoridan baholaymiz. (8.3.5$_1$) formuladan

$$\|\boldsymbol{x}^1(t)-\boldsymbol{x}^0(t)\|=\left\|\int_{t_0}^{t}\boldsymbol{f}(s,\boldsymbol{x}^0)ds\right\|\le M|t-t_0|\,. \qquad (8.3.7_1)$$

Agar L bilan Lipshits doimiysini belgilasak, ya'ni ixtiyoriy $(t,\boldsymbol{x})^T\in S$ va $(t,\tilde{\boldsymbol{x}})^T\in S$ nuqtalar uchun $\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\tilde{\boldsymbol{x}})\|\le L\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|$ bo'lsa, matematik induktsiya prinsipi yordamida ko'rsatish mumkinki,

$$\|\boldsymbol{x}^k(t)-\boldsymbol{x}^{k-1}(t)\|\le ML^{k-1}\frac{|t-t_0|^k}{k!},\ t\in[t_0-h;t_0+h],\ k\in\mathbb{N}\,, \qquad (8.3.7_k)$$

tengsizliklar o'rinli bo'ladi. (8.3.7$_k$) tengsizlikdan tekis yaqinlashish haqidagi Veyershtrass alomatiga ko'ra (8.3.6) qatorning $|t-t_0|\le h$ bo'lganda tekis yaqinlashishi kelib chiqadi, chunki $\|\boldsymbol{x}^k(t)-\boldsymbol{x}^{k-1}(t)\|$larni majorirlovchi sonli qator yaqinlashuvchi: $\sum_{k=1}^{+\infty}ML^{k-1}\frac{h^k}{k!}=\sum_{k=1}^{+\infty}\frac{M}{L}\frac{(Lh)^k}{k!}=\frac{M}{L}\left(e^{Lh}-1\right)$. Shunday qilib, $\boldsymbol{x}^k(t)$ funksional ketma-ketlik $[t_0-h;t_0+h]$ segmentda tekis yaqinlashuvchi. Uning limitini $\boldsymbol{\varphi}(t)$ bilan belgilaylik :

$$\lim_{k\to\infty}\boldsymbol{x}^k(t)=\boldsymbol{\varphi}(t)\ ,\ t\in[t_0-h,t_0+h]. \qquad (8.3.8)$$

Uzluksiz funksiyalarning tekis limiti sifatida $\boldsymbol{\varphi}(t)$funksiya uzluksiz bo'ladi.

$\boldsymbol{\varphi}(t)$ ning $[t_0-h;t_0+h]$ da (8.3.3) integral tenglamani qanoatlantirishini isbotlaylik. Buning uchun

$$\boldsymbol{x}^{k+1}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}^k(s))ds$$

tenglikda $k \to \infty$ deb limitga o‘tamiz. $\boldsymbol{f}$ uzluksiz, $\boldsymbol{x}^k(s)$ funksional ketma-ketlik $s \in [t_0 - h; t_0 + h]$ da $\boldsymbol{\varphi}(s)$ ga tekis intilgani uchun $\int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}^k(s))ds$ funksional ketma-ketlik $t \in [t_0 - h; t_0 + h]$ da $\int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{\varphi}(s))ds$ ga tekis intiladi va limitda

$$\boldsymbol{\varphi}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{\varphi}(s))ds$$

tenglikni olamiz. Shunday qilib, $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ funksiya (8.3.3) integral tenglamanining $t \in [t_0 - h; t_0 + h]$ da yechimi.

Endi (8.3.3) ning $[t_0 - h; t_0 + h]$ segmentda $\boldsymbol{\varphi}(t)$ dan boshqa yechimi yo‘qligini ko‘rsataylik. $\boldsymbol{x} = \boldsymbol{\psi}(t)$ (8.3.3)ning $[t_0 - h; t_0 + h]$ segmentda aniqlangan ixtiyoriy yechimi bo‘lsin:

$$\boldsymbol{\psi}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{\psi}(s))ds, \ \boldsymbol{\psi}(t) \in C\left([t_0 - h; t_0 + h], \mathbb{R}^n\right)$$

Yuqorida qurilgan $\mathbf{x}^k(t)$ ketma-ket yaqinlashishlar bilan $\boldsymbol{\psi}(t)$ orasidagi farqni baholaymiz.

$$\left\|\boldsymbol{\psi}(t) - \boldsymbol{x}^0(t)\right\| = \left\|\boldsymbol{\psi}(t) - \boldsymbol{x}^0\right\| = \left\|\int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{\psi}(s))ds\right\| \le M\left|t - t_0\right|$$

Matematik induksiya prinsipi yordamida

$$\left\|\boldsymbol{\psi}(t) - \boldsymbol{x}^k(t)\right\| \le ML^k \frac{\left|t - t_0\right|^{k+1}}{(k+1)!}, \ t \in [t_0 - h, t_0 + h], \ k \in \mathbb{N} \qquad (8.3.9)$$

ekanligini topamiz. (8.3.9) da $k \to \infty$ deb, (8.3.1) ga ko‘ra $\left\|\boldsymbol{\psi}(t) - \boldsymbol{\varphi}(t)\right\| \le 0$ tengsizlikni hosil qilamiz. Oxirgi tengsizlik $\forall t \in [t_0 - h, t_0 + h]$ uchun $\boldsymbol{\psi}(t) = \boldsymbol{\varphi}(t)$ bo‘lishini ko‘rsatadi. Teoremaning yagonalik qismi ham isbotlandi. ✍

Eslatma. $\boldsymbol{\psi}(t) = \boldsymbol{\varphi}(t), |t - t_0| \le h,$ bo‘lgani uchun (8.3.9) dan ketma-ket yaqinlashishlarning xatoligini baholovchi tengsizlikni hosil qilamiz:

$$\left\|\boldsymbol{\varphi}(t)-\boldsymbol{x}^k(t)\right\| \le ML^k \frac{\left|t-t_0\right|^{k+1}}{(k+1)!}, \ |t-t_0|\le h .$$

Teorema 2. Aytaylik, $D\subset\mathbb{R}^{1+n}$ – soha, $\boldsymbol{f}:D\to\mathbb{R}^n$ uzluksiz va u D da lokal Lipshits shartini qanoatlantirsin. U holda D sohaning ixtiyoriy $(t_0,\boldsymbol{x}^0)$ nuqtasidan (8.3.1) tenglamaning integaral chizig‘i o‘tadi. Bunda $(t_0,\boldsymbol{x}^0)$ nuqta orqali o‘tuvchi ixtiyoriy ikki yechim ularning umumiy aniqlanish oralig‘ida ustma-ust tushadi.

⊶ $(t_0,\boldsymbol{x}^0)\in D$ nuqtadan o‘tuvchi integral chiziqning mavjudligini isbotlaylik. $(t_0,\boldsymbol{x}^0)$ nuqta uchun $a>0$ va $b>0$ sonlarini shunday kichik tanlaylikki, ularga ko‘ra qurilgan ushbu

$$S=\left\{(t,\boldsymbol{x})\in\mathbb{R}^{1+n}\,\middle|\,|t-t_0|\le a\ ,\ \|\,\boldsymbol{x}-\boldsymbol{x}^0\,\|\le b\right\}$$

silindr D da joylashsin: $S\subset D$. Agar kerak bo‘lsa, a va b larni kichraytirib, $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya S da $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantiradi deb, hisoblaymiz. (Aslida bunga hojat yo‘q, chunki § 8.2 dagi lemma 2 ga ko‘ra D dagi har qanday kompaktda $\mathbf{f}$ funksiya $\mathbf{x}$ bo‘yicha Lipshits shartini qanoatlantiradi (albatta har bir kompaktda o‘zining Lipshits konstantasi bilan). Endi shu S ga tatbiq etilgan Koshi-Pikar-Lindelyof teoremasidan $(t_0,\boldsymbol{x}^0)\in D$ nuqta orqali o‘tuvchi integral chiziqning mavjudligi ravshan.

Faraz qilaylik, I_1 oraliqda aniqlangan $\boldsymbol{\varphi}_1(t)$ va I_2 oraliqda aniqlangan $\boldsymbol{\varphi}_2(t)$ yechimlar $(t_0,\boldsymbol{x}^0)\in G$ nuqta orqali o‘tsin, $\boldsymbol{\varphi}_1(t_0)=\boldsymbol{\varphi}_2(t_0)=\mathbf{x}^0$ ($t_0\in I_1\bigcap I_2$). Biz $I_1\bigcap I_2$ oraliqda $\boldsymbol{\varphi}_1(t)=\boldsymbol{\varphi}_2(t)$ ekanligini ko‘rsatishimiz kerak. t_0 ning o‘ng tomonidagi $t\in I_1\bigcap I_2$ nuqtalarda $\boldsymbol{\varphi}_1(t)=\boldsymbol{\varphi}_2(t)$ ekanligini isbotlaymiz. t_0 ning chap tomoni uchun isbot shunga o‘xshash bajariladi.

Agar t_0 dan o‘ngdaga biror $\tilde{t}\in I_1\bigcap I_2$ nuqtada $\boldsymbol{\varphi}_1(\tilde{t})\ne\boldsymbol{\varphi}_2(\tilde{t})$ bo‘lsa, $\tau=\inf\{s\mid\forall t\in(s,\tilde{t}]\ \boldsymbol{\varphi}_1(t)=\boldsymbol{\varphi}_2(t)\}$ deymiz. Ravshanki, $t_0<\tau<\tilde{t}$ va $\boldsymbol{\varphi}_1(\tau)=\boldsymbol{\varphi}_2(\tau)$, chunki, agar $\boldsymbol{\varphi}_1(\tau)\ne\boldsymbol{\varphi}_2(\tau)$ bo‘lganda edi, $\boldsymbol{\varphi}_1(t)$ va $\boldsymbol{\varphi}_2(t)$ uzluksiz bo‘lgani uchun τ nuqtaning chap tomonidagi unga yaqin t nuqtalarda ham $\boldsymbol{\varphi}_1(t)\ne\boldsymbol{\varphi}_2(t)$ bo‘lardi. Bu esa τ ning $\inf$ ekanliliga zid.

Shunday qilib, $\boldsymbol{\varphi}_1(\tau)=\boldsymbol{\varphi}_2(\tau)=\tilde{\boldsymbol{x}}$, lekin τ ning o‘ng tomonidagi τ ga yetarlicha yaqin barcha t lar uchun $\boldsymbol{\varphi}_1(t)\ne\boldsymbol{\varphi}_2(t)$. Bu $(\tau,\tilde{\mathbf{x}})\in G$ nuqtadan ikkita integral chiziq chiqqanligini anglatadi. Bu esa shu nuqta uchun tatbiq etilgan Koshi-Pikar-Lindelyof teoremasining yechimning yagonaligi haqidagi xulosasiga zid. ✍

Umumiy holda $(t_0, \boldsymbol{x}^0) \in D$ nuqta orqali o‘tuvchi yechimning Koshi-Pikar-Lindelyof teoremasi ta’minlovchi mavjudlik segmenti $[t_0 - h, t_0 + h]$ ning uzunligi $2h$ shu $(t_0, \boldsymbol{x}^0)$ nuqtaga bog‘liq bo‘ladi. Ya’ni turli $(t_0, \boldsymbol{x}^0) \in D$ nuqtalar uchun bu segmentning uzunligi umumiy holda har xil. Shu munosabat bilan quyidagi teoremani keltiramiz.

Teorema 3. $D \subset \mathbb{R}^{1+n}$ – soha , $\boldsymbol{f}(t, \boldsymbol{x}) \in C(D, \mathbb{R}^n)$ va $\boldsymbol{f}$ funksiya D da ($\boldsymbol{x}$ bo‘yicha) lokal Lipshits shartini qanoatlantirsin. U holda D da yotuvchi ixtiyoriy K kompakt uchun shunday $\delta = \delta(K) > 0,$ son topiladiki, K ning ixtiyoriy $(t_0, \boldsymbol{x}^0)$ nuqtasidan kamida $[t_0 - \delta, t_0 + \delta]$ segmentda aniqlangan yechim o‘tadi (bu yerda δ soni $(t_0, \boldsymbol{x}^0) \in K$ nuqtaga bog‘liq emas, u K ga bog‘liq xolos).

🗝 $K \subset D$ ixtiyoriy kompakt bo‘lsin. K dan D ning chegarasi ∂D gacha bo‘lgan masofani d deylik: $d = dist(K, \partial D)$. K to‘plam D sohada yotuvchi kompakt bo‘lgani uchun $K \bigcap \partial D \neq \varnothing$ va, demak, $d > 0$ bo‘ladi. $D_0 = \{\boldsymbol{x} \in D | \mathrm{dist}(\boldsymbol{x}, K) < d/2\}$ deylik. Ravshanki, D_0 – ochiq va $K \subset D_0 \subset \bar{D}_0 \subset G.$ $\boldsymbol{f}$ funksiya D da uzluksiz bo‘lgani uchun u D ning qismi $\bar{D}_0$ kompaktda ham uzluksiz hamda chegaralangan:

$$\| \boldsymbol{f}(t, \boldsymbol{x}) \| \leq M, \ (t, \boldsymbol{x}) \in \bar{D}_0$$

Ixtiyoriy $(t_0, \boldsymbol{x}^0) \in K$ nuqta uchun $S = \{(t, \boldsymbol{x}) \in \mathbb{R}^n \big| |t - t_0| \leq d/2, \| \boldsymbol{x} - \boldsymbol{x}^0 \| \leq d/2\}$ silindrni qaraylik. Tushunarliki, $S \subset \bar{D}_0$. Demak, S da $\| \boldsymbol{f}(t, \boldsymbol{x}) \| \leq M$ va $\boldsymbol{f}(t, \boldsymbol{x})$ funksiya $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantiradi. Endi Koshi-Pikar-Lindelyof teoremasidan ravshanki, $\delta = \min\{\frac{d}{2}, \frac{d}{2M}\} > 0$ soni hamma $(t_0, \boldsymbol{x}^0) \in K$ nuqtalar uchun umumiy $h = \delta$ bo‘lib xizmat qiladi. ✍

Misol 1. Ushbu

$$\frac{dx}{dt} = 2x + y, \ \frac{dy}{dt} = x + 2y, x(0) = 1, \ y(0) = -1; \ (x = x_1, y = x_2)$$

Koshi masalasini yechishga ketma-ket yaqinlashishlar metodini tatbiq etaylik.

Ixtiyoriy $S=\{(t,x,y)\in\mathbb{R}^3 \,\big|\, |t|<a,\ (x-1)^2+(y+1)^2<b^2\}\ (a>0, b>0)$ silindrda MYaTning shartlari qanoatlanadi: $f_1(x,y)=2x+y,\ f_2(x,y)=x+2y$ funksiyalari S da uzluksiz va x,y o‘zgaruvchilari bo‘yicha Lipshits shartini qanoatlantiradi.

Berilgan masalaga ekvivalent integral tenglamalar sistemasini yozamiz:

$$x(t)=1+\int_0^t [2x(s)+y(s)]ds$$

$$y(t)=-1+\int_0^t [x(s)+2y(s)]ds$$

Nolinchi yaqinlashish: $x^0(t)=1,\ y^0(t)=-1.$

Birinchi yaqinlashish:

$$x^1(t)=1+\int_0^t [2x^0(s)+y^0(s)]ds=1+\int_0^t [2-1]ds=1+t\,,$$

$$y^1(t)=-1+\int_0^t [x^0(s)+2y^0(s)]ds=-1+\int_0^t [1-2]ds=-1-t\,.$$

Ikkinchi yaqinlashish:

$$x^2(t)=1+\int_0^t [2x^1(s)+y^1(s)]ds=1+\int_0^t [2(1+s)-1-s]ds=1+t+\frac{t^2}{2},$$

$$y^2(t)=-1+\int_0^t [x^1(s)+2y^1(s)]ds=-1+\int_0^t [1+s+2(-1-s)]ds=-1-t-\frac{t^2}{2}.$$

…. … … … …

k – yaqinlashish ham shunga o‘xshash topiladi. Undan esa

$$\lim_{k\to\infty} x^k(t)=e^t\ ,\ \lim_{k\to\infty} y^k(t)=-e^t$$

ekanligi hosil bo‘ladi. Bevosita tekshirib ko‘rish mumkinki, $x=e^t$, $y=-e^t$ funksiyalar berilgan Koshi masalasining yechimi.

Misol 2. Ushbu

$$\begin{cases} x'=x^2\sin y+ty^2 \\ y'=x^3y \end{cases}$$

sistemaning o‘ng tomoni $f_1(t,x,y)=x+2yx^2\sin y+ty^2$, $f_2(t,x,y)=x^3y$ barcha $(t,x,y)\in\mathbb{R}^3$ nuqtalarda uzluksiz va ixtiyoriy kompaktda chegaralangan $\frac{\partial f_1}{\partial x}=2x\sin y$, $\frac{\partial f_1}{\partial y}=x^2\cos y+2ty$, $\frac{\partial f_2}{\partial x}=3x^2y$, $\frac{\partial f_2}{\partial y}=x^3$ xususiy hosilalarga ega, ya'ni f_1 va f_2 lar $\mathbb{R}^3$da lokal Lipshits shartini qanoatlantiradi. Demak, teorema 2 ga ko‘ra ixtiyoriy $(t_0,x_0,y_0)\in\mathbb{R}^3$ nuqtadan bu sistemaning yagona integral chizig‘i o‘tadi.

***Teorema 4* (*yechimning global mavjudligi to‘g‘risida*).** Faraz qilaylik, (8.3.3) sistemaning o‘ng tomonidagi $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya $\forall t\in[a;b]$, $\forall\boldsymbol{x}\in\mathbb{R}^n$ bo‘lganda aniqlangan va uzluksiz, hamda $\boldsymbol{x}$ vektor o‘zgaruvchi bo‘yicha Lipshits shartini qanoatlantirsin. U holda $\forall t_0\in[a;b]$ va $\forall x^0\in\mathbb{R}^n$ uchun (K) masalasining birato‘la $[a;b]$ da aniqlangan yechimi mavjud va yagonadir.

🗝 Yana (8.3.5) ketma-ket yaqinlashishlarni tuzaylik. $(5_1),\ldots,(5_k),\ldots$ integrallar endi $\forall t\in[a;b]$ uchun ma'noga ega, chunki $f(s,\boldsymbol{x}(s))$ vektor funksiya $\forall s\in[a;b]$ uchun aniqlangan. Teoremaning shartiga ko‘ra

$$\left\| f(t,\boldsymbol{x})-f(t,\boldsymbol{x}^0)\right\|\le L\left\|\boldsymbol{x}-\boldsymbol{x}^0\right\|.$$

Bundan

$$\| f(t,\boldsymbol{x})\|\le\left\| f(t,\boldsymbol{x}^0)\right\|+L\left\|\boldsymbol{x}-\boldsymbol{x}^0\right\|.$$

yoki

$$\| f(t,\boldsymbol{x})\|\le A+L\left\|\boldsymbol{x}-\boldsymbol{x}^0\right\|,\ A\overset{def}{=}\max_{t\in[a;b]}\left\| f(t,\boldsymbol{x}^0)\right\|. \qquad (8.3.10)$$

Endi (8.3.10) baholashdan foydalanib, (8.3.5) ketma-ket yaqinlashishlar uchun quyidagilarni hosil qilamiz.

$$\left\|\boldsymbol{x}^1(t)-\boldsymbol{x}^0(t)\right\|\le A\cdot|t-t_0|$$

$$\left\|\boldsymbol{x}^2(t)-\boldsymbol{x}^1(t)\right\|\le\left\|\int_{t_0}^{t}\left[f(s,\boldsymbol{x}^1(s))-f(s,\boldsymbol{x}^0(s))\right]ds\right\|\le$$

$$\left|L\int_{t_0}^{t}\left\|\boldsymbol{x}^1(s)-\boldsymbol{x}^0(s)\right\|ds\right|\le L\cdot\left|\int_{t_0}^{t}A|s-t_0|ds\right|=LA\frac{|t-t_0|^2}{2!},$$

....

$$\left\|\boldsymbol{x}^k(t)-\boldsymbol{x}^{k-1}(t)\right\| \le \left| L\int_{t_0}^{t}\left\|\boldsymbol{x}^{k-1}(s)-\boldsymbol{x}^{k-2}(s)\right\|ds \right| \le L^{k-1}A\frac{|t-t_0|^k}{k!},$$

....

Hosil qilingan bu baholashlarda $t\in[a;b],\, t_0\in[a;b]$ bo‘lishi kerak. Demak, $|t-t_0|\le b-a$ va (8.3.6) funksional qator $t\in[a;b]$ da tekis yaqinlashuvchi. Qolgan fikr yuritishlar Koshi-Pikar-Lindelyof teoremasining isbotidagi kabidir. ✍

Koshi–Pikar–Lindelyof teoremasidan yuqori tartibli hosilaga nisbatan yechilgan skalyar noma’lum funksiya uchun Koshi masalasi yechimining mavjudligi va yagonaligi to‘g‘risidagi teoremani keltirib chiqaraylik (§ 3.1 ga qarang).

Skalyar t o‘zgaruvchining skalyar noma’lum funksiyasi $y=y(t)$ ga nisbatan ushbu

$$y^{(n)}=g(t,y,y',...,y^{(n-1)}) \tag{8.3.11}$$

$n-$ tartibli yuqori hosilaga nisbatan yechilgan differensial tenglamani qaraylik. Quyidagi boshlang‘ich shartlarni qo‘yaylik:

$$y\big|_{t_0}=y_0,\; y'\big|_{t_0}=y'_0,...,\; y^{(n-1)}\big|_{t_0}=y_0^{(n-1)} \tag{8.3.12}$$

(8.3.11) tenglamani

$$y=x_1,\; y'=x_2,...,\; y^{(n-1)}=x_n \tag{8.3.13}$$

o‘zgaruvchilarga nisbatan birinchi tartibli differensial tenglamalar sistemasiga keltiramiz:

$$\begin{cases} x'_1=x_2\\ x'_2=x_3\\ \quad\vdots\\ x'_{n-1}=x_n\\ x'_n=g(t,x_1,...,x_n)\end{cases} \tag{8.3.14}$$

Bu sistemaning vektor ko‘rinishi

$$\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x}),\;\; \boldsymbol{f}(t,\boldsymbol{x})=\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_n\\ g(t,x_1,...,x_n)\end{pmatrix}. \tag{8.3.15}$$

(8.3.12) boshlang‘ich shartlar

$$x_1\Big|_{t_0}=y_0\,,\ x_2\Big|_{t_0}=y_0',\ldots,\ x_n\Big|_{t_0}=y_0^{(n-1)} \qquad (8.3.16)$$

ko‘rinishga o‘tadi. Uni

$$\boldsymbol{x}\Big|_{t_0}=\boldsymbol{x}^0\ \ (x_1^0=y_0,\, x_2^0=y_0'\ldots,\, x_n^0=y_0^{(n-1)}) \qquad (8.3.17)$$

vektor ko‘rinishida yozamiz.

Shunday qilib, (8.3.15), (8.3.17) Koshi masalasi hosil bo‘ldi. Agar $(t_0,\boldsymbol{x}^0)$ nuqtaning biror atrofida $g(t,\boldsymbol{x})$ haqiqiy funksiya uzluksiz va u $x_1,x_2,\ldots,x_n$ o‘zgaruvchilarga nisbatan Lipshits shartini qanoatlantirsa, u holda (8.3.15) dagi $\boldsymbol{f}(t,\boldsymbol{x})$ vektor funksiya $\boldsymbol{x}$ vektor o‘zgaruvchi bo‘yicha shu atrofda Lipshits shartini qanoatlantiradi. Demak, bu holda normal sistema uchun Koshi–Pikar–Lindelyof teoremasini qo‘llab (8.3.11), (8.3.12) Koshi masalasi yechimining mavjudligi va yagonaligini ko‘ramiz.

Normal sistema uchun Koshi masalasi (K) uchun Eyler siniq chiziqlari $n=1$ holidagidek kiritiladi. Ularning $n=1$ holidagi xossalari qaralayotgan umumiy holda ham saqlanadi.

Teorema 5. Aytaylik, $\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})$ sistemada $\boldsymbol{f}(t,\boldsymbol{x})\in C(D,\mathbb{R}^n)$va $\boldsymbol{f}(t,\boldsymbol{x})$ funksiya D da ($\boldsymbol{x}$ bo‘yicha) lokal Lipshits shartini qanoatlantirsin. U holda berilgan normal sistema ixtiyoriy $(t_0,\boldsymbol{x}^0)\in D$ nuqtaning biror kichik atrofida $\boldsymbol{x}=\boldsymbol{\varphi}(t,c_1,c_2,\ldots,c_n)$ umumiy yechimga ega.

Agar (8.3.1) sistemaning o‘ng tomonidan faqat uzluksizlik talab qilinsa, quyidagi mavjudlik teoremasi o‘rinli bo‘ladi.

Teorema 6 (Peano). Agar $D\subset\mathbb{R}^{1+n}$ sohada $\boldsymbol{f}(t,\boldsymbol{x})\in C(D,\mathbb{R}^n)$ bo‘lsa, ixtiyoriy $(t_0,\boldsymbol{x}^0)\in D$uchun (8.3.1), (8.3.2) Koshi masalasi kamida bitta yechimga ega bo‘ladi.

Agar (8.3.1) sistemaning o‘ng tomonidan $\boldsymbol{f}(t,\boldsymbol{x})\in C(D,\mathbb{R}^n)$ dan boshqa shart talab qilinmasa, yechim yagona bo‘lmasligi mumkin.

Masalalar bo‘limidagi 4-, 5- masalalarda yechimning yagonaligi uchun yetarli shartlar keltirilgan.

Masalalar

1. $\mathbb{R}^n$ fazoda quyidagi qismiy tartibni kiritaylik:

$$\boldsymbol{x}\le\boldsymbol{y}\Leftrightarrow x^j\le y^j\,,\ j=\overline{1,n}\ ;$$

$$\boldsymbol{x}<\boldsymbol{y}\Leftrightarrow x^j<y^j\,,\ j=\overline{1,n}\ .$$

Berilgan $\boldsymbol{f}:(a,b]\to\mathbb{R}^n$ funksiya uchun Dini hosilalari

$$D^{-}\boldsymbol{f}(t)=\overline{\lim_{h\to 0-}}\frac{\boldsymbol{f}(t+h)-\boldsymbol{f}(t)}{h}\ \text{(yuqori chap hosila)}$$

$$D_{-}\boldsymbol{f}(t)=\underline{\lim_{h\to 0-}}\frac{\boldsymbol{f}(t+h)-\boldsymbol{f}(t)}{h}\ \text{(quyi chap hosila)}$$

formulalar bilan aniqlanadi. Quyidagi tasdiqlarni isbotlang:

Faraz qilaylik, $\boldsymbol{f}\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ va $\forall t\in[a,b]$ uchun $\boldsymbol{x}\le\boldsymbol{y},\ x^j=y^j$ ekanligidan $f^j(t,\boldsymbol{x})\le f^j(t,\boldsymbol{y})$ tengsizlik kelib chiqsin ($j=\overline{1,n}$). Bundan tashqari, $\boldsymbol{\varphi}(t)$ uchun

$\boldsymbol{\varphi}'(t)=\boldsymbol{f}(t,\boldsymbol{\varphi}(t)),\ t\in[a,b]$, bo‘lsin. U holda, agar $\mathbf{u}\in C([a,b],\mathbb{R}^n)$ va

$$\begin{cases} D^{-}\boldsymbol{u}(t)>\boldsymbol{f}(t,\boldsymbol{u}(t)),\ a<t\le b,\\ \boldsymbol{u}(a)>\boldsymbol{\varphi}(a)\end{cases}$$

bo‘lsa, $\boldsymbol{u}(t)>\boldsymbol{\varphi}(t),\ t\in[a,b],$ baholash o‘rinli; agar $\boldsymbol{v}\in C([a,b],\mathbb{R}^n)$ va

$$\begin{cases} D_{-}\boldsymbol{v}(t)<\boldsymbol{f}(t,\boldsymbol{v}(t)),\ a<t\le b,\\ \boldsymbol{v}(a)<\boldsymbol{\varphi}(a)\end{cases}$$

bo‘lsa esa, $\boldsymbol{v}(t)<\boldsymbol{\varphi}(t),\ t\in[a,b],$ tengsizlik o‘rinli bo‘ladi. Bu tasdiqlarni isbotlang.

2. Faraz qilaylik, $f\in C(\mathbb{R},\mathbb{R})$ hamda $x=x(t)$ funksiya $x'=f(x)$ tenglamaning $t\in[a,b]$ segmentda aniqlangan yechimi bo‘lsin. Agar $x(a)=x(b)$ bo‘lsa, $x(t)=\text{const}$ ekanligini isbotlang. Bu tasdiq $\boldsymbol{f}\in C(\mathbb{R}^n,\mathbb{R}^n)$, $n>1$, holida o‘rinli emas. Lekin, agar $\boldsymbol{f}\in C(\mathbb{R}^n,\mathbb{R}^n)$ funksiya biror $\varphi\in C^1(\mathbb{R}^n,\mathbb{R})$ skalyar funksiyaning gradiyentidan iborat, ya’ni $\boldsymbol{f}=\text{grad}\,\varphi$ bo‘lsa, yuqorida keltirilgan tasdiq o‘rinli bo‘ladi. Shularni isbotlang.

3. Faraz qilaylik, $\boldsymbol{f}\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ va

$$\big(\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\boldsymbol{y})\big)\cdot(\boldsymbol{x}-\boldsymbol{y})\le\theta\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{t-a},\ a<t\le b,\ \{\boldsymbol{x},\boldsymbol{y}\}\subset\mathbb{R}^n,\ 0<\theta<1,$$

shart o‘rinli bo‘lsin. U holda

$$\begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})\\ \boldsymbol{x}|_{t_0}=\boldsymbol{x}^0\ \ (t_0\in[a,b],\ \boldsymbol{x}^0\in\mathbb{R}^n)\end{cases}$$

boshlang‘ich masalaning ko‘pi bilan bitta yechimi borligini isbotlang.

4. $D\subset\mathbb{R}^{n+1}-$ soha, $(t_0,\boldsymbol{x}^0)\in D$ va $\boldsymbol{f}\in C(D;\mathbb{R}^n)$ funksiya quyidagi shartni qanoatlantirsin

$$\forall t\ge t_0\ \ \forall(t,\boldsymbol{x})\in D\ \ \forall(t,\boldsymbol{y})\in D$$

$$(\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\boldsymbol{y}))\cdot(\boldsymbol{x}-\boldsymbol{y})\le 2|\boldsymbol{x}-\boldsymbol{y}|\cdot\varphi(|\boldsymbol{x}-\boldsymbol{y}|);$$

bu yerda $\varphi\in C([0;+\infty);\mathbb{R}_+)$ o‘suvchi funksiya, $\varphi(0)=0$ va $\int_0^r\frac{ds}{\varphi(s)}=+\infty\ (r>0).$

U holda

$$\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x}),\boldsymbol{x}|_{t_0}=\boldsymbol{x}^0\quad(K)$$

Koshi masalasi $t \ge t_0$ da yagona yechimga ega bo‘lishini ko‘rsating. Shu tasdiqqa o‘xshash tasdiq t_0 dan chap tomonda, ya’ni $t \le t_0$ da ham o‘rinli. Shularni isbotlang.

5. $D \subset \mathbb{R}^{n+1}$ – soha, $\boldsymbol{f} \in C(D;\mathbb{R}^n)$ funksiya uchun

$$\forall (t,\boldsymbol{x}) \in D \ \forall (t,\boldsymbol{y}) \in D \ \left|\boldsymbol{f}(t,\boldsymbol{x}) - \boldsymbol{f}(t,\boldsymbol{y})\right| \le \varphi(|\boldsymbol{x}-\boldsymbol{y}|).$$

bo‘lsin. Bu yerda $\varphi \in C([0;+\infty);\mathbb{R}_+)$ – o‘suvchi funksiya, $\varphi(0)=0$ va

$$\int_0^r \frac{ds}{\varphi(s)} = +\infty. \ (r>0).$$

U holda $\forall (t_0, \boldsymbol{x}^0) \in D$ uchun

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{x}\big|_{t_0} = \boldsymbol{x}^0 \end{cases}$$

Koshi masalasi yagona yechimga ega bo‘ladi (Bu – Osgud teoremasi). Shuni isbotlang.

§ 8.4. Yechimlarni davom ettirish

Ushbu

$$\boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \tag{8.4.1}$$

sistemani qaraylik, bu paragrafda $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya $D \subset \mathbb{R}^{1+n}$ sohada uzluksiz ($\boldsymbol{f} \in C(D,\mathbb{R}^n)$) va D da joylashgan har qanday kompaktda $\boldsymbol{x}$ vektor o‘zgaruvchi bo‘yicha Lipshits shartini qanoatlantiradi deb faraz qilinadi. Bu farazimizga ko‘ra $\forall (t_0, \boldsymbol{x}^0) \in D$ uchun ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{x}\big|_{t_0} = \boldsymbol{x}^0 \end{cases} \tag{8.4.2}$$

Koshi masalasi biror $[t_0, t_0 + h_0]$ $(h_0 > 0)$ segmentda aniqlangan yagona $\boldsymbol{x} = \varphi(t)$ yechimga ega.

Agar $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ funksiya (8.4.1) differensial tenglamaning $I = [a;b]$ oraliqda, $\boldsymbol{x} = \boldsymbol{\psi}(t)$ funksiya esa uning $J = [a;b^*], b \le b^*$, yoki $J = [a;b^*), b < b^*$, oraliqda aniqlangan yechimi bo‘lib, ular I da ustma-ust ham tushsa, u holda $\boldsymbol{x} = \boldsymbol{\psi}(t)$ yechim $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimning I dan J gacha o‘ngga davomi (davom ettirilishi) deb ataladi.

Yechimning boshqa tur oraliqlardan o‘ngga hamda chapga davomi shunga o‘xshash aniqlanadi.

Yechimning o‘ngga davom ettirishni amalga oshirishdan avval yechimlarni yelimlash (biriktirish) bilan bog‘liq bo‘lgan bir jumlani keltiramiz ($n=1$ holi § 3.4 da qaralgan edi).

Jumla 1. Agar $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ funksiya (8.4.1) differensial tenglamaning $[t_0,t_1]$ segmentda, $\boldsymbol{x}=\boldsymbol{\varphi}_1(t)$ esa uning $[t_1,t_2]$ segmentda aniqlangan yechimlari bo‘lib, $\boldsymbol{\varphi}(t_1)=\boldsymbol{\varphi}_1(t_1)$ shart ham bajarilsa, u holda bu yechimlarning yelimlanishi (biriktirilishi) bo‘lgan

$$\boldsymbol{\psi}(t)=\begin{cases}\boldsymbol{\varphi}(t), \text{ agar } t\in[t_0,t_1] \text{ bo'lsa}\\ \boldsymbol{\varphi}_1(t), \text{ agar } t\in[t_1,t_2] \text{ bo'lsa}\end{cases}$$

funksiya (8.4.1) differensial tenglamaning $[t_0,t_2]$ segmentda aniqlangan yechimini beradi, ya’ni $\boldsymbol{x}=\boldsymbol{\psi}(t)$ yechim $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimning $[t_0,t_1]$ segmentdan $[t_0,t_2]$ segmentgacha davomidan iborat.

🗝 Isboti $n=1$ holidagidek. 👍

Yechimning o‘ng uchi bo‘lmish $(t_1;\boldsymbol{x}^1)\overset{def}{=}(t_0+h_0,\varphi(t_0+h_0))\in D$ nuqtaga ko‘ra

$$\begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})\\ \boldsymbol{x}\big|_{t_1}=\boldsymbol{x}^1\end{cases}$$

Koshi masalasini yechib, $[t_1,t_1+h_1]$ $(h_1>0)$ segmentda aniqlangan yagona $\boldsymbol{x}=\varphi_1(t)$ yechimni topamiz. Yuqoridagi ikki yechimni yelimlanishi (biriktirilishi) dan ushbu

$$\boldsymbol{\varphi}^*(t)=\begin{cases}\boldsymbol{\varphi}(t), \text{ agar } t\in[t_0,t_1] \text{ bo'lsa,}\\ \boldsymbol{\varphi}_1(t), \text{ agar } t\in[t_1,t_1+h_1] \text{ bo'lsa}\end{cases}\quad (t_1=t_0+h_0,\varphi(t_1)=\varphi_1(t_1)=\boldsymbol{x}^1)$$

funksiyani quramiz. Jumla 1 ga ko‘ra bu $\varphi^*(t)$ funksiya (8.4.2) masalaning $[t_0,t_1]\cup[t_1,t_1+h_1]=[t_0,t_1+h_1]$ segmentda aniqlangan yechimidir. U $[t_0,\,t_0+h_0]$ da aniqlangan $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimning $[t_0,t_1+h_1]$ segmentgacha (o‘ngga) davomi. Bu yechimni (funksiyani) yana $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ bilan belgilaymiz; endi bu $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ funksiya (8.4.2) masalaning $[t_0,t_1+h_1]$ segmentda aniqlangan yechimidir. Yechimning bu davomi bir qiymatli aniqlanadi. Endi bu yechimni yana o‘ngga davom ettiramiz va hokazo.

Yechimning chapga davomi yuqoridagiga o‘xshash amalga oshiriladi.

Endi davomsiz yechim tushunchasimi kiritamiz.

Monoton o‘sib D ga intiluvchi K_j kompaktlar ketma-ketligini qaraylik:

$$K_1 \subset K_2 \subset ... \subset K_j \subset ..., \ \bigcup_{j=1}^{\infty} K_j = D. \qquad (8.4.3)$$

Masalan,

$$K_j = \left\{ (t, \boldsymbol{x}) \in D \middle| \ \text{dist}((t, \boldsymbol{x}), \partial D) \ge \frac{1}{j}, |t| \le j, \|\boldsymbol{x}\| \le j \right\}$$

deyish mumkin; agar $\partial D = \varnothing$ ($D = \mathbb{R}^{1+n}$) bo‘lsa, $\text{dist}((t, \boldsymbol{x}), \partial D) = +\infty$ deb hisoblaymiz.

Berilgan $(t_0, \boldsymbol{x}^0) \in D$ nuqta K_{j_1} da yotsin, $(t_0, \boldsymbol{x}^0) \in K_{j_1}$. § 8.3. dagi teorema 3 ga ko‘ra (8.4.2) masala yechimini o‘ngga o‘zgarmas qadam uzunliga bilan davom ettirib, chekli qadamdan so‘ng $(t_1, \boldsymbol{\varphi}(t_1)) \notin K_{j_1}$ ($t_1 > t_0$) nuqtani hosil qilamiz (yechim $t = t_1$ da K_{j_1} kompaktdan tashqarida). Aytaylik, $(t_1, \boldsymbol{\varphi}(t_1)) \in K_{j_2}$ bo‘lsin ($j_2 < j_1$ bo‘lishi ham mumkin; buning ahamiyati yo‘q). Endi yechimni t_1 dan o‘ngga K_{j_2} dan chiqqunga qadar davom ettiramiz va hokazo. Natijada monoton o‘suvchi $t_1, t_2, ..., t_m, ...$ ketma-ketlikni hosil qilamiz. Demak, chekli yoki cheksiz $T = \lim\limits_{j \to \infty} t_j$ mavjud va $[t_0, t_1] \cup [t_1, t_2] \cup ... \cup [t_{m-1}, t_m] \cup ... = [t_0, T)$ bo‘ladi. Davom ettirish natijasida biz (8.4.2) masalaning $[t_0, T)$ oraliqda aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimini hosil qilamiz. Bu yechim garafigi D da joylashgan har qanday kompaktdan chiqib ketadi. Haqiqatan ham, agar $\left\{ (t, \boldsymbol{\varphi}(t)) \in \mathbb{R}^{1+n} \mid t_0 \le t < T \right\}$ grafik biror $K \subset D$ kompaktda joylashgan bo‘lganda edi, u holda bu grafik K ni qoplagan biror K_j da yotardi; bunday bo‘lishi mumkin emas, chunki t_0 dan boshlangan yechim o‘ngga davom ettirilishi natijasida chekli qadamdan so‘ng o‘sha K_j dan tashqariga chiqishi kerak edi.

Yechim t_0 dan chapga ham shu yo‘sinda davom ettiriladi. Natijada $(\tau; T)$ intervalda aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim hosil bo‘ladi ($\tau = -\infty$ yoki/va $T = +\infty$ bo‘lishi mumkin). Bu yechim (K) masalaning (D sohadagi) **davomsiz yechimi** deyiladi.

Davomsiz yechim $\boldsymbol{\varphi}(t)$ning (8.4.3) dagi K_j ($j \in \mathbb{N}$) kompaktlarning tanlanishiga bog‘liq emasligini ko‘rsatish qiyin emas.

Faraz qilaylik, $\tilde{K}_j$ kompaktlar ham monoton o‘sib D ga intiluvchi, ya’ni

$$\tilde{K}_1 \subset \tilde{K}_2 \subset ... \subset \tilde{K}_j \subset ..., \quad \bigcup_{j=1}^{\infty} \tilde{K}_j = D$$

xususiyatga ega va ularga ko‘ra qurilgan (8.4.2) Koshi masalasining davomsiz yechimi $(\tilde{\tau};\tilde{T})$ intervalda aniqlangan $\tilde{\varphi}(t)$ funksiyadan iborat bo‘lsin.

Jumla 2. $\boldsymbol{\varphi}(t)$va $\tilde{\varphi}(t)$ davomsiz yechimlar ustma-ust tushadi, ya’ni $(\tau;T)=(\tilde{\tau};\tilde{T})$ va $\forall t \in (\tau;T)=(\tilde{\tau};\tilde{T})$ uchun $\varphi(t)=\tilde{\varphi}(t)$.

🗝 Ikkala $\varphi(t)$va $\tilde{\boldsymbol{\varphi}}(t)$davomsiz yechim ham bitta (8.4.2) Koshi masalasining yechimi bolgani uchun yechimning yagonalik xossasiga ko‘ra ular aniqlanish sohalarining tengligidan $((\tau;T)=(\tilde{\tau};\tilde{T}))$ bu yechimlarnig tengligi, ya’ni jumlaning isboti kelib chiqadi. Demak, $\tau=\tilde{\tau}$ va $T=\tilde{T}$ ekanligini isbotlashimiz kifoya. Biz $T=\tilde{T}$ tenglikni ko‘rsatamiz, $\tau=\tilde{\tau}$ ekanligi shunga o‘xshash isbotlanadi.

Faraz qilaylik, $T \neq \tilde{T}$ bo‘lsin. Aniqlik uchun $T<\tilde{T}$ deylik. Tushunarliki, $T-$chekli son va $\forall t \in [t_0,T)$uchun $\boldsymbol{\varphi}(t)=\tilde{\boldsymbol{\varphi}}(t)$. Demak, $\lim\limits_{t\to T-0}\boldsymbol{\varphi}(t)=\tilde{\boldsymbol{\varphi}}(T)$ limit mavjud va $(T,\tilde{\boldsymbol{\varphi}}(T)\in D$. Ravshanki, K_j $(j\in\mathbb{N})$ kompaktlarning birortasi, masalan K_{j_0}, ushbu $\left\{(t,\boldsymbol{\varphi}(t))\in\mathbb{R}^{1+n} \mid t_0 \le t < T\right\}$grafikni qoplaydi: $K_{j_0} \supset \left\{(t,\boldsymbol{\varphi}(t))\in\mathbb{R}^{1+n} \mid t_0 \le t < T\right\}$. Lekin $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ davomsiz yechimning qurilishiga ko‘ra uning biror $t\in[t_0,T)$ dagi qiymati K_{j_0}kompaktdan tashqarida bo‘lishi kerak edi. Hosil bo‘lgan ziddiyat farazimizning noto‘g‘ri va, demak, $T=\tilde{T}$ ekanligini isbotlaydi. 👍

Shunday qilib, (8.4.2) Koshi masalasining davomsiz yechimi bir qiymatli aniqlangan.

Davomsiz yechimning xususiyatini quyidagi teorema ochadi.

Teorema. Faraz qilaylik, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya $D\subset\mathbb{R}^{n+1}$ sohada uzluksiz ($\boldsymbol{f}\in C(D,\mathbb{R}^n)$ va D da joylashgan har qanday kompaktda $\boldsymbol{x}$ bo‘yicha Lipshits shartini qanoatlantirsin hamda $(t_0,\boldsymbol{x}^0)\in D$ uchun qo‘yilgan (8.4.2) Koshi masalasining $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ davomsiz yechimi $(\tau;T)$ intervalda aniqlangan bo‘lsin. U holda ixtiyoriy $K\subset D$ kompakt uchun shunday chekli $\tau_1 \in (\tau;T)$ va $T_1 \in (\tau;T)$ lar mavjudki, har qanday $\tilde{t} \in (\tau;\tau_1)$ uchun $(\tilde{t};\boldsymbol{\varphi}(\tilde{t})) \notin K$ va har qanday $t \in (T_1,T)$ uchun $(t;\boldsymbol{\varphi}(t)) \notin K$ bo‘ladi.

🗝 Ixtiyoriy $K \subset D$ kompakt berilgan bo‘lsin. Teoremani T_1 uchun isbotlaymiz. τ_1 uchun isbot shunga o‘xshash bo‘ladi. (8.4.3) munosabatlarga ko‘ra K_j $(j \in \mathbb{N})$ kompaktlar orasida K kompaktni qoplovchi K_{j_0} mavjud, $K_{j_0} \supset K$. Agar barcha $t \in (\tau;T)$ lar uchun $\boldsymbol{\varphi}(t) \notin K_{j_0}$ bo‘lsa, teorema isbot bo‘ldi; chunki bu holda, masalan, $T_1 = t_0$ olish mumkin. Endi faraz qilaylik, shunday $t_* \in (\tau,T)$ mavjud bo‘lsinki, uning uchun $\boldsymbol{\varphi}(t_*) \in K_{j_0}$ bo‘lsin. U holda ushbu $\Delta \overset{def}{=} \left\{t \in (\tau,T) | (t,\boldsymbol{\varphi}(t) \in K_{j_0}\right\}$ to‘plam bo‘shmas $(t_* \in \Delta)$ va chegaralangan (chunki K_{j_0} - kompakt). Demak, chekli $\sup\Delta$ mavjud. $T_1 = \sup\Delta$ deylik. U holda supremum ta’rifiga ko‘ra barcha $t \in (T_1,T)$ lar uchun $(t;\boldsymbol{\varphi}(t)) \notin K_{j_0}$ va, demak, $(t;\boldsymbol{\varphi}(t)) \notin K$ ham bo‘ladi. ✍

Biz yuqorida (8.4.2) Koshi masalasining davomsiz yechimini qurdik. (8.4.1) differensial tenglamaning biror yechimini davom ettirish va davomsiz yechim tushunchasi yuqoridagidan bevosita kelib chiqadi, chunki (8.4.1) tenglamaning I oraliqda aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimi uhbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{\varphi}(t_0) \quad (t_0 \in I) \end{cases}$$

Koshi masalasining yechimi demakdir. Bu masala yechimining I dan tashqariga davomi (u $t_0 \in I$ ga bog‘liq emas) I oraliqda aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimning davomidir.

Masalalar

1. Ushbu

$$\frac{dx}{dt} = \frac{y}{t}, \; \frac{dy}{dt} = -\frac{x}{t}, \; x(1) = 1, \; y(1) = 0$$

masalaning davomsiz yechimi $(0,+\infty)$ intervalda aniqlangan ekanligini ko‘rsating.

2. Faraz qilaylik, $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ tenglamaning o‘ng tomoni $\boldsymbol{x} \in G$ larda ($G \subset \mathbb{R}^n$ – soha) aniqlangan va tenglama uchun yechimning mavjudlik va yagonalik xossasi o‘rinli bo‘lsin. Agar bu tenglamaning $[t_0;t_1]\ (t_0<t_1)$ segmentda aniqlangan $x=\varphi(t)$ yechimi uchun $\varphi(t_0)=\varphi(t_1)$ bo‘lsa, bu yechim $t\in(-\infty;+\infty)$ oraliqqa davom ettirilishi mumkinligini ko‘rsating.

3. Ushbu $y'=y^2+x^2,\ y(0)=0,$ masala $[0;2,6)$ oraliqda aniqlangan yechimga ega emas (yechim $[0;2,6)$ gacha davom etmas) ligini isbotlang.

4. $\{f,g\}\subset C(\mathbb{R},\mathbb{R})$ va $G(x)=\int_0^x g(s)ds$ funksiyalar uchun

$$\exists m>0\ \forall x\in\mathbb{R}\ G(x)\ge mx^2 \text{ va } \forall y\in\mathbb{R}\ \ y f(y)\ge 0$$

shartlar o‘rinli bo‘lsin. Ushbu

$$\begin{cases} x''+f(x')+g(x)=0 \\ x(0)=x_0,\ x'(0)=v_0\ \ (\{x_0,v_0\}\subset\mathbb{R}) \end{cases}$$

boshlang‘ich masalaning yechimi o‘ngga $[0,+\infty)$ gacha davom etishini ko‘tsating.

§ 8.5. Yechimning boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligi

Yechimning parametrlarga uzluksiz bog‘liqligi. Dastlab skalyar noma’lum funksiya $x=x(t)$ uchun

$$\begin{cases} \dfrac{dx}{dt}+p(t,\mu)x=q(t,\mu) \\ x(\tau)=\xi \end{cases}$$

chiziqli boshlang‘ich masalani qaraylik; bu yerda $\mu-$haqiqiy parametr, $\mu\in(\mu_1,\mu_2),\tau\in I$ va $\{p,q\}\subset C(I\times(\mu_1,\mu_2))$. Tushunarliki, bu masalaning yechimi na faqat $t$ga, balki u boshlang‘ich ma’lumot τ,ξ va μ parametrlarga bog‘liq. U, ma’lumki,

$$x(t;\tau,\xi,\mu)=\xi\cdot e^{-\int_\tau^t p(s,\mu)ds}+e^{-\int_\tau^t p(s,\mu)ds}\cdot\int_\tau^t q(\tilde{s},\mu)\cdot e^{\int_\tau^{\tilde{s}} p(s,\mu)ds}d\tilde{s}$$

formula bilan kvadraturalarda ifodalanadi. Bu formula ko‘rinishidan ravshanki, qo‘yilgan shartlarda yechim $C(I\times I\times\mathbb{R}\times(\mu_1,\mu_2))$ sinfga tegishli; xususan, yechim boshlang‘ich ma’lumotlar va parametrga uzluksiz bog‘liq. Umumiy holda differensial tenglama chiziqli emas va qo‘yilgan masala yechimning oshkor formulasi mavjud bo‘lmaganligi sababli unig boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligini o‘rganish oson emas.

Endi ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu}) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \end{cases} \tag{8.5.1}$$

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \in M$ $(M \subset \mathbb{R}^m -$soha) $\boldsymbol{\mu}$ parametr(lar)ga bo‘gliq bo‘lgan Koshi masalasini qaraylik. Faraz qilaylik, $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu})$ vektor-funksiya $(t, \boldsymbol{x}) \in D$, $\boldsymbol{\mu} \in M$ bo‘lganda aniqangan va (barcha argumentlari bo‘yicha) uzluksiz ($\boldsymbol{f} \in C(D \times M, \mathbb{R}^n)$) hamda $D \times M$ sohada $\mathbf{x}$ vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin, ya’ni har qanday $(t, \boldsymbol{x}, \boldsymbol{\mu}) \in D \times M$ nuqtaning yetarlicha kichik atrofi uchun shunday $L > 0$ son mavjudki, shu atrofdagi barcha $(t, \boldsymbol{x}^1, \boldsymbol{\mu})$ va $(t, \boldsymbol{x}^2, \boldsymbol{\mu})$ nuqtalar uchun

$$\| \boldsymbol{f}(t, \boldsymbol{x}^2, \boldsymbol{\mu}) - \boldsymbol{f}(t, \boldsymbol{x}^1, \boldsymbol{\mu}) \| \le L \| \boldsymbol{x}^2 - \boldsymbol{x}^1 \|$$

tengsizlk o‘rinli. Oxirgi shart bajarilishi uchun, masalan, ixtiyoriy $(t, \boldsymbol{x}, \boldsymbol{\mu}) \in D \times M$ nuqtaning biror atrofida $\left| \dfrac{\partial f_i}{\partial x_j} \right| \le \text{const}$ bo‘lishi yetarli. Qo‘yilgan shartlarda har qanday $(t_0, \boldsymbol{x}^0, \boldsymbol{\mu}) \in D \times M$ uchun 1 masala yagona davomsiz $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$, $t \in I$, yechimga ega. Bu davomsiz yechimning aniqlanish intervali, tushunarliki, tayinlangan $(t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ qiymatlarga bog‘liq bo‘ladi, $I = I(t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$. Demak, $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ yechim $(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu}) \in I \times D \times M \subset R^{2+n+m}$ sohada aniqlangan. Agar $(t_0, \boldsymbol{x}^0)$ tayinlangan bo‘lsa, u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ yozuv o‘rniga $\boldsymbol{x} = \boldsymbol{\varphi}(t; \boldsymbol{\mu})$ yozuvni ishlatamiz.

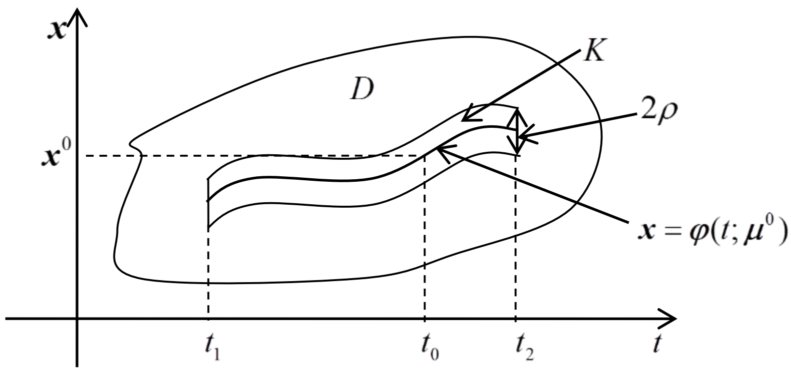
Matematik modellar tuzilganda odatda t_0 va $\boldsymbol{x}^0$ boshlang‘ich ma’lumotlar hamda o‘ng tomondagi $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu})$ vektor-funksiya kichik xatolikka ega bo‘lgan har xil o‘lchashlar yoki hisoblashlar yordamida topilgan bo‘ladi. Shuning uchun “bu kichik xatoliklar hisobiga yechim chekli (katta) qiymatga o‘zgarib ketmaydimi?” degan savolga javob berish muhim amaliy ahamiyatga ega. Albatta, amaliyotga tatbiq nuqtai nazaridan kichik xatoliklar yechimni ko‘p o‘zgartirmasligi, ya’ni boshlang‘ich ma’lumotlar va parametrlarning kichik o‘zgarishi yechimning kichik o‘zgarishiga olib kelishi kerak.

Biz bu bandda yechimning boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligini o‘rganamiz. Bu bog‘lanishning silliqligini esa § 12.1 da tekshiramiz.

Dastlab yechimning parametrlarga uzluksiz bog‘liqligini ifodalovchi teoremani isbotlaymiz, so‘ngra esa yechimning boshlang‘ich ma’lumotlarga uzlksiz bog‘liqligini ana shu teoremadan keltirib chiqaramiz.

Teorema 1 **(*yecimning parametrlarga uzluksiz bog‘liqligi*).** Faraz qilaylik, $\boldsymbol{f} \in C(D\times M, \mathbb{R}^n)$ bo‘lsin va u $D\times M$ sohada $\boldsymbol{x}$ vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin hamda $\boldsymbol{\mu}=\boldsymbol{\mu}^0$ bo‘lganda (8.5.1) masala $t\in[t_1,t_2]$ ($t_0\in[t_1,t_2]$, $(t_0,\boldsymbol{x}^0,\boldsymbol{\mu}^0)\in D\times M$) segmentda aniqlangan $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0)$ yechimga ega bo‘lsin. U holda shunday yetarlicha kichik $\delta>0$ son mavjudki, $\|\boldsymbol{\mu}-\boldsymbol{\mu}^0\|<\delta$ bo‘lganda $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})\ (=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu}))$ yechim barcha $t\in[t_1,t_2]$ larda aniqlangan va $(t;\boldsymbol{\mu})$ o‘zgaruvchilar bo‘yicha uzluksiz vektor-funksiyadan iborat bo‘ladi, $\boldsymbol{\varphi}(t;\boldsymbol{\mu})\in C\big([t_1,t_2]\times B_\delta(\boldsymbol{\mu}^0)\big)$.

🗝 Tushunarliki, berilgan yechim grafigi bo‘lmish $\{(t,\boldsymbol{x})\in\mathbb{R}^{1+n}\mid t_1\le t\le t_2,\ \boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0)\}$ to‘plam $\mathbb{R}^{1+n}$ fazoda yopiq va chegaralangan hamda u D sohada joylashgan. Xususan, biror yetarlicha kichik $\rho>0$ uchun ushbu $K=\{(t,\boldsymbol{x})\in\mathbb{R}^{1+n}\mid t_1\le t\le t_2,\ \|\boldsymbol{x}-\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0)\|\le\rho\}$ nay kompakt to‘plamdan iborat bo‘ladi va u ham D sohada yotadi.



Yetarlicha kichik $\sigma>0$ uchun $\overline{B}_\sigma(\boldsymbol{\mu}^0)=\{\boldsymbol{\mu}\in\mathbb{R}^m\mid \|\boldsymbol{\mu}-\boldsymbol{\mu}^0\|\le\sigma\}\subset M$ bo‘ladi. $\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})$ vektor-funksiya $K\times\overline{B}_\sigma(\boldsymbol{\mu}^0)\subset D\times M$ kompaktda $\boldsymbol{x}$ vektor o‘zgaruvchi bo‘yicha Lipshits shartini qanoatlantiradi, ya’ni shunday $L>0$ son mavjudki, ixtiyoriy $(t,\boldsymbol{x}^1,\boldsymbol{\mu})\in K\times\overline{B}_\sigma(\boldsymbol{\mu}^0)$ va $(t,\boldsymbol{x}^2,\boldsymbol{\mu})\in K\times\overline{B}_\sigma(\boldsymbol{\mu}^0)$ nuqtalar uchun

$$\| \boldsymbol{f}(t,\boldsymbol{x}^2,\boldsymbol{\mu}) - \boldsymbol{f}(t,\boldsymbol{x}^1,\boldsymbol{\mu}) \| \le L \| \boldsymbol{x}^2 - \boldsymbol{x}^1 \|$$

tengsizlik bajariladi. Ushbu $\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu})$ vektor-funksiya $(t,\boldsymbol{\mu}) \in [t_1,t_2] \times \overline{B}_\sigma(\boldsymbol{\mu}^0)$ kompaktda uzluksiz, demak, tekis uzluksiz ham. Shuning uchun ixtioriy $\varepsilon > 0$, $\varepsilon \le \rho$, songa ko‘ra shunday $\delta = \delta(\varepsilon) > 0$, $\delta \le \sigma$, topiladiki, ixtiyoriy $t \in [t_1,t_2]$ va $\| \boldsymbol{\mu} - \boldsymbol{\mu}^0 \| < \delta$ shartni qanoatlantiruvchi barcha $\boldsymbol{\mu}$ lar uchun $\| \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu}) - \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu}^0) \| < \varepsilon$ bo‘ladi. Ixtiyoriy $\boldsymbol{\mu}$, $\| \boldsymbol{\mu} - \boldsymbol{\mu}^0 \| < \delta$, uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t;\boldsymbol{\mu})$ yechim biror $t \in I \subset [t_1,t_2]$ oraliqda aniqlangan. Bu yerdagi $\delta > 0$ sonini kichraytirib, mos $\boldsymbol{x} = \boldsymbol{\varphi}(t;\boldsymbol{\mu})$, $\boldsymbol{\mu} \in B_\delta(\boldsymbol{\mu}^0)$, yechimlarni $t \in [t_1,t_2]$ oraliqqacha davom ettirish mumkinligini ko‘rsatamiz. Yechimning grafigi K nayda yotgan t paytlar uchun (grafik K dan chiqib ketmagunga qadar)

$$\| \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}),\boldsymbol{\mu}) - \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu}^0) \| \le$$
$$\le \| \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}),\boldsymbol{\mu}) - \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu}) \| + \| \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}),\boldsymbol{\mu}) - \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0),\boldsymbol{\mu}^0) \| \le$$
$$\le L \| \boldsymbol{\varphi}(t;\boldsymbol{\mu}) - \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0) \| + \varepsilon$$

va, demak,

$$\| \boldsymbol{\varphi}(t;\boldsymbol{\mu}) - \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0) \| = \left\| \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\boldsymbol{\mu}),\boldsymbol{\mu})ds - \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\boldsymbol{\mu}^0),\boldsymbol{\mu}^0)ds \right\| \le$$
$$\le \left| \int_{t_0}^{t} \| \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\boldsymbol{\mu}),\boldsymbol{\mu}) - \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\boldsymbol{\mu}^0),\boldsymbol{\mu}^0)) \| ds \right| \le \left| \int_{t_0}^{t} (L \| \boldsymbol{\varphi}(s;\boldsymbol{\mu}) - \boldsymbol{\varphi}(s;\boldsymbol{\mu}^0) \| + \varepsilon) ds \right| \le$$
$$\le \varepsilon |t - t_0| + L \left| \int_{t_0}^{t} \| \boldsymbol{\varphi}(s;\boldsymbol{\mu}) - \boldsymbol{\varphi}(s;\boldsymbol{\mu}^0) \| ds \right|.$$

Gronuoll-Bellman tengsizligiga ko‘ra (§ 3.2. ga qarang)

$$\| \boldsymbol{\varphi}(t;\boldsymbol{\mu}) - \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0) \| \le \frac{\varepsilon}{L}(e^{L|t-t_0|} - 1).$$

Endi $\varepsilon > 0$ sonni shunday kichik tanlaylikki, uning uchun

$$\frac{\varepsilon}{L} \sup_{t \in [t_1,t_2]} (e^{L|t-t_0|} - 1) < \rho$$

bo‘lsin. Shu $\varepsilon > 0$ songa ko‘ra topilgan $\delta > 0$, $\delta \le \sigma$, sonni tayinlaylik. U holda $\| \boldsymbol{\mu} - \boldsymbol{\mu}^0 \| < \delta$ shartni qanoatlantiruvchi ixtiyoriy $t \in [t_1,t_2]$ va ixtiyoriy $\boldsymbol{\mu} \in B_\delta(\boldsymbol{\mu}^0)$ uchun $\| \boldsymbol{\varphi}(t;\boldsymbol{\mu}) - \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0) \| < \rho$ bo‘ladi, ya’ni

$\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ yechim $t\in[t_1,t_2]$ paytlarda K nayning yon sirtiga yetib borolmaydi va, demak, $[t_1,t_2]$gacha davom etadi. Endi $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ vektor-funksiyaning aytilgan t va $\boldsymbol{\mu}$ lar bo‘yicha uzluksiz ekanligini ko‘rsatishimiz qoldi. Analizdan ma’lumki, agar bu funksiya $\boldsymbol{\mu}\in B_\delta(\boldsymbol{\mu}^0)$ tayinlanganda t bo‘yicha $[t_1,t_2]$da uzluksiz (bizda esa uning t bo‘yicha hosilasi ham uzluksiz) va $\boldsymbol{\mu}$ bo‘yicha $t\in[t_1,t_2]$ga nisbatan tekis uzluksiz bo‘lsa, $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ vektor-funksiya $(t;\boldsymbol{\mu})\in[t_1;t_2]\times B_\delta(\boldsymbol{\mu}^0)$ o‘zgaruvchilar majmuasi bo‘yicha uzluksiz bo‘ladi. Demak, biz quyidagini ko‘rsatishimiz kifoya: har qanday $\tilde{\boldsymbol{\mu}}\in B_\delta(\boldsymbol{\mu}^0)$ uchun yetarlicha kichik ixtioriy $\eta>0$ son berilganda ham shunday $\omega>0$ topiladiki, $\|\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}\|<\omega$ shartni qanoatlantiruvchi barcha $\boldsymbol{\mu}\in B_\delta(\boldsymbol{\mu}^0)$ lar va barcha $t\in[t_1,t_2]$lar uchun $\|\boldsymbol{\varphi}(t;\boldsymbol{\mu})-\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}})\|<\eta$ tengsizlik o‘rinli bo‘ladi. Ixtiyoriy $\tilde{\boldsymbol{\mu}}\in B_\delta(\boldsymbol{\mu}^0)$ parametrni tayinlab, $(t,\boldsymbol{\mu})\in[t_1,t_2]\times\overline{B}_\sigma(\boldsymbol{\mu}^0)$ ning ushbu $\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}}),\boldsymbol{\mu})$ vektor-funksiyasini qaraylik. Yuqoridagiga o‘xshash fikr yuritib, $\|\boldsymbol{\varphi}(t;\boldsymbol{\mu})-\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}})\|$ ni baholaymiz. $\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}}),\boldsymbol{\mu})$ funksiya $[t_1,t_2]\times\overline{B}_\sigma(\boldsymbol{\mu}^0)$ kompaktda uzluksiz bo‘lgani uchun u shu yerda tekis uzluksiz hamdir. Demak, ixtioriy $\theta>0$ son berilganda ham shunday $\omega=\omega(\theta,\tilde{\boldsymbol{\mu}})>0$ topiladiki, $\|\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}\|<\omega$ shartni qanoatlantiruvchi barcha $\boldsymbol{\mu}\in\overline{B}_\delta(\boldsymbol{\mu}^0)$ lar va ixtiyoriy $t\in[t_1,t_2]$ uchun $\|\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}}),\boldsymbol{\mu})-\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}}),\boldsymbol{\mu}^0)\|<\theta$ bo‘ladi. Endi yuqoridagi baholashlarga o‘xshash ushbu

$$\|\boldsymbol{\varphi}(t;\boldsymbol{\mu})-\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}})\|\le\frac{\theta}{L}(e^{L|t-t_0|}-1),\ t\in[t_1,t_2],$$

tengsizlikni topamiz. Agar $\theta>0$ sonni

$$\frac{\theta}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)<\eta$$

shartdan tanlab, unga mos $\omega=\omega(\theta,\tilde{\boldsymbol{\mu}})>0$ sonni topsak, u holda $\|\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}\|<\omega$ ekanligidan barcha $t\in[t_1,t_2]$ lar uchun

$$\|\boldsymbol{\varphi}(t;\boldsymbol{\mu})-\boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}})\|\le\frac{\theta}{L}(e^{L|t-t_0|}-1)\le\frac{\theta}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)<\eta$$

bo‘lishi kelib chiqadi. ♙

Biz teoremaning shartlarida quyidagi tenglikning o‘rinli ekanligini isbotladik:

$$\boldsymbol{\varphi}(t;\boldsymbol{\mu}) = \boldsymbol{\varphi}(t;\tilde{\boldsymbol{\mu}}) + \boldsymbol{r}_0(t;\boldsymbol{\mu}^0),$$

bunda $\boldsymbol{r}_0(t;\boldsymbol{\mu})$ qoldiq (farq) $\boldsymbol{\mu}$ o‘zgaruvchi $\boldsymbol{\mu}^0$ ga intilganda $t \in [t_1, t_2]$ga nisbatan tekis nolga intiladi, ya’ni

$$\boldsymbol{r}_0(t;\boldsymbol{\mu}) \underset{\mu\to\mu^0}{\overset{t\in[t_1,t_2]}{\rightrightarrows}} 0 \ (\boldsymbol{r}_0(t;\boldsymbol{\mu}) \text{ qoldiq tekis } o(\boldsymbol{\mu} - \boldsymbol{\mu}^0) \text{dan iborat}).$$

Demak, $\boldsymbol{\mu}^0$ ga yaqin $\boldsymbol{\mu}$larda $\boldsymbol{\varphi}(t;\boldsymbol{\mu}) \approx \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0)$ desak, bunda qilingan xato $\boldsymbol{r}_0(t;\boldsymbol{\mu}) = o(\boldsymbol{\mu} - \boldsymbol{\mu}^0)$, bo‘ladi, ya’ni

$$\boldsymbol{\varphi}(t;\boldsymbol{\mu}) = \boldsymbol{\varphi}(t;\boldsymbol{\mu}^0) + o(\boldsymbol{\mu} - \boldsymbol{\mu}^0),\ \boldsymbol{\mu} \to \boldsymbol{\mu}^0.$$

Keltirilgan tenglikda yechimlar ayirmasining chegaralangan va tayinlangan $[t_1, t_2]$ segmentda qaralayotganligi (baholanayotganligi) muhimdir. Buni quyidagi misollar asoslaydi.

Misol 1. Ushbu

$$x' = (x + \mu)^2 \ \ (\mu > 0)$$

skalyar tenglamani qaraylik; bunda $\mu -$ kichik musbat parametr. Uning $\mu = 0$ $(\mu_0 = 0)$ bo‘lganda $x(0) = 0$ boshlang‘ich shartni qanoatlantiruvchi yechimi $x = \varphi(t;0) = 0,\ 0 \le t < +\infty$. Lekin qaralayotgan tenglamaning o‘sha $x(0) = 0$ boshlang‘ich shartni qanoatlantiruvchi yechimini, topish qiyin emas. U ushbu

$$x = \varphi(t;\mu) = \frac{\mu}{1 - \mu t} - \mu$$

formula bilan aniqlanadi. Bu yechim $0 \le t < 1/\mu$ $(\mu > 0)$ oralig‘ida aniqlangan va μ parametr nolga intilganda bu oraliq cheksiz kengayadi va $x = \varphi(t;\mu)$ yechim $x = \varphi(t;0)$ yechimga shu oraliqda tekis intilmaydi; aslida

$$\sup_{0\le t<1/\mu} \left|\varphi(t;\mu) - \varphi(t;0)\right| = \sup_{0\le t<1/\mu} \frac{\mu^2 t}{1 - \mu t} = +\infty.$$ ☝

Misol 2. Ushbu

$$\begin{cases} x' = y \\ y' = -\mu y - \omega^2 x \end{cases} \ (\mu \ge 0,\ \omega > 0)$$

sistemani qaraylik. U elastik prujinaga berkitilgan moddiy nuqtaning harakatiga tezlikka proporsional qarshilik kuchi bilan to‘sqinlik qiluvchi muhitdagi harakatini ifodalaydi:

$$x'' = -\mu x' - \omega^2 x\,. \qquad (8.5.2)$$

Qarshilik yo‘qolganda $\mu = 0$ va garmonik ossilyator tenglamasi hosil bo‘ladi:

$$\begin{cases} x' = y \\ y' = -\omega^2 x \end{cases} \quad \text{yoki} \quad x'' = -\omega^2 x;$$

uning umumiy yechimi $x = \varphi(t;0) = A\cos(\omega t + \alpha_0)$ ($A, \alpha_0 - const$).

Qarshilik kichik, aniqrog‘i $0 < \mu < 2\omega$ bo‘lganda qaralayotgan (8.5.2) tenglamaning umumiy yechimi

$$x = \varphi(t;\mu) = \tilde{A}e^{-t\mu/2}\cos(\tilde{\omega}t + \tilde{\alpha}_0),\ \tilde{\omega} = \frac{\sqrt{4\omega^2 - \mu}}{2} \quad (\tilde{A}, \tilde{\alpha}_0 - const)$$

formula bilan beriladi. Ixtiyoriy chegaralangan vaqt oralig‘ida bir xil boshlang‘ich shartli $x = \varphi(t;\mu)$ va $x = \varphi(t;0)$ yechimlar kichik μ larda yaqin bo‘ladi. Lekin cheksiz vaqtlar uchun ular yaqin bo‘lmaydi, chunki $\varphi(t;\mu)\xrightarrow[t\to+\infty]{}0$, $\varphi(t;0)$ esa (o‘zgarmas amplitudali, so‘nmas) garmonik tebranishlarni ifodalaydi. 👍

Yechimning boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligi. Yechimnig boshlang‘ich ma’lumotlarga bog‘liqligini o‘rganish maqsadida Koshi masalasini ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu}) \\ \boldsymbol{x}|_{\tau} = \boldsymbol{\xi} \end{cases} \qquad (8.5.3)$$

ko‘rinishda yozib olaylik, bunda $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu})$ vekor-funksiya yuqorida aytilgan shartlarni qanoatlantiradi deb faraz qilinadi. Bu masalaning yechimi $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau,\boldsymbol{\xi},\boldsymbol{\mu})$ kabi yoziladi. Yangi $s = t - \tau, \mathbf{y} = \mathbf{x} - \boldsymbol{\xi}$ o‘zgaruvchilarga o‘tamiz. Natijada ushbu

$$\begin{cases} \dfrac{d\boldsymbol{y}}{ds} = \boldsymbol{f}(s + \tau, \boldsymbol{y} + \boldsymbol{\xi}, \boldsymbol{\mu}) \\ \boldsymbol{y}|_0 = 0 \end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi $(\tau, \boldsymbol{\xi}, \boldsymbol{\mu})$ o‘zgaruvchilar parametrlar rolini o‘ynaydi:

$$\begin{cases} \dfrac{d\boldsymbol{y}}{ds} = \boldsymbol{g}(s, \boldsymbol{y}, \tau, \boldsymbol{\xi}, \boldsymbol{\mu}) \\ \boldsymbol{y}|_0 = 0 \end{cases} \quad (\text{ bunda } \boldsymbol{g}(s, \boldsymbol{y}, \tau, \boldsymbol{\xi}, \boldsymbol{\mu}) \equiv \boldsymbol{f}(s + \tau, \boldsymbol{y} + \boldsymbol{\xi}, \boldsymbol{\mu})) \quad (8.5.4)$$

Yechimni $\boldsymbol{y} = \boldsymbol{\psi}(s, \tau, \boldsymbol{\xi}, \boldsymbol{\mu})$ bilan belgilaymiz. Bunda ravshanki, eski $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau,\boldsymbol{\xi},\boldsymbol{\mu})$ va yangi $\boldsymbol{y} = \boldsymbol{\psi}(s, \tau, \boldsymbol{\xi}, \boldsymbol{\mu})$ yechimlar orasida

$\varphi(t;\tau,\xi,\mu)=\xi+\psi(t-\tau,\tau,\xi,\mu)$ bog‘lanish o‘rinli. Isbotlangan teoremani (8.5.4) masalaga, ya’ni $y=\psi(s,\tau,\xi,\mu)$ yechimga qo‘llab, oxirgi munosabatga ko‘ra quyidagi teoremani hosil qilamiz.

***Teorema* 2 (*yechimning boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligi*).** Faraz qilaylik, $f\in C(D\times M,\mathbb{R}^n)$ va bu $f(t,x,\mu)$ funksiya $D\times M$ sohada x vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin hamda $\mu=\mu^0$ bo‘lganda (8.5.1) masala $t\in[t_1,t_2]$ ($t_0\in[t_1,t_2],(t_0,x^0,\mu^0)\in D\times M$) segmentda aniqlangan $x=\varphi(t;t_0,x^0,\mu^0)$ yechimga ega bo‘lsin. U holda shunday yetarlicha kichik $\delta>0$ soni mavjudki, ushbu $|\tau-t_0|<\delta$, $\|\xi-x^0\|<\delta$ va $\|\mu-\mu^0\|<\delta$ shartlar bajarilganda $x=\varphi(t;\tau,\xi,\mu)$ yechim barcha $t\in[t_1,t_2]$ larda aniqlangan va $(t;\tau,\xi,\mu)$ (t yechimning argumenti, (τ,ξ) boshlang‘ich ma’lumotlar, μ parametrlar) o‘zgaruvchilar bo‘yicha uzluksiz vektor-funksiyadan iborat bo‘ladi, ya’ni

$$\varphi(t;\tau,\xi,\mu)\in C\big([t_1,t_2]\times(t_0-\delta,t_0+\delta)\times B_\delta(x^0)\times B_\delta(\mu^0)\big).$$

Mustaqil isbotlash uchun

Teorema (yechimning boshlang‘ich qiymat bo‘yicha Lipshits shartini qanoatlantirishi). Aytaylik, $f(t,x)$ vektor-funksiya $(t,x)\in D\ (D\subset\mathbb{R}^{1+n})$ sohada uzluksiz va x bo‘yicha Lipshits shartini qanoatlantirsin hamda $(t_0,\xi^0)\in D$ uchun ushbu

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=\xi^0\end{cases}$$

boshlang‘ich masalaning $x=\varphi(t;\xi^0)$ yechimi $t\in[t_1;t_2]$ oraliqda aniqlangan bo‘lsin. U holda shunday $\delta>0$ soni mavjudki, $\|\xi-\xi^0\|<\delta$ bo‘lganda

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=\xi\end{cases}$$

masalaning $x=\varphi(t;\xi)$ yechimi ham $t\in[t_1;t_2]$ oraliqda aniqlangan va ξ boshlang‘ich qiymat bo‘yicha Lipshits shartini qanoatlantiradi, ya’ni shunday $L>0$ soni mavjudki, $t\in[t_1;t_2]$, $\|\xi'-\xi^0\|<\delta$, $\|\xi''-\xi^0\|<\delta$ ekanligidan $\|\varphi(t;\xi')-\varphi(t;\xi'')\|\le L\|\xi'-\xi''\|$ tengsizlik kelib chiqadi.

MODUL 9. NORMAL KO‘RINISHDAGI CHIZIQLI DIFFERENSIAL TENGLAMALAR SISTEMASI

Biz bu bobda berilgan chiziqli sistemadagi berilgan koeffitsientlar va ozod hadlarni biror oraliqda uzluksiz deb faraz qilamiz. Vektorlarni ustun ko‘rinishda yozamiz.

§ 9.1. Chiziqli differensial tenglamalar normal sistemasining umumiy xossalari

1. Biz bu paragrafda n- tartibli chiziqli normal sistema yechimlarining umumiy xossalarini o‘rganamiz. Qulaylik uchun bu sistemani vektorli ko‘rinishda yozamiz:

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t), \tag{9.1.1}$$

bu yerda

$$A(t) = \left\| a_{ij}(t) \right\| \in C\left(I; \mathbb{M}_{n\times n}(\mathbb{R})\right),\ \boldsymbol{g}(t) = \left(g_1(t), g_2(t), \ldots, g_n(t)\right)^T \in C(I, \mathbb{R}^n),$$

$$\boldsymbol{x} = \left(x_1, x_2, \ldots, x_n\right)^T,\ \boldsymbol{x} = \boldsymbol{x}(t),\ t \in I,$$

deb hisoblanadi. Bu shartlarda global yechimning mavjudligi va yagonaligi haqidagi teoremaga ko‘ra ushbu

$$\begin{cases} \boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t) \\ \boldsymbol{x}\big|_{t_0} = \boldsymbol{x}^0 \quad (t_0 \in I, \boldsymbol{x}^0 \in \mathbb{R}^n) \end{cases}$$

Koshi masalasi birato‘la I oraliqda aniqlangan yagona yechimga ega.

Jumla 1 (Superpozitsiya prinsipi). *Agar*

$\boldsymbol{x} = \boldsymbol{x}^1(t)$ *vektor-funksiya* $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}^1(t)$ *sistemaning,*

$\boldsymbol{x} = \boldsymbol{x}^2(t)$ *vektor-funksiya esa* $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}^2(t)$ *sistemaning yechimlari bo‘lsa,* $\boldsymbol{x} = \lambda_1\boldsymbol{x}^1(t) + \lambda_2\boldsymbol{x}^2(t)$ *(*$\lambda_1, \lambda_2 - \text{const}$*) vektor-funksiya ushbu* $\boldsymbol{x}' = A(t)\boldsymbol{x} + \lambda_1\boldsymbol{g}^1(t) + \lambda_2\boldsymbol{g}^2(t)$ *sistemaning yechimi bo‘ladi.*

🗝 Isboti oson. Berilganga ko‘ra

$$\frac{d\boldsymbol{x}^1}{dt} = A(t)\boldsymbol{x}^1 + \boldsymbol{g}^1(t) \text{ va } \frac{d\boldsymbol{x}^2}{dt} = A(t)\boldsymbol{x}^2 + \boldsymbol{g}^2(t).$$

Bu ayniyatlarning birinchisini λ_1 ga, ikkinchisini esa λ_2 ga ko‘paytirib hadma-had qo‘shsak, jumla isbot bo‘ladi:

$$\frac{d(\lambda_1\boldsymbol{x}^1 + \lambda_2\boldsymbol{x}^2)}{dt} = A(t)(\lambda_1\boldsymbol{x}^1 + \lambda_2\boldsymbol{x}^2) + \lambda_1\boldsymbol{g}^1(t) + \lambda_2\boldsymbol{g}^2(t).$$ 👍

(9.1.1) ga mos bir jinsli chiziqli sistema deb

$$\boldsymbol{x}' = A(t)\boldsymbol{x} \qquad (9.1.2)$$

differensial tenglamalar sistemasiga aytiladi.

Jumla 2. (9.1.1) *sistemaning barcha yechimlari (umumiy yechimi) uning biror tayin (xususiy) yechimiga mos bir jinsli sistema* (9.1.2) *ning barcha yechimlarini (umumiy yechimini) qo'shishdan hosil bo'ladi.*

🗝 $\boldsymbol{x} = \boldsymbol{x}_{xus}(t)$ vektor-funksiya (9.1.1) sistemaning biror tayin yechimi bo'lsin, $\frac{d\boldsymbol{x}_{xus}}{dt} = A(t)\boldsymbol{x}_{xus} + \boldsymbol{g}(t)$. Uning ixtiyoriy $\boldsymbol{x} = \boldsymbol{\psi}(t)$ yechimini olaylik, $\frac{d\boldsymbol{\psi}}{dt} = A(t)\boldsymbol{\psi} + \boldsymbol{g}(t)$. Superpozitsiya prinsipidan ravshanki, $\boldsymbol{x} = \boldsymbol{x}_{b.j.}(t) \overset{def}{=} \boldsymbol{\psi}(t) - \boldsymbol{x}_{xus}(t)$ vektor-funksiya (9.1.2) bir jinsli sistemaning yechimi: $\frac{d\boldsymbol{x}_{b.j.}}{dt} = A(t)\boldsymbol{x}_{b.j.}$. Demak, $\boldsymbol{\psi}(t) = \boldsymbol{x}_{xus}(t) + \boldsymbol{x}_{b.j.}(t)$, ya'ni (9.1.1) sistemaning ixtiyoriy $\boldsymbol{x} = \boldsymbol{\psi}(t)$ yechimi uning $\boldsymbol{x} = \boldsymbol{x}_{xus}(t)$ xususiy yechimiga mos bir jinsli sistema (9.1.2) ning $\boldsymbol{x} = \boldsymbol{x}_{b.j.}(t)$ yechimini qo'shishdan hosil bo'lgan. Ikkinchi tomondan, yana superpozitsiya prinsipidan ravshanki, (9.1.1) sistemaning yechimiga mos bir jinsli sistema (9.1.2) ning yechimini qo'shib, yana (9.1.1) sistemaning yechimini hosil qilamiz. 👍

2. Yechimning yagonalik xossasidan kelib chiquvchi quyidagi natijani alohida e'tirof etaylik:

agar $\boldsymbol{x} = \boldsymbol{x}(t)$ vektor-funksiya (9.1.2) bir jinsli sistemaning I oraliqda yechimi va biror $t_0 \in I$ nuqtada $\boldsymbol{x}(t_0) = 0$ bo'lsa, u holda I oraliqda yechim aynan nolga teng, ya'ni $\boldsymbol{x}(t) \equiv 0$, bo'ladi.

n- tartibli bir jinsli sistema (9.1.2) ning barcha yechimlari to'plamini V_n bilan belgilaylik:

$$V_n = \left\{ \boldsymbol{x}(t) \in C^1(I;\mathbb{R}^n) \,\middle|\, \boldsymbol{x}'(t) \equiv A(t)\boldsymbol{x}(t),\, t \in I \right\}.$$

Bu V_n to'plamda ikki elementni (vektor-funksiyani) qo'shish va elementni songa ko'paytirish amallari odatdagidek, ya'ni nuqtama-nuqta aniqlanadi. Agar V_n to'plam bu amallarga nisbatan yopiq bo'lsa, tushunarliki, u chiziqli (vektor) fazoni tashkil etadi. Bu holda u $C^1(I;\mathbb{R}^n)$ chiziqli (vektor) fazoning qismfazosi ham bo'ladi.

Teorema. *Bir jinsli sistema* (9.1.2) *ning ixtiyoriy ikki* $\boldsymbol{x} = \boldsymbol{x}^1(t)$ *va* $\boldsymbol{x} = \boldsymbol{x}^2(t)$ *yechimining ixtiyoriy* $\boldsymbol{x} = \lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t)$ $(\lambda_1, \lambda_2 - const)$ *chiziqli kombinatsiyasi ham shu sistemaning yechimidir. Demak,* V_n *chiziqli fazo.*

🗝 Berilganga ko‘ra $\dfrac{d\boldsymbol{x}^1}{dt} = A(t)\boldsymbol{x}^1$ va $\dfrac{d\boldsymbol{x}^2}{dt} = A(t)\boldsymbol{x}^2$. Superpozitsiya prinsipiga ko‘ra har qanday $\lambda_1$va λ_2 sonlar uchun $\boldsymbol{x} = \lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t)$ vektor-funksiya ham (9.1.2) ning yechimi. Demak,

$$\{\boldsymbol{x}^1, \boldsymbol{x}^2\} \subset V_n \Rightarrow \lambda_1 \boldsymbol{x}^1 + \lambda_2 \boldsymbol{x}^2 \in V_n .$$ 👍

V_n fazoda nol-vektor $\boldsymbol{x}(t) \equiv 0$ trivial yechimdan iborat. $\boldsymbol{x}(t) \in V_n$ vektorning qarama-qarshisi $(-1)\boldsymbol{x}(t) = -\boldsymbol{x}(t) \in V_n$ vektordir.

Masalalar

1. $V_n = \left\{ \boldsymbol{x}(t) \in C^1(I;\mathbb{R}^n) \mid \boldsymbol{x}'(t) \equiv A(t)\boldsymbol{x}(t),\, t \in I \right\} \left(A(t) \in C\left(I;\mathbb{M}_{n\times n}(\mathbb{R})\right)\right)$ ning chiziqli fazo ekanligini bevosita va qat’iy isbotlang.

2. $V_n$chiziqli fazoni $\mathbb{R}^n$chiziqli fazoga akslantirishni quyidagicha kiritamiz.

$t_0 \in I$ nuqtani tayinlab, har bir $\boldsymbol{x}(t) \in V_n = \left\{ \boldsymbol{x}(t) \in C^1(I;\mathbb{R}^n) \mid \boldsymbol{x}'(t) \equiv A(t)\boldsymbol{x}(t),\, t \in I \right\}$ yechimga uning $\boldsymbol{x}(t_0) \in \mathbb{R}^n$ qiymatini mos qo‘yaylik: $V_n \to \mathbb{R}^n,\ \boldsymbol{x}(t) \to \boldsymbol{x}(t_0)$.

Bu akslantirishning chiziqli fazolar izomorfizmi ekanligini (in’yektivligini, syuryektivligini va chiziqli amallarni saqlashini) ko‘rsating. Izomorf chiziqli fazolarning o‘lchamlari teng bo‘lishidan $\dim V_n = \dim \mathbb{R}^n = n$ ekanligini asoslang.

§ 9.2. Chiziqli erkli va chiziqli bog‘langan vektor-funksiyalar. Vronskian

Qiymatlari tayin $\mathbb{R}^n$ fazoga tegishli bo‘lgan $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$, $t \in I$, vektor-funksiyalar berilgan bo‘lsin. Ularning chiziqli kombinatsiyasi deb ushbu

$$\lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t) + \cdots + \lambda_m \boldsymbol{x}^m(t)$$

ifodaga aytiladi; bu yerda $\lambda_1, \lambda_2, \ldots, \lambda_m -$ sonlar, ular chiziqli kombinatsiyaning koeffitsientlari deb ataladi. Koeffitsientlar $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ bo‘lganda trivial chiziqli kombinatsiya hosil bo‘ladi. Ravshanki, trivial chiziqli kombinatsiya nol-vektordan iborat.

Agar $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$, $t \in I$, vektor-funksiyalarning biror notrivial chiziqli kombinatsiyasi I oraliqda nol-vektorga teng, ya'ni kamida bittasi noldan farqli bo'lgan $\lambda_1, \lambda_2, \ldots, \lambda_m$ ($|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \neq 0$) sonlar mavjud bo'lib, ular uchun

$$\lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t) + \cdots + \lambda_m \boldsymbol{x}^m(t) \equiv \boldsymbol{0},\ t \in I,$$

ayniyat o'rinli bo'lsa, u holda bu $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$ vektor-funksiyalar I oraliqda **chiziqli bog'langan** deyiladi. Aks holda, ya'ni berilgan vektor-funksiyalarning faqat trivial chiziqli kombinatsiyasigina nol-vektordan iborat bo'lsa, ular **chiziqli erkli** (chiziqli bog'lanmagan) vektor-funksiyalar deb ataladi. Demak, $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$ funksiyalarning I oraliqda chiziqli erkliligi ushbu

$$\lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t) + \cdots + \lambda_m \boldsymbol{x}^m(t) \equiv \boldsymbol{0}\ (t \in I) \Rightarrow$$

$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$$

implikatsiyaning rostligini anglatadi.

Agar biror $\lambda_1, \lambda_2, \ldots, \lambda_m$ sonlar uchun

$$\boldsymbol{x}(t) = \lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t) + \cdots + \lambda_m \boldsymbol{x}^m(t),\ t \in I,$$

tenglik o'rinli bo'lsa, $\boldsymbol{x}(t)$ vektor-funksiya $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$ lar orqali (I da) chiziqli ifodalangan deyiladi.

Ravshanki, $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$ vektor-funksiyalar I oraliqda chiziqli bog'langan bo'lishi uchun ularning birortasi qolganlari orqali I da chiziqli ifodalanishi yetarli va zarurdir.

$n \times 1$ o'lchamli n dona

$$\boldsymbol{x}^1(t) = \begin{pmatrix} x_1^1(t) \\ x_2^1(t) \\ \vdots \\ x_n^1(t) \end{pmatrix}, \boldsymbol{x}^2(t) = \begin{pmatrix} x_1^2(t) \\ x_2^2(t) \\ \vdots \\ x_n^2(t) \end{pmatrix}, \ldots, \boldsymbol{x}^n(t) = \begin{pmatrix} x_1^n(t) \\ x_2^n(t) \\ \vdots \\ x_n^n(t) \end{pmatrix}$$

vektor-funksiyalarning **vronskiani (Vronskiy determinanti)** deb ushbu

$$W[\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n] = \begin{vmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix} \qquad (9.1.3)$$

determinantga aytiladi.

Teorema 2. *Agar* $n\times 1$ *o'lchamli* $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\dots,\boldsymbol{x}^n(t)$ *vektor-funksiyalar* I *oraliqda chiziqli bog'langan bo'lsa, ularning vronskiani* I *da aynan nolga teng.*

🗝 Berilganga ko'ra kamida bittasi noldan farqli $\lambda_1,\lambda_2,\dots,\lambda_n$ sonlar ($|\lambda_1|+|\lambda_2|+\dots+|\lambda_n|\neq 0$) uchun

$$\lambda_1\begin{pmatrix}x_1^1(t)\\x_2^1(t)\\\vdots\\x_n^1(t)\end{pmatrix}+\lambda_2\begin{pmatrix}x_1^2(t)\\x_2^2(t)\\\vdots\\x_n^2(t)\end{pmatrix}+\dots+\lambda_n\begin{pmatrix}x_1^n(t)\\x_2^n(t)\\\vdots\\x_n^n(t)\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\ t\in I,$$

ayniyat o'rinli. Bu tenglik $W[\boldsymbol{x}^1,\boldsymbol{x}^2,\dots,\boldsymbol{x}^n]$ vronskian ustunlari orasida ixtiyoriy $t\in I$ nuqtada chiziqli bog'lanish mavjudligini anglatadi. Algebradan ma'lum teoremaga ko'ra, bu determinant ixtiyoriy $t\in I$ nuqtada nolga teng. 👍

Natija. *Agar* $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\dots,\boldsymbol{x}^n(t)$ *vektor-funksiyalarning vronskiani* I *oraliqning biror nuqtasida noldan farqli bo'lsa, bu vektor-funksiyalar* I *da chiziqli erkli.*

🗝 Haqiqatan ham, agar berilgan funksiyalar chiziqli bog'liq bo'lganda edi, u holda isbotlangan teoremaga ko'ra vronskian aynan nolga teng bo'lardi; bu esa berilganga zid. 👍

Umumiy holda vronskianning nolga tengligidan mos vektor-funksiyalarning chiziqli bog'langanligi kelib chiqmaydi. Lekin (9.1.2) sistemaning yechimi bo'lgan funksiyalar uchun – kelib chiqadi. Bu quyidagi teoremada keltirilgan.

Teorema 3. n- *tartibli bir jinsli sistema* (9.1.2) *ning* n *dona* $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\dots,\boldsymbol{x}^n(t),\ t\in I$, *yechimlari berilgan va ularning vronskiani* $W(t)$ *bo'lsin. Quyidagi alternativa o'rinli:*

yo $W(t)$ *biror nuqtada ham nolga aylanmaydi va bu holda yechimlar chiziqli erkli,*

yoki $W(t)$ *aynan nolga teng va bu holda yechimlar chiziqli bog'langan.*

🗝 $W(t)$ biror nuqtada ham nolga aylanmasin. U holda yuqoridagi natijaga ko'ra berilgan $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\dots,\boldsymbol{x}^n(t),\ t\in I$, yechimlar chiziqli erkli.

Endi faraz qilaylik, $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\dots,\boldsymbol{x}^n(t),\ t\in I$, yechimlarning $W(t)$ vronskiani biror $t=t_0\in I$ nuqtada nolga teng bo'lsin. Demak, ushbu

$$\begin{cases} \lambda_1 x_1^1(t_0)+\lambda_2 x_1^2(t_0)+\cdots+\lambda_n x_1^n(t_0)=0 \\ \lambda_1 x_2^1(t_0)+\lambda_2 x_2^2(t_0)+\cdots+\lambda_n x_2^n(t_0)=0 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \lambda_1 x_n^1(t_0)+\lambda_2 x_n^2(t_0)+\cdots+\lambda_n x_n^n(t_0)=0 \end{cases}$$

chiziqli bir jinsli algebraik sistema biror notrivial $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(|\lambda_1|+|\lambda_2|+\cdots+|\lambda_n|\neq 0)$ yechimga ega. Ana shu notrivial yechimga ko‘ra $\boldsymbol{x}(t)=\lambda_1\boldsymbol{x}^1(t)+\lambda_2\boldsymbol{x}^2(t)+\cdots+\lambda_n\boldsymbol{x}^n(t)$ vektor-funksiyani tuzaylik. Yechimlarning chiziqli kombinatsiyasi sifatida $\boldsymbol{x}(t)$ ham (9.1.2) bir jinsli sistemaning yechimi. $\lambda_1, \lambda_2, \ldots, \lambda_n$ larning tanlanishiga ko‘ra $\boldsymbol{x}(t_0)=0$. Yechimning yagonalik xossasiga ko‘ra $\boldsymbol{x}(t)\equiv\boldsymbol{0}$, ya’ni $\lambda_1\boldsymbol{x}^1(t)+\lambda_2\boldsymbol{x}^2(t)+\cdots+\lambda_n\boldsymbol{x}^n(t)\equiv\boldsymbol{0},\ t\in I$. Bu ayniyat $|\lambda_1|+|\lambda_2|+\cdots+|\lambda_n|\neq 0$ bo‘lgani uchun $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^n(t)$, $t\in I$, yechimlarning chiziqli bog‘langanligini isbotlaydi. Demak, ularning vronskiani aynan nolga teng. ✍

Shunday qilib, yechimlarning chiziqli erkli yoki chiziqli bog‘langan ekanligini vronskian to‘laligicha hal qiladi.

Masalalar

1. I oraliqda aniqlangan vektor-funksiyalar berilgan bo‘lsin. Agar bu funksiyalar

a) biror $I^*\subset I$ oraliqda chiziqli erkli bo‘lsa, ular I oraliqda ham chiziqli erkli bo‘ladi;

b) I oraliqda chiziqli bog‘langan bo‘lsa, ular ixtiyoriy $\tilde{I}\subset I$ oraliqda ham chiziqli bog‘langan bo‘ladi.

Shu tasdiqlarni isbotlang.

2. Ushbu

$$H(t)=\begin{cases} 0, \text{ agar } t<0 \text{ bo`lsa} \\ 1, \text{ agar } t\geq 0 \text{ bo`lsa} \end{cases}$$

funksiya yordamida aniqlangan

$$\boldsymbol{x}^1(t)=\begin{pmatrix} H(t) \\ H(t) \end{pmatrix},\ \boldsymbol{x}^2(t)=\begin{pmatrix} 1-H(t) \\ 1-H(t) \end{pmatrix}$$

vektor-funksiyalarni chiziqli erklilikka tekshiring.

§ 9.3. Fundamental matritsa. Chiziqli bir jinsli normal sistema umumiy yechimining tuzilishi

n- tartibli (9.1.2) bir jinsli normal sistemaning n dona chiziqli erkli yechimlari fundamental (bazis) yechimlar yoki yechimlarning fundamental sistemasi (fundamental sistema) deb ataladi. $\boldsymbol{x}=\boldsymbol{\varphi}^1(t), \boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots, \boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ fundamental (bazis) yechimlarning koordinatalarini ustunlar bo‘ylab yozishdan hosil bo‘lgan ushbu

$$\Phi(t)=[\boldsymbol{\varphi}^1(t)\vdots\boldsymbol{\varphi}^2(t)\vdots\ldots\vdots\boldsymbol{\varphi}^n(t)]=\begin{pmatrix} \varphi_1^1(t) & \varphi_1^2(t) & \cdots & \varphi_1^n(t) \\ \varphi_2^1(t) & \varphi_2^2(t) & \cdots & \varphi_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n^1(t) & \varphi_n^2(t) & \cdots & \varphi_n^n(t) \end{pmatrix} \qquad (9.3.1)$$

matritsa fundamental matritsa deb ataladi (bu yerda va bundan keyin ushbu $\vdots$ belgi bo‘sh joyni anglatadi). Ravshanki, fundamental matritsaning determinanti mos yechimlarning vronskianidan iborat. $\det\Phi(t)=W(t)\neq 0$ bo‘lgani uchun fundamental matritsa teskarilanuvchi, ya’ni $\Phi^{-1}(t)$ teskari matritsa mavjud: $\Phi^{-1}(t)\Phi(t)=\Phi(t)\Phi^{-1}(t)=E$, $E-n\times n$ o‘lchamli birlik matritsa.

Agar $\boldsymbol{x}=\boldsymbol{\varphi}^1(t), \boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots, \boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ ($t\in I$) yechimlardan tuzilgan $\Phi(t)=[\boldsymbol{\varphi}^1(t)\vdots\boldsymbol{\varphi}^2(t)\vdots\ldots\vdots\boldsymbol{\varphi}^n(t)]$ matritsa biror $t=t_0$ nuqtada teskarilanuvchi, ya’ni $\det\Phi(t_0)\neq 0$ bo‘lsa, u holda barcha $t\in I$ nuqtalarda ham $\det\Phi(t)=W(t)\neq 0$ va $\boldsymbol{x}=\boldsymbol{\varphi}^1(t), \boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots, \boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ yechimlar chiziqli erkli, berilgan $\Phi(t)$ matritsa esa fundamental matritsdan iborat bo‘ladi.

Bazis (fundamental) yechimlarning (fundamental matritsaning) mavjudligini va umumiy yechimning ko‘rinishini quyidagi teorema ifodalaydi.

Теорема. (9.1.2) *bir jinsli sistema bazis yechimlarga ega va uning umumiy yechimi biror fundamental sistemasining ixtiyoriy chiziqli kombinatsiyasi sifatida ifodalanadi, ya’ni yechimlar fazosining o‘lchami sistemaning tartibiga teng:* $\dim V_n = n$.

🗝 $\mathbb{R}^n$ fazoning standart bazisini odatdagidek $\boldsymbol{e}^1=(1,0,0,\ldots,0)^T$, $\boldsymbol{e}^2=(0,1,0,\ldots,0)^T,\ldots,\boldsymbol{e}^n=(0,0,\ldots,0,1)^T$ bilan belgilab, ushbu

$$\begin{cases} \boldsymbol{x}'=A(t)\boldsymbol{x} \\ \boldsymbol{x}(t_0)=\boldsymbol{e}^j \end{cases}, \quad j=\overline{1,n},$$

n dona Koshi masalasini qaraylik. Bu masalalarning har biri I oraliqda aniqlangan yagona yechimga ega. Yechimlarni mos ravishda

$\boldsymbol{x}=\boldsymbol{\varphi}^1(t), \boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots,\boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ bilan belgilaylik. Bu yechimlar chiziqli erkli, chunki ularning vronskiani t_0 nuqtada noldan farqli (birga teng). Shunday qilib, topilgan yechimlar (9.1.2) sistemaning bazis yechimlaridir.

Endi faraz qilaylik, (9.1.2) sistemaning $\boldsymbol{x}=\boldsymbol{\varphi}^1(t)$, $\boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots,\boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ fundamental (bazis) yechimlari berilgan bo‘lsin; bu yechimlar yuqorida qurilgan yechimlardan iborat bo‘lishi shart emas. (9.1.2) sistemaning umumiy yechimi $\boldsymbol{x}=c_1\boldsymbol{\varphi}^1(t)+c_2\boldsymbol{\varphi}^2(t)+\cdots+c_n\boldsymbol{\varphi}^n(t)$ formula bilan ifodalanishini ko‘rsatishimiz kerak; bunda $c_1,c_2,\ldots,c_n$ – ixtiyoriy o‘zgarmaslar. Birinchidan, bu formula o‘zgarmaslarning ixtiyoriy qiymatida (9.1.2) sistemaning yechimi (yechimlarning chiziqli kombinatsiyasi sifatida). Ikkinchidan, (9.1.2) ning har qanday yechimi shu ko‘rinishda ekanligini ko‘rsatish kerak. (9.1.2) ning ixtiyoriy $\boldsymbol{x}=\boldsymbol{x}(t)$ yechimi berilgan bo‘lsin. Biror $t_0\in I$ nuqtani tayinlab, $\boldsymbol{x}(t_0)\in\mathbb{R}^n$ vektorni $\boldsymbol{\varphi}^1(t_0),\boldsymbol{\varphi}^2(t_0),\ldots,\boldsymbol{\varphi}^n(t_0)$ vektorlarning chiziqli kombinatsiyasi ko‘rinishida ifodalaylik:

$$\boldsymbol{x}(t_0)=c_1\boldsymbol{\varphi}^1(t_0)+c_2\boldsymbol{\varphi}^2(t_0)+\cdots+c_n\boldsymbol{\varphi}^n(t_0).$$

Bu yerdagi $c_1,c_2,\ldots,c_n$ sonlar bir qiymatli aniqlanadi, chunki $\boldsymbol{x}=\boldsymbol{\varphi}^1(t), \boldsymbol{x}=\boldsymbol{\varphi}^2(t),\ldots,\boldsymbol{x}=\boldsymbol{\varphi}^n(t)$ chiziqli erkli yechimlarning vronskiani noldan farqli:

$$\boldsymbol{x}(t_0)=\Phi(t_0)\boldsymbol{c},\ \boldsymbol{c}=(c_1,c_2,...,c_n)^T \Rightarrow \boldsymbol{c}=\Phi^{-1}(t_0)\boldsymbol{x}(t_0).$$

Shu $c_1,c_2,\ldots,c_n$ sonlarga ko‘ra $\tilde{\boldsymbol{x}}(t)=c_1\boldsymbol{\varphi}^1(t)+c_2\boldsymbol{\varphi}^2(t)+\cdots+c_n\boldsymbol{\varphi}^n(t)$ funksiyani tuzaylik. U (9.1.2) ning yechimi (yechimlarning chiziqli kombinatsiyasi sifatida) va $\tilde{\boldsymbol{x}}(t_0)=\boldsymbol{x}(t_0)$ ($c_1,c_2,\ldots,c_n$ larning tanlanishiga ko‘ra). Shuning uchun yechimning yagonalik xosasidan $\tilde{\boldsymbol{x}}(t)\equiv\boldsymbol{x}(t)$, ya’ni $\boldsymbol{x}(t)=c_1\boldsymbol{\varphi}^1(t)+c_2\boldsymbol{\varphi}^2(t)+\cdots+c_n\boldsymbol{\varphi}^n(t)$ kelib chiqadi. Shunday qilib, (9.1.2) ning ixtiyoriy $\boldsymbol{x}=\boldsymbol{x}(t)$ yechimi berilgan bazis yechimlarning chiziqli kombinatsiyasi ko‘rinishida bir qiymatli ifodalandi:

$$\boldsymbol{x}=\Phi(t)\boldsymbol{c}, \tag{9.3.2}$$

bu yerda $\Phi(t)=[\boldsymbol{\varphi}^1(t)\vdots\boldsymbol{\varphi}^2(t)\vdots\ldots\vdots\boldsymbol{\varphi}^n(t)]$, $\boldsymbol{c}=(c_1,c_2,\ldots,c_n)^T\in\mathbb{R}^n$. ☝

Demak, (9.1.2) sistemaning barcha yechimlarini (umumiy yechimini) topish uchun uning n dona chiziqli erkli yechimlarini, ya’ni fundamental matritsasini topish yetarli.

Bu yerda shuni e’tirof etish kerakki, umumiy holda bazis yechimlarni qurish algoritmi (usuli) mavjud emas. Biz ularning mavjudligini isbotladik xolos.

Masalalar

1. Berilgan o‘lchamlari moslangan $\underset{n\times k}{B},\ \underset{k\times m_1}{C_1},\ \underset{k\times m_2}{C_2},\ \dots,\ \underset{k\times m_l}{C_l}$ matritsalar uchun (matritsa tagida uning o‘lchami ko‘rsatilgan), quyidagi tenglikni tekshiring:

$$B[C_1 \vdots C_2 \vdots \dots \vdots C_l] = [BC_1 \vdots BC_1 \vdots \dots \vdots BC_l].$$

2. $\{\boldsymbol{b}^j\}_{j=1}^n - \mathbb{R}^n$ fazoning bazisi bo‘lsin. Ushbu

$$\begin{cases} \boldsymbol{x}' = A(t)\boldsymbol{x} \\ \boldsymbol{x}(t_0) = \boldsymbol{b}^j \end{cases},\quad j = \overline{1,n}\ ,$$

n dona Koshi masalalarining yechimlari $\boldsymbol{x} = \boldsymbol{\psi}^j(t)$dan tuzilgan $\{\boldsymbol{\psi}^j(t)\}_{j=1}^n$ funksiyalar $\boldsymbol{x}' = A(t)\boldsymbol{x}$ sistemaning bazis yechimlarini tashkil etadi. Shu tasdiqni isbotlang.

§ 9.4. Fundamental matritsa xossalari. Liuvill formulasi

Jumla 1. *Fundamental matritsa* $\Phi = \Phi(t)$ *ushbu*

$$\Phi' = A(t)\Phi \tag{9.4.1}$$

matritsali differensial tenglamani qanoatlantiradi.

🗝 Isboti matritsalarni ko‘paytirishning xossalaridan osongina kelib chiqadi:

$$\begin{aligned}\Phi'(t) &= [(\boldsymbol{\varphi}^1)'(t) \vdots (\boldsymbol{\varphi}^2)'(t) \vdots \dots \vdots (\boldsymbol{\varphi}^n)'(t)] = \\ &= [A(t)\boldsymbol{\varphi}^1(t) \vdots A(t)\boldsymbol{\varphi}^2(t) \vdots \dots \vdots A(t)\boldsymbol{\varphi}^n(t)] = \\ &= A(t)[\boldsymbol{\varphi}^1(t) \vdots \boldsymbol{\varphi}^2(t) \vdots \dots \vdots \boldsymbol{\varphi}^n(t)] = \\ &= A(t)\Phi(t).\end{aligned}$$ 🖒

Jumla 2. *Ushu*

$$\Phi' = A(t)\Phi,\ \Phi(t_0) = \Phi_0, \det\Phi_0 \neq 0$$

matritsali Koshi masalasining yechimi (9.1.2) *sistemaning fundamental matritsasidir*.

🗝 $\Phi = \Phi(t) = [\boldsymbol{\varphi}^1(t) \vdots \boldsymbol{\varphi}^2(t) \vdots \dots \vdots \boldsymbol{\varphi}^n(t)]$ matritsa $\Phi' = A(t)\Phi$ tenglamani qanoatlantirgani uchun uning ustunlari (9.1.2) sistemaninig yechimlaridan iborat bo‘ladi. $\det\Phi(t_0) \neq 0$ bo‘lgani uchun esa $\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), \dots, \boldsymbol{\varphi}^n(t)$ lar chiziqli erkli yechimlarni tashkil etadi. 🖒

Natija. *Agar* $\det C \neq 0$ *bo'lsa,* $\Phi(t)$ *bilan birgalikda* $\Phi(t)C$ *ham fundamental matritsadir*.

Teorema. (9.1.2) *bir jinsli sistemaning ixtiyoriy ikki fundamental matritsasidan biri ikkinchisini biror teskarilanuvchi o'zgarmas matritsaga o'ngdan ko'paytirishdan hosil bo'ladi.*

🗝 $\Phi = \Phi(t)$ va $\tilde{\Phi} = \tilde{\Phi}(t)$ fundamental matritsalar berilgan bo'lsin:

$$\Phi' = A(t)\Phi,\ \det\Phi(t) \neq 0 \text{ va } \tilde{\Phi}' = A(t)\tilde{\Phi},\ \det\tilde{\Phi}(t) \neq 0. \quad (9.4.2)$$

Ravshanki, $\Phi(t), \tilde{\Phi}(t) \in C^1$. Biz biror teskarilanuvchi o'zgarmas C matritsa uchun $\tilde{\Phi}(t) = \Phi(t)C$ bo'lishini ko'rsatishimiz kerak.

Dastlab $\Phi^{-1}(t)$ teskari matritsaning hosilasini hisoblaylik. Ravshanki,

$$\frac{d\Phi^{-1}(t)}{dt} = \lim_{h\to 0}\frac{\Phi^{-1}(t+h) - \Phi^{-1}(t)}{h} = \lim_{h\to 0}\Phi^{-1}(t+h)\frac{\Phi(t) - \Phi(t+h)}{h}\Phi^{-1}(t) =$$

$$= -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t).$$

Demak, teskari matritsa hosilasi uchun quyidagi formula o'rinli:

$$\frac{d\Phi^{-1}(t)}{dt} = -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t). \quad (9.4.3)$$

Endi $\Phi^{-1}(t)\tilde{\Phi}(t)$ matritsaning hosilasi nol-matritsadan iborat ekanligini (9.4.3) va (9.4.2) formulalardan foydalanib, ko'rsatamiz:

$$\frac{d\left(\Phi^{-1}(t)\tilde{\Phi}(t)\right)}{dt} = \frac{d\Phi^{-1}(t)}{dt}\tilde{\Phi}(t) + \Phi^{-1}(t)\frac{d\tilde{\Phi}(t)}{dt} =$$

$$-\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t)\tilde{\Phi}(t) + \Phi^{-1}(t)\frac{d\tilde{\Phi}(t)}{dt} =$$

$$= -\Phi^{-1}(t)A(t)\Phi(t)\Phi^{-1}(t)\tilde{\Phi}(t) + \Phi^{-1}(t)A(t)\tilde{\Phi}(t) = 0.$$

Demak, $\Phi^{-1}(t)\tilde{\Phi}(t)$ matritsa o'zgarmas, ya'ni ixtiyoriy $t \in I$ uchun

$$\Phi^{-1}(t)\tilde{\Phi}(t) = C \quad (9.4.4)$$

bunda $C = \Phi^{-1}(t_0)\tilde{\Phi}(t_0)$, $\det C = \det\tilde{\Phi}(t_0)/\det\Phi(t_0)$, $t_0 \in I$. Nihoyat, (9.4.4) formuladan $\tilde{\Phi}(t) = C\Phi(t)$ ekanligini topamiz. ✍

Endi ushbu

$$\begin{cases} \boldsymbol{x}' = A(t)\boldsymbol{x} \\ \boldsymbol{x}\big|_{t_0} = \boldsymbol{x}^0,\ (t_0 \in I,\ \boldsymbol{x}^0 \in \mathbb{R}^n) \end{cases} \quad (9.4.5)$$

Koshi masalasini yechaylik. Umumiy yechim formulasi (9.1.4) $\boldsymbol{x} = \Phi(t)\boldsymbol{c}$ ga ko‘ra boshlang‘ich shart qanoantlanishi uchun $\boldsymbol{x}^0 = \Phi(t_0)\boldsymbol{c}$, ya’ni $\boldsymbol{c} = \Phi^{-1}(t_0)\boldsymbol{x}^0$ bo‘lishi kerakligini topamiz. Demak, (9.4.5) masala yechimi $\boldsymbol{x} = \Phi(t)\Phi^{-1}(t_0)\boldsymbol{x}^0$ ko‘rinishda bo‘ladi.

Ushbu

$$\Phi(t,t_0) = \Phi(t)\Phi^{-1}(t_0) \tag{9.4.6}$$

$t = t_0$ nuqtada normalangan ($\Phi(t_0,t_0) = \Phi(t_0)\Phi^{-1}(t_0) = E$) fundamental matritsani kiritib, (9.4.5) Koshi masalasining yechimini

$$\boldsymbol{x} = \Phi(t,t_0)\boldsymbol{x}^0 \tag{9.4.7}$$

ko‘rinishda ifodalaymiz.

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$$\begin{cases} x_1' = \dfrac{2}{t}x_1 + \dfrac{5}{t}x_2 \\ x_2' = \dfrac{1}{t}x_1 - \dfrac{2}{t}x_2 \end{cases} \quad (t > 0)$$

sistemaning barcha yechimlarini topaylik.

🗝 Berilgan sistemaning ko‘rinishidan uning darajali funksiya sifatidagi yechimlari mavjud bo‘lishi mumkinligi ko‘rinib turibdi. Yechimni $x_1 = \alpha t^k, x_2 = \beta t^k$ ko‘rinishda izlaymiz. Bu funksiyalarni sistemaga qo‘yib, quyidagi munosabatlarga kelamiz.

$$\begin{cases} \alpha k = 2\alpha + 5\beta = 0 \\ \beta k = \alpha - 2\beta \end{cases} \Leftrightarrow \begin{cases} (2-k)\alpha + 5\beta = 0 \\ \alpha - (2+k)\beta = 0 \end{cases} \tag{9.4.8}$$

α, β larga nisbatan bu chiziqli bir jinsli algebraik tenglamalar sistemasi notrivial yechimga ega bo‘lishi uchun uning determinanti nolga teng bo‘lishi kerak:

$$\begin{vmatrix} 2-k & 5 \\ 1 & -2-k \end{vmatrix} = 0 \Leftrightarrow k = \pm 3.$$

Yuqoridagi (9.4.8) chiziqli sistemadan $k = 3$ va $k = -3$ larga mos α, β larni aniqlaymiz:

$$k = 3:\ \ \alpha = 5, \beta = 1\ ;\ \ k = -3:\ \alpha = 1, \beta = -1.$$

Shunday qilib, biz berilgan sistemaning

$$\boldsymbol{x}^1(t) = \begin{pmatrix} 5t^3 \\ t^3 \end{pmatrix} \quad \text{va} \quad \boldsymbol{x}^2(t) = \begin{pmatrix} t^{-3} \\ -t^{-3} \end{pmatrix}$$

yechimlarini topdik. Ular chiziqli erkli, chunki mos vronskian

$$W\left[\boldsymbol{x}^1(t),\boldsymbol{x}^2(t)\right]\equiv\begin{vmatrix}5t^3 & t^{-3}\\ t^3 & -t^{-3}\end{vmatrix}=-6\neq 0.$$

Demak, qaralayotgan sistemaning umumiy yechimi topilgan yechimlarning chiziqli kombinatsiyasidan iborat:

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=c_1\begin{pmatrix}5t^3\\ t^3\end{pmatrix}+c_2\begin{pmatrix}t^{-3}\\ -t^{-3}\end{pmatrix},\ c_1,\,c_2-\text{const}.$$ ☝

Teorema (*Liuvill formulasi*). *Agar* n *ta vektor-funksiya* n*- tartibli* (9.1.2) *bir jinsli sistemaning* I *oraliqda yechimlarii bo‘lsa, ularning* $W(t)$ *vronskiani uchun* I *da ushbu*

$$W(t)=W(t_0)\exp\left(\int_{t_0}^{t}\mathrm{tr}A(s)\,ds\right)\ (t_0\in I) \qquad (9.4.9)$$

Liuvill formulasi o‘rinli; bu yerda $\mathrm{tr}A(s)=\sum_{j=1}^{n}a_{jj}(s)$ *miqdor* $A(s)$ *matritsaning izi.*

🗝 (9.1.2) sistemaning $\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\ldots,\boldsymbol{x}^n(t)$ yechimlari vronskiani

$$W(t)=\begin{vmatrix}x_1^1(t) & x_1^2(t) & \cdots & x_1^n(t)\\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t)\\ \vdots & \vdots & \ddots & \vdots\\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t)\end{vmatrix}$$

ning hosilasini hisoblaylik. Determinantni differensiallash qoidasiga ko‘ra

$$W'(t)=W_1(t)+W_2(t)+\cdots+W_n(t), \qquad (9.4.10)$$

bunda $W_j(t)$ determinant $W(t)$ dan uning j- satridagi elementlarini ularning hosilasi bilan almashtirishdan hosil bo‘lgan. $W_1(t)$ ni hisoblaymiz ($\dot{x}_i^j(t)\equiv\dfrac{dx_i^j(t)}{dt}$):

$$W_1(t)=\begin{vmatrix}\dot{x}_1^1(t) & \dot{x}_1^2(t) & \cdots & \dot{x}_1^n(t)\\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t)\\ \vdots & \vdots & \ddots & \vdots\\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t)\end{vmatrix}. \qquad (9.4.11)$$

$\boldsymbol{x}^1(t),\boldsymbol{x}^2(t),\ldots,\boldsymbol{x}^n(t)$ funksiyalar (9.1.2) sistemaning yechimi bo‘lgani uchun

$$\dot{x}_1^1(t)=\sum_{k=1}^{n}a_{1k}(t)x_k^1(t),\ \dot{x}_1^2(t)=\sum_{k=1}^{n}a_{1k}(t)x_k^2(t),\ \dots,\ \dot{x}_1^n(t)=\sum_{k=1}^{n}a_{1k}(t)x_k^n(t).$$

Bu formulalarni hisobga olib, (9.4.11) determinantning 2- satrini $(-a_{12}(t))$ga, 3- satrini $(-a_{13}(t))$ga va h.k., n- satrini $(-a_{1n}(t))$ga ko‘paytirib 1-satrga qo‘shamiz. Bunda determinantning qiymati o‘zgarmaydi va natijada

$$W_1(t)=\begin{vmatrix} a_{11}(t)x_1^1(t) & a_{11}(t)x_1^2(t) & \cdots & a_{11}(t)x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix}=a_{11}(t)W(t)$$

munosabatga kelamiz. Shunga o‘xshash almashtirishlarni bajarib, qolgan $W_j(t)$ determinantlarni ham hisoblaymiz:

$$W_2(t)=a_{22}(t)W(t),\ \dots,\ W_n(t)=a_{nn}(t)W(t).$$

Hisoblangan $W_j(t)$ larni (9.4.10) formulaga qo‘yib,

$$W'(t)=\big(a_{11}(t)+a_{22}(t)+\dots+a_{nn}(t)\big)W(t),$$

ya’ni

$$W'(t)=\mathrm{tr}A(t)\cdot W(t)$$

ekanligini topamiz. Oxirgi tenglik $W(t)$ga nisbatan birinchi tartibli chiziqli differensial tenglamadir. Undan Liuvill formulasi ravshan. ♘

Liuvill formulasidan bizga ma’lum bo‘lgan quyidagi tasdiq o‘z-o‘zidan kelib chiqadi: agar (9.1.2) bir jinsli sistema yechimlarining vronskiani biror $t_0 \in I$ nuqtada nolga teng bo‘lsa, u barcha $t \in I$ nuqtalarda ham nolga teng.

Chiziqli bir jinsli sistemani uning bazis yechimlariga ko‘ra tiklash

Biz yuqorida (9.1.2) bir jinsli sistema uchun bazis yechimlarning mavjudligini ko‘rsatdik. Endi bazis yechimlariga ko‘ra mos bir jinsli sistemani tiklash masalasini qaraymiz.

Teorema. *Aytaylik,* $\{\boldsymbol{\varphi}^1(t),\boldsymbol{\varphi}^2(t),\dots,\boldsymbol{\varphi}^n(t)\}\subset C^1(I;\mathbb{R}^n)$ *funksiyalarning* $W(t)=W[\boldsymbol{\varphi}^1(t),\boldsymbol{\varphi}^2(t),\dots,\boldsymbol{\varphi}^n(t)]$ *vronskiani* I *oraliqda nolga aylanmasin. U holda bazis yechimlari shu funksiyalardan iborat bo‘lgan* $\boldsymbol{x}'=A(t)\boldsymbol{x}$ *ko‘rinishdagi normal sistema mavjud, yagona va u* $\boldsymbol{x}'=\Phi'(t)\Phi^{-1}(t)\boldsymbol{x}$, *bunda* $\Phi(t)=[\boldsymbol{\varphi}^1(t)\vdots\boldsymbol{\varphi}^2(t)\vdots\dots\vdots\boldsymbol{\varphi}^n(t)]$, *sistemadan iborat.*

🗝 Dastlab teoremaning yagonalik qismini isbotlaylik. Faraz qolaylik, $\boldsymbol{x}' = A(t)\boldsymbol{x}$ ko‘rinishdagi sistema $\mathbf{x} = \boldsymbol{\varphi}^j(t),\ j = 1, 2, \ldots, n,$ yechimlarga ega bo‘lsin. Demak,

$$\dot{\boldsymbol{\varphi}}^j(t) = A(t)\boldsymbol{\varphi}^j(t),\ t \in I,\ j = \overline{1, n}$$

ayniyatlar o‘rinli. Ularni bitta matritsaviy ayniyat $\Phi'(t) = A(t)\Phi(t),\ t \in I,\quad (\Phi(t) = [\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), \ldots, \boldsymbol{\varphi}^n(t)])$ ko‘rinishida yozib, $A(t)$ matritsaning $A(t) = \Phi'(t)\Phi^{-1}(t),\ t \in I,$ formula bilan bir qiymatli aniqlanishini topamiz.

Endi teoremaning mavjudlik qismini isbotlaymiz. Buning uchun ushbu $\boldsymbol{x}' = \Phi'(t)\Phi^{-1}(t)\boldsymbol{x}$ ciziqli normal sistemaning bazis yechimlari $\mathbf{x} = \boldsymbol{\varphi}^j(t),\ j = 1, 2, \ldots, n,$ ekanligini ko‘rsatiish kifoya. Ixtiyoriy o‘zgarmas $\boldsymbol{c} \in \mathbb{R}^n$ vektor uchun $\mathbf{x} = \Phi(t)\boldsymbol{c}$ funksiya qaralayotgan $\boldsymbol{x}' = \Phi'(t)\Phi^{-1}(t)\boldsymbol{x}$ sistemaning yechimi:

$$\left.\begin{aligned} \boldsymbol{x}' &= \Phi'(t)\boldsymbol{c}, \\ \Phi'(t)\Phi^{-1}(t)\boldsymbol{x} &= \Phi'(t)\Phi^{-1}(t)\Phi(t)\boldsymbol{c} = \Phi'(t)\boldsymbol{c} \end{aligned}\right\} \Rightarrow \boldsymbol{x}' = \Phi'(t)\Phi^{-1}(t)\boldsymbol{x}.$$

$\boldsymbol{c}$ o‘rniga $\boldsymbol{e}^j \in \mathbb{R}^n,\ j = 1, 2, ..., n,$ bazis vektorlarni olib,
$\mathbf{x} = \boldsymbol{\varphi}^j(t),\ j = 1, 2, \ldots, n,$ funksiyalar yechim ekanligini ko‘ramiz.
$\det\Phi(t) = W(t)$ noldan farqli bo‘lgani uchun bu yechimlar bazis yechimlarni tashkil etadi. 👍

Eslatma. Topilgan $\boldsymbol{x}' = \Phi'(t)\Phi^{-1}(t)\boldsymbol{x}$ normal sistemani quyidagi ko‘rinishda ham yozish mumkin:

$$\frac{1}{W(t)}\begin{vmatrix} \dot{x}_i & \dot{\varphi}_i^1(t) & \cdots & \dot{\varphi}_i^j(t) & \cdots & \dot{\varphi}_i^n(t) \\ x_1 & \varphi_1^1(t) & \cdots & \varphi_1^j(t) & \cdots & \varphi_1^n(t) \\ x_2 & \varphi_2^1(t) & \cdots & \varphi_2^j(t) & \cdots & \varphi_2^n(t) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \varphi_n^1(t) & \cdots & \varphi_n^j(t) & \cdots & \varphi_n^n(t) \end{vmatrix} = 0,\ i = \overline{1, n}. \qquad (9.4,13)$$

Masalalar

1. Aytaylik, $A(t) \in C\left(I; \mathbb{M}_{n\times n}(\mathbb{R})\right)$ matritsa ushbu $A^T(t) = -A(t),\ t \in I,$ shartni qanoatlantirsin. Agar $\boldsymbol{x}' = A(t)\boldsymbol{x}$ sistemaning $\Phi(t)$ fundamental matritsasi biror $t_0 \in I$ nuqtada ortogonal ($\Phi(t_0)\Phi^T(t_0) = E$) bo‘lsa, u ixtiyoriy $t \in I$ nuqtada ham ortogonal bo‘lishini isbotlang.

2. Faraz qilaylik, $A(t)\in C\left(I;\mathbb{M}_{n\times n}(\mathbb{R})\right)$matritsa har qanday $t\in I$ nuqtada simmetrik, ya'ni $A^T(t)=A(t)$ bo'lsin. Agar $\boldsymbol{x}'=A(t)\boldsymbol{x}$ sistemaning $\Phi(t)$fundamental matritsasi biror $t_0\in I$ nuqtada simmetrik bo'lsa, u ixtiyoriy $t\in I$ nuqtada ham simmetrik bo'lishini ko'rsating.

§ 9.5. Bir jinsli bo'lmagan chiziqli normal sistemani yechish

Bir jinsli bo'lmagan chiziqli differensial tenglamalar sistemasi (9.1.1) ga qaytaylik.

§ 9.1 dagi jumla 2 da isbotlagan edikki, (9.1.1) sistemaning umumiy yechimi uning biror (xususiy) yechimiga mos bir jinsli sistema (9.1.2) ning umumiy yechimini qo'shishdan hosil bo'ladi. (9.1.1) ning xususiy yechimini esa mos bir jiinsli sistema (9.1.2) ning fundamental matritsasi $\Phi(t)$orqali topish mumkin. Buning uchun bir jinsli sistema (9.1.2) ning umumiy yechimidagi ixtiyoriy o'zgarmaslar $(c_1,c_2,\dots,c_n)^T=\boldsymbol{c}$ ni variatsiyalaymiz (Lagranj metodi) va (9.1.1) ning xususiy yechimini

$$\boldsymbol{x}(t)=\Phi(t)\boldsymbol{u}(t) \tag{9.5.1}$$

ko'rinishda izlaymiz, bu yerda $\boldsymbol{u}(t)-$hozircha noma'lum vektor-funksiya. (9.5.1) dan (9.4.1) ga ko'ra yozamiz

$$\boldsymbol{x}'(t)=\Phi'(t)\boldsymbol{u}(t)+\Phi(t)\boldsymbol{u}'(t)=A(t)\Phi(t)\boldsymbol{u}(t)+\Phi(t)\boldsymbol{u}'(t).$$

Buni va (9.5.1)ni (9.1.1) sistemaga qo'yib, $\boldsymbol{u}(t)$ noma'lum vektor-funksiya uchun

$$A(t)\Phi(t)\boldsymbol{u}(t)+\Phi(t)\boldsymbol{u}'(t)=A(t)\Phi(t)\boldsymbol{u}(t)+\boldsymbol{g}(t),$$

ya'ni

$$\boldsymbol{u}'(t)=\Phi^{-1}(t)\boldsymbol{g}(t)$$

tenglamani hosil qilamiz. Oxirgi tenglamaning

$$\boldsymbol{u}(t)=\int_{t_0}^{t}\Phi^{-1}(s)\boldsymbol{g}(s)ds \quad (t_0,t\in I)$$

xususiy yechimni olamiz va uni (9.5.1) ga qo'yib, (9.1.1)ning izlangan

$$\boldsymbol{x}(t)=\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\boldsymbol{g}(s)ds \tag{9.5.2}$$

xususiy yechimini topamiz. Bir jinsli bo'lmagan (9.1.1) sistemaning (9.5.2) xususiy yechimiga unga mos bir jinsli sistemaning umumiy yechimni qo'shib, (9.1.1) sistemaning umumiy yechimi uchun ushbu

$$\boldsymbol{x}(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{g}(s)ds + \Phi(t)\boldsymbol{c} \qquad (9.5.3)$$

formulani topamiz, bunda $\boldsymbol{c} \in \mathbb{R}^n$ $-$ixtiyoriy o‘zgarmas vektor.

Endi (9.5.3) umumiy yechimdan foydalanib, ushbu

$$\begin{cases} \boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \ \ (t_0 \in I, \boldsymbol{x}^0 \in \mathbb{R}^n) \end{cases} \qquad (9.5.4)$$

Koshi masalasining yechimi

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}^0 + \int_{t_0}^{t} \Phi(t,s)\boldsymbol{g}(s)ds \qquad (9.5.5)$$

ko‘rinishda bo‘lishini osongina topamiz. Bu (9.5.5) formula Koshi formulasi deb ataladi.

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$$\begin{cases} x_1' = \frac{2}{t}x_1 + \frac{5}{t}x_2 + g_1(t) \\ x_2' = \frac{1}{t}x_1 - \frac{2}{t}x_2 + g_2(t) \end{cases}, \{g_1(t), g_2(t)\} \subset C\big((0;+\infty);\mathbb{R}\big),$$

chiziqli sistemaning $x_1(1) = 1,\ x_2(1) = -1$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini topaylik.

⊶Berilgan sistemaga mos bir jinsli sistemaning bazis yechimlarini yuqorida (**§** 9.4 dagi misol) topgan edik. Unga ko‘ra

$$\Phi(t) = \begin{pmatrix} 5t^3 & t^{-3} \\ t^3 & -t^{-3} \end{pmatrix} \ (t > 0).$$

Normalangan fundamental matritsa $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ ni hisoblash uchun $\Phi^{-1}(s)$ teskari matritsani topish kerak. Hisoblashlar

$$\Phi^{-1}(s) = \tfrac{1}{6}\begin{pmatrix} s^{-3} & s^{-3} \\ s^3 & -5s^3 \end{pmatrix} \ (s > 0)$$

ekanligini ko‘rsatadi. Demak $(t > 0, s > 0)$,

$$\Phi(t,s) = \Phi(t)\Phi^{-1}(s) = \tfrac{1}{6}\begin{pmatrix} 5t^3s^{-3} + t^{-3}s^3 & 5t^3s^{-3} - 5t^{-3}s^3 \\ t^3s^{-3} - t^{-3}s^3 & t^3s^{-3} + 5t^{-3}s^3 \end{pmatrix}.$$

Endi (9.5.5) Koshi formulasidan izlangan yechimni topamiz $(t_0 = 1)$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5t^3 + t^{-3} & 5t^3 - 5t^{-3} \\ t^3 - t^{-3} & t^3 + 5t^{-3} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} +$$

$$+\frac{1}{6}\int_1^t \begin{pmatrix} 5t^3 s^{-3} + t^{-3}s^3 & 5t^3 s^{-3} - 5t^{-3}s^3 \\ t^3 s^{-3} - t^{-3}s^3 & t^3 s^{-3} + 5t^{-3}s^3 \end{pmatrix}\begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds$$

yoki soddalashtirishlardan keyin uni

$$x_1(t) = \frac{1}{t^3} + \frac{1}{6}\int_1^t \left(5t^3 \frac{g_1(s) + g_2(s)}{s^3} + \frac{s^3(5g_1(s) - g_2(s))}{5t^3}\right) ds,$$

$$x_2(t) = -\frac{1}{t^3} + \frac{1}{6}\int_1^t \left(t^3 \frac{g_1(s) + g_2(s)}{s^3} + \frac{s^3(-5g_1(s) + g_2(s))}{5t^3}\right) ds$$

(bu yerda $t > 0$) skalyar ko‘rinishga keltiramiz. ✍

Masalalar

1. Agar n- tartibli berilgan $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$ sistemaning $\boldsymbol{x} = \boldsymbol{\psi}(t)$ va mos bir jinsli $\boldsymbol{x}' = A(t)\boldsymbol{x}$ sistemaning $\boldsymbol{x} = \boldsymbol{\varphi}^j(t), j = \overline{1,n}$, $W(t) = \det[\boldsymbol{\varphi}^1(t) \vdots \boldsymbol{\varphi}^2(t) \vdots \ldots \vdots \boldsymbol{\varphi}^n(t)] \neq 0$, yechimlari ma’lum bo‘lsa, berilgan sistemaning umumiy yechimi formulasini yozing.

2. Ixtiyoriy o‘zgarmaslarni variatsiyalash usulini skalyar ko‘rinishda bajaring.

§ 9.6. Sistemani komplekslashtirish

Biz yuqorida $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$ sistemani haqiqiy sohada o‘rgandik, ya’ni berilgan funksiyalar va yechim haqiqiy edi. Ba’zan bunday sistemalarni kompleks yechimlarini qarashga to‘g‘ri keladi.

Kompleks sonlar maydoni ustida qurilgan $\mathbb{C}^n = \{\mathbf{w} = (w_1, w_2, \ldots, w_n)^T \mid w_j \in \mathbb{C}, j = \overline{1,n}\}$ chiziqli (vektor) fazoni ushbu $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ to‘g‘ri yig‘indi sifatida tasvirlaylik, ya’ni $\mathbb{C}^n = \{\mathbf{w} = \mathbf{u} + i\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^n\}$ deylik; bu yerda $\mathrm{Re}\mathbf{w} = \mathbf{u} \in \mathbb{R}^n$, $\mathrm{Im}\mathbf{w} = \mathbf{v} \in \mathbb{R}^n$ haqiqiy vektorlar $\mathbf{w} \in \mathbb{C}^n$ kompleks vektorning mos ravishda haqiqiy va mavhum qismlari deb ataladi. Bunda $\lambda = \alpha + i\beta$ $(\{\alpha, \beta\} \subset \mathbb{R})$ kompleks son va $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^n$ kompleks vektor uchun

$$\lambda\mathbf{w} = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}) = \alpha\mathbf{u} - \beta\mathbf{v} + i(\alpha\mathbf{v} + \beta\mathbf{v})$$

va $\mathbf{w}^1 = \mathbf{u}^1 + i\mathbf{v}^1 \in \mathbb{C}^n$, $\mathbf{w}^2 = \mathbf{u}^2 + i\mathbf{v}^2 \in \mathbb{C}^n$ kompleks vektorlar uchun

$$\mathbf{w}^1+\mathbf{w}^2=(\mathbf{u}^1+i\mathbf{v}^1)+(\mathbf{u}^2+i\mathbf{v}^2)=\mathbf{u}^1+\mathbf{u}^2+i(\mathbf{v}^1+\mathbf{v}^2)\in\mathbb{C}^n$$

bo‘ladi. $\mathbb{C}^n$ fazoning bunday tasvirlanishi $\mathbb{R}^n$ fazoning komplekslashtirilishi deb ataladi. $\mathbb{C}^n$ fazoning $\mathrm{Re}\mathbb{C}^n=\mathbb{R}^n\oplus i0$ qismi $\mathbb{R}^n$ fazo bilan tenglashtiriladi. $\mathbf{w}=\mathbf{u}+i\mathbf{v}\in\mathbb{C}^n$ kompleks vektorning qo‘shmasi deb $\overline{\mathbf{w}}=\overline{\mathbf{u}+i\mathbf{v}}=\mathbf{u}-i\mathbf{v}\in\mathbb{C}^n$ kompleks vektorga aytiladi.

Jumla. $\mathbb{R}^n$ *fazoning ixtiyoriy bazisi uning komplekslashtirilishi bo‘lgan* $\mathbb{C}^n$ *fazoning ham bazisidir.*

🗝 $\mathbf{b}^1,\mathbf{b}^2,\ldots,\mathbf{b}^n$ vektorlar $\mathbb{R}^n$ fazoning ixtiyoriy bazisi bo‘lsin. Ixtiyoriy $\mathbf{w}=\mathbf{u}+i\mathbf{v}\in\mathbb{C}^n$ $(\mathbf{u}\in\mathbb{R}^n,\mathbf{v}\in\mathbb{R}^n)$ vektorni olaylik. $\mathbf{u}=\alpha_1\mathbf{b}^1+\alpha_2\mathbf{b}^2+\cdots+\alpha_n\mathbf{b}^n$ va $\mathbf{v}=\beta_1\mathbf{b}^1+\beta_2\mathbf{b}^2+\cdots+\beta_n\mathbf{b}^n$ bo‘lgani uchun

$$\mathbf{w}=\alpha_1\mathbf{b}^1+\alpha_2\mathbf{b}^2+\cdots+\alpha_n\mathbf{b}^n+i(\beta_1\mathbf{b}^1+\beta_2\mathbf{b}^2+\cdots+\beta_n\mathbf{b}^n)=$$
$$=(\alpha_1+i\beta_1)\mathbf{b}^1+(\alpha_2+i\beta_2)\mathbf{b}^2+\cdots+(\alpha_n+i\beta_n)\mathbf{b}^n,$$

ya’ni ixtiyoriy $\mathbf{w}=\mathbf{u}+i\mathbf{v}\in\mathbb{C}^n$ vektor $\mathbf{b}^1,\mathbf{b}^2,\ldots,\mathbf{b}^n$ vektorlar orqali chiziqli ifodalanadi. Endi $\mathbf{b}^1,\mathbf{b}^2,\ldots,\mathbf{b}^n$ vektorlarning $\mathbb{C}^n$ fazoda chiziqli erkli ekanligini ko‘rsatamiz. Agar $(\alpha_1+i\beta_1)\mathbf{b}^1+(\alpha_2+i\beta_2)\mathbf{b}^2+\cdots+(\alpha_n+i\beta_n)\mathbf{b}^n=0+i0$ bo‘lsa, u holda $\alpha_1\mathbf{b}^1+\alpha_2\mathbf{b}^2+\cdots+\alpha_n\mathbf{b}^n+i(\beta_1\mathbf{b}^1+\beta_2\mathbf{b}^2+\cdots+\beta_n\mathbf{b}^n)=0+i0$ va, bundan

$$\left.\begin{array}{l}\alpha_1\mathbf{b}^1+\alpha_2\mathbf{b}^2+\cdots+\alpha_n\mathbf{b}^n=0\\ \beta_1\mathbf{b}^1+\beta_2\mathbf{b}^2+\cdots+\beta_n\mathbf{b}^n=0\end{array}\right\}\Rightarrow\alpha_1=\alpha_2=\cdots\alpha_n=\beta_1=\beta_2=\cdots=\beta_n=0.$$

Demak, $\mathbf{b}^1,\mathbf{b}^2,\ldots,\mathbf{b}^n$ vektorlar $\mathbb{C}^n$ fazoning ham bazisi. 👍

$n\times n$ o‘lchamli A matritsaning elementlari kompleks sonlardan iborat bo‘lsa ($A\in\mathbb{M}_{n\times n}(\mathbb{C})$), u holda $A=\mathrm{Re}A+i\,\mathrm{Im}A$ deb yozish mumkin, bunda $\{\mathrm{Re}A,\mathrm{Im}A\}\subset\mathbb{M}_{n\times n}(\mathbb{R})$ matritsalar haqiqiy elementlardan tuzilgan va $n\times n$ o‘lchamli, ular A ning (mos ravishda) haqiqiy va mavhum qismlari deb ataladi.

$t\in I$ haqiqiy o‘zgaruvchining kompleks qiymatli vektor-funksiyasi $\mathbf{w}:I\to\mathbb{C}^n$ akslantirishni anglatadi. Bu funksiya har bir $t\in I$ haqiqiy songa $\mathbf{w}(t)=\mathbf{u}(t)+i\mathbf{v}(t)\in\mathbb{C}^n$ kompleks vektorni mos keltiradi, bunda $\mathbf{u}:I\to\mathbb{R}^n$, $\mathbf{v}:I\to\mathbb{R}^n$ – haqiqiy vektor-funksiyalar; ular $\mathbf{w}:I\to\mathbb{C}^n$ kompleks vektor-funksiyaning (mos ravishda) haqiqiy va mavhum qismlari deb ataladi. Agar $\mathrm{Re}\mathbf{w}(t)=\mathbf{u}(t)=(u_1(t),u_2(t),\ldots,u_n(t))^T\in\mathbb{R}^n$, $\mathrm{Im}\mathbf{w}(t)=\mathbf{v}(t)=(v_1(t),v_2(t),\ldots,v_n(t))^T\in\mathbb{R}^n$ desak, $\mathbf{w}:I\to\mathbb{C}^n$ kompleks vektor-funksiyani $2n$ dona $u_j(t), v_j(t)$ $(j=\overline{1,n})$ haqiqiy funksiyalar (koordinata funksiyalari) orqali berish mumkin. $A: I \to \mathbb{M}_{n\times n}(\mathbb{C})$ matritsaviy qiymatli funksiya har bir $t \in I$ haqiqiy songa $A(t) = \operatorname{Re} A(t) + i \operatorname{Im} A(t) \in \mathbb{M}_{n\times n}(\mathbb{C})$ $(\{\operatorname{Re} A(t), \operatorname{Im} A(t)\} \subset \mathbb{M}_{n\times n}(\mathbb{R}))$ matritsani mos keltiradi. Analizning kompleks vektor-funksiyalar (kompleks matritsaviy qiymatli funksiyalar) uchun limit, uzluksizlik, hosila, integral va h.k. tushunchalari odatdagidek kiritiladi. Masalan, $\mathbf{w}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$ kompleks vektor-funksiyaning hosilasi

$$\mathbf{w}'(t) = \lim_{h\to 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} + i \lim_{h\to 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} = \lim_{h\to 0} \frac{\mathbf{w}(t+h) - \mathbf{w}(t)}{h}$$

formula yordamida aniqlanadi.

$C(I;\mathbb{C}^n)$ bilan barcha $\mathbf{w}: I \to \mathbb{C}^n$ uzluksiz kompleks vektor-funksiyalar, $C(I;\mathbb{M}_{n\times n}(\mathbb{C}))$ bilan esa barcha $A: I \to \mathbb{M}_{n\times n}(\mathbb{C})$ uzluksiz matritsaviy funksiyalar sinfini belgilaymiz.

Endi

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$$

kompleks chiziqli sistemani o‘rganish mumkin, bu yerda $A(t) \in C\left(I;\mathbb{M}_{n\times n}(\mathbb{C})\right)$, $\boldsymbol{g}(t) = \left(g_1(t), g_2(t), \ldots, g_n(t)\right)^T \in C(I,\mathbb{C}^n)$, $\boldsymbol{x} = \boldsymbol{x}(t) -$noma’lum kompleks vektor-funksiya. Bu sistemning nazariyasi haqiqiy sohadagiga juda ham o‘xshash. Haqiqiy holdagi barcha (tengsizlik bilan bog‘liq bo‘lmagan) teoremalar kompleks holda ham o‘z kuchini saqlaydi. Endi faqat $\mathbb{R}^n$ chiziqli fazo o‘rnida $\mathbb{C}^n$ chiziqli fazoni ishlatish kerak, xolos. Masalan, $\boldsymbol{x}' = A(t)\boldsymbol{x}$ n- tartibli bir jinsli sistemaning yechimlari to‘plami n o‘lchamli kompleks chiziqli fazoni tashkil etadi.

Haqiqiy sohada berilgan $\boldsymbol{x}' = A(t)\boldsymbol{x}$, $A(t) \in C\left(I;\mathbb{M}_{n\times n}(\mathbb{R})\right)$sistemani qaraylik. Bu sistemaning kompleks yechimlarini izlash sistemani komplekslashtirish deb ataladi. Agar bu sistemaning kompleks yechimi topilgan bo‘lsa, uning haqiqiy va mavhum qismlari ham shu sistemaning yechimi bo‘ladi. Haqiqatan ham, faraz qilaylk, $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t) -$ haqiqiy va mavhum qismlari ajratilgan kompleks yechim bo‘lsin. Demak,

$$\mathbf{u}'(t) + i\mathbf{v}'(t) = A(t)\mathbf{u}(t) + iA(t)\mathbf{v}(t), t \in I.$$

$A(t) -$haqiqiy matritsa bo‘lgani uchun oxirgi tenglikdan

$$\mathbf{u}'(t) = A(t)\mathbf{u}(t),\ \mathbf{v}'(t) = A(t)\mathbf{v}(t), t \in I,$$

ayniyatlarni hosil qilamiz. Ular $\mathbf{u}(t)$ va $\mathbf{v}(t)$ larning yechim ekanligini anglatadi.

Masalalar

1. Haqiqiy sohada $\boldsymbol{x}' = A(t)\boldsymbol{x}$ va $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$ sistemalar uchun § 9.1 da keltirilgan tasdiqlarni kompleks holga o‘tkazing.

2. Agar $\boldsymbol{x} = \boldsymbol{x}(t)$ kompleks vektor-funksiya ushbu $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$, bunda $A(t) \in C\big(I; \mathbb{M}_{n\times n}(\mathbb{R})\big)$(haqiqiy) va $\boldsymbol{g}(t) \in C(I; \mathbb{C}^n)$ (kompleks),sistemaning yechimi bo‘lsa, u holda $\boldsymbol{x} = \operatorname{Re}\boldsymbol{x}(t)$ $(\boldsymbol{x} = \operatorname{Im}\boldsymbol{x}(t))$ haqiqiy vektor-funksiya $\boldsymbol{x}' = A(t)\boldsymbol{x} + \operatorname{Re}\boldsymbol{g}(t)$ (mos ravishda $\boldsymbol{x}' = A(t)\boldsymbol{x} + \operatorname{Im}\boldsymbol{g}(t)$) haqiqiy sistemaning yechimi ekanligini isbotlang.

MODUL 10. CHIZIQLI O‘ZGARMAS KOEFFITSIENTLI NORMAL SISTEMALAR

§ 10.1. $x' = Ax$ sistemani matritsaning Jordan kanonik ko‘rinishidan foydalanib yechish

Ushbu

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases}$$

normal ko‘rinishdagi chiziqli o‘zgarmas koeffitsientli bir jinsli sistemaning umumiy yechimini topaylik. Bu sistemaning vektorli ko‘rinishi

$$\boldsymbol{x}' = A\boldsymbol{x}, \tag{10.1.1}$$

bu yerda $A = [a_{kl}] \in \mathbb{M}_{n\times n}(\mathbb{C})$ berilgan matritsa, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ $-t$ argumantning noma’lum vektor-funksiyasi.

Chiziqli normal sistemalar nazariyasidan ma’lumki, (10.1.1) bir jinsli sistemaning umumiy yechimi uning n dona chiziqli erkli (bazis) yechimlarining ixtiyoriy chiziqli kombinatsiyasi kabi ifodalanadi. Bazis yechimlarni topish maqsadida (10.1.1) sistemada

$$\boldsymbol{x} = S\boldsymbol{y},\ \det S \neq 0, \tag{10.1.2}$$

almashtirish bajaraylik. Bu yerdagi S matritsani keyinroq tanlaymiz. Yangi noma’lum $\boldsymbol{y}$ vektor-funksiyaga nisbatan $S\boldsymbol{y}' = SA\boldsymbol{y}$, ya’ni

$$\boldsymbol{y}' = S^{-1}AS\boldsymbol{y} \tag{10.1.3}$$

sistemani hosil qilamiz. Chiziqli algebradan ma’lumki, har qanday A kvadrat matritsani biror teskarilanuvchi S matritsa yordamida Jordan kanonik ko‘rinishiga keltirish mumkin, ya’ni

$$S^{-1}AS = J \tag{10.1.4}$$

katakli-diagonal Jordan matritsasidan iborat bo‘ladi:

$$J = \mathrm{diag}(J_{\lambda_1,n_1},\dots,J_{\lambda_2,n_2},\dots,J_{\lambda_s,n_s}) = \begin{pmatrix} J_{\lambda_1,n_1} & & & & \\ & \ddots & & & \\ & & J_{\lambda_2,n_2} & & \\ & & & \ddots & \\ & & & & J_{\lambda_s,n_s} \end{pmatrix};$$

(10.1.5)

bunda J Jordan (katakli-diagonal) matritsasining diagonali bo‘ylab Jordan kataklari, boshqa o‘rinlarda esa nollar joylashgan bo‘lib, u quyidagicha tuziladi. Faraz qilaylik, A matritsaning turli xos (xarakteristik) sonlari $\lambda_1,\lambda_2,\dots,\lambda_s$ $(s \le n)$ mos ravishda $k_1,k_2,\dots,k_s$ karrali $(k_1 + k_2 + \dots + k_s = n)$ hamda λ_q, $q = \overline{1,s}$, xos songa mos kelgan chiziqli erkli xos vektorlar soni p_q, ya’ni $\dim\{\mathbf{x} \mid (A - \lambda_q E)\mathbf{x} = 0\} = p_q$ $(p_q = n - \mathrm{rank}(A - \lambda_q E))$ bo‘lsin. U holda λ_q xos songa p_q dona

$$J_{\lambda_q,d_{qj}} = \begin{pmatrix} \lambda_q & 1 & & & \\ & \lambda_q & 1 & & \\ & & \lambda_q & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_q \end{pmatrix} \in \mathbb{M}_{d_{qj}\times d_{qj}}(\mathbb{C}), \qquad (10.1.6)$$

$$j = \overline{1, p_q},\ (d_{q1} + d_{q2} + \dots + d_{q\,p_q} = k_q)$$

$d_{qj} \times d_{qj}$ o‘lchamli Jordan kataklari mos keladi; bu yerda bo‘sh o‘rinlarda nollar yozilgan deb tushunish kerak. Ravshanki, λ_q ga mos kelgan Jordan kataklarining eng katta o‘lchamlisi uchun $\tilde{k}_q \overset{def}{=} \max\{d_{q1}, d_{q2}, \dots, d_{q\,p_q}\} \le k_q$ bo‘ladi. (10.1.5) formulada $J_{\lambda_1,d_{11}} = J_{\lambda_1,n_1},\dots,J_{\lambda_1,d_{1p_1}},\ J_{\lambda_2,d_{21}} = J_{\lambda_2,n_2},\dots,\ J_{\lambda_s,d_{s\,p_s}} = J_{\lambda_s,n_s}$.

S matritsani yuqorida aytilgandek tanlaylik. U holda (10.1.3) sistemani Jordan kataklariga mos ravishda mustaqil sistemalarga ajratish mumkin. Ularning birinchisini kengaytirilgan skalyar ko‘rinishda yozaylik:

$$\begin{cases} y_1' = \lambda_1 y_1 + y_2, \\ y_2' = \lambda_1 y_2 + y_3, \\ \dots\dots\dots\dots\dots\dots\dots \\ y_{n_1-1}' = \lambda_1 y_{n_1-1} + y_{n_1}, \\ y_{n_1}' = \lambda_1 y_{n_1}, \end{cases} \qquad (10.1.7)$$

Boshqa Jordan kataklariga mos sistemalar ham shu (10.1.7) sistemaga o‘xshash ko‘rinishda bo‘ladi. (10.1.7) sistemadagi tenglamalarni $\mu = e^{-\lambda_1 t}$ integrallovchi ko‘paytuvchiga ko‘paytirib, sistemani quyidagi ko‘rinishga keltiramiz:

$$\begin{cases} \left(y_1 e^{-\lambda_1 t}\right)' = y_2 e^{-\lambda_1 t}, \\ \left(y_2 e^{-\lambda_1 t}\right)' = y_3 e^{-\lambda_1 t}, \\ \dots\dots\dots\dots\dots\dots\dots \\ \left(y_{n_1-1} e^{-\lambda_1 t}\right)' = y_{n_1} e^{-\lambda_1 t}, \\ \left(y_{n_1} e^{-\lambda_1 t}\right)' = 0, \end{cases} \qquad (10.1.8)$$

Oxirgi sistema pastdan yuqoriga qarab osongina yechiladi: $y_{n_1} e^{-\lambda_1 t} = c_{n_1}$, ... Quyidagilar hosil bo‘ladi:

$$\begin{cases} y_1 = \left(c_1 + \frac{c_2}{1!} t + \ldots + \frac{c_{n_1-1}}{(n_1-2)!} t^{n_1-2} + \frac{c_{n_1}}{(n_1-1)!} t^{n_1-1} \right) e^{\lambda_1 t}, \\ y_2 = \left(c_2 + \frac{c_3}{1!} t + \ldots + \frac{c_{n_1}}{(n_1-2)!} t^{n_1-2} \right) e^{\lambda_1 t}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{n_1-1} = \left(c_{n_1-1} + \frac{c_{n_1}}{1!} t \right) e^{\lambda_1 t}, \\ y_{n_1} = c_{n_1} e^{\lambda_1 t}, \end{cases} \qquad (10.1.9)$$

Bu yechimlarda $c_1, c_2, \ldots, c_{n_1}$ ixtiyoriy o‘zgarmaslar o‘rniga bittasiga 1, qolganlariga esa 0 qo‘yamiz va $y_j = 0,\ j = \overline{n_1 + 1, n}$, deb, (10.1.3) sistemaning J_{λ_1, n_1} Jordan katagiga mos kelgan quyidagi yechimlarini (n_1 dona) topamiz:

$$\begin{cases} y_1 = e^{\lambda_1 t}, \\ y_2 = 0, \\ \vdots \\ y_{n_1} = 0, \\ y_j = 0,\ j = \overline{n_1 + 1, n}. \end{cases}; \quad \begin{cases} y_1 = \dfrac{t}{1!} e^{\lambda_1 t}, \\ y_2 = e^{\lambda_1 t}, \\ y_3 = 0 \\ \vdots \\ y_{n_1} = 0, \\ y_j = 0,\ j = \overline{n_1 + 1, n}. \end{cases};$$

$$\begin{cases} y_1 = \dfrac{t^2}{2!} e^{\lambda_1 t}, \\ y_2 = \dfrac{t}{1!} e^{\lambda_1 t}, \\ y_3 = e^{\lambda_1 t}, \\ \vdots \\ y_{n_1} = 0, \\ y_j = 0,\ j = \overline{n_1 + 1, n}. \end{cases}; \ \ldots\ ; \quad \begin{cases} y_1 = \dfrac{t^{n_1 - 1}}{(n_1 - 1)!} e^{\lambda_1 t}, \\ y_2 = \dfrac{t^{n_1 - 2}}{(n_1 - 2)!} e^{\lambda_1 t}, \\ y_3 = \dfrac{t^{n_1 - 3}}{(n_1 - 3)!} e^{\lambda_1 t}, \\ \vdots \\ y_{n_1} = e^{\lambda_1 t}, \\ y_j = 0,\ j = \overline{n_1 + 1, n}. \end{cases} \tag{10.1.10}$$

Bu yechimlar chiziqli erkli bo‘ladi (nega?).

Ma’lumki, (10.1.4) formuladagi S keltiruvchi matritsa $S = [\boldsymbol{s}^1 \vdots \boldsymbol{s}^2 \vdots \cdots \vdots \boldsymbol{s}^n]$ ko‘rinishda bo‘ladi; bu yerda $\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^n$ vektorlar chiziqli erkli (Jordan bazisi). (10.1.10) ga ko‘ra $\boldsymbol{x} = S\boldsymbol{y}$ almashtirish formulasidagi ko‘paytirishni bajarib, berilgan (10.1.1) sistemaning

$$\boldsymbol{x}^1 = \boldsymbol{s}^1 e^{\lambda_1 t}, \ \boldsymbol{x}^2 = \left(\boldsymbol{s}^1 \frac{t}{1!} + \boldsymbol{s}^2\right) e^{\lambda_1 t}, ...,$$
$$\boldsymbol{x}^{n_1} = \left(\boldsymbol{s}^1 \frac{t^{n_1-1}}{(n_1-1)!} + \boldsymbol{s}^2 \frac{t^{n_1-2}}{(n_1-2)!} + ... + \boldsymbol{s}^{n_1-1}\frac{t}{1!} + \boldsymbol{s}^{n_1}\right) e^{\lambda_1 t} \tag{10.1.11}$$

n_1 dona chiziqli erkli yechimlarini topamiz; bu yerdagi vektor koeffitsientli ko‘phadning darajasi $n_1 - 1$ J_{λ_1, n_1} katakning n_1 o‘lchamidan bittaga kam: $n_1 - 1 \le k_1 - 1$. λ_1 xarakteristik songa mos kelgan boshqa Jordan kataklari orqali qurilgan yechimlardagi ko‘phadlarning darajalari ham mos katakning tartibidan bittaga kam va $\le k_1 - 1$ bo‘ladi.

(10.1.3) sistemaning qolgan Jordan kataklariga mos kelgan chiziqli erkli yechimlarini shunga o‘xshash topib, ularni birlashtirib, bu sistemaning n dona chiziqli erkli yechimlarini hosil qilamiz Bu yechimlarga ko‘ra $\boldsymbol{x} = S\boldsymbol{y}$ formula orqali (10.1.1) sistemaning bazis yechimlarini quramiz.

Masalalar yechganda A matritsaning k_j karrali λ_j $(j = \overline{1,s})$ xarakteristik soniga mos kelgan Jordan kataklarining o‘lchamlarini topmasdan, yechimni birdaniga

$$\boldsymbol{x} = (\boldsymbol{c}^1 + \boldsymbol{c}^2 t + ... + \boldsymbol{c}^{k_j} t^{k_j - 1}) e^{\lambda_j t} \tag{10.1.12}$$

ko‘rinishda izlash mumkin. Bu yerdagi $\boldsymbol{c}^1, \boldsymbol{c}^2, ..., \boldsymbol{c}^{k_j}$ noma’lum vektor-koeffitsientlarni aniqlash uchun (10.1.12) ifodani (10.1.1) tenglamaga qo‘yamiz; hosil bo‘lgan tenglikni $e^{\lambda_j t}$ ga qisqartirib, va t ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, topamiz:

$$(\boldsymbol{c}^2 + 2\boldsymbol{c}^3 t + ... + (k_j - 1)\boldsymbol{c}^{k_j} t^{k_j - 2}) e^{\lambda_j t} + (\boldsymbol{c}^1 + \boldsymbol{c}^2 t + ... + \boldsymbol{c}^{k_j} t^{k_j - 1}) \lambda_j e^{\lambda_j t} =$$
$$= A(\boldsymbol{c}^1 + \boldsymbol{c}^2 t + ... + \boldsymbol{c}^{k_j} t^{k_j - 1}) e^{\lambda_j t},$$

(o‘ngdan chapga)

$$A\boldsymbol{c}^{k_j} t^{k_j - 1} + A\boldsymbol{c}^{k_j - 1} t^{k_j - 2} + \ldots + A\boldsymbol{c}^2 t + A\boldsymbol{c}^1 =$$
$$= \lambda_j \boldsymbol{c}^{k_j} t^{k_j - 1} + \left(\lambda_j \boldsymbol{c}^{k_j - 1} + (k_j - 1)\boldsymbol{c}^{k_j}\right) t^{k_j - 2} + \ldots + \left(\lambda_j \boldsymbol{c}^2 + 2\boldsymbol{c}^3\right) t + \left(\lambda_j \boldsymbol{c}^1 + \boldsymbol{c}^2\right),$$

$$\begin{cases} t^{k_j-1}: \quad A\boldsymbol{c}^{k_j} = \lambda_j \boldsymbol{c}^{k_j}; \\ t^{k_j-2}: \quad A\boldsymbol{c}^{k_j-1} = \lambda_j \boldsymbol{c}^{k_j-1} + (k_j-1)\boldsymbol{c}^{k_j}; \\ \cdots\cdots \\ t \ : \ A\boldsymbol{c}^2 = \lambda_j \boldsymbol{c}^2 + 2\boldsymbol{c}^3; \\ t^0 \ : \ A\boldsymbol{c}^1 = \lambda_j \boldsymbol{c}^1 + \boldsymbol{c}^2. \end{cases}$$

Oxirgi algebraik sistemani yuqoridan pastga qarab yechamiz. Bunda $\boldsymbol{c}^1, \boldsymbol{c}^2, ..., \boldsymbol{c}^{k_j}$ noma'lumlar k_j dona ixtiyoriy skalyar parametr orqali ifodalanadi (yechimlar k_j o‘lchamli chiziqli fazo tashkil etadi). Barcha λ_j xarakteristik sonlar uchun ularga mos keluvchi yechimlarni topib, (10.1.1) ning bazis yechimlarini quramiz. Umumiy yechim endi bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi ko‘rinishida yoziladi.

E’tirof etaylikki, agar $\lambda_j, j = \overline{1,n}$, $- A$ matritsaning oddiy (bir karrali) xos sonlari va $\boldsymbol{s}^j$ – matritsaning shu λ_j xos sonlariga mos kelgan xos vektorlari, ya’ni $A\boldsymbol{s}^j = \lambda_j \boldsymbol{s}^j, \boldsymbol{s}^j \neq \boldsymbol{0}$, bo‘lsa, $\boldsymbol{x}^j = e^{\lambda_j t}\boldsymbol{s}^j$ – (10.1.1) sistemaning bazis yechimlarini tashkil etadi.

Agar (10.1.1) sistemadagi barcha koeffitsientlar haqiqiy sonlar bo‘lsa, odatda haqiqiy yechimlarni topish talab etiladi. Bu holda berilgan sistemani kopmplekslashtirib, kompleks yechimlarni qurib, so‘ngra ulardan foydalanib Eyler formulalariga ko‘ra haqiqiy bazis yechimlarni qurish kerak bo‘ladi.

Misol 1. Ushbu

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ A = \begin{pmatrix} 0 & -4 & -2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \qquad (10.1.14)$$

sistemaning umumiy yechimini quraylik.

🗝Berilgan sistemaning haqiqiy yechimlarini topish kerak (chunki sistamaning barcha koeffitsientlari haqiqiy sonlar). Berilgan misolda $n = 3$. A matritsaning xarakteristik ko‘phadi

$$\det(A-\lambda E)=\begin{vmatrix}-\lambda & -4 & -2\\ -1 & -\lambda & -1\\ 1 & 2 & 3-\lambda\end{vmatrix}=-(\lambda+1)(\lambda-2)^2,$$

xarakteristik (xos) sonlari $\lambda_1=-1, \lambda_2=\lambda_3=2$.

Oddiy xarakteristik son $\lambda=-1$ ga berilgan sistemaning

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=e^{-t}\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} \qquad (10.1.15)$$

ko‘rinishdagi yechimi mos keladi, bunda $s_1, s_2, s_3 -$ hozircha noma’lum sonlar. Ularni topish uchun (10.1.15) ni berilgan sistemaga qo‘yib, uning qanoatlanishini talab qilamiz:

$$\begin{cases}s_1-4s_2-2s_3=0\\ -s_1+s_2-s_3=0\\ s_1+2s_2+4s_3=0\end{cases} \Rightarrow s_1=-2s_3, s_2=-s_3 .$$

Demak, $s_3=c_1$ deb, quyidagi bir parametrli yechimlar oilasini topamiz:

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=c_1e^{-t}\begin{pmatrix}2\\ 1\\ -1\end{pmatrix} \text{ yoki } \begin{cases}x=2c_1e^{-t},\\ y=c_1e^{-t},\\ z=-c_1e^{-t}.\end{cases} \qquad (10.1.16)$$

Endi $k=2$ ikki karrali $\lambda=2$ xos songa mos kelgan yechimlarni quramiz. Yechimni $(k-1=1)$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=e^{2t}\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}t+\begin{pmatrix}a\\ b\\ c\end{pmatrix}\right) \qquad (10.1.17)$$

ko‘rinishda izlaymiz. Buni berilgan sistemaga qo‘yib, $\alpha, \beta, \lambda, a, b, c$ noma’lum koeffitsientlarni aniqlaymiz:

$$2e^{2t}\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}t+\begin{pmatrix}a\\ b\\ c\end{pmatrix}\right)+e^{2t}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=Ae^{2t}\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}t+\begin{pmatrix}a\\ b\\ c\end{pmatrix}\right);$$

bundan

$$A\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=2\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix},\quad A\begin{pmatrix}a\\ b\\ c\end{pmatrix}=2\begin{pmatrix}a\\ b\\ c\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.$$

Sistemalarni yechamiz:

$$A\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=2\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\Rightarrow\begin{cases}-4\beta-2\gamma=2\alpha,\\ -\alpha-\gamma=2\beta,\\ \alpha+2\beta+3\gamma=2\gamma.\end{cases}\Rightarrow\begin{cases}\alpha=-2\beta-\gamma\\ (\beta,\gamma-\text{ixtiyoriy})\end{cases};$$

$$A\begin{pmatrix}a\\ b\\ c\end{pmatrix}=2\begin{pmatrix}a\\ b\\ c\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\Rightarrow\begin{cases}-4b-2c=2a-\alpha\\ -a-c=2b+\beta\\ a+2b+3c=2c+\gamma\end{cases}\Rightarrow\begin{cases}2a+4b+2c=2\beta+\gamma\\ a+2b+c=-\beta\\ a+2b+c=\gamma\end{cases}$$

Oxirgi sistemalardan quyidagini topamiz:

$$\alpha=\beta=\gamma=0 \text{ va } \begin{cases}a=c_2,\\ b=c_3,\\ c=-c_2-2c_3.\end{cases}\quad (c_2,c_3-\text{ixtiyoriy o‘zgarmaslar})$$

Demak, (10.1.14) sistemaning ikki karrali $\lambda=2$ xos songa mos kelgan yechimlari (10.1.17) ga ko‘ra

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=e^{2t}\begin{pmatrix}c_2\\ c_3\\ -c_2-2c_3\end{pmatrix}=c_2e^{2t}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+c_3e^{2t}\begin{pmatrix}0\\ 1\\ -2\end{pmatrix}\qquad(10.1.18)$$

formula bilan beriladi. Endi oldin topilgan (10.1.16) yechimga (10.1.18) yechimni qo‘shamiz va berilgan sistema (10.1.14) ning umumiy yechimini topamiz:

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}2e^{-t} & e^{2t} & 0\\ e^{-t} & 0 & e^{2t}\\ -e^{-t} & -e^{2t} & -2e^{2t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}$$

yoki skalyar ko‘rinishda

$$\begin{cases}x=2c_1e^{-t}+c_2e^{2t},\\ y=c_1e^{-t}+c_3e^{2t},\\ z=-c_1e^{-t}-(c_2+2c_3)e^{2t}.\end{cases}$$

Qaralgan misolda fundamental matritsa, ravshanki, quyidagi ko‘rinishga ega:

$$\Phi(t)=\begin{pmatrix} 2e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & e^{2t} \\ -e^{-t} & -e^{2t} & -2e^{2t} \end{pmatrix}. \qquad (10.1.19)$$

Izoh. Ikki karrali ($k=2$) $\lambda=2$ xos songa mos kelgan yechimlarni (10.1.17) ko‘rinishda izlab, $\alpha=\beta=\gamma=0$ ekanligini topdik. Bu natija A matritsaning ikki karrali $\lambda=2$ xos soniga uning ikki dona chiziqli erkli xos vektori mos kelishini anglatadi; bu xos vektorlar (10.1.18) formulaning o‘ng tomonida qatnashgan.

Eslatma. Berilgan sistemani boshqacha usulda yechib, boshqacha ko‘rinishdagi fundamental matritsani topish mumkin. Lekin, ixtiyoriy ikki fundamental matritsaning biri ikkinchisini o‘ng tomondan teskarilanuvchi o‘zgarmas matritsaga ko‘paytirishdan hosil bo‘ladi. Qaralayotgan holda (10.1.14) sistemaning har qanday $\Psi(t)$ fundamental matritsasi biz topgan (10.1.19) fundamental matritsa orqali quyidagicha ifodalanadi:

$$\Psi(t)=\Phi(t)\mathrm{C},\ \det\mathrm{C}\neq 0.$$

Misol 2. Ushbu

$$\begin{cases} x'=3x+2y-z \\ y'=-x+3y+z \\ z'=y+2z \end{cases} \qquad (10.1.20)$$

sistemani yeching.

🔑 Berilgan sistemaning tartibi $n=3$, matritsasi

$$A=\begin{pmatrix} 3 & 2 & -1 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Xarakteristik sonlar:

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix}=0 \Rightarrow \lambda_1=2, \lambda_{2,3}=3\pm i$$

$\lambda=2$ xarakteristik songa

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=e^{2t}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

ko‘rinishdagi yechim mos keladi. Buni berilgan sistema (10.1.20) ga qo‘yib, s_1, s_2, s_3 larni aniqlaymiz ($s_3 = 1$ tanlangan):

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Demak, $\lambda = 2$ xarakteristik songa mos yechim

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{10.1.21}$$

Endi $\lambda = 3 + i$ xos songa mos kelgan

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

yechimlarni topamiz. Buni berilgan sistemaga qo‘yamiz va a, b, c larni topamiz:

$$\begin{pmatrix} -i & 2 & -1 \\ -1 & -i & 1 \\ 0 & 1 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = 2 - i, \\ b = 1 + i, \\ c = 1 (\text{tanlangan}). \end{cases}$$

Demak, sistema yechimi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} 2-i \\ 1+i \\ 1 \end{pmatrix} = e^{3t} e^{it} \begin{pmatrix} 2-i \\ 1+i \\ 1 \end{pmatrix}$$

yoki, $e^{it} = \cos t + i \sin t$ Eyler formulasiga ko‘ra haqiqiy va mavhum qismlarni ajratsak,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix} + i e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix}.$$

Yechimning haqiqiy va mavhum qismlari ham yechim bo‘lgani uchun bundan $\lambda = 3 \pm i$ xarakteristik sonlarga mos kelgan ikkita haqiqiy yechimni aniqlaymiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix}. \qquad (10.1.22)$$

Endi sistemaning umumiy yechimini (10.1.21) va (10.1.22) bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi sifatida yozamiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix},$$

bunda c_1, c_2, c_3 – ixtiyoriy haqiqiy o‘zgarmaslar. ✍

Masalalar

Sistemalarni yeching:

1. $\begin{cases} x_1' = x_1 - x_2, \\ x_1' = x_1 + x_2. \end{cases}$ **2.** $\begin{cases} x' = 3x - 2y + z, \\ y' = x + z, \\ z' = 2x + 2y + z. \end{cases}$ **3.** $\begin{cases} x' = 2x - y + z, \\ y' = 2y - z, \\ z' = y + 2z. \end{cases}$

§ 10.2. $\boldsymbol{x}' = A\boldsymbol{x}$ sistemani eksponensial matritsa yordamida yechish

Ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} \ \ (A \in \mathbb{M}_{n\times n}(\mathbb{C}) - \text{ berilgan o‘zgarmas matritsa}) \qquad (10.2.1)$$

o‘zgarmas koeffitsientli bir jinsli sistemani boshqacha usulda yechish maqsadida **eksponensial matritsa** tushunchasini kiritamiz

Ma’lumki,

$$\begin{cases} x' = ax \\ x(0) = x_0 \end{cases} \quad (a, x_0 - \text{berilgan o‘zgarmas sonlar})$$

skalyar Koshi masalasining $(-\infty, +\infty)$da aniqlangan yagona yechimi

$$x = e^{at}x_0 = \left(1 + at + \frac{a^2}{2!}t^2 + \ldots + \frac{a^n}{n!}t^n + \ldots\right)x_0, \ t \in \mathbb{R},$$

formula bilan beriladi.

Ushbu

$$\begin{cases} \boldsymbol{x}' = A\boldsymbol{x} \\ \boldsymbol{x}(0) = \boldsymbol{x}^0 \end{cases} \quad (\boldsymbol{x}^0 \in \mathbb{R}^n - \text{berilgan vektor}) \qquad (10.2.2)$$

Koshi masalasining yechimini shunga o‘xshash formula bilan aniqlash maqsadida

$$E + At + \frac{A^2}{2!}t^2 + \ldots + \frac{A^k}{k!}t^k + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k \qquad (10.2.3)$$

$(A^0 = E - n \times n$ o‘lchamli birlik matritsa) ko‘rinishdagi matritsaviy koeffitsientli darajali qatorni qaraylik.

Teorema 1. *Ushbu* $\sum_{k=0}^{\infty} \frac{A^k}{k!}t^k$ *darajali qator ixtiyoriy* $|t| \le \delta$ ($\delta > 0$) *oraliqda tekis va absolyut yaqinlashuvchi.*

🗝 Tekis yaqinlashish uchun Koshi mezonidan foydalanamiz. Ma’lumki, ushbu $\sum_{k=0}^{\infty} \frac{(\|A\| \cdot \delta)^k}{k!}$ sonli qator yaqinlashuvchi (uning yig‘indisi $\exp(\|A\|\delta)$ ga teng). Demak, u fundamental, ya’ni ixtiyoriy $\varepsilon > 0$ songa ko‘ra shunday ν natural son topiladiki, barcha $m > \nu$ nomerlar va ixtiyoriy $p \in \mathbb{N}$ uchun $\sum_{k=m+1}^{m+p} \frac{(\|A\| \cdot \delta)^k}{k!} < \varepsilon$ bo‘ladi.

$\sum_{k=0}^{\infty} \frac{A^k}{k!}t^k$ va $\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!}t^k \right\|$ funksional qatorlar $|t| \le \delta$ oraliqda tekis yaqinlashishning Koshi mezonini qanoatlantiradi. Haqiqatan ham, o‘sha ε, ν, m, p sonlar uchun

$$\left\| \sum_{k=m+1}^{m+p} \frac{A^k}{k!}t^k \right\| \le \sum_{k=m+1}^{m+p} \left\| \frac{A^k}{k!}t^k \right\| = \sum_{k=m+1}^{m+p} \frac{\|A^k\| \cdot |t|^k}{k!} \le$$

$$\le \sum_{k=m+1}^{m+p} \frac{\|A\|^k \cdot |t|^k}{k!} \le \sum_{k=m+1}^{m+p} \frac{\|A\|^k \cdot \delta^k}{k!} < \varepsilon. \text{ 👍}$$

Demak, $\sum_{k=0}^{\infty} \frac{A^k}{k!}t^k$ qatorning yig‘indisi $\forall t \in \mathbb{R}$ uchun ma'noga ega.

Endi eksponensial matritsaning ta’rifini kiritamiz:

$$e^{At} \overset{def}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k, \; t \in \mathbb{R}. \qquad (10.2.4)$$

Teorema 2. Quyidagi differensiallash formulasi o‘rinli:

$$\frac{d}{dt}e^{At} = Ae^{At}. \qquad (10.2.5)$$

🗝Ixtiyoriy chegaralangan oraliqda $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ qator tekis yaqinlashuvchi. Bundan tashqari, qator hadlarining hosilalaridan tuzilgan ushbu

$$\sum_{k=1}^{\infty}\frac{A^k}{(k-1)!}t^{k-1}=\sum_{k=0}^{\infty}\frac{A^{k+1}}{k!}t^k=A\cdot\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$$

qator ham ixtiyoriy chegaralangan oraliqda tekis yaqinlashuvchi bo‘lgani uchun analizdan ma’lun teoremaga ko‘ra $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ qatorni hadma-had differensiallash mumkin:

$$\frac{d}{dt}e^{At}=\frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k=\sum_{k=1}^{\infty}\frac{A^k}{(k-1)!}t^{k-1}=A\cdot\sum_{m=0}^{\infty}\frac{A^m}{m!}t^m=A\cdot e^{At}.$$ 👍

Natija. $X:\mathbb{R}\to\mathbb{M}_{n\times n}(\mathbb{C}),\ X=\exp(At),$ *matritsaviy funksiya ushbu*

$$\begin{cases}X'=AX\\ X(0)=E\end{cases}$$

boshlang‘ich masalaning yagona yechimini, $\boldsymbol{x}=\exp(At)\boldsymbol{x}^0$ *vektor-funksiya esa*

$$\begin{cases}\boldsymbol{x}'=A\boldsymbol{x}\\ \boldsymbol{x}(0)=\boldsymbol{x}^0\end{cases}$$

Koshi masalasining yagona yechimini ifodalaydi. Demak, $\exp(At)$ – (10.2.1) *sistemaning fundamental matritsasi.*

Bu natijadan foydalanib $\exp(At)$ eksponensial matritsni hisoblash mumkin. Buning uchun $\boldsymbol{x}'=A\boldsymbol{x}$ sistemaning bazis yechimlarini topib, $\Phi(t)$ fundamental matritsani quramiz. $\exp(At)$ ham fundamental matritsa, u $\Phi(t)$ fundamental matritsani o‘ngdan biror teskarilanuvchi C matritsaga ko‘paytirishdan hosil bo‘lishi kerak: $\exp(At)=\Phi(t)C$. Lekin, $\exp(A\cdot 0)=E$. Bundan $C=\Phi^{-1}(0)$ ni topamiz. Demak, $\mathbf{x}'=A\mathbf{x}$ sistemaning $\Phi(t)$ fundamental matritsasi orqali $\exp(At)$ eksponensial matritsa ushbu

$$\exp(At)=\Phi(t)\Phi^{-1}(0) \tag{10.2.6}$$

formula bilan hisoblanadi.

Bundan oldingi paragrafda bazis yechimlarning va fundamental matritsaning tuzilishini aniqlagan edik. Bunga va (10.2.6) formulaga ko‘ra eksponensial matritsa elementlarining tuzilishi to‘g‘risidgi quyidagi teoremalarni hosil qilamiz.

Teorema 3. *Har qanday* $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ *kompleks matritsa uchun* e^{tA} $(t \in \mathbb{R})$ *eksponensial matritsaning barcha elementlari* $\sum_{j=1}^{s} p_j(t) e^{\lambda_j t}$ *ko‘rinishga ega, bunda* $p_j(t)-$ *kompleks koeffitsientli ko‘phadlar,* $\deg p_j(t) \le \tilde{k}_j - 1$, $\tilde{k}_j$ *bilan* A *matritsaning* k_j *karrali* λ_j *xos qiymatiga mos kelgan Jordan kataklarining eng katta tartibi belgilangan* ($\tilde{k}_j \le k_j$), $s-$ (*turli*) *xos qiymatlar soni.*

Agar A matritsa haqiqiy bo‘lsa, $e^{t(\alpha+i\beta)} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$ Eyler formulasini hisobga olib, eksponensial matritsaning elementlari haqiqiy ekanligini ko‘ramiz.

Teorema 4. *Ixtiyoriy* $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ *haqiqiy matritsa uchun* e^{tA} $(t \in \mathbb{R})$ *eksponensial matritsaning barcha elementlari ushbu*

$$\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right) \qquad (10.2.7)$$

ko‘rinishga ega. Bu yerda $\alpha_j + i\beta_j,\ j = \overline{1,s},$ – A *matritsaning turli xos qiymatlari;* $p_j(t)$ *va* $q_j(t)-$ *haqiqiy koeffitsientli ko‘phadlar; agar* $\alpha_j + i\beta_j$ *xos songa mos kelgan Jordan kataklarining eng katta tartibi* $\tilde{k}_j$ *bo‘lsa, u holda bu* $p_j(t)$ *va* $q_j(t)$ *ko‘phadlarning darajalari* $\tilde{k}_j - 1$ *dan oshmaydi.*

e^{tA} matritsaning ko‘rinishini bilgan holda uni hisoblash mumkin. Buning uchun uning (10.2.5) elementlarini noma’lum koeffitsientlar orqali yozib, ushbu $(e^{tA})' = Ae^{tA},\ e^{0\cdot A} = E$, ayniyatdan noma’lum koeffitsientlar uchun chiziqli tenglamalarni tuzib, ularni yechish kerak. Lekin bu usul umumiy holda uzoq hisoblashlarni talab etadi.

e^{tA} matritsani hisoblashning yana bir usuli to‘g‘risida 4- va 5-masalalarga qarang.

Misol. Ushbu

$$A=\begin{pmatrix}0 & -4 & -2\\ -1 & 0 & -1\\ 1 & 2 & 3\end{pmatrix}$$

matritsa uchun e^{At} matritsani hisoblang.

🗝 Hisoblashni (10.2.6) formulaga ko‘ra bajaramiz. Ushbu

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}'=A\begin{pmatrix}x\\ y\\ z\end{pmatrix},\ A=\begin{pmatrix}0 & -4 & -2\\ -1 & 0 & -1\\ 1 & 2 & 3\end{pmatrix} \qquad (10.2.8)$$

sistemani tuzamiz va uning $\Phi(t)$ fundamental matritsasini qurib, $\exp(At)=\Phi(t)\Phi^{-1}(0)$ formulaga ko‘ra izlangan matritsani topamiz. Tuzilgan (10.2.8) sistemani yechib (oldinga paragrafdagi misol 1), fundamental matritsani tuzgan edik:

$$\Phi(t)=\begin{pmatrix}2e^{-t} & e^{2t} & 0\\ e^{-t} & 0 & e^{2t}\\ -e^{-t} & -e^{2t} & -2e^{2t}\end{pmatrix}.$$

Endi quyidagi matritsalarni ketma-ket hisoblaymiz:

$$\Phi(0)=\begin{pmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ -1 & -1 & -2\end{pmatrix} \quad \text{va} \quad \Phi^{-1}(0)=\frac{1}{3}\begin{pmatrix}1 & 2 & 1\\ 1 & -4 & -2\\ -1 & 1 & -1\end{pmatrix}$$

$$e^{At}=\Phi(t)\Phi^{-1}(0)=\begin{pmatrix}\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t} & -\frac{4}{3}e^{2t}+\frac{4}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{2}{3}e^{-t}\\ -\frac{1}{3}e^{2t}+\frac{1}{3}e^{-t} & \frac{1}{3}e^{2t}+\frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t}+\frac{1}{3}e^{-t}\\ \frac{1}{3}e^{2t}-\frac{1}{3}e^{-t} & \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & \frac{4}{3}e^{2t}-\frac{1}{3}e^{-t}\end{pmatrix}. \ 👍$$

Masalalar

1. Berilgan A matritsaga ko‘ra $\exp(tA)$ ni hisoblang

a) $A=\begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$ **b)** $A=\begin{pmatrix}3 & 2 & -1\\ -1 & 3 & 1\\ 0 & 1 & 2\end{pmatrix}$

2. Formulani isbotlang:

$$e^A=\lim_{k\to\infty}\left(E+\frac{1}{k}A\right)^k \quad (k\in\mathbb{N}, A\in\mathbb{M}_{n\times n}(\mathbb{C}), E\in\mathbb{M}_{n\times n}(\mathbb{C})-\text{birlik matritsa}).$$

3. $e^{\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)}=\mathrm{diag}(e^{\lambda_1},e^{\lambda_2},\ldots,e^{\lambda_n})$ tenglikning to‘g‘riligini ko‘rsating.

4. Matritsa eksponentasining quyidagi xossalarini isbotlang:

1⁰. agar $\{A,B,S\}\subset\mathbb{M}_{n\times n}(\mathbb{C})$ matritsalar uchun $A=SBS^{-1}$ bo‘lsa, $e^A=Se^BS^{-1}$ tenglik o‘rinli;

2⁰. agar $AB=BA$ bo‘lsa, $e^{A+B}=e^Ae^B$ bo‘ladi;

3⁰. $(e^A)^{-1}=e^{-A}$.

5. A matritsa S yordamida Jordan ko‘rinishi J ga keltirilsin: $A=SJS^{-1}$. J matritsani tashkil etuvchi Jordan kataklarini $J_1,J_2,\ldots,J_m$ deylik: $J=\mathrm{diag}(J_1,J_2,\ldots,J_m)$.

a) Ushbu $e^{At}=S\cdot\mathrm{diag}(e^{J_1t},e^{J_2t},\ldots,e^{J_mt})\cdot S^{-1}$ formulani isbotlang.

b) Jordan katagi

$$J_{\mu,p}=\begin{pmatrix}\mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \\ & & & & \mu\end{pmatrix}_{p\times p}$$

ning k - darajasini hisoblang ($k>p$).

c)
$$e^{tJ_{\mu,p}}=e^{\mu t}\begin{pmatrix}1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{p-1}}{(p-1)!} \\ 0 & 1 & \frac{t}{1!} & \cdots & \frac{t^{p-2}}{(p-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{t}{1!} \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$
formulani usbotlang.

6. Agar $\{A,B\}\subset\mathbb{M}_{n\times n}(\mathbb{C})$ matritsalar uchun $e^{A+B}=e^Ae^B$ bo‘lsa, $AB=BA$ ekanligini ko‘rsating.

7. $t>0$ uchun Quyidagi tengsizlikni isbotlang (matritsa normasi uchun $\|E\|=1$) :

$$e^{-\|A\|t}\le\left\|e^{At}\right\|\le e^{\|A\|t}.$$

8. Aytaylik, $f:(-r,r)\to\mathbb{R}$ funksiya darajali qatorga yoyilsin:

$$f(t)=a_0+a_1t+a_2\frac{t^2}{2!}+\ldots+a_k\frac{t^k}{k!}+\ldots\ ,\ |t|<r,$$

$A\in\mathbb{M}_{n\times n}(\mathbb{R})$ matritsaning barcha $\lambda=\lambda_j$ xos sonlari uchun esa $|\lambda_j|<r$ tengsizlik o‘rinli bo‘lsin. U holda

$$f(A)=a_0+a_1A+\frac{a_2}{2!}A^2+\ldots+\frac{a_k}{k!}A^k+\ldots\ ,$$

matritsaviy qatorning absolyut yaqinlashuvchi ekanligini ko‘rsating.

9. $A\in\mathbb{M}_{n\times n}(\mathbb{C})$ va $t\in\mathbb{R}$ bo‘lsin.

1^0. $\cos tA=\sum_{j=0}^{\infty}(-1)^j\frac{t^{2j}}{(2j)!}A^{2j}$ va $\sin tA=\sum_{j=0}^{\infty}(-1)^j\frac{t^{2j+1}}{(2j+1)!}A^{2j+1}$ matritsaviy funksiyalarning aniqlangan ekanligini asoslang;

2^0. $\frac{d}{dt}(\cos tA)$ va $\frac{d}{dt}(\sin tA)$ hosilalarni hisoblang;

3^0. ushbu

$$\begin{cases}\boldsymbol{x}'=-A\boldsymbol{y}\\ \boldsymbol{y}'=A\boldsymbol{x}\end{cases}\quad (\boldsymbol{x}=\boldsymbol{x}(t)\in\mathbb{R}^n,\boldsymbol{y}=\boldsymbol{y}(t)\in\mathbb{R}^n)$$

sistemani yeching.

10. Ushbu $\boldsymbol{x}'=a(t)A\boldsymbol{x}$ $\left(a(t)\in C(I,\mathbb{R}),A\in\mathbb{M}_{n\times n}(\mathbb{C})\right)$ sistema uchun fundamental matritsa $\Phi(t)=\exp\left(\int a(t)dtA\right)$ bo‘lishini ko‘rsating.

11. $\exp(At)=\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$, $A\in\mathbb{M}_{n\times n}(\mathbb{C})$, matritsviy darajali qatorning ixtiyoriy $|t|\le\delta$ $(\delta>0)$ oraliqda tekis va absolyut yaqinlashuvchiligini shu matritsani tashkil etuvchi barcha skalyar darajali qatorlarning tekis yaqinlashubchiligi orqali isbotlang.

12. Ushbu

$$\boldsymbol{x}'=\frac{1}{t}A\boldsymbol{x},\ A\in A\subset\mathbb{M}_{n\times n}(\mathbb{C}),$$

Eyler sistemasida $\tau=\ln t\ (t>0)$ almashtirish bajaring va uni yeching.

13. Ushbu

$$\begin{cases}X'=AX+XB\\ X(0)=C\end{cases},\ \{A,B,C\}\subset\mathbb{M}_{n\times n}(\mathbb{C}),$$

Koshi masalasining yechimi $X=e^{tA}Ce^{tB}$ formula bilan berilishini isbotlang.

14. Agar $X(t)$ kvadrat matritsa $t=0$ nuqtada differensiallanuvchi, $\det X(0)\ne 0$ va $X(t+s)=X(t)X(s),\{t,s\}\subset\mathbb{R},$ bo‘lsa, $X(t)=e^{tX'(0)}$ bo‘lishini ko‘rsating. Polia bu tasdiqning uzluksiz $X(t)$ matritsa uchun ham o‘rinli ekanligini isbotlagan.

§ 10.3. Bir jinsli bo‘lmagan sistemalarni yechish

Bir jinsli bo‘lmagan ushbu

$$\boldsymbol{x}'=A\boldsymbol{x}+\boldsymbol{f}(t)\ \left(A\in\mathbb{M}_{n\times n}(\mathbb{C}),\ \boldsymbol{f}(t)\in C(I,\mathbb{C}^n)\right)\qquad(10.3.1)$$

sistemani qaraylik. Bu sistemani osongina yechish mumkin. Yechimni $\boldsymbol{x}=e^{At}\boldsymbol{z}(t)$ ko‘rinishda izlasak,

$$Ae^{At}\boldsymbol{z}(t)+e^{At}\boldsymbol{z}'(t)=Ae^{At}\boldsymbol{z}(t)+\boldsymbol{f}(t) \Rightarrow e^{At}\boldsymbol{z}'(t)=\boldsymbol{f}(t) \Rightarrow \boldsymbol{z}'(t)=e^{-At}\boldsymbol{f}(t),$$

$$\boldsymbol{z}=\int_{t_0}^{t} e^{-sA}\boldsymbol{f}(s)\,ds+\boldsymbol{c}$$

ekanligini topamiz. Demak, (10.3.1) sistemaning umumiy yechimi e^{At} eksponensial matritsa yordamida quyidagicha ifodalanadi:

$$\boldsymbol{x}=\int_{t_0}^{t} e^{(t-s)A}\boldsymbol{f}(s)\,ds+e^{tA}\boldsymbol{c} \quad (t_0 \in I- \text{ tayinlangan}) . \tag{10.3.2}$$

Bu yerdagi $\int_{t_0}^{t} e^{(t-s)A}\boldsymbol{f}(s)\,ds$ qo‘shiluvchi (10.3.1) tenglamaning xususiy yechimi, $e^{tA}\boldsymbol{c}$ qo‘shiluvchi esa mos bir jinsli tenglamaning umumiy yechimi ($\boldsymbol{c}-$ixtiyoriy o‘zgarmas vektor. Bu xususiy yechimni topish uchun integrallash amali ishlatiladi. Lekin, agar $\boldsymbol{f}(t)$ ozod had kvaziko‘phaddan iborat bo‘lsa, xususiy yechimni integrallash amalini ishlatmasdan turib, noma’lum koeffitsientlar metodi yordamida qurish mumkin. Kvaziko‘phad (kompleks sohada) deb

$$\boldsymbol{f}(t)=e^{\gamma t}\boldsymbol{p}(t) \tag{10.3.3}$$

ko‘rinishdagi vektor-funksiyaga aytiladi; bu yerda $\gamma \in \mathbb{C}$ va $\boldsymbol{p}(t)=\boldsymbol{b}^m t^m+\boldsymbol{b}^{m-1}t^{m-1}+...+\boldsymbol{b}^1 t+\boldsymbol{b}^0$ – (kompleks) vektor koeffitsientli ko‘phad.

Noma’lum koeffitsientlar metodini asoslash maqsadida qaralayotgan

$$\boldsymbol{x}'=A\boldsymbol{x}+e^{\gamma t}\boldsymbol{p}(t) \tag{10.3.4}$$

sistemada $\boldsymbol{x}=S\boldsymbol{y}$ almashtirish bajaramiz; bu yerda $S-$ A matritsani Jordan kanonik ko‘rinishiga keltiruvchi matritsa: $S^{-1}AS=J-$ Jordan matritsasi. Natijada (10.3.4) sistema quyidagi sodda ko‘rinishga keladi:

$$\boldsymbol{y}'=J\boldsymbol{y}+e^{\gamma t}\tilde{\boldsymbol{p}}(t)\ ,\ \tilde{\boldsymbol{p}}(t)=S^{-1}\boldsymbol{p}(t), \left(\deg\tilde{\boldsymbol{p}}(t)=\deg\boldsymbol{p}(t)\right). \tag{10.3.5}$$

Bu yerdagi tenglamalar Jordan kataklariga mos ravishda bir-biridan mustaqil kichik sistemalarga ajraladi.

Tipik kichik sistema quyidagi ko‘rinishda bo‘ladi:

$$\begin{cases} y_1' = \lambda_1 y_1 + y_2 + e^{\gamma t}\tilde{p}_1(t), \\ y_2' = \lambda_1 y_2 + y_3 + e^{\gamma t}\tilde{p}_2(t), \\ \dots\dots\dots\dots\dots\dots\dots \\ y_{n_1-1}' = \lambda_1 y_{n_1-1} + y_{n_1} + e^{\gamma t}\tilde{p}_{n_1-1}(t), \\ y_{n_1}' = \lambda_1 y_{n_1} + e^{\gamma t}\tilde{p}_{n_1}(t). \end{cases}$$

Bu yerdagi tenglamalarni $\mu = e^{-\lambda_1 t}$ integrallovchi ko‘paytuvchiga ko‘paytiramiz va shakl almashtiramiz:

$$\begin{cases} \left(y_1 e^{-\lambda_1 t}\right)' = e^{-\lambda_1 t} y_2 + e^{(\gamma-\lambda_1)t}\tilde{p}_1(t), \\ \left(y_2 e^{-\lambda_1 t}\right)' = e^{-\lambda_1 t} y_3 + e^{(\gamma-\lambda_1)t}\tilde{p}_2(t), \\ \dots\dots\dots\dots\dots\dots\dots \\ \left(y_{n_1-1} e^{-\lambda_1 t}\right)' = e^{-\lambda_1 t} y_{n_1} + e^{(\gamma-\lambda_1)t}\tilde{p}_{n_1-1}(t), \\ \left(y_{n_1} e^{-\lambda_1 t}\right)' = e^{(\gamma-\lambda_1)t}\tilde{p}_{n_1}(t). \end{cases} \quad (10.3.6)$$

Oxirgi sistema quyidan yuqoriga qarab osongina yechiladi. Lekin, topiladigan xususiy yechim ko‘rinishi $\gamma \neq \lambda_1$ va $\gamma = \lambda_1$ hollarida har xil bo‘ladi.

Dastlab faraz qilaylik, $\gamma \neq \lambda_1$ bo‘lsin. U holda (10.3.6) sistemaning oxirgi tenglamasidan

$$y_{n_1} = e^{-\lambda_1 t}\int e^{(\gamma-\lambda_1)t}\tilde{p}_{n_1}(t)dt \quad (10.3.7)$$

ni topamiz. Bu yerdagi inegralni keraklicha marta $e^{(\gamma-\lambda_1)t}$ ni differensial ostiga kiritib, bo‘laklab integrallaymiz va y_{n_1} ni topamiz:

$$y_{n_1} = e^{\gamma t}\hat{p}_{n_1}(t); \quad (10.3.8)$$

bu yerda, ravshanki, $\hat{p}_{n_1}(t)$ ko‘phadning darajasi $\tilde{p}_{n_1}(t)$ ko‘phadning darajasi bilan bir xil bo‘ladi, $\deg\hat{p}_{n_1}(t) = \deg\tilde{p}_{n_1}(t) \leq \deg\boldsymbol{p}(t)$. (10.3.6) sistemaning oxiridan ikkinchi tenglamasi endi

$$\left(y_{n_1-1} e^{-\lambda_1 t}\right)' = e^{(\gamma-\lambda_1)t}\hat{p}_{n_1}(t) + e^{(\gamma-\lambda_1)t}\tilde{p}_{n_1-1}(t)$$

ko‘rinishga keladi. Bu tenglamani yechib, y_{n_1-1} ni topamiz:

$$y_{n_1-1} = e^{\gamma t}\,\widehat{p}_{n_1-1}(t),\ \deg \widehat{p}_{n_1-1}(t) \le \deg \boldsymbol{p}(t).$$

(10.3.6) sistemadagi qolgan tenglamalarni ham shunga o‘xshash yechib, topamiz:

$$y_j = e^{\gamma t}\,\widehat{p}_j(t),\ \deg \widehat{p}_j(t) \le \deg \boldsymbol{p}(t),\ j = \overline{1, n_1}, \qquad (10.3.8)$$

Yana faraz qilaylikki, γ A matritsaning xarakteristik soni bo‘lmasin, ya’ni barcha j lar uchun $\gamma \ne \lambda_j$. U holda (10.3.5) sistemadagi barcha mustaqil kichik sistemalar (10.3.6) ga o‘xshash yechiladi. Natijada

$$\boldsymbol{y} = e^{\gamma t}\,\widehat{\boldsymbol{p}}(t),\ \deg \widehat{\boldsymbol{p}} \le \deg \boldsymbol{p}(t),$$

yechim hosil bo‘ladi.

$\boldsymbol{x} = S\boldsymbol{y}$ o‘zgaruvchiga qaytib, γ son A matritsaning xarakteristik soni bo‘lmasa, (10.3.4) sistema

$$\boldsymbol{x} = e^{\gamma t}\boldsymbol{q}(t),\ \deg \boldsymbol{q}(t) \le \deg \boldsymbol{p}(t) \qquad (10.3.9)$$

ko‘rinishdagi yechimga ega bo‘lishini aniqlaymiz.

Endi faraz qilaylik, $\gamma = \lambda_1$ (yoki biror j uchun $\gamma = \lambda_j$) bo‘lsin. U holda (10.3.6) sistemaning oxirgi tenglamasi $\left(y_{n_1} e^{-\lambda_1 t}\right)' = \tilde{p}_{n_1}(t)$ ko‘rinishga keladi va u quyidagi ko‘rinishdagi yechimga ega bo‘ladi:

$$y_{n_1} = e^{\gamma t}\,\widehat{p}_{n_1}(t),\ \deg \widehat{p}_{n_1}(t) = \deg \tilde{p}_{n_1}(t) + 1 \le \deg \boldsymbol{p}(t) + 1.$$

(10.3.6) sistemaning qolgan tenglamalarini ham yechib (pastdan yuqoriga qarab), quyidagi ko‘rinishdagi yechimni hosil qilamiz:

$$y_j = e^{\gamma t}\,\widehat{p}_j(t),\ \deg \widehat{p}_j(t) \le \deg \boldsymbol{p}(t) + n_1,\ j = \overline{1, n_1};$$

bu yerda $n_1 \le \tilde{k}_1$ va $\tilde{k}_1$ son λ_1 xos songa mos kelgan Jordan kataklarining eng katta o‘lchami, tushunarliki, u λ_1 ning karralilik darajasi k_1 dan katta emas, $\tilde{k}_1 \le k_1$.Endi (10.3.5) sistemaning qolgan mustaqil kichik sistemalarini yechamiz va uning

$$\boldsymbol{y} = e^{\gamma t}\,\widehat{\boldsymbol{p}}(t),\ \deg \widehat{\boldsymbol{p}} \le \deg \boldsymbol{p}(t) + k_1,$$

ko‘rinishdagi yechimini topamiz.

Dastlabki $\boldsymbol{x} = S\boldsymbol{y}$ o‘zgaruvchiga qaytib, γ son k_1 karrali xarakteristik son bo‘lganda dastlabki (10.3.4) sistemaning

$$\boldsymbol{x} = e^{\gamma t}\boldsymbol{q}(t),\ \deg \boldsymbol{q}(t) \le \deg \boldsymbol{p}(t) + k_1$$

yechimini topamiz.

Tekshirishlarimizni xulosalab, xususiy yechimni topish algoritmini keltiramiz:
agar ozod hadi kvaziko‘phaddan iborat bo‘lgan (10.3.4) sistemadagi γ son xarakteristik son bo‘lmasa, (10.3.4) sistema

$$\boldsymbol{x} = e^{\gamma t}\boldsymbol{q}(t),\ \deg\boldsymbol{q}(t) \le \deg\boldsymbol{p}(t)\ ,$$

ko‘rinishdagi, γ son k karrali xarakteristik son bo‘lganda esa, bu sistema

$$\boldsymbol{x} = e^{\gamma t}\boldsymbol{q}(t),\ \deg\boldsymbol{q}(t) \le \deg\boldsymbol{p}(t) + k$$

ko‘rinishdagi yechimga ega.

Masalalar yechganda yechimning ko‘rinishini noma’lum koeffitsientlar orqali yozib, noma’lumlarni sistemaning qanoatlanishidan aniqlash kerak.

Agar haqiqiy sohada berilgan (10.3.4) sistemada $\boldsymbol{f}(t)$ ozod had

$$\boldsymbol{f}(t) = e^{\alpha t}\left(\boldsymbol{p}(t)\cos\beta t + \boldsymbol{q}(t)\sin\beta t\right)$$

(haqiqiy) kvaziko‘phaddan iborat bo‘lsa $\big(A-$haqiqiy matritsa, $\alpha,\beta \in \mathbb{R}$, $\boldsymbol{p}(t), \boldsymbol{q}(t)-$ vektor koeffitsientli ko‘phadlar$\big)$, u holda Eyler formulasidan topilgan

$$e^{\alpha t}\cos\beta t = \mathrm{Re}(e^{(\alpha+i\beta)t})\ ,\ \sin\beta t = \mathrm{Re}(-ie^{(\alpha+i\beta)t})$$

munosabatlarga ko‘ra

$$\boldsymbol{f}(t) = \boldsymbol{p}(t)e^{\alpha t}\cos\beta t + \boldsymbol{q}(t)e^{\alpha t}\sin\beta t = \boldsymbol{p}(t)\mathrm{Re}(e^{(\alpha+i\beta)t}) + \boldsymbol{q}(t)\mathrm{Re}(-ie^{(\alpha+i\beta)t}) =$$
$$= \mathrm{Re}\big(e^{(\alpha+i\beta)t}(\boldsymbol{p}(t) - i\boldsymbol{q}(t))\big)$$

ifodalashdan kelib chiqib, haqiqiy sohada berilgan (10.3.4) sistema o‘rniga

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\left(\boldsymbol{p}(t) - i\boldsymbol{q}(t)\right),\ \gamma = \alpha + i\beta,$$

sistemani tuzib, uning xususiy yechimini yuqorida aytilgan algoritmga ko‘ra topib, topilgan yechimning haqiqiy qismini ajratish kerak.

Bu holda xususiy yechimni kompleks sohaga chiqmasdan to‘g‘ridan to‘g‘ri qursa ham bo‘ladi. (10.3.4) sistemaning xususiy yechimini

$$\boldsymbol{x}(t) = e^{\alpha t}\left(\boldsymbol{m}(t)\cos\beta t + \boldsymbol{n}(t)\sin\beta t\right)$$

ko‘rinishda izlash mumkin. Bu yerda $\boldsymbol{m}(t),\boldsymbol{n}(t)-$vektor koeffitsientli ko‘phadlar va $\max\left(\deg\boldsymbol{m},\deg\boldsymbol{n}\right) = k + \max\left(\deg\boldsymbol{p},\deg\boldsymbol{q}\right)$, k bilan A matritsaning $\alpha + i\beta$ xarakteristik sonining karralilik darajasi belgilangan; $\alpha + i\beta$ xarakteristik son bo‘lmaganda esa $k = 0$ deb hisoblanadi. Bunday ko‘rinishdagi yechimni topish uchun uni (10.3.4) tenglamaga qo‘yib, hosil bo‘lgan ayniyatdan noma’lum koeffitsientlarni topish kerak.

Agar berilgan sistemadaagi ozod had ikki (yoki undan ortiq) kvaziko‘phad yig‘indisidan iborat, ya’ni

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\boldsymbol{p}(t) + e^{\omega t}\boldsymbol{r}(t)$$

bo‘lsa, bu sistemadan ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\boldsymbol{p}(t), \ \boldsymbol{x}' = A\boldsymbol{x} + e^{\omega t}\boldsymbol{r}(t)$$

sistemalarni tuzib, ularning xususiy yechimlarini qo‘shib, berilgan sistemaning xususiy yechimini topish mumkin.

Misol 10. Quyidagi sistemani yeching:

$$\begin{cases} x' = 2x - 2y + 18te^t, \\ y' = -x + 3y - 3e^t. \end{cases} \qquad (10.3.10)$$

🗝 Chiziqli sistemaning umumiy yechimi uning xususiy yechimiga mos bir jinsli sistemaning umumiy yechimini qo‘shishdan hosil bo‘ladi.

Dastlab mos bir jinsli sistemani tuzib, umumiy yechimini topamiz:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}; \qquad (10.3.11)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \qquad (10.3.12)$$

Endi (10.3.10) bir jinsli bo‘lmagan sistemaning xususiy yechimini topamiz. Bizda

$\begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix} + e^t\boldsymbol{p}(t)$; $\gamma = 1$, $\boldsymbol{p}(t) = t\begin{pmatrix} 18 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} -$ birinchi darajali ko‘phad ($m = 1$).

$\gamma = 1-$ bir karrali ($k = 1$) xos son bo‘lgani uchun xususiy yechimni

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t\boldsymbol{q}(t), \ \boldsymbol{q}(t) = \begin{pmatrix} a \\ b \end{pmatrix} t^2 + \begin{pmatrix} c \\ d \end{pmatrix} t + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad (10.3.13)$$

ikkinchi darajali ko‘phad ($m+k=1+1=2$) ko‘rinishida izlaymiz; bu yerda $a,b,c,d,\alpha,\beta-$ hozircha noma’lum sonlar. (10.3.13) ni qaralayotgan sistemaga qo‘yamiz:

$$e^t\left(\begin{pmatrix}a\\b\end{pmatrix}t^2+\begin{pmatrix}c\\d\end{pmatrix}t+\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right)+e^t\left(\begin{pmatrix}a\\b\end{pmatrix}2t+\begin{pmatrix}c\\d\end{pmatrix}\right)=Ae^t\left(\begin{pmatrix}a\\b\end{pmatrix}t^2+\begin{pmatrix}c\\d\end{pmatrix}t+\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right)+$$
$$+e^t\left(t\begin{pmatrix}18\\0\end{pmatrix}+\begin{pmatrix}0\\-3\end{pmatrix}\right).$$

Bu tenglik $t\in\mathbb{R}$ ga nisbatan ayniyat bo‘lishi kerak. Bu tenglikni e^t bo‘lib, t ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, quyidagi tenglamalarni hosil qilamiz:

$$\begin{pmatrix}a\\b\end{pmatrix}=A\begin{pmatrix}a\\b\end{pmatrix},\ \begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}2=A\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}18\\0\end{pmatrix},\ \begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=A\begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}0\\-3\end{pmatrix}.$$

Birinchi vektorli tenglamadan

$$\begin{pmatrix}a\\b\end{pmatrix}=A\begin{pmatrix}a\\b\end{pmatrix}\Rightarrow a=2b. \qquad (10.3.14)$$

Ikkinchi vektorli tenglamadan

$$\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}2=A\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}18\\0\end{pmatrix}\Rightarrow\begin{cases}c+2a=2c-2d+18\\d+2b=-c+3d\end{cases}\overset{(10.3.14)}{\Rightarrow}$$

$$\begin{cases}2d-c=18-4b\\2d-c=2b\end{cases}\Rightarrow\begin{cases}b=3\\2d-c=6\end{cases} \qquad (10.3.15)$$

Uchinchi vektorli tenglamadan

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=A\begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}0\\-3\end{pmatrix}\Rightarrow\begin{cases}\alpha+c=2\alpha-2\beta\\\beta+d=-\alpha+3\beta-3\end{cases}\overset{(10.3.15)}{\Rightarrow}$$

$$\begin{cases}2\beta-\alpha=6-2d\\2\beta-\alpha=3+d\end{cases}\Rightarrow\begin{cases}d=1\\2\beta-\alpha=4\end{cases} \qquad (10.3.16)$$

$\alpha=0$ tanlab, (10.3.16)–(10.3.14) tenglamalardan $\beta=2$, $d=1$, $c=-4$, $b=3$, $a=6$ qiymatlarni topamiz. Topilgan qiymatlarni (10.3.13) ga qo‘yib, (10.3.10) sistemaning quyidagi xususiy yechimini topamiz:

$$\begin{pmatrix}x\\y\end{pmatrix}=e^t\left(\begin{pmatrix}6\\3\end{pmatrix}t^2+\begin{pmatrix}-4\\1\end{pmatrix}t+\begin{pmatrix}0\\2\end{pmatrix}\right). \qquad (10.3.17)$$

Endi (10.3.17) xususiy yechimni mos bir jinsli tenglamaning umumiy yechimi (10.3.12) ga qo‘shib, berilgan (10.3.10) sistemaning umumiy yechimini topamiz

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t\left(\begin{pmatrix} 6 \\ 3 \end{pmatrix}t^2 + \begin{pmatrix} -4 \\ 1 \end{pmatrix}t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) + c_1e^t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Masalalar

Berilgan **s**istemalarning xususiy yechimi ko‘rinishini yozing.

1. $\begin{cases} x' = x + 2y + 2t^2e^t, \\ y' = 2x + 2y - te^t. \end{cases}$

2. $\begin{cases} x' = x + 2y + e^{5t}, \\ y' = 2x + 2y - 3e^{5t}. \end{cases}$

3. $\begin{cases} x' = 2x - y + z + 5e^{2t}, \\ y' = 2y - z - 2te^{2t}, \\ z' = y + 4z - 3e^{2t}. \end{cases}$

4. $\begin{cases} x' = 2x - y + z + e^{2t}(t\cos t - \sin t), \\ y' = 2y - z - 2e^{2t}(2\cos t + t\sin t, \\ z' = y + 2z + 3e^{2t}t\sin t. \end{cases}$

MODUL 11. AVTONOM SISTEMALAR

§ 11.1. Avtonom sistema yechimlarining umumiy xossalari

Ushbu

$$\begin{cases} x_1' = f_1(x_1, x_2, ..., x_n) \\ x_2' = f_2(x_1, x_2, ..., x_n) \\ \dots\dots\dots\dots\dots\dots \\ x_n' = f_n(x_1, x_2, ..., x_n) \end{cases} \quad (11.1.1)$$

differensial tenglamalar sistemasini qaraylik. Bu yerda sistemaning o‘ng tomoni erkli haqiqiy o‘zgaruvchi t ga bog‘liq emas. Shu sababli bu sistema **avtonom (muxtor) sistema** deb ataladi.

Harakati avtonom sistemalar bilan tavsiflanadigan mexanik sistemalarni boshqaruvchi qonunlar va ta’sir etuvchi kuchlar vaqt (t) o‘tishi bilan o‘zgarmaydi, ya’ni ular t vaqtga bog‘liq bo‘lmaydi.

(11.1.1) sistemada erkli o‘zgaruvchi t vaqt deb tushunilganda, bu sistema **dinamik sistema** deb yuritiladi.

(11.1.1) avtonom sistemani

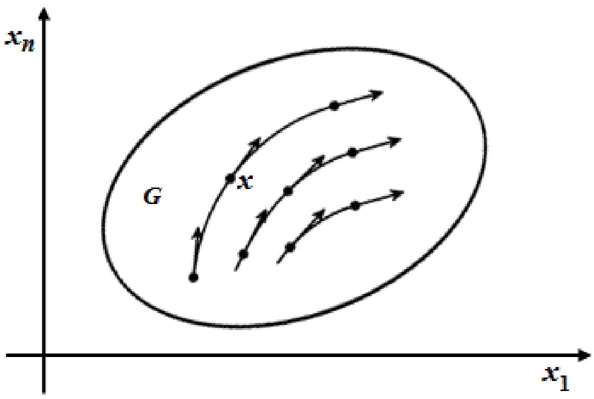
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \quad (11.1.2)$$

vektorli ko‘rinishda ifodalaylik; bu yerda

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \boldsymbol{x}' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \ \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix}, \ \boldsymbol{x} \in G \subset \mathbb{R}^n,$$

$f_j : G \to \mathbb{R}, \ j = \overline{1, n},$

$\boldsymbol{x} = \boldsymbol{\varphi}(t)$ noma’lum funksiya $n \times 1$ o‘lchamli vektor-funksiyadan iborat. Sistemaning o‘ng tomoni $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya aniqlangan $G \subset \mathbb{R}^n$ soha shu sistemaning **fazalar fazosi** deb ataladi. (11.1.2) avtonom sistema G fazalar fazosida $\boldsymbol{f}(\boldsymbol{x})$ vektor maydonni (tezliklar maydonini)



11.1- rasm.

aniqlaydi (11.1-rasm). $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechim fazalar fazosida parametrik ko‘rinishda berilgan egri chiziqni ifodalaydi. Bu egri chiziq (11.1.2) avtonom sistema uchun **trayektoriya** deb ataladi. Traektoriyaning $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ nuqtasidagi tezlik vektori shu nuqtadagi $\boldsymbol{f}(\boldsymbol{x})$ vektorga teng, $\boldsymbol{\varphi}'(t)=\boldsymbol{f}(\boldsymbol{\varphi}(t))$. Traektoriya o‘zining har bir $\boldsymbol{x}$ nuqtasida shu nuqtadagi $\boldsymbol{f}(\boldsymbol{x})$ vektorga urinadi.

Biz bu paragrafda avtonom (dinamik) sistema trayektoriyalarining umumiy xossalarini o‘rganamiz. G sohada $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya o‘zining barcha birinchi tartibli xususiy hosilalari bilan birgalikda uzluksiz, ya’ni $\boldsymbol{f}\in C^1(G;\mathbb{R}^n)$ deb hisoblaymiz. U holda ixtiyoriy $\boldsymbol{x}^0\in G$ uchun $\boldsymbol{x}|_{t_0}=\boldsymbol{x}^0$ shartni qanoatlantiruvchi yagona yechim $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ (yoki aniqroq $\boldsymbol{x}=\boldsymbol{\varphi}(t,t_0,\boldsymbol{x}^0)$) davomsiz yechim mavjud bo‘ladi. Bu $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechim aniqlovchi trayektoriya $t=t_0$ da $\boldsymbol{x}=\boldsymbol{x}^0\in G$ nuqtadan o‘tadi. Vaqt o‘tishi bilan $\boldsymbol{\varphi}(t)\in G$ nuqta traektoriya bo‘ylab harakat qiladi; bu harakat yo‘nalishini rasmda odatda strelka bilan ko‘rsatiladi. (11.1.2) avtonom sistemaning barcha fazaviy traektoriyalari uning fazaviy tasviri (portreti, manzarasi) deyiladi.

Agar $\boldsymbol{a}\in G$ nuqta uchun $\boldsymbol{f}(\boldsymbol{a})=0$ bo‘lsa, $\boldsymbol{a}$ nuqta $\boldsymbol{f}(\boldsymbol{x})$ vektor maydonning maxsus (kritik) nuqtasi deyiladi. Bunda hosil bo‘luvchi $\boldsymbol{x}(t)=\boldsymbol{a}$ o‘zgarmas yechimning fazaviy traektoriyasi $\boldsymbol{a}$ nuqtadan iborat bo‘ladi va u (11.1.2) sistemaning muvozanat nuqtasi yoki muvozanat (statsionar) holati deb ataladi.

Shuni ta’kidlaylikki, integral chiziq yechimning $1+n$ o‘lchamli $\mathbb{R}^{1+n}=\left\{(t,\boldsymbol{x})\,|\,t\in\mathbb{R},\boldsymbol{x}\in\mathbb{R}^n_{\boldsymbol{x}}\right\}$ fazodagi grafigidan iborat bo‘lib, uning $\mathbb{R}^n_{\boldsymbol{x}}$ $(=\mathbb{R}^n)$ (o‘lchami bittaga kam!) fazodagi ortogonal proyeksiyasi fazaviy traektoriyani aniqlaydi. Shuning uchun avtonom sistemani uning traektoriyalari orqali o‘rganish qulay.

Eslatma. Avtonom bo‘lmagan

$$\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})$$

sistemaga qo‘shimcha $y=t$ funksiyani kiritib, uni

$$\begin{cases} y'=1 \\ \boldsymbol{x}'=\boldsymbol{f}(y,\boldsymbol{x}) \end{cases}$$

avtonom ko‘rinishga keltirish mumkin. Lekin bu yerda $y = t$ va $x_1, ..., x_n$ o‘zgaruvchilarning mohiyatlari har xil va fazalar fazosi $(y, \mathbf{x})$ nuqtalar to‘plami bo‘lmish $1+n$ o‘lchamli $\mathbb{R}^{1+n}$ fazo qismidan iborat bo‘ladi.

Avtonom sistema (11.1.2) ning yechimlari va traektoriyalarining ba’zi xossalarini keltiramiz.

1⁰. *Agar $\boldsymbol{x} = \boldsymbol{\varphi}(t)$, $t \in (a;b)$, yechim bo‘lsa, ixtiyoriy $c \in \mathbb{R}$ uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t+c), t \in (a-c; b-c)$, funksiya ham yechim hamda ularning traektoriyalari ustma-ust tushadi.*

🗝 Shartga ko‘ra

$$\boldsymbol{\varphi}'(t) = \boldsymbol{f}(\boldsymbol{\varphi}(t)) \ (t \in (a;b)) \Rightarrow \boldsymbol{\varphi}'(t+c) = \boldsymbol{f}(\boldsymbol{\varphi}(t+c)) \ (t \in (a-c; b-c))$$

va, demak,

$$\frac{d}{dt}\boldsymbol{\varphi}(t+c) = \boldsymbol{\varphi}'(t+c) \cdot 1 = \boldsymbol{f}(\boldsymbol{\varphi}(t+c)) \ (t \in (a-c; b-c)).$$

Bu $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ va $\boldsymbol{x} = \boldsymbol{\varphi}(t+c)$ yechimlarning traektoriyalari bir xil bo‘ladi, chunki $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ traektoriyaning $\boldsymbol{\varphi}(\tilde{t})$ nuqtasidan $\boldsymbol{x} = \boldsymbol{\varphi}(t+c)$ yechim $t = \tilde{t} - c$ bo‘lganda, $\boldsymbol{x} = \boldsymbol{\varphi}(t+c)$ traektoriyaning $\boldsymbol{\varphi}(\tilde{t}+c)$ nuqtasidan esa $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim $t = \tilde{t} + c$ bo‘lganda o‘tadi. 👍

2⁰. *Agar $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ va $\boldsymbol{x} = \boldsymbol{\psi}(t)$ yechimlar uchun $\boldsymbol{\varphi}(t_1) = \boldsymbol{\psi}(t_2)$ bo‘lsa, $\boldsymbol{\varphi}(t) = \boldsymbol{\psi}(t + t_2 - t_1)$ bo‘ladi. Demak, umumiy nuqtaga ega bo‘lgan traektoriyalar ustma – ust tushadi, ya’ni fazalar fazosining har bir nuqtasidan yagona fazaviy traektoriya o‘tadi.*

🗝 Yuqoridagi 1^0 xossaga ko‘ra $\boldsymbol{x} = \boldsymbol{\psi}(t)$ yechim bo‘lganligi uchun $\boldsymbol{x} = \boldsymbol{\psi}(t + t_2 - t_1)$ ham yechim va ularning traektoriyalari bir xil. $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ va $\boldsymbol{x} = \boldsymbol{\psi}(t + t_2 - t_1)$ yechimlarning qiymatlari $t = t_1$ da teng bo‘lganligi uchun yechimning yagonalik xossasiga ko‘ra bu yechimlar ustma-ust tushadi, ya’ni $\boldsymbol{\varphi}(t) = \boldsymbol{\psi}(t + t_2 - t_1)$. 👍

Demak, $\boldsymbol{x} = \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}^0)$ fazaviy traektoriyani $\boldsymbol{x} = \boldsymbol{\varphi}(t - t_0, \boldsymbol{x}^0)$ ko‘rinishda yozish mumkin, ya’ni avtonom sistema uchun vaqt boshi t_0 ahamiyatsiz.

Natija. *Ixtiyoriy ikki traektoriya yo umumiy nuqtaga ega emas, yoki ustma-ust tushadi.*

3^0. *Agar* $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ *traektoriya uchun* $\lim_{t\to+\infty}\boldsymbol{\varphi}(t)=\boldsymbol{a}$ *(*$\boldsymbol{a}=(a_1,a_2,\ldots,a_n)^T\in G\subset\mathbb{R}^n$*) bo'lsa,* $\boldsymbol{a}$ *nuqta muvozanat nuqtadir.*

🗝 Teskarisini faraz qilaylik, ya'ni $\boldsymbol{a}$ nuqta muvozanat (maxsus, kritik) nuqta bo'lmasin. U holda $\boldsymbol{f}(\boldsymbol{a})\neq 0$ bo'ladi. Aniqlik uchun $\boldsymbol{f}(\boldsymbol{a})$ vektorning birinchi koordinatasi noldan farqli, ya'ni $f_1(\boldsymbol{a})\neq 0$ deylik. Fikrlashni yanada aniqlashtirish maqsadida $f_1(\boldsymbol{a})=\alpha>0$ deb ham hisoblaymiz. $\boldsymbol{f}$ uzluksiz bo'lgani uchun $\lim_{t\to+\infty} f_1(\boldsymbol{\varphi}(t))= f_1(\boldsymbol{a})=\alpha>0$. Limitning ta'rifiga ko'ra shunday t_* topiladiki, har qanday $t\geq t_*$ uchun $f_1(\boldsymbol{\varphi}(t))>\alpha/2$ bo'ladi. Demak, shu $t\geq t_*$ har uchun $\varphi_1'(t)=f_1(\boldsymbol{\varphi}(t))>\alpha/2$. Buni integrallab, $\varphi_1(t)-\varphi_1(t_*)>(t-t_*)\alpha/2$, $t\geq t_*$, tengsizlikni topamiz. Oxirgi tengsizlikdan $\lim_{t\to+\infty}\varphi_1(t)=+\infty$. Lekin berilganga ko'ra $\lim_{t\to+\infty}\varphi_1(t)=a_1\in\mathbb{R}$ bo'lishi kerak edi. Hosil bo'lgan ziddiyat isbotni tugallaydi. 👍

4^0. *Agar o'zgarmasdan (muvozanat nuqtadan) farqli biror* $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ *yechim uchun* $\boldsymbol{\varphi}(t_1)=\boldsymbol{\varphi}(t_2)$, $t_1\neq t_2$, *bo'lsa, bu yechim eng kichik musbat* $\tau_0>0$ *davrga ega bo'lgan davriy funksiyadan, uning trayektoriyasi esa sodda (o'z-o'zini kesmaydigan) yopiq chiziqdan iborat bo'ladi.*

🗝 Aniqlik uchun $\tau=t_2-t_1>0$ deylik. Yuqoridagi **2^0.** xossaga ko'ra $\boldsymbol{\varphi}(t)=\boldsymbol{\varphi}(t+\tau)$. Bu tenglikdan foydalanib yechimni o'ngga va chapga cheksiz davom ettiramiz va $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimning $(-\infty;+\infty)$ oraliqda aniqlangan $\tau>0$ davrli funksiyadan iborat ekanligini topamiz. Endi bu yechimning eng kichik musbat davrga ega ekanligini isbotlaymiz. T bilan $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ funksiyaning barcha musbat davrlari to'plamini belgilaylik:

$$T=\{\theta>0\mid\forall t\in\mathbb{R}\ \boldsymbol{\varphi}(t+\theta)=\boldsymbol{\varphi}(t)\}.$$

$T\neq\varnothing$, chunki $\tau\in T$. T quyidan nol bilan chegaralangan, demak, T aniq quyi chegaraga ega. $\tau_0=\inf T$ deylik. Ravshanki, $0\leq\tau_0\leq\tau$. Aniq quyi chegara ta'rifiga ko'ra $\tau_0$ga intiluvchi musbat davrlar ketma-ketligi θ_j, $j\in\mathbb{N}$, mavjud: $\theta_j\to\tau_0,\theta_j\in T$. $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ funksiya uzluksiz bo‘lgani uchun $\varphi(t+\theta_j)=\varphi(t)$ $(\theta_j \in T)$ munosabatda limitga o‘tib, $\varphi(t+\tau_0)=\varphi(t)$ ayniyatni topamiz. Agar $\tau_0 \neq 0$ bo‘lishini ko‘rsatsak, τ_0 eng kichik musbat davr ekanligini isbotlagan bo‘lamiz. Ko‘rsatrilishi kerak bo‘lgan tasdiqning teskarisini faraz qlaylik, ya’ni $\tau_0 = 0$ deylik. Demak, infimumning ta’rifiga ko‘ra, xohlagancha kichik musbat davrlar mavjud. Berilgabga ko‘ra $\varepsilon \overset{def}{=} \|\boldsymbol{\varphi}(t_1)-\boldsymbol{\varphi}(t_2)\| > 0$ bo‘ladi. $\boldsymbol{x}=\boldsymbol{\varphi}(t)$funksiya uzluksiz bo‘lgani uchun esa $t_1 \in \mathbb{R}$ nuqtaning shunday $B(t_1)$ atrofini topamizki, bu atrofdagi barcha t lar uchun $\|\varphi(t)-\varphi(t_1)\| < \varepsilon$ bo‘ladi. Yetarlicha kichik musbat θ davrni va $k \in \mathbb{Z}$ sonni tanlash evaziga $t = t_2 + k\theta$ nuqtani $B(t_1)$ atrofga tushirish mumkin. Ana shunday $t = t_2 + k\theta \in B(t_1)$ nuqta uchun $\varepsilon = \|\boldsymbol{\varphi}(t_2)-\boldsymbol{\varphi}(t_1)\| = \|\boldsymbol{\varphi}(t_2+k\theta)-\boldsymbol{\varphi}(t_1)\| < \varepsilon$ ziddiyat hosil bo‘ladi. Demak, farazimiz noto‘g‘ri va τ_0 eng kichik musbat davr. Bundan $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ $(0 \le t \le \tau_0)$ traektoriyaning sodda yopiq chiziq ekanligi kelib chiqadi, chunki $\boldsymbol{\varphi}(0)=\boldsymbol{\varphi}(\tau_0)$. Agar bu yopiq chiziq o‘z-o‘zini kesganda, ya’ni $[0;\tau_0]$ oraliqdagi biror t_1 va t_2 $(t_2 > t_1)$ har ($\{t_1,t_2\} \subset [0;\tau_0],\, t_2 - t_1 < \tau_0$) uchun $\boldsymbol{\varphi}(t_1)=\boldsymbol{\varphi}(t_2)$ bo‘lganda edi, u holda $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechim $\theta = t_2 - t_1 < \tau_0$ musbat davrga ega bo‘lardi. Bu esa τ_0 ning eng kichik musbat davr ekanligiga zid. Demak, $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ (yopiq) traektoriya o‘z-o‘zini kesmaydi.

Osongina ko‘rsatish mumkinki, $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimning ixtiyoriy T davri $\tau_0$ga karrali bo‘ladi. Faraz qilaylik, $T>0$ davr $\tau_0$ga karrali bo‘lmasin, ya’ni ixtiyoriy $k \in \mathbb{N}$ uchun $T \neq k\tau_0$ bo‘lsin. Tushunarliki, $T > \tau_0$ bo‘lishi kerak. $k\tau_0 < T$ tengsizlikni qanoatlantiruvchi eng katta $k \in \mathbb{N}$ ni m deylik, ya’ni $m\tau_0 < T$, $(m+1)\tau_0 > T$ $(m \in \mathbb{N})$. Demak, $T - m\tau_0 < \tau_0$ davr mavjud. Bu esa $\tau_0$ning eng kichik musbat davr ekanligiga zid. Shunday qilib, farazimiz noto‘g‘ri va har qanday davr $\tau_0$ga karrali. ☝

Teorema. *Yuqorida aytilgan* $\boldsymbol{f} \in C^1(G;\mathbb{R}^n)$ *shart bajarilganda* $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ *avtonom sistemaning har qanday traektoriyasi* (*yechimi*) *quyidagi uch turning bittasiga mansub bo‘ladi:*

— *nuqta, ya'ni muvozanat nuqtasi* (*yechimning davri ixtiyoriy son*)*;*

— *o'z-o'zini kesmaydigan yopiq chiziq* (*eng kichik musbat davrli yechim*)*;*

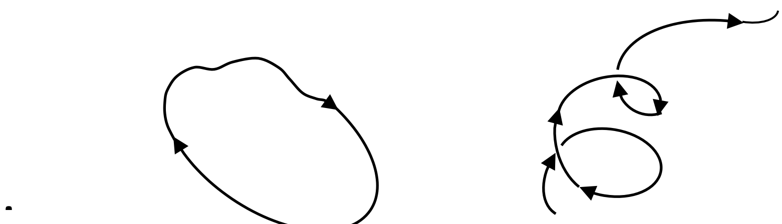
— *o'z-o'zini kesmaydigan yopiqmas chiziq* (*davrsiz yechim*).

🗝 Ixtiyoriy $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimni qaraylik. Mantiqan quyidagi uch hol bo'lishi mumkin xolos.

1). $\varphi(t) = \text{const}$; bu holda traektoriya nuqtadan iborat.

2). $\varphi(t) \neq \text{const}$, lekin biror t_1 va $t_2 \neq t_1$ har uchun $\varphi(t_1) = \varphi(t_2)$; bu holda 3° xossaga ko'ra traektoriya o'z-o'zini kesmaydigan yopiq chiziqdan iborat.

3). Barcha t_1 va $t_2 \neq t_1$ har uchun $\varphi(t_1) \neq \varphi(t_2)$; bu holda traektoriya o'z-o'zini kesmaydigan va yopiq bo'lmagan chiziqdan iborat (11.2- rasm). 👍



11.2- rasm.

5⁰. Avtonom sistema yechimlarining gruppaviy xossasi. Bu bandda (11.1.2) avtonom sistemaning barcha yechimlarini $(-\infty;+\infty)$ oraliqda aniqlangan deb hisoblaymiz.

$\boldsymbol{x} = \boldsymbol{\varphi}(t,\boldsymbol{\xi})$ bilan $\boldsymbol{x}(0) = \boldsymbol{\xi}$ boshlang'ich shartni qanoatlantiruvchi yechimni belgilab, ixtiyoriy $t \in \mathbb{R}$ uchun $g^t : G \to G$ akslantirishni $g^t(\boldsymbol{\xi}) = \boldsymbol{\varphi}(t,\boldsymbol{\xi})$ $(\boldsymbol{\xi} \in G,\ t - \text{parametr})$ formula bilan kiritaylik. U holda bir parametrli $g^t : G \to G$ akslantirishlar oilasi hosil bo'ladi. $\boldsymbol{f} \in C^1(G;\mathbb{R}^n)$ bo'lganligi uchun $g^t(\boldsymbol{\xi}) = \boldsymbol{\varphi}(t,\boldsymbol{\xi}) \in C^1(\mathbb{R} \times G; G)$ ham bo'ladi.

Jumla. $g^t : G \to G$ *almashtirishlar oilasi kompozitsiya amaliga nisbatan Abel gruppasini tashkil etadi, ya'ni*

1). $(g^r \circ g^s) \circ g^t = g^r \circ (g^s \circ g^t)$, $\{r,s,t\} \subset \mathbb{R}$, (*assotsiativlik*)*;*

2). $g^s \circ g^t = g^t \circ g^s = g^{t+s}$ (*kommutativlik*)*;*

3). $g^0 \circ g^t = g^t \circ g^0$ (*birlik elementning mavjudligi*);

4). $g^{-t} \circ g^t = g^t \circ g^{-t} = g^0$ (*teskari elementning mavjudligi*).

🗝 Assotsiativlik xossasi har qanday almashtirishlar uchun o‘rinli. Agar $g^t \circ g^s = g^{t+s}$ munosabatni isbotlasak, undan isbotlanashi kerak bo‘lgan boshqa xossalar bevosita kelib chiqadi. $g^t \circ g^s = g^{t+s}$ tenglik $\boldsymbol{\varphi}(t, \boldsymbol{\varphi}(s, \boldsymbol{\xi})) = \boldsymbol{\varphi}(t+s, \boldsymbol{\xi})$ ($\boldsymbol{\xi} \in G$) ekanligini anglatadi. Oxirgi tenglik $\boldsymbol{x} = \boldsymbol{\varphi}(t, \boldsymbol{\varphi}(s, \boldsymbol{\xi}))$ va $\boldsymbol{x} = \boldsymbol{\varphi}(t+s, \boldsymbol{\xi})$ yechimlarning $t = 0$ nuqtada tengligi va yechimning yagonalik xossasidan ravshan. 👍

Agar $g^t : G \to G, t \in \mathbb{R},$ bir parmetrli almashtirishlar majmuasi uchun ushbu

– $g^0 = 1 -$ Gni ayniy almashtirish, ya’ni $g^0(\boldsymbol{\xi}) = \boldsymbol{\xi}, \boldsymbol{\xi} \in G$;

– $g^{t+s} = g^t \circ g^s, \{t, s\} \subset \mathbb{R}$;

– $g^t(\boldsymbol{\xi}) \in C(\mathbb{R} \times G; G)$

shartlar o‘rinli bo‘lsa, u holda G da $g^t : G \to G, t \in \mathbb{R},$ **dinamik sistema** berilgan deb ataladi. Agar bundan tashqari $g^t(\boldsymbol{\xi}) \in C^1(\mathbb{R} \times G; G)$ ham bo‘lsa, qaralayotgan $g^t : G \to G, t \in \mathbb{R},$ dinamik sistema **fazaviy oqim** deyiladi.

Shunday qilib, yuqorida kiritilgan
$g^t : G \to G, \; g^t(\boldsymbol{\xi}) = \boldsymbol{\varphi}(t, \boldsymbol{\xi}), \; t \in \mathbb{R},$ almashtirishlar fazaviy oqimni tashkil etadi.

(11.1.2) avtonom sistema G da aniqlangan g^t fazaviy oqim orqali bir qiymatli tiklanadi.

Jumla. *Ushbu*

$$\left. \frac{dg^t}{dt} \right|_{t=0} = \boldsymbol{f}$$

formula o‘rinli.

🗝 Ixtiyoriy $\boldsymbol{\xi} \in G$ uchun quyidagi hisoblashlarni bajaramiz:

$$\left. \frac{dg^t}{dt} \right|_{t=0} (\boldsymbol{\xi}) = \left. \frac{dg^t(\boldsymbol{\xi})}{dt} \right|_{t=0} = \left. \frac{d\boldsymbol{\varphi}(t, \boldsymbol{\xi})}{dt} \right|_{t=0} = \boldsymbol{f}(\boldsymbol{\xi}).$$ 👍

Masalalar

1. Bir o‘lchamli $x' = f(x)$ avtonom sistema uchun $f(x) \in C^1(\mathbb{R})$ va $f(x)$ ikkita nolga ega bo‘lsin: $f(a) = 0, f(b) = 0$ $(a < b)$. Bu sistemaning har qanday traektoriyasi $(-\infty;a),\{a\},(a;b),\{b\},(b;+\infty)$ to‘plamlarning biridan iborat bo‘lishini isbotlang.

2. Faraz qilaylik, berilgan $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ avtonom sistemaning o‘ng tomoni $\boldsymbol{x} \in \mathbb{R}^n$ da aniqlangan bo‘lsin. Bu sistemaning $\boldsymbol{x} = \boldsymbol{x}(t)$ traektoriyasidagi t parametr o‘rniga $\tau = \tau(t)$ parametrni ushbu $\frac{d\tau}{dt} = \sqrt{1+\|\boldsymbol{f}(\boldsymbol{x})\|^2}$ tenglamaning yechimi sifatida kiritaylik. U holda $\frac{d\boldsymbol{x}}{d\tau} = \frac{\boldsymbol{f}(\boldsymbol{x})}{\sqrt{1+\|\boldsymbol{f}(\boldsymbol{x})\|^2}}$. Oxirgi sistemaning yechimlari $\tau \in (-\infty;+\infty)$ oraliqda aniqlangan va uning fazaviy tasviri berilgan avtonom sistemaning fazaviy tasviri bilan bir xil ekanligini ko‘rsating.

3. Faraz qilaylik, $f \in C(\mathbb{R},(0,+\infty))$ funksiya $\tau > 0$ davrga ega bo‘lsin. Agar $x = x(t)$ funksiya $x' = f(x)$ tenglamaning yechimi va

$$T = \int_0^\tau \frac{1}{f(x)}\,dx$$

bo‘lsa, u holda har qanday $t \in \mathbb{R}$ uchun $x(T+t) - x(t) = \tau$ bo‘lishini isbotlang. f funksiya davriy va ishorasini almashtiruvchi bo‘lgan holni ham tekshiring.

§ 11.2. Tekislikda avtonom sistemalar

Bu paragrafda ikki o‘lchamli ushbu

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad (x,y) \in G \subset \mathbb{R}^2 \tag{11.2.1}$$

avtonom sistemani qaraymiz; bu yerda $\{f,g\} \subset C^1(G;\mathbb{R})$. Sistemaning $\big(f(x,y),g(x,y)\big)$ vektor maydonning) maxsus nuqtalari to‘plamini G_0, $G_0 = \{(x,y) \in G \mid f^2(x,y) + g^2(x,y) = 0\}$, bilan belgilaylik. U holda $\tilde{G} = G \setminus G_0$ ochiq to‘plamda (aniqrog‘i, uning bog‘lanishli komponentalarida) ushbu

$$g(x,y)dx = f(x,y)dy,\ (x,y) \in \tilde{G}, \tag{11.2.2}$$

differensial tenglamani hosil qilamiz.

Jumla. *Faraz qilaylik,* $\{f,g\} \subset C^1(G;\mathbb{R})$ *bo‘lsin. U holda* (11.2.1) *sistemaning muvozanat holatidan farqli har qanday traektoriyasi* (11.2.2) *tenglamaning integral chizig‘idan iborat va aksincha, ya’ni* (11.2.2) *tenglamaning ixtiyoriy integral chizig‘i* (11.2.1) *sistemaning muvozanat holatidan farqli traektoriyasidan iborat bo‘ladi.*

🗝 (11.2.1) sistemaning muvozanat holatidan farqli

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad (11.2.3)$$

traektoriyasini qaraylik. Bu traektoriya muvozanat nuqtadan farqli bo‘lganligi uchun u G_0 bilan umumiy nuqtaga ega emas. Traektoriya $t = t_0$ paytda $(x_0, y_0) \in \tilde{G}$ nuqtadan o‘tgan bo‘lsin. Aniqlik uchun $f(x_0, y_0) \neq 0$ deylik. U holda $x'(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0) \neq 0$ va, demak, (11.2.3) sistemadan $(x_0, y_0) \in \tilde{G}$ nuqtaning yetarlicha kichik atrofida y ni $x$ning $y = y(t(x))$ funksiyasi sifatida ifodalash mumkin hamda

$$\frac{dy}{dx} = \frac{dy(t(x))}{dx} = \frac{y'(t)}{x'(t)} = \frac{g(x, y(t))}{f(x, y(t))} \ (t = t(x)).$$

Bu tenglik $y = y(t(x))$ funksiyaning (11.2.2) tenglama yechimi ekanligini anglatadi. Shunday qilib, (11.2.3) fazaviy traektoriya o‘zining ixtiyory $(x_0, y_0) \in \tilde{G}$ nuqtasi atrofida (11.2.2) tenglamaning integral chizig‘idan iborat.

Endi (11.2.2) tenglamaning ixtiyoriy $\gamma \subset \tilde{G}$ integral chizig‘ini qaraylik. Biz uning (11.2.1) avtonom sistemaning muvozanat nuqtadan farqli bo‘lgan traektoriyasi ekanligini ko‘rsatishimiz kerak. γ ning nuqtadan farqli ekanligi ravshan, chunki u o‘zining ixtiyoriy $(x_0, y_0) \in \gamma \subset \tilde{G}$ nuqtasi atrofida $f(x_0, y_0) \neq 0$ bo‘lganda $\frac{dy}{dx} = f(x, y)$ tenglama, ya’ni $y = y(x)$ ($y(x_0) = y_0$) yoki $g(x_0, y_0) \neq 0$ bo‘lganda $\frac{dx}{dy} = g(x, y)$ tenglama, ya’ni $x = x(y)$ ($x(y_0) = x_0$) ko‘rinishda ifodalanadi. Aniqlik uchun $f(x_0, y_0) \neq 0$ deylik. U holda $t_0 \in \mathbb{R}$ nuqtaning kichik atrofida $x = x(t)$ funksiyani ushbu $\frac{dx}{dt} = f(x, y(x)),\ x(t_0) = x_0,$ masalaning yechimi sifatida aniqlab, $y = y(x(t))$ funksiya uchun

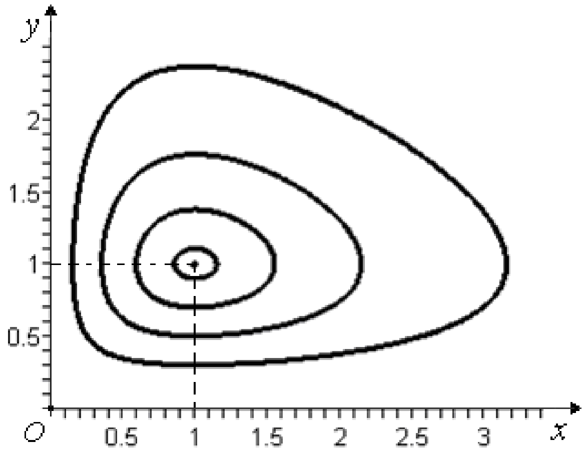
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{g(x, y)}{f(x, y)} f(x, y) = g(x, y),\ x = x(t),\ y(x(t_0)) = y(x_0) = y_0,$$

munosabatlarni hosil qilamiz. Demak, qurilgan $x = x(t)$ va $y = y(x(t))$ funksiyalar (11.2.1) sistemaning yechimi, ya’ni (11.2.2) tenglamaning ixtiyoriy $\gamma \subset \tilde{G}$ integral chizig‘i o‘zining ixtiyoriy (x_0, y_0) nuqtasining yetarlicha kichik atrofida (11.2.1) sistemaning shu nuqta orqali o‘tuvchi traektoriyasi bilan ustma-ust tushadi. γ integral chiziqni shunaqa atroflar bilan qoplab va yechimning yagonalik xossasidan foydalaninb, γ integral chiziqning to‘laligicha (11.2.1) sistema traektoriyasidan iborat ekanligiga ishonch hosil qilamiz. 👍

Misol 1. Ushbu

$$\begin{cases} x' = (1-y)x \\ y' = \alpha(x-1)y \end{cases} \quad (\alpha > 0 - \text{ o‘zgarmas, } x > 0, y > 0 \ )$$

Volterra-Lotka sistemasini qaraylik. Bu sistemadan $\dfrac{dy}{dx} = \dfrac{\alpha(x-1)y}{(1-y)x}$ tenglamani yozib, va uni yechib, traektoriyalarning $\dfrac{yx^{\alpha}}{e^{y}e^{\alpha x}} = c$, $c-$ o‘zgarmas son, tenglama bilan oshkormas ko‘rinishda berilishini topamiz. Ravshanki, $x = 1, y = 1$ o‘zgarmas yechim; u $(1;1)$ muvozanat nuqtani aniqlaydi. Traektoriyalar ($\alpha = 1/2$ holida) 11.3- rasmda keltirilgan (1- masalaga qarang).



11.3-rasm.

Misol 2. Tekislikda ushbu

$$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases} \qquad (11.2.4)$$

avtonom sistemani qaraylik. Bu sistemani tekshirish uchun fazaviy tekislikda $(r, \varphi), r \geq 0,$ qutb koordinatalarini kiritamiz:

$x = r\cos\varphi, y = r\sin\varphi.$ (11.2.4) sistemada bu almashtirishlarni bajaramiz:

$$\begin{cases} r'\cos\varphi - r\varphi'\sin\varphi = -r\sin\varphi + r(1-r^2)\cos\varphi, \\ r'\sin\varphi + r\varphi'\cos\varphi = r\cos\varphi + r(1-r^2)\sin\varphi. \end{cases}$$

Bundan

$$\begin{cases} r' = r(1-r^2), \\ \varphi' = 1. \end{cases}$$

Bu sistemaning tenglamalari ajralgan. Ularni alohoda-alohida yechib, topamiz:

$$r = 0, \quad \begin{cases} r = \dfrac{1}{\sqrt{1+c_1e^{-2t}}}, \\ \varphi = t + c_2. \end{cases}$$

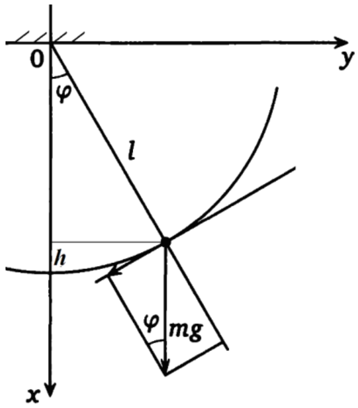
Demak, (11.2.4) sistemaning traektoriyalari:

$x = y = 0$ - muvozanat nuqtasi hamda

$\left\{ x = \dfrac{\cos(t+c_2)}{\sqrt{1+c_1e^{-2t}}},\ y = \dfrac{\sin(t+c_2)}{\sqrt{1+c_1e^{-2t}}};\ t \in \mathbb{R} \right\}$ - spirallar ($c_1 \neq 0$) va birlik aylana ($c_1 = 0$). Bundan ravshanki, bitta yopiq traektoriya $x^2 + y^2 = 1$ ($c_1 = 0$ da hosil bo‘luvchi) mavjud bo‘lib, muvozanat nuqtasidan boshqa barcha traektoriyalar vaqt o‘tishi bilan shu davraga intiladi.

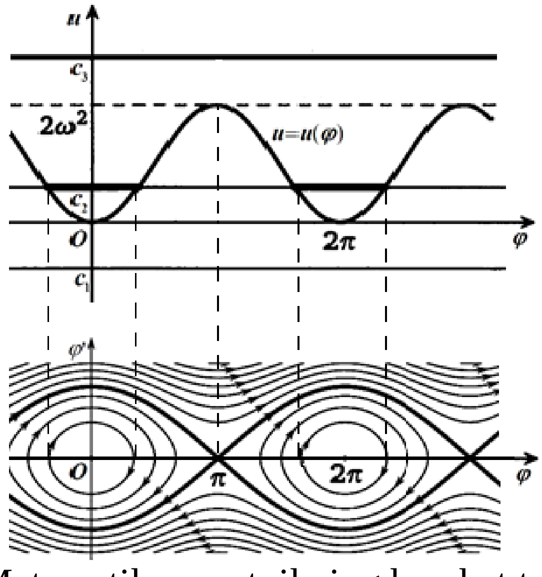
Misol 3. Matematik mayatnik harakati. Matematik mayatnik O nuqtada osilgan vaznsiz, cho‘zilmas, bukilmas l uzunlikdagi sterjen va unga biriktirilgan m massali moddiy nuqtadan tuzilgan sistemadir. Bu mayatnik O nuqta orqali o‘tuvchi vertikal tekislikda harakat qiladi. Mayatnikning holatini pastga yo‘nalgan vertikal o‘q bilan sterjen orasidagi burchak φ aniqlaydi (11.4 - rasm), $\varphi = \varphi(t)$ ($t-$ vaqt). Fazalar fazosi (φ, φ'), $\varphi' = \dfrac{d\varphi}{dt}$, nuqtalardan tashkil topadi. Matematik mayatnikning to‘la mexanik energiyasi harakat davomida saqlanadi. To‘la mexanik energiya E kinetik va potensial energiyalar yig‘indisidan iborat ($g-$ erkin tushish tezlanishi):

$$E = \frac{mv^2}{2} + mgh = \frac{ml^2\varphi'^2}{2} + mgl(1-\cos\varphi),$$
$$E = ml^2\left(\frac{\varphi'^2}{2} + \omega^2(1-\cos\varphi)\right),\ \omega = \sqrt{\frac{g}{l}}\ .$$
Demak, harakat davomida
$$\frac{\varphi'^2}{2} + u(\varphi) = c,\ u(\varphi) = \omega^2(1-\cos\varphi)\ \ (c = \text{const}).$$



11.4- rasm.

Bundan
$$\varphi' = \pm\sqrt{2\left(c-u(\varphi)\right)}\ .$$
Oxirgi tenglik trayektoriyalarni aniqlaydi (11.5- rasm). Qulaylik uchun kiritilgan ushbu $u(\varphi) = \omega^2(1-\cos\varphi)$ funksiya o‘zgarmas ko‘paytuvchi aniqligida matematik mayatnikning potensial energiyasini ifodalaydi.



11.5-rasm. Matematik mayatnikning harakat tryektoriyalari.

Agar $c = c_1 < 0$ bo‘lsa, trayektoriya (harakat) mavjud emas (energiya manfiy bo‘lmaydi). $c = 0$ bo‘lganda $\varphi = \pi k, k \in \mathbb{Z}$, muvozanat holatlari topiladi.

Agar $0 < c < 2\omega^2$ (rasmda $c = c_2$) bo‘lsa, trayektoriyalar $\varphi = 2\pi k, k \in \mathbb{Z}$, muvozanat nuqtalarini qurshab olgan yopiq chiziqlardan iborat (davriy yechim); bu - harakatning tebranishlardan iborat ekanligini anglatadi. $c > 2\omega^2$ hollarda trayektoriyalar yopiq bo‘lmagan chiziqlardir.

Masalalar

1. Volterra-Lotka sistemasining traektoriyalarini turli $c > 0$ larda tekshiring:

$$\frac{yx^{\alpha}}{e^{y}e^{\alpha x}} = c \ \ (x, y > 0)$$

2. Matematik mayatnikning harakat tryektoriyalarini quring.

3. Ushbu $x'' + x' + 2x = 0$ tenglamaning har qanday $x = x(t)$ yechimi $t \in (-\infty, +\infty)$ oraliqqacha davom etishi va uning davriy bo`lishini isbotlang.

§ 11.3. Tekislikda chiziqli avtonom sistemalar fazaviy portreti

Ushbu

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \text{ yoki } \begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (11.3.1)$$

sistmaning yechimlari tabiatini (x, y) fazalar tekisligida o‘rganamiz. Bunda a, b, c, d koeffitsientlar haqiqiy sonlar va dastlab bu sistema yagona maxsus nuqtaga ega, ya’ni $\det A \neq 0$ deb faraz qilinadi. Demak, (11.3.1) sistema $x = x(t) = 0, y = y(t) = 0$ bir dona muvozanat nuqtasiga ega.

A matritsaning xos (xarakteristik) sonlari

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \text{ yoki } \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

tenglamadan topiladi. Xarakteristik sonlarni λ_1 va λ_2 bilan belgilaylik. $\det A = ad - bc \neq 0$ bo‘lgani uchun $\lambda_1 \neq 0$ va $\lambda_2 \neq 0$.

Dastlab xarakteristik sonlar haqiqiy bo‘lgan holni qaraymiz. Bunda $\lambda_1 \neq \lambda_2$ yoki $\lambda_1 = \lambda_2$ bo‘ladi. Xos sonlar turli bo‘lsin. Ma’lumki, bu turli xos sonlarga mos keluvchi $\boldsymbol{a}_1$ va $\boldsymbol{a}_2$ xos vektorlar ($A\boldsymbol{a}_1 = \lambda_1\boldsymbol{a}_1, \ A\boldsymbol{a}_2 = \lambda_2\boldsymbol{a}_2$) chiziqli erkli. Bu vektorlar koordinatalarini ustunlar bo‘ylab yozib, $S=[\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]$ matritsani tuzaylik. $\boldsymbol{a}_1$ va $\boldsymbol{a}_2$ vektorlar chiziqli erkli bo‘lgani uchun $\det S \neq 0$, ya’ni S matritsa teskarilanuvchi. Ravshanki,

$$AS = A[\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]=[A\boldsymbol{a}_1 \vdots A\boldsymbol{a}_2]=[\lambda_1\boldsymbol{a}_1 \vdots \lambda_2\boldsymbol{a}_2]=[\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=S\Lambda\ ,\ \Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

Demak,

$$A=S^{-1}\Lambda S. \qquad (11.3.2)$$

Yangi u,v noma’lumlarni

$$\begin{pmatrix}u\\ v\end{pmatrix}=S^{-1}\begin{pmatrix}x\\ y\end{pmatrix}\ \left(\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}\right) \qquad (11.3.3)$$

formula yordamida kiritaylik. Bu chiziqli almashtirish natijasida (11.3.1) sistema

$$\begin{pmatrix}u\\ v\end{pmatrix}'=\Lambda\begin{pmatrix}u\\ v\end{pmatrix}\ \text{yoki}\ \begin{cases}u'=\lambda_1 u\\ v'=\lambda_2 v\end{cases} \qquad (11.3.4)$$

ko‘rinishga o‘tadi. Oxirgi (11.3.4) sistemaning yechimlari osongina topiladi:

$$u=c_1e^{\lambda_1 t}, v=c_2e^{\lambda_2 t}\ (c_1,c_2 - \text{ixtiyoriy o‘zgarmaslar}). \qquad (11.3.5)$$

Bu formulalar fazaviy traektoriyalarning parametrik tenglamasini ifodalaydi. t ni yo‘qotib, traektoriyalarni ushbu

$v=c_2\left(\frac{u}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}}$ $(c_1 \neq 0)$ va $u=0$ $(c_1=0)$, bunda $v>0$ yoki $v<0$,

o‘shkor ko‘rinishda ham yozish mumkin.

(11.3.1) sistemaning umumiy yechimi

$$\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}=[\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]\begin{pmatrix}c_1e^{\lambda_1 t}\\ c_2e^{\lambda_2 t}\end{pmatrix}=c_1e^{\lambda_1 t}\boldsymbol{a}_1+c_2e^{\lambda_2 t}\boldsymbol{a}_2$$

formula bilan beriladi.

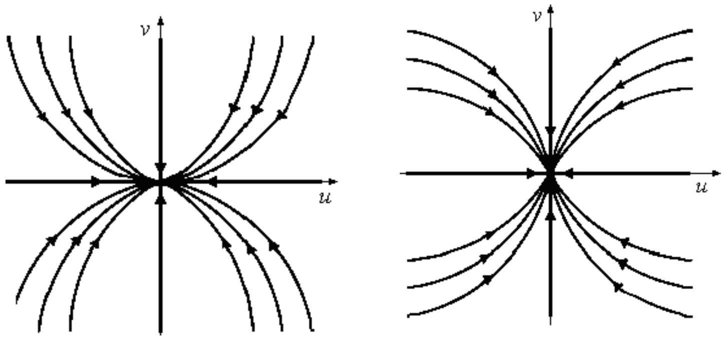
Dastlab λ_1,λ_2 xos sonlar bir xil ishorali, ya’ni $\lambda_1\cdot\lambda_2>0$ holni qaraylik. Bu holda $(0,0)$ maxsus nuqta ***tugun*** deb ataladi. $\lambda_2/\lambda_1>1$ $(|\lambda_2|>|\lambda_1|)$ bo‘lganda $v=c_2\left(\frac{u}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}}$ traektoriyalar Ou o‘qiga urinadi, $\lambda_2/\lambda_1<1$ $(|\lambda_2|<|\lambda_1|)$ bo‘lganda esa ular Ov o‘qiga urinadi. (11.3.3) chiziqli almashtirishda Ou o‘qi $\boldsymbol{a}_1$ xos vektor, Ov o‘qi esa $\boldsymbol{a}_2$ xos vektor orqali o‘tgan o‘qqa almashinadi:

$$\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} u \\ 0 \end{pmatrix} = [\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]\begin{pmatrix} u \\ 0 \end{pmatrix} = u\boldsymbol{a}_1 \ , \ \begin{pmatrix} x \\ y \end{pmatrix} = [\boldsymbol{a}_1 \vdots \boldsymbol{a}_2]\begin{pmatrix} 0 \\ v \end{pmatrix} = v\boldsymbol{a}_2 \, .$$

Demak, Oxy tekisligida traektoriyalar moduli bo‘yicha kichik xos songa mos kelgan xos vektorga urinadi.

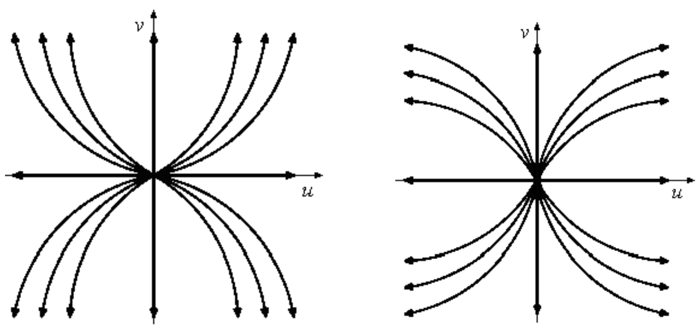
Agar xos sonlarning ikkalasi ham manfiy bo‘lsa $(\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2)$, (11.3.5) yechimlarning (u, v) fazalar tekisligidagi tasviri 11.6- rasmda ko‘rsatilgan tabiatli bo‘ladi. Vaqt o‘tishi bilan fazaviy nuqta koordinatalar boshiga (muvozanat nuqtaga) intiladi: $\lim\limits_{t\to+\infty} u = \lim\limits_{t\to+\infty} c_1 e^{\lambda_1 t} = 0$, $\lim\limits_{t\to+\infty} v = \lim\limits_{t\to+\infty} c_2 e^{\lambda_2 t} = 0$. Bu $(0,0)$ maxsus nuqta ***turg‘un tugun*** deb ataladi.

Turg‘un tugunga intiluvchi (kiruvchi) to‘rt dona koordinata yarim o‘qlaridan iborat bo‘lgan traektoriyalar ham mavjud.



11.6- rasm. Turg‘un tugun.

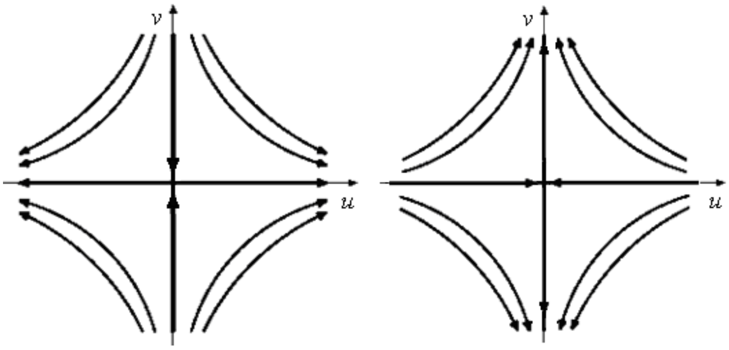
Agar ikkala xos son ham musbat bo‘lsa $(\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2)$, (11.3.5) yechimlarning fazaviy tasviri 11.7- rasmda keltirilgan. Bu holda $(0,0)$ maxsus nuqta ***noturg‘un tugun*** deb yuritiladi. Noturg‘un tugundan ($t = -\infty$ da) chiquvchi to‘rt dona koordinata yarim o‘qlaridan iborat bo‘lgan traektoriyalar mavjud.



Noturg‘un tugun, $\lambda_2 > \lambda_1 > 0$ **Noturg‘un tugun,** $\lambda_1 > \lambda_2 > 0$

11.7- rasm. Noturg‘un tugun.

Agar xos sonlar turli ishorali bo‘lsa ($\lambda_1\lambda_2 < 0, \lambda_1 \neq \lambda_2$), faza tasviri, masalan, 11.8- rasmdagidek bo‘ladi. Bu holda $(0,0)$ maxsus nuqta ***egar*** deb ataladi.

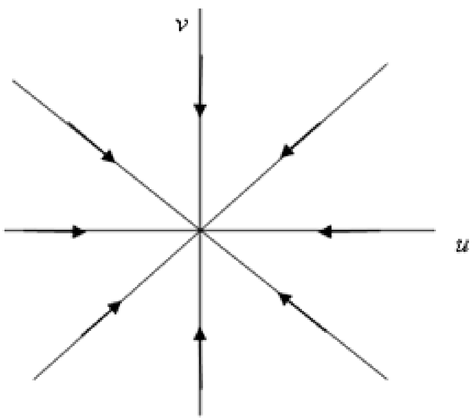


Egar, $\lambda_2 < 0 < \lambda_1$ Egar, $\lambda_1 < 0 < \lambda_2$

11.8- rasm. Egar.

Egardan chiquvchi yoki unga kiruvchi hamda traektoriyalar oilasini to‘rt qismga ajratuvchi to‘rt dona traektoriya (koordinata yarim o‘qlari) ***separatrisalar*** deb yuritiladi.

Endi A matritsa karrali xos sonlarga ega bo‘lgan $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ holni qaraymiz. Agar A matritsaning bu xos soniga ikki dona chiziqli erkli xos vektorlar mos kelsa, ya’ni A matritsa diagonallashtiriluvchi bo‘lsa, bir xil ishorali turli xos sonlar ($\lambda_1\lambda_2 > 0$) holidagi fikr yuritishlar bu holda ham o‘z kuchini saqlaydi. Bu holda maxsus nuqta ***dikritik tugun*** deyiladi (11.9- rasm). Traektoriyalar maxsus nuqtaga intiluvchi (kiruvchi) (***turg‘un dikritik tugun***) yoki undan chiquvchi (***noturg‘un dikritik tugun***) nurlardan iborat bo‘ladi.



11.9- rasm. Dikritik tugun

$\lambda_1 = \lambda_2 < 0$, chiziqli erkli xos

sonlar 2ta.

Endi A matritsa diagonallashtiriluvchi bo‘lmasin deylik. Bu holda, algebradan ma’lumki, $\boldsymbol{a} \neq 0$ xos vektorga ($A\boldsymbol{a} = \lambda\boldsymbol{a}$) chiziqli bog‘liq o‘lmagan shunday $\boldsymbol{b} \neq 0$ vektor topiladiki, uning uchun $A\boldsymbol{b} = \lambda\boldsymbol{b} + \boldsymbol{a}$ bo‘ladi; bunda E – 2×2 o‘lchamli birlik matritsa. $\boldsymbol{a}$ va $\boldsymbol{b}$ vektorlar chiziqli erkli bo‘lgani uchun ushbu $S = [\boldsymbol{a} \vdots \boldsymbol{b}]$ matritsa teskarilanuvchi. Quyidagilarga egamiz:

$$AS = A[\boldsymbol{a} \vdots \boldsymbol{b}] = [A\boldsymbol{a} \vdots A\boldsymbol{b}] = [\lambda\boldsymbol{a} \vdots \lambda\boldsymbol{b} + \boldsymbol{a}] = [\boldsymbol{a} \vdots \boldsymbol{b}]\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = S\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

$$A = S^{-1}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}S.$$

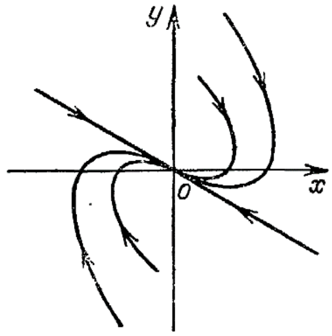
Endi (11.3.1) sistemada (11.3.3) almashtirishni bajarib, uni

$$\begin{cases} u' = \lambda u + v \\ v' = \lambda v \end{cases}$$

ko‘rinishga keltiramiz. Oxirgi sistemaning yechimi osongina topiladi:

$$\begin{cases} u = (c_1 t + c_2)e^{\lambda t} \\ v = c_1 e^{\lambda t} \end{cases}$$

Qaralayotgan ($\lambda_1 = \lambda_2 = \lambda$) holdagi maxsus nuqta ***aynigan tugun*** deb ataladi. Ouv tekisligida traektoriyalar ($t = +\infty$ yoki $t = -\infty$ da) Ou o‘qiga, Oxy tekisligida esa ular $\boldsymbol{a}$ xos vektorga urinadi. Fazaviy traektoriyalar 11.10- rasmda ko‘rsatilgan.



11.10- rasm. Aynigan tugun $\lambda_1 = \lambda_2 < 0$,
chiziqli erkli xos sonlar 1ta.
($\lambda_1 = \lambda_2 < 0$ holda yo`nalishlar teskari)

$\lambda_1 = \lambda_2 = \lambda < 0$ bo‘lganda $(0,0)$ maxsus nuqta turg‘un aynigan tugundan (11.10- rasm), $\lambda_1 = \lambda_2 = \lambda > 0$ bo‘lganda esa u noturg‘un aynigan tugundan iborat.

Endi xarakteristik sonlar kompleks bo‘lgan holni qaraylik. A matritsa haqiqiy bo‘lgani uchun uning λ_1 va λ_2 xos sonlari o‘zaro qo‘shma bo‘ladi: $\lambda_{1,2} = \alpha \pm i\beta, \{\alpha, \beta\} \subset \mathbb{R}, \beta \neq 0$; aniqlik uchun $\beta > 0$ deb hisoblaymiz. Mos xos vektorlar $\boldsymbol{a} \mp i\boldsymbol{b}$ ham o‘zaro qo‘shma ($\boldsymbol{a}, \boldsymbol{b}$ - haqiqiy vektorlar). U holda

$$A(\boldsymbol{a} - i\boldsymbol{b}) = (\alpha + i\beta)(\boldsymbol{a} - i\boldsymbol{b}) \text{ yoki } \begin{cases} A\boldsymbol{a} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \\ A\boldsymbol{b} = -\beta\boldsymbol{a} + \alpha\boldsymbol{b} \end{cases}.$$

Demak, agar $\boldsymbol{a}, \boldsymbol{b}$ vektorlar koordinatalarini ustunlar bo‘ylab yozib, $T = [\boldsymbol{a}, \boldsymbol{b}]$ matritsani tuzsak, u holda

$$AT = A[\boldsymbol{a} \vdots \boldsymbol{b}] = [A\boldsymbol{a} \vdots A\boldsymbol{b}] = [\alpha\boldsymbol{a} + \beta\boldsymbol{b} \vdots -\beta\boldsymbol{a} + \alpha\boldsymbol{b}] = [\boldsymbol{a} \vdots \boldsymbol{b}]\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = T\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

bo‘ladi. Bundan T teskarilanuvchi (turli xos sonlarga chiziqli erkli xos sonlar mos keladi, bundan esa $\boldsymbol{a}$ va $\boldsymbol{b}$ vektorlarning chiziqli erkliligi kelib chiqadi) bo‘lgani uchun

$$T^{-1}AT = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{11.3.6}$$

tenglik kelib chiqadi. Endi (11.3.1) sistemada

$$\begin{pmatrix} u \\ v \end{pmatrix} = T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \left(\text{ya'ni } \begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix} \right) \tag{11.3.7}$$

chiziqli almashtirishni bajarib,

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ yoki } \begin{cases} u' = \alpha u - \beta v \\ v' = \beta u + \alpha v \end{cases} \tag{11.3.8}$$

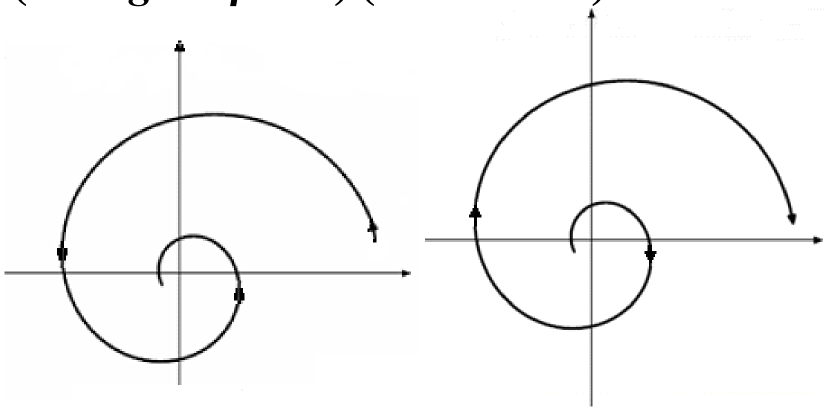
sistemani hosil qilamiz. (u, v) fazalar tekisligida (r, ϕ) qutb koordinatalarini ma’lum $u = r\cos\varphi, v = r\sin\varphi$ formulalar bilan kiritib, (11.3.8) sistemani ushbu

$$\begin{cases} r' = \alpha r \\ \varphi' = \beta \end{cases}$$

sodda ko‘rinishga keltiramiz. Bu sistema osongina yechiladi:

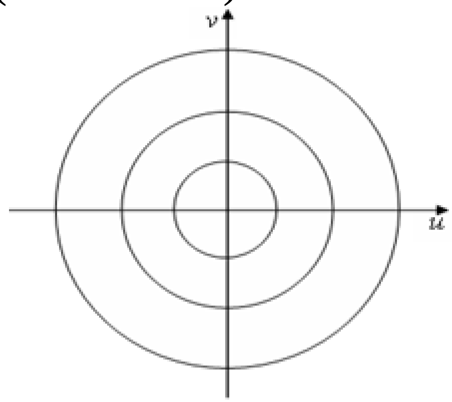
$$\begin{cases} r = r_0 e^{\alpha t} \\ \varphi = \beta t + \varphi_0 \end{cases} \quad (r_0 \geq 0, \phi_0 - \text{ixtiyoriy o‘zgarmaslar}). \tag{11.3.9}$$

Vaqt o‘tishi bilan harakatlanuvchi nuqtaning φ qutb koordinatasi ortadi ($\beta > 0$). Agar $\alpha = \mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 \neq 0$ bo‘lsa, (11.3.9) yechim ***spiral*** deb ataluvchi traektoriyalarni aniqlaydi. Bu holda $(0,0)$ maxsus nuqta ***fokus*** deb ataladi. $\alpha < 0$ bo‘lganda bu spiralning radiusi vaqt o‘tishi bilan kamayadi (***turg‘un fokus***), $\alpha > 0$ bo‘lganda esa – ortadi (***noturg‘un fokus***) (11.11- rasm).



11.11- rasm. Turg‘un va noturg‘un fokuslar.

Agar $\alpha = 0$, ya’ni xos sonlar sof mavhum bo‘lsa, traektoriyalar markazlari $\mathrm{O}(0,0)$nuqtada joylashgan aylanalar oilasidan ($r = r_0, r_0$ – const) iborat bo‘ladi (11.12- rasm).



11.12- rasm.

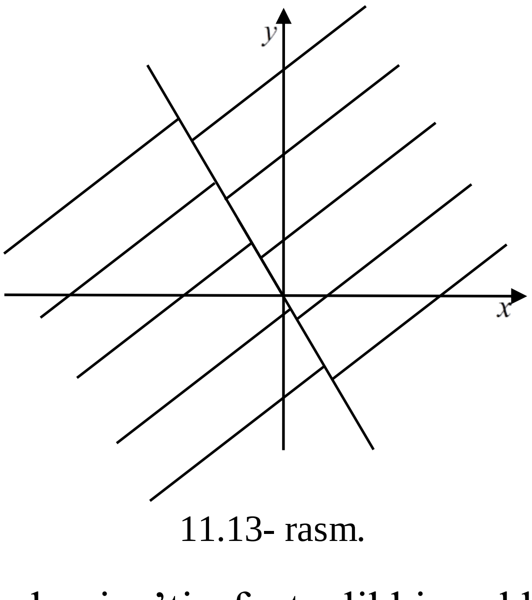
Oxy tekisligida esa traektoriyalar $\mathrm{O}(0,0)$ markazli ellipslar kabi tasvirlanadi. Bu holda $(0,0)$ maxsus nuqta ***markaz*** deb yuritiladi.

Oxy faza(lar) tekisligidagi traektoriyalar mazmunan *Ouv* faza tekisligidagi traektoriyalarga o‘xshash bo‘ladi, chunki (x, y) va (u, v) o‘zgaruvchilar chiziqli almashtirish bilan o‘zaro bog‘langan. Bunda *Ouv* tekisligida yarim to‘g‘ri chiziqdan iborat bo‘lgan traektoriyalar *Oxy* tekisligida ham yarim to‘g‘ri chiziqdan iborat bo‘lgan traektoriyalar sifatida tasvirlanadi. Traektoriyalar bo‘ylab harakat yo‘nalishi berilgan sistemaga bog‘liq. Shunday qilib, Oxy tekisligidagi traektoriyalar joylashishi, ya’ni maxsus nuqtaning tipi A matritsaning λ_1, λ_2 xos sonlari bilan aniqlanadi.

Biz yuqorida (11.3.1) sistemada $\det A \neq 0$ deb hisoblagan edik. Endi $\det A = 0$ holni qaraylik. By holda A matritsaning ranggi $\text{rang}A = 0$ yoki $\text{rang}A = 1$ bo‘ladi. $\text{rang}A = 0$ ($A-$ nol-matritsa) holida muvozanat nuqtalari

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

sistemadan topiladi va ular butun tekislikni to‘ldiradi, $\text{rang}A = 1$ holda esa muvozanat nuqtalar bir o‘lchamli fazoni tashkil etadi. Masalan $\det A = ad - bc = 0$, lekin $|a| + |b| \neq 0$ bo‘lsa, muvozanat nuqtalari butun bir to‘g‘ri chiziq $ax + by = 0$ni tashkil etadi, boshqa traektoriyalar ushbu $\frac{dy}{dx} = \text{const}$ tenglamani qanoatlantiradi va ular uchi $ax + by = 0$ to‘g‘ri chiziqda joylashgan o‘zaro parallel nurlardan iborat bo‘ladi (11.13- rasm).



11.13- rasm.

Izoh. Bu yerda shuni e’tirof etaylikki, ushbu $(x, y) \to (\mu x, \mu y)$ yoki $(u, v) \to (\mu u, \mu v)$ gomotetik almashtirishda traektoriya yana traektoriyaga akslanadi. Demak, traektoriyalar o‘zaro gomotetik chiziqlardan iborat bo‘ladi. Bu izoh ba’zan fazaviy portretni qurishda qo‘l keladi. Bundan, masalan, (11.3.1) chiziqli avtonom sistema yakkalangan (alohida, ajralgan) yopiq traektoriyaga ega emasligi kelib chiqadi.

Misol 1. Ushbu

$$\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$$

sistemaning maxsus nuqtasini tekshiring va maxsus nuqta atrofida traektoriyalarini chizing.

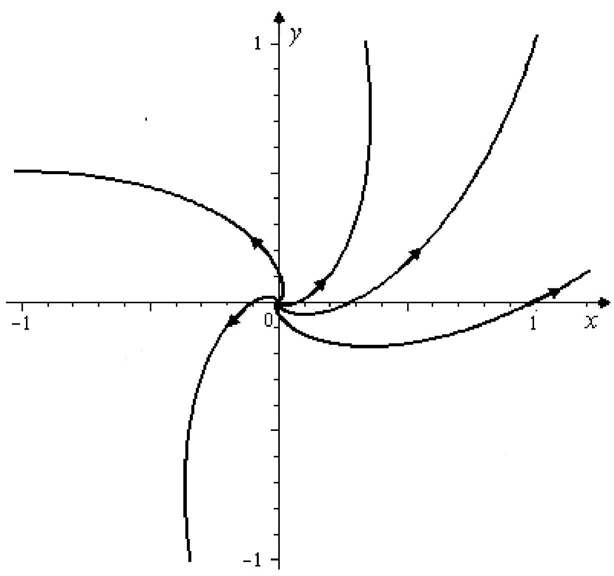
🗝 Xarakteristik sonlarni topamiz

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0\,,\ \lambda_{1,2} = 2 \pm i\,.$$

$\operatorname{Re}\lambda_1 > 0$ bo‘lganligi uchun $(0;0)$ maxsus nuqta noturg‘in fokusdan iborat. Traektoriyalarning (spirallarning) buralish yo‘nalishini aniqlash uchun, masalan, $(1;0)$ nuqtada tezlik vektorini quramiz:

$x' = 2,\ y' = 1.$

Demak, traektoriyalar bo‘ylab harakatlanuvchi nuqta soat mili aylanishiga teskari yo‘nalishda harakatlanadi va u $(0;0)$ nuqtadan uzoqlashadi (11.14-. rasm). 👍



11.14-rasm.

Misol 2. Ushbu

$$\begin{cases} x' = 2x + 3y \\ y' = x + 4y \end{cases}$$

sistemaning maxsus nuqtasini tekshiring va maxsus nuqta atrofida traektoriyalarini chizing.

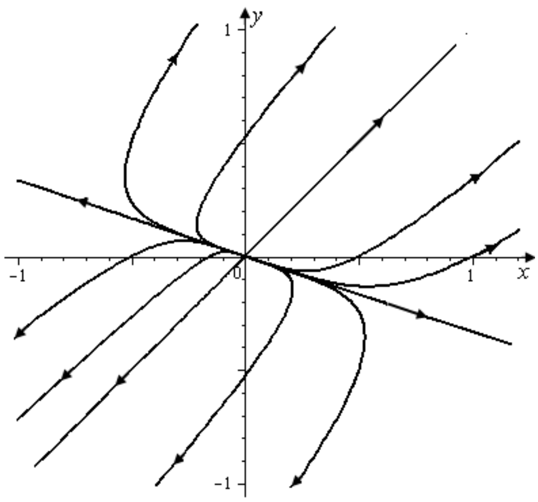
🗝 Berilgan sistemaning xarakteristik sonlari

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0;\ \lambda_1 = 1,\ \lambda_2 = 5.$$

$(0;0)$ muvozanat nuqta – tugun. Endi yarim to‘gri chiziqlardan iborat bo‘lgan $x = t,\ y = kt$ ($t > 0$ yoki $t < 0$) traektoriyalarni aniqlaymiz. Bularni berilgan sistemaga qo‘yib,

$$\begin{cases} 1 = 2t + 3kt \\ k = t + 4kt \end{cases};\ 3k^2 - 2k - 1 = 0;\ k_1 = 1,\ k_2 = -1/3$$

ekanligini topamiz. Demak, $y = x,\ y = -x/3,\ x > 0$ yoki $x < 0$, yarim to‘g‘ri chiziqlar izlangan traektoriyalarni ifodalaydi. Traektoriyalar bo‘ylab harakat yo‘nalishini topish uchun $(x_1; y_1) = (1;0)$ va $(x_2; y_2) = (-0,5; -0,5)$ nuqtalarda tezlik vektorlarini hisoblaymiz: $(x'; y') = (2;1)$ va $(x'; y') = (-2,5; -2,5)$. Traektoriyalar portreti 11.15-rasmda tasvirlangan. 👍



11.15- rasm.

Masalalar

1. $A - 3\times 3$ o‘lchamli haqiqiy matritsa bo‘lsin. Teskarilanuvchi shunday S matritsa topish mumkinki, uning uchun quyidagi tengliklarning biri o‘rinli bo‘ladi:

$$SAS^{-1}=\begin{pmatrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & \lambda\end{pmatrix},\ SAS^{-1}=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{pmatrix},\ SAS^{-1}=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix},$$

$$SAS^{-1}=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\ (\alpha,\beta,\lambda,\mu,\nu - \text{haqiqiy sonlar va } \beta\neq 0).$$

2. 1- masaladan foydalanib ushbu

$$\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=A\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

uch o‘lchamli avtonom sistemaning trayektoryalarini tekshiring.

§ 11.4. Tekislikda nochiziqli avtonom sistemalarning traektorialar manzarasi

Endi ushbu

$$\begin{cases}x' = f(x,y),\\ y' = g(x,y)\end{cases} \tag{11.4.1}$$

nochiziqli avtonom sistemani qaraylik. Bu yerda, soddalik uchun, f va g funksiyalarni ikki marta uzluksiz differensiallanuvchi ($\{f,g\}\subset C^2$) deb hisoblaymiz.

Sistemaning maxsus nuqtalari

$f(x,y)=0$, $g(x,y)=0$

tenglamalardan topiladi. (x_0,y_0) maxsus nuqtani tekshirish uchun bu nuqta atrofida $f(x,y)$ va $g(x,y)$ funksiyalarni Teylor formulasiga ko‘ra $r=\sqrt{(x-x_0)^2+(y-y_0)^2}\to 0$ bo‘lganda

$$f(x,y)=a(x-x_0)+b(y-y_0)+O(r^2),$$

$$a=\frac{\partial f(x_0,y_0)}{\partial x},\ b=\frac{\partial f(x_0,y_0)}{\partial y},$$

$$g(x,y)=c(x-x_0)+d(y-y_0)+O(r^2),$$

$$c=\frac{\partial g(x_0,y_0)}{\partial x},\ d=\frac{\partial g(x_0,y_0)}{\partial y},$$

ko‘rinishda tasvirlab, (11.4.1) sistemada $x = x_0 + u$, $y = y_0 + v$ almashtirishni bajaramiz, ya’ni koordinatalar boshini (x_0, y_0) maxsus nuqtaga ko‘chiramiz. Natijada

$$\begin{cases} u' = au + bv + O(\rho^2) \\ v' = cu + dv + O(\rho^2) \end{cases}, \quad \rho = \sqrt{u^2 + v^2} \to 0$$

tenglamalarni topamiz. Bu yerdagi yuqori tartibli cheksiz kichik miqdorlarni tashlab yuborishdan hosil bo‘luvchi ushbu

$$\begin{cases} u' = au + bv \\ v' = cu + dv \end{cases} \text{yoki} \begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix}, \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \qquad (11.4.2)$$

chiziqli avtonom sistema (11.4.1) nochiziqli avtonom sistemaning $(0,0)$ maxsus nuqta atrofida chiziqlilashtirilishi (yoki birinchi yaqinlashishi) deyiladi. (11.4.2) sistemaning $(0,0)$ maxsus nuqtasi, ya’ni (11.4.1) sistemaning (x_0, y_0) maxsus nuqtasi tabiatini quyidagi teorema ochadi.

Teorema. *Agar A matritsaning xos sonlari uchun* $\mathrm{Re}\,\lambda_1 \neq 0$ *va* $\mathrm{Re}\,\lambda_2 \neq 0$ *bo‘lsa,* (11.4.2) *nochiziqli sistema* $(0,0)$ *maxsus nuqtasining tipi* (*turi*) *chiziqlilashtirilgan* (11.4.2) *sistemaning maxsus nuqtasi tipi* (*turi*) *bilan bir xil. Bunda traektoriyalarning buralish va maxsus nuqtaga yaqinlashish yoki undan uzoqlashish yo‘nalishlari hamda turg‘unlik tabiatlari saqlanadi.*

Bu teoremaga qaraganda (kuchliroq) umumiyroq teoremaning isboti [9] da keltirilgan. 👍

Barcha maxsus nuqtalarning tabiatini tekshirib, ba’zi sohalarda tezliklar maydoni yo‘nalishlarini aniqlab, sistemaning traektoriyalari manzarasini quramiz.

Misol 1. Ushbu

$$\begin{cases} x' = 2x + y^2 - 1 \\ y' = 6x - y^2 + 1 \end{cases}$$

sistemaning traektoriyalar manzarasini quring.

🗝 Sistemaning muvozanat (kritik) nuqtalarini

$$\begin{cases} 2x + y^2 - 1 = 0 \\ 6x - y^2 + 1 = 0 \end{cases}$$

sistemani yechib topamiz. Ular ikkita: $(0,-1)$ va $(0,1)$. har bir kritik nuqta atrofida berilgan sistemani chiziqlilashtiramiz.

$(0,-1)$ nuqta atrofida chiziqlashtirilgan sistema

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_1 \begin{pmatrix} u \\ v \end{pmatrix}, A_1 = \left.\begin{pmatrix} (2x+y^2-1)'_x & (2x+y^2-1)'_y \\ (6x-y^2+1)'_x & (6x-y^2+1)'_y \end{pmatrix}\right|_{(0;-1)} = \begin{pmatrix} 2 & -2 \\ 6 & 2 \end{pmatrix},$$

$(0,1)$ nuqta atrofida chiziqlashtirilgan sistema esa

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_2 \begin{pmatrix} u \\ v \end{pmatrix}, A_2 = \left.\begin{pmatrix} (2x+y^2-1)'_x & (2x+y^2-1)'_y \\ (6x-y^2+1)'_x & (6x-y^2+1)'_y \end{pmatrix}\right|_{(0;1)} = \begin{pmatrix} 2 & 2 \\ 6 & -2 \end{pmatrix}$$

ko‘rinishga ega.

A_1 matritsaning xarakteristik sonlari kompleks $\lambda_1 = 2 + i2\sqrt{3}$, $\lambda_2 = 2 - i2\sqrt{3}$ va $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 2 > 0$. Demak, $(0,-1)$ muvozanat nuqta – noturg‘un fokus.

A_2 matritsaning xarakteristik sonlari turli ishorali: $\lambda_1 = -4, \lambda_2 = 4$. Demak, $(0,1)$ muvozanat nuqta – egar. Bu nuqtaga yaqinlashuvchi yoki undan uzoqlashuvchi to‘rtta traektoriya yo‘nalishini aniqlaymiz. Buning uchun

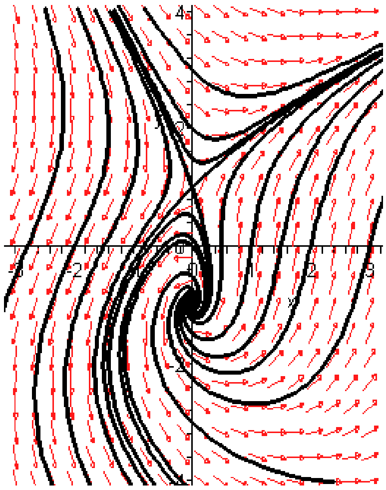
$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_2 \begin{pmatrix} u \\ v \end{pmatrix}, \text{ ya'ni } \begin{cases} u' = 2u + 2v \\ v' = 6u - 2v \end{cases}$$

sistemada $v = ku$ deymiz va noma’lum k sonni topamiz:

$$\begin{cases} u' = 2u + 2ku \\ kv' = 6u - 2ku \end{cases}, \; k^2 + 2k - 3 = 0, \; \{k_1 = 1, k_2 = -3\}.$$

Demak, $(0,1)$ muvozanat nuqtadan traektoriyalar $\alpha_1 = \mathrm{arctg}(1) = 45^0$ va $\alpha_2 = \mathrm{arctg}(-3) \approx -72^0$ burchak ostida ”o‘tadi”.

$2x + y^2 - 1 = 0$ va $6x - y^2 + 1 = 0$ parabolalar tekislikni beshta bo‘lakka ajratadi. har bir bo‘lakda tezlik vektorlarini tasvirlab, ularga urintirib bir nechta traektoriyalarni quramiz (11.16- rasm).



11.16- rasm.

Misol 2. Ushbu

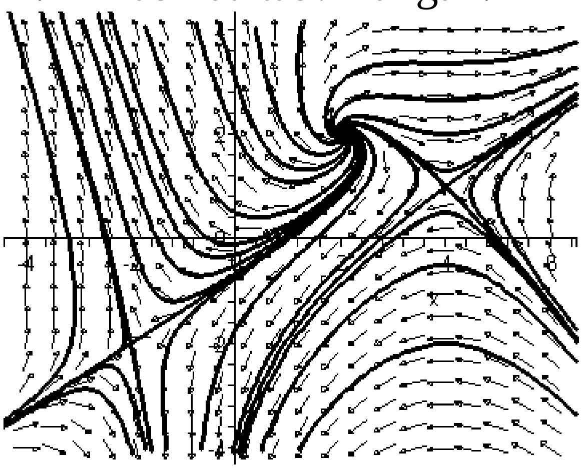
$$\begin{cases} x' = xy - 4 \\ y' = (x-4)(y-x) \end{cases}$$

sistemaning traektoriyalar manzarasi (portreti) ni tasvirlang.

Muvozanat nuqtalari:

$$(4;1) - \text{egar}, (2;2) - \text{ noturg‘un fokus}, \ (-2;-2) - \text{egar}.$$

Traektoriyalar 11.17- rasmda tasvirlangan.



11.17- rasm.

Masalalar

Quyidagi sistemalarning traektoriyalar manzarasini quring:

1. $\begin{cases} x' = (x-1)(y-x), \\ y' = x^2 + y^2 - 2. \end{cases}$ **2.** $\begin{cases} x' = 2x + 2y - xy - 3, \\ y' = x^2 - y^2. \end{cases}$ **3.** $\begin{cases} x' = (1-y)(y-2), \\ y' = \dfrac{1}{2}(x-1)(x-3). \end{cases}$

4. $\begin{cases} x' = y(7 - x^2 - y^2), \\ y' = 6 - x(7 - x^2 - y^2). \end{cases}$ **5.** $\begin{cases} x' = \sin y \\ y' = \sin x \end{cases}$

§ 11.5. Tekislikda avtonom sistemalarning sikllari (davralari)

Tekislikda ushbu

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}, \{f,g\} \subset C^1(G;\mathbb{R}), \qquad (11.5.1)$$

avtonom sistemani qaraylik.

Dastlab ba'zi tushuncha va tasdiqlarni eslaylik.

O'z-o'zini kesmaydigan yopiq traektoriyani sikl (davra) deb atagan edik. Sodda, ya'ni o'z-o'zini kesmaydigan va yopiq uzluksiz chiziq Jordan chizig'i deb ataladi. Jordan chizig'i aylananing uzluksiz biyektiv aksidan iborat bo'ladi. Jordan teoremasiga ko'ra har qanday Jordan chizig'i tekislikni chegaralari shu chiziqdan iborat bo'lgan ikkita chegaralangan va chegaralanmagan sohaga ajratadi; chegaralangan soha berilgan Jordan chizig'ining ichki qismi, chegaralanmagan soha esa uning tashqi qismi deb ataladi. Agar $G \subset \mathbb{R}^2$ sohada joylashgan har qanday Jordan chizig'ining ichki qismi to'laligicha Gda joylashsa, Gsoha **bir bog'lamli soha** deb ataladi.

Sikllarning mavjudmasligini isbotlashda quyidagi teoremadan foydalanish mumkin.

Teorema (Puankare). *Agar G – bir bog'lamli soha, G–* (11.5.1) *sistemaning Gda joylashgan yopiq traektoriyasi bo'lsa, u holda Γ ning ichki qismida kamida bitta kritik* (*statsionar*) *nuqta mavjud.*

Qat'iy va to'la isbot topologiya elementlarini bilishni talab qiladi [1]. Biz isbotning asosiy g'oyalarini keltiramiz. G – bir bog'lamli bo'lgani uchun $G \subset G$ yopiq traektoriyani uning ichidagi ixtiyoriy O^*nuqtaga uzluksiz deformatsiyalash mumkin, ya'ni $s \in [0;1]$ parametrga uzluksiz bog'liq bo'lgan shunday Γ_s sodda yopiq chiziqlar oilasi mavjudki, uning uchun $\Gamma_1 = \Gamma$ va $\Gamma_0 = O^*$ bo'ladi. Faraz qilaylik, (f,g) vektor-maydon G ning ichki qismida nolga aylanmasin. U holda Γda va uning ichidagi har qanday nuqtada $(f,g) \neq (0,0)$bo'lgani uchun shu vektor bilan Ox o'qi orasidagi θ burchakni uzluksiz o'zgaruvchi funksiya sifatida aniqlash mumkin. Γ_s ning biror nuqtasidan boshlab Γ_s bo'ylab harakatlaninib yana shu nuqtaga qaytib kelib to'xtasak, θ burchak biror $2\pi k, k \in \mathbb{Z}$ $(k \neq 0)$, orttirma oladi. Bu yerdagi k butun sonni $\Gamma_s$ning (f,g) vektor-maydonga nisbatan tartibi deymiz va uni T_s bilan belgilaymiz. T_s butun qiymatlar qabul qiladi va u $s\in[0;1]$ ning uzluksiz funksiyasi. Demak, T_s –o‘zgarmas, ya’ni $T_s=\text{const}=k_0\in\mathbb{Z}$, $s\in[0;1]$, bo‘lishi kerak. Lekin $\Gamma=\Gamma_1$ traektoriyaning ixtiyoriy nuqtasidagi (f,g) tezlik vektori shu nuqtada Γga urinadi, demak, shu traektoriya bo‘ylab soat mili yo‘nalishida (yoki teskari yo‘nalishda) bir marta to‘la aylanib chiqilganda θ burchak -2π (mos ravishda $+2\pi$) orttirma oladi, ya’ni $T_s=-1$ (mos ravishda $T_s=+1$). T_0 esa 0ga teng, chunki $\Gamma_0=O^*$ nuqtaning yetarlicha kichik atrofida (f,g) uzluksiz vektor-maydon deyarli o‘zgarmaydi. Hosil bo‘lgan ziddiyat farazimizning noto‘g‘riligini va (f,g) vektor-maydonning Γ ning ichki qismida kamida bitta kritik (statsionar) nuqtaga ega ekanligini isbotlaydi. ✍

Natija. *Agar bir bog‘lamli G sohada* (11.5.1) *sistemaning kritik nuqtasi bo‘lmasa, uning Gda yopiq traektoriyasi ham mavjud emas*.

Sikllarning mavjud emasligini isbotlashda quyidagi teoremadan ham foydalanish mumkin.

Teorema (Bendikson-Dyulak). *Agar G bir bog‘lamli soha bo‘lib, biror $h\in C^1(G)$ funksiya uchun Gsohada*

$$\frac{\partial(hf)}{\partial x}+\frac{\partial(hg)}{\partial y}>0$$

tengsizlik bajarilsa, u holda (11.5.1) *sistema Gsohada siklga ega emas.*

🗝Teskarisini faraz qilaylik, ya’ni (11.5.1) sistemaning Γ sikli mavjud bo‘lsin. G ning ichki qismini G^* bilan belgilaylik. U holda analizdan ma’lum bo‘lgan Grin formulasiga ko‘ra

$$\iint\limits_{G^*}\left(\frac{\partial(hf)}{\partial x}+\frac{\partial(hg)}{\partial y}\right)dxdy=\int\limits_{\Gamma}h(gdx-fdy)\,.$$

Bu tenglikning chap tomoni musbat son, o‘ng tomoni esa nolga teng, chunki G – (11.5.1) sistemaning traektoriyasi, ya’ni G da $gdx-fdy=0$. Hosil bo‘lgan ziddiyat siklning mavjud emasligini isbotlaydi. ✍

Limit davralar. (11.5.1) sistemaning limit davrasi deb uning (yakkalangan) ajratilgan sikliga aytiladi. Aniqrog‘i, agar G siklning yetarlicha kichik atrofida Gdan boshqa sikl mavjud bo‘lmasa, u holda G limit sikl (yoki limit davra) deb ataladi. Turli traektoriyalar umumiy nuqtaga ega bo‘lolmaganligi sababli Jordan teoremasiga ko‘ra G sikldan farqli har qanday traektoriya to‘laligicha yo Gning ichki qismida, yoki uning tashqi qismida joylashadi.

Misol. Ushbu

$$\begin{cases} x' = -y \\ y' = \ x \end{cases}$$

sistema cheksiz ko‘p yakkalanmagan sikllarga ega:

$$\begin{cases} x = c_1 \cos(t + c_2) \\ y = c_1 \sin(t + c_2) \end{cases}, \ x^2 + y^2 = c_1^2 .$$

Qutb koordinatalariga o‘tib, ushbu

$$\begin{cases} x' = -y + x\sin(x^2 + y^2) \\ y' = x + y\sin(x^2 + y^2) \end{cases}$$

sistema cheksiz ko‘p

$$\begin{cases} x = \sqrt{k\pi} \cos t \\ y = \sqrt{k\pi} \sin t \end{cases}, \ k = 1, 2, \ldots, \ (x^2 + y^2 = k\pi)$$

yakkalangan sikllarga (limit davralarga) ega ekanligini va traektoriyalarining tabiatini aniqlash mumkin. ☝

Teorema. *G (11.5.1) sistemaning limit davrasi bo‘lsin. U holda Γning ichki (tashqi) qismidagi unga yetarlicha yaqin nuqtadan boshlangan har qanday traektoriya yo $t \to +\infty$da, yoki $t \to -\infty$da G ga spiralsimon o‘raladi.*

Agar Γ limit davraning yetarlicha yaqinidan boshlangan tashqi va ichki traektoriyalar $t \to +\infty$da Gga o‘ralsa, u holda G limit davra ***turg‘un limit davra*** deb ataladi. Aks holda, ya’ni agar G limit davra turg‘un limit davra bo‘lmasa, u ***noturg‘un limit davra*** deb ataladi.

Noturg‘un limit davralar ikki xil bo‘ladi:

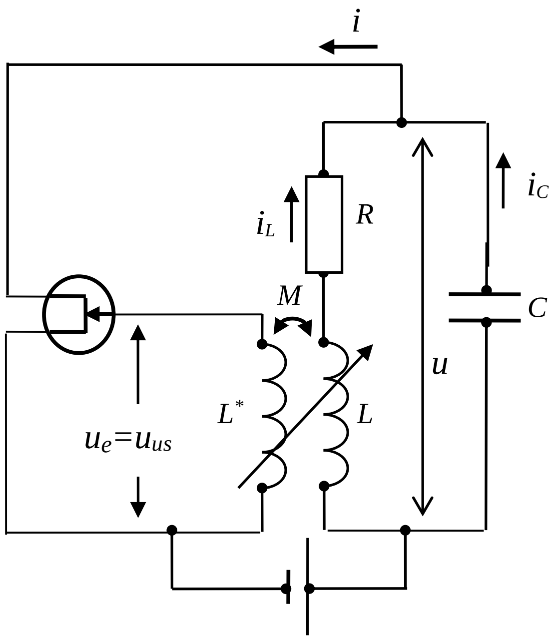
yarimturg‘un – tashqi traektoriyalar $t \to +\infty$da (yoki $t \to -\infty$da) ichki traektoriyalar esa $t \to -\infty$da (mos ravishda $t \to +\infty$da) Gga o‘raladi;

to‘la noturg‘un – tashqi va ichki traektoriyalar $t \to -\infty$da Gga o‘raladi.

Teoremaning isboti va traektoriyalarning limit davraga “o‘ralishining” to‘la tavsifi ***ergash funksiya*** tushunchasiga tayanadi. Biz teoremani isbotlamaymiz.

Avtotebranishlar generatori. Ba’zi fizik sistemalarda tashqi ta’sirsiz doimiy ravishda takrorlanib turuvchi (davriy) o‘zgarishlar (harakatlar) kuzatiladi. Masalan, mayatnikli soat, yuqori chastotali elektr tebranishlar generatori bunaqa sistemalarga misol bo‘la oladi. Elektr tebranishlar generatori triodlar yoki tranzistorlar asosida tuzilishi mumkin.

Eng oddiy tranzistorli elektr tebranishlar generatori LCR tebranishlar konturi, L ga induktiv bog‘liq bo‘lgan va tranzistorga ulangan L^* g‘altak ($M -$induksiya koeffitsienti) hamda elektr manbasidan iborat (11.18- rasm).



11.18- rasm.

Kirrxgoff qonuniga ko‘ra $i = i_L + i_C$. Ma’lumki, kondensatordagi tok $i_C = C\dfrac{du}{dt}$. Yana Kirxgof qonuniniga ko‘ra $u = Ri_L + L\dfrac{di_L}{dt}$. Demak,

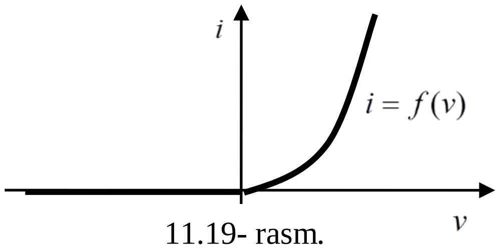
$$i = i_L + CR\frac{di_L}{dt} + CL\frac{d^2 i_L}{dt^2}. \qquad (11.5.6)$$

Bundan tashqari,

$$u_{us} = M\frac{di_L}{dt},\; i = f(u_{us}) = f(M\frac{di_L}{dt}); \qquad (11.5.7)$$

bu yerda $i = f(v)$ – tranzistorning xarakteristikasi, u i tokning v kuchlanishga bog‘lanish qonuniyatini ifodalaydi. Bu bog‘lanishning tipik grafigi 11.19- rasmda keltirilgan. (11.5.7) ni (11.5.6) ga qo‘yib, i_L tok uchun quyidagi ikkinchi tartibli differensial tenglamani topamiz:

$$CL\frac{d^2 i_L}{dt^2} + CR\frac{di_L}{dt} - f(M\frac{di_L}{dt}) + i_L = 0. \qquad (11.5.8)$$



11.19- rasm.

Bu yerda shuni e’tirof etaylikki, uch elektrodli elektron lampali generatordagi anod toki ham (11.5.8) ko‘rinishdagi tenglamani qanoatlantiradi [14].

Qulaylik uchun (11.5.8) tenglamada $t\sqrt{LC} = \tau$, $i_L(t) - f(0) = x(\tau)$ almashtirish bajaraylik. U holda ushbu

$$\frac{d^2 x}{d\tau^2} + F(\frac{dx}{d\tau}) + x = 0, \qquad (11.5.9)$$

tenglamaga kelamiz; bu yerda

$$F(y) = R\sqrt{\frac{C}{L}}y - f(\frac{M}{\sqrt{LC}}y) + f(0),\ F(0) = 0. \qquad (11.5.10)$$

Odatdagicha (11.5.9) tenglamadan ushbu

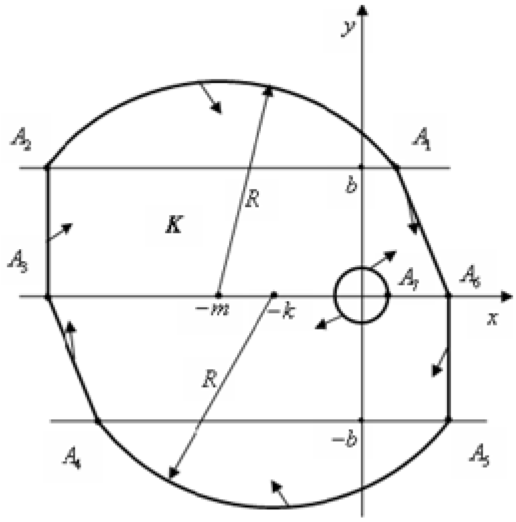
$$\frac{dx}{d\tau} = y,\ \frac{dy}{d\tau} = -x - F(y) \qquad (11.5.11)$$

normal sistemaga o‘tamiz. Ravshanki, (11.5.11) sistema yagona kritik (muvozanat) nuqtaga ega: $x = 0, y = 0$.

Jumla. *Faraz qilaylik,* $F \in C^1$, $F(0) = 0, F'(0) < 0$, $y \geq b$ $(b > 0)$ *bo‘lganda* $F(y) > m$, $y \leq -b$ *bo‘lganda esa* $F(y) < k$ $(k < m)$ *bo‘lsin. U holda* (11.5.11) *sistema yopiq traektoriyaga ega.* (*Agar* f – *chegaralangan,* $\in C^1$ *hamda* $Mf'(0) > RC$ *bo‘lsa,* (11.5.10) *dagi* F *funksiya keltirilgan faraz shartlarini qanoarlantiradi.*)

🗝 Oxy tekislikda shunday yopiq xalqasimon soha K ni quramizki, har qanday yechim uning ichidan tashqarisiga chiqib ketmaydi.

K ning ichki chegarasini $x^2 + y^2 = r^2$ aylana ko'rinishida tanlaymiz. $r > 0$ ni shunday kichik tanlaymizki, uning uchun $0 < |y| \le r$ bo'lganda $yF(y) < 0$ bo'lsin. $|y| \le r$ bo'lganda (11.5.11) sistemaning yechimi uchun $\frac{d}{d\tau}(x^2 + y^2) = -2yF(y) \ge 0,$ demak, yechim $x^2 + y^2 = r^2$ aylanadan uning ichiga kirolmaydi.



11.20- rasm.

K ning tashqi chegarasini bir necha qismdan iborat qilib tuzamiz. Uning $y \ge b$ yarim tekislikdagi chegarasi $(x+m)^2 + y^2 = R^2$ aylananing $\overset{\cup}{A_1A_2}$ yoyidan iborat bo'lsin, R ni keyinqoq tanlaymiz (11.20- rasm). Bu yoyda (11.5.11) sistemaning yechimi uchun

$$\frac{d}{d\tau}((x+m)^2 + y^2) = 2y(m - F(y)) < 0,$$

chunki $y \ge b$ bo'lganda $F(y) > m$. $y \le -b$ yarim tekislikda $(x+k)^2 + y^2 = R^2$ aylananing $\overset{\cup}{A_4A_5}$ yoyini olamiz. A_2A_3 va A_5A_6 vertikal kesmalarda mos ravishda $x' = y > 0$ va $x' = y < 0$ va, demak, ular orqali traektoriyalar K ga kiradi.

A_6A_1 va A_5A_4 kesmalarning burchak koeffitsienti $-b/(m-k)$. Bu kesmalarni kesuvchi traektoriyalarda $\frac{dy}{dx} = -\frac{x + F(y)}{y}$. $R > 0$ ni yetarlicha katta tanlash evaziga $|x|$ ni kattalashtiramiz va $\frac{dy}{dx} < -\frac{b}{m-k}$ tengsizlikning bajarilishini ta'minlaymiz. U holda traektoriyalar A_6A_1 va A_5A_4 kesmalar orqali K ga kiradi.

Shunday qilib, qurilgan yopiq xalqasimon K sohada maxsus nuqta yo'q va (11.5.11) sistemaning traektoriyalari K dan chiqib ketmaydi. Bundan esa K da (11.5.11) sistemaning yopiq traektoriyasi mavjud ekanligi kelib chiqadi.

Izoh. Hilbertning 16- muammosi tekislikda

$$\begin{cases} x' = A(x, y) \\ y' = B(x, y) \end{cases}$$, bunda $A(x, y)$ va $B(x, y)$ ko‘phadlar

polinomial sistemaning maksimal limit davralar sonini va ularning o‘zaro joylashuvini aniqlash bilan bog‘liq. $A(x, y)$ va $B(x, y)$ ko‘phadlar darajalarining kattasini n bilan, sistemaning maksimal limit davralar sonini H_n bilan belgilaylik. Ma’lumki, $H_0 = 0$, $H_1 = 0$, $H_2 \geq 4$, $H_3 \geq 8$, toq $n$lar uchun $H_n \geq (n-1)/2$, hamda $H_n < +\infty$. Lekin hatto $H_2 = 4$ degan (gipoteza) taxmin ham hanuzgacha to‘la isbotlanmagan.

Masalalar

1. Ushbu

$$\begin{cases} x' = y \\ y' = -p(y)y - x \end{cases}, \quad p(y) \in C(\mathbb{R}),\ p(y) > 0,$$

sistema limit siklga ega emasligini isbotlang.

Quyidagi sistemalar davriy yechimga ega emasligini isbotlang

2. $\begin{cases} x' = y, \\ y' = -ax - by + \alpha x^2 + \beta y^2. \end{cases}$

3. $\begin{cases} x' = x(a_1 x + b_1 y + c_1), \\ y' = y(a_2 x + b_2 y + c_2). \end{cases}$

4. $\begin{cases} x' = y + x(1 + \beta y)(x^2 + y^2 + 1), \\ y' = -x + (y - \beta x^2)(x^2 + y^2 + 1). \end{cases}$

5. Paragraph oxiridagi jumlani to‘la isbotlang.

MODUL 12. YECHIMNING BOSHLANG`ICH MA'LUMOTLAR VA PARAMETRLARGA SILLIQ BOG`LIQLIGI VA UNING TATBIQLARI

§ 12.1. Yechimning boshlang'ich ma'lumotlar va parametrlar bo'yicha differensiallanuvchligi

Ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu}) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \end{cases} \tag{12.1.1}$$

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \in M$ $(M \subset \mathbb{R}^m -$soha) $\boldsymbol{\mu}$ parametr(lar)ga bog`liq bo'lgan Koshi masalasini qaraylik, bunda $(t, \boldsymbol{x}) \in D \subset \mathbb{R}^{1+n}$. Faraz qilaylik, har bir $(t_0, \boldsymbol{x}^0, \boldsymbol{\mu}) \in D \times M$ uchun (12.1.1) masala $t \in I$ intervalda aniqlangan yagona davomsiz yechimga ega bo'lsin. Bu yechim nafaqat t ga, balki tayinlangan $(t_0, \boldsymbol{x}^0, \boldsymbol{\mu}) \in D \times M$ qiymatlarga ham bog'liq bo'ladi va uni biz $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ ko'rinishda belgilaymiz. Davomsiz yechimning aniqlanish intervali tayinlangan $(t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ qiymatlarga bog'liq bo'lgani ($I = I(t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$) uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ yechim $(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu}) \in I \times D \times M \subset R^{2+n+m}$ sohada aniqlangan. Agar $(t_0, \boldsymbol{x}^0)$ tayinlangan bo'lib, faqat $\boldsymbol{\mu}$ parametr turli qiymatlar qabul qilsa, u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0, \boldsymbol{\mu})$ yozuv o'rniga qisqaroq $\boldsymbol{x} = \boldsymbol{\varphi}(t; \boldsymbol{\mu})$ yozuvni ishlatamiz. Biz yechimning parametrlarga uzluksiz bog'liqligini § 8.5 da o'rgangan edik. Endi uning differensiallanuvchiligini o'rganamiz.

Soddalik uchun parametrlar sonini birga teng deb hisoblaymiz. Bu holda $m = 1$ va $M -$sonli interval, $\boldsymbol{\mu} = \mu \in M$.

Teorema 1 (yechimning parametr bo'yicha differensiallanuvchiligi). *Aytaylik,* $\boldsymbol{f}(t, \boldsymbol{x}, \mu)$, $\dfrac{\partial \boldsymbol{f}(t, \boldsymbol{x}, \mu)}{\partial x_j}$ $(j = 1, \ldots, n)$, $\dfrac{\partial \boldsymbol{f}(t, \boldsymbol{x}, \mu)}{\partial \mu}$ *funksiyalar* $(t, \boldsymbol{x}, \mu) \in D \times M$ *sohada uzluksiz,* (12.1.1) *masalaning* $\boldsymbol{x} = \boldsymbol{\varphi}(t; \mu)$ *yechimi esa har bir* $\mu \in M$ *uchun* $t \in [t_1, t_2]$ $(t_0 \in [t_1, t_2])$ *segmentda aniqlangan bo'lsin. U holda bu* *yechimning* $\boldsymbol{u} \equiv \dfrac{\partial \boldsymbol{\varphi}(t;\mu)}{\partial \mu}$ *(*$\boldsymbol{u} = \boldsymbol{u}(t;\mu)$*) hosilasi* $(t,\mu) \in [t_1, t_2] \times M$ *bo'lganda uzluksiz va u variatsiya uchun tenglama deb ataluvchi ushbu*

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{u} + \frac{\partial \boldsymbol{f}}{\partial \mu}, \ \boldsymbol{u}\big|_{t=t_0} = 0, \qquad (12.1.2)$$

chiziqli tenglamani qanoatlantiradi, bunda xususiy hosilalar $\boldsymbol{x} = \boldsymbol{\varphi}(t;\mu)$ *bo'lganda hisoblangan, ya'ni*

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial \boldsymbol{x}}\bigg|_{\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)}, \quad \frac{\partial \boldsymbol{f}}{\partial \mu} = \frac{\partial \boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial \mu}\bigg|_{\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)}.$$

Variatsiya uchun (12.1.2) *vektorli tenglamaning skalyar ko'rinishi quyidagi variatsiyalar uchun tenglamalar sistemasidan iborat:*

$$\frac{du_i}{dt} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} u_j + \frac{\partial f_i}{\partial \mu}, \ u_i\big|_{t=t_0} = 0 \ (i = 1,...,n). \qquad (12.1.3)$$

🗝 Teoremaning shartlariga ko'ra har qanday $\mu \in M$ uchun (12.1.2) chiziqli masalaning $t \in [t_1, t_2]$ segmentda aniqlangan yagona $\boldsymbol{u} = \boldsymbol{u}(t;\mu)$ yechimi mavjud va yechimning parametrlarga uzluksiz bog'liqligi to'g'risidagi teotremaga ko'ra $\boldsymbol{u}(t;\mu) \in C([t_1, t_2] \times M)$. Bu yechim, ravshanki, ushbu

$$\boldsymbol{u}(t;\mu) = \int_{t_0}^{t} \frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \boldsymbol{x}} \boldsymbol{u}(s;\mu)\,ds + \int_{t_0}^{t} \frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \mu}\,ds \qquad (12.1.4)$$

tenglamani qanoatlantiradi. Teoremani isbot qilish uchun hosilaning ta'rifiga ko'ra $\mu \in M$ tayinlanganda

$$\boldsymbol{\psi}(t;\bar{\mu}) \overset{def}{=} \boldsymbol{\varphi}(t;\bar{\mu}) - \boldsymbol{\varphi}(t;\mu) - \boldsymbol{u}(t;\mu)(\bar{\mu} - \mu) \ (\bar{\mu} \in M) \qquad (12.1.5)$$

funksiya uchun

$$\|\boldsymbol{\psi}(t;\bar{\mu})\| = o(\bar{\mu} - \mu), \ \bar{\mu} \to \mu,$$

asimptotik tenglikning o'rinli ekanligini ko'rsatamiz. Bunda $\boldsymbol{x} = \boldsymbol{\varphi}(t;\bar{\mu})$ vektor-funksiya (12.1.1) masalaning parametr $\bar{\mu}$ ga teng bo'lgandagi yechimi, ya'ni

$$\begin{cases} \dfrac{d\boldsymbol{\varphi}(t;\bar{\mu})}{dt} = \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\bar{\mu}),\bar{\mu}) \\ \boldsymbol{\varphi}(t;\bar{\mu})|_{t=t_0} = \boldsymbol{x}^0. \end{cases} \tag{12.1.6}$$

Demak,

$$\boldsymbol{\varphi}(t;\bar{\mu}) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\bar{\mu}),\bar{\mu})\,ds\ . \tag{12.1.7}$$

Shunga o‘xshash

$$\boldsymbol{\varphi}(t;\mu) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)\,ds\ . \tag{12.1.8}$$

Endi (12.1.7), (12.1.8) va (12.1.4) formulalarga ko‘ra (12.1.5) dan quyidagini hosil qilamiz:

$$\boldsymbol{\psi}(t;\bar{\mu}) = \int_{t_0}^{t} \boldsymbol{\Psi}(s;\bar{\mu})\,ds\,, \tag{12.1.9}$$

bu yerda qisqalik uchun

$$\boldsymbol{\Psi}(s;\bar{\mu}) \overset{def}{=} \boldsymbol{F}(s,\bar{\mu}) - (\bar{\mu}-\mu)\frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \boldsymbol{x}}\boldsymbol{u}(s;\mu) -$$

$$-(\bar{\mu}-\mu)\frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \mu}, \tag{12.1.10}$$

$$\boldsymbol{F}(s,\bar{\mu}) \overset{def}{=} \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\bar{\mu}),\bar{\mu}) - \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu) \tag{12.1.11}$$

deb belgilangan. $\boldsymbol{F}(s) = \left(F_1(s), F_2(s), ..., F_n(s)\right)^T$ vektor-funksiya koordinata-larining ko‘rinishini Lagranj formulasi va (12.1.5) dan topilgan $\boldsymbol{\varphi}(t;\bar{\mu}) - \boldsymbol{\varphi}(t;\mu) = \boldsymbol{\psi}(t;\bar{\mu}) + (\bar{\mu}-\mu)\boldsymbol{u}(t;\mu)$ tenglikka ko‘ra quyidagicha almashtiramiz:

$$F_j(s,\bar{\mu}) = (f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\bar{\mu}) - f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\mu)) + (f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\mu) -$$

$$- f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)) = \frac{\partial f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\mu^{j*})}{\partial \mu}(\bar{\mu}-\mu) +$$

$$+\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(s;\bar{\mu}) - \boldsymbol{\varphi}(s;\mu)) = \frac{\partial f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\mu^{j*})}{\partial \mu}(\bar{\mu}-\mu) +$$

$$+\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial \boldsymbol{x}}(\boldsymbol{\psi}(s;\bar{\mu}) + (\bar{\mu}-\mu)\boldsymbol{u}(s;\mu))\ ,$$

$$\mu^{j*}=\mu+\theta_{1j}(\bar{\mu}-\mu),\ \boldsymbol{x}^{j*}=\boldsymbol{\varphi}(s;\mu)+\theta_{2j}(\boldsymbol{\varphi}(s;\bar{\mu})-$$
$$-\boldsymbol{\varphi}(s;\mu)),\ 0<\theta_{1j},\theta_{2j}<1,\ j=1,...,n. \qquad 12.1.12)$$

Shunday qilib,

$$F_j(s,\bar{\mu})=\left(\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\bar{\mu}),\mu^{j*})}{\partial\mu}+\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial\boldsymbol{x}}\boldsymbol{u}(s;\mu)\right)(\bar{\mu}-\mu)+$$
$$+\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial\boldsymbol{x}}\boldsymbol{\psi}(s;\bar{\mu}). \qquad (12.1.13)$$

Endi (12.1.10) dan (12.1.11) va (12.1.13) ga ko‘ra quyidagini hosil qilamiz:

$$\Psi_j(s;\bar{\mu})=(\bar{\mu}-\mu)[(\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu^{j*})}{\partial\mu}-\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial\mu})+$$
$$+(\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial\boldsymbol{x}}-\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial\boldsymbol{x}})\boldsymbol{u}(s;\mu)]+\frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial\boldsymbol{x}}\boldsymbol{\psi}(s;\bar{\mu}).$$
$$(12.1.14)$$

$\bar{\mu}\in M$ o‘zgaruvchining (tayinlangan) $\mu\in M$ ga yaqin qiymatlarida (12.1.14) formulaning o‘ng tomondagi o‘rta qavs ichidagi birinchi va ikkinchi qo‘shiluvchilarni xohlagancha kichik qilish mumkinligini ko‘rsatamiz.

Ixtiyoriy $\varepsilon>0$ soni berilgan bo‘lsin. Uzluksiz funksiyalar kompozitsiyasi sifatida $g_j(s,\bar{\mu})=\dfrac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\bar{\mu})}{\partial\mu}$ $(j=1,...,n)$ funksiya $s\in[t_1,t_2]$ va μ ga yetarlicha yaqin $\bar{\mu}$ har uchun tekis uzluksiz bo‘ladi (Kantor teoremasiga ko‘ra). Demak, $\varepsilon>0$ soniga ko‘ra shunday $\delta=\delta(\varepsilon)>0$ topish mumkinki, $|\bar{\mu}-\mu|<\delta$ ekanligidan barcha $s\in[t_1,t_2]$ lar uchun

$$\left|\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\bar{\mu})}{\partial\mu}-\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial\mu}\right|<\varepsilon$$

bo‘lishi kelib chiqadi. $|\bar{\mu}-\mu|<\delta$ bo‘lganda (12.1.12) ga ko‘ra $|\mu^{j*}-\mu|<|\bar{\mu}-\mu|<\delta$ va, demak, $s\in[t_1,t_2]$ lar uchun

$$\left|\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu^{j*})}{\partial\mu}-\frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial\mu}\right|<\varepsilon \qquad (12.1.15)$$

va,demak, vektorning normasi uchun

$$\left\| \frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu^{j*})}{\partial \mu} - \frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \mu} \right\| < n\varepsilon \qquad (12.1.16)$$

tengsizlik ham o‘rinli bo‘ladi.

Endi $h_k^j(s,\boldsymbol{x}) = \dfrac{\partial f_j(s,\boldsymbol{x},\mu)}{\partial x_k}$ $(j,k=1,...,n)$ funksiyani qaraylik. U D sohada joylashgan ixtiyoriy kompaktda tekis uzluksiz. Uni D da yotuvchi va $\boldsymbol{\varphi}(s;\mu)$, $s\in[t_1,t_2]$, funksiyaning grafigini o‘z ichiga oluvchi kompaktda qaraymiz. Demak, shunday $\sigma=\sigma(\varepsilon)>0$ mavjudki, $\|\bar{\boldsymbol{x}}-\boldsymbol{x}\|<\sigma$ tengsizlikdan barcha $s\in[t_1,t_2]$ lar uchun

$$\left| \frac{\partial f_j(s,\bar{\boldsymbol{x}},\mu)}{\partial x_k} - \frac{\partial f_j(s,\boldsymbol{x},\mu)}{\partial x_k} \right| < \varepsilon \qquad (12.1.17)$$

ekanligi kelib chiqadi. $\boldsymbol{\varphi}(s;\bar{\mu})$ funksiya $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ (δ_0 – yetarlicha kichik musbat son) bo‘lganda tekis uzluksiz. Demak, shunday $\delta_1=\delta_1(\varepsilon)>0$ topiladiki, $|\bar{\mu}-\mu|<\delta_1$ bo‘lishidan barcha $s\in[t_1,t_2]$ lar uchun

$$\|\boldsymbol{\varphi}(s;\bar{\mu})-\boldsymbol{\varphi}(s;\mu)\|<\sigma \qquad (12.1.18)$$

tengsizlik kelib chiqadi. $\delta_1=\delta$ deb hisoblaymiz (har doim ularni kichiklashtirish mumkin). Shunday qilib, $|\bar{\mu}-\mu|<\delta$ bo‘lganda barcha $s\in[t_1,t_2]$ lar uchun (12.1.12) ga ko‘ra $\|\boldsymbol{x}^{j*}-\boldsymbol{\varphi}(s;\mu)\|<\|\boldsymbol{\varphi}(s;\bar{\mu})-\boldsymbol{\varphi}(s;\mu)\|<\sigma$ va (12.1.18) va (12.1.17) tengsizliklarga asosan

$$\left| \frac{\partial f_j(s,\boldsymbol{x}^{j*},\mu)}{\partial x_k} - \frac{\partial f_j(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial x_k} \right| < \varepsilon$$

va, demak, matritsaning normasi uchun

$$\left\| \frac{\partial \boldsymbol{f}(s,\boldsymbol{x}^{j*},\mu)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\mu),\mu)}{\partial \boldsymbol{x}} \right\| < n\varepsilon \qquad (12.1.19)$$

bo‘ladi. $\boldsymbol{u}(s;\mu)$ funksiya $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ bo‘lganda chegaralangan, ya’ni biror $\tilde{L}>0$ soni uchun

$$\|\boldsymbol{u}(s;\mu)\|\le\tilde{L} \qquad (12.1.20)$$

baholash o‘rinli; shunga o‘xshash biror $L>0$ va barcha $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ lar uchun

$$\left\| \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^*, \mu)}{\partial \boldsymbol{x}} \right\| \le L \qquad (12.1.21)$$

tengsizlik ham o‘rinli bo‘ladi.

Nihoyat, ixtiyoriy $\bar{\mu} \in B_\delta(\mu)$ va ixtiyoriy $s \in [t_1, t_2]$ uchun (12.1.16), (12.1.19), (12.1.20) va (12.1.21) tengsizliklarga ko‘ra (12.1.14) formuladan Koshi-Bunyakovskiy tengsizligidan foydalanib, quyidagi baholashlarni amalga oshiramiz:

$$\|\Psi(s;\bar{\mu})\| \le |\bar{\mu} - \mu| \cdot \left(\left\| \frac{\partial \boldsymbol{f}(s, \boldsymbol{\varphi}(s;\mu), \mu^{j*})}{\partial \mu} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{\varphi}(s;\mu), \mu)}{\partial \mu} \right\| + \right.$$

$$\left. + \left\| \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^{j*}, \mu)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{\varphi}(s;\mu), \mu)}{\partial \boldsymbol{x}} \right\| \cdot \|\boldsymbol{u}(s;\mu)\| \right) +$$

$$+ \left\| \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^{j*}, \mu)}{\partial \boldsymbol{x}} \right\| \cdot \|\boldsymbol{\psi}(s;\bar{\mu})\| \le$$

$$\le |\bar{\mu} - \mu| \cdot (n\varepsilon + n\varepsilon\tilde{L}) + L\|\boldsymbol{\psi}(s;\bar{\mu})\|.$$

Oxirgi tengsizlikka ko‘ra (12.1.9) formuladan barcha $\bar{\mu} \in B_\delta(\mu)$ va $t \in [t_1, t_2]$ lar uchun ushbu

$$\|\boldsymbol{\psi}(t;\bar{\mu})\| \le |\bar{\mu} - \mu| \cdot (1 + \tilde{L}) \cdot n \cdot |t - t_0| \varepsilon + L \left| \int_t^t \|\boldsymbol{\psi}(s;\bar{\mu})\| ds \right|$$

baholashni topamiz. Bundan Gronuoll-Bellman tengsizligiga ko‘ra (§ 3.2)

$$\|\boldsymbol{\psi}(t;\bar{\mu})\| \le |\bar{\mu} - \mu| n \frac{1 + \tilde{L}}{L} (e^{L|t - t_0|} - 1)\varepsilon.$$

Bu tengsizlikdan $t \in [t_1, t_2]$ ga nisbatan tekis baho(lash)ni ham topish mumkin:

$$\|\boldsymbol{\psi}(t;\bar{\mu})\| \le |\bar{\mu} - \mu| n \frac{1 + \tilde{L}}{L} \sup_{t \in [t_1, t_2]} (e^{L|t - t_0|} - 1)\varepsilon \le$$

$$|\bar{\mu} - \mu| n \frac{1 + \tilde{L}}{L} (e^{L(t_2 - t_1)} - 1) \cdot \varepsilon. \quad (12.1.22)$$

Shunday qilib, ixtiyoriy $\varepsilon > 0$ soniga ko‘ra shunday $\delta > 0$ soni topildiki, $|\bar{\mu} - \mu| < \delta$ tengsizlikdan (12.1.22) tengsizlik kelib chiqdi. Bu

$$\|\boldsymbol{\psi}(t;\bar{\mu})\| = o(\bar{\mu}-\mu),\ \bar{\mu}\to\mu,$$

ya'ni

$$\boldsymbol{\psi}(t;\bar{\mu}) = \boldsymbol{\psi}(t;\mu) + \boldsymbol{u}(t;\mu)(\bar{\mu}-\mu) + o(\bar{\mu}-\mu)\ \left(o(\bar{\mu}-\mu) \underset{t\in[t_1,t_2]}{\rightrightarrows} 0,\ \bar{\mu}\to\mu\right).$$

ekanligini anglatadi. ☝

Teorema parametrlar soni bittadan ko'p bo'lganda ham o'rinli. Bu holda $\boldsymbol{x} = \boldsymbol{\varphi}(t;\mu_1,\mu_2,\ldots,\mu_m)$ yechimning har bir $\boldsymbol{u}^j \equiv \dfrac{\boldsymbol{\varphi}(t;\mu)}{\partial\mu_j}$ $(j=1,\ldots,m)$ xususiy hosilasi variatsiya uchun mos (12.1.2) chiziqli tenglamani qanoatlantiradi:

$$\frac{d\boldsymbol{u}^j}{dt} = \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}\boldsymbol{u}^j + \frac{\partial\boldsymbol{f}}{\partial\mu_j},\ \ \boldsymbol{u}^j\Big|_{t=t_0} = 0\ (j=1,\ldots,m).$$

Keltirilgan teoremani quyidagicha qisqaroq (lekin noaniqroq) ifodalash mumkin:

Agar $\boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})$ sistemaning o'ng tomoni $\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})\in C^1$ bo'lsa, uning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ yechimi ham C^1 sinfga tegishli bo'ladi.

Variatsiyalar uchun (7.4.24) tenglamalar sistemasini (yoki uning (12.1.2) vektor ko'rinishini) hosil qilish uchun ushbu

$$\frac{d\varphi_i(t,\mu)}{dt} = f_i(t,\varphi_1(t,\mu),\ldots,\varphi_n(t,\mu),\mu),\ \ (i=1,\ldots,n)$$

ayniyatlarni μ bo'yicha differensiallash va aralash hosilalarda differensiallash tartibini almashtirish kerak. Agar $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechim biror μ da ma'lum bo'lsa, μ ning shu qiymatida yechimning μ bo'yicha hosilasi $\boldsymbol{u} = \dfrac{\boldsymbol{\varphi}(t;\mu)}{\partial\mu}$ ni (12.1.2) (yoki(12.1.3)) masalani yechib aniqlash mumkin.

Misol 1. Ushbu

$$x' = x + \mu x^3,\ x(0) = 1$$

masalaning $x=\varphi(t;\mu)$ yechimi uchun $\dfrac{\partial\varphi(t;0)}{\partial\mu}$ hosilani hisoblang.

Berilgan tenglamaning o'ng tomoni $f(t,x,\mu) = x + \mu x^3 \in C^1(\mathbb{R}^3)$, aslida $\in C^\infty(\mathbb{R}^3)$. Demak. isbotlangan teoremani qo'llash mumkin. Tenglamaga $x=\varphi(t;\mu)$ yechimni qo'yib, hosil bo'lgan ayniyatni μ bo'yicha differensiallaymiz va variatsiya uchun tenglamani topamiz

$\left(u(t;\mu)=\dfrac{\partial\varphi(t;\mu)}{\partial\mu}\right.$ kattalik yechimning parametr o‘zgarishi bilan o‘zgarishini (variatsiyasini) xarakterlaydi):

$$\frac{du(t;\mu)}{dt}=u(t;\mu)+\varphi^3(t;\mu)+3\mu\varphi^2(t;\mu)u(t;\mu),\ u(0;\mu)=\frac{\partial\varphi(0;\mu)}{\partial\mu}=0.$$

Biz $u(t;0)=\dfrac{\partial\varphi(t;0)}{\partial\mu}$ ni hisoblashimiz kerak. Oxirgi masalada (tenglamada) $\mu=0$ deb, topamiz:

$$\frac{du(t;0)}{dt}=u(t;0)+\varphi^3(t;0),\ u(0;0)=0.$$

Bu yerdagi $\varphi(t;0)$ funksiya berilgan masalada $\mu=0$ deb topiladi:

$$x'=x,\ x(0)=1,\ \text{ ya'ni }\ \varphi_t'(t;0)=\varphi(t;0),\ \varphi(0;0)=1.$$

Bu masalani yechib, $\varphi(t;0)=e^t$ ekanligini aniqlaymiz. Demak, $u(t;0)$ uchun

$$\frac{du(t;0)}{dt}=u(t;0)+e^{3t},\ u(0;0)=0,$$

masala hosil bo‘ldi. Bu masalani yechib, $u(t;0)=\dfrac{1}{3}(e^{3t}-e^t)$ ni topamiz. Shunday qilib, berilgan masalaning $x=\varphi(t;\mu)$ yechimi uchun $\dfrac{\partial\varphi(t;0)}{\partial\mu}=u(t;0)=\dfrac{1}{3}(e^{3t}-e^t)$.

Endi yechimni boshlang‘ich qiymatlar bo‘yicha differensiallash masalasi bilan shug‘ullanamiz. Buning uchun

$$\begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})\\ \boldsymbol{x}|_{t_0}=\boldsymbol{\xi}\end{cases}\tag{12.1.23}$$

ko‘rinishdagi Koshi masalasini qaraylik; bunda $(t,\boldsymbol{x})\in D$ $((t_0,\boldsymbol{\xi})\in D)$, $D-\mathbb{R}^{1+n}$ fazodagi soha. Bu masalaning yechimini $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\xi})$ $(\boldsymbol{\varphi}(t_0;\boldsymbol{\xi})=\boldsymbol{\xi})$ ko‘rinishda belgilaymiz (boshlang‘ich payt t_0 tayinlangan).

Teorema 2 (yechimning boshlang‘ich qiymatlar bo‘yicha differensiallanuvchiligi). *Aytaylik, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya va uning $\dfrac{\partial\boldsymbol{f}(t,\boldsymbol{x})}{\partial\boldsymbol{x}}$ xususiy hosilasi D sohada uzluksiz hamda* (12.1.23) *masalaning* $\boldsymbol{\xi}=\boldsymbol{\xi}^0$ *dagi* $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\xi}^0)$ *yechimi* $t\in[t_1,t_2]$ *oraliqda aniqlangan bo'lsin. U holda* $\boldsymbol{\xi}^0$ *nuqtaning biror* $B_{\delta_0}(\boldsymbol{\xi}^0)$ *atrofiga tegishli bo'lgan barcha* $\boldsymbol{\xi}$ *lar uchun* $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\xi})$ *yechimning boshlang'ich qiymatlar bo'yicha* $\boldsymbol{w}^j=\dfrac{\partial\boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial\xi_j}$ $(j=1,...,n)$ *hosilalari* $(t,\boldsymbol{\xi})\in[t_1,t_2]\times B_{\delta_0}(\boldsymbol{\xi}^0)$ *to'plamda uzluksiz va ular quyidagi masalalar yechimlaridir:*

$$\frac{d\boldsymbol{w}^j}{dt}=\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}\boldsymbol{w}^j\ ,\ \boldsymbol{w}^j|_{t_0}=\boldsymbol{e}^j\ ;\quad \left(\boldsymbol{e}^j=(0,...,0,\underset{j-\text{o'rin}}{1},0,...0)^T;\ j=1,...,n\right)$$

bunda $\dfrac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}=\left.\dfrac{\partial\boldsymbol{f}(t,\boldsymbol{x})}{\partial\boldsymbol{x}}\right|_{x=\varphi(t;\xi)}$.

🗝 Yangi $\mathbf{y}=\mathbf{x}-\boldsymbol{\xi}$ noma'lumga o'tamiz. Natijada ushbu

$$\begin{cases}\dfrac{d\boldsymbol{y}}{dt}=\boldsymbol{f}(t,\boldsymbol{y}+\boldsymbol{\xi})\\ \boldsymbol{y}|_{t=t_0}=0\end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi $\boldsymbol{\xi}=(\xi_1,\xi_2,...,\xi_n)^T$ kattaliklar parametrlar rolini o'ynaydi:

$$\begin{cases}\dfrac{d\boldsymbol{y}}{dt}=\boldsymbol{g}(t,\boldsymbol{y},\boldsymbol{\xi})\\ \boldsymbol{y}|_{t=t_0}=0\end{cases}\ (\text{ bunda}\,\boldsymbol{g}(t,\boldsymbol{y},\boldsymbol{\xi})=\boldsymbol{f}(t,\boldsymbol{y}+\boldsymbol{\xi}))\ . \qquad (12.1.24)$$

Yechimni $\mathbf{y}=\boldsymbol{\psi}(t;\boldsymbol{\xi})$ $(\boldsymbol{\psi}(t_0;\boldsymbol{\xi})=0)$ bilan belgilaymiz. Ravshanki, eski $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\xi})$ va yangi $\mathbf{y}=\boldsymbol{\psi}(t;\boldsymbol{\xi})$ yechimlar orasida $\boldsymbol{\varphi}(t;\boldsymbol{\xi})=\boldsymbol{\xi}+\boldsymbol{\psi}(t;\boldsymbol{\xi})$ bog'lanish mavjud. Yechimning parametrlar bo'yicha differensiallanuvchiligi haqidagi teoremani (12.1.24) masalaga, ya'ni $\mathbf{y}=\boldsymbol{\psi}(t;\boldsymbol{\xi})$ yechimga qo'llab, oxirgi $\boldsymbol{\varphi}(t;\boldsymbol{\xi})=\boldsymbol{\xi}+\boldsymbol{\psi}(t;\boldsymbol{\xi})$ munosabatga ko'ra teoremani isbotlaymiz. 👍

Natija. *Yechimning boshlang'ich qiymatlar bo'yicha differensiallanuvchiligi haqidagi teorema shartlarida ushbu*

$$\det \frac{\partial \boldsymbol{\varphi}(t;t_0,\boldsymbol{\xi}^0)}{\partial \boldsymbol{\xi}} = \exp \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial f_j(s,\boldsymbol{\varphi}(s;t_0,\boldsymbol{\xi}^0))}{\partial x_j} ds$$

formula o'rinli.

🗝 Teoremadan ravshanki,

$$\boldsymbol{x}' = \frac{\partial \boldsymbol{f}(t,\boldsymbol{\varphi}(t;t_0,\boldsymbol{\xi}^0))}{\partial \boldsymbol{x}} \boldsymbol{x}$$

chiziqli sistemaning fundamental matritsasi ushbu

$$\Phi = [\boldsymbol{w}^1, \boldsymbol{w}^2, ..., \boldsymbol{w}^n] = \left[\frac{\partial \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \xi_1}, \frac{\partial \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \xi_2}, ..., \frac{\partial \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \xi_n} \right] =$$

$$= \frac{\partial \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}$$

matritsadan iborat. Bundan tashqari $\Phi|_{t=t_0} = E$. Endi Liuvill formulasi isbotni tugatadi. ☝

Nihoyat, yechimning boshlang'ich payt bo'yicha differensiallanuvchiligini qarab chiqamiz. Ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{x}|_{\tau} = \boldsymbol{x}^0 \end{cases} \qquad (12.1.25)$$

boshlang'ich masalani qaraylik; bunda $(t,\boldsymbol{x}) \in D$ $((\tau,\boldsymbol{x}^0) \in D)$. Bu masalaning yechimini $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau)$ $(\boldsymbol{\varphi}(\tau;\tau) = \boldsymbol{x}^0)$ ko'rinishda belgilaymiz (boshlang'ich qiymat $\boldsymbol{x}^0$ tayinlangan).

Teorema 3 (yechimning boshlang'ich payt bo'yicha differensiallanuvchiligi). *Aytaylik, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya va uning $\frac{\partial \boldsymbol{f}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}$ xususiy hosilasi D sohada uzluksiz hamda* (12.1.25) *masalaning $\tau = t_0$ bo'lgandagi $\boldsymbol{x} = \boldsymbol{\varphi}(t;t_0)$ yechimi $t \in [t_1,t_2]$ oraliqda aniqlangan bo'lsin. U holda t_0 nuqtaning biror yetarli kichik $(t_0-\delta_0, t_0+\delta_0)$ atrofiga tegishli bo'lgan barcha τ har uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau)$ yechimning boshlang'ich payt bo'yicha $\boldsymbol{w} = \frac{\partial \boldsymbol{\varphi}(t;\tau)}{\partial \tau}$ hosilalasi $(t,\tau) \in [t_1,t_2] \times (t_0-\delta_0, t_0+\delta_0)$ to'plamda uzluksiz va u ushbu*

$$\begin{cases} \dfrac{d\boldsymbol{w}}{dt} = \boldsymbol{f}'_x(t,\boldsymbol{\varphi}(t;\tau))\boldsymbol{w} \\ \boldsymbol{w}|_{t=\tau} = -\boldsymbol{f}(\tau,\boldsymbol{x}^0) \end{cases} \tag{12.1.26}$$

masalaning yechimidan iborat bo'ladi.

🗝 Yangi $s = t-\tau$ erkli o'zgaruvchini kiritamiz. Natijada (12.1.25) masala o'rniga ushbu

$$\begin{cases} \dfrac{d\boldsymbol{x}}{ds} = \boldsymbol{f}(s+\tau,\boldsymbol{x}) \\ \boldsymbol{x}|_{s=0} = \boldsymbol{x}^0 \end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi τ kattalik parametr rolini o'ynaydi:

$$\begin{cases} \dfrac{d\boldsymbol{x}}{ds} = \boldsymbol{g}(s,\boldsymbol{x},\tau) \\ \boldsymbol{x}|_{s=0} = \boldsymbol{x}^0 \end{cases} \text{(bunda } \boldsymbol{g}(s,\boldsymbol{x},\tau) = \boldsymbol{f}(s+\tau,\boldsymbol{x})) \tag{12.1.27}$$

Yechimni $\boldsymbol{x} = \boldsymbol{\psi}(s;\tau)$ $(\boldsymbol{\psi}(0;\tau) = \boldsymbol{x}^0)$ bilan belgilaymiz:

$$\begin{cases} \dfrac{d\boldsymbol{\psi}(s;\tau)}{ds} = \boldsymbol{f}(s+\tau,\boldsymbol{\psi}(s;\tau)) \\ \boldsymbol{\psi}(s;\tau)|_{s=0} = \boldsymbol{x}^0 \end{cases}$$

Bunda, ravshanki, eski $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau)$ va yangi $\boldsymbol{x} = \boldsymbol{\psi}(s;\tau)$ yechimlar orasida $\boldsymbol{\varphi}(t;\tau) = \boldsymbol{\psi}(t-\tau,\tau)$ (yoki $\boldsymbol{\varphi}(s+\tau;\tau) = \boldsymbol{\psi}(s,\tau)$) bog'lanish mavjud. Yechimning parametr bo'yicha differensiallanuvchiligi haqidagi teoremani (12.1.27) masalaga qo'llab, $\boldsymbol{x} = \boldsymbol{\psi}(s;\tau)$ yechim τ parametrning t_0 ga yaqin qiymatlarida ($|\tau - t_0| < \delta_0$ $(\delta_0 > 0)$) mavjud, uning $\tilde{\boldsymbol{w}}(s,\tau) = \dfrac{\partial \boldsymbol{\psi}(s,\tau)}{\partial \tau}$ hosilasi uzluksiz hamda $\tilde{\boldsymbol{w}}(s,\tau)|_{s=0} = 0$ ekanligini topamiz. $\boldsymbol{\varphi}(t;\tau) = \boldsymbol{\psi}(t-\tau,\tau)$ formulaga ko'ra

$$\boldsymbol{w}(t;\tau) = \frac{\partial \boldsymbol{\varphi}(t;\tau)}{\partial \tau} = \frac{\partial \boldsymbol{\psi}(t-\tau,\tau)}{\partial s}\cdot(-1) + \frac{\partial \boldsymbol{\psi}(t-\tau,\tau)}{\partial \tau} =$$
$$= -\boldsymbol{f}(t,\boldsymbol{\psi}(t-\tau,\tau)) + \tilde{\boldsymbol{w}}(t-\tau,\tau) \tag{12.1.28}$$

hosila ham uzluksiz ($t \in [t_1,t_2]$, $|\tau - t_0| < \delta_0$) va

$$\boldsymbol{w}|_{t=\tau} = -\boldsymbol{f}(\tau,\boldsymbol{\varphi}(\tau,\tau)) + \tilde{\boldsymbol{w}}(0,\tau) = -\boldsymbol{f}(\tau,\boldsymbol{x}^0).$$

Bu (12.1.26) dagi ikkinchi munosabat. $\boldsymbol{x}=\boldsymbol{\varphi}(t;\tau)$ yechim bo‘lgani uchun, u (12.1.25) dagi differensial tenglamani qanoatlantiradi: $\boldsymbol{\varphi}'_t(t;\tau)=\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\tau))$. Bu tenglikni τ boyicha differensiallaymiz: $\boldsymbol{\varphi}''_{\tau t}(t;\tau)=\boldsymbol{f}'_{\boldsymbol{x}}(t,\boldsymbol{\varphi}(t;\tau))\boldsymbol{\varphi}'_\tau(t;\tau)$. Bu tenglikning o‘ng tomoni uzluksiz. Demak, uning chap tomonidagi $\boldsymbol{\varphi}''_{\tau t}(t;\tau)$ aralash hosila ham uzluksiz. Differensiallash tartibini o‘zgartirib, (12.1.26) dagi birinchi tenglikni hosil qilamiz. ✍

Eslatma. $\boldsymbol{\xi}\to\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{\xi})$ akslantirish $\boldsymbol{\xi}^0$ nuqta atrofida teskarilanuvchi va

$\boldsymbol{\xi}=\boldsymbol{\varphi}(t_0;t,\boldsymbol{x})$; bu yechimning yagonaligidan ravshan. Yuqoridagi teorema shartlarida to‘g‘ri va teskari akslantirishlar barcha o‘zgaruvchilar bo‘yicha lokal C^1 sinfga tegishli bo‘ladi.

Agar $\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})$ tenglamaning o‘ng tomoni $\boldsymbol{x}$ va μ bo‘yicha m marta uzluksiz differensiallanuvchi bo‘lsa, uning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechimi ham μ bo‘yicha m marta uzluksiz differensiallanuvchi bo‘ladi. Bu tasdiqning aniq ifodalanishi quyidagi teoremada keltirilgan.

Teorema 4. *Yechimning parametr bo‘yicha differensiallanuvchiligi to`g`risidagi teorema 1 shartlariga qo‘shimcha holda $\boldsymbol{f}(t,\boldsymbol{x},\mu)$ funksiya $x_1,...,x_n,\mu$ har bo‘yicha C^m sinfga tegishli bo‘lsin. U holda $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechimning t,μ bo‘yicha birinchi tartibli, μ bo‘yicha esa $m-$ tartibgacha hosilalari uzluksiz bo‘ladi.*

⊶ m bo‘yicha matematik induksiya metodini qo‘llaymiz. $m=1$ holi teorema 1da qaralgan. Faraz qilaylik, teorema $m=1,2,...,k-1$ ($k\geq 2$) qiymatlar uchun o‘rinli bo‘lsin. Teoremani $m=k$ uchun isbotlash kerak. Ravshanki, $\dfrac{\partial^k\boldsymbol{\varphi}(t;\mu)}{\partial\mu^k}=\dfrac{\partial^{k-1}\boldsymbol{u}}{\partial\mu^{k-1}}$ $\left(\boldsymbol{u}=\dfrac{\partial\boldsymbol{\varphi}(t;\mu)}{\partial\mu}\right)$ va $\boldsymbol{u}$ funksiya (12.1.2) masalaning yechimi, ya’ni

$$\boldsymbol{u}'_t=\boldsymbol{f}'_{\boldsymbol{x}}(t,\boldsymbol{\varphi}(t;\mu))\boldsymbol{u}+\boldsymbol{f}'_\mu(t,\boldsymbol{\varphi}(t;\mu)),\ \boldsymbol{u}\big|_{t=t_0}=0\ .$$

Bu yerdagi differensial tenglamaning o‘ng tomoni $u_1,...,u_n,\mu$ har bo‘yicha C^{k-1} sinfga tegishli ekanligini ko‘rsatish kifoya, chunki u holda $m=k-1$ uchun induksiya farazini yuqoridagi masalaga qo‘llab, $\boldsymbol{u}$ yechimning μ bo‘yicha $m=k-1$ tartibli $\frac{\partial^{k-1}\boldsymbol{u}}{\partial\mu^{k-1}}=\frac{\partial^k\boldsymbol{\varphi}(t;\mu)}{\partial\mu^k}$ hosilasi uzluksiz ekanligini topamiz. Birinchidan, teoremaning shartiga ko‘ra $\boldsymbol{f}$ funksiya $x_1,...,x_n,\mu$ har bo‘yicha C^k sinfga tegishli. Demak, $\boldsymbol{f}'_x$ va $\boldsymbol{f}'_\mu$ xususiy hosilalar $x_1,...,x_n,\mu$ har bo‘yicha $k-1$ marta uzluksiz differensiallanuvchi. Ikkinchidan, induksiya faraziga ko‘ra $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechim μ bo‘yicha C^{k-1} sinfga tegishli. Shuning uchun $\boldsymbol{f}'_{\boldsymbol{x}}(t,\boldsymbol{\varphi}(t;\mu))$ va $\boldsymbol{f}'_\mu(t,\boldsymbol{\varphi}(t;\mu))$ murakkab funksiyalar μ bo‘yicha, $\boldsymbol{f}'_{\boldsymbol{x}}(t,\boldsymbol{\varphi}(t;\mu))\boldsymbol{u}+\boldsymbol{f}'_\mu(t,\boldsymbol{\varphi}(t;\mu))$ funksiya esa $u_1,...,u_n,\mu$ har bo‘yicha C^{k-1} sinfga tegishli ekanligi ravshan.. ☝

Masalalar

Ushbu

$$\begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})\\ \boldsymbol{x}|_{t_0}=\boldsymbol{x}^0\end{cases}\quad \text{va}\quad \begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x})+\boldsymbol{r}(t,\boldsymbol{x})\\ \boldsymbol{x}|_{t_0}=\boldsymbol{x}^0\end{cases}$$

masalalarning yechimlarini $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)$ va (mos ravishda) $\boldsymbol{x}=\boldsymbol{\psi}(t;t_0,\boldsymbol{x}^0)$ bilan belgilaylik; bunda $\{\mathbf{f}(t,\mathbf{x}),\,\mathbf{f}'_{\mathbf{x}}(t,\mathbf{x}),\,\mathbf{r}(t,\mathbf{x}),\,\mathbf{r}'_{\mathbf{x}}(t,\mathbf{x})\}\subset C(\mathbb{R}^{1+n},\mathbb{R}^n)$ deb faraz qilinadi. Quyidagi belgilashni kiritaylik:

$$\Phi(t;t_0,\boldsymbol{x}^0)=\frac{\partial\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)}{\partial\boldsymbol{x}^0}.$$

Quyidagilarni isbotlang:

1. $\boldsymbol{\psi}(t;t_0,\boldsymbol{x}^0)=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)+\int_{t_0}^{t}\Phi(t;s,\boldsymbol{\psi}(s;t_0,\boldsymbol{x}^0))\,\boldsymbol{f}(s,\boldsymbol{\psi}(s;t_0,\boldsymbol{x}^0)ds$

(V. A. Alekseev formulasi).

2. $\boldsymbol{\varphi}(t;t_0,\boldsymbol{y}^0)=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)+\int_0^1\Phi\big(t;t_0,\boldsymbol{x}^0+s(\boldsymbol{y}^0-\boldsymbol{x}^0)\big)ds(\boldsymbol{y}^0-\boldsymbol{x}^0)$.

3. $\boldsymbol{\psi}(t;t_0,\boldsymbol{y}^0)=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)+\int_0^1\Phi\big(t;t_0,\boldsymbol{x}^0+s(\boldsymbol{y}^0-\boldsymbol{x}^0)\big)ds(\boldsymbol{y}^0-\boldsymbol{x}^0)+$

$$+\int_{t_0}^{t}\Phi(t;s,\boldsymbol{\psi}(s;t_0,\boldsymbol{y}^0))\,\boldsymbol{f}(s,\boldsymbol{\psi}(s;t_0,\boldsymbol{y}^0)ds\,.$$

§ 12.2. Kichik parametr metodi

Differensial tenglamalarning taqribiy yechimlarini topishda, amaliy masalalarni yechishda kichik parametr metodi muhim o‘rin tutadi. § 11.1 dagi ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \mu) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \end{cases} \tag{12.1.1}$$

nochiziqli masalaning μ skalyar parametr qiymati $\mu = 0$ bo'lgandagi yechimi $\boldsymbol{x} = \boldsymbol{\varphi}^0(t)$ ma'lum bo'lsin. U holda μ parametrning 0ga yaqin (kichik) qiymatlarida bu masalaning taqribiy yechimini kichik parametr metodi yordamida qurish mumkin.

Teorema. *Aytaylik,* § 12.1 dagi *teorema 4 ning shartlari* $\{(t, \boldsymbol{x}, \mu) | (t, \boldsymbol{x}) \in D,\ \mu \in (-\varepsilon, \varepsilon)\}$ $(\varepsilon > 0)$ *sohada o'rinli,* $\mu = 0$ *bo'lgandagi* (*12.1.1*) *masalaning* $\boldsymbol{x} = \boldsymbol{\varphi}^0(t)$ *yechimi* $t \in [t_1, t_2]$ *oraliqda aniqlangan bo'lsin. U holda* (*12.1.1*) *masalaning* $\boldsymbol{x} = \boldsymbol{\varphi}(t;\mu)$ $(t \in [t_1, t_2])$ *yechimi uchun*

$$\boldsymbol{\varphi}(t;\mu) = \boldsymbol{\varphi}^0(t) + \boldsymbol{\varphi}^1(t)\mu + \boldsymbol{\varphi}^2(t)\mu^2 + \ldots + \boldsymbol{\varphi}^m(t)\mu^m + o(\mu^m),\ \mu \to 0, \tag{12.2.1}$$

asimptotik yoyilma o'rinli; bundan tashqari, bu yerdagi kichik o $t \in [t_1, t_2]$*ga nisbatan tekis ham bo'ladi.*

🗝 Bu teorema § 11.1 dagi teorema 4 ning bevosita natijasidir. 👍

Konkret masalalar yechilganda (12.2.1) yoyilmani, ya'ni $\boldsymbol{\varphi}^0(t), \boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), \ldots, \boldsymbol{\varphi}^m(t)$ vektor-funksiyalarni aniqlash uchun bu yoyilmani qaralayotgan tenglamaga qo'yib,

$$\frac{d\boldsymbol{\varphi}^0(t)}{dt} + \frac{d\boldsymbol{\varphi}^1(t)}{dt}\mu + \frac{d\boldsymbol{\varphi}^2(t)}{dt}\mu^2 + \\ \ldots + \frac{d\boldsymbol{\varphi}^m(t)}{dt}\mu^m + o(\mu^m) = \boldsymbol{f}(t, \boldsymbol{\varphi}(t;\mu), \mu),\ \mu \to 0,$$

o'ng tomonni μning darajalari bo'yicha yoyib,

$$\frac{d\boldsymbol{\varphi}^0(t)}{dt} + \frac{d\boldsymbol{\varphi}^1(t)}{dt}\mu + \frac{d\boldsymbol{\varphi}^2(t)}{dt}\mu^2 + \ldots + \frac{d\boldsymbol{\varphi}^m(t)}{dt}\mu^m + o(\mu^m) =$$

$$= \boldsymbol{f}(t, \boldsymbol{\varphi}(t;0), 0) + \left(\frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}}\boldsymbol{\varphi}^1(t) + \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \mu}\right)\mu +$$

$$+ \ldots + \left(\frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}} m!\boldsymbol{\varphi}^m(t) + \ldots\right)\mu^m + o(\mu^m),\ \mu \to 0,$$

hosil bo'lgan tenglikning chap va o'ng tomonlaridagi μning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish kerak:

$$\mu^0: \ \frac{d\boldsymbol{\varphi}^0(t)}{dt} = \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)$$

$$\mu^1: \ \frac{d\boldsymbol{\varphi}^1(t)}{dt} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^1(t) + \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \mu}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mu^m: \frac{d\boldsymbol{\varphi}^m(t)}{dt} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}} m! \boldsymbol{\varphi}^m(t) + \ldots \ .$$

Bunda $\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), ..., \boldsymbol{\varphi}^m(t)$ funksiyalar uchun chiziqli tenglamalar hosil bo‘ladi.

Boshlang‘ich shartdan

$$\boldsymbol{\varphi}(t;\mu)|_{t=t_0} = \boldsymbol{x}^0 = \boldsymbol{\varphi}^0(t)|_{t=t_0} + \boldsymbol{\varphi}^1(t)|_{t=t_0} \mu + \boldsymbol{\varphi}^2(t)|_{t=t_0} \mu^2 +$$
$$+ \ldots + \boldsymbol{\varphi}^m(t)|_{t=t_0} \mu^m + o(\mu^m)|_{t=t_0}, \ \mu \to 0,$$

ya’ni

$$\boldsymbol{\varphi}^0(t)|_{t=t_0} = \boldsymbol{x}^0, \ \boldsymbol{\varphi}^1(t)|_{t=t_0} = 0, \ \boldsymbol{\varphi}^2(t)|_{t=t_0} = 0, \ldots, \boldsymbol{\varphi}^m(t)|_{t=t_0} = 0$$

shartlar hosil bo‘ladi.

Hosil qilingan tenglamalardan $\boldsymbol{\varphi}^0(t)$ dan boshlab ketma-ket $\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), ..., \boldsymbol{\varphi}^m(t)$
yechimlarni mos boshlang‘ich shartlarga ko‘ra topish kerak.

Misol. Ushbu

$$\begin{cases} x' = 3y + \mu x \\ y' = 2t + \mu xy \\ x|_{t=1} = 1, \ y|_{t=1} = 1 \end{cases}$$

masala yechimining kichik μ parametr bo‘yicha yoyilmasidagi dastlabki uchta hadni quring.

🗝Berilgan sistemaning o‘ng tomoni $(t, x, y) \in D = \mathbb{R}^3$, $|\mu| < +\infty$ sohada xohlagancha marta uzluksiz differensiallanuvchi. Demak, teoremaning shartlari ixtiyoriy m uchun o‘rinli. Biz

$$\begin{cases} x = \varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2) \\ y = \psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2) \end{cases}, \ \mu \to 0,$$

yoyilmalardagi koeffitsientlarni topishimiz kerak. Bu yoyilmalarni berilgan sistema va boshlang‘ich shartlarga qo‘yamiz ($o(\mu^2)$ miqdorlar $\mu \to 0$ da tushuniladi):

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=3(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2))+$$
$$+\mu(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2)),$$
$$\psi_0'(t)+\psi_1'(t)\mu+\psi_2'(t)\mu^2+o(\mu^2)=2t+\mu(\varphi_0(t)+\varphi_1(t)\mu$$
$$+\varphi_2(t)\mu^2+o(\mu^2))(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2)),$$
$$(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2))|_{t=1}=1,$$
$$(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2))|_{t=1}=1.$$

Bu yerdagi birinchi va ikkinchi tenglamaning o‘ng tomonini μ ning darajalari bo‘ylab yoyamiz (qavslarni ochib, tartiblari μ^2 gacha bo‘lgan hadlarni saqlaymiz):

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=3\psi_0(t)+$$
$$+(3\psi_1(t)+\varphi_0(t))\mu+(3\psi_2(t)+\varphi_1(t))\mu^2+o(\mu^2),$$
$$\psi_0'(t)+\psi_1'(t)\mu+\psi_2'(t)\mu^2+o(\mu^2)=2t+\varphi_0(t)\psi_0(t)\mu+$$
$$+\varphi_0(t)\psi_1(t)+\varphi_1(t)\psi_0(t))\mu^2+o(\mu^2),$$
$$\varphi_0(1)+\varphi_1(1)\mu+\varphi_2(1)\mu^2+o(\mu^2)|_{t=1}=1,$$
$$\psi_0(1)+\psi_1(1)\mu+\psi_2(1)\mu^2+o(\mu^2))|_{t=1}=1.$$

Endi μ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, quyidagi masalalarni tuzamiz:

$$\mu^0:\ \begin{cases}\varphi_0'(t)=3\psi_0(t)\\ \psi_0'(t)=2t\\ \varphi_0(1)=1,\ \psi_0(1)=1\end{cases},$$
$$\mu^1:\ \begin{cases}\varphi_1'(t)=3\psi_1(t)+\varphi_0(t)\\ \psi_1'(t)=\varphi_0(t)\psi_0(t)\\ \varphi_1(1)=0,\ \psi_1(1)=0\end{cases},$$
$$\mu^2:\ \begin{cases}\varphi_2'(t)=3\psi_2(t)+\varphi_1(t)\\ \psi_2'(t)=\varphi_0(t)\psi_1(t)+\varphi_1(t)\psi_0(t)\\ \varphi_2(1)=0,\ \psi_2(1)=0\end{cases}$$

Bu masalalarni birinchisidan boshlab ketma-ket yechamiz va quyidagilarni topamiz:

$$\begin{cases}\varphi_0(t)=t^3\\ \psi_0(t)=t^2\end{cases},\quad \begin{cases}\varphi_1(t)=\dfrac{t^7}{14}+\dfrac{t^4}{4}\\ \psi_1(t)=\dfrac{t^6}{6}\end{cases},\quad \begin{cases}\varphi_2(t)=\dfrac{t^{11}}{140}+\dfrac{5t^8}{192}+\dfrac{t^5}{20}\\ \psi_2(t)=\dfrac{t^{10}}{42}+\dfrac{t^7}{28}\end{cases}$$

Demak, berilgan masala yechimi uchun ushbu

$$\begin{cases}x=t^3+\left(\dfrac{t^7}{14}+\dfrac{t^4}{4}\right)\mu+\left(\dfrac{t^{11}}{140}+\dfrac{5t^8}{192}+\dfrac{t^5}{20}\right)\mu^2+o(\mu^2)\\ y=t^2+\dfrac{t^6}{6}\mu+\left(\dfrac{t^{10}}{42}+\dfrac{t^7}{28}\right)\mu^2+o(\mu^2)\end{cases},\ \mu\to 0,$$

asimptotik yoyilmalar o‘rinli.

Masalalar

1. Quyidagi masalani qarang:

$$\begin{cases}x'-x+\mu x^2=0,\\ x|_{t=0}=1.\end{cases}$$

a) Masala yechimining kichik μ parametr bo‘yicha yoyilmasidagi dastlabki uchta hadni quring.

b) Aniq yechimni toping (Bernulli tenglamasi).

d) Aniq yechimning kichik μ bo‘yicha yoyilmasini toping va uni a) banddagi yoyilma bilan taqqoslang.

Quyidagi masalalar yechimining kichik μ parametr bo‘yicha yoyilmasidagi dastlabki uchta hadni aniqlang

2. $\begin{cases}x'=y\\ y'=-x-\mu x^2\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}$ $(x''=-x-\mu x^2)$.

3. $\begin{cases}x'=y\\ y'=-x-\mu x^3\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}$ ($x''=-x-\mu x^3$ — Dyuffing tenglamasi) .

4. $\begin{cases}x'=y\\ y'=-x+\mu(1-x^2)y\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}$ ($x''=-x+\mu(1-x^2)x'$ — Van-der-Pol tenglamasi) .

12.3. Birinchi integrallar

Quyidagi sistemani qaraylik:

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t,\boldsymbol{x}). \qquad (12.3.1)$$

Biz bu yerda $\boldsymbol{f}\in C^1(D,\mathbb{R}^n)$ deb hisoblaymiz ($D\subset\mathbb{R}^{1+n}$ – soha), $\boldsymbol{f}=(f_1,...,f_n)^T$. Avvalgidek, (12.3.1) sistemaning $\boldsymbol{x}|_{t=t_0}=\boldsymbol{x}^0$ boshlang'ich shartni qanoatlantiruvchi yechimini $\boldsymbol{x}=\boldsymbol{\varphi}(t,t_0,\boldsymbol{x}^0)$ bilan belgilaymiz.

O'zgarmasdan farqli $u=u(t,\boldsymbol{x})\in C^1(D,\mathbb{R})$ funksiyani qaraylik. Agar (12.3.1) sistemaning (D da joylashgan) ixtiyoriy $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimida (yechimi bo`ylab) $u(t,\boldsymbol{x})$ funksiya o'zgarmasga aylansa, ya'ni $u(t,\boldsymbol{\varphi}(t))=\text{const}$ bo'lsa, u holda $u(t,\boldsymbol{x})$ funksiya (12.3.1) sistemaning (D sohada aniqlangan) **birinchi integrali** deyiladi.

Misol 1. Ushbu

$$\begin{cases} x_1'=x_2 \\ x_2'=-x_1 \end{cases}$$

sistemaning birinchi integrali $u=x_1^2+x_2^2$, chunki ixtiyoriy $x_1=x_1(t)$, $x_2=x_2(t)$ yechim bo'ylab $\frac{1}{2}\frac{d}{dt}\left(x_1^2+x_2^2\right)=x_1x_1'+x_2x_2'=x_1x_2+x_2(-x_1)=0$; demak, $x_1^2+x_2^2=\text{const}$.

Misol 2. $H=H(p_1,...,p_n,q_1,...,q_n)\in C^2$ funksiyaga ko'ra tuzilgan ushbu

$$\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}\ ,\ \frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\ i=1,...,n \qquad (12.3.2)$$

sistemani qaraylik. (12.3.2) differensial tenglamalar sistemasi Hamiltonning kanonik tenglamalar sistemasi deb ataladi. Bu yerdagi H funksiya Hamilton funksiyasi deyiladi. Fizikada uchraydigan ko'p jarayonlar (12.3.2) sistema bilan (boshqariladi) ifodalanadi. Hamiltonning H funksiyasi (12.3.2) kanonik tenglamalar sistemasi uchun birinchi integraldir.

Haqiqatdan ham, ixtiyoriy $p_i=p_i(t)$, $q_i=q_i(t)$ yechim bo'ylab

$$\frac{dH(p_1(t),...,p_n(t),q_1(t),...,q_n(t))}{dt}=\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{dp_i}{dt}+\sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\ \frac{dq_i}{dt}=$$

$$=-\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\cdot\frac{\partial H}{\partial q_i}+\sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}=0$$

va, demak, $H=\text{const}$ bo'ladi.

Teorema 1. *O'zgarmasdan farqli* $u \in C^1(D,\mathbb{R})$ *funksiya* (12.3.1) *sistemaning birinchi integrali bo'lishi uchun* D *sohada*

$$\frac{\partial u(t,\boldsymbol{x})}{\partial t} + \sum_{i=1}^{n} \frac{\partial u(t,\boldsymbol{x})}{\partial x_i} \cdot f_i(t,\boldsymbol{x}) = 0 \qquad (12.3.3)$$

tenglikning o'rinli bo'lishi yetarli va zarurdir.

🗝 **Yetarliligi.** O`zgarmasdan farqli $u \in C^1$ funksiya (12.3.3) shartni qanoatlantirsin. (12.3.1) sistemaning ixtiyoriy $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimida $u = u(t,\boldsymbol{x})$ funksiya o'zgarmasga aylanadi, chunki (12.3.3) ga ko'ra uning hosilasi nolga teng:

$$\frac{du(t,\boldsymbol{\varphi}(t))}{dt} = \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t)} + \sum_{i=1}^{n} \left.\frac{\partial u}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t)} \cdot \frac{dx_i}{dt} =$$

$$= \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t)} + \sum_{i=1}^{n} \left.\frac{\partial u}{\partial x_i} \cdot f_i\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t)} = 0$$

Zarurligi. O`zgarmasdan farqli $u \in C^1$ funksiya (12.3.1) sistemaning birinchi integrali bo'lsin. $\forall(\tau,\xi) \in D$ nuqtada (12.3.3) munosabatning o'rinli ekanligini ko'rsatishimiz kerak. Buning uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t,\tau,\boldsymbol{\xi})$ $(\boldsymbol{\xi} = \boldsymbol{\varphi}(\tau,\tau,\boldsymbol{\xi}))$ yechimni olib, $G(t) = u(t,\boldsymbol{\varphi}(t,\tau,\boldsymbol{\xi}))$ funksiyani tuzaylik. Bu funksiya t ga bog'liq emas (u birinchi integral bo'lgani uchun). Demak, uning t bo'yicha hosilasi nolga teng

($t = \tau$ nuqtada ham):

$$0 = \left.\frac{dG(t)}{dt}\right|_{t=\tau} = \left.\frac{\partial u}{\partial t}\right|_{\substack{\boldsymbol{x}=\boldsymbol{\varphi}(t,\tau,\xi)\\ t=\tau}} + \sum_{i=1}^{n} \left.\frac{\partial u}{\partial x_i} \cdot f_i\right|_{\substack{\boldsymbol{x}=\boldsymbol{\varphi}(t,\tau,\xi)\\ t=\tau}} =$$

$$= \frac{\partial u(\tau,\boldsymbol{\xi})}{\partial t} + \sum_{i=1}^{n} \frac{\partial u(\tau,\boldsymbol{\xi})}{\partial x_i} \cdot f_i(\tau,\boldsymbol{\xi}) \quad ;$$

biz bu yerda $\boldsymbol{\varphi}(\tau,\tau,\boldsymbol{\xi}) = \boldsymbol{\xi}$ ekanligidan foydalandik. 👍

Eslatma. $u(t,\boldsymbol{x})$ funksiyaning sath to'plami (chizig'i, sirti) deb $\{(t,\boldsymbol{x}) \mid u(t,\boldsymbol{x}) = c - \text{const}\} \subset \mathbb{R}^{1+n}$ to'plamga aytiladi. Demak, $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim grafigi (integral chiziq) birinchi integralning bitta sath to'plamida to'laligicha joylashadi.

Agar $u_1(t,x),...,u_k(t,x)$ $,k \le n,$ birinchi integrallar uchun ushbu

$$\frac{\partial(u_1,...,u_k)}{\partial(x_1,...,x_n)}=\left\|\frac{\partial u_i}{\partial x_j}\right\|=\begin{pmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n}\\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_k}{\partial x_1} & \frac{\partial u_k}{\partial x_2} & \cdots & \frac{\partial u_k}{\partial x_n}\end{pmatrix}$$

$k\times n$ o`lchamli Yakobi matritsasining berilgan nuqtadagi rangi $k$ ga teng bo‘lsa, u holda $u_1,...,u_k$ birinchi integrallar qaralayotgan nuqtada (funksional) erkli deb ataladi. Tushunarliki, nuqtada erkli birinchi integrallar shu nuqtaning biror atrofida ham erkli bo`ladi.

n- tartibli (12.3.1) sistemaning n dona erkli $u_1,u_2,...,u_n$ birinchi integrallari birinchi integrallarning to‘la sistemasi deyiladi. Bu holda $\left\|\frac{\partial u_i}{\partial x_j}\right\|$ kvadrat matritsaning determinanti noldan farqli. Birinchi integrallarning to‘la sistemasi $u_1,...,u_n$ uchun (12.3.3) shartlar

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}\frac{\partial u_i}{\partial x_j}f_j=0,\ i=1,...,n,\ \text{(ya'ni } \frac{\partial \boldsymbol{u}}{\partial t}+\frac{d\boldsymbol{u}}{d\boldsymbol{x}}\boldsymbol{f}=0\ \text{)}$$

(12.3.1) sistemaning o‘ng tomonini bir qiymatli aniqlaydi:

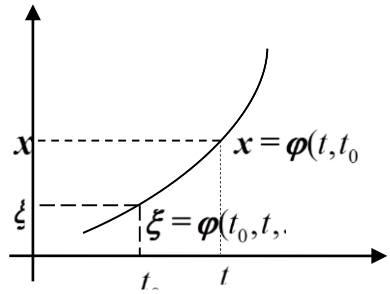
$$\boldsymbol{f}=-\left(\frac{d\boldsymbol{u}}{d\boldsymbol{x}}\right)^{-1}\frac{\partial \boldsymbol{u}}{\partial t}; \tag{12.3.4}$$

bu yerda $\boldsymbol{u}=(u_1,...,u_n)^T$; $\frac{d\boldsymbol{u}}{d\boldsymbol{x}}=\frac{\partial(u_1,...,u_n)}{\partial(x_1,...x_n)}=\left\|\frac{\partial u_i}{\partial x_j}\right\|-n\times n$ o`lchamli Yakobi matritsasi.

Teorema 2 (birinchi integrallarning to‘la sistemasi to`g`risida). *Agar* $\boldsymbol{f}\in C^1(D,\mathbb{R}^n)$ *bo‘lsa,* (12.3.1) *sistema ixtiyoriy* $(t_0,\boldsymbol{x}^0)\in D$ *nuqtaning yetarlicha kichik atrofida birinchi integrallarning to‘la sistemasiga ega. U* $\boldsymbol{u}(t,\boldsymbol{x})=\boldsymbol{\varphi}(t_0,t,\boldsymbol{x})$ *formula bilan aniqlanadi.* ((12.3.1) *sistema birinchi integrallarining to‘la sistemasini aniqlash uchun uning yechimini beruvchi* $\mathbf{x}=\boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi})$ *tenglikdan* $\boldsymbol{\xi}$ *ni topish kerak; yechimning yagonalik xossasiga ko‘ra* $\boldsymbol{\xi}=\boldsymbol{u}(t,\boldsymbol{x})=\boldsymbol{\varphi}(t_0,t,\mathbf{x})$ *hosil bo‘ladi*).

🗝 $\boldsymbol{f}$ ga nisbatan qo‘yilgan $\boldsymbol{f}\in C^1(D,R^n)$ shartda $\boldsymbol{x}=\boldsymbol{\varphi}(t,\tau,\boldsymbol{\xi})$ yechim $(t,\tau,\boldsymbol{\xi})=(t_0,t_0,\boldsymbol{x}^0)\in\mathbb{R}^{2+n}$ nuqtaning yetarlicha kichik atrofida

C^1 sinfga tegishli ekanligi hamda $\left.\dfrac{d\boldsymbol{\varphi}(t,\tau,\boldsymbol{\xi})}{d\boldsymbol{\xi}}\right|_{t=\tau}=E$ bo‘lishi bizga ma’lum. Demak, $\det\left.\dfrac{d\boldsymbol{\varphi}}{d\boldsymbol{\xi}}\right|_{t=\tau}=1\neq 0$ va $\det\dfrac{d\boldsymbol{\varphi}}{d\boldsymbol{\xi}}$ yakobianning qiymati $(t_0,t_0,\boldsymbol{x}^0)$ nuqtaning yetarlicha kichik atrofida ham nolga aylanmaydi.



t_0 paytda $\boldsymbol{\xi}$ nuqtada bo’lgan yechim t paytda $\boldsymbol{x}=\boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi})$ nuqtada bo’ladi

Teskari funksiya haqidagi teoremaga ko‘ra $(t_0,\boldsymbol{x}^0)\in D$ nuqtaning yetarlicha kichik atrofida $\boldsymbol{u}(t,\boldsymbol{x})\equiv\boldsymbol{\varphi}(t_0,t,\boldsymbol{x})\in C^1$ va $\dfrac{d\boldsymbol{u}}{d\boldsymbol{x}}$ matritsaning rangi n ga teng, ya’ni u teskarilanuvchi. Endi $\boldsymbol{u}(t,\boldsymbol{x})=(u_1(t,\boldsymbol{x}),...,u_n(t,\boldsymbol{x}))^T$ vektor-funksiya (12.3.1) sistemaning ixtiyoriy yechimida o‘zgarmasga aylanishini ko‘rsatish qoldi. $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimni olaylik Aytaylik, $\boldsymbol{\varphi}(t)=\boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi})$ bo‘lsin. Yechimning yagonalik xossasiga ko‘ra $\boldsymbol{u}(t,\boldsymbol{\varphi}(t))=\boldsymbol{\varphi}(t_0,t,\boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi}))=\boldsymbol{\xi}=\text{const}$ ekanligi ravshan. 👍

Shunday qilib, agar (12.3.1) sistema uchun birinchi integrallarning to‘la sistemasi $\boldsymbol{u}(t,\boldsymbol{x})=(u_1(t,\boldsymbol{x}),...,u_n(t,\boldsymbol{x}))^T$ topilgan bo‘lsa, u holda $\boldsymbol{u}(t,\boldsymbol{x})=\boldsymbol{c}$ ($\mathbf{c}\in\mathbb{R}^n-$o‘zgarmas vektor) tenglikni $\mathbf{x}$ ga nisbatan yechib, (12.3.1) sistemaning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{c})$ yechimlarini hosil qilamiz:

$$\boldsymbol{u}(t,\boldsymbol{x})=\boldsymbol{c}\ \Rightarrow\ \frac{\partial\boldsymbol{u}}{\partial t}+\frac{d\boldsymbol{u}}{d\boldsymbol{x}}\cdot\frac{d\boldsymbol{x}}{dt}=0\ \Rightarrow\ \frac{d\boldsymbol{x}}{dt}=-\left.\left(\frac{d\boldsymbol{u}}{d\boldsymbol{x}}\right)^{-1}\frac{\partial\boldsymbol{u}}{\partial t}\right|_{x=\varphi(t;\boldsymbol{c})}. \quad (12.3.5)$$

Bundan $\boldsymbol{u}(t,\boldsymbol{x})-$to‘la sistema bo‘lgani uchun (12.3.4) ga ko`ra .

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t,\boldsymbol{x}),\ \boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{c}).$$

Bu yerda topilgan $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{c})$ yechim (lokal) umumiy yechimni ifodalaydi.

Ko‘pincha $u(t,\boldsymbol{x})$ birinchi integral orqali yozilgan $u(t,\boldsymbol{x})=c$ munosabat ham birinchi integral deb ataladi.

Teorema 3. *Agar $\boldsymbol{u}=(u_1,...u_n)^T$ birinchi integrallarning to‘la sistemasi, $v(t,\boldsymbol{x})$ esa ixtiyoriy birinchi integral bo‘lsa, u holda ixtiyoriy $(t_0,\boldsymbol{x}^0)\in D$ nuqtaning yetarlicha kichik atrofida $v(t,\boldsymbol{x})=\varphi(u_1(t,\boldsymbol{x}),...,u_n(t,\boldsymbol{x}))$ munosabat o‘rinli bo‘ladi, ya’ni har qanday birinchi integral (lokal) integrallarning to‘la sistemasi orqali ifodalanadi; bu yerdagi φ funksiya $\boldsymbol{u}(t_0,\boldsymbol{x}^0)\in\mathbb{R}^n$ nuqtaning biror atrofida C^1 sinfga tegishli.*

🗝 $(t_0,\boldsymbol{x}^0)\in D$ nuqtaning yetarlicha kichik atrofida $\boldsymbol{x}$ o‘rniga yangi noma’lum $\mathbf{y}$ funksiyani

$$\boldsymbol{y}=\boldsymbol{u}(t,\boldsymbol{x}) \qquad (12.3.6)$$

formula bilan kiritamiz. U holda (12.3.1) sistema $\boldsymbol{y}'=0$ ko‘rinishga keladi. (12.3.6) dan $\boldsymbol{x}$ ni $\boldsymbol{y}$ orqali ifodalaymiz: $\boldsymbol{x}=\boldsymbol{u}^{-1}(t,\boldsymbol{y})$. $v(t,\boldsymbol{x})$ funksiyani ham $\boldsymbol{y}$ orqali ifodalab, $\varphi(t,\boldsymbol{y})$ funksiyani hosil qilamiz: $v(t,\boldsymbol{x})=v(t,\boldsymbol{u}^{-1}(t,\boldsymbol{y}))=\varphi(t,\boldsymbol{y})$. $v(t,\mathbf{x})$ funksiya (12.3.1) sistemaning birinchi integrali bo‘lgani uchun $\frac{dv}{dt}=\frac{d\varphi}{dt}=0$ Bundan $\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{n}\frac{\partial\varphi}{\partial y_i}\frac{dy_i}{dt}=0$. Demak, $\frac{\partial\varphi}{\partial t}=0,$ ya’ni φ funksiya t ga oshkor holda bog‘liq emas: $u(t,\boldsymbol{x})=\varphi(\boldsymbol{y})=\varphi(\boldsymbol{u})$. 👍

Misol 3. Ushbu

$$\frac{dx}{dt}=\frac{t-y}{y-x}, \qquad \frac{dy}{dt}=\frac{x-t}{y-x} \qquad (12.3.7)$$

sistema birinchi integrallarining to`la sistemasini (ya’ni umumiy integralini) topaylik.

🗝 Tenglamalarni hadma-had qo‘shamiz.

$$\frac{d(x+y)}{dt}=\frac{x-y}{y-x}\Rightarrow\frac{d(x+y)}{dt}=-1$$

Demak,

$$x+y=-t+c_1. \qquad (12.3.8)$$

Birinchi tenglamaning har ikkala tomonini x ga, ikkinchisini y ga ko‘paytirib, hosil bo‘lgan tengliklarni hadma-had qo‘shamiz:

$$\frac{d}{dt}\left(\frac{x^2}{2}+\frac{y^2}{2}\right)=\frac{xt-yt}{y-x}\Rightarrow\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right)=-t$$

Oxirgi integrallanuvchi kombinatsiyadan yana bir dona

$$x^2+y^2=-t^2+c_2 \qquad (12.3.9)$$

birinchi integralni topamiz.

(12.3.8) va (12.3.9) birinchi integrallar (12.3.7) sistemaning umumiy integralini beradi (ularning erkli ekanligini tekshirish o‘quvchiga havola etiladi). ✍

Avtonom sistema birinchi integrallarining to‘la sistemasida to‘xtalaylik.

Avtonom sistema

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(\boldsymbol{x}) \qquad (12.3.10)$$

vektor ko‘rinishda berilgan bo‘lsin. Biz uning t ga bog`liq bo`lmagan, ya’ni $u=u(\boldsymbol{x})$ ko`rinishdagi birinchi integrali bilan qiziqamiz. Eslaylikki, agar $\boldsymbol{f}(\boldsymbol{b})=0$ bo‘lsa, $\boldsymbol{b}\in\mathbb{R}^n$ nuqta (12.3.10) sistemaning muvozanat nuqtasi bo‘ladi. Biz (12.3.10) sistemani uning muvozanat (statsionar) nuqtasi bo‘lmagan nuqta atrofida tekshiramiz.

Teorema 4. *Faraz qilaylik,* $\boldsymbol{f}(\boldsymbol{b})\neq 0$ *va* $\boldsymbol{b}\in\mathbb{R}^n$ *nuqtaning biror atrofida* $\boldsymbol{f}\in C^1$ *bo‘lsin. U holda* $\boldsymbol{b}$ *nuqtaning biror kichik atrofida* (12.3.10) *sistemaning* $(n-1)$ *ta erkli birinchi integrallari mavjud.*

⊶ $\boldsymbol{f}(\boldsymbol{b})=(f_1(\boldsymbol{b}),...,(f_n(\boldsymbol{b}))^T\neq 0$ bo‘lgani uchun $f_k(\boldsymbol{b})=0,\ k=\overline{1,n},$ qiymatlarning birortasi noldan farqli. Aniqlik uchun $f_1(\boldsymbol{b})\neq 0$ deylik. $f_1\in C^1$ bo‘lgani uchun $\boldsymbol{b}$ nuqtaning biror atrofida ham $f_1(\boldsymbol{x})\neq 0$

Shu atrofda

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1,...,x_n) \\ \dfrac{dx_2}{dt} = f_2(x_1,...,x_n) \\ \dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = f_n(x_1,...,x_n) \end{cases}$$

va $\dfrac{dx_1}{dt} = f_1 \neq 0$ bo‘lgani uchun t o‘rniga x_1 erkli o‘zgaruvchini kiritamiz hamda

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1},\ \frac{dx_3}{dx_1} = \frac{f_3}{f_1},\ \dots\dots\dots,\ \frac{dx_n}{dx_1} = \frac{f_n}{f_1}$$

sistemani hosil qilamiz. Oxirgi sistema uchun yuqoridagi teoremaga ko‘ra birinchi integrallarning to‘la sistemasi mavjud. Bu birinchi integrallar (12.3.10) sistemaning t o‘zgaruvchiga bog‘liq bo‘lmagan $(n-1)$ ta erkli birinchi integrallarini tashkil etadi. ✍

Agar (12.3.1) sistemaning k ta ($k < n$) erkli birinchi integrallari $u_1(t,\boldsymbol{x}),...,u_k(t,\boldsymbol{x})$ topilgan bo‘lsa, u holda

$$\begin{cases} u_1(t,x_1,...,x_n) = c_1 \\ u_2(t,x_1,...,x_n) = c_2 \\ \dots\dots\dots\dots\dots \\ u_k(t,x_1,...,x_n) = c_k \end{cases} \tag{12.3.10}$$

sistemadan $x_1,...,x_n$ noma’lumlarning k tasini qolganlari orqali ifodalash mumkin (oshkormas funksiya haqidagi teoremaga ko`ra). Aniqlik uchun $x_1,...,x_k$ o‘zgaruvchilar $x_{k+1},...,x_n$ ($c_1,...c_k$ hamda t) orqali ifodalansin deylik

$$\begin{aligned} x_1 &= \varphi_1(x_{k+1},...,x_n,c_1,...,c_k,t) \\ x_2 &= \varphi_2(x_{k+1},...,x_n,c_1,...,c_k,t) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_k &= \varphi_k(x_{k+1},...,x_n,c_1,...,c_k,t) \end{aligned}$$

Bu munosabatlarni (12.3.1) sistemaning keyingi $(n-k)$ ta tenglamalariga qo‘yib $x_{k+1},...,x_n$ noma’lumlarga nisbatan ushbu

$$\begin{cases} \dfrac{dx_{k+1}}{dt} = f_{k+1}(\varphi_1,...,\varphi_k, x_{k+1},...,x_n) \\ \dfrac{dx_{k+2}}{dt} = f_{k+2}(\varphi_1,...,\varphi_k, x_{k+1},...,x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = f_n(\varphi_1,...,\varphi_k, x_{k+1},...,x_n) \end{cases} \qquad (12.3.11)$$

sistemani hosil qilamiz.

(12.3.11) sistema uchun birinchi integrallarning to‘la sistemasi va (12.3.10) munosabatlar birgalikda (12.3.1) sistema birinchi integrallarining to‘la sistemasini beradi, ya’ni (12.3.1) sistemani yechish masalasini hal qiladi.

Birinchi integrallarni topishning umumiy usuli yo‘q. Ko‘p hollarda berilgansistema tenglamalarini almashtirish yordamida osonginaintegrallanuvchi differensial tenglama hosil qilish mumkin. Bunnday tenglama integrallanuvchi kombinatsiya deb ataladi.

Berilgan differensial tenglamalar sistemasining integrallanuvchi kombinatsiyalarini tuzib, sistemaning (o‘zaro bog‘liq bo‘lmagan) erkli birinchi integrallarini topish mumkin.

Misol 4. Ushbu

$$\begin{cases} x' = (1-y)x \\ y' = \alpha(x-1)y \end{cases} \quad (\alpha = \text{const} > 0)$$

Volterra-Lotka sistemasini qaraylik ($x > 0, y > 0$).

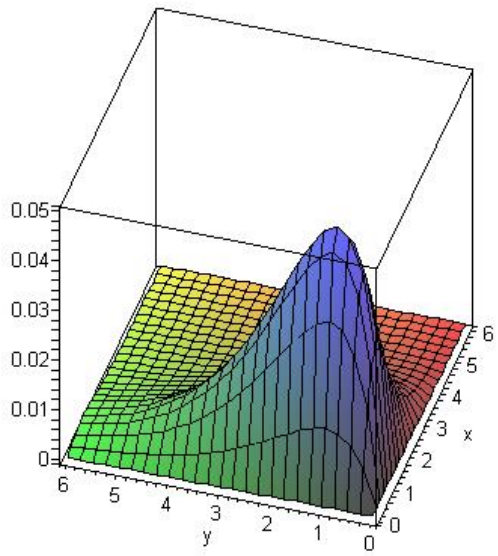
🔑Bu sistemaning birinchi integralini topish uchun ikkinchi tenglamani birinchisiga hadma-had bo‘laylik:

$$\frac{dy}{dx} = \frac{\alpha(x-1)y}{(1-y)x}.$$

Bu o‘zgaruvchilari ajraluvchi tenglamani osongina yechamiz:

$$\frac{yx^\alpha}{e^y e^{\alpha x}} = c \ .$$

Demak, $u(x,y) = \dfrac{x^\alpha y}{e^{\alpha x} e^y}$ –birinchi integral. Bu $u = u(x,y) = \dfrac{x^\alpha y}{e^{\alpha x} e^y}$ funksiyaning grafigi 12.1- rasmda keltirilgan.



12.1- rasm.

Bu yerda shuni e'tirof etish mumkinki, agar $x(0)=x_0$, $y(0)=y_0$ boshlang‘ich qiymatlar va x ning t paytdagi qiymati $x(t)$ ma'lum (u yoki bu usul yordamida topilgan) bo‘lsa, u holda $y(t)$ qiymatni berilgan differensial tenglamalar sistemasini yechmasdan turib, birinchi integraldan foydalanib topish mumkin. Buning uchun

$$\frac{yx^{\alpha}(t)}{e^{y}e^{\alpha x(t)}}=\frac{y_0x_0^{\alpha}}{e^{y_0}e^{\alpha x_0}}$$

transendent tenglamani $y(=y(t))$ga nisbatan yechish kerak (biror sonli usul yordamida). ☝

Birinchi integralning differensial tenglamani tekshirishdagi tadbig‘i sifatida to‘g‘ri chiziq bo‘ylab (inersial sanoq sistemasida) berilgan $F(x)$ kuch ta'sirida harakat qiluvchi, massasi birga teng bo‘lgan moddiy nuqta harakatini o‘rganamiz. Bu holda Nyutonning ikkinchi qonuni

$$\frac{d^2x}{dt^2}=F(x) \qquad (12.3.12)$$

tenglamani beradi; bu yerda $x=x(t)$ nuqtaning t paytdagi koordinatasi, $\frac{d^2x}{dt^2}$ – uning tezlanishi; biz $F(x)$ funksiyani biror intervalda differensiallanuvchi deb faraz qilamiz.

(12.3.12) tenglamada $x_1 = x,\ x_2 = \dfrac{dx}{dt}$ deb uni quyidagi sistemaga keltiramiz:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = F(x_1) \end{cases} \qquad (12.3.13)$$

(12.3.12) yoki (12.3.13) sistema uchun

$T = \dfrac{1}{2}\left(\dfrac{dx}{dt}\right)^2 = \dfrac{x_2^2}{2} -$ kinetik energiya, $U = -\int_{x_0}^{x} F(s)ds$ potensial energiya $\left(F(x) = -\dfrac{dU}{dx}\right)$, $E = T + U -$ to‘la (mexanik) energiya deb ataladi.

Teorema 5 (energiyaning saqlanish qonuni). *To‘la energiya* E (12.3.13) *sistemaning birinchi integralidir* (*har qanday harakatda to‘la energiya saqlanadi*).

🗝 Isboti oson:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{x_2^2(t)}{2} + U(x_1(t))\right) = x_2 \cdot x_2' + U' \cdot x_1' =$$

$$= x_2 F(x_1) - F(x_1) \cdot x_2 = 0. \ 👍$$

(12.3.13) sistema traektoriyasining har biri energiyaning sath to‘plamida joylashadi. Energiyaning

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \frac{x_2^2}{2} + U(x_1) = E = \text{const}\right\}$$

sath to‘plami (chizig‘i) sistemaning muvozanat holatidan, ya’ni

$$\{(x_1, x_2) \mid F(x_1) = 0,\ x_2 = 0\}$$

nuqta(lar)dan boshqa barcha nuqtalar atrofida silliq chiziqdan iborat bo‘ladi, chunki bunday nuqtalarda

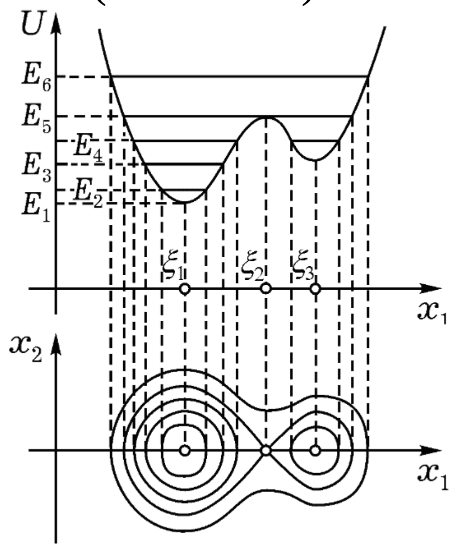
$$\frac{\partial E}{\partial x_1} = -F(x_1),\ \frac{\partial E}{\partial x_2} = x_2$$

hosilalarning kamida biri 0 ga teng emas va oshkormas funksiya haqidagi teoremaga ko‘ra sath to‘plami bu nuqtalar atrofida

$x_1 = x_1(x_2)$ yoki $x_2 = x_2(x_1)$ differensiallanuvchi funksiyaning grafigidan iborat bo‘ladi. Energiyaning sath chizig‘i (to‘plami)

$$|x_2| = \sqrt{2(E - U(x_1))} \tag{12.3.14}$$

tenglama bilan beriladi. $U(x_1)$ funksiyaning grafigiga ko‘ra (12.3.14) chiziqni qurish qiyin emas (12.2- rasm).



12.2- rasm.

Endi 12.3- rasmda keltirilgan fazaviy traektoriyani qaraylik. Birinchi integraldan

$$x' = \pm\sqrt{2(E - U(x))} \tag{12.3.15}$$

tenglamani topamiz. Bu tenglamada o‘zgaruvchilar ajraladi.

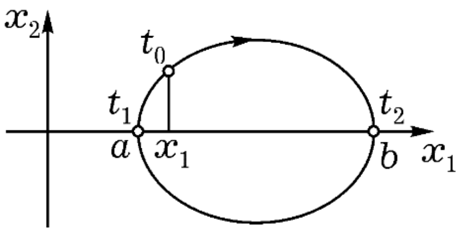
(12.3.15) ning $x(t_0) = x_1, \quad x'(t_0) = x_2 > 0$ shartni qanoatlantiruvchi $x = x(t)$ yechimi

$$t - t_0 = \int_{x_1}^{x(t)} \frac{ds}{\sqrt{2(E - U(s))}} \tag{12.3.16}$$

tenglikdan aniqlanadi. $U'(a) \neq 0,\ U'(b) \neq 0$ bo‘lgani uchun

$$\frac{\omega}{2} = \int_a^b \frac{ds}{\sqrt{2(E - U(s))}}$$

chekli sondan iborat (integral yaqinlashuvchi). Demak, (12.3.16) formula (12.3.12) ning $x = x(t)$ yechimini biror $t_1 \le t \le t_2$ oraliqda aniqlaydi, bunda $x(t_1) = a,\ \ x(t_2) = b,\ \ t_2 - t_1 = \omega / 2$ bo‘ladi.



12.3- rasm. $U'(a)\neq 0, U'(b)\neq 0, U(a)=U(b)=E$.

Endi $x(t)$ yechimni $[t_1,t_2]$ segmentdan $[t_2,t_2+\omega/2]$ segmentgacha

$x(t_2+\tau)=x(t_2-\tau)$, $0\le\tau\le\omega/2$ formulaga ko'ra davom ettiramiz. So'ngra $x(t+\omega)\equiv x(t)$ munosabat bilan uni $-\infty<t<\infty$ oraliqqa davriy davom ettiramiz. Hosil bo'lgan $x=x(t)$ funksiya (12.3.12) tenglamani $\forall t\in\mathbb{R}$ nuqtada qanoatlantiradi hamda $x(t_0)=x_1$, $x'(t_0)=x_2$ bo'ladi. Qurilgan $x=x(t)$ yechim ω davrli; uning fazaviy traektoriyasi 12.3- rasmda ko'rsatilgan silliq yopiq egri chiziqdan iborat.

Masalalar

1. Sistemani yeching

$$x'=x^2+y^2, y'=2xy.$$

2. Ushbu

$$x'=-x, y'=-y$$

sistema ychun quyidagi tasdiqlarni isbotlang:

a) Sistema ixtiyoriy $\delta>0$ uchun $B_\delta=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<\delta^2\}$ doirada aniqlangan birinchi integralga ega emas.

b) Sistema $x>0$ yarim tekislikda birinchi integalga ega.

3. Ushbu

$$x'=1+3y^2,\ y'=xyz,\ z'=-xz^2$$

sistemaning ikkita erkli birinchi integralini toping.

4. Quyidagi avtonom sistemaning ikkita erkli birinchi integralini toping va ular yordamida sistemaning traektoriyalarini tekshiring:

$$x_1'=x_2-x_3,\ x_2'=x_3-x_1,\ x_3'=x_1-x_2.$$

MODUL 13. TURG‘UNLIK NAZARIYASI ELEMENTLARI

§ 13.1. Turg‘unlik tushunchasi

Ko‘plab jarayonlarning kechishi oddiy differensial tenglamalar bilan tavsiflanadi. Bu tenglamalar cheksiz ko‘p yechimga ega bo‘lsa-da, tegishli jarayon bitta yechim bilan aniqlanadi va u ma’lum bir boshlang‘ich ma’lumotlarga mos keladi. Boshlang‘ich qiymatlar sal o‘zgarganda hosil bo‘luvchi yechim vaqt o‘tishi bilan dastlabki yechimga yaqinligicha qoladimi (turg‘un yechim) yoki undan uzoqlashib ketadimi (noturg‘un yechim)? degan savolning javobini bilish juda katta amaliy ahamiyatga ega. Chunki odatda boshlang‘ich qiymatlar xatolikka ega bo‘luvchi o‘lchashlar, taqribiy hisoblashlar orqali aniqlanadi va bu qiymatlarning sal o‘zgarishining yechimga ta’sirini bilish nihoyatda muhimdir. Agar yechim vaqt (t) o‘tishi bilan dastlabkisidan uzoqlashib ketsa, o‘rganilayotgan jarayonning tabiatini katta t larda oldindan aytib bo‘lmaydi.

Turg‘unlikning turli ta’riflarini Puasson, Lagranj, Lyapunov va boshqalar kiritishgan. Biz Lyapunov ma’nosidagi turg‘unlik bilan tanishamiz.

Differensial tenglamalarning quyidagi normal sistemasini qaraylik:

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}); \qquad (13.1.1)$$

bu yerda $\boldsymbol{f} \in C(\mathbb{R}_+ \times D; \mathbb{R}^n)$ ($\mathbb{R}_+ = [0, +\infty)$, $D \subset \mathbb{R}^n$ −soha) va $\boldsymbol{f}(t, \boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantiradi deb hisoblanadi. Bu shartlarda $\forall (t_0, \boldsymbol{x}^0) \in \mathbb{R}_+ \times D$ uchun ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}^0 \end{cases}$$

masala $t \in [t_0, T)$ oraliqda aniqlangan (o‘ngga davomsiz) yagona $\boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0)$ yechimga ega. Biz o‘ngga cheksiz davom etgan ($T = +\infty$), ya’ni $t \in [t_0, +\infty)$ oraliqda aniqlangan yechimlarning turg‘unlik xossalarini o‘rganamiz.

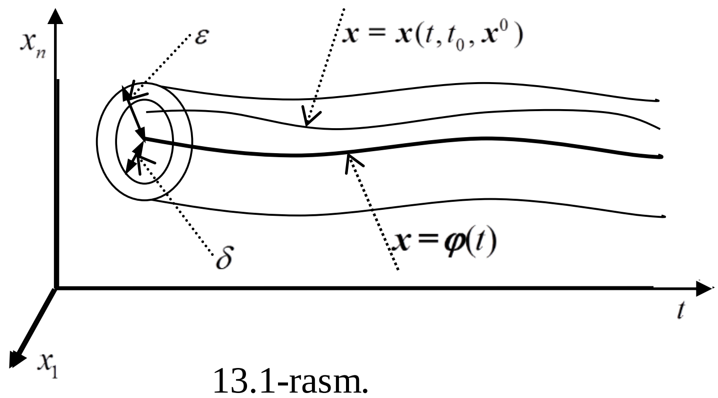
Bizga (13.1.1) tenglamaning $\mathbb{R}_+$ da aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$, $\boldsymbol{\varphi} : \mathbb{R}_+ \to D$, yechimi berilgan bo‘lsin. Agar ixtiyoriy $t_0 \in \mathbb{R}_+$ va $\varepsilon > 0$ sonlariga ko‘ra shunday $\delta > 0$ soni topilsaki, (13.1.1) tenglamaning $\boldsymbol{x}(t_0) = \boldsymbol{x}^0$ boshlang‘ich qiymatli $\boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0)$

yechimlari $\|\boldsymbol{x}^0-\boldsymbol{\varphi}(t_0)\|<\delta$ bo‘lganda mavjud va o‘ngga $+\infty$ gacha davom ettirilib, barcha $t\in[t_0,+\infty)$ paytlarda $\|\boldsymbol{x}(t,t_0,\boldsymbol{x}^0)-\boldsymbol{\varphi}(t)\|<\varepsilon$ bo‘lsa, u holda $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechim Lyapunov ma’nosida (yoki Lyapunovga ko‘ra) turg‘un yechim deb ataladi. Keltirilgan shart boshlang‘ich qiymatlarning yaqinligidan ($\|\boldsymbol{x}^0-\boldsymbol{\varphi}(t_0)\|<\delta$) barcha keyingi paytlarda ham yechimlarning yaqinligi

($\|\boldsymbol{x}(t,t_0,\boldsymbol{x}^0)-\boldsymbol{\varphi}(t)\|<\varepsilon,\ t\in[t_0,+\infty)$) kelib chiqishini anglatadi (13.1-rasm).

Turg‘un yechim ta’rifidagi “barcha $t\in[t_0,+\infty)$ paytlarda $\|\boldsymbol{x}(t,t_0,\boldsymbol{x}^0)-\boldsymbol{\varphi}(t)\|<\varepsilon$ bo‘lsa” shartini ushbu

“ $\sup\limits_{t\in[t_0,+\infty)}\|\boldsymbol{x}(t,t_0,\boldsymbol{x}^0)-\boldsymbol{\varphi}(t)\|<\varepsilon$ bo‘lsa” shart bilan almashtirish mumkin.



13.1-rasm.

Oldindan beriladigan ixtiyoriy $\varepsilon>0$ sonni har doim yetarlicha kichik deb hisoblash mumkin, chunki biror $\varepsilon_0>0$ ga ko‘ra topilgan $\delta=\delta_0>0$ son har qanday $\varepsilon\geq\varepsilon_0$ son uchun ham δ bo‘lib xizmat qiladi. Demak, barcha $\varepsilon\in(0;\varepsilon_0]$ sonlar, ya’ni yetarlicha kichik εlar uchun ularga mos δ larni topish kerak xolos.

Umumiy holda topiladigan $\delta>0$ son tayinlangan $t_0\in\mathbb{R}_+$ va berilgan $\varepsilon>0$ sonlarga bog‘liq bo‘ladi, ya’ni $\delta=\delta(t_0,\varepsilon)$. Agar turg‘unlik ta’rifidagi $\delta>0$ sonni $t_0\in[0,+\infty)$ ga bog‘liqsiz holda tanlash mumkin, ya’ni $\delta=\delta(\varepsilon)$ bo‘lsa, u holda yechim ($t_0\in[0,+\infty)$ ga nisbatan) **tekis turg‘un yechim** deb ataladi.

Turg‘un bo‘lmagan yechim noturg‘un yechim deyiladi.

Agar

1) $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim turg‘un va

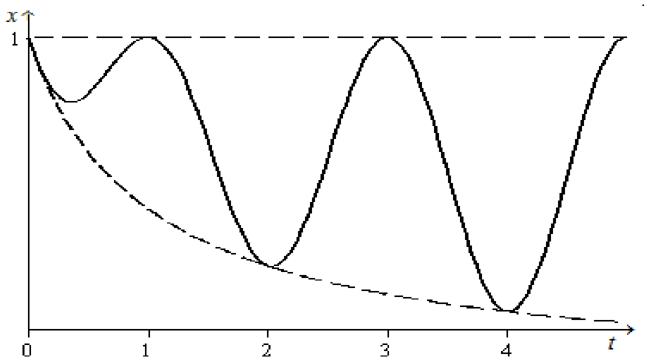
2) shunday $\delta_0 > 0$ mavjud bo‘lib, $\left\|\boldsymbol{x}^0 - \boldsymbol{\varphi}(t_0)\right\| < \delta_0$ ekanligidan

$\lim\limits_{t\to+\infty} \| \boldsymbol{x}(t,t_0,\boldsymbol{x}^0) - \boldsymbol{\varphi}(t) \| = 0$ bo‘lishi ham kelib chiqsa,

u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim ***asimptotik turg‘un yechim*** deb ataladi.

Umumiy holda turg‘unlikdan tekis turg‘unlik kelib chiqmaydi; bundan tashqari, turg‘unlikdan asimptotik turg‘unlik ham kelib chiqmaydi. Bu tasdiqlarni quyidagi misol asoslaydi.

Misol. Ushbu $f(t) = \dfrac{t}{t+1}\sin^2\dfrac{\pi}{2}t + \dfrac{1}{t+1},\ t \ge 0,$ funksiyani aniqlaylik. Ravshanki, $f(t) \in C^1([0;+\infty);\mathbb{R})$ hamda $1/(t+1) \le f(t) \le 1,\ t \ge 0$ (13.2- rasm).



13.2- rasm.

Endi ushbu

$$x' = \frac{f'(t)}{f(t)}x,\ t \ge 0, \qquad (13.1.2)$$

tenglamani qaraylik. Uning $t = t_0\ (t_0 \ge 0)$ da x_0 ga aylanuvchi yechimi

$$x = x(t;t_0,x_0) = \frac{x_0}{f(t_0)}f(t) \qquad (13.1.3)$$

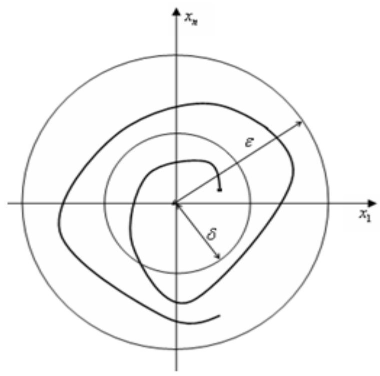
ko‘rinishga ega. (13.1.2) tenglamaning $x(t) \equiv 0$ yechimi turg‘un, chunki $|x(t;t_0,x_0)| \le \sup\limits_{t\ge t_0}|x(t;t_0,x_0)| = |x_0|/f(t_0) < \varepsilon$ bo‘lishi uchun δ sifatida $\delta = \delta(t_0,\varepsilon) = \varepsilon \cdot f(t_0)$ ni tanlash kifoya. Tushunarliki, bu δ ni yaxshilab, ya’ni kichiklashtirib bo‘lmaydi. Lekin bu $x(t) \equiv 0$ yechim $t_0 \ge 0$ ga nisbatan tekis turg‘un emas, chunki

$\inf\limits_{t_0 \ge 0} \delta(t_0, \varepsilon) = \varepsilon \cdot \inf\limits_{t_0 \ge 0} f(t_0) = 0$. Umumiy yechim uchun (13.1.3) formuladan $x(t) \equiv 0$ yechimning asimptotik turg‘un emasligi ham kelib chiqadi. Haqiqatan ham, agar u asimptotik turg‘un bo‘lganda edi, u holda yetarli kichik $|x_0|$ $(x_0 \ne 0)$ har uchun $|x(t; t_0, x_0)| = \dfrac{|x_0|}{f(t_0)} f(t) \xrightarrow[t \to +\infty]{} 0$ bo‘lardi. Lekin $t = 2n-1, n \in \mathbb{N}$, da $|x(2n-1; t_0, x_0)| = \dfrac{|x_0|}{f(t_0)} f(2n-1) = \dfrac{|x_0|}{f(t_0)} > 0$ va ziddiyat hosil bo‘lardi.✍

Berilgan tenglamaning ixtiyoriy tayinlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t),\ t \ge 0$, yechimini turg‘unlikka tekshirishni **boshqa bir tenglamaning trivial**, ya’ni nolga teng yechimini turg‘unlikka tekshirishga keltirish mumkin. Buning uchun (13.1.1) tenglamada $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{\varphi}(t)$ almashtirishni bajarish kerak. Yangi noma’lum $\boldsymbol{y}$ quyidagi tenglamani qanoatlantiradi:

$$\boldsymbol{y}' = \boldsymbol{g}(t, \boldsymbol{y}),\ \boldsymbol{g}(t, \boldsymbol{y}) \equiv \boldsymbol{f}(t, \boldsymbol{y} + \boldsymbol{\varphi}(t)) - \boldsymbol{f}(t, \boldsymbol{\varphi}(t)),\ \ \boldsymbol{g}(t, 0) = 0.$$

Oxirgi differensial tenglama $\boldsymbol{y} = 0$ trivial yechimga ega. Bu yechimning turg‘unligi (asimptotik turg‘unligi) (13.1.1) tenglamaning $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimini turg‘unligiga (asimptotik turg‘unligiga) teng kuchlidir. Nol yechimning turg‘unligi quyidagini anglatadi: Fazalar fazosida ixtiyoriy B_ε sharni olmaylik, shunday yetarlicha kichik $\delta > 0$ radiusli B_δ shar topiladiki, $t = t_0$ da bu shar ichidan chiquvchi ixtiyoriy traektoriya $t \ge t_0$ paytlarda to‘laligicha B_ε shar ichida qoladi (13.3- rasm).



13.3- rasm.

(13.1.1) ning o‘ng tomonidan talab qilingan shartlarda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim boshlang‘ich qiymatlarga chegaralangan segmentda uzluksiz bog‘liq, ya’ni $\forall [\alpha, \beta] \subset [t_0, +\infty)$ segment va $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ topiladiki, $\gamma \in [\alpha, \beta]$ paytda $\boldsymbol{z}^0$ qiymat qabul qiluvchi $\boldsymbol{x} = \boldsymbol{x}(t, \gamma, \boldsymbol{z}^0)$ yechim, agar $\| \boldsymbol{x}(\gamma, \gamma, \boldsymbol{z}^0) - \boldsymbol{\varphi}(\gamma) \| = \| \boldsymbol{z}^0 - \boldsymbol{\varphi}(\gamma) \| < \delta$ bo‘lsa, barcha $t \in [\alpha, \beta]$ larda aniqlangan va $\forall t \in [\alpha, \beta]$ uchun $\left\| \boldsymbol{x}(t, \gamma, \boldsymbol{z}^0) - \boldsymbol{\varphi}(t) \right\| < \varepsilon$ tengsizlikni qanoatlantiradi. Bu xossadan ravshanki, $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimning turg‘unligi (asimptotik turg‘unligi) boshlang‘ich payt $t_0 \in [0, +\infty)$ ning tanlanishiga bog‘liq emas, ya’ni agar biror $t_0 \in [0, +\infty)$ uchun $\boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{\varphi}(t_0))$ yechim turg‘un (asimptotik turg‘un) bo‘lsa, u holda ixtiyoriy $\tilde{t}_0 \in [0, +\infty)$ uchun ham mos $\boldsymbol{x} = \boldsymbol{x}(t, \tilde{t}_0, \boldsymbol{\varphi}(\tilde{t}_0))$ yechim turg‘un (asimptotik turg‘un) bo‘ladi.

Masalalar

1. Avtonom sistema uchun yechimning turg‘unligidan uning (boshlang‘ich payt bo‘yicha) tekis turg‘unligi kelib chiqishini isbotlang**.**

2. Agar normal sistemaning o‘ng tomoni davriy, ya’ni $\boldsymbol{f}(t+T, \boldsymbol{x}) \equiv \boldsymbol{f}(t, \boldsymbol{x}), T > 0$, bo‘lsa, bunday sistema yechimining turg‘unligidan uning boshlang‘ich payt bo‘yicha tekis turg‘unligi kelib chiqadi. Shuni isbotlang.

3. Ushbu

$$g(t) = \left(1 - e^{-t}\right)\cos^2\left(\frac{\pi}{2}t\right) + e^{-t}, \ t \geq 0,$$

funksiyaga ko‘ra tuzilgan

$$x' = \frac{g'(t)}{g(t)}x, \ t \geq 0,$$

differensial tenglamaning $x(t) \equiv 0$ yechimini turg‘unlikka, tekis turg‘unlikka va asimptotik turg‘unlikka tershiring.

4. Volterra-Lotka sistemasining muvozanat holatini turg‘unlikka tekshiring.

§ 13.2. Chiziqli sistemalarning turg‘unligi

Bu paragrafda ushbu

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{b}(t) \qquad (13.2.1)$$

chiziqli sistemaning turg‘unligini tekshiramiz; bunda $A(t) \in C\left([0; +\infty); \mathrm{M}_{n \times n}(\mathbb{R})\right)$ va $\boldsymbol{b}(t) \in C\left([0; +\infty); \mathbb{R}^n\right)$ deb hisoblanadi. Demak, ixtiyoriy $\boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \in \mathbb{R}^n$, $t_0 \in [0; +\infty)$, boshlang‘ich shart uchun birato‘la $[0; +\infty)$ oraliqda aniqlangan $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$ yagona yechim mavjud. (13.2.1) ga mos bir jinsli sistema

$$\boldsymbol{y}' = A(t)\boldsymbol{y} \qquad (13.2.2)$$

ko‘rinishda bo‘ladi.

Teorema 1. (13.2.1) *chiziqli sistemaning har qanday yechimining turg'unligi* (*tekis yoki asimptotik turg'unligi*) *mos bir jinsli sistema* (13.2.2) *ning bitta* $\boldsymbol{y}=0$ *trivial yechimining turg'unligiga* (*mos ravishda tekis yoki asimptotik turg'unligiga*) *teng kuchli.*

🗝 Teoremaning turg'unlikka oid qismini isbotlaymiz. Uning tekis yoki asimptotik turg'unlikka oid qismlari shunga o'xshash isbotlanadi. (13.2.1) sistemaning ixtiyoriy bir $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ turg'un yechimini olaylik. Demak, $t\in[t_0,+\infty)$ uchun

$$\boldsymbol{\varphi}'(t)=A(t)\boldsymbol{\varphi}(t)+\boldsymbol{b}(t),$$

$$\left\|\boldsymbol{x}^0-\boldsymbol{\varphi}(t_0)\right\|<\delta \Rightarrow \|\,\boldsymbol{x}(t,t_0,\boldsymbol{x}^0)-\boldsymbol{\varphi}(t)\,\|<\varepsilon; \qquad (13.2.3)$$

bu yerda $\delta>0$ son oldindan berilgan ixtiyoriy $\varepsilon>0$ va $t_0\geq 0$ sonlarga ko'ra turg'unlik ta'rifidan topilgan. Biz (13.2.2) sistemaning $\boldsymbol{y}=\boldsymbol{y}(t;t_0,\boldsymbol{y}^0)$ yechimi uchun

$$\|\boldsymbol{y}^0\|<\delta \Rightarrow \|\,\boldsymbol{y}(t;t_0,\boldsymbol{y}^0)\,\|<\varepsilon \qquad (13.2.4)$$

implikatsiyaning o'rinliligini ko'rsatishimiz kifoya, chunki bu holda (13.2.2) ning $\boldsymbol{y}=0$ yechimi turg'un bo'ladi. Agar $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{\varphi}(t)$ desak, u holda bu yerdagi $\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{\varphi}(t)$ funksiya (13.2.1) sistemaning yechimi bo'ladi va (13.2.4) implikatsiya (13.2.3) implikatsiyadan bevosita kelib chiqadi.

Endi faraz qilaylik, (13.2.2) ning $\boldsymbol{y}=0$ yechimi turg'un, ya'ni (13.2.4) implikatsiya o'rinli bo'lsin. (13.2.1) ning ixtiyoriy $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimi turg'un ekanligini isbotlash kerak. Bu esa yana o'sha $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{\varphi}(t)$ almashtirish yordamida yuqoridagiga o'xshash asoslanadi. 👍

Shunday qilib, quyidagi alternativa o'rinli:

yo (13.2.1) chiziqli sistemaning barcha yechimlari turg'un (tekis yoki asimptotik turg'un); bu holda (13.2.1) sistema ***turg'un sistema*** (mos ravishda **tekis** yoki ***asimptotik turg'un sistema***) deb ataladi,

yoki uning barcha yechimlari noturg'un; bu holda esa (13.2.1) sistema ***noturg'un sistema*** deb ataladi.

Biz endi (13.2.2) sistemaning trivial yechimini turg'unlikka tekshiramiz. Bu turg'unlik (13.2.1) va (13.2.2) sistemalarning turg'unligiga teng kuchli. (13.2.2)sistemada noma'lumni odatdagidek $\boldsymbol{x}=\boldsymbol{x}(t)$ bilan belgilab, uni

$$\boldsymbol{x}' = A(t)\boldsymbol{x} \qquad (13.2.5)$$

ko'rinishda yozib olamiz.

Ma'lumki, (13.2.5) sistemaning $t = t_0$ paytda $\boldsymbol{x}^0$ga aylanuvchi yechimi $\boldsymbol{x} = \boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$ ushbu

$$\boldsymbol{x}(t; t_0, \boldsymbol{x}^0) = \Phi(t, t_0)\boldsymbol{x}^0 \qquad (13.2.6)$$

formula bilan ifodalanadi. Bu yerdagi $\Phi(t, t_0)$matritsa (13.2.5) sistemaning normalangan fundamental matritsasi, ya'ni uning ustunlari (13.2.5)ning n dona chiziqli erkli yechimlaridan tashkil topgan va $\Phi(t_0, t_0) = E -$birlik matritsa.

(13.2.6) formuladan ravshanki, ixtiyoriy λ va μ sonlar va ixtiyoriy $\boldsymbol{a} \in \mathbb{R}^n$ va $\boldsymbol{b} \in \mathbb{R}^n$ vektorlar uchun

$$\boldsymbol{x}(t; t_0, \lambda\boldsymbol{a} + \mu\boldsymbol{b}) = \lambda\boldsymbol{x}(t; t_0, \boldsymbol{a}) + \mu\boldsymbol{x}(t; t_0, \boldsymbol{b}).$$

Bu formula (13.2.5) sistemaning $\boldsymbol{x} = \boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$yechimi $\boldsymbol{x}^0$ boshlang'ich qiymatga nisbatan chiziqli funksiya ekanligini anglatadi.

Teorema 2. *Chiziqli bir jinsli tenglamalar sistemasi* (13.2.5) *ning* $t = t_0 \geq 0$ *nuqtada normalangan fundamental matritsasi* $\Phi(t, t_0)$ *bo'lsin. U holda* (13.2.5) *sistemaning **turg'un** bo'lishi uchun ushbu*

$$\forall t_0 \geq 0 \ \exists m(t_0) > 0 \ \forall t \geq t_0 \ \ \| \Phi(t, t_0) \| \leq m(t_0) \qquad (13.2.7)$$

shartning (*tayinlangan har bir* $t_0 \geq 0$ *uchun* $\Phi(t, t_0)$ *matritsa chegaralangan*), ***tekis turg'un*** *bo'lishi uchun esa ushbu*

$$\exists m > 0 \ \forall t_0 \geq 0 \ \forall t \geq t_0 \ \ \|\Phi(t, t_0)\| \leq m \qquad (13.2.8)$$

shartning ($\Phi(t, t_0)$*matritsalar oilasi bir tekis chegaralangan*) *bajarilishi yetarli va zarurdir.*

🗝 Teoremaning turg'unlikka oid qismini isbotlaylik. $\boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$ yechim uchun (13.2.6) formuladan (13.2.7) ga ko'ra

$$\| \boldsymbol{x}(t; t_0, \boldsymbol{x}^0) \| = \| \Phi(t, t_0)\boldsymbol{x}^0 \| \leq \| \Phi(t, t_0) \| \cdot \| \boldsymbol{x}^0 \| \leq m(t_0) \| \boldsymbol{x}^0 \|.$$

Demak, $\forall \varepsilon > 0$ va $\forall t_0 \geq 0$ sonlar uchun $\delta = \varepsilon / m(t_0)$ desak, u holda $\| \boldsymbol{x}^0 \| < \delta$ ekanligidan $\forall t \geq t_0$ uchun

$$\| \boldsymbol{x}(t; t_0, \boldsymbol{x}^0) \| \leq m(t_0) \| \boldsymbol{x}^0 \| < m(t_0)\delta = \varepsilon$$

bo'lishi kelib chiqadi. Bu esa (13.2.5) ning trivial yechimining turg'unligini anglatadi. Demak, (13.2.5) sistema ham turg'un.

Endi faraz qilaylik, (13.2.5) sistema turg‘un bo‘lsin. Demak, xususan, uning trivial yechimi ham turg‘un. Demak, $\varepsilon = 1$ son uchun $\forall t_0 \geq 0$ songa ko‘ra shunday $\delta_0 = \delta_0(t_0) > 0$ topamizki, $\| \boldsymbol{x}^0 \| < \delta_0$ bo‘lganda (13.2.5)ning $\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)$ yechimi uchun

$$\forall t \geq t_0 \text{ paytda } \| \boldsymbol{x}(t;t_0,\boldsymbol{x}^0) \| < 1 \qquad (13.2.9)$$

bo‘ladi. $\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n$ vektorlar $\mathbb{R}^n$ning standart bazisi bo‘lsin. Ularga ko‘ra qurilgan $\boldsymbol{x}(t;t_0,\boldsymbol{e}^j),\ j = \overline{1,n}$, yechimlar (13.2.5) ning fundamental sistemasini tashkil etadi. Ravshanki, ulardan $t = t_0$ da normalangan fundamental matritsa $\Phi(t,t_0)$ tuziladi. Yechimning boshlang‘ich qiymatga nisbatan chiziqlilik xossasidan

$$\boldsymbol{x}(t;t_0,(\delta_0/2)\boldsymbol{e}^j) = (\delta_0/2)\boldsymbol{x}(t;t_0,\boldsymbol{e}^j),\ j = \overline{1,n}.$$

Bundan, $\| (\delta_0/2)\boldsymbol{e}^j \| = \delta_0/2 < \delta_0$ bo‘lgani uchun, (13.2.9) ga ko‘ra

$$\| (\delta_0/2)\boldsymbol{x}(t;t_0,\boldsymbol{e}^j) \| < 1, \text{ ya'ni } \| \boldsymbol{x}(t;t_0,\boldsymbol{e}^j) \| < 2/\delta_0,\ t \geq t_0,\ j = \overline{1,n}.$$

Shunday qilib, qurilgan bazis yechimlar va, demak, ulardan tuzilgan $\Phi(t,t_0)$normalangan fundamental matritsa ham chegaralangan.

Teoremaning turg‘unlikka oid qismi isbot bo‘ldi. Teoremaning tekis turg‘unlikka oid qismining isboti uning turg‘unlikka oid qismining isbotidan bevosita ravshan. 👍

Teorema 3. *Chiziqli bir jinsli differensial tenglamalar sistemasi* (13.2.5) *ning asimptotik turg‘un bo‘lishi uchun uning biror* (*va, demak ixtiyoriy*) *fundamental matritsasining* $t \to +\infty$ *dagi limiti nol-matritsadan iborat bo‘lishi yetarli va zarurdir*.

🔑 Faraz qilaylik, nol yechim asimptotik turg‘un bo‘lsin. Demak, $\| \boldsymbol{x}^0 \| < \delta_0$ boshlang‘ich qiymatlar uchun

$$\boldsymbol{x}(t;t_0,\boldsymbol{x}^0) \underset{t \to +\infty}{\longrightarrow} 0. \qquad (13.2.9)$$

Biz biror fundamental matritsaning nolga intilishini ko‘rsatishimiz kifoya. Ushbu

$$\boldsymbol{x}(t;t_0,(\delta_0/2)\boldsymbol{e}^j) = (\delta_0/2)\boldsymbol{x}(t;t_0,\boldsymbol{e}^j),\ j = \overline{1,n},$$

yechimlarni qaraylik. Ularning boshlang‘ich qiymatlari $\| (\delta_0/2)\boldsymbol{e}^j \| < \delta_0$bo‘lgani uchun (13.2.9) farazimizga ko‘ra

$$\boldsymbol{x}(t;t_0,(\delta_0/2)\boldsymbol{e}^j) \underset{t \to +\infty}{\longrightarrow} 0$$

Demak, ana shu $\boldsymbol{x}(t;t_0,(\delta_0/2)\boldsymbol{e}^j)$, $j=\overline{1,n}$, yechimlardan tuzilgan fundamental matritsa nolga intiladi.

Endi faraz qilaylik, biror fundamental matritsa $t\to+\infty$da nolga intilsin. Demak, $\Phi(t,t_0)$ normalangan fundamental matritsa ham nolga intiladi:

$$\|\Phi(t,t_0)\|\underset{t\to+\infty}{\to}0.$$

Ixtiyoriy $\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)$ yechim uchun $\|\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)\|=\|\Phi(t,t_0)\boldsymbol{x}^0\|\le\|\Phi(t,t_0)\|\cdot\|\boldsymbol{x}^0\|\underset{t\to+\infty}{\to}0.$

Demak, $\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)\underset{t\to+\infty}{\to}0$. Bundan esa nol-yechimning asimptotik turg‘unligi kelib chiqadi. ✍

Endi chiziqli o‘zgarmas koeffitsientli sistemalarning turg‘unligini o‘rganamiz. O‘zgarmas koeffitsientli ushbu

$$\boldsymbol{x}'=A\boldsymbol{x} \tag{13.2.10}$$

sistemani qaraylik, bunda $A-$haqiqiy sonlardan tuzilgan $n\times n$ matritsa, ya’ni $A\in\mathrm{M}_{n\times n}(\mathbb{R})$. Bu sistemaning turg‘unligi (noturg‘unligi) A maritsaning xos sonlari bilan aniqlanadi.

Teorema 4.

1^0 . *Agar xos sonlarning hammasi manfiy haqiqiy qismga ega bo‘lsa, u holda* (13.2.10) *sistema asimptotik turg‘un bo‘ladi.*

2^0 . *Agar xos sonlarning barchasi nomusbat haqiqiy qismga ega bo‘lib, haqiqiy qismi nol bo‘lgan xos sonlarga faqat 1- tartibli Jordan kataklari mos kelsa, u holda (13.*2.10) *sistema turg‘un bo‘ladi.*

3^0 . *Agar xos sonlarning birortasi musbat haqiqiy qismga ega bo‘lsa, yoki haqiqiy qismi nol bo‘lgan xos sonlarning birortasiga kamida ikkinchi tartibli Jordan katagi mos kelsa, u holda* (13.2.10) *sistema noturg‘un bo‘ladi.*

⊶ A matritsaning (turli) xos sonlarini $\lambda_1,\lambda_2,\ldots,\lambda_s$ $(s\le n)$ bilan, $\tilde{k}_j$ bilan esa λ_j ga mos kelgan Jordan kataklarining eng katta tartibini belgilaylik. $\lambda_j\,(j=\overline{1,s})$ xos sonlarning haqiqiy va mavhum qismlarini ajrataylik: $\lambda_j=\alpha_j+i\beta_j$. U holda fundamental matritsaning elementlari

$$\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t)+q_j(t)\sin(\beta_j t)\right) \tag{13.2.11}$$

ko‘rinishda yoziladi, bunda $p_j(t)$ va $q_j(t)$ ko‘phadlarning darajalari $\tilde{k}_j-1$ dan kichik yoki unga teng.

Analizdan ma’lumki, ixtiyoriy $\alpha<0$ son va ixtiyoriy $p(t)$ ko‘phad uchun $\lim\limits_{t\to+\infty} p(t)e^{\alpha t}=0$. Demak, 1^0 holda barcha $\alpha_j=\mathrm{Re}\lambda_j<0$ bo‘lgani uchun

$$\left|\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t)+q_j(t)\sin(\beta_j t)\right)\right|\le$$
$$\le\sum_{j=1}^{s}|p_j(t)|e^{\alpha_j t}+\sum_{j=1}^{s}|q_j(t)|e^{\alpha_j t}\xrightarrow[t\to+\infty]{}0,$$

ya’ni fundamental matritsaning hamma elementlari $t\to+\infty$ da nolga intiladi. Shuning uchun 1^0 holda (13.2.10) sistema asimptotik turg‘un.

2^0 holda (13.2.11) dagi $\alpha_j=\mathrm{Re}\lambda_j<0$ sonlarga mos keluvchi qo‘shiluvchilar ixtioriy $[t_0;+\infty)$ oraliqda chegaralangan (1^0 holdagi singari), $\alpha_j=\mathrm{Re}\lambda_j=0$ sonlarga mos keluvchi qo‘shiluvchilar ham chegaralangan, chunki ular nolinchi darajali ko‘phadlardan (o‘zgarmaslardan) iborat. Demak, fundamental matritsaning barcha elementlari $[t_0;+\infty)$ da chegaralangan va (13.2.10) sistema turg‘un.

Endi 3^0 holni qaraylik. Agar biror $\alpha_j=\mathrm{Re}\lambda_j>0$ bo‘lsa, u holda (13.2.11) dagi shu songa mos kelgan qo‘shiluvchilar va , demak, fundamental matritsa ham $[t_0;+\infty)$ da chegaralanmagan. Agar $\alpha_j=\mathrm{Re}\lambda_j=0$ va $\tilde{k}_j\ge 2$ bo‘lsa, u holda (13.2.11) dagi shu songa mos kelgan

$$e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t)+q_j(t)\sin(\beta_j t)\right)=p_j(t)\cos(\beta_j t)+q_j(t)\sin(\beta_j t)$$

qo‘shiluvchida $\deg p_j(t)\ge 1$, $\deg q_j(t)\ge 1$ bo‘lgani uchun u $[t_0;+\infty)$ da chegaralanmagan. Demak, fundamental matritsa ham chegaralanmagan. Shuning uchun (13.2.10) sistema turg‘un emas. 👍

Izoh. Teoremada A matritsaning o‘zgarmas ekanligi muhim. Quyidagi misol bu tasdiqni asoslaydi. Ushbu

$$\begin{cases} x' = (-3+4\cos^2 3t)x + (-3+2\sin 6t)y \\ y' = (3+2\sin 6t)x + (-3+4\sin^2 3t)y \end{cases}$$

o‘zgaruvchi koeffitsientli sistemani qaraylik. Bu sistemaning matritsasi:

$$A(t) = \begin{pmatrix} -3+4\cos^2 3t & -3+2\sin 6t \\ 3+2\sin 6t & -3+4\sin^2 3t \end{pmatrix}.$$

Uning xos sonlari: $\lambda_{1,2} = -1 \pm i\sqrt{5}$ (t ga bog‘liq emas).

Osongina tekshirib ko‘rish mumkinki, qaralayotgan sistema

$$\begin{cases} x = ce^t \cos 3t \\ y = ce^t \sin 3t \end{cases} \quad (c \neq 0)$$

ko‘rinishdagi yechimga ega. Bu yechim $\mathrm{Re}\lambda_{1,2} < 0$ bo‘lishiga qaramasdan $t \to +\infty$ da chegaralanmagan. Bundan qaralayotgan sistema trivial yechimining turg‘un emasligi kelib chiqadi.

Shunday qilib, umumiy holda koeffitsientlari o‘zgaruvchi bo‘lgan chiziqli sistema matritsasining xos sonlari uning turg‘unligini aniqlamaydi.

Misol 1. Ushbu

$$x' = 2y - z, y' = 3x - 2z, z' = 5x - 4y$$

sistemani turg‘unlikka tekshiraylik.

🗝 Sistemaning xarakteristik tenglamasi

$$\begin{vmatrix} -\lambda & 2 & -1 \\ 3 & -\lambda & -2 \\ 5 & -4 & -\lambda \end{vmatrix} = -\lambda^3 + 9\lambda - 8 = 0.$$

Ravshanki, $\lambda = 1 > 0$ bu tenglamaning ildizi. Demak, (boshqa xarakteristik sonlarning qiymatlaridan qat’iy nazar) berilgan sistema noturg‘un. 👍

Misol 2. Ushbu

$$x' = -x + y - z,\ y' = 2x - \frac{1}{3}y - \frac{7}{3}z,\ z' = x + \frac{4}{3}y - \frac{8}{3}z$$

sistemani turg‘unlikka tekshiring.

🗝 Sistemaning xarakteristik sonlari

$$\begin{vmatrix} -1-\lambda & 1 & -1 \\ 2 & -1/3-\lambda & -7/3 \\ 1 & 4/3 & -8/3-\lambda \end{vmatrix} = 0 \ \Rightarrow \ \lambda_1 = -2, \lambda_{2,3} = -1 \pm i .$$

Barcha xarakteristik sonlarning haqiqiy qismi manfiy bo‘lgani uchun sistema asimptotik turg‘un. 👍

Ushbu

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0 = 0, \ a_n > 0, \qquad (13.2.12)$$

haqiqiy koeffitsientli algebraik tenglama ildizlarining haqiqiy qismi manfiy bo‘lishini aniqlash uchun foydalaniladigan mezonni isbotsiz keltiramiz. Dastlab ushbu

$$\begin{vmatrix} a_n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 & \cdots & 0 & 0 \\ a_{n-6} & a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 \\ & & & & \ddots & & & & & \\ & & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

determinantni tuzaylik. Uning bosh diagonalida (13.2.12) ko‘phadning $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ koeffitsientlari, satrlarida esa a_j lar indeksning o‘sish tartibida joylashgan bo‘lib, bunda $j < 0$ yoki $j > n$ indekslar uchun $a_j = 0$ deb hisobanadi. Bu determinantning bosh diagonal minorlarini

$$\Delta_1 = a_n, \ \Delta_2 = \begin{vmatrix} a_n & 0 \\ a_{n-2} & a_{n-1} \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_n & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_n & 0 & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-6} & a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}, \ . \ . \ .$$

bilan belgilaylik.

Teorema (L'yenar–Shipar mezoni). (13.2.12) *tenglama barcha ildizlarining haqiqiy qismi manfiy bo'lishi uchun ushbu*

1) *barcha* a_j *lar musbat, ya'ni*

$$a_n > 0,\ a_{n-1} > 0,\ a_{n-2} > 0,\ \ldots,\ a_1 > 0,\ a_0 > 0;$$

2) $\Delta_3 > 0,\ \Delta_5 > 0,\ \Delta_7 > 0,\ \ldots$

shartlarning bir vaqtda bajarilishi yetarli va zarurdir.

Masalalar

1. Ushbu

$$x' = a(t)x,\ a(t) \in C([0,+\infty)),$$

skalyar chiziqli tenglamani qaraylik. Quyidagilarni isbotlang:

a) Tenglamaning turg'un bo'lishi uchun biror $k(s), s \geq 0,$ funksiya uchun

$$\int_{t_0}^{t} a(u)du \leq k(t_0), t \geq t_0 \geq 0,$$

sartning bajarilishi yetarli va zarurdir.

b) Tenglamaning tekis turg'un bo'lishi uchun biror $m \in \mathbb{R}$ son uchun

$$\int_{t_0}^{t} a(u)du \leq m,\ t \geq t_0 \geq 0,$$

shartning bajarilishi yetarli va zarurdir.

c) Tenglamaning asimptopik turg'un bo'lishi uchun

$$\lim_{t \to +\infty} \int_{0}^{t} a(u)du = -\infty$$

shartning bajarilishi yetarli va zarurdir.

2. Tenglamani turg'unlikka tekshiring $x' = -2x + 3\cos t$.

Turg'unlikka tekshiring

3. $x' = y - z, y' = z - x, z' = x - y$. **4.** $x' = y + z, y' = z + x, z' = x + y$.

§ 13.3. Lyapunov funksiyalari yordamida turg'unlikka tekshirish

Bu paragrafda ushbu

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}),\ \boldsymbol{f}(t, 0) \equiv 0,\ t \geq 0, \tag{13.3.1}$$

sistemaning $\boldsymbol{x}(t) \equiv 0$ yechimini (muvozanat holatini) turg'unlikka tekshirishda Lyapunovning to'g'ri metodi, ya'ni Lyapunov funksiyalaridan foydalanish bilan tanishamiz. (13.3.1) sistemada $\boldsymbol{f} \in C(\mathbb{R}_+ \times B_\rho; \mathbb{R}^n)$, bunda $\mathbb{R}_+ = [0, +\infty)$, $B_\rho = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| < \rho\}$)}

($\rho > 0$) va $\boldsymbol{f}(t, \boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantiradi deb hisoblanadi.

Bir misoldan boshlaylik. m massali ($m > 0$) moddiy nuqta x har o‘qida harakat qilsin va u x nuqtada bo‘lganda unga uni koordinatalar boshiga qaytaruvchi $F_{el} = -kx$ ($k = \text{const} > 0$) elastiklik kuchi ta’sir etsin. Nuqtaning harakat tenglamasi, Nyutonning ikkinchi qonuniga ko‘ra, $mx'' + kx = 0$ ko‘rinishdagi garmonik ossilyator tenglamasidan iborat bo‘ladi. Harakatdagi nuqtaning to‘la mexanik energiyasi v uning $k\frac{x^2}{2}$ potensial va $m\frac{x'^2}{2}$ kinetik energiyalarining yig‘indisidan iborat, ya’ni $v = k\frac{x^2}{2} + m\frac{x'^2}{2}$. Harakat tenglamasini normal sistema ko‘rinishiga o‘tkazsak,

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x \end{cases}$$

hosil bo‘ladi. Harakat davomida $v = v(x, y) = k\frac{x^2}{2} + m\frac{y^2}{2}$ to‘la energiya, ma’lumki, saqlanadi (ixtiyoriy yechimda uning hosilasi nolga teng). Endi faraz qilaylik, harakatlanuvchi nuqtaga $F_q = -\mu(x, x')x'$ ($\mu = \mu(x, x') \ge 0,\ \mu \in C^1$) qarshilik kuchi ta’sir etsin. U holda harakat tenglamasi

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x - \dfrac{\mu(x, y)}{m}y \end{cases}$$

ko‘rinishda ifodalanadi. Ixtiyoriy $x = x(t),\ y = y(t)$ harakatni qaraylik. Shu harakat davomida to‘la mexanik energiya $v(t) = v(x(t), y(t)) = k\frac{x^2(t)}{2} + m\frac{y^2(t)}{2}$ bog‘lanishga muvofiq o‘zgaradi. Uning o‘zgarish tezligi

$$\frac{dv(t)}{dt} = \frac{\partial v(x(t), y(t))}{\partial x}\frac{dx(t)}{dt} + \frac{\partial v(x(t), y(t))}{\partial y}\frac{dy(t)}{dt} =$$

$$= kx(t)x'(t) + my(t)y'(t) = kx(t)y(t) + my(t)\Big(-\frac{k}{m}x(t) - \frac{\mu(x(t), y(t))}{m}y(t)\Big) =$$

$$= -\mu(x(t), y(t)) \cdot y^2(t) \le 0.$$

Demak, harakat davomida v to‘la energiya ortmaydi, ya’ni barcha $t \ge 0$ paytlarda $v(t) = k\frac{x^2(t)}{2} + m\frac{y^2(t)}{2} \le v|_{t=0}$ bo‘ladi. Bundan $x = x(t),\ y = y(t)$ yechimning (harakatning) chegaralanganligi va barcha $t \ge 0$ paytlarda mavjudligi kelib chiqadi. v energiyaning ortmaganligi va quyidan nol bilan chegaralanganligi uchun $\lim\limits_{t\to+\infty} v(t) = r \ge 0$ mavjud. Agar $r > 0$ bo‘lsa, $x = x(t),\ y = y(t)$ harakat vaqt o‘tishi bilan $k\frac{x^2}{2} + m\frac{y^2}{2} = r$ ellipsga yaqinlashadi ($\lim\limits_{t\to+\infty} x(t)$ va $\lim\limits_{t\to+\infty} y(t)$ limitlar mavjud, chunki $x'(t)$va $y'(t)$ lar chegaralangan), $r = 0$ bo‘lganda esa $x(t)$ va $y(t)$ lar nolga intiladi. Bu yerda shuni ta’kidlaylikki, qaralgan $v = v(x, y)$ funksiya (to‘la mexanik energiya) yechimlarning tabiatini (ularni topmagan bo‘lsak-da) ochishga yordam berdi.

Tekshirilgan misolning juda ham uzoqqa boruvchi umumlashishi Lyapunovning ikkinchi metodini tashkil etadi. Bu metodda yechimlarning xususiyatlari Lyapunov funksiyalari deb ataluvchi (misoldagi $v = v(x, y)$ ga o‘xshash) funksiyalar orqali o‘rganiladi. Lyapunov bu metodini o‘zining 1982-yilda yozgan doktorlik dissertatsiyasida bayon qilgan. Hozirgi zamon turg‘unlik nazariyasi ana shu ishdan boshlangan deb hisoblanadi.

Biror $v(t, \boldsymbol{x}) = v(t, x_1, x_2, \ldots, x_n),\ v(t, \boldsymbol{x}) \in C^1(\mathbb{R}_+ \times B_\rho; \mathbb{R})$ funksiya berilgan bo‘lsin. Bu funksiya (13.3.1) sistemaning $\boldsymbol{x} = \boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ yechimida ($x'(t)_i = f_i(t, \mathbf{x}(t)),\ i = \overline{1,n}$) t o‘zgaruvchining $v(t, \boldsymbol{x}(t)) = v(t, x_1(t), x_2(t), \ldots, x_n(t))$ funksiyasiga aylanadi. Uning hosilasi

$$\frac{d}{dt}v(t, x_1(t), x_2(t), \ldots, x_n(t)) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial v}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial v}{\partial x_n}\cdot\frac{dx_n}{dt} =$$

$$= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1}\cdot f_1 + \frac{\partial v}{\partial x_2}\cdot f_2 + \cdots + \frac{\partial v}{\partial x_n}\cdot f_n\ .$$

formula bilan hisoblanadi. Shundan kelib chiqib, $v(t,\mathbf{x})$ **funksiyaning** (13.3.1) **sistemaga ko'ra hosilasi** deb ushbu

$$\left.\frac{dv}{dt}\right|_{(13.3.1)} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1}\cdot f_1 + \frac{\partial v}{\partial x_2}\cdot f_2 + \cdots + \frac{\partial v}{\partial x_n}\cdot f_n \qquad (13.3.2)$$

funksiyaga aytiladi. Agar v funksiya t ga bog'liq bo'lmay, faqat $\mathbf{x}$ ga bog'liq, ya'ni $v = v(\boldsymbol{x})$ bo'lsa, bu funksiyaning (13.3.1) sistemaga ko'ra hosilasi

$$\left.\frac{dv}{dt}\right|_{(13.3.1)} = \frac{\partial v}{\partial x_1}\cdot f_1 + \frac{\partial v}{\partial x_2}\cdot f_2 + \cdots + \frac{\partial v}{\partial x_n}\cdot f_n =$$
$$= \mathrm{grad} v\cdot \boldsymbol{f}, \ \ \mathrm{grad} v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n}\right), \qquad (13.3.3)$$

formula bilan topiladi.

Bundan keyin t o'zgaruvchi nomanfiy qiymatlar qabul qiladi deb hisoblaymiz, agar boshqasi aytilgan bo'lmasa.

Agar $\omega(u)\in C([0,\rho),\mathbb{R}_+)$, funksiya keng ma'noda o'suvchi (kamaymaydigan) va faqat $u=0$ da nol qiymat qabul qilsa ($\omega(0)=0$ va $u>0 \Rightarrow \omega(u)>0$), uni Xan funksiyasi deb ataymiz va bunday funksiyalar sinfini $\mathrm{X}([0,\rho))$ (xi) bilan belgilaymiz.

Teorema 1 (M. A. Lyapunov, turg'unlik to'g'risidagi). (13.3.1*) sistema berilgan bo'lsin. Agar biror* $v(t,\boldsymbol{x})\in C^1(\mathbb{R}_+\times B_\rho;\mathbb{R})$, $v(t,\mathbf{0})\equiv 0$, *funksiya uchun shunday Xan funksiyasi* $\omega(u)\in \mathrm{X}([0,\rho))$ *topilib, ular uchun*

$$\omega(|\boldsymbol{x}|)\le v(t,\boldsymbol{x}), \qquad (13.3.4)$$

$$\left.\frac{dv}{dt}\right|_{(13.3.1)} \le 0 \qquad (13.3.5)$$

shartlar qanoatlansa, u holda (13.3.1) *sistemaning* $\boldsymbol{x}(t)\equiv 0$ *yechimi turg'un bo'ladi.*

⊶ Ixtiyoriy $\varepsilon, 0<\varepsilon<\rho$, va $t_0\in\mathbb{R}_+$ sonlar berilgan bo'lsin. $v(t_0,\boldsymbol{x})$ funksiyaning $\boldsymbol{x}=0$ nuqtadagi uzluksizligining ta'rifiga ko'ra $\omega(\varepsilon)>0$ son uchun shunday $\delta(\varepsilon,t_0)>0$ son topiladiki, $\left\|\boldsymbol{x}^0\right\|<\delta(\varepsilon,t_0)$ tengsizlikdan $v(t_0,\boldsymbol{x}^0)<\omega(\varepsilon)$ ($v(t,\boldsymbol{x})\ge 0$) kelib chiqadi. Demak,

$$\sup_{\|\boldsymbol{x}^0\|<\delta(\varepsilon,t_0)} v(t_0,\boldsymbol{x}^0)\le\omega(\varepsilon).$$

Ixtiyoriy $\boldsymbol{x}=\boldsymbol{x}(t)=\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)$, $\|\boldsymbol{x}^0\|<\delta(\varepsilon,t_0)$, yechimini qaraylik. Teoremaning (13.3.5) shartiga ko‘ra bu yechim bo‘ylab $v(t,\mathbf{x})$ funksiya o‘smaydi. Demak, barcha $t\ge t_0$ lar uchun

$$\omega(|\boldsymbol{x}(t)|)\le v(t,\boldsymbol{x}(t))\le v(t_0,\boldsymbol{x}^0)\le\sup_{\|\boldsymbol{x}^0\|<\delta(\varepsilon,t_0)} v(t_0,\boldsymbol{x}^0)\le\omega(\varepsilon).$$

Bundan $\omega(u)$ning monotonligiga ko‘ra qaralayotgan yechim $[t_0,+\infty)$ oraliqqacha davom etishi va $\|\boldsymbol{x}(t)\|=\|\boldsymbol{x}(t;t_0,\boldsymbol{x}^0)\|\le\varepsilon$ bo‘lishi kelib chiqadi. ✍

(13.3.4) shartni qanoatlantiruvchi $v(t,\boldsymbol{x})$, $v(t,\boldsymbol{x})\in C^1(\mathbb{R}_+\times B_\rho;\mathbb{R})$, $v(t,\mathbf{0})\equiv 0$, funksiya aniq musbat funksiya deb ataladi.

Masalan, $v(t,x,y)=x^2+y^2-2\varepsilon xy\cos t$ funksiya $|\varepsilon|<1$ bo‘lganda aniq musbat, chunki

$$v(t,x,y)=x^2+y^2-2\varepsilon xy\cos t\ge x^2+y^2-2|\varepsilon|\cdot|x|\cdot|y|\ge$$

$$\ge x^2+y^2-|\varepsilon|(x^2+y^2)=(1-|\varepsilon|)(x^2+y^2)=\omega(\sqrt{x^2+y^2}),$$

$$\omega(u)=(1-|\varepsilon|)u^2\in \mathrm{X}\big([0,+\infty)\big).$$

Lekin, shu funksiya $|\varepsilon|=1$ bo‘lganda nomanfiy bo‘lsa-da, aniq musbat emas: $v(t,x,y)\ge 0$, biroq

$$v(t,x,x)=2x^2\mp 2x^2\cos t=2x^2(1\mp\cos t)=0,\cos t=\pm 1.$$

Jumla. Agar $v(\boldsymbol{x})$ (t ga bog‘liq bo‘lmagan) funksiya $\boldsymbol{x}=0$ nuqtaning biror atrofida C^1 sinfga tegishli, $v(0)=0$ va shu atrofdagi barcha $\boldsymbol{x}\ne\mathbf{0}$ nuqtalarda $v(\boldsymbol{x})>0$ bo‘lsa, bu $v(\boldsymbol{x})$ funksiya **aniq musbat funksiya** bo‘ladi.

🗝 B_ρ berilgan atrofda joylashgajn bo‘lsin. Quyidagi funksiyani aniqlaylik:

$$\omega(u)=\begin{cases}\inf\limits_{u\le\|\boldsymbol{x}\|\le\rho} v(\boldsymbol{x}), \text{ agar } 0\le u\le\rho \text{ bo`lsa},\\ \omega(r), \text{ agar } u>\rho \text{ bo`lsa.}\end{cases}$$

Bu $\omega(u)$ funksiyaning uzluksiz va keng ma’noda o‘suvchi ekanligi osongina tekshiriladi; bundan tashqari, $\omega(0)=0$ va $u>0$ da $\omega(u)>0$ hamda $\omega(\|\boldsymbol{x}\|)\le v(\boldsymbol{x})$. ✍

Ravshanki, $v(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$ yoki umumiyroq $v(\boldsymbol{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$, bunda $\lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_n > 0$, funksiya (kvadratik forma) aniq musbat. Lekin, $v(x, y) = (x - y)^2$ funksiya nomanfiy bo‘lsa-da, aniq musbat emas.

Tushunarliki, $v(\boldsymbol{x})$ aniq musbat funksiya $\boldsymbol{x} = \boldsymbol{0}$ nuqtada minimumga ega. Demak, uning shu nuqtadagi xususiy hosilalari nolga teng:

$$\frac{\partial v(\boldsymbol{0})}{\partial x_1} = \frac{\partial v(\boldsymbol{0})}{\partial x_2} = \dots = \frac{\partial v(\boldsymbol{0})}{\partial x_n} = 0.$$

Faraz qilaylik, $v(\boldsymbol{x})$ funksiya $\boldsymbol{x} = \boldsymbol{0}$ nuqtaning biror atrofida C^2 sinfga tegishli, hamda

$$v(\boldsymbol{0}) = 0, \text{ va } \frac{\partial v(\boldsymbol{0})}{\partial x_1} = \frac{\partial v(\boldsymbol{0})}{\partial x_2} = \dots = \frac{\partial v(\boldsymbol{0})}{\partial x_n} = 0$$

bo‘lsin. U holda Teylor formulasiga ko‘ra

$$v(\boldsymbol{x}) = \frac{1}{2} \sum_{k,j=1}^{n} a_{kj} x_k x_j + \alpha(\boldsymbol{x}) \|\boldsymbol{x}\|^2, \; a_{kj} = \frac{\partial^2 v(\boldsymbol{0})}{\partial x_k \partial x_j}, \; \alpha(\boldsymbol{x}) \xrightarrow[\mathbf{x} \to 0]{} 0.$$

Bu formuladan ravshanki, agar $\frac{1}{2} \sum_{k,j=1}^{n} a_{kj} x_k x_j$ kvadratik forma aniq musbat, ya’ni $\frac{1}{2} \sum_{k,j=1}^{n} a_{kj} x_k x_j \geq \lambda_0 \|\boldsymbol{x}\|^2$ $(\lambda_0 > 0)$ bo‘lsa, u holda $v(\boldsymbol{x})$ funksiya ham aniq musbat bo‘ladi. Agar bu kvadratik forma nomanfiy bo‘lsa, $v(\boldsymbol{x})$ning aniq musbatligini yoyilmadagi yuqori tartibli had, ya’ni $\alpha(\boldsymbol{x}) \|\boldsymbol{x}\|^2$ aniqlaydi.

Kvadratik formani aniq musbatlikka tekshirish uchun algebradan ma’lum bo‘lgan **Sil’vestr mezoni**dan foydalanish mumkin. Bu mezonga ko‘ra $\sum_{k,j=1}^{n} a_{kj} x_k x_j$ kvadratik forma aniq musbat bo‘lishi uchun uning matritsasining barcha bosh diagonal minorlari

$$\Delta_1 = a_{11}, \; \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \dots, \Delta_n = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

musbat bo‘lishi yetarli va zarurdir.

Misol 1. Ushbu $v = v(x,y) = 1 + x^2 + x^3 - \cos(x-y)$ funksiyani aniq musbatlikka tekshiraylik.

🗝 Teylor formulasiga ko‘ra

$$v = \frac{1}{2}(3x^2 - 2xy + y^2) + \ldots,$$

bunda $\ldots$ bilan yuqori tartibli hadlar belgilangan. Endi $3x^2 - 2xy + y^2$ kvadratik formaning matritsasini tuzib, uning bosh diagonal minorlarini hisoblaymiz:

$$\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \ \Delta_1 = 3 > 0, \ \Delta_2 = \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2 > 0.$$

Demak, Sil’vestr mezoniga ko‘ra $3x^2 - 2xy + y^2$ kvadratik forma, va, demak, berilgan $v(x,y) = 1 + x^2 + x^3 - \cos(x-y)$ funksiya ham aniq musbat. 👍

Izoh. $3x^2 - 2xy + y^2$ kvadratik formaning aniq musbatligini Sil’vestr mezonisiz ham asoslash mumkin. Buning uchun bu kvadratik formadan to‘la kvadrat ajratish kifoya: $3x^2 - 2xy + y^2 = (x-y)^2 + 2x^2$.

Teorema 2 (K. P. Persidskiy, tekis turg‘unlik to‘g‘risidagi). (13.3.1) *sistema berilgan bo‘lsin. Agar* $v(t,\boldsymbol{x}) \in C^1(\mathbb{R}_+ \times B_\rho; \mathbb{R})$, $v(t,\boldsymbol{0}) \equiv 0$, va $\omega(u), \mu(u) \in \mathrm{X}([0,\rho))$ *Xan funksiyalari mavjud bo‘lib, ular uchun*

$$\omega(|\boldsymbol{x}|) \le v(t,\boldsymbol{x}) \le \mu(|\boldsymbol{x}|), \qquad (13.3.6)$$

$$\left.\frac{dv}{dt}\right|_{(13.3.1)} \le 0 \qquad (13.3.7)$$

shartlar bajarilsa, u holda (13.3.1) *sistemaning* $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ *yechimi tekis turg‘un bo‘ladi.*

🗝 Ixtiyoriy $\varepsilon, 0 < \varepsilon < \rho,$ son berilgan bo‘lsin. $\delta > 0$ sonni $\omega(\varepsilon) = \mu(\delta)$ shartdan tanlaylik; tushunarliki, $\delta = \delta(\varepsilon)$ bo‘ladi. (13.3.6), (13.3.7) shartlarga va $\mu(u)$ ning monotonligiga ko‘ra $t \ge t_0$ va $\|\boldsymbol{x}^0\| < \delta(\varepsilon)$ bo‘lganda $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t;t_0,\boldsymbol{x}^0)$ yechim uchun quyidagilar o‘rinli:

$$\omega(|\boldsymbol{x}(t)|) \le v(t,\boldsymbol{x}(t)) \le v(t_0,\boldsymbol{x}^0) \le \mu(|\boldsymbol{x}^0|) \le \mu(\delta) = \omega(\varepsilon).$$

Bundan $\omega(u)$ning monotonligiga ko‘ra barcha $t \geq t_0$ lar va $\|\boldsymbol{x}^0\| < \delta(\varepsilon)$ uchun $\|\mathbf{x}(t)\| = \|\mathbf{x}(t;t_0,\mathbf{x}^0)\| \leq \varepsilon$ kelib chiqadi. ✍

Isbotlangan teorema 2 dagi ikkinchi shartni kuchaytirib, asimptotik turg‘unlik haqidagi teoremani hosil qilish mumkin.

Teorema 3 (M. A. Lyapunov, asimptotik turg‘unlik to‘g‘risidagi). *Agar* (13.3.1) *sistema uchun* $v(t,\boldsymbol{x}) \in C^1(\mathbb{R}_+ \times B_\rho; \mathbb{R})$ va $\omega(u), \mu(u), \nu(u) \in \mathrm{X}([0,\rho))$ *Xan funksiyalari topilib, ular uchun*

$$\omega(|\boldsymbol{x}|) \leq v(t,\boldsymbol{x}) \leq \mu(|\boldsymbol{x}|), \tag{13.3.8}$$

$$\left.\frac{dv}{dt}\right|_{(13.3.1)} \leq -\nu(|\boldsymbol{x}|) \tag{13.3.9}$$

shartlar qanoatlansa, u holda (13.3.1) *sistemaning* $\boldsymbol{x}(t) \equiv \mathbf{0}$ *yechimi asimptotik turg‘un bo‘ladi.*

🗝 Bundan oldingi teoremaning isbotida yetarlicha kichik ixtiyoriy $\varepsilon > 0$ va ixtiyoriy $t_0 \geq 0$ sonlar uchun shunday $\delta = \delta(\varepsilon) > 0$ topdikki, boshlang‘ich qiymati $|\boldsymbol{x}^0| = |\boldsymbol{x}(t_0)| < \delta$ shartni qanoatlantiruvchi ixtiyoriy $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t,t_0,\boldsymbol{x}^0)$ yechim uchun barcha $t \geq t_0$ paytlarda $\|\boldsymbol{x}(t)\| < \varepsilon$ bo‘ldi. ε ga ko‘ra topilgan mos δ ni tayinlaylik. Boshlang‘ich qiymati $|\boldsymbol{x}^0| = |\boldsymbol{x}(t_0)| < \delta$ shartni qanoatlantiruvchi barcha $\boldsymbol{x} = \boldsymbol{x}(t)$ yechimlar uchun $\lim\limits_{t\to+\infty} \boldsymbol{x}(t) = \mathbf{0}$ ham bo‘lishini ko‘rsatamiz va $\boldsymbol{x}(t) \equiv \mathbf{0}$ yechimning asimptotik turg‘unligini isbotlaymiz.

Dastlab aytilgan $\boldsymbol{x} = \boldsymbol{x}(t)$, $|\boldsymbol{x}(t_0)| < \delta$, yechim uchun $\lim\limits_{t\to+\infty} v(t,\boldsymbol{x}(t)) = 0$ ekanligini ko‘rsatamiz. Berilganga ko‘ra $v(t,\boldsymbol{x}(t)) \geq 0$ va $\dfrac{dv(t,\boldsymbol{x}(t))}{dt} \leq 0$ bo‘lgani uchun $v(t,\boldsymbol{x}(t))$ funksiya keng ma’noda kamayuvchi va chekli $\lim\limits_{t\to+\infty} v(t,\boldsymbol{x}(t)) = a$, $a \geq 0$, limitga ega. Biz $a > 0$ bo‘la olmasligini isbotlaymiz. Teskarisini faraz qilaylik, ya’ni $a > 0$ bo‘lsin. U holda limitning ta’rifiga ko‘ra shunday t_* topamizki, barcha $t \geq t_*$ lar uchun $v(t,\boldsymbol{x}(t)) > a/2$ bo‘ladi. Berilgan (13.3.8) shartning o‘ng qismiga ko‘ra $\mu(|\boldsymbol{x}(t)|) \geq v(t,\boldsymbol{x}(t)) > a/2, t \geq t_0$. Bundan $\mu(u) \in \mathrm{X}$ va $\nu(u) \in \mathrm{X}$ ga ko‘ra biror $b>0$ uchun $|\boldsymbol{x}(t)|>b, t\geq t_*$, va $\nu(|\boldsymbol{x}(t)|)\geq \nu(b)\equiv c>0, t\geq t_*$ kelib chiqadi.

Endi (13.3.9) shartga ko‘ra $\dfrac{dv(t,\boldsymbol{x}(t))}{dt}\leq -\nu(|\boldsymbol{x}(t)|)\leq -c,\ t\geq t_*$. Bu tengsizlikni $[t_*,t]$ segmentda integrallab, topamiz: $v(t,\boldsymbol{x}(t))\leq v(t_*,\boldsymbol{x}(t_*))-c\cdot(t-t_*), t\geq t_*$. Bundan $t\to+\infty$ deb ziddiyatga kelamiz, chunki $v(t,\boldsymbol{x})$ funksiyaning qiymatlari nomanfiy. Demak, farazimiz noto‘g‘ri va $\lim\limits_{t\to+\infty} v(t,\boldsymbol{x}(t))=0$. Endi (13.3.8) shartning chap qismini $\boldsymbol{x}=\boldsymbol{x}(t)$ uchun yozaylik: $0\leq \omega(|\boldsymbol{x}(t)|)\leq v(t,\boldsymbol{x}(t))$. Bundan $\lim\limits_{t\to+\infty} v(t,\boldsymbol{x}(t))=0$ ga ko‘ra $\lim\limits_{t\to+\infty}\boldsymbol{x}(t)=\boldsymbol{0}$ kelib chiqadi. ♨

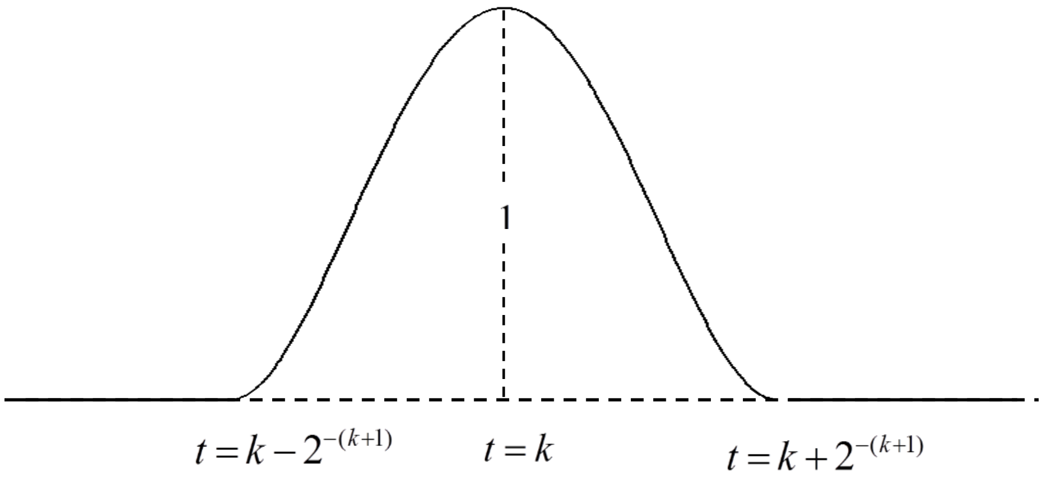
Yiqoridagi teoremalarda ishlatilgan $v(t,\mathbf{x})$ funksiyalar Lyapunov funksiyalari deb ataladi. Lyapunov funksiyalarini qurishning umumiy metodi mavjud emas. Konkret sistemalar uchun ularning tuzilishidan kelib chiqib, Lyapunov funksiyalarini u yoki bu ko‘rinishda tanlashga harakat qilish mumkin. Ba’zan Lyapunov funksiyasini kvadratik forma ko‘rinishida qurish mumkin bo‘ladi.

Bu yerda shuni e’tirof etaylikki, tekis turg‘unlik va asimptotik turg‘unlik to‘g‘risidagi teoremalardagi $v(t,\mathbf{x})\leq \mu(|\mathbf{x}|)$ shartni tashlab yuborsak, ularning xulosalari saqlanmaydi. Bu tasdiqni asoslovchi misol keltiraylik.

Misol 2. Quyidagi funksiyani qaraylik:

$$h(t)=\begin{cases}\dfrac{1}{\alpha_k^4}\left(1-(t-k)^2\right)\left(t-k+\alpha_k\right)^2\left(t-k-\alpha_k\right)^2,\ t\in[k-\alpha_k,k+\alpha_k],\\ 0,\ t\in[k-1,k-\alpha_k]\bigcup[k+\alpha_k,k+1]\end{cases}$$

bu yerda $\alpha_k=2^{-(k+1)}$, $k=1,2,3,....$ (rasmga qarang).



Endi $f(t)\overset{def}{=}(1-e^{-t})h(t)+e^{-t}$ funksiyani kiritaylik. Ko‘rish qiyin emaski, bu funksiya $[0,+\infty)$ oraliqda aniqlangan va uzluksiz differensiallanuvchi hamda $0<f(t)\le 1,\ t\in[0,+\infty)$.

Ushbu

$$x'=\frac{f'(t)}{f(t)}x \tag{13.3.10}$$

chiziqli differensial tenglamaning nol yechimini turg‘unlikka tekshirish maqsadida

$$v(t,x)=\frac{x^2}{f^2(t)}\left(4-\int_0^t f^2(s)ds\right)$$

funksiyani qaraylik. Bu funksiya aniq musbat, chunki

$$\int_0^t f^2(s)ds=\int_0^t\left((h(s)-e^{-s}h(s)+e^{-s}\right)^2ds\le\int_0^t\left((h(s)+e^{-s}\right)^2ds\le$$

$$\left|(a+b)^2\le 2(a^2+b^2)\right|\le 2\int_0^t(h^2(s)+e^{-2s})ds\le$$

$$\le 2\int_0^{+\infty}h^2(s)ds+2\int_0^{+\infty}e^{-2s}ds=2\sum_{k=1}^{\infty}\int_{2(k-1)}^{2k}h^2(s)ds+1\le$$

$$\le 2\sum_{k=1}^{\infty}2\alpha_k\cdot 1+1=2\sum_{k=1}^{\infty}2\cdot 2^{-(k+1)}+1=3,$$

$$v(t,x)=\frac{x^2}{f^2(t)}\left(4-\int_0^t f^2(s)ds\right)\ge\frac{x^2}{f^2(t)}\ge x^2.$$

Qaralayotgan $v=v(t,x)$ funksiyaning berilgan tenglamaga ko‘ra hosilasini bevosita hisoblab topamiz:

$$\left.\frac{dv}{dt}\right|_{(13.3.10)}=-x^2.$$

Berilgan tenglamaning nol-yechimi turg‘un, lekin tekis turg‘un ham emas, asimptotik turg‘un ham emas (§ 13.1 dagi misolga qarang). Demak, tekis turg‘unlik teoremasidagi $v(t,\boldsymbol{x}) \le \mu(|\boldsymbol{x}|)$ shart muhim. Bu shartni asimptotik turg‘unlik to‘g‘risidagi teoremadan ham olib tashlash mumkin emas.

Misol 3. Ushbu

$$\begin{cases} x' = y + 4x^2y^2 - 4x^5 \\ y' = -x - 2y^3 - 4x^3y \end{cases} \tag{13.3.11}$$

sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimini (muvozanat holatini) turg‘unlikka tekshiraylik.

🗝 Lyapunov funksiyasi sifatida $v = v(x,y) = \frac{1}{4}(x^2 + y^2)$ kvadratik formani tanlaymiz. Uning aniq musbat ekanligi ravshan. v ning berilgan sistemaga ko‘ra hosilasi

$$\left.\frac{dv}{dt}\right|_{(13.3.10)} = \frac{1}{4}(2xx' + 2yy') =$$

$$= \frac{1}{2}(x(y + 4x^2y^2 - 4x^5) + y(-x - 2y^3 - 4x^3y)) = -(2x^6 + y^4).$$

Demak, $(2x^6 + y^4)$ funksiya aniq musbat bo‘lganligi uchun Lyapunovning asimptotik turg‘unlik to‘g‘risidagi teoremasining barcha shartlari bajariladi. Bu teoremaga ko‘ra (13.3.11) sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimi asimptotik turg‘un.

N. G. Chetayevning quyidagi teoremasi Lyapunovning noturg‘un yechim to‘g‘risidagi teoremasining muhim umumlashishidir.

Teorema 4 (N. G. Chetayev, noturg‘un yechim to‘g‘risidagi). *Aytaylik,* (13.3.1) *sistema uchun quyidagi shartlarni qanoatlantiruvchi* U *soha va* $v = v(\mathbf{x})$ *funksiya* (*Lyapunov funksiyasi*) *mavjud bo‘lsin:*

1^0. *U soha $\mathbf{0} \in \mathbb{R}^n$ nuqtaning biror B_ρ atrofida yotadi, ya’ni $U \subset B_\rho$ va $\mathbf{0} \in \partial U$;*

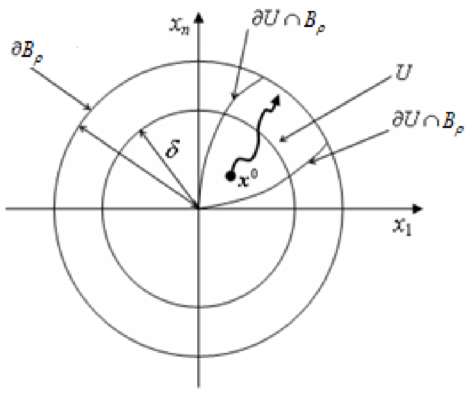
2^0. *$v \in C(U \cup \partial U)$, U sohada $v > 0$, lekin ∂U ning B_ρ dagi qismida $v(\mathbf{x})|_{\partial U \cap B_\rho} = 0$;*

3^0. *$v \in C^1(U)$ va biror $w \in C(U \cup \partial U)$ funksiya uchun $t \in [0;+\infty)$, $\boldsymbol{x} \in U$ bo‘lganda*

$$\left.\frac{dv}{dt}\right|_{(1)} \geq w(\boldsymbol{x}) > 0 .$$

U holda (13.3.1) *sistemaning* $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ *yechimi noturg'un bo'ladi.*

🗝 Teskarisini faraz qilaylik, ya'ni $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ yechim turg'un bo'lsin. U holda ta'rifga asosan $\rho > 0$ songa ko'ra shunday $\delta > 0$ topiladiki, boshlang'ich qiymati $\| \boldsymbol{x}^0 \| = \| \boldsymbol{x}(t_0) \| < \delta$ shartni qanoatlantiruvchi har qanday $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$ yechim uchun barcha $t \geq t_0$ paytlarda $\| \boldsymbol{x}(t) \| < \rho$ ($\boldsymbol{x}(t) \in B_\rho$) bo'ladi.



VI.6- rasm.

$\boldsymbol{0} \in \partial U$ bo'lganligi uchun $\boldsymbol{x}^0 = \boldsymbol{x}(t_0) \in U, \| \boldsymbol{x}^0 \| < \delta,$ tanlab, shunaqa boshlang'ich qiymatli $\boldsymbol{x} = \boldsymbol{x}(t)$ yechimlar uchun barcha $t \geq t_0$ larda ham $\| \boldsymbol{x}(t) \| < \rho$ bo'lavermasligini ko'rsatamiz. $\boldsymbol{x}(t) \in U$ tegishlilik saqlangunga qadar teorema shartiga ko'ra $\frac{dv(\boldsymbol{x}(t))}{dt} = \left.\frac{dv}{dt}\right|_{(1)} > 0$, ya'ni $v(\boldsymbol{x}(t))$ o'suvchi bo'ladi. Demak, bunday t lar uchun $v(\boldsymbol{x}(t)) > v(\boldsymbol{x}(t_0)) = v_0 > 0$. Endi $\tilde{U} = \{ \boldsymbol{x} \in U \cup \partial U \mid v(\boldsymbol{x}) \geq v_0 \}$ to'plamni qaraylik. Bu $\tilde{U}$ to'plam yopiq, chunki $U \cup \partial U$ yopiq, $v(\boldsymbol{x})$ esa uzluksiz bo'lganligi uchun $\tilde{U}$ ning ixtiyoriy $\boldsymbol{y}$ limit nuqtasi uchun $\boldsymbol{y} \in U \cup \partial U$ va $v(\boldsymbol{y}) \geq v_0$, ya'ni $\boldsymbol{y} \in \tilde{U}$. $\tilde{U}$ to'plam chegaralangan hamdir, chunki uning nuqtalari uchun $\| \boldsymbol{y} \| \leq \rho$. Demak, $\tilde{U} \subset U \cup \partial U -$kompakt va $\tilde{U}$ da $w(\boldsymbol{x}) \geq \beta > 0$ hamda $v(\boldsymbol{x})$ yuqoridan chegaralangan. Qaralayotgan $\boldsymbol{x}(t)$ yechim $\tilde{U}$ to'plamning chegarasigacha yetib kelolmaydi: $\partial \tilde{U}$ ning $B_\rho$dagi qismida $v(\boldsymbol{x}) \geq v_0 > 0$, lekin $v(\boldsymbol{x})|_{\partial U \cap B_\rho} = 0$; $\partial \tilde{U}$ ning ∂B_ρ dagi qismida $\| \boldsymbol{x} \| = \rho$, yechim uchun esa barcha $t \geq t_0$ paytlarda $\| \boldsymbol{x}(t) \| < \rho$.

Teoremaning shartiga ko‘ra $\boldsymbol{x}(t)$ yechim uchun $\frac{dv(\boldsymbol{x}(t))}{dt} \geq w(\boldsymbol{x}(t)) \geq \beta > 0$.Bundan

$v(\boldsymbol{x}(t)) \geq v(\boldsymbol{x}(t_0)) + \beta(t - t_0) \xrightarrow[t\to+\infty]{} +\infty$. Bu munosabat $v(\mathbf{x})$ning $\tilde{U}$ da chegaralanganligiga zid. Demak, farazimiz noto‘g‘ri va teorema isbot bo‘ldi. ✍

Misol 4. Ushbu

$$x' = y - 2x^2,\ y' = 2xy + y^3$$

sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimini turg‘unlikka tekshiraylik.

⊶ U soha sifatida birinchi chorakni olib, $v(x,y) = xy$ funksiyani qaraylik. Ravshanki, $(0,0) \in \partial U$ va U sohaning chegarasida $v(x,0) = v(0,y) = 0$. $v(x,y) = xy$ funksiyaning berilgan sistemaga ko‘ra hosilasi U sohada

$$\frac{dv}{dt} = x'y + xy' = (y - 2x^2)y + x(2xy + y^3) = y^2 + xy^3,$$

$$\frac{dv}{dt} = w(x,y) > 0,\ w(x,y) = y^2 + xy^3,\ x > 0,\ y > 0.$$

Chetayev teoremasining barcha shartlari bajarildi. Demak, berilgan sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimi noturg‘un. ✍

Masalalar

1. Ushbu

$$\omega(u) = \begin{cases} 0,\ u = 0 \\ 2u - 1/(n+1),\ 1/(n+1) \leq u < (2n+1)/(2n(n+1)) \\ 1/n,\ (2n+1)/(2n(n+1)) \leq u < 1/n \end{cases}$$

funksiya Xan sinfi $\mathrm{X}([0,1))$ ga tegishlimi (grafigini quring)?

Quyidagi sistemalarning nol-yechimini turg‘unlikka tekshiring:

2. $x' = -y + x^3, y' = x + y^3$. **3.** $x' = y + xy, y' = -x - xy$.

4. $x' = y - x^3, y' = -x^3 - y^3$. **5.** $x' = y + kx(x^2 + y^2), y' = -x + ky(x^2 + y^2)$

§ 13.4. Birinchi yaqinlashishga ko‘ra turg‘unlik

(13.3.1) sistema ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}) \qquad (13.4.1)$$

ko‘rinishda bo‘lsin, bu yerda $A \in \mathbb{M}_{n\times n}(\mathbb{R})$, $\boldsymbol{g} \in C(\mathbb{R}_+ \times B_\rho; \mathbb{R}^n)$, $\boldsymbol{g}(t, \boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantiradi va

$\| \boldsymbol{g}(t,\boldsymbol{x})\| \le \alpha(\boldsymbol{x})\|\boldsymbol{x}\|, \quad \alpha(\boldsymbol{x}) \xrightarrow[x\to 0]{} 0$, deb hisoblanadi. Xususan, $\boldsymbol{g}(t,0)\equiv 0$ va (13.4.1) sistema $\boldsymbol{x}(t)\equiv 0$ yechimga ega.

Agar (13.4.1) sistemada $\mathbf{x}\to 0$ da yuqori tartibli cheksiz kichik miqdor $\boldsymbol{g}(t,\boldsymbol{x})$ni tashlab yuborsak, **birinchi yaqinlashish sistemasi** deb ataluvchi

$$\boldsymbol{x}' = A\boldsymbol{x} \tag{13.4.2}$$

sistemani hosil qilamiz. Oxirgi (13.4.2) chiziqli sistema (13.4.1) nochiziqli sistemaning chiziqlilashtirilishi deb ham yuritiladi.

Dastlab (13.4.2) birinchi yaqinlashish sistemasi uchun Lyapunov funksiyasini barcha xarakteristik sonlarining haqiqiy qismi manfiy bo‘lganda quraylik.

Lemma. *Agar A matritsaning barcha $\lambda_j$xarakteristik sonlari uchun $\mathrm{Re}\lambda_j < 0$ bo‘lsa, u holda Lyapunovning asimptotik turg‘unlik haqidagi teoremasi shartlarini qanoatlantiruvchi Lyapunov funksiyasi mavjud.*

🗝 Lyapunov funksiyasini

$$v(\boldsymbol{x}) = (\boldsymbol{x}, Q\boldsymbol{x}) = \boldsymbol{x}^T Q\boldsymbol{x} \ \ (Q\in \mathbb{M}_{n\times n}(\mathbb{R}) - \text{simmetrik matritsa})$$

kvadratik forma ko‘rinishida izlaymiz. Uning (13.4.2) sistemaga ko‘ra hosilasi

$$\left.\frac{dv}{dt}\right|_{(13.4.2)} = \frac{d\boldsymbol{x}^T}{dt}Q\,\boldsymbol{x} + \boldsymbol{x}^T Q\frac{d\boldsymbol{x}}{dt} = (A\boldsymbol{x})^T Q\,\boldsymbol{x} + \boldsymbol{x}^T QA\boldsymbol{x} = .$$

$$= \boldsymbol{x}^T A^T Q\boldsymbol{x} + \boldsymbol{x}^T QA\boldsymbol{x} = \boldsymbol{x}^T (A^T Q + QA)\,\boldsymbol{x}$$

Demak, agar Q matritsani ushbu

$$A^T Q + QA \ = -E \tag{13.4.3}$$

shartdan tanlasak, u holda

$$\left.\frac{dv}{dt}\right|_{(13.4.2)} = -\boldsymbol{x}^T\boldsymbol{x} = -\|\boldsymbol{x}\|^2 \quad (\nu(\|\boldsymbol{x}\|) = \|\boldsymbol{x}\|^2\ )$$

bo‘ladi. (13.4.3) tenglamani qanoatlantiruvchi Q matritsani topish uchun ushbu

$$\frac{dX}{dt} = A^T X + XA, \quad X(0) = E \tag{13.4.4}$$

matritsaviy Koshi masalasini qaraylik. Bu masala

$$X(t) = E + A^T \cdot \int_0^t X(\tau)d\tau + \int_0^t X(\tau)d\tau \cdot A \qquad (13.4.5)$$

integral tenglamaga ekvivalent. Agar uning $t \to +\infty$ da nol-matritsaga intiluvchi yechimi mavjud va mos xosmas integrallar yaqinlashuvchi bo‘lsa, u holda

$$A^T \cdot \int_0^{+\infty} X(\tau)d\tau + \int_0^{+\infty} X(\tau)d\tau \cdot A = -E$$

tenglik o‘rinli bo‘ladi, ya’ni (13.4.3) tenglamaning

$$Q = \int_0^{+\infty} X(\tau)d\tau$$

yechimi topiladi. (13.4.4) Koshi masalasining yechimini

$$X(t) = Y(t)Z(t)$$

ko‘rinishda izlaymiz. Buni (13.4.4) ga qo‘yib,

$$Y'(t)Z(t) + Y(t)Z'(t) = A^T Y(t)Z(t) + Y(t)Z(t)A, \ Y(0)Z(0) = E,$$

munosabatlar qanoatlanishi uchun

$$Y'(t) = A^T Y(t), \ Z'(t) = Z(t)A \ , \ Y(0) = E, \ Z(0) = E,$$

deymiz. Bularni yechib,

$$Y(t) = e^{tA^T}, \ Z(t) = e^{tA}$$

ekanligini topamiz. Demak, (13.4.4) Koshi masalasining yechimi ushbu

$$X(t) = e^{tA^T} e^{tA}$$

simmetrik matritsaviy funksiyadan iborat.

Endi e^{tA} matritsaning elementlarini baholaymiz. $\alpha_j = \mathrm{Re}\lambda_j$, $\beta_j = \mathrm{Im}\lambda_j$ deb, e^{tA} matritsaning elementlarini

$$\sum_{j=1}^{s} e^{\alpha_j t} \left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t) \right)$$

ko‘rinishda yozamiz, bunda $p_j(t), q_j(t) -$haqiqiy koeffitsientli ko‘phadlar. Shartga ko‘ra $\max\limits_j \alpha_j < 0$. Ushbu $\max\limits_j \alpha_j < -\alpha < 0$ shartni qanoatlantiruvchi ixtiyoriy $\alpha > 0$ sonni olaylik. $\max\limits_j \alpha_j + \alpha < 0$ bo‘lgani uchun shunday $c > 0$ son topiladiki, uning uchun

$$e^{(\alpha_j + \alpha)t} |p_j(t)| \le c \, , \ e^{(\alpha_j + \alpha)t} |q_j(t)| \le c \ \ (t \ge 0, \, j = 1, 2, \ldots, s)$$

bo‘ladi. Endi $e^{tA} = \| \varphi_{kl}(t) \|$ matritsaning elementlari quyidagicha baholanadi:

$$|\varphi_{kl}(t)| = \left| \sum_{j=1}^{s} e^{\alpha_j t} \left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t) \right) \right| \le$$

$$\le e^{-\alpha t} \left(\sum_{j=1}^{s} e^{(\alpha_j + \alpha)t} |p_j(t)| + \sum_{j=1}^{s} e^{(\alpha_j + \alpha)t} |q_j(t)| \right) \le 2sce^{-\alpha t} \ (t \ge 0). \tag{13.4.6}$$

$(e^{tA})^T = e^{tA^T}$ bo‘lgani uchun e^{tA^T} matritsaning elementlari uchun ham shu (13.4.6) baholashlar o‘rinli. Demak, $X(t) = e^{tA^T} e^{tA}$ matritsaning $x_{kl}(t)$ elementlari uchun

$$|x_{kl}(t)| \le \text{const} \cdot e^{-2\alpha t} \text{ va } \int_0^{+\infty} |x_{kl}(t)| dt \le \text{const} \cdot \int_0^{+\infty} e^{-2\alpha t} dt < +\infty .$$

Shuning uchun (13.4.5) tenglikda $t \to +\infty$ deb limitga o‘tish mumkin. Natijada

$$Q = \int_0^{+\infty} X(t)dt = \int_0^{+\infty} e^{tA^T} e^{tA} dt \ \ (Q^T = Q)$$

deb,

$$v(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x} = \int_0^{+\infty} \boldsymbol{x}^T e^{tA^T} e^{tA} \boldsymbol{x}\, dt = \int_0^{+\infty} (e^{tA}\boldsymbol{x}, e^{tA}\boldsymbol{x}) dt = \int_0^{+\infty} \| e^{tA}\boldsymbol{x} \|^2 dt$$

funksiyani topamiz. Oxirgi formuladan ravshanki, $v(\mathbf{x}) -$ aniq musbat. Qurilishiga ko‘ra

$$\left. \frac{dv}{dt} \right|_{(13.4.2)} = -\nu(\|\mathbf{x}\|), \ \nu(\|\mathbf{x}\|) = \|\mathbf{x}\|^2 .$$

Shunday qilib, qurilgan $v(\mathbf{x}) = \int_0^{+\infty} \| e^{tA}\mathbf{x} \|^2 dt$ izlangan Lyapunov funksiyasidir. 👍

Teorema (Birinchi yaqinlashishga ko‘ra asimptotik turg‘unlik to‘g‘risidagi). *Agar A matritsaning barcha λ_j xarakteristik sonlari uchun* $\mathrm{Re}\lambda_j < 0$ *bo‘lsa,* (13.4.1) *sistemaning* $\mathbf{x}(t) \equiv \mathbf{0}$ *yechimi asimptotik turg‘un bo‘ladi.*

🗝 Asimptotik turg‘unlik to‘g‘risidagi Lyapunov teoremasidan foydalanamiz. Lyapunov funksiyasi sifatida birinchi yaqinlashish sistemasi uchun qurilgan

$$v(\boldsymbol{x}) = \sum_{k,l=1}^{n} q_{kl} x_k x_l = \boldsymbol{x}^T Q \boldsymbol{x} = \int_0^{+\infty} (e^{tA}\boldsymbol{x}, e^{tA}\boldsymbol{x}) dt,$$

$$Q = \int_0^{+\infty} e^{tA^T} e^{tA} dt = \| q_{kl} \| \ .$$

funksiyani olamiz. Uning (13.4.1) sistemaga ko‘ra $\frac{dv}{dt}$ hosilasini hisoblaymiz

$$\left.\frac{dv}{dt}\right|_{(13.4.1)} = \text{grad}v \cdot (A\boldsymbol{x} + \mathbf{g}(t,\boldsymbol{x})) = \text{grad}v \cdot A\boldsymbol{x} + \text{grad}v \cdot \mathbf{g}(t,\boldsymbol{x}). \qquad (13.4.7)$$

Bu yerdagi birinchi qo‘shiluvchi uchun lemmaning isbotida

$$\text{grad}v \cdot A\boldsymbol{x} = -\|\boldsymbol{x}\|^2 \qquad (13.4.8)$$

ekanligi ko‘rsatilgan edi. Ikkinchi qo‘shiluvchini baholaymiz. Tushunarliki,

$$\frac{\partial v}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k,l=1}^{n} q_{kl} x_k x_l = 2\sum_{l=1}^{n} q_{jl} x_l \ \ (j = 1,2,\ldots,n) \ .$$

Demak, Koshi-Bunyakovskiy tengsizligiga ko‘ra

$$\left|\frac{\partial v}{\partial x_j}\right| = 2\left|\sum_{l=1}^{n} q_{jl} x_l\right| \le 2\sqrt{\sum_{l=1}^{n} q^2_{jl}} \ \|\boldsymbol{x}\| \le c\|\boldsymbol{x}\|, \ j = 1,2,\ldots,n;$$

bunda $c = 2\max\limits_j \sqrt{\sum_{l=1}^{n} q^2_{jl}}$. Demak,

$$\|\text{grad}v\| = \sqrt{\sum_{j=1}^{n} \left|\frac{\partial v}{\partial x_j}\right|^2} \le nc\|\boldsymbol{x}\| \ .$$

Yana Koshi-Bunyakovskiy tengsizligiga ko‘ra

$$\begin{aligned} &\text{grad}v \cdot \mathbf{g}(t,\boldsymbol{x}) \le \|\text{grad}v\| \cdot \|\mathbf{g}(t,\boldsymbol{x})\| \le \\ &\le nc\|\boldsymbol{x}\| \cdot \alpha(\boldsymbol{x})\|\boldsymbol{x}\| = nc\alpha(\boldsymbol{x})\|\boldsymbol{x}\|^2 \end{aligned} \qquad (13.4.9)$$

Endi (13.4.9) va (13.4.8) munosabatlardan foydalanib, (13.4.7) dan $\alpha(\boldsymbol{x}) \le \frac{1}{2nc}$ shartni qanoatlantiruvchi barcha $\boldsymbol{x}$ lar va ixtiyoriy $t \ge 0$ uchun

$$\left.\frac{dv}{dt}\right|_{(13.4.1)} \leq -\|\boldsymbol{x}\|^2 + nc\alpha(\boldsymbol{x})\|\boldsymbol{x}\|^2 \leq -\frac{1}{2}\|\boldsymbol{x}\|^2.$$

ekanligini topamiz. Demak, (13.4.1) sistema uchun asimptotik turg‘unlik to‘g‘risidagi Lyapunov teoremasiga ko‘ra $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ yechim asimptotik turg‘un. ☝

Misol 1. Ushbu

$$x'' + h'(x)x' + x = 0$$

L’yenard tenglamasini qaraylik, bunda $h \in C^1(\mathbb{R})$ va $h(0) = 0$. Bu tenglama $x(t) \equiv 0$ nol yechimga ega. Uning turg‘unligi deganda mos

$$\begin{cases} x' = y - h(x) \\ y' = -x \end{cases}$$

normal sistema nol yechimining turg‘unligi tushuniladi. Oxirgi sistemani chiziqlilashtiramiz:

$$\begin{cases} x' = -h'(0)x + y \\ y' = -x \end{cases}$$

Bu chiziqli sistemaning xarakteristik tenglamasi

$$\lambda^2 + h'(0)\lambda + 1 = 0.$$

Ravshanki, agar $h'(0) > 0$ bo‘lsa, xarakteristik sonlarning haqiqiy qismi manfiy. Shuning uchun yuqoridagi chiziqli sistemaning va, demak, L’yenard tenglamasining nol yechimi asimptotik turg‘un.

$h(x) = x - x^3/3$ bo‘lganda L’yenard tenglamasi ushbu

$$x'' + (1 - x^2)x' + x = 0$$

Van der Pol tenglamasiga aylanadi. Van der Pol tenglamasining nol yechimi ham turg‘undir. ☝

Teorema (Birinchi yaqinlashishga ko‘ra noturg‘unlik to‘g‘risidagi). *Agar A matritsaning biror λ_j xarakteristik soni uchun* $\operatorname{Re}\lambda_j > 0$ *bo‘lsa,* (13.4.1) *sistemaning* $\boldsymbol{x}(t) \equiv 0$ *yechimi noturg‘un bo‘ladi.*

Bu teoremani isbotsiz qabul qilamiz.

Agar $\max\limits_j \alpha_j = \max\limits_j \operatorname{Re}\lambda_j = 0$ bo‘lsa, turg‘unlik yoki noturg‘unlik birinchi yaqinlashishga ko‘ra hal qilinmaydi. Bu holda $\boldsymbol{x}(t) \equiv 0$ yechimning turg‘unligi yoki noturg‘unligi (13.4.1) sistemadagi yuqori tartibli had $\boldsymbol{g}(t,\boldsymbol{x})$ $(\|\boldsymbol{g}(t,\boldsymbol{x})\| \le \alpha(\boldsymbol{x})\|\boldsymbol{x}\|,\ \alpha(\boldsymbol{x}) \xrightarrow[\boldsymbol{x}\to 0]{} 0)$ bilan aniqlanadi.

Misol 2. Ushbu

$$\begin{cases} x' = y + 4x^2y^2 - 4x^5 \\ y' = -x - 2y^3 - 4x^3y \end{cases}$$

sistemaning nol yechimi asimptotik turg‘un. (§ 13.3 dagi misol 2 ga qarang). Uning chiziqli qismi

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

sistemadan iborat. Xarakteristik sonlar $\lambda_{1,2} = \pm i$. Demak, birinchi yaqinlashishga ko‘ra nol yechimni turg‘unlikka tekshirib bo‘lmaydi.

Misol 3. Ushbu

$$\begin{cases} x' = y - 2x^2 \\ y' = 2xy + y^3 \end{cases}$$

sistemaning nol yechimi, bizga ma’lumki, noturg‘un (§ 13.3 dagi misol 3 ga qarang). Sistemaning birinchi yaqinlashishi

$$\begin{cases} x' = y \\ y' = 0 \end{cases}$$

uchun xatakteristik sonlar $\lambda_1 = \lambda_2 = 0$. Demak, birinchi yaqinlashishga ko‘ra nol yechimning turg‘unligi haqida hech narsa deb bo‘lmaydi. ☝

Masalalar

1. Agar A o‘zgarmas matritsaning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j < 0$ bo‘lsa, ushbu $v(\boldsymbol{x}) = \int_0^{+\infty} (e^{tA}\boldsymbol{x}, e^{tA}\boldsymbol{x})dt$ funksiya ma’noga ega (xosmas integral yaqinlashuvchi) va uning $\boldsymbol{x}' = A\boldsymbol{x}$ sistemaga ko‘ra hosilasi uchun $\left.\dfrac{dv}{dt}\right|_{x'=Ax} = -\|\boldsymbol{x}\|^2$ bo‘lishini bevosita isbotlang.

2. Faraz qilaylik, $\{A,B,C\} \subset \mathrm{M}_{n\times n}(\mathbb{R})$ bo‘lsin. Agar A va B matritsalarning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j < 0$ bo‘lsa, u holda

$$X = \int_0^{\infty} e^{tA}Ce^{tB}dt$$

matritsa aniqlangan (xosmas integral yaqinlashuvchi) hamda u

$$AX + XB = -C$$

tenglamaning yechimi bo‘lishini ko‘rsating.

3. Muvozanat holatlarni toping va turg‘ulikka tekshiring:

a) $\begin{cases} x' = x^2(y-1)(4-x^2), \\ y' = y^2(x-1)(y^2-x). \end{cases}$ b) $\begin{cases} x' = (1-x^2)(y+x(1-x^2)), \\ y' = -x+(1-x^2)y. \end{cases}$

4. Ushbu

$$\begin{cases} x' = -\sigma x + \sigma y \\ y' = rx - y - xz \\ z' = -bz + xy \end{cases}$$

Lorens differensial tenglamalar sistemasi berilgan bo‘sin; bunda $\sigma, r, b -$ musbat o‘zgarmaslar. Bu sistemani Lorens (Edward N. Lorentz – Massachusets texnologiya institutida metereolog olim) atmosferadagi havo oqimlarining matematik modeli sifatida hosil qilgan va $\sigma = 10, b = 8/3,$ $r = 28$ holida o‘rgangan. Lorens sistemasining muvozanat holatlarini turg‘unlikka tekshiring

MODUL 14. IKKINCHI TARTIBLI CHIZIQLI TENGLAMALARNI QATORLAR YORDAMIDA YECHISH. YECHIMLAR NOLLARINING TABIATI

§ 14.1. Analitik koeffitsientli tenglamalarni darajali qatorlar yordamida yechish

Differensial tenglamalarning yechimlari har doim ham kvadraturalarda ifodalanavermaydi. Tenglamada qatnashgan funksiyalarning barchasi analitik bo‘lganda, yechimlarni darajali qator yig‘indisi, ya’ni analitik funksiya sifatida topish mumkin. Shu munosabat bilan dastlab matematik analiz kursidan ma’lum bo‘lgan analitik funksiya ta’rifi va uning ba’zi xossalarini eslaylik.

Agar bir o‘zgaruvchining $y = f(x)$ funksiyasi x_0 nuqtaning biror $(\delta > 0)$ atrofida biror $\sum_{n=0}^{+\infty} a_n (x - x_0)^n$ (x_0 markazli) darajali qatorning yig‘inidisi sifatida tasvirlansa, ya’ni

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \quad (\, |x - x_0| < \delta \,) \tag{14.1.1}$$

bo‘lsa, u holda $y = f(x)$ funksiya x_0 nuqtada analitik funksiya deyiladi.

Matematik analizdan ma’lumki, masalan, $\sin x, \cos x, e^x$ funksiyalar ixtiyoriy $x_0 \in \mathbb{R}$ nuqtada analitik. Agar funksiya (a,b) intervalning har bir nuqtasida analitik bo‘lsa, bu funksiya (a,b) intervalda analitik deyiladi.

(14.1.1) tenglik aslida $|x - x_0| < R$ yaqinlashish intervalida o‘rinli bo‘ladi. (14.1.1) dagi darajali qatorni uning yaqinlashish intervalida xohlagancha marta hadma-had differensiallash mumkin; bunda qatorning R yaqinlashish radiusi o‘zgarmaydi. x_0 nuqtada analitik funksiya (14.1.1) darajali qatorning yaqinlashish intervalida (shu intervalning har bir nuqtasida) ham analitik bo‘ladi. (14.1.1) formuladagi darajali qator koeffitsientlari uchun

$$a_n = \frac{f^{(n)}(x_0)}{n!},\ n = 0,1,2,\ldots\ (0! = 1;\ 1! = 1;\ n! = 1 \cdot 2 \cdot \ldots \cdot n,\ n \in \mathbb{N}),$$

formulalar o‘rinli, ya’ni analitik funksiya o‘zining Teylor qatori yig‘indisidan iborat. Analitik funksiyalar yig‘indisi, ayirmasi, ko‘paytmasi va nisbati (bo‘luvchi nolga aylanmaganda) yana analitiik bo‘ladi.

Differensial tenglama yechimni darajali qator yig‘indisi sifatida topish, ya’ni analitik yechimni qurish jarayoni bilan ikkinchi tartibli chiziqli differensial tenglama misolida tanishamiz. Birinchi yoki yuqori tartibli chiziqli tenglama yoki chiziqli tenglamalar sistemasi uchun ham analitik yechimni topish jarayoni shunga o‘xshash amalga oshiriladi. Analitik yechimni qurish (topish) uchun tenglama(lar)dagi barcha koeffitsientlar analitik bo‘lishi kerak.

Aniqrog‘i, ushbu

$$y'' + p_1(x)y' + p_0(x)y = q(x) \tag{14.1.2}$$

tenglamani qaraylik. Bu yerdagi $p_1(x),\ p_0(x),\ q(x)$ funksiyalar x_0 nuqtada, ya’ni x_0 nuqtaning biror atrofida analitik deb hisoblanadi. Yechimni x_0 markazli hozircha noma’lum koeffitsientli darajali qator yig‘indisi sifatida yozamiz:

$$y = \sum_{n=0}^{+\infty} a_n (x - x_0)^n,\ |x - x_0| < R_1\ \ (R_1 > 0), \tag{14.1.3}$$

bu yerda $a_n\ (n = 0,1,2,...)$ – noma’lum koeffitsientlar. Qatorni hadma-had ikki marta differensiallaymiz:

$$y' = \sum_{n=1}^{+\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} (x - x_0)^n,\ |x - x_0| < R_1, \tag{14.1.4}$$

$$\begin{aligned} y'' &= \sum_{n=1}^{+\infty} n(n+1) a_{n+1} (x - x_0)^{n-1} = \\ &= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} (x - x_0)^n,\ |x - x_0| < R_1. \end{aligned} \tag{14.1.5}$$

Tenglamada berilgan koeffitsientlarni ham x_0 markazli darajali qatorlarga yoyamiz:

$$p_1(x) = \sum_{n=0}^{+\infty} b_n (x - x_0)^n,\ |x - x_0| < R_2, \tag{14.1.6}$$

$$p_0(x) = \sum_{n=0}^{+\infty} c_n (x - x_0)^n,\ |x - x_0| < R_2, \tag{14.1.7}$$

$$q(x)=\sum_{n=0}^{+\infty} d_n (x-x_0)^n,\ |x-x_0|<R_2, \qquad (14.1.8)$$

(bu yerda $p_1(x), p_0(x), q(x)$ uchun darajali qatorlar yaqinlashish radiuslarining eng kichigini R_2 bilan belgiladik). Bu qatorlarni qaralayotgan (14.1.2) differensial tenglamaga qo‘yib, zarur soddalashtirishlarni bajaramiz:

$$\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n+\sum_{n=0}^{+\infty}b_n(x-x_0)^n\cdot\sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-x_0)^n+$$

$$+\sum_{n=0}^{+\infty}c_n(x-x_0)^n\cdot\sum_{n=0}^{+\infty}a_n(x-x_0)^n=\sum_{n=0}^{+\infty}d_n(x-x_0)^n;$$

$$\sum_{n=0}^{+\infty}\left((n+1)(n+2)a_{n+2}+\sum_{k=0}^{n}((k+1)a_{k+1}b_{n-k}+a_kc_{n-k})\right)(x-x_0)^n=$$

$$=\sum_{n=0}^{+\infty}d_n(x-x_0)^n.$$

Bu tenglikning chap va o‘ng tomonidagi $(x-x_0)$ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, $a_n\ (n=0,1,2,...)$ noma’lumlarni topish uchun cheksiz algebraic sistemani hosil qilamiz:

$$(x-x_0)^0:\ 1\cdot 2\cdot a_2+a_1b_0+a_0c_0=d_0,$$

$$(x-x_0)^1:\ 2\cdot 3\cdot a_3+a_1b_1+a_0c_1+2a_2b_0+a_1c_0=d_1,$$

. . . .

$$(x-x_0)^n:\ (n+1)(n+2)a_{n+2}+\sum_{k=0}^{n}((k+1)a_{k+1}b_{n-k}+a_kc_{n-k})=d_n,$$

. . . .

Agar $a_0=y(x_0)$ va $a_1=y'(x_0)$ qiymatlarni ixtiyoriy tanlasak (Koshi masalasida ular berilgan bo‘ladi), u holda bu yerdagi birinchi tenglamadan a_2, ikkinchisidan a_3 va h.k., barcha qolgan a_n lar bir qiymatli aniqlanadi. Topilgan a_n larga ko‘ra (14.1.3) darajali qatorning yaqinlashish radiusini hisoblaymiz va qurilgan analitik funksiya yaqinlashish intervalida (14.1.2) differensial tenglamaning yechimi ekanligini asoslaymiz. Ko‘rsatish mumkinki, $R_1\ge R_2$ bo‘ladi, ya’ni qurilgan (14.1.3) funksiya (14.1.2) tenglamada qatnashgan analitik funksiyalarining umumiy analitiklik intervalida shu tenglamaning (analitik) yechimi bo‘ladi. Shunday qilib, (14.1.2) tenglamaning a_0, a_1 ikki parametrli analitik yechimlar oilasini hosil qilamiz. Tushunarliki, bu umumiy yechimni ifodlaydi.

Yuqorida bajarilgan ishlarning umumiy holda qonuniy ekanligini quyidagi teorema asoslaydi.

Teorema 1. *Faraz qilaylik,* $p_1(x), p_0(x)$ *va* $q(x)$ *funksiyalar* $(x_0 - R, x_0 + R)$ *intervalda analitik bo'lsin. U holda ixtiyoriy* a_0 *va* a_1 *sonlar uchun ushbu*

$$y'' + p_1(x)y' + p_0(x)y = q(x),\ y(x_0) = a_0,\ y'(x_0) = a_1,$$

Koshi masalasi $(x_0 - R, x_0 + R)$ *intervalda aniqlangan yagona analitik yechimga ega* (§ 17.4 ga qarang).

Misol 1. Ushbu

$$y'' - y = 0 \tag{14.1.9}$$

differensial tenglamaning

$$y = \sum_{n=0}^{+\infty} a_n x^n \quad (x_0 = 0) \tag{14.1.10}$$

ko'rinishdagi yechimini quring.

🗝 Bu yerdagi $a_n\,(n = 0,1,2,...)-$ noma'lum sonlarni aniqlashimiz kerak. (14.1.10) dan y'' ni hisoblaymiz:

$$y'' = \sum_{n=2}^{+\infty}(n-1)na_n x^{n-2} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n. \tag{14.1.11}$$

(14.1.11) va (14.1.10) formulalarni (14.1.9) ga qo'yamiz:

$$\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n = \sum_{n=0}^{+\infty}a_n x^n \ .$$

x ning bir xil darajalari koeffitsienlarini tenglashtiramiz:

$$x^0:\ 1\cdot 2\cdot a_2 = a_0,$$
$$x^1:\ 2\cdot 3\cdot a_3 = a_1,$$
$$x^2:\ 3\cdot 4\cdot a_4 = a_2,$$
$$\dots\dots\dots\dots$$
$$x^n:\ (n+1)(n+2)a_{n+2} = a_n,$$
$$\dots\dots\dots\dots$$

a_0, a_1 larni ixtiyoriy tayinlaymiz. U holda yuqoridagi tenglamalarning toq nomerlilaridan

$$a_2 = \frac{a_0}{1\cdot 2},\ a_4 = \frac{a_2}{3\cdot 4} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4} = \frac{a_0}{4!},\ \dots,\ a_{2n} = \frac{a_0}{(2n)!}\ (n = 1,2,...)\ ,$$

juft nomerlilaridan esa

$$a_3=\frac{a_1}{2\cdot 3},\ a_5=\frac{a_3}{4\cdot 5}=\frac{a_1}{2\cdot 3\cdot 4\cdot 5}=\frac{a_1}{5!},\ \ldots,\ a_{2n+1}=\frac{a_1}{(2n+1)!}\ \ (n=1,2,...)$$

tengliklarni topamiz. Bularni (14.1.10) ga qo‘yib, formal ravishda yechimni topamiz:

$$y=a_0\sum_{n=1}^{+\infty}\frac{1}{(2n)!}x^{2n}+a_1\sum_{n=1}^{+\infty}\frac{1}{(2n+1)!}x^{2n+1}. \qquad (14.1.12)$$

Bu formuladagi darajali qatorlar $(-\infty,+\infty)$ oraliqda absolyut yaqinlashuvchi (masalan, Dalamber alomatiga asosan). Demak, yuqorida qilingan ishlar (hadma-had differensiallash va h.k.) qonuniy va (14.1.12) formula (14.1.9) tenglamaning $(-\infty,+\infty)$ oraliqda aniqlangan analitik yechimini beradi.

Matematik analizdan ma’lumki, giperbolik kosinus va giperbolik sinus funksiyalari uchun

$$\mathrm{ch}x=\sum_{n=1}^{+\infty}\frac{1}{(2n)!}x^{2n},\ \mathrm{sh}x=\sum_{n=1}^{+\infty}\frac{1}{(2n+1)!}x^{2n+1}.$$

Shunday qilib, (14.1.9) tenglamaning ushbu

$$y=a_0\mathrm{ch}x+a_1\mathrm{sh}x\ \ (a_0,a_1-\text{ ixtiyoriy o‘zgarmaslar})$$

ikki parametrli analitik yechimlar oilasini (umumiy yechimini) topdik. 👍

Yuqoridagi (14.1.2) tenglamaning (14.1.3) yechimidagi $a_n\,(n=0,1,2,...)$ koeffitsientlarni

$$a_n=\frac{y^{(n)}(x_0)}{n!}\ \ (n=0,1,2,...)$$

formulalarga ko‘ra topish ham mumkin. Bunda $a_0=y(x_0)$, $a_1=y'(x_0)$ va qolgan koeffitsientlar (14.1.2) tenglamadan aniqlanadi. $y=y(x)$ yechim bo‘lgani uchun, u x_0 nuqtaning biror atrofida (14.1.3) tenglamani ayniyatga aylantiradi:

$$y''(x)+p_1(x)y'(x)+p_0(x)y(x)=q(x). \qquad (14.1.13)$$

Bu ayniyatda $x=x_0$ deb $y''(x_0)$ni, demak, $a_2=\frac{y''(x_0)}{2!}$ ni topamiz. (14.1.13) ayniyatni ketma-ket differensiallab va $x=x_0$ deb, $y'''(x_0), y^{IV}(x_0),...$ hamda $a_3,a_4,...$larni hisoblaymiz:

$$y'''(x)=q'(x)-p_1'(x)y'(x)-p_1(x)y''(x)-p_0'(x)y(x)-p_0(x)y'(x),$$

bundan $y'''(x_0)$ va $a_3 = \frac{y'''(x_0)}{3!}$ lar;

$$y^{IV}(x) = (q'(x) - p_1'(x)y'(x) - p_1(x)y''(x) -$$
$$- p_0(x)y(x) - p_0(x)y'(x))' = q''(x) - \dots ,$$

bundan $y^{IV}(x_0)$ va $a_4 = \frac{y^{IV}(x_0)}{4!}$ lar; va h.k. topiladi.

Biz yuqorida chiziqli tenglamalarning analitik yechimlarini topish bilan shug'ullandik. Nochiziqli tenglamalarning analitik yechimlari ham yuqoridagiga o'xshash quriladi. Bunda quyidagi teorema asos bo'lib xizmat qiladi.

Teorema 2. *Aytaylik,*

$$y'' = f(x, y, y'), y(x_0) = y_0, \; y'(x_0) = z_0$$

$(x_0, y_0, z_0 -$*berilgan sonlar*$)$ *Koshi masalasi berilgan bo'lsin. Agar* $f(x, y, z)$ *funksiya* $(x_0, y_0, z_0) \in \mathbb{R}^3$ *boshlang'ich ma'lumotlar nuqtasining biror atrofida analitik, ya'ni biror absolyut yaqinlashuvchi* $\sum\limits_{k,l,m=0}^{+\infty} a_{k,l,m}(x - x_0)^k (y - y_0)^l (z - z_0)^m$ *karrali qatorning yig'indisi sifatida ifodalansa, u holda berilgan masala* $x_0 \in \mathbb{R}$ *nuqtaning biror atrofida analitik bo'lgan yagona yechimga ega.*

Shunga o'xshash teorema yuqori tartibli differesial tenglama, yoki tenglamalar sistemasi uchun ham o'rinli. Umumiy holda tenglamalarning analitik yechimlari to'g'risida § 17.4 ga qarang.

$y = \sum\limits_{n=0}^{+\infty} a_n (x - x_0)^n$ analitik yechimning $a_n \; (n = 0, 1, 2, \dots)$ koeffitsientlarini topishni yuqorida aytilgan ikki usuldan biri yordamida amalga oshirish mumkin.

Misol 2. Ushbu

$$y'' = x + y^2, \; y(1) = 1, \; y'(1) = 0,$$

Koshi masalasining analitik yechimi topilsin.

🗝 Yechimni

$$y = \sum_{n=0}^{+\infty} a_n (x-1)^n \quad (x_0 = 1),$$

ko‘rinishda izlaymiz. $a_n\,(n=0,1,2,...)$ koeffitsientlarni, $a_n=\dfrac{y^{(n)}(1)}{n!}$ formulaga ko‘ra, berilgan tenglamani differensiallash yordamida topamiz. Quyidagi hisoblashlarni bajaramiz:

$$a_0=y(1)=1,\; a_1=y'(1)=0;$$

$$y''=x+y^2\,, y''(1)=1+y^2(1)=1,\; a_2=\frac{y''(1)}{2!}=\frac{1}{2};$$

$$y'''=1+2yy'=1+2y(1)y'(1)=1,\; a_3=\frac{y'''(1)}{3!}=\frac{1}{6};$$

$$y^{IV}=2y'^2+2yy',\, y^{IV}(1)=1,\; a_4=\frac{y^{IV}(1)}{4!}=\frac{1}{24};$$

$$y^{V}=6y'y''+2yy''',\, y^{V}(1)=2,\; a_5=\frac{y^{V}(1)}{5!}=\frac{1}{60};$$

$$y^{VI}=6y''^2+8y'y'''+2yy^{IV}\,,\, y^{VI}(1)=14,\; a_6=\frac{y^{VI}(1)}{6!}=\frac{7}{360};$$

. . . .

Demak, izlangan yechim

$$y=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+$$

$$+\frac{1}{60}(x-1)^5+\frac{7}{360}(x-1)^6+...$$

ko‘rinishga ega. Bu darajali qatorning yaqinlashish radiusi qat’iy musbat bo‘ladi. ☝

Masalalar

1. Ushbu

$$y''-xy=0$$

Eyri tenglamasining yechimlari (analitik)

$$y=a_0\Big(1+\sum_{n=1}^{+\infty}\frac{x^{3n}}{2\cdot3\cdot...\cdot(3n-1)\cdot(3n)}\Big)+a_1\Big(x+\sum_{n=1}^{+\infty}\frac{x^{3n+1}}{3\cdot4\cdot...\cdot(3n)\cdot(3n+1)}\Big)$$

(a_0, a_1-ixtiyoriy o‘zgarmaslar) ko‘rinishda va bu yerdagi qatorlarning yaqinlashish radiuslari $R=+\infty$ ekanligini isbotlang.

2. Ermit tenglamasi deb ataluvchi

$$x''-2tx'+2\lambda x=0\;(\;\lambda-\text{parametr})$$

tenglamani qaraylik (bu tenglama matematikaning ba’zi sohalarida va kvant mexanikasida uchraydi).

1) Quyidagi belgilashlarni kiritaylik:

$$x_1(t) = 1 - \lambda t^2 + \frac{2^2\lambda(\lambda-2)}{4!}t^4 - \frac{2^3\lambda(\lambda-2)(\lambda-4)}{6!}t^6 + \ldots,$$

$$x_2(t) = t - \frac{2(\lambda-1)}{3!}t^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}t^5 - \frac{2^3(\lambda-1)(\lambda-3)(\lambda-5)}{7!}t^7 + \ldots .$$

Bu funksiyalar $-\infty < t < +\infty$ oraliqda aniqlangan, chiziqli erkli va ixtiyoriy a_0, a_1 o‘zgarmas sonlar uchun

$$x = a_0 x_1(t) + a_1 x_2(t)$$

funksiya Ermit tenglamasining yechimi bo‘lishini isbotlang.

2) Ermit tenglamasining $n-$darajali ko‘phaddan iborat bo‘lgan yechimi, agar t^n oldidagi koeffitsient 2^n ga teng bo‘lsa, $(n-$darajali$)$ Ermit ko‘phadi deb ataladi va $H_n(t)$ bilan belgilanadi. $H_0(t)=1,\ H_1(t)=2t,\ H_2(t)=4t^2-2,\ \ H_3(t)=8t^3-12$ tengliklarni isbotlang.

§ 14.2. Regular maxsus nuqta atrofida yechimniumumlashgan darajali qator yordamida qurish. Frobenius metodi

Endi (14.1.2) tenglamaga qaraganda umumiyroq

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x) \qquad (14.2.1)$$

differensial tenglamani qaraylik. Bu yerdagi $p_2(x),\ p_1(x),\ p_0(x),\ q(x)$ funksiyalar x_0 nuqtaning biror atrofida analitik deb hisoblanadi. Agar $p_2(x_0) \neq 0$, ya’ni x_0-tenglamaning Regular nuqtasi bo‘lsa, u holda (14.2.1) tenglama x_0 nuqtaning yetarlicha kichik atrofida ushbu

$$y'' + \frac{p_1(x)}{p_2(x)}y' + \frac{p_0(x)}{p_2(x)}y = \frac{q(x)}{p_2(x)}$$

analitik koeffitsientli tenglamaga ekvivalent. Oxirgi tenglamaning x_0 nuqtada analitik yechimini topish bilan yuqorida tanishdik.

Faraz qilaylik, $p_2(x_0)=0$, ya’ni x_0- (14.2.1) tenglamaning maxsus nuqtasi bo‘lsin. Bu holda (14.2.1) tenglama, umuman olganda, x_0 nuqtada analitik yechimga ega bo‘lmasligi mumkin. Lekin, ba’zi hollarda yechimni umumlashgan darajali qator yig‘indisi ko‘rinishida ifodalash mumkin.

Ushbu

$$(x-x_0)^2 y'' + (x-x_0)p_1(x)y' + p_0(x)y = 0$$

yoki

$$y'' + \frac{p_1(x)}{x - x_0} y' + \frac{p_0(x)}{(x - x_0)^2} y = 0 \qquad (14.2.2)$$

tenglamani qaraylik; bu yerda $p_1(x), p_0(x) -$ x_0 nuqtada analitik funksiyalar. Bu holda x_0 nuqta (14.2.2) tenglama uchun **regular maxsus nuqta** deyiladi.

Bundan keyin qisqalik uchun $x_0 = 0$ deb hisoblaymiz. har doim $s = x - x_0$ almashtirish yordamida x_0 nuqtani 0 nuqtaga o‘tkazish mumkin, bunda koeffitsientlarning analitikligi saqlanadi.

Quyidagi teoremani isbotsiz keltiramiz.

Teorema 1. *Agar* $x_0 = 0$ *nuqta (14.2.2) tenglama uchun regular maxsus nuqta bo‘lsa, u holda (14.2.2) tenglama*

$$y = x^{\mu} \sum_{n=0}^{+\infty} a_n x^n \ \ (\mu, a_n \ (n = 0,1,2,...) - o\text{‘}zgarmas\ sonlar) \qquad (14.2.3)$$

ko‘rinishdagi kamida bitta yechimga ega; bu umumlashgan darajali qator biror $x \in (0, \rho)$ $(\rho > 0)$ *intervalda yaqinlashuvchi bo‘ladi.*

Teoremada aytilgan $\mu, a_n \ (n = 0,1,2,...)$ sonlarni topish uchun, Frobenius metodiga ko‘ra, ushbu ($x > 0$)

$$p_0(x) = \sum_{n=0}^{+\infty} c_n x^n \ , \ p_1(x) = \sum_{n=0}^{+\infty} b_n x^n \ ;$$

$$y = x^{\mu} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n x^{n+\mu} , \ y' = \sum_{n=0}^{+\infty} (n+\mu) a_n x^{n+\mu-1} ,$$

$$y'' = \sum_{n=0}^{+\infty} (n+\mu)(n+\mu-1) a_n x^{n+\mu-2} \ ;$$

$$p_0(x) y = \sum_{n=0}^{+\infty} c_n x^n \cdot \sum_{n=0}^{+\infty} a_n x^{n+\mu} = \sum_{n=0}^{+\infty} \sum_{k=1}^{n} a_k c_{n-k} x^{n+\mu} ;$$

$$x p_1(x) y' = \sum_{n=0}^{+\infty} b_n x^n \cdot \sum_{n=0}^{+\infty} (n+\mu) a_n x^{n+\mu} = \sum_{n=0}^{+\infty} \sum_{k=1}^{n} (k+\mu) a_k b_{n-k} x^{n+\mu} \ ;$$

$$x^2 y'' = \sum_{n=0}^{+\infty} (n+\mu)(n+\mu-1) a_n x^{n+\mu} \ ;$$

yoyilmalarni (14.2.2) tenglamaga qo‘yib, uni quyidagi ko‘rinishga keltiramiz:

$$\sum_{n=0}^{+\infty} ((n+\mu)(n+\mu-1) a_n + \sum_{k=0}^{n} a_k \left((k+\mu) b_{n-k} + c_{n-k}\right)) x^{n+\mu} = 0$$

yoki

$$a_0\left(\mu(\mu-1)+\mu b_0+c_0\right)x^{\mu}+\sum_{n=1}^{+\infty}(a_n((n+\mu)(n+\mu-1)+(n+\mu)b_0+c_0)+$$
$$+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right))x^{n+\mu}=0$$

yoki yana qisqaroq

$$a_0A(\mu)x^{\mu}+\sum_{n=1}^{+\infty}(a_nA(n+\mu)+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right))x^{n+\mu}=0;\qquad(14.2.4)$$

bu yerda

$$A(\mu)=\mu(\mu-1)+\mu b_0+c_0. \qquad (14.2.5)$$

(14.2.4) tenglik ayniyat bo‘lishi uchun x ning darajalari oldidagi koeffitsientlar nolga teng bo‘lishi kerak:

$$a_0A(\mu)=0,\ a_nA(n+\mu)+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right)=0\ \ (n\geq 1).\qquad(14.2.6)$$

Bu yerdagi birinchi shartdan a_0 ni ixtiyoriy deb, μ ga nisbatan kvadrat tenglama hosil qilamiz: $A(\mu)=0$ yoki (14.2.5) ga ko‘ra

$$\mu(\mu-1)+\mu b_0+c_0=0 \qquad (14.2.7)$$

Bu kvadrat tenglama (14.2.2) differensial tenglamaning ***aniqlovchi tenglamasi*** deyiladi. Agar berilgan μ uchun $A(1+\mu)\neq 0,\ A(2+\mu)\neq 0,...,A(n+\mu)\neq 0,...$ bo‘lsa, u holda (14.2.6) tenglamadan tayinlangan $a_0\neq 0$ ga ko‘ra rekurrent usulda barcha $a_1=a_1(\mu), a_2=a_2(\mu), ..., a_n=a_n(\mu),...$ koeffitsientlarni bir qiymatli topamiz.

Aniqlovchi kvadrat tenglamaning μ_1 va μ_2 ildizlari haqiqiy bo‘lgan hol bilan chegaralanamiz. Aniqlik uchun $\mu_1\geq\mu_2$ deylik. Demak, $A(\mu_1)=0$ va $\mu>\mu_1$ bo‘lganda $A(\mu)\neq 0$. Shuning uchun (14.2.6) rekurrent munosabatdan $a_0=1$ deb, barcha $a_n=a_n(\mu_1)$ $(n\geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Shunday qilib, bu holda (14.2.2) tenglamaning bitta

$$y_1(x)=x^{\mu_1}\sum_{n=0}^{+\infty}a_n(\mu_1)x^n=x^{\mu_1}(1+\sum_{n=1}^{+\infty}a_n(\mu_1)x^n) \qquad (14.2.8)$$

yechimini hosil qilamiz. (14.2.2) tenglamaning umumiy yechimini topish uchun uning (14.2.8) yechimga chiziqli bog‘liq bo‘lmagan yana bir $y_2(x)$ yechimini topishimiz kerak. Bu $y_2(x)$ yechimni qurish usuli μ_1 va μ_2 ildizlariga bog‘liq. Quyidagi hollar bo‘lishi mumkin.

1^0. $\mu_1-\mu_2$ ayirma butun son bo‘lmasin. U holda, ravshanki, $A(1+\mu_2)\neq 0,\ A(2+\mu_2)\neq 0,\ldots,A(n+\mu_2)\neq 0,\ldots$ va $a_0=1$ deb, (14.2.6) rekurrent munosabatdan barcha $a_n=a_n(\mu_2)$ $(n\geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Demak, $y_2(x)$ yechim sifatida quyidagi funksiyani olish mumkin:

$$y_2(x)=x^{\mu_2}\sum_{n=0}^{+\infty}a_n(\mu_2)x^n=x^{\mu_2}(1+\sum_{n=1}^{+\infty}a_n(\mu_2)x^n). \qquad (14.2.9)$$

2^0. $\mu_1=\mu_2$ bo‘lsin. Bu holda $A(\mu)=(\mu-\mu_1)^2$. $\mu\neq\mu_1$ ni o‘zgaruvchi deb hisoblaymiz va ixtiyoriy $a_0\neq 0$ ni tayinlab, (14.2.6) rekurrent munosabatdan barcha $a_n=a_n(\mu)$ $(n\geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Bu koeffitsientlarga ko‘ra (14.2.3) formuladagi

$$y=y(x,\mu)=x^{\mu}\sum_{n=0}^{+\infty}a_nx^n=x^{\mu}(a_0+\sum_{n=1}^{+\infty}a_n(\mu)x^n) \qquad (14.2.10)$$

funksiyani tuzaylik. Bu funksiyaning qurilishiga ko‘ra

$$x^2y''+xp_1(x)y'+p_0(x)y=a_0A(\mu)x^{\mu}+\sum_{n=1}^{+\infty}\Big(a_nA(n+\mu)+$$

$$+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right)\Big)x^{n+\mu}=a_0A(\mu)x^{\mu}+\sum_{n=1}^{+\infty}\Big(a_nA(n+\mu)+$$

$$+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right)\Big)x^{n+\mu}=a_0A(\mu)x^{\mu}\ .$$

Demak, $x^2y''+xp_1(x)y'+p_0(x)y=a_0(\mu-\mu_1)^2x^{\mu}$.

Oxirgi tenglikni μ bo‘yicha differensiaymiz. x va μ bo‘yicha differensiallash tartibini almashtirib,

$$x^2\left(\frac{\partial y}{\partial\mu}\right)''+xp_1(x)\left(\frac{\partial y}{\partial\mu}\right)'+p_0(x)\frac{\partial y}{\partial\mu}=2a_0(\mu-\mu_1)x^{\mu}+a_0(\mu-\mu_1)^2x^{\mu}\ln x$$

ekanligini topamiz (bizda $x>0$). Bu tenglikda $\mu\to\mu_1$ deb, (14.2.10) formula bilan aniqlangan funksiyaning ushbu

$$y_2(x) = \frac{\partial y(x,\mu)}{\partial \mu}\bigg|_{\mu=\mu_1}$$

hosilasi (14.2.2) tenglamaning yechimi bo‘lishini topamiz. (14.2.10) formulaga ko‘ra bu ikkinchi chiziqli erkli yechim

$$y_2(x) = y_1(x)\ln x + x^{\mu_1}\sum_{n=0}^{+\infty} a_n'(\mu_1)x^n \qquad (14.2.11)$$

ko‘rinishga ega bo‘ladi. Bu yerdagi $a_n'(\mu_1)$ larni noma’lum koeffitsientlar metodi yordamida, ya’ni (14.2.11) ni (14.2.2) tenglamaga qo‘yib, tenglamaning qanoatlanishi shartidan aniqlash mumkin.

3^0. $\mu_1 - \mu_2$ ayirma natural son bo‘lsin. Bu holda ikkinchi chiziqli erkli $y_2(x)$ yechimni qurish murakkabroq kechadi. Birdaniga natijani keltiramiz. Bu holda $y_2(x)$ yechim

$$y_2(x) = \tilde{a}_{-1}y_1(x)\ln x + x^{\mu_2}\sum_{n=0}^{+\infty} \tilde{a}_n(\mu_2)x^n \qquad (14.2.12)$$

ko‘rinishda bo‘ladi. Bu yechimni qurish uchun dastlab Frobenius metodidan foydalanib, μ_2 ga ko‘ra (14.2.2) tenglamaning yechimini qurish kerak. Agar qurilgan $y_2(x)$yechim $y_1(x)$ (14.2.5) yechimga chiziqli bog‘liq bo‘lmasa, u (14.2.8) ko‘rinishda bo‘ladi (bunda $\tilde{a}_{-1} = 0$). Aks holda, ya’ni $y_2(x)$yechim $y_1(x)$ ga chiziqli bog‘liq (proprtsional) bo‘lib chiqsa, 2^0 banddagidek ish tutib,

$$y_2(x) = \frac{\partial}{\partial \mu}\big((\mu - \mu_2)y(x,\mu)\big)\bigg|_{\mu=\mu_2}$$

izlangan yechimni topish kerak. Tushunarliki, (14.2.12) ko‘rinishdagi yechimni noma’lum koefftsientlar metodidan foydalanib ham topish mumkin.

Agar aniqlovchi tenglamaning μ_1, μ_2 ildizlari kompleks bo‘lsa, u holda $\mu_2 = \bar{\mu}_1$ bo‘ladi (tenglamadagi koeffitsienlar haqiqiy ekanligi uchun), $x^\mu = e^{\mu \ln x}$ va $e^{\alpha \pm i\beta} = e^\alpha(\cos\beta \pm i\sin\beta)$ Eyler formulalaridan foydalanib chiziqli erkli haqiqiy $y_1(x)$ va $y_2(x)$ yechimlarni qurish mumkin.

Misol 1. Ushbu

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (\nu = \text{const} \geq 0) \qquad (14.2.13)$$

Bessel tenglamasining chiziqli erkli yechimlarini quring.

🗝Ravshanki, Bessel tenglamasi uchun $x=0$ nuqta regular maxsus nuqtadir. Frobenius metodidan kelib chiqib, yechimni

$$y = x^{\mu}\sum_{n=0}^{+\infty} a_n x^n = a_0 x^{\mu} + a_1 x^{1+\mu} + a_2 x^{2+\mu} + ... + a_n x^{n+\mu} + ... \tag{14.2.14}$$

ko‘rinishda izlaymiz; bu yerda $\mu, a_n\ (n=0,1,2,...)$ – hozircha noma’lum sonlar. Bu funksiyani, uning birinchi va ikkinchi hosilalarni (14.2.9) tenglamaga qo‘yamiz:

$$x^2(\mu(\mu-1)a_0x^{\mu-2} + (1+\mu)\mu a_1 x^{\mu-1} + ... + (n+\mu)(n+\mu-1)a_n x^{n+\mu-2} + ...) +$$
$$+x(\mu a_0 x^{\mu-1} + (1+\mu)a_1x^{\mu} + (2+\mu)a_2x^{\mu+1} ... + (n+\mu)a_nx^{n+\mu-1} + ...) +$$
$$+(x^2-\nu^2)(a_0x^{\mu} + a_1x^{1+\mu} + a_2x^{2+\mu} + ... + a_nx^{n+\mu} + ...) = 0 .$$

Qavslarni ochib chiqib, o‘xshash hadlarni ixchamlaymiz:

$$a_0(\mu^2-\nu^2)x^{\mu} + a_1\left((\mu+1)^2-\nu^2\right)x^{\mu+1} +$$
$$+ ... + \left(a_n\left((\mu+n)^2-\nu^2\right) + a_{n-2}\right)x^{\mu+n} + ... = 0.$$

x ning darajalari oldidagi koeffitsientlarni nolga tenglashtirib, noma’lumlarni topish uchun quyidagi sistemani hosil qilamiz:

$$\begin{cases} a_0(\mu^2-\nu^2) = 0, \\ a_1\left((\mu+1)^2-\nu^2\right) = 0, \\ \dots\dots\dots\dots\dots\dots\dots \\ a_n\left((\mu+n)^2-\nu^2\right) + a_{n-2} = 0 \ \ (n \ge 2), \\ \dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{14.2.15}$$

a_0 ni ixtiyoriy deb, ushbu $\mu^2-\nu^2=0$ aniqlovchi tenglamani topamiz. Uning yechimlari ν musbat bo‘lganda ikkita $\mu_1=\nu, \mu_2=-\nu$.

Dastlab $\mu=\nu$ holni qaraymiz. (14.2.15) ning ikkinchi tenglamasida $\mu=\nu$ deb, $a_1=0$ ekanligin topamiz. (14.2.15) sistemadan ushbu $a_n\left((\mu+n)^2-\nu^2\right)+a_{n-2}=0,$ ya’ni

$$a_n = -\frac{1}{(\mu+n+\nu)(\mu+n-\nu)}a_{n-2} \quad (n \ge 2) \tag{14.2.16}$$

munosabatni ham topamiz. Bu rekurrent formulani ketma-ket qo‘llab, barcha toq indeksli koeffitsientlarning nolga teng ekanligini hosil qilamiz: $0 = a_1 = a_3 = a_5 = \ldots$. Juft indeksli koeffitsientlar uchun esa

$$a_{2n} = (-1)^n \frac{a_0}{2^{2n} n!(\nu+1)(\nu+2)\ldots(\nu+n)} \quad (n \geq 1)$$

formulalarni topamiz. Endi topilgan qiymatlarni (14.2.14) formulaga qo‘yib, Bessel tenglamasining

$$y = a_0 x^\nu \left(1 + \sum_{n=1}^{+\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(\nu+1)(\nu+2)\ldots(\nu+n)}\right)$$

yechimini hosil qilamiz. Bu yerdagi a_0 o‘zgarmasni qulaylik uchun $a_0 = \dfrac{1}{2^\nu \Gamma(\nu+1)}$ ($\Gamma -$gamma funksiya) deb tanlanadi va bunda hosil bo‘lgan $y = J_\nu(x)$ yechim ν- tartibli birinchi tur Bessel funksiyasi deb ataladi. Kerakli shakl almashtirishlarni bajarib, $J_\nu(x)$ Bessel funksiyasini quyidagi qator ko‘rinishda ifodalaymiz:

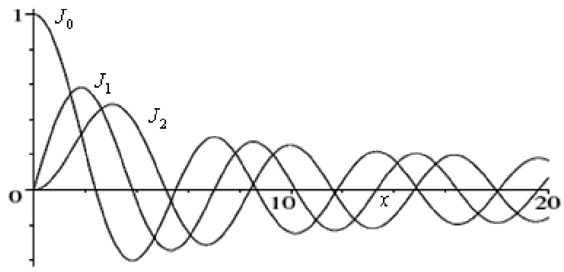
$$J_\nu(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \quad (14.2.17)$$

Osongina ko‘rsatish mumkinki, $x^{-\nu} J_\nu(x)$ uchun qator ixtiyoriy $x \in [a,b]$ segmentda absolyut va tekis yaqinlashuvchi.

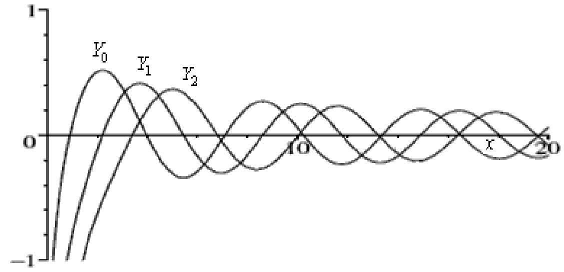
Endi $\mu = -\nu$ bo‘lsin. Bunda Bessel tenglamasining $J_{-\nu}(x)$ yechimini hosil qilamiz. Ravshanki, agar ν son butun bo‘lmasa, $J_\nu(x)$ va $J_{-\nu}(x)$ yechimlar chiziqli erkli bo‘ladi. Lekin, agar ν butun son bo‘lsa, ular chiziqli bog‘langan bo‘ladi ($J_{-\nu}(x) = (-1)^\nu J_\nu(x)$) va bu holda ushbu

$$Y_\nu(x) = \frac{\cos\nu\pi \cdot J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \quad (14.2.18)$$

funksiya kiritiladi. Agar $\nu = k$ butun son bo‘lsa, (14.2.18) formulani $Y_k(x) = \lim\limits_{\nu \to k} Y_\nu(x)$ deb tushunish kerak. Bunda har doim $J_\nu(x)$ va $Y_\nu(x) -$chiziqli erkli yechimlar bo‘ladi (14.1- va 14.2- rasmlar). Bu yerdagi $Y_\nu(x)$ funksiya ν- tartibli ikkinchi tur Bessel funksiyasi (yoki Neyman funksiyasi) deb ataladi. ☝



14.1- rasm. Birinchi tur 0-, 1- va 2- tartibli Bessel funksiyalari.



14.2- rasm. Ikkinchi tur 0-, 1- va 2- tartibli Bessel funksiyalari.

Bessel funksiyasi uchun (14.2.17) formuladan quyidagi ikki formula osongina kelib chiqadi:

$$\begin{cases} xJ'_\nu(x) + \nu J_\nu(x) = xJ_{\nu-1}(x), \\ xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x). \end{cases} \qquad (14.2.19)$$

Ba’zi tenglamalarning yechimini maxsus almashtirishlar yordamida Bessel funksiyalari orqali ifodalash mumkin bo‘ladi.

Masalan, ushbu

$$y' = x + y^2$$

maxsus Rikkati tenglamasini qaraylik. $y = y(x)$ noma’lum funksiya o‘rniga yangi $u = u(x)$ noma’lum funksyani

$$y = -\frac{1}{u} \cdot \frac{du}{dx} \qquad (14.2.20)$$

formula orqali kiritib, u funksiya uchun

$$u'' + xu = 0 \qquad (14.2.21)$$

chiziqli tenglamani hosil qilamiz. Bu tenglamani ushbu

$$w = \frac{u}{\sqrt{x}}, \quad z = \frac{2}{3}x^{3/2}, \; w = w(z) \;\; (x > 0)$$

almashtirishlar yordamida

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + \left[z^2 - \left(\frac{1}{3}\right)^2 \right] w = 0$$

Bessel tenglamasiga keltiramiz. $-1/3$ va $1/3$ tartibli birinchi tur $J_{-1/3}(z)$ va $J_{1/3}(z)$ Bessel funksiyalari oxirgi tenglamaning chiziqli erkli yechimlari. Demak, (14.2.21) tenglamaning umumiy yechimi

$$u = c_1\sqrt{x}J_{-1/3}\Big(\frac{2}{3}x^{3/2}\Big) + c_2\sqrt{x}J_{1/3}\Big(\frac{2}{3}x^{3/2}\Big) \quad (x>0)$$

ko‘rinishda bo‘ladi. Endi (14.2.20) almashtirish va (14.2.19) formulalarga ko‘ra berilgan maxsus Rikkati tenglamasining Bessel funksiyalari orqali ifodalangan ushbu

$$y = -\sqrt{x}\,\frac{J_{-2/3}\Big(\frac{2}{3}x^{3/2}\Big) - cJ_{2/3}\Big(\frac{2}{3}x^{3/2}\Big)}{cJ_{-1/3}\Big(\frac{2}{3}x^{3/2}\Big) + J_{1/3}\Big(\frac{2}{3}x^{3/2}\Big)} \quad (x>0)$$

yechimini topamiz.

Masalalar

1. Agar k butun son bo‘lsa, $J_{\pm\frac{k}{2}}(x)$ Bessel funksiyalarining elementar funksiyalar orqali ifodalanishini ko‘rsating.

2. Ushbu

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0$$

tenglama Lejandr tenglamasi deb ataladi (λ – berilgan son, parametr).

1) Lejandr tenglamasining umumiy yechimi quyidagi ko‘rinishda bo‘lishini isbotlang:

$y = a_0y_1(x) + a_1y_2(x)$ (a_0, a_1 – ixtiyoriy o‘zgarmaslar), bu yerda

$$y_1(x) = 1 - \frac{\lambda(\lambda+1)}{2!}x^2 + \frac{(\lambda-2)\lambda(\lambda+1)(\lambda+3)}{4!}x^4 -$$
$$-\frac{(\lambda-4)(\lambda-2)\lambda(\lambda+1)(\lambda+3)(\lambda+5)}{6!}x^6 + \dots ,$$
$$y_2(x) = x - \frac{(\lambda-1)(\lambda+2)}{3!}x^3 + \frac{(\lambda-3)(\lambda-1)(\lambda+2)(\lambda+4)}{5!}x^5 -$$
$$-\frac{(\lambda-5)(\lambda-3)(\lambda-1)(\lambda+2)(\lambda+4)(\lambda+6)}{7!}x^7 + \dots .$$

2) Bu qatorlarning yaqinlashish radiuslari $R \ge 1$ ekanligini ko‘rsating. Ularni $x = \pm 1$ nuqtalarda yaqinlashishga tekshiring. λ butun son bo‘lgan taqdirdagina $y_1(x)$ va $y_2(x)$ yechimlar $[-1,1]$ segmentda aniqlangan bo‘lishini ko‘rsating.

3) Agar λ nomanfiy juft son bo‘lsa, u holda $y_1(x)$ ko‘phadga aylanadi; bu holda

$$P_\lambda(x) = \frac{y_1(x)}{y_1(1)}$$

ko‘phadni kiritamiz. Agar λ musbat toq son bo‘lsa, u holda $y_2(x)$ ko‘phaddan iborat bo‘ladi; bu holda

$$P_\lambda(x) = \frac{y_2(x)}{y_2(1)}$$

deymiz. Kiritilgan $P_\lambda(x)$ $(\lambda = 0,1,2,...)$ ko‘phadlar Lejandr ko‘phadlari deb ataladi. $P_0(x), P_1(x), P_2(x), P_3(x)$ va $P_4(x)$ ko‘phadlarning standart ko‘rinishini toping. Ushbu

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n \quad (n = 0,1,2,...)$$

formulalarni isbotlang.

3. Ushbu

$$(1-x^2)y'' - xy' + \lambda^2 y = 0$$

differensial tenglama Chebishyov tenglamasi deb ataladi ($\lambda -$berilgan son). Bu tenglamaning $x=0$ da analitik yechimlarini toping. λ ning qanday qiymatlarida yechimlar ko‘phadlardan iborat bo‘ladi? Bu ko‘phadlar Chebishyov ko‘phadlari deb ataladi.

4. Ushbu

$$x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$$

differensial tenglama gipergeometrik tenglama yoki Gauss tenglamasi deb ataladi; bunda $\alpha, \beta, \gamma -$berilgan sonlar. Quyidagi tasdiqlarni isbotlang:

1) Bu tenglama uchun $x = 0 -$ Regular maxsus nuqta.

2) Agar γ soni $0, -1, -2, -3, ...$sonlaridan farqli bo‘lsa, ushbu

$$F(\alpha, \beta, \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} x^k, \ |x| < 1, \ \text{(G)}$$

funksiya gipergeometrik tenglamaning yechimi bo‘ladi; bunda

$$(a)_0 = 1; \ (a)_n = a(a+1)(a+2)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \ n \in \mathbb{N}.$$

3) Agar $\gamma = -n, n = 0,1,2,....,$ va $\alpha \neq -p, \beta \neq -p, p \in \mathbb{N}, p < n,$ bo‘lsa, (G) qator aniqlanmagan, lekin

$$\lim_{\gamma \to -n} \frac{F(\alpha, \beta, \gamma; x)}{\Gamma(\gamma)} = \frac{(\alpha)_{n+1} (\beta)_{n+1}}{(n+1)!} x^{n+1} F(\alpha+n+1, \beta+n+1, n+2; x)$$

limit mavjud va u Gauss tenglamasining yechimi.

4) Agar $\alpha = -n$ yoki $\beta = -n$ $(n = 0,1,2,....)$ va $\gamma = -p,$ bunda $p = n, \ n+1, \ n+2, ...,$ bo‘lsa,

$$F(-n, \beta, -p; x) = \sum_{k=0}^{n} \frac{(-n)_k (\beta)_k}{(-p)_k k!} x^k \ \text{ yoki } \ F(\alpha, -n, -p; x) = \sum_{k=0}^{n} \frac{(\alpha)_k (-n)_k}{(-p)_k k!} x^k$$

ko‘phad gipergeometrik tenglama yechimi bo‘ladi.

Aniqlangan $F(\alpha, \beta, \gamma; x)$ funksiya gipergeometrik funksiya yoki Gauss funksiyasi deb ataladi.

5) Gauss tenglamasining ikkinchi yechimni topish uchun unda $y = x^{1-\gamma}u$ almashtirishni bajaring va hosil bo‘lgan tenglamaning kerakli yechimidan foydalaning.

6) Gauss tenglamasi uchun $x = 1$ nuqta ham Regular maxsus nuqta. Bu nuqta atrofida yechimni topish uchun tenglamada $x = 1 - t$ almashtirishni bajaring. Hosil bo‘lgan gipergeometrik tenglamaning kerakli yechimlarini quring.

5. Ushbu

$$\begin{cases} x^2 y' - y + x = 0 \\ y(0) = 0 \end{cases}$$

boshlang‘ich masalani yeching. U analitik yechimga egami?

6. Ushbu

$$x^3 y' - 2y = 0$$

differensial tenglamani yeching. har qanday yechimning barcha hosilalari nol nuqtada nolga teng ekanligini ko‘rsating. Tenglamaning nol nuqtada analitik bo‘lgan yechimlari nechta?

§ 14.3. Yechimlarning nollari. Tebranuvchi va tebranmas yechimlar

Ushbu

$$y'' + a_1(x)y' + a_0(x)y = 0,\ a_j(x) \in C\big((a,b)\big),\ j = 0,1, \qquad (14.3.1)$$

ikkinchi tartibli chiziqli bir jinsli tenglama berilgan bo‘lsin. Biz bu paragrafda uning $y = y(x)$ yechimi ishoralarining o‘zgarishini o‘rganamiz.

Dastlab (14.3.1) tenglamani ushbu

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0 \qquad (14.3.2)$$

o‘z-o‘ziga qo‘shma differensial tenglama deb ataluvchi tenglama ko‘rinishiga keltiraylik. Buning uchun (14.3.1) ning har ikkala tomonini biror silliq $\mu(x)$ funksiyaga ko‘paytiramiz va hosil bo‘lgan tenglamaning o‘z-o‘ziga qo‘shma bo‘lishi shartidan kerakli $\mu(x)$ ni topamiz:

$$\mu(x)y'' + \mu(x)a_1(x)y' + \mu(x)a_0(x)y = 0.$$

Bu tenglama o‘z-o‘ziga qo‘shma bo‘lishi uchun

$$\mu'(x) = \mu(x)a_1(x),\ \text{ya'ni}\ \mu(x) = \exp\left[\int_{x_0}^{x} a_1(s)ds\right]$$

bo‘lishi kifoya. $\mu(x)$ uchun bu ifodani tegishli tenglamaga qo‘yib, ushbu

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=0 \qquad (14.3.2)$$

o‘z-o‘ziga qo‘shma tenglamani hosil qilamiz; bu yerda

$$p(x)=\exp\left[\int_{x_0}^{x}a_1(s)ds\right]>0\ ,\ q(x)=a_0(x)\exp\left[\int_{x_0}^{x}a_1(s)ds\right] \qquad (14.3.3)$$

Shunday qilib, agar $a_1(x)$, $a_0(x)$ koeffitsiyentlar (a,b) intervalda uzluksiz bo‘lsa, u holda (14.3.1) tenglamani (14.3.2) o‘z-o‘ziga qo‘shma ko‘rinishga keltirish mumkin; bu yerda $p\in C^1((a,b))$ va $q\in C((a,b))$

Endi (14.3.2) tenglamada x argument o‘rniga $\xi=\xi(x)$ o‘zgaruvchini ushbu

$$\frac{d\xi}{dx}=\frac{1}{p(x)}$$

tenglamaning yechimi kabi kiritamiz. U holda (14.3.2) tenglama

$$\frac{1}{p(x)}\frac{d}{d\xi}\left[\frac{dy}{d\xi}\right]+q(x)y=0$$

yoki

$$\frac{d^2y}{d\xi^2}+Q(\xi)y=0 \qquad (14.3.4)$$

ko‘rinishni oladi; bu yerda $Q(\xi)=p(x)q(x)\big|_{x=x(\xi)}$ ya’ni

$$Q(\xi)=a_0(x)\exp\left[2\int_{x_0}^{x}a_1(s)ds\right]\Bigg|_{x=x(\xi)}$$

(bu yerda $x=x(\xi)$ funksiya $\xi=\xi(x)$ ga teskari funksiyani belgilaydi; u mavjud va uzluksiz differensiallanuvchi, chunki $\xi'(x)=1/p(x)>0$).

Endi ikkinchi tartibli chiziqli tenglamaning nollarini o‘rganamiz.

Odatdagidek, agar $y(x_0)=0$ bo‘lsa, x_0 ni $y=y(x)$ funksiyaning noli deb ataymiz. Bundan buyon «soddalashtirilgan»

$$y''+q(x)y=0 \qquad (14.3.5)$$

tenglamani qaraymiz; bu yerda $q(x)\in C((a,b))$.

Teorema 1. *(14.3.5) tenglamaning har qanday notrivial yechimi ixtiyoriy* $[a_1,b_1]\subset(a,b)$ *segmentda chekli sondagi nollarga ega xolos.*

🗝Teskarisini faraz qilaylik. U holda biror $y=y(x)$ notrivial yechim biror $[a_1,b_1]\subset(a,b)$ segmentda cheksiz ko‘p nollarga ega bo‘ladi. Bu nollardan $x_1,x_2,...,x_n,...$ larini ajrataylik: $y(x_n)=0,\ x_n\in[a_1,b_1],\ n\in\mathbb{N}$.

Chegaralangan $\{x_n\}$ ketma-ketlikdan Veyyershtrass teoremasiga ko‘ra yaqinlashuvchi qismiy ketma-ketlik $\{x_{n_k}\}$ ni ajratamiz; $x_{n_k}\to x_0\in[a_1,b_1]$ bo‘ladi. $y(x)$ yechimning uzluksizligiga ko‘ra $y(x_0)=0$. Hosila ta’rifidan

$$y'(x_0)=\lim_{x\to x_0}\frac{y(x)-y(x_0)}{x-x_0}=\lim_{k\to\infty}\frac{y(x_{n_k})-y(x_0)}{x_{n_k}-x_0}=0$$

Demak, $x_0\in[a_1,b_1]$ nuqtada $y(x_0)=0$, $y'(x_0)=0$. Yechimning yagonalik xossasiga ko‘ra $y(x)\equiv0$. Bu berilganga zid. Farazimiz noto‘g‘ri va teorema isbot bo‘ldi. 👍

Natija. *Faraz qilaylik,* $y(x)$ *funksiya (14.3.5) tenglamaning notrivial yechimi bo‘lsin. Agar* $y(x)$ *yechim* $(a;b)$ *intervalda kamida ikkita nolga ega va ularning biri* x_0 *bo‘lsa* ($y(x_0)=0$)*, u holda* $y(x)$ *yechimning shunday* $x_1\neq x_0$ *noli mavjudki,* (x_0,x_1) $(x_0<x_1)$ *yoki* (x_1,x_0) $(x_1<x_0)$ *intervalda* $y(x)$ *ning noli bo‘lmaydi.*

Bu x_0 va x_1 nollar $y(x)$ yechimning **qo‘shni (ketma-ket kelgan) nollari** deb ataladi.

Quyidagi tenglamalarni qaraylik

$$y''+q_1(x)y=0, \tag{14.3.6}$$

$$u''+q_2(x)u=0; \tag{14.3.7}$$

bu yerda $\{q_1(x),q_2(x)\}\subset C((a;b))$.

Teorema 2 (Shturmning taqqoslash teoremasi). *Faraz qilaylik,* (a,b) *intervalda* $q_1(x)\le q_2(x)$, x_0 *va* x_1 *har esa (14.3.6) tenglamaning notrivial yechimi bo‘lmish* $y=y(x)$ *funksiyaning ketma-ket kelgan ikkita noli bo‘lsin* $(x_0<x_1)$. *U holda (14.3.7) tenglamaning har qanday* $u=u(x)$ *notrivial yechimi* (x_0,x_1)

intervalda kamida bitta nolga ega, yoki aks holda $u(x_0)=u(x_1)=0$ *va* (x_0,x_1) *intervalda* $q_1(x)=q_2(x)$.

Faraz qilaylik, (14.3.7) tenglamaning notrivial $u(x)$ yechimi (x_0,x_1) intervalda nolga ega bo‘lmasin. Demak, $u(x)$ uzluksiz funksiya bu intervalda o‘z ishorasini saqlaydi. Umumiylikni buzmasdan (x_0,x_1) intervalda $u(x)>0$ deb hisoblaymiz
(aks holda $-u(x)$ yechimni qaraymiz).

Teorema shartiga ko‘ra $y(x_0)=y(x_1)=0$ va $\forall x\in(x_0,x_1)$ uchun $y(x)\neq 0$. Umumiylikni buzmasdan (x_0,x_1) intervalda $y(x)>0$ deb hisoblaymiz. Demak, xususan, (x_0,x_1) intervalda $yu>0$.

(14.3.6) va (14.3.7) dan

$$y''u-u''y=(q_2(x)-q_1(x))yu \qquad (14.3.8)$$

tenglik ravshan. Uni x_0 dan x_1 gacha integrallaylik:

$$\int_{x_0}^{x_1} y''u dx-\int_{x_0}^{x_1} u''y dx=\int_{x_0}^{x_1}\left[(q_2(x)-q_1(x))\right]y(x)u(x)dx$$

Chap tomondagi integrallarni bo‘laklab integrallaymiz. $y(x_0)=y(x_1)=0$ ekanligini hisobga olib quyidagini topamiz:

$$y'(x_1)u(x_1)-y'(x_0)u(x_0)=\int_{x_0}^{x_1}\left[(q_2(x)-q_1(x))\right]y(x)u(x)dx. \qquad (14.3.9)$$

(x_0,x_1) da $y(x)>0$ bo‘lgani uchun, $y'(x_0)\geq 0$. Lekin $y'(x_0)=0$ bo‘la olmaydi, chunki aks holda (a,b) da $y(x)\equiv 0$ bo‘lar edi. Demak, $y'(x_0)>0$. Shunga o‘xshash $y'(x_1)<0$ ekanligi isbotlanadi. Farazimizga ko‘ra (x_0,x_1) intervalda $u(x)>0$. Bundan u ning uzluksizligiga ko‘ra $u(x_0)\geq 0$ va $u(x_1)\geq 0$ ekanligi kelib chiqadi. Shuning uchun (14.3.9) tenglikning chap tomoni musbat emas, ya’ni $y'(x_1)u(x_1)-y'(x_0)u(x_0)\leq 0$. Teorema shartiga ko‘ra $q_2(x)-q_1(x)\geq 0$ va yuqorida e’tirof etilganiga ko‘ra (x_0,x_1) intervalda $yu>0$. Demak, (14.3.9) tenglikning o‘ng tomoni manfiy emas. Agar $u(x_0)=0$, $u(x_1)=0$ va (x_0,x_1) intervalda $q_1(x)=q_2(x)$ tengliklarning birortasi o‘rinli bo‘lmasa, ziddiyat hosil bo‘ladi.

Misol. Ushbu

$$u'' + \left(1 + \frac{1}{x^2}\right)u = 0 \qquad (14.3.10)$$

tenglamani

$$y'' + y = 0$$

tenglama bilan taqqoslaylik. Oxirgi tenglamaning $y = \sin x$ yechimi $(0;+\infty)$ intervalda $x_k = k\pi$, $k \in \mathbb{N}$ nollarga ega. Shturm teoremasiga ko‘ra (14.3.10) tenglamaning har qanday yechimi $[k\pi;(k+1)\pi]$, $k \in \mathbb{N}$, segmentda kamida bitta nolga ega. ✍

Teorema 3 (nollarning navbatlashishi to‘g‘risidagi Shturm teoremasi). *Faraz qilaylik,* $y_1(x)$ *va* $y_2(x)$ *funksiyalar (14.3.5) tenglamaning chiziqli erkli yechimlari,* x_0 *va* x_1 *har esa bu* $y_1(x)$ *yechimning qo‘shni nollari bo‘lsin. U holda* (x_0, x_1) *intervalda* $y_2(x)$ *yechim yagona nolga ega, ya’ni (14.3.5) tenglama chiziqli erkli yechimlarining nollari navbatlashadi.*

🗝 Shturmning taqqoslash teoremasidan $y_2(x)$ yechimning $[x_0, x_1]$ segmentda kamida bitta nolga ega ekanligi kelib chiqadi. x_0 va x_1 nuqtalar $y_2(x)$ ning nollari bo‘la olmaydi, chunki aks holda vronskian

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$x = x_0$ yoki $x = x_1$ nuqtada nolga aylanardi va, demak, $y_1(x)$ va $y_2(x)$ yechimlar chiziqli bog‘liq bo‘lar edi.

Shunday qilib, $y_2(x)$ yechim (x_0, x_1) intervalda kamida bitta nolga ega. Endi shu nolning yagona ekanligini isbotlaymiz.

Faraz qilaylik, $y_2(x)$ yechim (x_0, x_1) intervalda kamida ikkita nolga ega bo‘lsin. U holda $y_2(x)$ yechimning (x_0, x_1) intervalda joylashgan qo‘shni x_0', x_1' nollari mavjud. Shturm teoremasiga ko‘ra $[x_0', x_1']$ segmentda $y_1(x)$ yechimning kamida bitta noli mavjud. Bu x_0, x_1 har $y_1(x)$ ning qo‘shni ikkita noli ekanligiga zid. Demak, farazimiz noto‘g‘ri va (x_0, x_1) intervalda $y_2(x)$ ning bir dona noli bor xolos. ✍

Teorema 4 (Knezer). *Agar (14.3.5) tenglamada* $q(x)$ *funksiya uchun* $(x_0;+\infty)$ $(x_0>0)$ *intervalda* $0<q(x)\le 1/(4x^2)$ *tengsizlik o'rinli bo'lsa, u holda (14.3.5) tenglamaning ixtiyoriy notrivial yechimi* $(x_0;+\infty)$ *intervalda ko'pi bilan bitta nolga ega bo'ladi. Agar* $(x_0;+\infty)$ *da* $q(x)\ge(1+\varepsilon)/(4x^2)$ $(0<\varepsilon-const)$ *tengsizlik o'rinli bo'lsa, u holda (14.3.5) tenglamaning ixtiyoriy notrivial yechimi* $(x_0;+\infty)$ *intervalda cheksiz ko'p nollarga ega bo'ladi.*

🗝 Ushbu

$$y''+\frac{1}{4x^2}y=0 \qquad (14.3.11)$$

tenglamani qaraylik. Bu tenglama $x>0$ da $y=\sqrt{x}$ yechimga ega.

Agar $(x_0;+\infty)$ $(x_0>0)$ intervalda $q(x)\le 1/(4x^2)$ bo'lsa, u holda Shturm teoremasiga ko'ra (14.3.10) tenglama ixtiyoriy yechimining qo'shni ikkita noli orasida (14.3.11) tenglama har qanday notrivial yechimining kamida bitta noli yotadi. Agar (14.3.10) tenglamaning biror yechimi $(x_0;+\infty)$ intervalda ikkita nolga ega bo'lganida edi, u holda (14.3.11) tenglamaning $y=\sqrt{x}$ yechimi bu nollar orasida kamida bir marta nolga aylangan bo'lar edi, lekin $y=\sqrt{x}$ funksiya $(x_0;+\infty)$ $(x_0>0)$ oraliqda nolga teng bo'la olmaydi.

Endi $(x_0;+\infty)$ intervalda $\frac{1+\varepsilon}{4x^2}\le q(x)$, $x>x_0$, $\varepsilon>0$, bo'lgan holni qaraylik. Bu holda taqqoslash uchun ushbu

$$y''+\frac{1+\varepsilon}{4x^2}y=0 \qquad (14.3.12)$$

tenglamani qaraymiz. Osongina tekshirib ko'rish mumkinki, bu tenglama $x>x_0$ oraliqda $y=\sqrt{x}\cdot\cos(\frac{\sqrt{\varepsilon}}{2}\ln x)$ yechimga ega (bu tenglamaning umumiy yechimini $t=\ln x$ almashtirish yordamida topish mumkin). Ravshanki, bu yechim cheksiz ko'p nollarga ega. Shturm teoremasiga ko'ra (14.3.12) tenglamaning $y=\sqrt{x}\cdot\cos(\frac{\sqrt{\varepsilon}}{2}\ln x)$ yechimi nollari orasida (14.3.5) tenglamaning har qanday yechimining kamida bitta noli mavjud. Demak, (14.3.5) tenglamaning har qanday yechimi $x>x_0$ oraliqda cheksiz ko'p nollarga ega. 👍

Masalalar

1. Agar (14.3.1) tenglamada $\{a_0(x),\, a_1(x),\, a_1'(x)\} \subset C((a,b))$ bo‘lsa, u holda $y=y(x)$ noma’lum funksiyani $y=z(x)\exp\left[-\frac{1}{2}\int\limits_{x_0}^{x} a_1(x)ds\right]$ almashtirish yordamida (14.3.1) tenglamani $z''+q(x)z=0$ «sodda» ko‘rinishga keltirish mumkinligini isbotlang. $q(x)$ni aniqlang.

2. Aytaylik, $q(x)\in C\left((a;b)\right)$ va biror $\omega>0$ son uchun

$$q(x)\le \omega^2 \;\left(\text{yoki } q(x)\ge \omega^2\right),\; x\in(a,b),$$

tengsizlik o‘rinli bo‘lsin. U holda $y''+q(x)y=0$ tenglamaning har qanday notrivial yechimining qo‘shni $x_0<x_1$ nollari uchun ushbu

$$x_1-x_0\ge \frac{\pi}{\omega}\;\left(\text{yoki mos ravishda } x_1-x_0\le \frac{\pi}{\omega}\right)$$

baholash o‘rinli bo‘lishini isbotlang.

3. Ushbu

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0,\; x>0\; (\nu=\mathrm{const}\ge 0),$$

Bessel tenglamasi har qanday notrivial yechimining qo‘shni nollari orasidagi masofa $0\le\nu<1/2$ bo‘lganda π dan kichik, $\nu=1/2$ holida π ga teng, $\nu>1/2$ bo‘lganda esa π dan katta. Shuni isbotlang.

4. Ushbu

$$x^2y''+xy'+(x^2-\nu^2)y=0,\; x>0\; (\nu=\mathrm{const}\ge 0),$$

Bessel tenglamasi ixtiyoriy notrivial yechimining ketma-ket x_n nollari orasidagi masofa uchun $\lim\limits_{n\to\infty}(x_n-x_{n-1})=\pi$ bo‘lishini isbotlang.

5. Ushbu $y''+xy=0$ tenglama ixtiyoriy yechimining ketma-ket x_n nollari orasidagi masofa uchun $\lim\limits_{n\to\infty}(x_n-x_{n-1})=0$ bo‘lishini ko‘rsating.

6. Aytaylik, $y=y(x)$ funksiya ushbu

$$y''+q(x)y=0,\; q(x)\in C\left([a,+\infty)\right),$$

tenglamaning notrivial yechimi bo‘lsin. Quyidagilarni isbotlang.

1^0. Agar $q(x)\ge q_0>0,\; x\in[a,+\infty),$ bo‘lsa, $y(x)$ cheksiz ko‘p nollarga ega; $q(x)<0,\; x\in[a,+\infty),$ bo‘lganda esa $y(x)$ ko‘pi bilan bitta nolga ega.

2^0. Agar $\lim\limits_{x\to+\infty} q(x)=q_\infty>0$ bo‘lsa, $y(x)$ning qo‘shni nollari orasidagi masofa nollar $+\infty$ ga intilganda $\pi/\sqrt{q_\infty}$ ga intiladi.

MODUL 15. CHEGARAVIY MASALALAR

§ 15.1. Chegaraviy masala tushunchasi. Bir jinsli chegaraviy masala

Ushbu

$$L[y] \stackrel{def}{=} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x) \qquad (15.1.1)$$

2-tartibli chiziqli differensial tenglamani qaraylik; bu yerda $\{p_2, p_1, p_0, f\} \subset C([a,b])$ va $[a,b]$ da $p_2(x) \geq \omega = \text{const} > 0$ deb hisoblanadi.

Biz (15.1.1) tenglama uchun Koshi masalasi bilan tanishdik; bunda tayinlangan $x_0 \in [a,b]$ nuqtada $y(x_0)$ va $y'(x_0)$ qiymatlar berilgan bo‘ladi va yechim biror $I \ni x_0$ oraliqda izlanadi.

Bu tenglama uchun berilgan $[a,b]$ segmentning chegaralari a va b nuqtalarda ham shartlar qo‘yish mumkin. (15.1.1) tenglamaning ushbu

$$l_1(y,a) \stackrel{def}{=} \alpha_1 y'(a) + \alpha_0 y(a) = \alpha,\ l_2(y,b) \stackrel{def}{=} \beta_1 y'(b) + \beta_0 y(b) = \beta \quad (15.1.2)$$

chiziqli shartlarni qanoatlantiruvchi $y = y(x) \in C^2([a,b])$ yechimini topish masalasini qaraylik; bu yerda $\alpha_1, \alpha_0, \alpha, \beta_1, \beta_0, \beta -$o‘zgarmaslar va $|\alpha_1| + |\alpha_0| \neq 0$, $|\beta_1| + |\beta_0| \neq 0$. (15.1.2) shartlar chegaraviy shartlar, (15.1.1),(15.1.2) masala esa - chegaraviy masala deb ataladi va quyidagicha yoziladi:

$$\begin{cases} L[y] = f(x), \\ l_1(y,a) = \alpha, l_2(y,b) = \beta. \end{cases} \qquad (15.1.1),(15.1.2)$$

Shuni ta’kidlaylikki, (15.1.2) chegaraviy shartlardan $y(x)$ va $y'(x)$ qiymatlar bir vaqtda na $x = a$, na $x = b$ nuqtada topiladi. Shuning uchun (15.1.1),(15.1.2) chegaraviy masala Koshi masalasiga bevosita keltirilmaydi.

(15.1.2) chegaraviy shartlar ajralgan chegaraviy shartlar deb ataladi: birinchi shart $x = a$, ikkinchisi esa $x = b$nuqtada qo‘yilgan. (15.1.1) tenglama uchun ajralmagan chegaraviy shartlar ham qo‘yilishi mumkin. Masalan, davriylik shartlari:

$$y(a) = y(b),\ y'(a) = y'(b).$$

(15.1.1),(15.1.2) chegaraviy masala yechimining mavjudligi va yagonaligi nazariya uchun ham, amaliyot uchun ham katta ahamiyatga ega. Variatsion hisob va matematik fizikaning bir qancha masalalari chegaraviy masalalarni yechishga keltiriladi.

Agar (15.1.2) chegaraviy shartlarda $\alpha_1 = \beta_1 = 0$ bo‘lsa, mos shartlar I tur (tip) shartlar (chegarada noma’lum funksiya qiymatlari berilgan); agar $\alpha_0 = \beta_0 = 0$ bo‘lsa, $-$II tur (chegarada noma’lum funksiya hosilasining qiymatlari berilgan); agar α_i va β_i har noldan farqli bo‘lsa, $-$III tur (aralash) shartlar deb ataladi. Bu shartlarga mos chegaraviy masalalar esa I, II va III chegaraviy masalalar deb yuritiladi.

Shunday qilib,

I chegaraviy masala:
$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \\ y(a) = \alpha/\alpha_0,\ y(b) = \beta/\beta_0\ ; \end{cases} \tag{I}$$

II chegaraviy masala:
$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \\ y'(a) = \alpha/\alpha_1,\ y'(b) = \beta/\beta_1\ ; \end{cases} \tag{II}$$

III chegaraviy masala:
$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \\ \alpha_1 y'(a) + \alpha_0 y(a) = \alpha\ \left(\alpha_1\alpha_0 \neq 0\right), \beta_1 y'(b) + \\ +\beta_0 y(b) = \beta\ \left(\beta_1\beta_0 \neq 0\right); \end{cases} \tag{III}$$

Bir jinsli bo‘lmagan (15.1.2) chegaraviy shartlarni noma’lum funksiyani almashtirish yordamida bir jinsli ko‘rinishga keltirish mumkin. (15.1.2) chegaraviy shartlarni qanoatlantiruvchi biror $y = u(x)$ funksiyani topaylik. Tekshirib ko‘rib ishonch hosil qilish mumkinki, agar $\alpha_0\beta_0 \neq \dfrac{\beta_0\alpha_1 - \beta_1\alpha_0}{b-a}$ bo‘lsa, $u(x)$ funksiya sifatida biror $u(x) = lx + m$ chiziqli funksiyani olish mumkin; aks holda esa $u(x)$ funksiyani $u(x) = kx^2 + m$ ko‘rinishdagi kvadratik funksiylar orasidan tanlasa bo‘ladi ($k, l, m -$o‘zgarmaslar). Endi (15.1.1),(15.1.2) masaladagi $y = y(x)$ noma’lum funksiya o‘rniga yangi $u(x) + y(x)$noma’lum funksiyani kiritib, (15.1.2) chegaraviy shartlarni bir jinsli, ya’ni

$$l_1(y,a) = 0,\ l_2(y,b) = 0 \tag{15.1.3}$$

ko‘rinishga keltirish mumkin. Bunda (15.1.1) tenglamaning o‘ng tomoni o‘zgaradi xolos:

$$L[y]=g(x)\ \left(p_2(x)y''+p_1(x)y'+p_0(x)y=g(x)\right). \qquad (15.1.4)$$

(15.1.1) va/yoki (15.1.4) chiziqli differensial tenglamaga mos bir jinsli differensial tenglama

$$L[y]=0\ \left(p_2(x)y''+p_1(x)y'+p_0(x)y=0\right) \qquad (15.1.5)$$

ko‘rinishda bo‘ladi.

(15.1.5) bir jinsli tenglamaning (15.1.3) bir jinsli chegaraviy shartlarni qanoatlantiruvchi yechimini topish masalasi, ya’ni

$$\begin{cases} L[y]=0, \\ l_1(y,a)=0,\ l_2(y,b)=0 \end{cases} \qquad (15.1.5),(15.1.3)$$

chegaraviy masala bir jinsli masaladir, chunki (15.1.5) differensial tenglama ham (15.1.3) chegaraviy shartlar ham bir jinsli. Bu masala (15.1.1),(15.1.2) (bir jinslimas) chegaraviy masalaga mos bir jinsli masala deb ataladi. Ravshanki, bir jinsli chegaraviy masala har doim trivial $y(x)\equiv 0$ yechimga ega.

Teorema. Quyidagi alternativa o‘rinli:

yo (15.1.5),(15.1.3) bir jinsli masala yagona trivial yechimga ega, bunda mos (15.1.1),(15.1.2) masala tenglama va chegaraviy shartlardagi o‘ng tomonlarning ixtiyoriy qiymatlarida yagona yechimga ega,

yo (15.1.5),(15.1.3) bir jinsli masala cheksiz ko‘p yechimga ega, bunda mos (15.1.1),(15.1.2) bir jinslimas masala o‘ng tomonlarning ba’zi qiymatlarida birorta ham yechimga ega emas, qolgan barcha qiymatlarida esa cheksiz ko‘p yechimga ega.

$L[y]=0$ (15.1.5) tenglamaning chiziqli erkli yechimlari $y_1=y_1(x)$ va $y_2=y_2(x)$ bo‘lsin. Uning umuniy yechimi

$$y=c_1y_1+c_2y_2 \qquad (15.1.6)$$

ko‘rinishda bo‘ladi. $L[y]=f(x)$ (15.1.1) tenglamaning biror xususiy yechimini $y_{xus}=y_{xus}(x)$ bilan belgilab, uning umumiy yechimini

$$y=c_1y_1+c_2y_2+y_{xus} \qquad (15.1.7)$$

ko‘rinishda ifodalaylik. Bir jinsli (15.1.5) ,(15.1.3) chegaraviy masalani yechish uchun (15.1.6) ni (15.1.3) shartlarga qo‘yib, c_1, c_2 noma’lumlarga nisbatan ushbu

$$\begin{cases} c_1 l_1(y_1,a) + c_2 l_1(y_2,a) = 0, \\ c_1 l_2(y_1,b) + c_2 l_2(y_2,b) = 0 \end{cases} \tag{15.1.7}$$

bir jinsli chiziqli algebraik tenglamalar sistemasini hosil qilamiz.

(15.1.1),(15.1.2) chegaraviy masalaning yechimini topish uchun esa (11) ni (15.1.2) chegaraviy shartlarga qo‘yamiz va c_1, c_2 noma’lumlarga nisbatan

$$\begin{cases} c_1 l_1(y_1,a) + c_2 l_1(y_2,a) = \alpha - l_1(y_{xus},a), \\ c_1 l_2(y_1,b) + c_2 l_2(y_2,b) = \beta - l_2(y_{xus},b) \end{cases} \tag{15.1.8}$$

chiziqli bir jinslimas algebraik tenglamalar sistemasiga kelamiz.

Algebradan ma’lumki, (15.1.8) va mos bir jinsli (15.1.7) chiziqli algebraik sistemalar yechimlarining soni ushbu

$$\Delta = \begin{vmatrix} l_1(y_1,a) & l_1(y_2,a) \\ l_2(y_1,b) & l_2(y_2,b) \end{vmatrix} \tag{15.1.9}$$

determinant qiymatining nolga teng yoki tengmasligi bilan aniqlanadi.

$\Delta \neq 0$ bo‘lganda (15.1.6) bir jinsli sistema ((15.1.5),(15.1.3) bir jinsli masala ham) faqat trivial yechimga ega, (15.1.8) sistema ((15.1.1),(15.1.2) masala) esa o‘ng tomoni ixtiyoriy bo‘lganda ham yagona yechimga ega. $\Delta = 0$ bo‘lganda (15.1.6) bir jinsli sistema ((15.1.5),(15.1.3) bir jinsli masala) notrivial yechimlarga ega, (15.1.8) sistema ((15.1.1),(15.1.2) masala) esa o‘ng tomonning ba’zi qiymatlarida birorta ham yechimga ega emas, qolgan barcha qiymatlarida esa cheksiz ko‘p yechimga ega. 👍

Masalalar

Ushbu

$$y'' + y = 0 \tag{*}$$

tenglama berilgan bo‘lsin. Bu tenglamaning quyidagi I tur chegaraviy shartlarni qanoatlantiruvchi yechimlarini toping:

1. $y(0)=0$, $y(\pi)=0$; **2.** $y(0)=0$, $y(\pi)=1$; **3.** $y(0)=0$, $y(1)=1$;

§ 15.2. Chegaraviy masala yechimining yagonaligi

Bu yerda yechimning yagonaligini o‘z-o‘ziga qo‘shma tenglama uchun integral tengliklar yordamida o‘rganamiz.

Ushbu

$$\mathcal{L}[y] \overset{def}{=} \frac{d}{dx}\left(p(x)\frac{dy}{dx} \right) + q(x)y, \tag{15.2.1}$$

bu yerda

$$\{p(x), p'(x), q(x)\} \in C([a,b]); p(x) \geq \omega = \\ = \text{const} > 0,\ q(x) \leq 0,\ x \in [a,b], \tag{15.2.2}$$

o‘z-o‘ziga qo‘shma operator orqali tuzilgan

$$\mathcal{L}[y] = g(x)\text{, ya'ni } \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = g(x) \tag{15.2.3}$$

chiziqli tenglama uchun (15.1.2) chegaraviy shartlarni qo‘yaylik. Mos bir jinsli tenglama

$$\mathcal{L}[y] = 0 \text{ ya'ni } \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0 \tag{15.2.4}$$

ko‘rinishga ega. Bu (15.2.4) bir jinsli tenglama uchun (15.1.3) bir jinsli chegaraviy shartlarni qo‘yib, mos bir jinsli chegaraviy masala (15.2.4),(15.1.3) ni hosil qilamiz:

$$\begin{cases} \mathcal{L}[y] = 0, \\ l_1(y,a) = 0,\ l_2(y,b) = 0. \end{cases} \tag{15.2.4),(15.1.3}$$

Tushunarliki, § 14.5 dagi teorema 1 L operatorni o‘z-o‘ziga qo‘shma operator $\mathcal{L}$ bilan almashtirganda ham o‘z kuchini saqlaydi.

(15.2.4),(15.1.3) bir jinsli masalaning trivial yechimdan boshqa yechimga ega bo‘lmasligi uchun yetarli shartlar quyidagi teoremada keltirilgan.

Teorema 1. Faraz qilaylik**,** (15.2.2) shartlar bajarilsin. U holda (15.2.4),(15.1.3) bir jinsli chegaraviy masala uchun quyidagi tasdiqlar o‘rinli:

1) agar $\alpha_1 = \beta_1 = 0$ bo‘lsa, bir jinsli I chegaraviy masala faqat trivial yechimga ega;

2) agar $\alpha_0 = \beta_0 = 0$ va (a,b) da $q(x) \not\equiv 0$ bo‘lsa, bir jinsli II chegaraviy masala faqat trivial yechimga ega;

3) agar $\alpha_0 = \beta_0 = 0$ va (a,b) da $q(x) \equiv 0$ bo‘lsa, bir jinsli II chegaraviy masalaning yechimlari o‘zgarmaslardan iborat bo‘ladi;

4) agar $\alpha_1\alpha_0 < 0$ va $\beta_1\beta_0 > 0$ bo‘lsa, bir jinsli III chegaraviy masala faqat trivial yechimga ega.

🗝 Aytaylik, $y = y(x) \in C^2([a,b])$ funksiya (15.2.4),(15.1.3) bir jinsli chegaraviy masalaning yechimi bo‘lsin. Ravshanki,

$$y\mathcal{L}[y]=y\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y^2=0,\ x\in[a,b].$$

Bu ayniyatni x bo‘yicha a dan b gacha integrallab topamiz:

$$\int_a^b y(x)\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dx+\int_a^b q(x)y^2(x)dx=0.$$

Oxirgi tenglikning chap tomonidagi birinchi integralni bo‘laklab integrallaymiz va quyidagi tenglikni hosil qilamiz:

$$\int_a^b\left(p(x)y'^2(x)-q(x)y^2(x)\right)dx+p(a)y(a)y'(a)-\\ -p(b)y(b)y'(b)=0. \qquad (15.2.5)$$

Agar $\alpha_1=\beta_1=0$ bo‘lsa, $y(a)=y(b)=0$ va $q(x)\le 0$ shartga ko‘ra (15.2.5) dan

$$\int_a^b p(x)y'^2(x)dx=0 \qquad (15.2.5)$$

tenglikni topamiz. Bu tenglikdan $p(x)\ge\omega=\text{const}>0$ shartga ko‘ra $y'(x)=0,\ x\in[a,b],$ ekanligini ko‘ramiz. Demak, $y(x)\equiv c=\text{const},$ $x\in[a,b].$ $y(x)\in C([a,b])$ va $y(a)=y(b)=0$ bo‘lgani uchun $c=0,$ ya’ni $y(x)\equiv 0,\ x\in[a,b].$ Bu bir jinsli I chegaraviy masala faqat trivial yechimga ega ekanligini isbotlaydi. 1) qism isbot bo‘ldi.

Agar $\alpha_0=\beta_0=0$ bo‘lsa, $y'(a)=y'(b)=0$chegaraviy shartlar qanoatlangan va (15.2.5) tenglikka ko‘ra $q(x)\le 0$ bo‘lganda yana (15.2.5) munosabat hosil bo‘ladi. Bundan $y(x)\equiv c=\text{const},\ x\in[a,b],$ ekanligi kelib chiqadi. $y(x)\equiv c$ni (15.2.4) tenglamaga qo‘yib, $q(x)c\equiv 0,\ x\in[a,b],$ shartga kelamiz. Bundan $q(x)\not\equiv 0$ bo‘lganda $c=0$ ekanligi kelib chiqadi, ya’ni yechim $y(x)\equiv c=0.$ 2) qism isbotlandi. $q(x)\equiv 0$ bo‘lganda yechim $y(x)\equiv c=\text{const},\ x\in[a,b]$ 3) qism isbotlandi.

Endi $\alpha_0\beta_0<0$ va $\alpha_1\beta_1>0$ bo‘lgan holni qaraylik. Bu holda (15.2.5) tenglikdan (15.1.3) chegaraviy shartlarga ko‘ra quyidagini topamiz:

$$\int_a^b\left(p(x)y'^2(x)-q(x)y^2(x)\right)dx-p(a)\frac{\alpha_0}{\alpha_1}y^2(a)+p(b)\frac{\beta_0}{\beta_1}y^2(b)=0.$$

Bundan $y(x)\equiv c=\text{const}$, $x\in[a,b]$, va $y(a)=y(b)=0$ ekanligi kelib chiqadi. $y(x)\in C([a,b])$ bo‘lgani uchun $y(x)\equiv c=0$. 4) qism isbotlandi. ☝

§ 15.3. Grin funksiyasi

Bu paragrafda chegaraviy masalani **Grin funksiyasi** yordamida o‘rganamiz.

Ushbu

$$\begin{cases} L[y]=g(x), \\ l_1(y,a)=0,\ l_2(y,b)=0 \end{cases} \qquad (15.1.4),(15.1.3)$$

chegaraviy masala uchun Grin funksiyasi deb, quyidagi uchta xossaga ega bo‘lgan $G(x,\xi)$ funksiyaga aytiladi:

1. $G(x,\xi)$ funksiya $x\in[a,b]$, $\xi\in[a,b]$ bo‘lganda aniqlangan va uzluksiz: $G(x,\xi)\in C([a,b]\times[a,b])$.

2. Tayinlangan $\xi\in(a,b)$ uchun $y(x)=G(x,\xi)$ funksiya x bo‘yicha $x\neq\xi$ nuqtalarda mos bir jinsli $L[y]=0$ tenglamani, $x=a$ va $x=b$ nuqtalarda esa (15.1.3) chegaraviy shartlarni qanoatlantiradi.

3. Tayinlangan $\xi\in(a,b)$ uchun $G(x,\xi)$ funksiyaning birinchi tartibli hosilasi $x=\xi$ nuqtada sakrashga ega va

$$\frac{\partial G(x,\xi)}{\partial x}\Big|_{x=\xi+0}-\frac{\partial G(x,\xi)}{\partial x}\Big|_{x=\xi-0}=\frac{1}{p_2(\xi)}\ . \qquad (15.3.1)$$

Terema 1 (***Grin funksiyasining mavjudligi to‘g‘risida***). *Agar ushbu*

$$\begin{cases} L[y]=0, \\ l_1(y,a)=0,\ l_2(y,b)=0 \end{cases} \qquad (15.1.5),(15.1.3)$$

bir jinsli chegaraviy masala faqat trivial yechimga ega bo‘lsa, u holda (15.1.4),(15.1.3) *masala uchun Grin funksiyasi mavjud va u quyidagi ko‘rinishga ega:*

$$G(x,\xi)=\begin{cases} c_1(\xi)y_1(x), \text{ agar } a\le x\le\xi \text{ bo‘lsa}, \\ c_2(\xi)y_2(x), \text{ agar } \xi\le x\le b \text{ bo‘lsa}, \end{cases} \qquad (15.3.2)$$

bu yerda y_1 va y_2 − bir jinsli tenglama (15.1.5) *ning mos ravishda $l_1(y_1,a)=0$ va $l_2(y_2,b)=0$ shartlarni qanoatlantiruvchi notrivial yechimlari,* $c_1(\xi)$ va $c_2(\xi)$*lar esa ushbu*

$$\begin{cases} c_1(\xi)y_1(\xi) - c_2(\xi)y_2(\xi) = 0, \\ c_2(\xi)y_2'(\xi) - c_1(\xi)y_1'(\xi) = 1/p_2(\xi) \end{cases} \tag{15.3.3}$$

sistemadan aniqlanadi.

Izoh. (15.3.3) dagi birinchi shart (15.3.2) Grin funksiyasining uzluksizligini, ikkinchi shart esa uning (15.3.1) xossasini ta'minlaydi.

y_1 va y_2 funksiyalarni quyidagi Koshi masalalarining yechimlari sifatida tanlaylik:

$$L[y_1] = 0,\ y_1(a) = \alpha_1,\ y_1'(a) = -\alpha_0;\ L[y_2] = 0,\ y_2(b) = \beta_1,\ y_1'(b) = -\beta_0.$$

Ravshanki, $l_1(y_1, a) = 0$ va $l_2(y_2, b) = 0$. Bundan tashqari, y_1 va y_2 yechimlar notrivial ($|\alpha_1| + |\alpha_0| \neq 0$, $|\beta_1| + |\beta_0| \neq 0$) va chiziqli erkli, chunki aks holda $y_2(x) \equiv cy_1(x)$

($c \neq 0,\ l_1(y_2, a) = cl_1(y_1, a) = 0$). Demak, teoremaning shartiga zid ravishda (15.1.5),(15.1.3) masala y_1 bilan birgalikda y_2 notrivial yechimga ham ega bo'lardi. Shunday qilib, $L[y] = 0$ tenglamaning ixtiyoriy yechimi $y = c_1y_1 + c_2y_2$ ko'rinishda bo'ladi. Grin funksiyasinng ta'rifidan uning (15.3.2) ko'rinishda bo'lishi kerakligi kelib chiqadi. y_1 va y_2 yechimlar chiziqli erkli bo'lgani uchun ularning vronskiani noldan farqli, va demak, (15.3.3) sistema $c_1(\xi)$ va $c_2(\xi)$larni bir qiymatli aniqlaydi, chunki mos determinant

$$\begin{vmatrix} y_1(\xi) & -y_2(\xi) \\ -y_1'(\xi) & y_2'(\xi) \end{vmatrix} = W[y_1(\xi), y_2(\xi)] \neq 0.$$

Ravshanki, (15.3.3) sistemadan topilgan $c_1(\xi)$ va $c_2(\xi)$ funksiyalar $\xi \in [a,b]$da uzluksiz. Demak, (15.3.3) dagi birinchi shartga ko'ra (15.3.2) formula bilan aniqlangan funksiya $G(x,\xi) \in C([a,b] \times [a,b])$. Tayinlangan $\xi \in (a,b)$ uchun $G(x,\xi)$ funksiya x bo'yicha $x \neq \xi$ nuqtalarda ikki marta differensiallanuvchi va uning birinchi tartibli hosilasi $x = \xi$ nuqtada (15.3.3) dagi 2-formulaga ko'ra qiymati $1/p_2(\xi)$ ga teng bo'lgan sakrashga ega.

Eslatma. Teoremaning shartlari bajarilganda Grin funksiyasi bir qiymatli aniqlanadi. Agar y_1 va y_2 yechimlar c_3y_1 va c_4y_2 bilan almashtirilsa, (15.3.3) dan ravshanki, (15.3.2) dagi $G(x,\xi)$ funksiya o'zgarmaydi.

Bu yerda yana shuni e'tirof etaylikki, biz teoremani konstruktiv isbotladik, ya'ni Grin funksiyasini qurish usulini keltirdik.

Yuqorida qurilgan Grin funksiyasi (15.1.4),(15.1.3) chegaraviy masalaning yechimini topishga imkon beradi.

Teorema 2. *Faraz qilaylik,* (15.1.5),(15.1.3) *masala faqat trivial yechimga ega,* $G(x,\xi)$ *uning Grin funksiyasi va* $g(x)\in C([a;b])$ *bo'lsin. U holda ushbu*

$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x), \\ \alpha_1 y'(a) + \alpha_0 y(a) = 0 \ \ (|\alpha_1| + |\alpha_0| \neq 0), \\ \beta_1 y'(b) + \beta_0 y(b) = 0 \ \ (|\beta_1| + |\beta_0| \neq 0), \end{cases} \tag{15.3.4}$$

chegaraviy masalaning yagona yechimi

$$y(x) = \int_a^b G(x,\xi)g(\xi)d\xi \tag{15.3.5}$$

formula bilan ifodalanadi.

⊶Biz (15.3.5) formula bilan aniqlangan funksiya (15.3.4) masalaning yechimi ekanligini ko'rsatishimiz kerak.

Ixtiyoriy $x\in(a,b)$ nuqta uchun

$$y(x) = \int_a^x G(x,\xi)g(\xi)d\xi + \int_x^b G(x,\xi)g(\xi)d\xi .$$

Bundan (integralni differensiallash haqidagi Leybnits formulasiga ko'ra)

$$y'(x) = G(x,\xi)g(\xi)\Big|_{\xi=x-0} + \int_a^x \frac{G(x,\xi)}{\partial x}g(\xi)d\xi - G(x,\xi)g(\xi)\Big|_{\xi=x+0} +$$

$$+\int_x^b \frac{G(x,\xi)}{\partial x}g(\xi)d\xi = \int_a^x \frac{G(x,\xi)}{\partial x}g(\xi)d\xi + \int_x^b \frac{G(x,\xi)}{\partial x}g(\xi)d\xi , \tag{15.3.6}$$

chunki $G(x,\xi)$ uzluksiz funksiya: $G(x,\xi)\Big|_{\xi=x-0} = G(x,\xi)\Big|_{\xi=x+0}$.

Endi ikkinchi tartibli hosilani hisoblaymiz ((15.3.6) dan foydalanamiz):

$$y'' = \int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + \left[\frac{\partial G}{\partial x}\Big|_{\xi=x-0} - \frac{\partial G}{\partial x}\Big|_{\xi=x+0} \right] g(x) =$$

$$= \int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + \frac{1}{p_2(x)} g(x), \qquad (15.3.7)$$

chunki (15.3.1) ga ko'ra $\frac{\partial G}{\partial x}\Big|_{\xi=x-0} - \frac{\partial G}{\partial x}\Big|_{\xi=x+0} = \frac{1}{p_2(x)}$.

(15.3.6) va (15.3.7) formulalarga ko'ra (15.3.5) formula bilan aniqlangan funksiya $L[y] = g(x)$ tenglamani qanoatlantirishi kelib chiqadi:

$$L[y] \equiv p_2(x)y'' + p_1(x)y' + p_0(x)y =$$

$$= p_2(x)\int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + g(x) + p_1(x)\int_a^b \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi +$$

$$+ p_0(x)\int_a^b G(x,\xi)g(\xi)d\xi = g(x).$$

Endi chegaraviy shartlarning qanoatlanganligini tekshiramiz. (15.3.5) formulalarga ko'ra

$$\alpha_1 y'(x) + \alpha_0 y(x) = \alpha_1 \int_a^x \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi + \alpha_1 \int_x^b \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi +$$

$$+ \alpha_0 \int_a^x G(x,\xi)g(\xi)d\xi + \alpha_0 \int_x^b G(x,\xi)g(\xi)d\xi$$

Bu tenglikda $x \to a+0$ deb limitga o'tamiz va Grin funksiyasining 2- xossasiga ko'ra (15.3.4) chegaraviy shartlar birinchisining bajarilishini isbotlaymiz:

$$\alpha_1 y'(a) + \alpha_0 y(a) = \alpha_1 \int_a^b \frac{\partial G(a,\xi)}{\partial x} g(\xi)d\xi + \alpha_0 \int_a^b G(a,\xi)g(\xi)d\xi =$$

$$= \int_a^b \left(\alpha_1 \frac{\partial G(a,\xi)}{\partial x} + \alpha_0 G(a,\xi)\right)g(\xi)d\xi = \int_a^b 0 \cdot g(\xi)d\xi = 0$$

Shunga o'xshash $\beta_1 y'(b) + \beta_0 y(b) = 0$ ekanlgi ham tekshiriladi. ✍

Misol. Ushbu

$$y'' + y = g(x) \ \left(g(x) \in C\left([0, \pi/2]\right)\right), \ y(0) = 0, \ y(\pi/2) = 0,$$

chegaraviy masala uchun Grin funksiyasini quring va yechimini yozing.

🗝 Mos bir jinsli chegaraviy masala

$$y'' + y = 0,\ y(0) = 0,\ y(\pi/2) = 0,$$

faqat trivial yechimga ega (tekshirib ko‘ring), ya’ni Grin funksiyasining mavjudlik sharti bajariladi. Bir jinsli $y'' + y = 0$ tenglamaning $y_1 = \sin x$ va $y_2 = \cos x$ yechimlari $y_1(0) = 0$ va $y_2(\pi/2) = 0$ shartlarni qanoatlantiradi. Demak, (15.3.2) ga ko‘ra izlanayotgan Grin funksiyasi

$$G(x,\xi) = \begin{cases} c_1(\xi)\sin x, \text{ agar } 0 \le x \le \xi \text{ bo‘lsa}, \\ c_2(\xi)\cos x, \text{ agar } \xi \le x \le \pi/2 \text{ bo‘lsa}, \end{cases}$$

ko‘rinishga ega. (15.3.3) sistemadan $c_1(\xi)$ va $c_2(\xi)$larni topishimiz kerak. Qaralayotgan holda bu sistema quyidagi ko‘rinishda:

$$\begin{cases} c_1(\xi)\sin\xi - c_2(\xi)\cos\xi = 0, \\ -c_2(\xi)\sin\xi - c_1(\xi)\cos\xi = 1. \end{cases}$$

Bu sistemani yechib, $c_1(\xi) = -\cos\xi,\ c_2(\xi) = \sin\xi$ ekanligini topamiz. Demak, berilgan
chegaraviy masala uchun Grin funksiyasi

$$G(x,\xi) = \begin{cases} -\cos\xi\sin x, \text{ agar } 0 \le x \le \xi \text{ bo‘lsa}, \\ -\sin\xi\cos x, \text{ agar } \xi \le x \le \pi/2 \text{ bo‘lsa}, \end{cases}$$

formula bilan beriladi. Berilgan masala yechimi endi (15.3.5) formulaga ko‘ra topiladi:

$$y(x) = \int_0^{\pi/2} G(x,\xi)g(\xi)d\xi = -\int_0^x \cos x g(\xi)\sin\xi d\xi - \int_x^{\pi/2} \sin x g(\xi)\cos\xi d\xi .$$ 👍

§ 15.4. Shturm-Liuvill masalasi haqida ma’lumot

Quyidagi λ parametrli bir jinsli chegaraviy masalani qaraylik:

$$\begin{cases} L[y] - \lambda y = 0, \\ \alpha_1 y'(a) + \alpha_0 y(a) = 0, \\ \beta_1 y'(b) + \beta_0 y(b) = 0 \end{cases}$$

Bu masala, ravshanki, λ parametrning har qanday qiymatida trivial $y = 0$ yechimga ega. Agar bu masala berilgan λ uchun notrivial yechimga ega bo‘lsa, bu λ qaralayotgan masalaning xos soni, notrivial yechim esa (shu xos songa mos) xos funksiyasi deyiladi. Barcha xos sonlar va mos xos funksiyalarni topish **Shturm-Liuvill masalasi** deb ataladi. Bu masalani o‘rganish matematik fizikada katta ahamiyatga ega.

Misol. Ushbu

$$\begin{cases} y'' - \lambda y = 0, \\ y'(0) = 0, \\ y'(\pi) = 0 \end{cases}$$

masalaning xos sonlari va xos funksiyalarini toping.

🗝 $y'' - \lambda y = 0$ o‘zgarmas koeffitsientli tenglamaning umumiy yechimi osongina quriladi:

$\lambda > 0$ bo‘lganda $y = c_1 \exp\left(\sqrt{\lambda} x\right) + c_2 \exp\left(-\sqrt{\lambda} x\right)$;

$\lambda = 0$ bo‘lganda $y = c_1 + c_2 x$.

Qo‘yilgan $y'(0) = 0, y'(\pi) = 0$ shartlardan ikkala holda ham $c_1 = c_2 = 0$ ekanligini, ya’ni yechimning trivial bo‘lishini topamiz. Demak, notrivial yechimlar $\lambda < 0$ bo‘lganda mavjud bo‘lishi mumkin xolos. $\lambda = -\mu^2, \mu > 0,$ deylik. U holda berilgan tenglamaning umumiy yechimi $y = c_1 \cos \mu x + c_2 \sin \mu x$ ko‘rinishda ifodalanadi. Chegaraviy shartlarga ko‘ra yechim noldan farqli bo‘lishi uchun $c_2 = 0$ va $\sin \mu\pi = 0 \Rightarrow \mu = \mu_k = k, k \in \mathbb{N},$ bo‘lishi kerakligini topamiz. Demak, xos sonlar $\lambda = \lambda_k = -\mu_k^2 = -k^2$, mos xos funksiyalar esa $y = y_k = c \cos kx,\ k = 1, 2, \ldots$. ✍

Izoh. Grin funksiyasi umumlashgan ma’noda ushbu

$$L_x[G(x,\xi)] = \delta(x - \xi),\ a < x < b,$$

tenglamani qanoatlantiradi. Bu yerda $\delta(x-\xi)$ – Dirakning delta-funksiyasi. Uni $x = \xi$ nuqtada qo‘yilgan intensivligi 1 ga teng bo‘lgan impuls deb tasavvur qilish mumkin. $\delta(x-\xi)$ ni noldan farqli qiymatlari ξ nuqtaning kichik atroflarida joylashgan uzluksiz va

$$\lim_{n\to\infty} \varphi_n(x-\xi) = 0, x \neq \xi, \int_a^b \varphi_n(x-\xi)dx = 1$$

shartlarni qanoatlantiruvchi $\varphi_n(x-\xi)$ funksiyalar limiti deb tushunish kerak. U holda ixtiyoriy $f(x) \in C\left([a,b]\right)$ funksiya uchun

$$\lim_{n\to\infty}\int_a^b \varphi_n(x-\xi)\, f(x)dx = \int_a^b \varphi_n(x-\xi)\, f(\xi)dx = f(\xi)$$

bo‘ladi. Delta-funksiya umumlashgan funksiyadir. Umumlashgan funksiyalar nazariyasi matematik fizikaning zamonaviy qurollaridan biri hisoblanadi.

Masalalar

1. Agar $y(a)=y(b)=0$ va barcha $x\in(a,b)$ lar uchun $y(x)>0$ va $y''(x)+y(x)>0$ bo‘lsa, $b-a>\pi$ bo‘lishini isbotlang.

2. Masala uchun Grin funksiyasini quring:

$$xy''+y'=g(x),\ y'(1)=0, y(2)=0.$$

3. Masalaning xos son va xos funksiyalarini toping (Shturm-Liuvill masalasini yeching): $(1+x)^2y''+2(1+x)y'=\lambda y\ \ (0<x<1),\ y(0)=y(1)=0$.

4. Berilgan masalani yechishni Grin funksiyasi yordamida integral tenglamani yechishga keltiring:

$$e^x y''+e^x y'=\lambda y,\ y(0)-y'(0)=0,\ y(1)+y'(1)=0.$$

5. Agar $x^2y''+2xy'-2y=f(x)$ tenglamaning yechimi $x\to 0+$ va $x\to+\infty$ da chegaralangan bo‘lsa, shu yechimni va uning hosilasini, $0\le f(x)\le m$, $f(x)\in C((0,+\infty))$, deb faraz qilib, baholang.

MODUL 16. XUSUSIY HOSILALI BIRINCHI TARTIBLI TENGLAMALAR

§ 16.1. Umumiy ma'lumotlar

Asosiy ta'riflar. Qisqalik uchun

$$\boldsymbol{x} = (x_0, x_1, ..., x_n)^T \in \mathbb{R}^{1+n}, \quad \boldsymbol{p} = (p_0, p_1, ..., p_n)^T \in \mathbb{R}^{1+n}, \quad u \in \mathbb{R}$$

belgilashlarni kiritaylik. Ushbu

$$F(\boldsymbol{x}, u, \boldsymbol{p}) \equiv F(x_0, x_1, ..., x_n, u, p_0, p_1, ..., p_n)$$

haqiqiy funksiya $(\boldsymbol{x}, u, \boldsymbol{p})$ vektor o'zgaruvchi bo'yicha biror $G \subset \mathbb{R}^{2n+3}$ sohada aniqlangan, $p_0, p_1, ..., p_n$ o'zgaruvchilar bo'yicha o'zining birinchi tartibli xusisiy hosilalari bilan birgalikda uzluksiz $(\in C^1)$ va G da $\sum_{i=0}^{n} \left| \frac{\partial F}{\partial p_i} \right| \neq 0$ ham bo'lsin.

Ushbu

$$F(x_0, x_1, ..., x_n, u, \frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}) = 0 \qquad (16.1.1)$$

tenglama $u = u(x_0, x_1, ..., x_n)$ noma'lum funksiyaga nisbatan **birinchi tartibli xususiy hosilali differensial tenglama** deyiladi.

Agar $u = u(x_0, x_1, ..., x_n)$ funksiya $D \subset \mathbb{R}^{1+n}$ sohada $\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}$ uzluksiz hosilalarga ega, ya'ni $u \in C^1(D, \mathbb{R})$ bo'lib, (16.1.1) tenglamani ayniyatga aylantirsa (qanoatlantirsa), u holda shu u funksiya (16.1.1) tenglamaning (D sohada aniqlangan) yechimi deyiladi. Tabiiyki, bu holda

$$\forall\, \boldsymbol{x} = (x_0, x_1, ..., x_n)^T \in D \text{ uchun } (x_0, x_1, ... x_n, u, \frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})^T \in G$$

bo'lishi ham kerak.

Misol 1. $u = xy + y \cdot \sqrt{x^2 + 1}$ funksiya ushbu

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0$$

ikki o'zgaruvchining $z = z(x, y)$ noma'lum funksiyasiga nisbatan xususiy hosilali differensial tenglamaning yechimi ekanligini asoslang.

🗝Bunga kerakli hisoblashlarni bajarib ishonch hosil qilamiz:

$$\frac{\partial u}{\partial x}=y+\frac{yx}{\sqrt{x^2+1}}\,,\ \frac{\partial u}{\partial y}=x+\sqrt{x^2+1}\ \left(u\in C^1(\mathbb{R}^2)\right)\,,$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}=x\Big(y+\frac{yx}{\sqrt{x^2+1}}\Big)+y\Big(x+\sqrt{x^2+1}\Big)-$$

$$-\Big(y+\frac{yx}{\sqrt{x^2+1}}\Big)\Big(x+\sqrt{x^2+1}\Big)=0.$$ 👍

$u=u(\boldsymbol{x})$ yechimning $(\boldsymbol{x},u)$ o‘zgaruvchilar fazosidagi ($\mathbb{R}^{2+n}$ fazodagi) grafigi tenglamaning integral sirti deb ataladi.

Agar $F(x_0,x_1,...,x_n,u,p_0,p_1,...,p_n)$ funksiya $p_0,p_1,...,p_n$ o‘zgaruvchilarga nisbatan chiziqli (aniqrog‘i, affin), ya’ni

$$F(\boldsymbol{x},u,\boldsymbol{p})=\sum_{i=0}^{n}a_i(\boldsymbol{x},u)p_i-b(\boldsymbol{x},u)$$

bo‘lsa, (16.1.1) tenglama kvazichiziqli tenglama deb ataladi. Demak, birinchi tartibli xususiy hosilali kvazichiziqli differensial tenglamaning umumiy ko‘rinishi quyidagicha:

$$\sum_{i=0}^{n}a_i(\boldsymbol{x},u(\boldsymbol{x}))\frac{\partial u}{\partial x_i}=b(\boldsymbol{x},u(\boldsymbol{x}))\qquad(16.1.2)$$

Agar $F(\boldsymbol{x},u,\boldsymbol{p})$ funksiya u va $p_0,p_1,...,p_n$ o‘zgaruvchilarning chiziqli (affin) funksiyasidan iborat bo‘lsa, u holda (16.1.1) tenglama chiziqli tenglama deyiladi.

Ciziqli tenglama

$$\sum_{i=0}^{n}a_i(\boldsymbol{x})\frac{\partial u}{\partial x_i}=b(\boldsymbol{x})u+c(\boldsymbol{x})\qquad(16.1.3)$$

ko‘rinishga ega.

Yechimlar majmuasi haqida umumiy ma’lumotlar. Birinchi tartibli oddiy differensial tenglamaning barcha yechimlar majmuasi umumiy holda (maxsus yechimlardan tashqari) bir parametrli yechimlar oilasidan iborat.

Birinchi tartibli xususiy hosilali tenglama holidagi vaziyat murakkabroq bo‘ladi. Bu holdagi tenglamaning yechimlari ba’zi maxsus yechimlarni hisobga olmaganda erkli o‘zgaruvchilardan tashqari ixtiyoriy funksiyaga ham bog‘liq bo‘ladi. Bu ixtiyoriy funksiyaning argumentlari soni tenglama yechimining argumentlari sonidan bittaga kam bo‘ladi (umumiy holda).

Misollar qaraylik.

Misol 1. $u = u(x, y)$ ikki argumentning funksiyasiga nisbatan

$$\frac{\partial u}{\partial y} = 0 \qquad (16.1.4)$$

tenglama berilgan bo‘lsin. Bu tenglama yechimning y ga bog‘liq emasligini anglatadi. Demak, berilgan (16.1.4) tenglamaning har qanday $u = u(x, y)$ yechimi

$$u = \varphi(x)$$

ko‘rinishda bo‘ladi; bunda $\varphi(x) -$ bir argumentning ixtiyoriy silliq funksiyasi.

Misol 2. Endi ushbu

$$f_y'(x, y, u) \cdot \frac{\partial u}{\partial x} - f_x'(x, y, u) \cdot \frac{\partial u}{\partial y} = 0 \qquad (16.1.5)$$

kvazichiziqli tenglamani qaraylik, bunda berilgan f funksiya nafaqat x, y erkli o‘zgaruvchilarga, balki u noma’lum funksiya $u = u(x, y)$ ga ham oshkor ko‘rinishda bog‘liq.

Bu tenglamadan ixtiyoriy $u(x, y)$ yechim va $\tilde{f}(x, y) = f(x, y, u(x, y))$ funksiyalarning yakobiani nolga teng ekanligi kelib chiqadi:

$$\frac{\partial(u, \tilde{f})}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial \tilde{f}}{\partial x} & \frac{\partial \tilde{f}}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} \right) - \frac{\partial u}{\partial y} \cdot \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} \right) =$$

$$= \frac{\partial f}{\partial y} \cdot \frac{\partial u}{\partial x} - \frac{\partial f}{\partial x} \cdot \frac{\partial u}{\partial y} = 0.$$

Demak, matematik analizdan ma’lum teoremaga ko‘ra, $u(x, y)$ va $f(x, y, u(x, y))$ funksiyalar (funksional) bog‘liq, ya’ni (16.1.5) tenglamaning $u(x, y)$ yechimi

$$u(x, y) = \varphi(f(x, y, u(x, y)))$$

munosabat bilan beriladi; bunda $\varphi(\circ) -$ ixtiyoriy silliq funksiya. Oxirgi tenglik $u(x, y)$ yechimni oshkormas ko‘rinishda aniqlaydi.

Masalan, (16.1.5) tenglamaning xususiy holi bo‘lgan ushbu

$$u_t' + u u_x' = 0 \qquad (16.1.6)$$

(bizda $f(t,x,u)=x-ut$ va $u'_t+uu'_x = f'_x u'_t - f'_t u'_x$) tenglamaning yechimi

$$u=\varphi(x-ut)$$

formula bilan oshkormas ko‘rinishda beriladi, bunda $\varphi(\circ)-$ixtiyoriy silliq funksiya.

(16.1.6) differensial tenglamaga quyidagicha ma’no berish mumkin. Aytaylik, zarrachalar to‘g‘ri chiziq bo‘ylab harakat qilayotgan bo‘lsin. Agar $u(t,x(t))$ ni t paytda to‘g‘ri chiziqning $x(t)$ nuqtasidagi zarrachaning tezligi deb tushunsak, u holda (16.1.6) differensial tenglama barcha zarrachalarning tezlanishi nolga teng ekanligini anglatadi:

$$\frac{du(t,x(t))}{dt}=u'_t+u'_x\cdot\frac{dx}{dt}=u'_t+uu'_x=0.$$

Masalalar

1. $g(x)$ — $\mathbb{R}$ da uzluksiz, lekin birorta nuqtada ham differensiallanuvchi bo‘lmasin. Ushbu

$$u'_t+u'_x=g(x-t)$$

tenglamani qaraylik. Bu tenglama tekislikning hech qanday sohasida $u\in C^1$ yechimga ega bo‘la olmasligini isbotlang.

§ 16.2. Chiziqli tenglamalar

Birinchi tartibli xususiy hosilali chiziqli differensial tenglama

$$a_0(\boldsymbol{x})\frac{\partial u}{\partial x_0}+a_1(\boldsymbol{x})\frac{\partial u}{\partial x_1}+...+a_n(\boldsymbol{x})\frac{\partial u}{\partial x_n}=b(\boldsymbol{x})u+c(\boldsymbol{x}) \qquad (16.2.1)$$

ni qaraylik; bu yerda $a_0(\boldsymbol{x}),a_1(\boldsymbol{x}),...,a_n(\boldsymbol{x}),b(\boldsymbol{x}),c(\boldsymbol{x})$ berilgan funksiyalar $D\subset\mathbb{R}^{1+n}$ sohada uzluksiz differensiallanuvchi $\left(\in C^1\right)$ hamda har bir $\mathbf{x}\in D$ nuqtada $a_0(\boldsymbol{x}),a_1(\boldsymbol{x}),...,a_n(\boldsymbol{x})$ koefftsiyentlarning kamida bittasi 0 dan farqli, ya’ni

$$a_0^2(\boldsymbol{x})+a_1^2(\boldsymbol{x})+...+a_n^2(\boldsymbol{x})>0$$

deb faraz qilinadi. Oxirgi shart (16.2.1) tenglamaning har bir $\mathbf{x}\in D$ nuqtada differensial tenglamadan iborat bo‘lishini ta’minlaydi. Biz aniqlik uchun D sohada $a_0(\boldsymbol{x})$ nolga aylanmaydi deb hisoblaymiz. Shu sohada (16.2.1) tenglamaning har ikkala tomonini $a_0(\boldsymbol{x})$ ga bo‘lib, x_0 o‘zgaruvchini t bilan belgilab, uni quyidagi ko‘rinishga keltiramiz:

$$\frac{\partial u}{\partial t} + f_1(t,\boldsymbol{x})\frac{\partial u}{\partial x_1} + ... + f_n(t,\boldsymbol{x})\frac{\partial u}{\partial x_n} = g(t,\boldsymbol{x})u + h(t,\boldsymbol{x}) \qquad (16.2.2)$$

Bu yerda endi $\boldsymbol{x} = (x_1,...,x_n)^T$ va $f_1,...,f_n,g,h$ funksiyalar $\in C^1$.

Ushbu

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t,\boldsymbol{x}) \\ \dots \quad \dots \quad \dots \\ \dfrac{dx_n}{dt} = f_n(t,\boldsymbol{x}) \end{cases}$$

yoki vektor ko‘rinishidagi

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t,\boldsymbol{x}), \ \ \boldsymbol{f}(t,\boldsymbol{x}) = (f_1(t,\boldsymbol{x}),\dots,f_n(t,\boldsymbol{x}))^T, \qquad (16.2.3)$$

oddiy differensial tenglamalar sistemasi (16.2.2) xususiy hosilali tenglamaning **xarakteristik sistemasi** deyiladi. (16.2.3) sistema yechimlarining $(t,\boldsymbol{x}) \in \mathbb{R}^{1+n}$ fazodagi grafiklari (16.2.2) ning **xarakteristikalari** deb ataladi.

Farazimizga ko‘ra $\boldsymbol{f}(t,\boldsymbol{x}) \in C^1(D,\mathbb{R}^n)$. Demak, D sohaning har bir $(t_0,\boldsymbol{\xi})$ nuqtasidan (16.2.2) tenglamaning yagona xarakteristikasi o‘tadi.

(16.2.3) sistemaning yechimini ((16.2.2) ning xarakteristikasini) ushbu

$$\boldsymbol{x} = \boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi}) \quad \left(\boldsymbol{\varphi}(t_0,t_0,\boldsymbol{\xi}) = \boldsymbol{\xi}\right) \qquad (16.2.4)$$

ko‘rinishda yozaylik. Agar

$\left|t - t_0\right| < a, \left|\boldsymbol{\xi} - \boldsymbol{x}^0\right| < b \quad ((t_0,\boldsymbol{x}^0) \in D; \ a,b -$yetarlicha kichik musbat sonlar) bo‘lsa, u holda (16.2.4) tenglamani $\boldsymbol{\xi}$ga nisbatan yechib, $\boldsymbol{\xi} = \boldsymbol{\varphi}(t_0,t,\boldsymbol{x})$ ekanligini topamiz. Ma’lumki,

$$\boldsymbol{\varphi}(t,\boldsymbol{x}) = \boldsymbol{\varphi}(t_0,t,\boldsymbol{x}) = \left(\varphi_1(t_0,t,\boldsymbol{x}),\dots,\varphi_n(t_0,t,\boldsymbol{x})\right)^T$$

vektor funksiyaning komponentalari (16.2.3)sistemaning erkli birinchi integrallar sistemasini aniqlaydi:

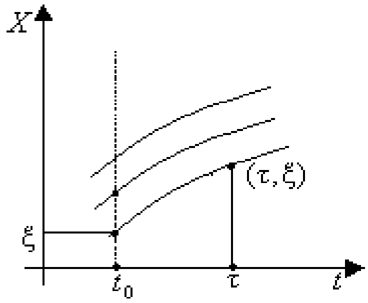
$$\frac{\partial \varphi_i}{\partial t} + \sum_{j=1}^{n} f_j \frac{\partial \varphi_i}{\partial x_j} = 0, \quad i = \overline{1,n} \qquad (16.2.5)$$

yoki vektorli yozuvda

$$\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial t}+\frac{d\boldsymbol{\varphi}(t,\boldsymbol{x})}{d\boldsymbol{x}}\boldsymbol{f}=0.$$

Ravshanki,

$$\left.\frac{d\boldsymbol{\varphi}(t,t_0,\boldsymbol{\xi})}{d\boldsymbol{\xi}}\right|_{t=t_0}=E \ (E-\text{birlik matritsa})$$



16.1- rasm.

Demak, t_0 ga yetarlicha yaqin t lar uchun $\frac{d\boldsymbol{\varphi}}{d\boldsymbol{\xi}}$ matritsa teskarilanuvchi hamda

$$\left.\frac{d\boldsymbol{\varphi}(t_0,t,\boldsymbol{\xi})}{d\boldsymbol{\xi}}\right|_{t=t_0}=E$$

bo‘ladi. Demak, $(t_0,\boldsymbol{x}^0)\in D$ nuqtaning yetarlicha kichik atrofida $(t,\boldsymbol{x})\in D$ o‘zgaruvchilari (koordinatalari) o‘rniga yangi $(\tau,\boldsymbol{\xi})$ o‘zgaruvchilarni (koordinatalarni)

$$\tau=t,\ \boldsymbol{\xi}=\boldsymbol{\varphi}(t,\boldsymbol{x})\equiv\boldsymbol{\varphi}(t_0,t,\boldsymbol{x})$$

formulalar yordamida kiritish mumkin (16.1- rasm). Bunda

$$t=\tau,\quad \boldsymbol{x}=\boldsymbol{\varphi}(\tau,t_0,\boldsymbol{\xi})$$

bo‘ladi. u noma’lum funksiyaning hosilalarini yangi o‘zgaruvchilar orqali ifodalaymiz:

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \tau}+\frac{du}{d\boldsymbol{\xi}}\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial t},\ \frac{du}{d\boldsymbol{x}}=\frac{du}{d\boldsymbol{\xi}}\frac{d\boldsymbol{\varphi}(t,\boldsymbol{x})}{d\boldsymbol{x}} \qquad (16.2.6)$$

Dastlab (16.2.2) ning xususiy holi bo‘lmish ushbu

$$\frac{\partial u}{\partial t}+f_1(t,\boldsymbol{x})\frac{\partial u}{\partial x_1}+...+f_n(t,\boldsymbol{x})\frac{\partial u}{\partial x_n}\equiv\frac{\partial u}{\partial t}+\frac{du}{d\boldsymbol{x}}\boldsymbol{f}=0 \qquad (16.2.7)$$

tenglamani yechaylik.

Almashtirish formulalari (16.2.6)ga ko‘ra (16.2.7) tenglama

$$\frac{\partial u}{\partial t}+\frac{du}{d\boldsymbol{x}}\boldsymbol{f}=\frac{\partial u}{\partial \tau}+\frac{du}{d\boldsymbol{\xi}}\left(\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial t}+\frac{d\boldsymbol{\varphi}(t,\boldsymbol{x})}{d\boldsymbol{x}}\boldsymbol{f}\right)=\frac{\partial u}{\partial \tau}=0$$

ko‘rinishni oladi. Oxirgi tenglamadan (16.2.7) tenglamaning har qanday yechimi τ ga bog‘liq bo‘lmay, balki faqat $\boldsymbol{\xi}=(\xi_1,\ldots\xi_n)^T$ ga bog‘liq bo‘lishi kelib chiqadi. Shunday qilib, (16.2.7) tenglamaning har qanday yechimi (16.2.3) xarakteristik sistema birinchi integrallari to‘la sistemasi $\xi_1 = \varphi_1(t, \boldsymbol{x}), ..., \xi_n = \varphi_n(t, \boldsymbol{x})$ ning funksiyasidan iborat, ya’ni

$$u = u(t, \boldsymbol{x}) = c(\varphi_1(t, \boldsymbol{x}), ..., \varphi_n(t, \boldsymbol{x}));$$

bu yerda $c(\xi_1, ..., \xi_n) -$ ixtiyoriy $\in C^1$ funksiya. Shunday qilib, (16.2.7) tenglamaning umumiy yechimi ixtiyoriy funksiya $c(\xi_1, ..., \xi_n)$ ga bog‘liq.

(16.2.3) sistemaning har qanday birinchi integrali birinchi integrallarning to‘la sistemasi bo‘lmish $\varphi_1(t, \boldsymbol{x}), ..., \varphi_n(t, \boldsymbol{x})$ larning funksiyasidan iborat bo‘lgani uchun quyidagi teorema isbot bo‘ldi.

Teorema 1. *Faraz qilaylik,* $(t_0, \boldsymbol{x}^0)$ *nuqtaning atrofida* $f_1(t, \boldsymbol{x}), ..., f_n(t, \boldsymbol{x})$ *funksiyalar uzluksiz differintsiallanuvchi bo‘lsin.* $\psi_1(t, \boldsymbol{x}), ..., \psi_n(t, \boldsymbol{x})$ *lar bilan* (16.2.3) *sistemaning* $(t_0, \boldsymbol{x}^0)$ *nuqta atrofida aniqlangan erkli birinchi integrallarini belgilaylik. U holda* (16.2.7) *tenglamaning yechimi* $(t_0, \boldsymbol{x}^0)$ *nuqtaning biror atrofida mavjud va har qanday* $u = u(t, \boldsymbol{x})$ *yechim* $\psi_1(t, \boldsymbol{x}), ..., \psi_n(t, \boldsymbol{x})$ *larning funksiyasi sifatida ifodalanadi:*

$$u = c(\psi_1(t, \boldsymbol{x}), ..., \psi_n(t, \boldsymbol{x})),$$

bunda $c(u_1, u_2, \ldots, u_n) -$*silliq funksiya.*

Misollar qaraylik**.**

Misol 1. $\mathbb{R}^3_{(x,y,z)}$ fazoda noldan farqli $\{a; b; c\}$ o‘zgarmas vektor berilgan bo‘lsin. Agar $u(x, y, z) = 0$ sirtning $\left\{\frac{\partial u}{\partial x}; \frac{\partial u}{\partial y}; \frac{\partial u}{\partial z}\right\}$ normal vektori berilgan vektorga perpendikulyar bo‘lsa, u holda $u = u(x, y, z)$ funksiya uchun

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} = 0 \qquad (16.2.8)$$

tenglama hosil qilamiz.

Mos xarakteristik sistemani tuzamiz.

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$$

yoki $cdx - adz = 0, \quad cdy - bdz = 0.$

Ikkita birinchi integral osongina topiladi:

$$c\,x - a\,z = c_1, \quad c\,y - b\,z = c_2$$

Xarakteristika berilgan $\{a;b;c\}$ vektorga parallel to‘g‘ri chiziqlardan iborat. (16.2.8) ning yechimlari ana shu xarakteristikalaridan tuziladi va u

$$\Phi(cx-az, cy-bz)=0$$

ko‘rinishda beriladi; Bu yerda $\Phi-$ikki o‘zgaruvchining ixtiyoriy silliq funksiyasi. Oxirgi tenglama yasovchilari $\{a;b;c\}$ vektorga parallel bo‘lgan silindrik sirt tenglamasini ifodalaydi. ✍

Misol 2. (x,y,z) nuqtadagi normal vektori berilgan $A(a;b;c)$ nuqtadan shu (x,y,z) nuqtaga o‘tkazilgan vektorga perpendikulyar bo‘lgan $u(x,y,z)=0$ sirt uchun

$$(x-a)\frac{\partial u}{\partial x}+(y-b)\frac{\partial u}{\partial y}+(z-c)\frac{\partial u}{\partial z}=0$$

tenglama hosil bo‘ladi.

Xarakteristik sistema

$$\frac{dx}{x-a}=\frac{dy}{y-b}=\frac{dz}{z-c}.$$

Uning birinchi integrallari

$$\frac{x-a}{z-c}=c_1 \ , \ \frac{y-b}{z-c}=c_2,$$

xarakteristikalari esa berilgan $A(a,b,c)$ nuqta orqali o‘tuvchi to‘g‘ri chiziqlar oilasidan iborat. Integral sirt ana shu to‘g‘ri chiziqlardan tuziladi. Uning tenglamasi

$$\Phi\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)=0$$

ko‘rinishda bo‘ladi. Bu tenglama uchi berilgan $A(a,b,c)$ nuqtada, joylashgan konik sirtni ifodalaydi. ✍

Endi (16.2.2) tenglamaning

$$u\big|_{t=t_0}=u_0(\boldsymbol{x}),\ u_0\in C^1, \tag{16.2.9}$$

shartni qanoatlantiruvchi yechimini topish masalasini, ya’ni **Koshi masalas**ini qaraylik. (16.2.7),(16.2.9) Koshi masalasining yechimi (16.2.4) ga ko‘ra $u=u_0(\boldsymbol{\varphi}(t_0,t,\boldsymbol{x}))$ ko‘rinishda ifodalanadi.

Misol 3. Ushbu

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=0 \quad (v=\text{const}) \tag{16.2.10}$$

tenglamani yechaylik. Bu tenglama uchun xarakteristik tenglama $\frac{dx}{dt} = v$ osongina yechiladi:

$$x = \varphi(t, t_0, \xi) = \xi + v(t - t_0), \quad \xi = \varphi(t_0, t, x) = x - v(t - t_0)$$

Demak, (16.2.10) tenglamaning umumiy yechimi

$$u = u(t, x) = u_0(x - v(t - t_0)) \qquad (16.2.11)$$

ko‘rinishda bo‘ladi. Bu yerda $u_0(x) = u\big|_{t=t_0}$ boshlang‘ich funksiya.

(16.2.11) formula x o‘qi bo‘ylab o‘zgarmas v tezlik bilan harakat qiluvchi to‘lqinni anglatadi.

Endi umumiy ko‘rinishdagi chiziqli tenglama (16.2.2) ni yechishga qaytaylik. (16.2.2) tenglama $(\tau, \boldsymbol{\xi})$ o‘zgaruvchilarga nisbatan quyidagi ko‘rinishni oladi:

$$\frac{\partial u}{\partial \tau} = g(\tau, \boldsymbol{\varphi}(\tau, t_0, \boldsymbol{\xi}))u + h(\tau, \boldsymbol{\varphi}(\tau, t_0, \boldsymbol{\xi})).$$

Bu tenglama osongina yechiladi:

$$u = \exp\left[\int_{t_0}^{\tau} g(s, \boldsymbol{\varphi}(\tau, t_0, \boldsymbol{\xi}))ds\right] c(\boldsymbol{\xi}) + \\ + \int_{t_0}^{\tau} \exp\left[-\int_{t_0}^{s} g(r, \boldsymbol{\varphi}(r, t_0, \boldsymbol{\xi}))dr\right] \cdot h(s, \boldsymbol{\varphi}(s, t_0, \boldsymbol{\xi}))ds \; ;$$

bu yerda $c(\boldsymbol{\xi}) = c(\xi_1, ..., \xi_n) -$ ixtiyoriy $\in C^1$ funksiya (integrallash "o‘zgarmasi"). Oxirgi tenglikda $(t, \mathbf{x})$ o‘zgaruvchilarga qaytamiz.

Bizga ma’lum

$$\boldsymbol{\varphi}(s, t_0, \boldsymbol{\varphi}(t_0, t, \boldsymbol{x})) = \boldsymbol{\varphi}(s, t, \boldsymbol{x})$$

munosabatdan foydalanib, (16.2.2) tenglamaning har qanday yechimi, agar u mavjud bo‘lsa, ushbu

$$u = \exp\left[\int_{t_0}^{t} g(s, \boldsymbol{\varphi}(s, t, \boldsymbol{x}))ds\right] c(\boldsymbol{\xi}) + \\ + \int_{t_0}^{t} \exp\left[-\int_{t_0}^{s} g(r, \boldsymbol{\varphi}(r, t, \boldsymbol{x}))dr\right] \cdot h(s, \boldsymbol{\varphi}(s, t, \boldsymbol{x}))ds \; ; \qquad (16.2.12)$$

ko‘rinishda ifodalanishini topamiz. $\boldsymbol{f}, g, h$ funksiyalari C^1 sinfga tegishli ekanligidan foydalanib, (16.2.12) formula bilan berilgan u

funksiyaning haqiqatdan ham (16.2.2) tenglama yechimi ekanligini tekshirib ko‘rish qiyin emas.

Endi umumiy yechim formulasi (16.2.12) dan (16.2.2),(16.2.9) Koshi masalasining yagona yechimini osongina topamiz:

$$u = \exp\left[\int_{t_0}^{t} g(s,\boldsymbol{\varphi}(s,t,\boldsymbol{x}))ds\right]u_0(\boldsymbol{\xi}) + \int_{t_0}^{t}\exp\left[-\int_{t_0}^{s} g(r,\boldsymbol{\varphi}(r,t,\boldsymbol{x}))dr\right]\cdot h(s,\boldsymbol{\varphi}(s,t,\boldsymbol{x}))ds \quad ; (16.2.13)$$

Shunday qilib, biz quyidagi teoremani isbotladik.

Teorema 2. *Faraz qilaylik* $(t_0,\mathbf{x}^0)\in\mathbb{R}^{1+n}$ *nuqtaning biror atrofida* $f_1(t,\boldsymbol{x}),...,f_n(t,\boldsymbol{x}),g(t,\boldsymbol{x}),h(t,\boldsymbol{x})$ *funksiyalar* C^1 *sinfga tegishli bo‘lsin. U holda shu nuqtaning yetarlicha kichik atrofida* (16.2.2) *tenglama yechimga ega va uning har qanday yechimi* (16.2.12) *formula bilan (*$c\in C^1$*) ifodalanadi;* (16.2.2),(16.2.9) *Koshi masalasi yagona yechimga ega va bu yechim* (16.2.13) *formula bilan aniqlanadi.*

Eslatma. Teoremadagi $\{f_1(t,\boldsymbol{x}),...,f_n(t,\boldsymbol{x}),g(t,\boldsymbol{x}),h(t,\boldsymbol{x})\}\subset C^1$ shart ahamiyatli. Agar biz funksiyalardan faqat uzluksizlikni talab qilsak, u holda (16.2.7),(16.2.9) Koshi masalasi (yoki (16.2.7) tenglama) birorta ham $u\in C^1$ yechimga ega bo‘lmasligi mumkin.

Aytaylik, $g(x)$ — sonlar o‘qi $\mathbb{R}$ da uzluksiz, lekin birorta nuqtada ham differensiallanuvchi bo‘lmasin. U holda ushbu

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=g(x-t)u \qquad (16.2.14)$$

tenglama notrivial yechimga ega bo‘lolmaydi. Teskarsini faraz qilaylik. $u\in C^1$, $u(t_0,x_0)\neq 0$, funksiya (16.2.14) tenglamaning yechimi bo‘lsin. Umumiylikni buzmasdan $u(t_0,x_0)>0$ deb hisoblaymiz. Yangi $\tau=t$, $\xi=x-t$ koordinatalarida (16.2.14) tenglama $u'_\tau=g(\xi)u$ ko‘rinishni oladi. Bu tenglamani yechib,

$$u=\exp\left[(\tau-t_0)g(\xi)\right]c(\xi)$$

yoki

$$u=\exp\big((t-t_0)g(x-t)\big)c(x-t) \qquad (16.2.15)$$

bo‘lishi kerakligini topamiz. Farazimizga ko‘ra $u\big|_{t=t_0} = u_0(x)$ funksiya $x_0 \in \mathbb{R}$ nuqtaning kichik atrofida musbat va uzluksiz differensiallanuvchi. (16.2.15) ga ko‘ra

$$u(t,x) = \exp\big((t-t_0)g(x-t)\big)u_0(x-t+t_0) \ . \qquad (16.2.16)$$

Bundan (t_0, x_0) nuqtaning yetarlicha kichik atrofida $u(t,x) > 0$ bo‘lgani uchun, shu atrofda

$$(t-t_0)g(x-t) = \ln\big(u_0(x-t+t_0)/u(t,x)\big)$$

tenglikni topamiz. Bu tenglikdan $t \neq t_0$ da $g(x-t)$ ning differensiyalanuvchi ekanligi kelib chiqadi. Bu esa berilganga zid. Hosil bo‘lgan ziddiyat (16.2.14) tenglamaning noldan farqli yechimga ega bo‘la olmasligini isbotlaydi.

Masalalar

1. Ushbu $u_t' + cu_x' = u \quad (c = \text{const} > 0)$ differensial tenglamaning har qanday notrivial yechimi $(t,x) \in \mathbb{R}^2$ da chegaralanmagan ekanligini isbotlang.

§ 16.3. Kvazichiziqli tenglamalar. Xarakteristikalar metodi

Yechimni qurish. Ushbu

$$\sum_{i=0}^{n} a_i(\boldsymbol{x},u)\frac{\partial u}{\partial x_i} = a_{n+1}(\boldsymbol{x},u) \qquad (16.3.1)$$

kvazichiziqli tenglamani qaraylik; bu yerda $a_i(\boldsymbol{x},u),\ i=\overline{0,n+1},$ funksiyalari biror $(\boldsymbol{x},u) \in G \subset \mathbb{R}^{2+n}$ sohada aniqlangan, C^1 sinfga tegishli va

$$\sum_{i=0}^{n} |a_i(\boldsymbol{x},u)| > 0$$

deb faraz qilinadi.

(16.3.1) kvazichiziqli tenglamani yechishni chiziqli tenglamani yechishga keltirish mumkin. (16.3.1) tenglamaning $u = u(\boldsymbol{x})$ yechimini

$$v(\boldsymbol{x},u) = 0 \qquad (16.3.2)$$

tenglama bilan berilgan oshkormas funksiya sifatida izlaylik. U holda (16.3.2) dan topilgan

$$\frac{\partial u}{\partial x_i} = -\frac{\partial v}{\partial x_i} \Big/ \frac{\partial v}{\partial u}; \quad i = \overline{0,n}, \qquad (16.3.3)$$

hosilalarni (16.3.1) ga qo‘yib,

$$\sum_{i=0}^{n} a_i(\boldsymbol{x},u)\frac{\partial v}{\partial x_i} + a_{n+1}(\boldsymbol{x},u)\frac{\partial v}{\partial u} = 0 \qquad (16.3.4)$$

tenglikni hosil qilamiz.(16.3.4) munosabat (16.3.2) bog‘lanish asosida o‘rinli bo‘lishi kerak, ya’ni bu yerda $(\boldsymbol{x},u)$ erkli o‘zgaruvchi emas. Biz (16.3.4) ni $x_0, x_1,..., x_{n+1} = u$ erkli o‘zgaruvchilarga bog‘liq bo‘lgan $v = v(x_0, x_1,..., x_n, u)$ funksiyaga nisbatan chiziqli tenglama sifatida qaraymiz. (16.3.4) tenglama uchun xarakteristik sistema

$$\begin{aligned} &\frac{dx_i}{d\tau} = a_i(\boldsymbol{x},u), \quad i = \overline{0,n} \\ &\frac{du}{d\tau} = a_{n+1}(\boldsymbol{x},u) \end{aligned} \qquad (16.3.5)$$

(16.3.5) sistemada τ –parametr; u additiv o‘zgarmas aniqligida kiritiladi. (16.3.5) sistemaning $(\boldsymbol{x},u) \in G \subset \mathbb{R}^{n+2}$ fazodagi traektoriyalari (16.3.1) tenglamaning xarakteristikalari (xarakteristik chiziqlari) deb yuritiladi.

Bu yerda shuni ta’kidlash lozimki, xususiy hosilali chiziqli differensial tenglamaning xarakteristikalari yuqorida kiritilishiga ko‘ra $\boldsymbol{x}$ lar fazosida joylashgan. Ular hozirgi ma’nodagi $(\boldsymbol{x},u)$ lar fazosida yotuvchi xarakteristikalarning $\boldsymbol{x}$ o‘zgaruvchilar fazosidagi ortogonal proyeksiyalaridan iborat bo‘ladi.

Faraz qilaylik, $\psi_1(\boldsymbol{x},u),\ldots,\psi_{n+1}(\boldsymbol{x},u)$ funksiyalar (16.3.5) xarakteristik sistema uchun birinchi integrallarning to‘la sistemasi bo‘lsin.

U holda (16.3.4) chiziqli tenglamaning umumiy yechimi

$$v = c(\psi_1(\boldsymbol{x},u),\ldots,\psi_{n+1}(\boldsymbol{x},u)) \equiv v(\boldsymbol{x},u) \qquad (16.3.6)$$

ko‘rinishda ifodalanadi. Agar

$$c(\psi_1(\boldsymbol{x},u),\ldots,\psi_{n+1}(\boldsymbol{x},u)) = 0 \quad (v(\boldsymbol{x},u) = 0)$$

tenglikdan $u = u(\boldsymbol{x})$ funksiya aniqlansa, hamda

$$\left.\frac{\partial v(\boldsymbol{x},u)}{\partial u}\right|_{(x,u)} \neq 0 \qquad (16.3.7)$$

shart bajarilsa, u holda (16.3.3) formulalarga ko‘ra topilgan hosilalarni (16.3.1) tenglamaga qo‘yib va (16.3.4) ni hisobga olib, (16.3.1) ning qanoatlanishini ko‘ramiz.

Shunday qilib, quyidagi teorema isbotlandi.

Teorema 3. *Faraz qilaylik,* (16.3.4) *chiziqli tenglamaning yechimi* $v(\boldsymbol{x},u)$ *uchun* $(\boldsymbol{x}^0,u_0)\in G$ *nuqtada*

$$v(\boldsymbol{x}^0,u_0)=0 \ va \ \left.\frac{\partial v(\boldsymbol{x},u)}{\partial u}\right|_{(x^0,u_0)}\neq 0$$

bo'lsin. U holda oshkormas funksiya haqidagi teoremaga ko'ra $v(\boldsymbol{x},u)=0$ *tenglama* $\boldsymbol{x}^0\in\mathbb{R}^{n+1}$ *nuqtaning biror atrofida* $u=u(\boldsymbol{x})$ *funksiyani aniqlaydi. Bu* $u=u(\boldsymbol{x})$ *funksiya ana shu atrofda kvazichiziqli tenglama* (16.3.1) *ning yechimini aniqlaydi.*

Xarakteristikalar metodi. Kvazichiziqli tenglama (16.3.1) uchun Koshi masalasini yechishda geometrik yondoshish va xarakteristik chiziq tushunchasi juda ham qo'l keladi.

Endi biz ana shu xarakteristikalar metodi deb ataluvchi metodda to'xtalaylik.

(16.3.1) tenglama $G\subset\mathbb{R}^{n+2}$ sohada silliq vektor maydon

$$\boldsymbol{a}(\boldsymbol{x},u)=\left[(a_0(\boldsymbol{x},u),a_1(\boldsymbol{x},u),...,a_{n+1}(\boldsymbol{x},u)\right]^T$$

ni aniqlaydi. Agar $u=u(\mathbf{x})$ funksiya (16.3.1) tenglamaning yechimi bo'lsa, bu yechimning grafigiga (integral sirtga) normal bo'lgan

$$\boldsymbol{n}=\boldsymbol{n}(\boldsymbol{x})=\left[\frac{\partial u}{\partial x_0},\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n},-1\right]^T$$

vektor $\boldsymbol{a}(\boldsymbol{x},u)$ vektor maydonga mos nuqtada ortogonal bo'ladi (bu vektorlarning skalyar ko'paytmasi nolga teng):

$$(\boldsymbol{a}(\boldsymbol{x},u),\boldsymbol{n}(\boldsymbol{x}))=0\,.$$

Bu shart $\boldsymbol{a}$ vektorning integral sirtga urinma ekanligini anglatadi.

Oxirgi tenglikdan quyidagi tasdiq kelib chiqadi.

Teorema 4. $u=u(\boldsymbol{x})\in C^1$ *funksiya* (16.3.1) *tenglamaning yechimi bo'lishi uchun uning grafigi o'zining har bir nuqtasida* $\boldsymbol{a}(\boldsymbol{x},u)$ *vektor maydonga urinishi yetarli va zarurdir.*

Xarakteristik sistema (16.3.5) yechimlarining ((16.3.1) tenglama xarakteristikalarining) $\boldsymbol{a}(\boldsymbol{x},u)$ vektor maydonga urinma ekanligi ravshan.

Agar $S\subset\mathbb{R}^{n+2}$ gipersirtning har bir nuqtasidan (16.3.5) sistemaning yagona yechimi o'tsa va u S sirtda joylashsa, S sirt (16.3.1) tenglamaning xarakteristikalaridan yoki (16.3.5) ning yechimlaridan tuzilgan deyiladi.

Teorema 5. $u = u(\boldsymbol{x}) \in C^1$ *funksiya* (16.3.1) *tenglamaning yechimi bo'lishi uchun uning grafigi shu tenglmaning xarakteristikalaridan tuzilgan bo'lishi yetarli va zarurdir.*

⊶Faraz qilaylik, $u = u(\boldsymbol{x}) \in C^1 -$ (16.3.1) tenglamaning yechimi, $S = \Gamma_u$ esa uning grafigi bo'lsin. Ixtiyoriy $(\boldsymbol{x}^0, u(\boldsymbol{x}^0)) \in \Gamma_u$ nuqtadan (16.3.1) ning yagona xarakteristikasi ((16.3.5) ning yechimi) o'tadi. Bu xarakteristikani χ bilan belgilab, uning to'laligicha Γ_u da yotishini ko'rsatamiz.

$\boldsymbol{x} = \boldsymbol{x}(\tau)$ bilan ushbu

$$\frac{dx_i}{d\tau} = a_i(\boldsymbol{x}, u(\boldsymbol{x})), \ i = \overline{0,n}, \ \boldsymbol{x}(0) = \boldsymbol{x}^0 \qquad (16.3.8)$$

masalaning yechimini belgilaylik. Parametrik tenglamasi $\boldsymbol{x} = \boldsymbol{x}(\tau), u = u(\boldsymbol{x}(\tau))$ bo'lgan χ^* chiziqni qaraylik. $\chi^* \subset \Gamma_u$ ekanligi ravshan. $u = u(\boldsymbol{x})$ funksiya (16.3.1) ning yechimi bo'lgani uchun tenglama $\boldsymbol{x} = \boldsymbol{x}(\tau)$ da ham qanoatlanadi. (16.3.8) va (16.3.1) ga ko'ra

$$\frac{du(\boldsymbol{x}(\tau))}{d\tau} = a_{n+1}(\boldsymbol{x}(\tau), u(\boldsymbol{x}(\tau))), \ u(\boldsymbol{x}(0)) = \boldsymbol{x}^0. \qquad (16.3.9)$$

(16.3.8) va (16.3.9) dan yechimning χ^* ning $\left(\boldsymbol{x}^0; u(\boldsymbol{x}^0)\right) \in \Gamma_u$ nuqtasidan o'tuvchi xarakteristika ekanligi kelib chiqadi. Bu xarakteristika yagona bo'lgani uchun $\chi = \chi^* \subset \Gamma_u$. Endi faraz qilaylik, $u = u(\boldsymbol{x})$ ning $\Gamma_u \subset \mathbb{R}^{n+2}$ grafigi (16.3.1) tenglama xarakteristikalaridan tuzilgan bo'lsin. $u = u(\boldsymbol{x})$ funksiya (16.3.1) tenglama yechimi ekanligini ko'rsatamiz. Ixtiyoriy $\left(\boldsymbol{x}^0; u(\boldsymbol{x}^0)\right) \in \Gamma_u$ nuqta orqali o'tgan xarakteristika ((16.3.5) ning yechimi) Γ_u da yotadi va u (16.3.5) ga ko'ra shu nuqtada $\boldsymbol{a}\left(\boldsymbol{x}^0; u(\boldsymbol{x}^0)\right)$ vektorga urinadi. Demak, ixtiyoriy $\left(\boldsymbol{x}^0; u(\boldsymbol{x}^0)\right) \in \Gamma_u$ nuqtada Γ_u sirt $\boldsymbol{a}\left(\boldsymbol{x}^0; u(\boldsymbol{x}^0)\right)$ vektorga urinadi . Bundan esa $u = u(\boldsymbol{x})$ ning (16.3.1) tenglama yechimi ekanligi kelib chiqadi.

Koshi masalasi. Endi (16.3.1) tenglama uchun Koshi masalasi bilan shug'ullanamiz. Dastlab (16.3.1) tenglama o'rniga ushbu

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n} f_i(t,\boldsymbol{x},u)\frac{\partial u}{\partial x_i}=f_{n+1}(t,\boldsymbol{x},u) \qquad (16.3.10)$$

keltirilgan tenglamani qaraymiz ($x_0=t$ deb belgiladik); bu yerda $\boldsymbol{x}=(x_1,...,x_n)^T\in D\subset\mathbb{R}^{n+2}$. (16.3.10) tenglamaning xarakteristikalari $(t,\boldsymbol{x},u)$ nuqtalar fazosida

$$\begin{aligned}&\frac{dx_i}{dt}=f_i(t,\boldsymbol{x},u),\quad i=\overline{1,n}\\&\frac{du}{dt}=f_{n+1}(t,\boldsymbol{x},u)\end{aligned} \qquad (16.3.11)$$

sistemadan aniqlanadi (bu holda xarakteristikada τ parametr o‘rniga erkli o‘zgaruvchi t ni oldik). (16.3.10) tenglamaning

$$u\big|_{t=t_0}=u_0(\boldsymbol{x})\quad (u_0(\boldsymbol{x})\in C^1(D)) \qquad (16.3.12)$$

boshlang‘ich shartni qanoatlantiruvchi yechimini topish Koshi masalasi deb ataladi.

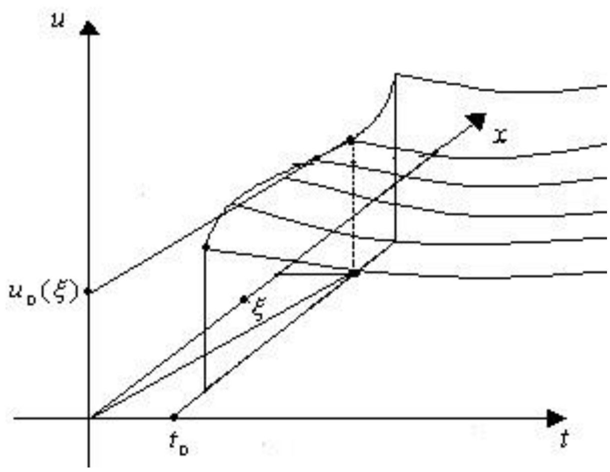
Yuqorida isbotlangan teorema (16.3.10),(16.3.12) Koshi masalasining yechimini qurishga imkon beradi. Izlanayotgan yechim grafigi ushbu $(t_0,\boldsymbol{\xi},u_0(\boldsymbol{\xi}))$ nuqtalar orqali o‘tkazilgan xarakteristikalardan tuziladi (16.1- rasm.).

Bu xarakteristikalar (16.3.11) sistemaning ushbu

$$\boldsymbol{x}\big|_{t_0}=\boldsymbol{\xi},\ u\big|_{t_0}=u_0(\boldsymbol{\xi})$$

boshlang‘ich shartli yechimlardan iborat. Bu yechimlarni Koshi ko‘rinishida yozaylik:

$$\boldsymbol{x}=\Phi(t,t_0,\boldsymbol{\xi},u_0(\boldsymbol{\xi})),\ u=\psi(t,t_0,\boldsymbol{\xi},u_0(\boldsymbol{\xi})). \qquad (16.3.13)$$



16.1- rasm.

Bu (16.3.13) tengliklar yechim grafigining parametrik tenglamasini beradi ($\boldsymbol{\xi}$-parametr, t_0-tayinlangan). Yechimning parametrlarga silliq bog‘liqligi haqidagi teoremaga ko‘ra Φ va ψ funksiyalari t va $\boldsymbol{\xi}$ o‘zgaruvchilarning uzluksiz differensiallanuvchi funksiyalaridan iborat. $u=u(t,\boldsymbol{x})\in C^1$ yechimni topish uchun

(16.3.13) dagi birinchi tenglikdan ξ ni $t, \boldsymbol{x}$ orqali $\xi = \Xi(t, \boldsymbol{x})$ funksiya sifatida ifodalab, (16.3.2) tenglikka qo‘yish kerak.

$$\det \frac{\partial \Phi(t, t_0, \xi, u_0(\xi))}{\partial \xi}\Big|_{t=t_0} = 1 \neq 0$$

bo‘lgani uchun teskari funksiya haqidagi teoremaga ko‘ra t_0 ga yetarlicha yaqin t larda (16.3.13) dan Ξ funksiya topiladi va u C^1 sinfga tegishli bo‘ladi. Shunday qilib, t_0 ga yetarlicha yaqin t larda (16.3.10), (16.3.12) Koshi masalasi

$$u = u(t, \boldsymbol{x}) \equiv \psi(t, t_0, \Xi(t, \boldsymbol{x}), u_0(\Xi(t, \boldsymbol{x})))$$

yagona yechimga ega.

Bu yerda shuni ta’kidlash lozimki, (16.3.10), (16.3.12) Koshi masalasining yechimi mavjud bo‘lgan vaqt oralig‘i silliq Ξ (16.3.13) funksiyaning mavjudlik shartidan aniqlanadi. Bu oraliq umumiy holda yechimning berilgan u_0 qiymatiga bog‘liq bo‘ladi.

Misol 1. Ushbu $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,\ u|_{t=0} = u_0(x)\ (u_0 \in C^1(\mathbb{R}))$ Koshi masalasini yechaylik.

🗝 Xarakteristik sistema (16.3.11) quyidagi ko‘rinishni oladi:

$$\frac{dx}{dt} = u;\ \frac{du}{dt} = 0.$$

Bu sistemani $x|_{t=0} = \xi,\ u|_{t=0} = u_0(\xi)$ boshlang‘ich shartlarda yechib, izlanayotgan yechim grafigini tashkil etuvchi xarakteriskalarni topamiz:

$$u = u_0(\xi);\ \ x = tu_0(\xi) + \xi.$$

$u = u(t,x)$ yechimni topish uchun $u = u_0(x - tu)$ tenglama kelib chiqadi. Agar berilgan (t,x) nuqtaga u ning turli qiymatlari mos kelsa, bir qiymatli yechim mavjud bo‘lmay qoladi, yechim buziladi. 👍

Misol 2. Ushbu

$$\frac{\partial u}{\partial t} - x_1\frac{\partial u}{\partial x_1} - 2x_2\frac{\partial u}{\partial x_2} = 3u,\ u|_{t=0} = x_1^2 + x_2^2$$

Koshi masalasini yeching.

🗝 Yechim grafigi $(0, \xi_1, \xi_2, \xi_1^2 + \xi_2^2) \in \mathbb{R}^4$ nuqtalardan o‘tgan xarakteristikalardan tuziladi. Ularni topish uchun

$$\frac{dx_1}{dt} = -x_1\ , \frac{dx_2}{dt} = -2x_2\ , \frac{du}{dt} = 3u$$

xarakteristik sistemaning

$$x_1\big|_{t=0} = \xi_1,\ x_2\big|_{t=0} = \xi_2,\ u\big|_{t=0} = \xi_1^2 + \xi_2^2$$

Boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish kerak. Bu masala osongina yechiladi:

$$x_1 = \xi_1 e^{-t},\ x_2 = \xi_2 e^{-2t},\ u = (\xi_1^2 + \xi_2^2)e^{3t}\ .$$

Bu yerdan ξ_1, ξ_2 parametrlarni yo‘qotib, yechimni oshkor ko‘rinishda olamiz:

$$u = \left((x_1 e^{t})^2 + (x_2 e^{2t})^2\right)e^{3t}\ \text{yoki}\ u = (x_1^2 + x_1^2 e^{2t})e^{5t}\ .$$ ✍

Endi ***umumiy holda Koshi masalasi***ni qaraymiz. $\boldsymbol{x} = (x_0, x_1, ..., x_n)$ nuqtalar fazosida regulyar gipersirt S berilgan bo‘lsin. S ning regulyarligi u o‘zining har bir $\boldsymbol{x}^0 \in S$ nuqtasi atrofida

$$F(\boldsymbol{x}) = 0 \qquad (16.3.14)$$

silliq tenglama bilan berilishi mumkinligini anglatadi; bunda $F(\mathbf{x})$ funksiya $\boldsymbol{x}^0 \in \mathbb{R}^{1+n}$ nuqtaning biror atrofida C^1 sinfiga tegishli hamda

$$\mathrm{grad}F\Big|_{\boldsymbol{x}^0} = \left[\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}\right]\Bigg|_{\boldsymbol{x}^0} \neq \mathbf{0} \qquad (16.3.15)$$

bo‘lishi kerak (S sirt uzluksiz o‘zgaruvchi normal vektorga ega).

Endi (16.3.1) tenglamaning S sirtda berilgan $g \in C^1$ funksiyaga aylanuvchi ya’ni

$$u\big|_S = g \qquad (16.3.16)$$

shartni qanoatlantiruvchi yechimini topish to‘g‘risidagi Koshi masalasini yechaylik. Aniqlik uchun

$$\frac{\partial F}{\partial x_0}\Bigg|_{\boldsymbol{x}^0} \neq 0$$

deb hisoblaymiz. U holda S sirt $\boldsymbol{x} = (x_0^0, x_1^0, ..., x_n^0)^T \in S$ nuqtaning biror atrofida

$$x_0 = s(x_1, ..., x_n) \qquad (s \in C^1) \qquad (16.3.17)$$

oshkor ko‘rinishda ifodalanadi. $(x_0, x_1, ..., x_n)$ o‘zgaruvchilar o‘rniga yangi $(t, y_1, ..., y_n)$ o‘zgaruvchilarni ushbu

$$t = F(x_0, x_1, .., x_n)$$
$$y_1 = x_1 - x_1^0$$
$$\dots \quad \dots \quad \dots$$
$$y_n = x_n - x_n^0$$

formula bilan kiritaylik. Almashtirish yakobiani $\boldsymbol{x}^0 \in \mathbb{R}^{1+n}$ nuqtada

$$\det \frac{\partial(t, y_1, ..., y_n)}{\partial(x_0, x_1, ..., x_n)}\bigg|_{x^0} = \frac{\partial F}{\partial x_0}\bigg|_{x^0} \neq 0$$

Demak, oshkormas funksiya haqidagi teoremaga ko‘ra $\boldsymbol{x}^0$ nuqtaning biror atrofida sistemadan $(x_0, x_1, ..., x_n)$ o‘zgaruvchilar $(t, y_1, ..., y_n)$ o‘zgaruvchilar orqali bir qiymatli ifodalanadi:

$$\begin{aligned} x_0 &= F^*(t, y_1, .., y_n) \\ x_1 &= y_1 + x_1^0 \\ &\dots \quad \dots \quad \dots \\ x_n &= y_n + x_n^0 \end{aligned} \qquad (16.3.18)$$

va bunda $F^* \in C^1$ bo‘ladi. Yangi $(t, y_1, ..., y_n)$ o‘zgaruvchilarda S sirt $t = 0$ tenglama bilan beriladi va (16.3.16) Koshi sharti ushbu

$$u\big|_{t=0} = g \qquad (16.3.19)$$

ko‘rinishni oladi; bu yerdagi g funksiya $(y_1, ..., y_n)$ o‘zgaruvchilar orqali ifodalangan. (16.3.18) o‘tish formulalariga ko‘ra quyidagi hosilalarni hisoblaymiz:

$$\frac{\partial u}{\partial x_0} = \frac{\partial u}{\partial t}\frac{\partial F}{\partial x_0}; \quad \frac{\partial u}{\partial x_j} = \frac{\partial u}{\partial t}\frac{\partial F}{\partial x_j} + \frac{\partial u}{\partial y_j}; \quad j = \overline{1, n}.$$

Buni (16.3.1)ga qo‘yib,

$$\sum_{i=0}^{n} a_i \frac{\partial F}{\partial x_i} \cdot \frac{\partial u}{\partial t} + \sum_{i=0}^{n} a_j \frac{\partial u}{\partial y_j} = a_{n+1} \qquad (16.3.20)$$

tenglikni hosil qilamiz. Shunday qilib, (16.3.1),(16.3.16) Koshi masalasi $(t, \mathbf{y})$ erkli o‘zgaruvchilarda (16.3.20),(16.3.19) masalani yechishga keltirildi.

Faraz qilaylik, $\boldsymbol{x}^0 \in S$ nuqtada S sirt $(\boldsymbol{x}^0, u(\boldsymbol{x}^0))$ nuqtadan o‘tgan xarakteriskaning $\mathbb{R}_{\boldsymbol{x}}^{n+1}$ fazodagi proyeksiyasiga urinmasin, ya’ni

$$\sum_{i=0}^{n} a_i(\boldsymbol{x}^0, u(\boldsymbol{x}^0)) \frac{\partial F}{\partial x_i} \neq 0 \qquad (16.3.21)$$

bo‘lsin. U holda $\boldsymbol{x}^0$ nuqtaning S sirtdagi biror atrofida ham (16.3.21) tengsizlik saqlanadi va $\boldsymbol{x}^0$ ning kichik atrofida (16.3.20) tenglama (16.3.10) ko‘rinishidagi tenglamaga keladi. Yuqorida isbotlanganga ko‘ra (16.3.20),(16.3.19) masala $\boldsymbol{x}^0$ ning kichik atrofida yagona yechimga ega bo‘ladi. Hosil bo‘lgan natijani teorema sifatida ifodalaylik.

Teorema 6. *S* (16.3.14) *regulyar sirtning* (16.3.21) *shartni qanoatlantiruvchi har qanday* $\boldsymbol{x}^0$*nuqtasining yetarlicha kichik atrofida* (16.3.1),(16.3.16) *Koshi masalasi yagona yechimga ega. Yechimning grafigi* $\{(\boldsymbol{x},u) \mid \boldsymbol{x}\in S,\ u=u(\boldsymbol{x})\}$ *sirt nuqtalaridan o‘tkazilgan xarakteristikalaridan tuzilgan.*

Agar $\boldsymbol{x}=\boldsymbol{x}^0$ *nuqtada*

$$\sum_{i=0}^{n} a_i(\boldsymbol{x},u(\boldsymbol{x}))\frac{\partial F}{\partial x_i}=0 \qquad (16.3.22)$$

bo‘lsa, u holda (16.3.20) *tenglik*

$$\sum_{j=1}^{n} a_j\frac{\partial u}{\partial y_j}=a_{n+1} \qquad (16.3.23)$$

ko‘rinishga keladi va u berilgan g funksiyaning hosilalari orasidagi bog‘lanishni ifodalaydi. Tabiiyki, umumiy holda bu shart hech qanday atrofda o‘rinli bo‘la olmaydi.

Agar (16.3.22) *va* (16.3.23) *tengliklar* $\boldsymbol{x}^0$ *nuqtaning S sirtdagi biror atrofida ham qanoatlansa, u holda* (16.3.1), (16.3.19) *Koshi masalasi cheksiz ko‘p yechimga ega bo‘ladi.*

Eslatma. Umumiy ko‘rinishdagi $F\left(x_1,x_2,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\ldots\frac{\partial u}{\partial x_n}\right)=0$ tenglama ham xarakteristikalar (aniqrog‘i xarakteristik polosa) usuli yordamida tekshirilishi mumkin.

Masalalar

1. Ushbu

$$u_t'+u_x'=u^2,\ u|_{t=0}=1$$

Koshi masalasining yechimi chekli paytda cheksizlikka ketib qolishini (yechimning portlashini) ko‘rsating.

2. Koshi masalasini yeching

$$u_t+(u_x)^2=0, u|_{t=0}=x^2.$$

MODUL 17. KOSHI-KOVALEVSKAYA TEOREMASI

§ 17.1. Karrali darajali qatorlar

Dastlab qulaylik uchun multiindeksli belgilashlarni kiritaylik. $\mathbb{Z}_+ \overset{def}{=} \{0,1,2,\ldots\}$ – barcha nomanfiy butun sonlar to'plami bo'lsin. Odatdagidek, $\mathbb{Z}_+^n = \mathbb{Z}_+ \times \mathbb{Z}_+ \times \ldots \times \mathbb{Z}_+$ (n marta) deymiz. $k = (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n$ – multiindeks deb ataladi. $k = (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n$ multiindeks va $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ vektor uchun quyidagi belgilashlarni ham kiritamiz:

$$k! = k_1! k_2! \ldots k_n!;\ \|k\| = k_1 + \ldots + k_n\ ;\ x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}.$$

Ushbu

$$(a+b)^m = \sum_{i+j=m} \frac{m!}{i!\,j!} a^i b^j \quad (a,b \in \mathbb{R};\ i,j,m \in \mathbb{Z}_+)$$

Nyuton binomi formulasining umumlashishi bo'lmish multinomial formula endi quyidagicha qisqa ko'rinishda yoziladi:

$$(x_1 + x_2 + ... + x_n)^m = \sum_{k_1 + k_2 + \ldots + k_n = m} \frac{m!}{k_1! k_2! \ldots k_n!} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} = \sum_{\|k\|=m} \frac{m!}{k!} x^k .$$

$\boldsymbol{x}^0 = (x_1^0, x_2^0, ..., x_n^0) \in \mathbb{R}^n$ tayinlangan, $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ esa o'zgaruvchi bo'lsin.

Ushbu

$$\sum_{\|k\|=0}^{\infty} a_k (\boldsymbol{x} - \boldsymbol{x}^0)^k \equiv \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1 \ldots k_n} (x_1 - x_1^0)^{k_1} ... (x_n - x_n^0)^{k_n} \qquad (17.1.1)$$

n karrali darajali qatorni qaraylik. Bu yerda $a_k = a_{k_1 ... k_n}$ $(k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n)$ berilgan haqiqiy sonlar. (17.1.1) karrali qator hadlarini biror usul yordamida natural sonlar bilan nomerlab (nomerlash - $\mathbb{Z}_+^n$ ni $\mathbb{N}$ ga biyektiv akslantirish demakdir), undan oddiy (bir karrali) qator tuzaylik. Bu oddiy qatorning yaqinlashishi va uning yig'indisi odatdagicha tushuniladi. Uning yaqinlashishi va yig'indisi, umumiy holda, dastlabki (17.1.1) qator hadlarining qanday usul bilan nomerlanganiga bog'liq. Lekin oddiy yaqinlashish o'rniga absolyut yaqinlashish qaralganda bunday bog'liqlik yo'qoladi. Agar (17.1.1) qatorning hadlarini biror usul bilan nomerlashdan tuzilgan oddiy qator absolyut yaqinlashsa, uning hadlarini boshqa usul bilan nomerlash orqali tuzilgan har qanday oddiy qator ham absolyut yaqinlashadi va uning yig‘indisi nomerlash usuliga bogliq bo‘lmaydi (bu analiz kursida isbotlanadi) hamda bu holda (17.1.1) qatorning yig‘indisi deb uning hadlarini biror bir usul bilan nomerlashdan hosil bo‘lgan absolyut yaqinlashuvchi qatorning yig‘indisiga aytiladi.

Demak, absolyut yaqinlashuvchi (17.1.1) darajali qatorni turli usullarda qo‘shib, quyidagilarni yozish mumkin:

$$\sum_{k_1,\dots,k_n=0}^{\infty} a_{k_1..k_n}(x_1-x_1^0)^{k_1}\dots(x_n-x_n^0)^{k_n} = \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} a_{k_1\dots k_n}(x_1-x_1^0)^{k_1}\dots(x_n-x_n^0)^{k_n} =$$

$$= \sum_{N=0}^{\infty}\sum_{k_1,\dots,k_n=0}^{N} a_{k_1\dots k_n}(x_1-x_1^0)^{k_1}\dots(x_n-x_n^0)^{k_n} = \sum_{j=0}^{\infty}\sum_{k_1+\dots+k_n=j} a_{k_1\dots k_n}(x_1-x_1^0)^{k_1}\dots(x_n-x_n^0)^{k_n}.$$

Soddalik uchun $x^0 = 0$ deb hisoblaylik, ya’ni

$$\sum_{\|k\|=0}^{\infty} a_k \mathbf{x}^k \,. \qquad (17.1.1_0)$$

ko‘rinishidagi darajali qatorlarni qaraylik.

$r_1 > 0,\dots,r_n > 0$ sonlar berilgan bo‘lsin.

Ushbu $\left\{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n \,\middle|\, |x_1| < r_1,\dots,|x_n| < r_n\right\}$ to‘plam ochiq parallelepiped (**0**-markazli) (poliinterval) deyiladi va $\overset{0}{\Pi}(r_1,\dots,r_n)$ bilan belgilanadi. Demak,

$$\overset{0}{\Pi}(r_1,\dots,r_n) = (-r_1;r_1)\times\dots\times(-r_n;r_n).$$

Ushbu $\Pi(r_1,\dots,r_n) = [-r_1;r]\times\dots\times[-r_n;r_n].$ to‘plam esa yopiq parallelepiped (polisegment) deb yuritiladi.

Teorema (Abel). Aytaylik, $(17.1.1_0)$ darajali qator $\mathbf{x} = \tilde{\mathbf{x}}$, $\tilde{x}_j \neq 0$, $j = \overline{1,n}$, nuqtada absolyut yaqinlashuvchi bo‘lsin. U holda bu qator $|x_1| \le |\tilde{x}_1|,\dots,|x_n| \le |\tilde{x}_n|$ shartlarni qanoatlantiruvchi barcha $\mathbf{x} = (x_1,\dots,x_n)$ nuqtalardan tuzilgan $\Pi(|\tilde{x}_1|,\dots,|\tilde{x}_n|)$ yopiq parallelepipedda tekis yaqinlashuvchi bo‘ladi.

🗝 $|x_1| \le |\tilde{x}_1|,\dots,|x_n| \le |\tilde{x}_n|$ shartlar bilan aniqlangan parallelepipedning $x = (x_1,\dots,x_n)$ nuqtalarida

$$|a_k|\cdot|x^k|=|a_k|\cdot|x_1|^{k_1}\dots|x_n|^{k_n}\le|a_k|\cdot|\tilde{x}_1|^{k_1}\dots|\tilde{x}_n|^{k_n}=|a_k|\cdot|\tilde{x}^k|$$

va berilganga ko‘ra $\sum_{|k|=0}^{\infty}|a_k|.|\tilde{x}^k|$ sonli qator yaqinlashuvchi bo‘lganligi uchun tekis yaqinlashish to‘g‘risidagi Veyershtrass alomatidan teoremaning isboti ravshan. 👍

Natija. Agar (17.1.1$_0$) darajali qator $\overset{0}{\Pi}(r_1,...,r_n)$ da absolyut yaqinlashsa, u shu $\overset{0}{\Pi}(r_1,...,r_n)$ da joylashgan ixtiyoriy $\Pi(\rho_1,...,\rho_n)$, $\rho_j<r_j$, $j=\overline{1,n}$, yopiq parallelepipedda tekis va absolyut yaqinlashadi.

Ma’lumki, ushbu $\sum_{j=0}^{\infty}q^j$ qator $|q|<1$ bo‘lganda absolyut yaqinlashuvchi va $\sum_{j=0}^{\infty}q^j=\dfrac{1}{1-q}$ tenglik o‘rinli (cheksiz kamayuvchi geometrik progressiya yig‘ndisi). Bundan $\forall x\in\overset{0}{\Pi}(1,...,1)$ uchun $\sum_{\|k\|=0}^{\infty}x^k$ karrali darajali qatorning absolyut yaqinlashuvchiligi kelib chiqadi va

$$\sum_{\|k\|=0}^{\infty}\boldsymbol{x}^k=\sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty}x_1^{k_1}\dots x_n^{k_n}=\frac{1}{1-x_1}\dots\frac{1}{1-x_n},\ |x_j|<1,\ j=\overline{1,n}.$$

Agar $|x_j|<\dfrac{\alpha}{n}$, $j=\overline{1,n}$ $(\alpha>0)$, bo‘lsa, multinomial formulaga ko‘ra

$$\frac{\alpha}{\alpha-(x_1+x_2+\dots x_n)}=\sum_{m=0}^{\infty}\left(\frac{x_1+x_2+\dots x_n}{\alpha}\right)^m=\sum_{m=0}^{\infty}\sum_{\|k\|=m}\frac{1}{\alpha^m}\frac{m!}{k!}\boldsymbol{x}^k=\sum_{\|k\|=0}^{\infty}\frac{\|k\|!}{\alpha^{\|k\|}k!}\boldsymbol{x}^k,$$

ya’ni $\sum_{\|k\|=0}^{\infty}\dfrac{\|k\|!}{\alpha^{\|k\|}k!}\boldsymbol{x}^k$ karrali darajali qator $\overset{0}{\Pi}(\alpha/n,...,\alpha/n)$ poliintervalda absolyut yaqinlashuvchi va

$$\sum_{\|k\|=0}^{\infty}\frac{\|k\|!}{\alpha^{\|k\|}k!}\boldsymbol{x}^k=\frac{\alpha}{\alpha-(x_1+x_2+\dots x_n)},\ x\in\overset{0}{\Pi}(\alpha/n,...,\alpha/n).$$

Darajali qatorlarning absolyut yaqinlashuvchiligini isbotlashda ba’zan majorantlar metodidan foydalanish mumkin. Ushbu

$$\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$$

darajali qator berilgan bo‘lsin. Agar A_k sonlar uchun

$$A_k \geq 0 \text{ va } |a_k| \leq A_k \ , k \in \mathbb{Z}_+^n,$$

bo‘lsa, $\sum_{\|k\|=0}^{\infty} A_k \boldsymbol{x}^k$ qator berilgan $\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$ qator uchun **majorant qator** deb ataladi. Masalan, $\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$ qator uchun $\sum_{\|k\|=0}^{\infty} |a_k| \boldsymbol{x}^k$ qator majorant qatordir. Ravshanki, agar berilgan qatorning majorant qatori (absolyut) yaqinlashuvchi bo‘lsa, berilgan qator ham absolyut yaqinlashuvchi bo‘ladi.

Agar berilgan $\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$ qatorning koeffitsientlari $|a_k| \leq 1$, $k \in \mathbb{Z}_+^n$, shartlarni qanoatlantirsa, bu qator uchun $|x_j| < 1, j = \overline{1,n}$, sohada absolyut yaqinlashuvchi bo‘lgan $\sum_{\|k\|=0}^{\infty} \boldsymbol{x}^k$ qator majorant qatordir. Demak, shu sohada berilgan $\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$ qator ham absolyut yaqinlashuvchi bo‘ladi.

Aytaylik, $\sum_{\|k\|=0}^{\infty} a_k \boldsymbol{x}^k$ qator $\overset{0}{\Pi}(R,...,R)$ $(R > 0)$ poliintervalda absolyut yaqinlashuvchi bo‘lsin. Uning uchun majoranta qatorni quyidagicha qurish mumkin. Biror $r \in (0, R)$ sonni olaylik. Berilgan qator $\boldsymbol{x} = (r, \ldots, r)$ nuqtada, ya’ni $\sum_{\|k\|=0}^{\infty} a_k r^{\|k\|}$ qator absolyut yaqinlashadi. Demak, $\exists M > 0 \ \forall k \in \mathbb{Z}_+^n \ \left|a_k r^{\|k\|}\right| \leq M$. Bundan

$$|a_k| \leq \frac{M}{r^{\|k\|}} \leq \frac{M\|k\|!}{k! r^{\|k\|}}, \ k \in \mathbb{Z}_+^n,$$

chunki

$$\frac{\|k\|!}{k!} = \frac{(k_1 + ... + k_n)!}{k_1! ... k_n!} \geq 1.$$

Endi $\overset{0}{\Pi}\left(r/n,...,r/n\right)$ poliintervalda absolyut yaqinlashuvchi ushbu

$$\sum_{\|k\|=0}^{\infty}\frac{M\,\|k\|!}{r^{\|k\|}k!}\boldsymbol{x}^k\ \left(=\frac{Mr}{r-(x_1+x_2+\ldots x_n)}\right)$$

darajali qator qaralayotgan qator uchun majorant qator ekanligi ravshan.

§ 17.2. Ko‘p o‘zgaruvchining analitik funksiyalari

Agar $f(\boldsymbol{x})=f(x_1,...,x_n)$ funksiya $\boldsymbol{x}^0\in\mathbb{R}^n$ nuqtaning biror atrofida absolyut yaqinlashuvchi karrali darajali qatorning yig‘indisi sifatida

$$f(x)=\sum_{\|k\|=0}^{\infty}a_k(\boldsymbol{x}-\boldsymbol{x}^0)^k, \qquad (17.2.1)$$

kabi tasvirlansa, u holda $f(\boldsymbol{x})$ funksiya $\boldsymbol{x}^0$ nuqtada analitik funksiya deyiladi. *D* sohaning har bir nuqtasida analitik funksiya *D* sohada analitik funksiya deb ataladi. $n=1$ holida, ya’ni bir o‘zgaruvchining analitik funksiyasi holida darajali qatordan absolyut yaqinlashishni talab qilish shart emas, chunki bu holda, ma’lumki, intervalda yaqinlashuvchi darajali qator absolyut yaqinlashuvchi ham bo‘ladi.

Yozuvda qisqalik uchun (17.2.1) da $x^0=0$ deb hisoblaymiz:

$$f(\boldsymbol{x})=\sum_{\|k\|=0}^{\infty}a_k\boldsymbol{x}^k \qquad (17.2.1_0)$$

$\Pi(r_1,...,r_n)$ yopiq parallelepipedda absolyut yaqinlashuvchi darajali qatorning yig‘indisi shu parallelepipedda uzluksizdir. Bu Abel teoremasi va uzluksiz funksiyalardan tuzilgan tekis yaqinlashuvchi qator yig‘indisining uzluksiz ekanligidan kelib chiqadi.

Matematik analiz kursida isbotlanadiki, $\overset{0}{\Pi}(r_1,...,r_n)$ parallelepipedda absolyut yaqinlashuvchi darajali qator ($17.2.1_0$) ning yig‘indisi shu parallilepipedda xohlagancha marta differensiallanuvchi va uning ixtiyoriy hosilasi qatorni hadma–had differentsiyalashdan hosil bo‘ladi.

Endi ($17.2.1_0$) qatorni hadma-had differensiallab hosil bo‘lgan qatorlarda $x=0$ deb

$$\frac{\partial^{\|k\|} f(0)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} = k_1! \dots k_n! a_{k_1 \dots k_n}$$

tenglikni hosil qilamiz. Bundan ($17.2.1_0$) qator oʻzining yigʻindisi uchun Teylor qatori ekanligi kelib chiqadi:

$$a_{k_1 \dots k_n} = \frac{1}{k_1! \dots k_n!} \frac{\partial^{\|k\|} f(0)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \tag{17.2.2}$$

Bundan esa analitik funksiya darajali qatorga yoyilmasining yagonaligi kelib chiqadi.

Quyidagi teoremalar matematik analiz kursida isbotlanadi.

Teorema 1. Agar $f(x)$ funksiya berilgan nuqtada analitik boʻlsa, u shu nuqtaning biror atrofida ham analitik boʻladi.

Ravshanki, darajali qator oʻz yaqinlashish sohasida joylashgan har qanday polisegmentda absolyut va tekis yaqinlashuvchidir. Demak, uning yigʻindisi yaqinlashish sohasida uzluksiz. Bundan tashqari, bu qatorning yigʻindisi bunday polisegmentda C^{∞} sinfga tegishli ham boʻladi.

Analitik funksiyalar ustida arifmetik amallar hamda analitik funksiyalar kompozitsiyasi yana analitik funksiyani beradi.

Teorema 2. Faraz qilaylik, $f(\boldsymbol{u}),\ \boldsymbol{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$, funksiya $\boldsymbol{u} = \boldsymbol{u}^0 = (u_1^0, \dots, u_m^0)$ nuqtada analitik, $g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}),\ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, funksiyalar esa $\boldsymbol{x} = \boldsymbol{x}^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ nuqtada analitik hamda $\boldsymbol{u}^0 = (g_1(\boldsymbol{x}^0), \dots, g_m(\boldsymbol{x}^0))$ boʻlsin. U holda bu funksiyalar kompozitsiyasi boʻlmish $\varphi(\boldsymbol{x}) = f(g_1(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))$ funksiya $\boldsymbol{x} = \boldsymbol{x}^0$ nuqtada analitik boʻladi.

Natija. Agar $g_1(\boldsymbol{x}) = g_1(x_1, \dots, x_n)$ va $g_2(\boldsymbol{x}) = g_2(x_1, \dots, x_n)$ funksiyalar $\boldsymbol{x} = \boldsymbol{x}^0$ nuqtada analitik boʻlsa, ularning yigʻindisi, ayirmasi va koʻpaytmasi ham shu nuqtada analitik boʻladi.

Isbot . Yuqorida keltirilgan teoremani $f(u_1, u_2) = u_1 + u_2$, $f(u_1, u_2) = u_1 - u_2$, $f(u_1, u_2) = u_1 \cdot u_2$ va $g_1(x_1, \dots, x_n)$, $g_2(x_1, \dots, x_n)$ funksiyalarga tatbiq etish kifoya.

Teorema 3. Agar $f_1(\boldsymbol{x}) = f_1(x_1,...,x_n)$va $f_2(\boldsymbol{x}) = f_2(x_1,...,x_n)$, $\boldsymbol{x} = (x_1,...,x_n) \in \mathbb{R}^n$, funksiyalar $\boldsymbol{x} = \boldsymbol{x}^0$ nuqtada analitik va $f_2(\boldsymbol{x}^0) \neq 0$ bo‘lsa, $\dfrac{f_1(x)}{f_2(x)}$ funksiya ham $\boldsymbol{x} = \boldsymbol{x}^0$ nuqtada analitik bo‘ladi.

§ 17.3. Xususiy hosilali tenglamalar uchun Koshi masalasi yechimining analitikligi

Ushbu $u = u(t,\boldsymbol{x}) = u(t, x_1,\ldots,x_n), \left(\boldsymbol{x} = (x_1, x_2,\ldots,x_n)\right)$, noma’lum funksiyaning $\dfrac{\partial u}{\partial t}$ hosilasiga nisbatan yechilgan birinchi tartibli xususiy hosilali ushbu

$$\frac{\partial u}{\partial t} = F\big(t, x_1,\ldots,x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2},\ldots,\frac{\partial u}{\partial x_n}\big) \tag{17.3.1}$$

differensial tenglama berilgan bo‘lsin. Biz bu tenglamaning

$$u\big|_{t=0} = \varphi(\boldsymbol{x}) \tag{17.3.2}$$

Koshi shartini qanoatlantiruvchi yechimini berilgan $(0,\boldsymbol{x}^0) \in \mathbb{R}^n$ nuqtada analitiklikka tekshiramiz. Bunda tenglamaning qanoatlanishidan kelib chiqib, tabiiy ravishda $\varphi(x)$ funksiya $\boldsymbol{x}^0 \in \mathbb{R}^n$ nuqtada, $F(t,\boldsymbol{x},u,p)$ $(\boldsymbol{p} = (p_1, p_2,\ldots p_n))$ funksiya esa $(0,\boldsymbol{x}^0,u_0,\boldsymbol{p}^0) \in \mathbb{R}^{2n+2}$ $\big(u_0 = \varphi(\boldsymbol{x}^0)$, $\boldsymbol{p}^0 = (p_1^0, p_2^0,\ldots,p_n^0)$, $p_j^0 = \varphi'_{x_j}(x^0)$, $j = \overline{1,n}\big)$ nuqtada analitik deb hisoblaymiz.

Koshi masalasini xususiy hosilali differensial tenglamalar sistemasi uchun ham qo‘yish mumkin. $u_1 = u_1(t,\boldsymbol{x})$, ..., $u_m = u_m(t,\boldsymbol{x})$ noma’lum funksiyalarga nisbatan quyidagi ko‘rinishdagi sistemani qaraylik:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = f_1\big(t,\boldsymbol{x},u_1,u_2,...,u_m,\dfrac{\partial u_1}{\partial \boldsymbol{x}},\dfrac{\partial u_2}{\partial \boldsymbol{x}},...,\dfrac{\partial u_m}{\partial \boldsymbol{x}}\big), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial u_m}{\partial t} = f_m\big(t,\boldsymbol{x},u_1,u_2,...,u_m,\dfrac{\partial u_1}{\partial \boldsymbol{x}},\dfrac{\partial u_2}{\partial \boldsymbol{x}},...,\dfrac{\partial u_m}{\partial \boldsymbol{x}}\big). \end{cases} \tag{17.3.3}$$

Bu sistemaning $t = 0$ da

$$u_1\big|_{t=0} = \varphi_1(\boldsymbol{x}),\ u_2\big|_{t=0} = \varphi_m(\boldsymbol{x}),.....,u_m\big|_{t=0} = \varphi_m(\boldsymbol{x}), \tag{17.3.4}$$

boshlang‘ich qiymatlarni qabul qiluvchi $u_1 = u_1(t, \boldsymbol{x}), ..., u_m = u_m(t, \boldsymbol{x})$ yechimini topish Koshi masalasi deyiladi. Biz bu masalada ham uchragan barcha funksiyalarni analitik funksiyalar deb hisoblaymiz.

Keltirilgan (17.3.1),(17.3.2) va (17.3.3),(17.3.4) masalalar maxsus ko‘rinishdagi kvazichiziqli tenglamalar sistemasi uchun Koshi masalasiga keltiriladi.

Bu keltirishni soddaroq (17.3.1), (17.3.2) nochiziqli Koshi masalasi uchun bajaraylik.

Faraz qilaylik, qaralayotgan Koshi masalasi analitik yechimga ega bo‘lsin. Ushbu

$$p_j = \frac{\partial u}{\partial x_j}, \; j = \overline{1, n},$$

belgilashlarni kiritaylik. U holda berilgan tenglamani x_j o‘zgaruvchilar bo‘yicha bir marta differensiallash natijasida quyidagi sistemaga kelamiz (qaralayotgan funksiyalar analitik bo‘lgani uchun aralash xususiy hosila uni hisoblash tartibiga bog‘liq emas):

$$\begin{cases} \dfrac{\partial u}{\partial t} = F(t, \boldsymbol{x}, u, \boldsymbol{p}), \\ \dfrac{\partial p_j}{\partial t} = F'_{x_j} + F'_u p_j + F'_{p_1} \dfrac{\partial p_1}{\partial x_j} + \ldots + F'_{p_n} \dfrac{\partial p_n}{\partial x_j}, \; j = \overline{1, n}. \end{cases} \tag{17.3.5}$$

Bu yerda

$$F'_{x_j} = \frac{\partial F(t, \boldsymbol{x}, u, \boldsymbol{p})}{\partial x_j}, F'_u = \frac{\partial F(t, \boldsymbol{x}, u, \boldsymbol{p})}{\partial u}, F'_{p_j} = \frac{\partial F(t, \boldsymbol{x}, u, \boldsymbol{p})}{\partial p_j}, \; j = \overline{1, n}.$$

Boshlang‘ich shartlar (17.3.2), ravshanki,

$$u\big|_{t=0} = \varphi(\boldsymbol{x}), \; p_j\big|_{t=0} = \varphi'_{x_j}(\boldsymbol{x}), \; j = \overline{1, n}, \tag{17.3.6}$$

ko‘rinishga o‘tadi. Tushunarliki, bu yerda uchragan funksiyalarning barchasi o‘z argumentlari bo‘yicha analitik.

Endi faraz qilaylikki, (17.3.5),(17.3.6) Koshi masalasi analitik yechimga ega bo‘lsin. (17.3.5) sistemadagi birinchi tenglikni x_j bo‘yicha differensiallab. (17.3.5) dan $\dfrac{\partial p_j(t, \boldsymbol{x})}{\partial t} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u(t, \boldsymbol{x})}{\partial x_j}\right)$

ekanligini topamiz. Bundan $p_j(t,\boldsymbol{x}) = \dfrac{\partial u(t,\boldsymbol{x})}{\partial x_j} + c_j(\boldsymbol{x})$, $j = \overline{1,n}$. Lekin (17.3.6) boshlang'ich shartlarga ko'ra $c_j(\boldsymbol{x}) = 0$. Demak, $p_j(t,\boldsymbol{x}) = \dfrac{\partial u(t,\boldsymbol{x})}{\partial x_j}$, $j = \overline{1,n}$. Bu tengliklarni (17.3.5) dagi birinchi tenglamaga qo'yib, $u = u(t,\boldsymbol{x})$ funksiya (17.3.1) tenglamani qanoatlantirishini ko'ramiz. Uning (17.3.2) boshlang'ich shartni qanoatlantirishi (17.3.6) dan ravshan.

Shunday qilib, analitik yechimning mavjudligi ma'nosida (17.3.1),(17.3.2) Koshi masalasi (17.3.5),(17.3.6) Koshi masalasiga ekvivalent.

(17.3.5),(17.3.6) masala $u_1(t,\boldsymbol{x}), u_2(t,\boldsymbol{x}), \ldots, u_m(t,\boldsymbol{x})$ noma'lum funksiyalarga nisbatan quyidagi umumiy ko'rinishdagi kvazichiziqli sistema uchun qo'yilgan Koshi masalasining xususiy holidir:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = G_1(t,\boldsymbol{x},\boldsymbol{u}) + \sum\limits_{i=1}^{n}\sum\limits_{l=1}^{m} G_{1;li}(t,\boldsymbol{x},\boldsymbol{u}) \dfrac{\partial u_l}{\partial x_i} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial u_m}{\partial t} = G_m(t,\boldsymbol{x},\boldsymbol{u}) + \sum\limits_{i=1}^{n}\sum\limits_{l=1}^{m} G_{m;li}(t,\boldsymbol{x},\boldsymbol{u}) \dfrac{\partial u_l}{\partial x_i} \end{cases} \quad \left(\boldsymbol{u} = (u_1, u_2, \ldots, u_m)\right) \quad (17.3.7)$$

$$u_j\big|_{t=0} = \varphi_j(\boldsymbol{x}),\ j = \overline{1,m}. \quad (17.3.8)$$

(17.3.7) kvazichiziqli tenglamalar sistemasidagi $G_j(t,\boldsymbol{x},\boldsymbol{u}), G_{j;li}(t,\boldsymbol{x},\boldsymbol{u})$ koeffitsientlar $u_1,...,u_m$ noma'lum funksiyalardan tashqari $t, x_1,...,x_n$ erkli o'zgaruvchilarga ham bog'liq. Yangi noma'lum funksiyalarni kiritish yordamida sistemadagi koeffitsientlarni erkli o'zgaruvchilarga oshkor ko'rinishda bog'liq bo'lmagan va $G_j \equiv 0$ holga keltirish mumkin.

Buning uchun ushbu

$$\tau = \tau(t, x_1,...,x_n),\ \xi_i = \xi_i(t, x_1,...,x_n),\ i = \overline{1,m},$$

funksiyalarni

$$\frac{\partial \tau}{\partial t} = \frac{\partial \xi_1}{\partial x_1}, \frac{\partial \xi_1}{\partial t} = 0, ..., \frac{\partial \xi_n}{\partial t} = 0; \quad (17.3.9)$$

$$\tau\big|_{t=0} = 0, \xi_1\big|_{t=0} = x_1, ..., \xi_n\big|_{t=0} = x_n \quad (17.3.10)$$

masala yechimi sifatida aniqlaylik. Ravshanki, (17.3.9),(17.3.10) masalaning yagona yechimi $\tau = t$, $\xi_1 = x_1, ..., \xi_n = x_n$ bo‘ladi.

$\frac{\partial \xi_1}{\partial x_1} = 1$ bo‘lgani uchun (17.3.7) kvazichiziqli sistema o‘rniga ushbu

$$\frac{\partial u_j}{\partial t} = G_j(\tau, \xi_1, ..., \xi_n, u_1, ..., u_m)\frac{\partial \xi_1}{\partial x_1} + \sum_{i=1}^{n}\sum_{l=1}^{m} G_{j;li}(\tau, \xi_1, ..., \xi_n, u_1, ..., u_m)\frac{\partial u_l}{\partial x_i},$$

$$j = \overline{1, m}; \quad \frac{\partial \tau}{\partial t} = \frac{\partial \xi_1}{\partial x_1}, \frac{\partial \xi_1}{\partial t} = 0, ..., \frac{\partial \xi_n}{\partial t} = 0$$

(17.3.11)

sistemani yozish mumkin. Bu sistema uchun boshlang‘ich shartlar quyidagicha:

$$u_i\big|_{t=0} = \varphi_i(x_1, ..., x_n), \ i = \overline{1, m};$$
$$\tau\big|_{t=0} = 0, \xi_1\big|_{t=0} = x_1, ..., \xi_n\big|_{t=0} = x_n. \qquad (17.3.12)$$

Ravshanki, (17.3.11) kvazichiziqli sistema ushbu

$$\frac{\partial u_j}{\partial t} = \sum_{i=1}^{n}\sum_{l=1}^{d} g_{j;li}(u_1, ..., u_d)\frac{\partial u_l}{\partial x_i}, \ j = \overline{1, d};$$

ko‘rinishga ega (bizda $d = m + 1 + n$, $u_{m+1} = \tau, u_{m+2} = \xi_1, \ldots, u_{m+1+n} = \xi_n$).

Shunday qilib, umumiy ko‘rinishdagi kvazichiziqli sistema uchun qo‘yilgan (17.3.7), (17.3.8) Koshi masalasi quyidagi maxsus ko‘rinishdagi kvazichiziqli sistema uchun Koshi masalasiga keltirildi:

$$\frac{\partial u_j}{\partial t} = \sum_{i=1}^{n}\sum_{l=1}^{m} q_{j;li}(u_1, ..., u_m)\frac{\partial u_l}{\partial x_i}, \ j = \overline{1, m}; \qquad (17.3.13)$$

$$u_j\big|_{t=0} = \varphi_j(x_1, ..., x_n), \ j = \overline{1, m}. \qquad (17.3.14)$$

Bu yerda kvazichiziqli tenglamalar sistemasidagi $q_{j;li}(u_1, ..., u_m)$ koeffitsientlar faqat $u_1, ..., u_m$ noma’lum funksiyalarga bog‘liq xolos; ular $t, x_1, ..., x_n$ erkli o‘zgaruvchilarga bog‘liq emas.

Bitta tenglama uchun (17.3.1),(17.3.2) Koshi masalasini (17.3.13),(17.3.14) ko‘rinishdagi Koshi masalasiga keltirilganidek (17.3.3),(17.3.4) Koshi masalasini ham shu (17.3.13),(17.3.14) ko‘rinishdagi Koshi masalasiga keltiriladi.

Eslaylikki, (17.3.13),(17.3.14) Koshi masalasidagi $q_{j;li}(u_1,...,u_m)$ berilgan koeffitsiyentlar va (17.3.14) boshlang‘ich shartlardagi $\varphi_j(x_1,...,x_n)$ berilgan funksiyalar mos ravishda $\boldsymbol{u}=\boldsymbol{u}^0=(u_1^0,u_2^0,\ldots,u_m^0)\left(u_j^0=\varphi_j(\boldsymbol{x}^0),\ j=\overline{1,m}\right)$ va $\boldsymbol{x}^0\in\mathbb{R}^n$ nuqtada analitik. Soddalik uchun $\boldsymbol{x}^0=0$ va $u_j^0=\varphi_j(0,...,0)=0$ deb faraz qilamiz. Bunga har doim $\boldsymbol{x}$ o‘rniga $\boldsymbol{x}-\boldsymbol{x}^0$ ni va u_j lar o‘rniga esa $u_j-\varphi_j(\boldsymbol{x}^0)$ni qo‘yish yordamida erishish mumkin. Bunda (17.3.13) sistema ko‘rinishi o‘zgarmaydi, berilgan koeffitsientlar va boshlang‘ich funksiyalar analitikligicha qoladi.

Dastlab (17.3.13),(17.3.14) masalaning analitik yechimi yagona bo‘lshini ko‘rsatamiz. Faraz qilaylik, qaralayotgan masalaning analitik yechimi $u_j(t,\boldsymbol{x}),\ j=\overline{1,m}$, mavjud bo‘lsin. Bu yechimning yagonaligini isbotlash uchun ana shu yechimning

$$\frac{\partial^{\|k\|}u_j}{\partial t^{k_0}\partial x_1^{k_1}...\partial x_n^{k_n}},\ j=\overline{1,m},\ k=(k_0,k_1,...,k_n)\in\mathbb{Z}_+^{n+1},\ \|k\|=k_0+k_1+...+k_n,$$

xususiy hosilalari $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtada bir qiymatli aniqlanishini ko‘rsatish kifoya, chunki

$$u_j=u_j(t,x_1,...,x_n)=\sum_{\|k\|=0}^{\infty}\frac{1}{k_0!k_1!...k_n!}\frac{\partial^{\|k\|}u_j(0,0,...,0)}{\partial t^{k_0}\partial x_1^{k_1}...\partial x_n^{k_n}}t^{k_0}x_1^{k_1}...x_n^{k_n}\ .$$

t o‘zgaruvchi qatnashmagan

$$\frac{\partial^{\|k\|}u_j(0,0,...,0)}{\partial x_1^{k_1}...\partial x_n^{k_n}},\ j=\overline{1,m},\ k_0=0,\ k_1,...,k_n=0,1,2,...,\|k\|=0+k_1+...+k_n,$$

ko‘rinishdagi barcha xususiy hosilalar (17.3.14) boshlang‘ich shartlardan bir qiymatli topiladi (hosila olish va unga $x_1=...=x_n=0$ qiymatlar qo‘yish yordamida):

$$\frac{\partial^{\|k\|}u_j(0,0,...,0)}{\partial x_1^{k_1}...\partial x_n^{k_n}}=\frac{\partial^{\|k\|}\varphi_j(0,0,...,0)}{\partial x_1^{k_1}...\partial x_n^{k_n}},\ j=\overline{1,m},\ k_0=0,\ k_1,...,k_n=0,1,2,...,.$$

t bo‘yicha hosila qatnashgan xususiy hosilalarni topish uchun berilgan tenglamalardan foydalanamiz. $\dfrac{\partial u_j(0,0,...,0)}{\partial t},\ j=\overline{1,m},$

xususiy hosilalar bevosita (17.3.13) tenglamalardan bir qiymatli topiladi:

$$\frac{\partial u_j(0,0,...,0)}{\partial t}=\sum_{l=1}^{n}\sum_{i=1}^{m} g_{j;li}(0,...,0)\frac{\partial u_l(0,0,...,0)}{\partial x_i},\ j=\overline{1,m}.$$

(17.3.13) tenglamalarni $x_k,\ k=\overline{1,n}$, bo‘yicha differensiallaylik:

$$\frac{\partial^2 u_j}{\partial t\partial x_k}=\sum_{i=1}^{n}\sum_{l=1}^{m}\left(\sum_{p=1}^{m}\frac{g_{j;li}}{\partial u_p}\frac{\partial u_p}{\partial x_k}\frac{\partial u_l}{\partial x_i}+g_{j;li}\frac{\partial^2 u_l}{\partial x_k\partial x_i}\right),\ j=\overline{1,m}\ ,\ k=\overline{1,n}$$

(17.3.15)

Bu tengliklarning o‘ng tomonidagi xususiy hosilalarning qiymatlari $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtada ma’lum. Unga ma’lum qiymatlarni qo‘yib, $\frac{\partial^2 u_i(0,0,...,0)}{\partial t\partial x_k}$ qiymatlarni bir qiymatli topamiz.

Endi (17.3.13) tenglamalarni t bo‘yicha differensiallaylik:

$$\frac{\partial^2 u_j}{\partial t^2}=\sum_{i=1}^{n}\sum_{l=1}^{m}\left(\sum_{p=1}^{m}\frac{\partial g_{j;li}}{\partial u_p}\frac{\partial u_p}{\partial t}\frac{\partial u_l}{\partial x_i}+g_{j;li}\frac{\partial^2 u_l}{\partial t\partial x_i}\right),\ j=\overline{1,m}. \qquad (17.3.16)$$

Ravshanki, bu tengliklardan yuqorida hisoblangan hosilalarga ko‘ra $\frac{\partial^2 u_j(0,0,...,0)}{\partial t^2}$ qiymatlar bir qiymatli topiladi.

Shunday qilib, biz noma’lum funksiyalarning barcha ikkinchi tartibli xususiy hosilalarini $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtada bir qiymatli topdik.

Yuqorida bajarilgan ishni davom ettiramiz. (17.3.15) sistemani $x_p,\ p=\overline{1,n}$, bo‘yicha differensiallab, t bo‘yicha birinchi tartibli hosilalar qatnashgan uchunchi tartibli $\frac{\partial^3 u_j}{\partial t\partial x_k\partial x_p}$ xususiy hosilalarni topamiz. (17.3.16) tengliklarni $x_p,\ p=\overline{1,n}$, bo‘yicha differensiallab, $\frac{\partial^3 u_j}{\partial t^2\partial x_p}$ xususiy hosilalarni topamiz. (17.3.16) tengliklarni t bo‘yicha differensiallab, $\frac{\partial^3 u_j}{\partial t^3}$ hosilalarni hisoblaymiz va h.k. Endi tushunarliki, differensiallash va o‘rniga qo‘yish yordamida barcha xususiy hosilalarning

$$\frac{\partial^{\|k\|}u_j(0,0,...,0)}{\partial t^{k_0}\partial x_1^{k_1}...\partial x_n^{k_n}},\ j=\overline{1,m},\ k_0,k_1,...,k_n=0,1,2,.....,\|k\|=k_0+k_1+...+k_n,$$

qiymatlarini bir qiymatli aniqlaymiz. Shunday qilib, berilgan boshlang‘ich shartlar va tenglamalar analitik yechimning barcha xususiy hosilalarni $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtada bir qiymatli aniqlaydi. Analitik yechimning yagonaligi isbot bo‘ldi.

Endi (17.3.13),(17.3.14) masala yechimining mavjudligini isbotlaymiz. Yuqorida topilgan

$$c_{k_0k_1...k_n}^{(j)}=\frac{\partial^{\|k\|}u_j(0,0,...,0)}{\partial t^{k_0}\partial x_1^{k_1}...\partial x_n^{k_n}},\ j=\overline{1,m},\ k=(k_0,k_1,...,k_n)\in\mathbb{Z}_+^{n+1},$$

sonlarga ko‘ra formal ravishda tuzilgan ushbu

$$u_j(t,x_1,...,x_n)=\sum_{\|k\|=0}^{\infty}c_{k_0k_1...k_n}^{(j)}t^{k_0}x_1^{k_1}\cdot...\cdot x_n^{k_n},$$

darajali qatorni qaraylik. Berilganga ko‘ra $\varphi_j(x_1,...,x_n),\ j=\overline{1,m},$ boshlang‘ich funksiyalar va $q_{j;li}(u_1,...,u_m)$ koeffitsientlar mos ravishda $(0,...,0)\in\mathbb{R}^n$ va $(0,...,0)\in\mathbb{R}^m$ nuqtada analitik, ya’ni ular mos nuqtalar atrofida absolyut yaqinlashuvchi darajali qatorga yoyiladi:

$$\varphi_j(x_1,...,x_n)=\sum_{\|k\|=1}^{\infty}a_{k_1...k_n}^{(j)}x_1^{k_1}\cdot...\cdot x_n^{k_n},\ |x_i|\le\rho\ (\rho>0),\qquad(17.3.17)$$

$$q_{j;li}(u_1,...,u_m)=\sum_{\|p\|=0}^{\infty}b_{p_1...p_m}^{(j;li)}u_1^{p_1}\cdot...\cdot u_m^{p_m},\ |u_l|\le r\ \ (r>0).\qquad(17.3.18)$$

Biz ushbu

$$u_j(t,x_1,...,x_n)=\sum_{\|k\|=0}^{\infty}c_{k_0k_1...k_n}^{(j)}t^{k_0}x_1^{k_1}\cdot...\cdot x_n^{k_n},\qquad(17.3.19)$$

formal qator $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtaning biror atrofida absolyut yaqinlashuvchi va qaralayotgan Koshi masalasining yechimi bo‘lishini isbotlaymiz. Yuqorida (17.3.19) qatorning koeffitsientlari differensial tenglamalar va boshlang‘ich shartlar orqali bir qiymatli aniqlanishini ko‘rsatdik. Agar (17.3.19) qator $(t,x_1,...,x_n)=(0,0,...,0)$ nuqtaning biror atrofida absolyut yaqinlashuvchi bo‘lsa, bu atrofda qatorni hadma-had xohlagancha marta differensiallash mumkin. Topilgan hosilalarni tenglamaga qo‘yib, hosil bo‘luvchi ifodalarni yana $t, x_1, ..., x_n$ bo‘ylab yoysak, chap va o‘ng tomondagi bir xil darajali qatorlar hosil bo‘ladi, chunki $c^{(j)}_{k_0 k_1 ... k_n}$ koeffitsientlar ana shu tenglik shartidan topilgan. Bundan tashqari boshlang‘ich shartlar ham qanoatlantiradi. Demak, agar (17.3.19) qator $(t, x_1, ..., x_n) = (0, 0, ..., 0)$ nuqtaning biror atrofida absolyut yaqinlashuvchi bo‘lsa, u (17.3.13),(17.3.14) masala yechimi bo‘ladi.

(17.3.19) qatorning absolyut yaqinlashuvchiligini majorantlar metodidan foydalanib isbotlaymiz.

Boshlang‘ich funksiyalar $\varphi_j(x_1, ..., x_n)$ va $g_{j;li}(u_1, ..., u_m)$ koeffitsientlar uchun $\psi(x_1, ..., x_n)$ va $G_{j;li}(u_1, ..., u_m)$ majorantalarni qaraylik:

$$\psi_j(x_1, ..., x_n) = \sum_{\|k\|=0}^{\infty} A^{(j)}_{k_1 ... k_n} x_1^{k_1} \cdot ... \cdot x_n^{k_n}, \; A^{(j)}_{k_1 ... k_n} \ge 0, \qquad (17.3.20)$$

$$\left| a^{(j)}_{k_1 ... k_n} \right| \le A^{(j)}_{k_1 ... k_n}$$

$$G_{j;li}(u_1, ..., u_m) = \sum_{\|p\|=0}^{\infty} B^{(j;li)}_{p_1 ... p_m} u_1^{p_1} \cdot ... \cdot u_m^{p_m}, \qquad (17.3.21)$$

$$B^{(j;li)}_{p_1 ... p_m} \ge 0, \; \left| b^{(j;li)}_{p_1 ... p_m} \right| \le B^{(j;li)}_{p_1 ... p_m}.$$

Qaralayotgan (17.3.13), (17.3.14) Koshi masalasidagi berilganlarni ularning majorantalari bilan almashtirib, quyidagi majorant masalani tuzaylik:

$$\frac{\partial v_j}{\partial t} = \sum_{i=1}^{n} \sum_{l=1}^{m} G_{j;li}(v_1, ..., v_m) \frac{\partial v_l}{\partial x_i}, \; j = \overline{1, m}; \qquad (17.3.22)$$

$$v_j \big|_{t=0} = \psi_j(x_1, ..., x_n), \; j = \overline{1, m}. \qquad (17.3.23)$$

Yuqorida keltirilgan usul bilan bu majorant masalaning mumkin bo‘lgan yechimi uchun

$$v_j(t, x_1, ..., x_n) = \sum_{\|k\|=0}^{\infty} C^{(j)}_{k_0 k_1 ... k_n} t^{k_0} \cdot x_1^{k_1} \cdot ... \cdot x_n^{k_n} \qquad (17.3.24)$$

darajali qatorlarni tuzaylik.

Tushunarliki, bunda $C^{(j)}_{k_0k_1\dots k_n}$ koeffetsientlar $A^{(j)}_{k_1\dots k_n}$ va $B^{(j;li)}_{p_1\cdots p_m}$ koeffetsiyentlar orqali $c^{(j)}_{k_0k_1\dots k_n}$ larning $a^{(j)}_{k_1\dots k_n}$ va $b^{(j;li)}_{p_1\cdots p_m}$ lar orqali ifodalangan formulalari yordamida topiladi. Bu ifodalanish jarayonini diqqat bilan ko‘zdan kechiraylik. $C^{(j)}_{k_0k_1\dots k_n}$ larni topishda biz darajali qatorlarni hadma-had differensiallash va hosil bo‘lgan qatorlarni berilgan differensial tenglamalarga qo‘yish hamda $t = x_1 = ... = x_m = 0$ da hisoblash amallarini bajaramiz. O‘rniga qo‘yishda qo‘shish va ko‘paytirish amallari ishlatiladi. Demak, $C^{(j)}_{k_0k_1\dots k_n}$ sonlar $A^{(j)}_{k_1\dots k_n}$ va $B^{(j;li)}_{p_1\cdots p_m}$ sonlarning nomanfiy butun koeffetsnentli ko‘phadidan iborat bo‘ladi. Endi ravshanki, (17.3.21) lar bilan birgalikda $\left|c^{(j)}_{k_0k_1\dots k_n}\right| \le C^{(j)}_{k_0k_1\dots k_n}$ tengsizliklar ham o‘rinli bo‘ladi. Demak, (17.3.24) darajali qatorlar (17.3.19) darajali qatorlar uchun majoranta bo‘ladi. Shuning uchun, agar biz (17.3.24) majoranta qatorlarning absolyut yaqinlashuvchiligini ko‘rsatsak, bundan (17.3.19) darajali qatorlarning ham absolyut yaqinlashuvchiligi kelib chiqadi.

Biz endi majoranta masalani shunday sodda tanlaymizki, natijada uning yechimi chekli ko‘rinishda topilib, majoranta qatorning yaqinlashuvchiligini bevosita tekshirish mumkin bo‘lsin.

Berilganga ko‘ra

$$\sum_{\|k\|=1}^{\infty} a^{(j)}_{k_1\dots k_n} \rho^{\|k\|} \quad \text{va} \quad \sum_{\|p\|=0}^{\infty} b^{(j;li)}_{p_1\cdots p_m} r^{\|p\|}$$

sonli qatorlar absolyut yaqinlashuvchi. Demak, shunday $M > 0$ mavjudki, uning uchun

$$\left|a^{(j)}_{k_1\dots k_n} \rho^{\|k\|}\right| \le M \quad \text{va} \quad \left|b^{(j;li)}_{p_1\cdots p_m} r^{\|p\|}\right| \le M,$$

ya’ni

$$\left|a^{(j)}_{k_1\dots k_n}\right| \le \frac{M}{\rho^{\|k\|}} \quad \text{va} \quad \left|b^{(j;li)}_{p_1\cdots p_m}\right| \le \frac{M}{r^{\|p\|}}, \qquad (17.3.25)$$

baholashlar o‘rinli bo‘ladi.

Endi ushbu

$$A^{(j)}_{k_1\dots k_n} = \frac{M}{\rho^{\|k\|}} \frac{(k_1 + \dots + k_n)!}{k_1!\dots k_n!}, \quad B^{(j;li)}_{p_1\cdots p_m} = \frac{M}{r^{\|p\|}} \frac{(p_1 + \dots + p_m)!}{p_1!\dots p_m!}$$

sonlarni kiritaylik. Ravshanki, ular nomanfiy va

$$\left|a^{(j)}_{k_1\ldots k_n}\right| \le A^{(j)}_{k_1\ldots k_n},\ \left|b^{(j;li)}_{p_1\ldots p_m}\right| \le B^{(j;li)}_{p_1\ldots p_m}.$$

Kiritilgan sonlarga ko'ra quyidagi darajali qatorlarni tuzaylik:

$$\psi_j(x_1,\ldots,x_n) = \sum_{\|k\|=1}^{\infty} A^{(j)}_{k_1\ldots k_n} x_1^{k_1}\cdot\ldots\cdot x_n^{k_n} = \sum_{\|k\|=1}^{\infty} \frac{M}{\rho^{\|k\|}}\frac{\|k\|!}{k_1!\ldots k_n!} x_1^{k_1}\cdot\ldots\cdot x_n^{k_n},$$

$$G_{j;li}(u_1,\ldots,u_m) = \sum_{\|p\|=0}^{\infty} B^{(j;li)}_{p_1\ldots p_m} u_1^{p_1}\cdot\ldots\cdot u_m^{p_m} = \sum_{\|p\|=0}^{\infty} \frac{M}{r^{\|p\|}}\frac{\|p\|!}{p_1!\ldots p_m!} u_1^{p_1}\cdot\ldots\cdot u_m^{p_m}.$$

Ravshanki (§ 17.1 ga qarang),

$$\psi_j(x_1,\ldots,x_n) = \sum_{\|k\|=0}^{\infty} \frac{M}{\rho^{\|k\|}}\frac{\|k\|!}{k_1!\ldots k_n!} x_1^{k_1}\cdot\ldots\cdot x_n^{k_n} - M =$$

$$= \frac{M}{1-\dfrac{x_1+\ldots+x_n}{\rho}} - M = M\frac{x_1+\ldots+x_n}{\rho-(x_1+\ldots+x_n)},$$

$$G_{j;li}(u_1,\ldots,u_m) = \sum_{\|p\|=0}^{\infty} \frac{M}{r^{\|p\|}}\frac{\|p\|!}{p_1!\ldots p_m!} u_1^{p_1}\cdot\ldots\cdot u_m^{p_m} = \frac{Mr}{r-(u_1+\ldots+u_m)}.$$

Shunday qilib, (17.3.22),(17.3.23) majorant masala quyidagi ko'rinishga ega:

$$\frac{\partial v_j}{\partial t} = \frac{Mr}{r-(v_1+\ldots+v_m)}\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{\partial v_l}{\partial x_i},\ j=\overline{1,m}; \qquad (17.3.26)$$

$$v_j\Big|_{t=0} = M\frac{x_1+\ldots+x_n}{\rho-(x_1+\ldots+x_n)},\ j=\overline{1,m}. \qquad (17.3.27)$$

Bu (17.3.26),(17.3.27) masalaning yechimi, ravshanki,

$$v_j(t,x_1,\ldots,x_n) = v(t,y),\ y = x_1+\ldots+x_n,$$

ko'rinishda bo'ladi. $v(t,y)$ funksiya uchun ushbu

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{Mr}{r-mv} mn\dfrac{\partial v}{\partial y}; \\ v\Big|_{t=0} = M\dfrac{y}{\rho-y}. \end{cases} \qquad (17.3.28)$$

Koshi masalasi hosil bo'ladi. Bu birinchi tartibli kvazichiziqli tenglama uchun masalani standart metod yordamida yechamiz. Xarakteristik sistema:

$$\frac{dt}{d\tau}=1,\ \frac{dy}{d\tau} = -\frac{Mmnr}{r-mv},\ \frac{dv}{d\tau}=0.$$

Demak, xarakteristikalar

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{Mmnr}{mv-r} \\ \dfrac{dv}{dt} = 0 \end{cases} \quad \text{yoki} \quad \begin{cases} y = \dfrac{Mmnr}{mv-r}t + \text{const} \\ v\text{=const} \end{cases}$$

formulalar bilan beriladi. Izlanayotgan Koshi masalasini yechish uchun, ma'lumki,

$$y\big|_{t=0} = \xi \,,\; v\big|_{t=0} = \frac{M\xi}{\rho - \xi}$$

xarakteristikalarni yozish kerak:

$$\begin{cases} y = \dfrac{Mmnr}{mv-r}t + \xi \\ v\text{=} \dfrac{M\xi}{\rho - \xi} \end{cases} \tag{17.3.29}$$

Bu formulalar yechimning parametrik ko'rinishini ifodalaydi (ξ – parametr). (17.3.29) dan ξ parametrni yo'qotib, v yechim uchun quyidagi munosabatni hosil qilamiz:

$$y = \frac{Mmnr}{mv-r}t + \frac{\rho v}{v+M}.$$

Oxirgi tenglik $v = v(t,y)$ yechimni oshkormas ko'rinishda aniqlaydi. U v ga nisbatan kvadrat tenglamadir:

$$m(\rho - y)v^2 + (Mnmrt + ry - Mmy - \rho r)v + Mr(Mnt + y) = 0 \tag{17.3.30}$$

Bu kvadrat tenglamaning diskriminanti

$$D(t,y) = (Mnmrt + ry - Mmy - \rho r)^2 - 4m(\rho - y)Mr(Mnt + y)$$

$t = y = 0$ nuqta atrofida uzluksiz va bu nuqtada musbat: $D(0,0) = \rho^2 r^2 > 0$. Demak, shu nuqtaning biror atrofida ham musbat. Biz izlayotgan v yechim (17.3.30) kvadrat tenglamaning $t = y = 0$ da 0 ga aylanuvchi yechimidir:

$$v = \frac{-(Mnmrt + ry - Mmy - \rho r) - \sqrt{(Mnmrt + ry - Mmy - \rho r)^2 - 4m(\rho - y)Mr(Mnt + y)}}{2m(\rho - y)}.$$

Bu formuladan $v = v(t,y)$ yechimning $(t,y) = (0,0)$ nuqtada analitikligi ko'rinib turibdi. Tushunarliki, $v(t,x_1,\ldots x_n) = = v(t, x_1 + ... + x_n)$ funksiya ham $t = 0, x_1 = 0, ..., x_n = 0$ nuqtada analitik.

Shunday qilib, (17.3.24) majorant qator koordinatalar boshining biror atrofida absolyut yaqinlashuvchi. Demak, (17.3.19) darajali qator ham absolyut yaqinlashuvchi. Isbotlangan tasdiq Koshi-Kovalevskaya teoremasi deb ataladi.

Teorema (Koshi-Koyealevskaya). *Aytaylik,* (17.3.13),(17.3.14) *Koshi masalasidagi berilganlar boshlang'ich funksiyalar va tenglama koeffitsientlari* $(t,x)=(0,0)$ *nuqta atrofida analitik funksiyalardan iborat bo'lsin, U holda bu masala* $(t,x)=(0,0)$ *nuqtaning biror atrofida analitik bo'lgan yagona yechimga ega*.

Bu teorema (17.3.1), (17.3.2) va (17.3.3), (17.3.4) ko'rinishdagi Koshi masalalari uchun ham o'rinlidir, chunki yuqorida aytilganidek, berilganlari analitik bo'lgan bunday masalalar ham (17.3.13),(17.3.14) ko`rinishdgi berilganlari analitik Koshi masalasiga keltiriladi.

Koshi-Kovalevskaya teoremasini Koshi sharti analitik gipersirtda berilgan holda ham isbot qilish mumkin. Bu holda faqat gipersirt "xarakteristik sirt" bo'lmasligi kerak. Bu masalalarda to'xtalmaymiz.

Masalalar

Quyidagi Koshi masalalari $(0,0)$ nuqta atrofida analitik yechimga egami?

1. $\dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x},\ u\Big|_{t=0}=\dfrac{1}{1+x^2}.$ **2.** $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2},\ u\Big|_{t=0}=\dfrac{1}{1+x^2}.$

§ 17.4. Oddiy differensial tenglamalar sistemalarining analitik yechimlari

Ushbu $\boldsymbol{x}=\boldsymbol{x}(t)$ noma'lum vektor-funksiyaga nisbatan

$$\begin{cases}\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x}),\\ \boldsymbol{x}(t_0)=\boldsymbol{x}^0\end{cases} \qquad (17.4.1)$$

Koshi masalasini qaraylik. Agar $\boldsymbol{f}(t,\boldsymbol{x})=\left(f_1(t,\boldsymbol{x}),f_2(t,\boldsymbol{x}),\ldots f_n(t,\boldsymbol{x})\right)^T$ vektor-funksiyaning barcha $f_j(t,\boldsymbol{x})=f_j(t,x_1,x_2,\ldots x_n)$, $j=\overline{1,n}$, koordinata funksiyalari analitik bo'lsa, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya analitik deb ataladi. Koshi-Kovalevskaya teoremasi, agar sistemaning o'ng tomoni analitik funksiyadan iborat bo'lsa, Koshi masalasi analitik yechimining mavjudligi va yagonaligini tasdiqlaydi.

Teorema (Normal sistema yechimining analitikligi to‘g‘-risida). *Agar* $\boldsymbol{f}(t,\boldsymbol{x})$ *vektor-funksiya* $(t_0,\boldsymbol{x}^0)\in\mathbb{R}^{1+n}$ *boshlang‘ich nuqtada analitik bo‘lsa, (17.4.1) Koshi masalasi* t_0 *nuqtada analitik bo‘lgan yagona yechimga ega.*

Teorema majorantlar metodidan foydalanib isbotlanadi.

Endi chiziqli normal sistema uchun quyidagi Koshi masalasini qaraylik:

$$\begin{cases}\boldsymbol{x}'=A(t)\boldsymbol{x}+\boldsymbol{g}(t)\\ \boldsymbol{x}(t_0)=\boldsymbol{x}^0\end{cases}\tag{17.4.2}$$

Agar matritsaviy funksiyaning barcha elementlari analitik bo‘lsa, bunday matritsaviy funksiya analitik deb ataladi.

Teorema (**chiziqli normal sistema yechimining analitiklik intervali to‘g‘risida**). *Agar (17.4.2) sistemadagi* $A(t)$ *va* $\boldsymbol{g}(t)$ *funksiyalar* t_0 *nuqtada analitik bo‘lsa, (17.4.2) Koshi masalasi* t_0 *nuqtada analitik bo‘lgan yagona* $\boldsymbol{x}=\boldsymbol{x}(t)$ *yechimga ega. Bundan tashqari agar* $A(t)$ *va* $\boldsymbol{g}(t)$ *funksiyalarning* t_0 *markazli darajali qatorlarga yoyilmasi* (t_0-R,t_0+R) *umumiy intervalda yaqinlashsa,* $\boldsymbol{x}=\boldsymbol{x}(t)$ *yechimning* t_0 *markazli darajali qatorga yoyilmasi ham shu* (t_0-R,t_0+R) *intervalda yaqinlashuvchi bo‘ladi.*

Masalalar

1. Normal sistema yechimining analitikligi to‘g‘risidagi teoremani isbotlang.

2. Chiziqli normal sistema yechimining analitiklik intervali to‘g‘risidagi teoremani isbotlang.

YECHIMLAR, KO‘RSATMALAR VA JAVOBLAR

§ 1.1

1. $a \neq b, a+b \neq 0$ bo‘lganda $y(x) = -\dfrac{\cos(a+b)x}{2(a+b)} - \dfrac{\cos(a-b)x}{2(a-b)} + c$,

$a = b \neq 0$ bo‘lganda $y(x) = -\dfrac{1}{4a}\cos(2ax) + c$, $a = b = 0$ bo‘lganda esa $y(x) = c$.

2. $x > 0$ oraliqda yechimlar $y = \int_1^x \frac{e^s}{s} ds + c$, $x < 0$ oraliqda yechimlar

$y = \int_{-1}^x \frac{e^s}{s} ds + c$.

3. $y = \frac{1}{2}x|x| + c$. **4.** Chunki $y \notin C^1((0;1))$ $(x = 1/2$ nuqtada y' hosila mavjud emas).

5. Berilgan funksiya berilgan tenglamani hech qanday oraliqda ayniyatga aylantirmasligini isbotlang.

6. Bevosita tekshiriladi**:**

a) $\varphi'(x) = f(x, \varphi(x)) \Rightarrow \varphi'(-x) = f(-x, \varphi(-x))$; demak,

$(\varphi(-x))' = -\varphi'(-x) = -f(-x, \varphi(-x)) = -(-f(x, \varphi(-x))) = f(x, \varphi(-x))$, ya’ni $y = \varphi(-x)$ ham yechim. b) va v) ham shunga o‘xshash tekshiriladi.

7. $f(x) + g(x) = \text{const} \Rightarrow f(x) + g(x) = f(0) + g(0) = \frac{\pi}{4}$, $x \in \mathbb{R}$;

$$f(+\infty) + g(+\infty) = \frac{\pi}{4} \Rightarrow \left(\int_0^{+\infty} e^{-s^2} ds\right)^2 = \frac{\pi}{4} \ .$$

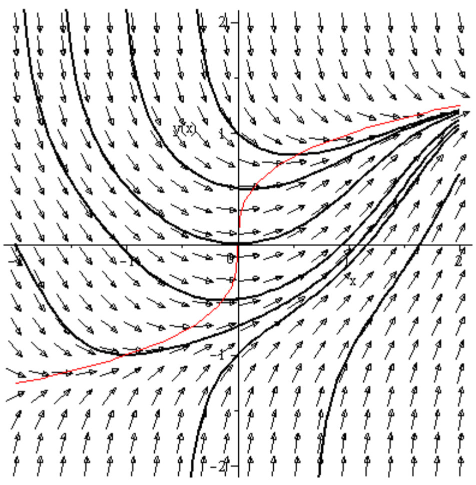
§ 1.2

1. Yechimlar cheksiz ko‘p: $y = 0,\ y = (x-c)^3$.

2. Integral chiziq $y = \dfrac{1}{1-x}$, yechim $x \in (-\infty, 0)$ da aniqlangan.

3. Tenglamani $x^2 - y^2 > 0$ va $x^2 - y^2 < 0$ to‘plamlarda qarang.

4. J.1- rasmga qarang.



J.1- rasm.

§ 1.3

1. Bevosita tekshiriladi. **2.** Yo‘nalishlar maydoni $\boldsymbol{v}=\boldsymbol{v}(x,y)=(y,x)$ kabi. $x=0, y=cx$.

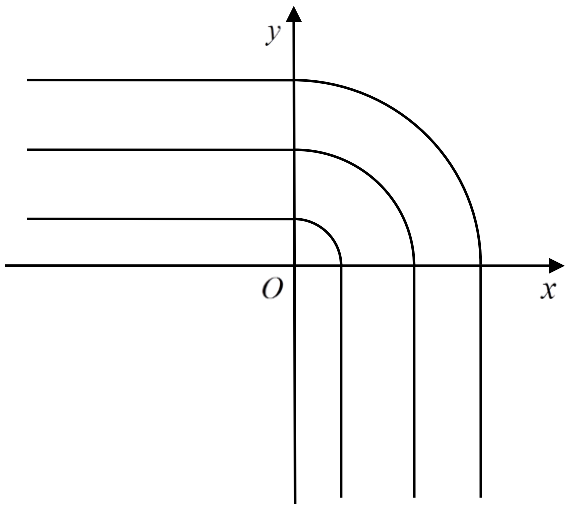
3. $x\geq 0$, $y\geq 0$ chorak tekislikda berilgan tenglama $2xdx+2ydy=0$ ko‘rinishga keladi. Uning yechimlari $x^2+y^2=c$;

$x>0$, $y\leq 0$ bo‘lganda berilgan tenglama $2xdx=0$, yechimlari $x=c$;

$x\leq 0$, $y>0$ bo‘lganda berilgan tenglama $2ydy=0$, yechimlari $y=c$;

$x\leq 0$, $y\leq 0$ bo‘lganda differensial tenglama aniqlanmagan.

Yechimlar grafiklari J.2- rasmda ko‘rsatilgan.



J.2- rasm.

§ 2.1

1. Teskarisini faraz qiling.

2. $y=c\exp\left(\frac{x|x|}{2}\right)$. **3.** $y(x)=\begin{cases} x^2/2\text{ ,agar } x\in[0;2] \text{ bo'lsa} \\ 2e^{x-2}\text{ ,agar } x\in(2;+\infty) \text{ bo'lsa}\end{cases}$

4. (*) da $v=0$ deylik. U holda

$$f(u)=\frac{f(u)+f(0)}{1-f(u)\cdot f(0)}\Rightarrow f(0)\cdot[1+f^2(u)]=0\Rightarrow f(0)=0.$$

Berilganga ko‘ra ushbu

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h}=k$$

hosila mavjud. Endi (*) ga ko‘ra $f'(x)$ ni hisoblaymiz.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)+f(h)}{1-f(x)\cdot f(h)}-f(x)\right)=$$

$$=\lim_{h\to 0}\frac{f(h)\cdot[1+f^2(x)]}{h\cdot[1-f(x)\cdot f(h)]}=k\cdot[1+f^2(x)],$$

chunki $f(h)\underset{h\to 0}{\to} f(0)=0$. ($f'(0)$ mavjud bo‘lagini uchun $f(x)$ funksiya $x=0$ nuqtada uzluksiz). Demak, $y=f(x)$ noma’lum funksiya $y'=k\cdot(1+y^2)$ differensial tenglamani qanoatlantiradi. Oxirgi tenglamani o‘zgaruvchilarni ajratib yechamiz:

$$\frac{dy}{1+y^2}=k\cdot dx \;\Rightarrow\; \text{arctg}y=kx+c$$

$y|_{x=0}=f(0)=0$ bo‘lgani uchun $c=0$ bo‘lishi kerak. Demak, agar noma’lum funksiya $y=f(x)$ mavjud bo‘lsa, u $\text{arctg}y=kx$ munosabatni qanoatlantiradi. Oxirgi tenglikdan $y=\text{tg}kx$ ekanligini topamiz. Tangens funksiyaning xossalariga ko‘ra bu $y=\text{tg}kx$ funksiyaning (*) funksional tenlamani qanoatlantirishini va $y'|_{x=0}=k$ hosilaga ega ekanligin ko‘rish qiyin emas. Shunday qilib, qo‘yilgan masalaning yechimlari $f(x) = \text{tg}kx$ $(k = \text{const})$ formula bilan beriladi. Yuqoridagi fikr yuritishlardan ravshanki, $-\frac{\pi}{2} < kx < \frac{\pi}{2}$, ya'ni $k \neq 0$ bo'lganda $f(x) = \text{tg}kx$ yechim $|x| < a = \frac{\pi}{2|k|}$ oraliqda aniqlangan. $k = f'(0) = 0$ bo'lganda esa yechim $f(x) \equiv 0$ va u $x \in (-\infty; +\infty)$ oraliqda aniqlangan.

5. $N = k/a, T = 1/k$. **6.** O'zgaruvchilari ajraladigan $y' = y^2 \cos x$ tenglamaning yechimlari $y = 1/(c - \sin x), y = 0$ formulalar bilan beriladi. Ularning orasida bittasi, ya'ni $y = 0$ davriy. Qolganlari davriy emas: o'zgarmasning $|c|>1$ qiymatlarida $(-\infty, +\infty)$ oraliqda aniqlangan davriy bo'lmagan yechimlar hosil bo'ladi; $|c| \leq 1$ bo'lganda yechim chegaralangan oraliqda aniqlangan bo'ladi, bu oraliqning chetlarida u cheksizlikka ketib qoladi.

Ushbu $y' = (y^2 + 1)(2 + \cos x)$ tenglamaning yechimlari $y = \text{tg}(2x + \sin x + c)$.

Ularning barchasi davriy.

Ushbu $y' = (y^2 - 1)(2 + \cos x)$ tenglamaning yechimlari $y = \text{th}(2x + \sin x + c), y = \pm 1$. Ularning ikkitasi ($y = \pm 1$) davriy , qolganlari $(-\infty, +\infty)$ oraliqda aniqlangan bo'lsa-da, davriy emas.

§ 2.2

1. $(x + y)\exp(x/(x + y)) = c$. (Tenglama o'zgaruvchilariga nisbatan bir jinsli).

2. $y = xu$ almashtirish bajaring.

3. $y = x^2\left(1 - \frac{1}{\ln x - c}\right)$. Tenglamada $y = z^m$ almashtirishni bajaring va $m = 2$ da o'zgaruvchilariga nisbatan bir jinsli tenglama hosil qiling.

§ 2.3

1. $y = e^x(x - 1)/x^2, x \in (0, +\infty)$. **2.** Umumiy yechim xususiy yechimga mos bir jinsli tenglamaning umumiy yechimini qo'shishdan hosil bo'ladi.

3. $v(t) = u(t) - x(t)$ deylik. Berilganga ko'ra $v' - p(t)v \geq 0, v(0) \geq 0$. Demak, $\frac{d}{dt}\left(v \cdot \exp\left(-\int_0^t p(s)ds\right)\right) \geq 0$. Buni integrallab, kerakli tengsizlikni topamiz: $v(t) \geq v(0) + \exp\left(\int_0^t p(s)ds\right) \geq 0$.

4. $y = \ln\frac{a - 1}{x + cx^a}$. $e^{-y} = z$ deng. **5.** $\cos y = \frac{c - x^3}{3(x^2 - 1)}$. $z = \cos y$ almashtirish bajaring.

6. $y = \exp(x)\int_x^{+\infty} \exp(-(1 + t^2))dt$

7. $y(x) = \begin{cases} -0,5 + e^{2x}, \text{ agar } x \leq -0,5\ln 2 \text{ bo'lsa}; \\ 0,5 - 0,25e^{-2x}, \text{ agar } x > -0,5\ln 2 \text{ bo'lsa}. \end{cases}$

§ 2.4

1. $1/x = 2 - y^2 + c\exp(-y^2/2)$ $(x = x(y)$ ga nisbatan Bernulli tenglamasi).

2. $z' = c(a-b)z - c$.

3. Ushbu

$$a(x) = \frac{y_2(x) - y(x)}{y_2(x) - y_1(x)},\ b(x) = \frac{y_3(x) - y_1(x)}{y_3(x) - y(x)}$$

funksiyalarni kiritib, $(a(x)b(x))' = a'(x)b(x) + a(x)b'(x)$ hosilaning nolga teng ekanligini bevosita tekshirish yo‘li bilan isbotlang. **4**. Oldingi masadan kelib chiqadi.

§ 2.5

1. Teskarisini faraz qiling va $du(x,y) = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ tenglikda $x = \cos\varphi, y = \sin\varphi$ deb, $du(\cos\varphi, \sin\varphi) = -d\varphi$ tenglikni hosil qiling. Oxirgi tenglikda φ ni 0 dan 2π gacha o‘zgartirib, ziddiyatga keling.

2. Aniqlangan $u(x,y)$ funksiyaning to‘la differensialini integral ostida differensiallash qoidasiga ko‘ra hisoblab, D da $du(x,y) = M(x,y)dx + N(x,y)dy$ bo‘lishini ko‘rsating.

3. $xy + x + y = c$. (to‘la differensialli tenglama).

4. $2x^2 + y^2 = cx + y$. ($u = y^2 - y$ almashtirish bajaring).

5. $xy - \ln xy^3 = c$ $(\mu = 1/(xy))$.

6. Bevosita tekshiriladi.

§ 3.1

1. Teskarisini faraz qiling. **2.** $y(0) = 0 \Rightarrow y(x) \equiv 0,\ x \in (-a; a)$.

3. $|y_1(x) - y_0(x)| \le M \Rightarrow |y_2(x) - y_1(x)| \le \sqrt[3]{M}\,|x|$,

$$|y_3(x) - y_2(x)| \le M^{1/9}\frac{3}{4}|x|^{4/3},\ |y_4(x) - y_3(x)| \le M^{1/27}\frac{3^2}{4\cdot 7}|x|^{7/3},....$$

baholashlarni hosil qiling. Ushbu $y_0(x) + (y_1(x) - y_0(x)) + (y_2(x) - y_1(x)) + \ldots$ qator va $\{y_n(x)\}$ ketma-ketlik tekis yaqinlashuvchi bo‘ladi. Limit funksiya berilgan Koshi masalasining yechimidan iborat.

4. MyaTning shartlari bajarilmaydi.

5. $y > 0$ va $y < 0$ yarim teksliklarsa sohada MYaT ni qo‘llash mumkin; $y = 0$ yechimning har bir $(x_0, 0)$ nutasidan boshqa bir $y = (x - x_0)^3$ yechim ham o‘tadi.

6. Bu matematik analizda isbotlanadi.

7. Ta’rifdan va chekli orttirmalar to‘g‘risidagi Lagranj formulasidan foydalaning.

8. $y(x) = y(x_0) + \int\limits_{x_0}^{x} y'(s)ds$ tenglikdan foydalaning.

§ 3.2

1. $v(x) = v_0 + \int\limits_{x_0}^{x} (\varphi_1(s)u(s) + \varphi_2(s))ds,\ x \in [x_0, b),$ deylik. U holda $u(x) \le v(x)$ va $\varphi_1(x) \ge 0$ bo‘lgani uchun

$$\frac{dv(x)}{dx} = \varphi_1(x)u(x) + \varphi_2(x) \le \varphi_1(x)v(x) + \varphi_2(x).$$

Oxirgi tengsizlikni $\exp\left(-\int_{x_0}^{x}\varphi_1(s)ds\right)$ ga ko‘paytirib, topamiz

$$\frac{d}{dx}\left(\exp\left(-\int_{x_0}^{x}\varphi_1(s)ds\right)\cdot v(x)\right)\le\varphi_2(x)\exp\left(-\int_{x_0}^{x}\varphi_1(s)ds\right).$$

Bu tengsizlikni integrallab va $v(x_0)=v_0$ ni hisobga olib, oddiy shakl almashtirishdan keyin izlangan tengsizlikni hosil qilamiz.

2. Berilgan integral tengsizlikdan $x=x_0$ da $u(x_0)<v(x_0)$ bo‘lishini ko‘ramiz. Uzluksiz funksiyalar xossasiga ko‘ra x_0 ga yetarlicha yaqin $\delta>x_0$ son mavjudki, barcha $x\in[x_0,\delta)$ nuqtalarda ham $u(x)<v(x)$ bo‘ladi. Faraz qilaylik, $u(x)<v(x)$ tengsizlik $[x_0,b)$ oraliqda bajarilmasin. U holda shunday $\xi\in(x_0,b)$ mavjudki, barcha $x\in(x_0,\xi)$ lar uchun $u(x)<v(x)$, va $u(\xi)=v(\xi)$. Berilgan integral tengsizlikka ko‘ra esa

$$u(\xi)<v_0(\xi)+\int_{x_0}^{\xi}F(\xi,s,u(s))ds\le v_0(\xi)+\int_{x_0}^{\xi}F(\xi,s,v(s))ds=v(\xi)$$

bo‘ladi. Bu esa farazimizga zid.

3. Yechimlar ayirmasining moduli uchun

$$|y(x)-u(x)|=\left|y_0-u_0+\int_{x_0}^{x}\left(f(s,y(s))-g(s,u(s))\right)ds\right|\le$$

$$\le|y_0-u_0|+\left|\int_{x_0}^{x}|f(s,y(s))-f(s,u(s))+f(s,u(s))-g(s,u(s))|ds\right|\le$$

$$\le|y_0-u_0|+\varepsilon|x-x_0|+L\left|\int_{x_0}^{x}|y(s)-u(s)|ds\right|.$$

Hosil bo‘lgan tengsizlikka Gronuoll-Bellman tipidagi tengsizlikni qo‘llang.

4. 3- masalaning yechilishiga qarang.

5. Gronuoll-Bellman tipidagi tengsizlik isbotiga qarang.

6. Ushbu $y(x)=y(x_0)+\int_{x_0}^{x}y'(s)ds$ formulaga ko‘ra

$$|y(x)|\le|y(x_0)|+\left|\int_{x_0}^{x}|y'(s)|ds\right|\le|y(x_0)|+\left|\int_{x_0}^{x}|\beta+\gamma|y(s)||ds\right|\le$$

$$\le|y(x_0)|+\beta|x-x_0|+\gamma\left|\int_{x_0}^{x}|y(s)|ds\right|.$$

Endi Gronuoll-Bellman tipidagi tengsizlikni qo‘llang.

§ 3.3

1. Paragrafda keltirilgan qurish va isbotlarni takrorlang.

2. Teskarisini faraz qiling.

3. Peano teoremasining isbotiga qarang.

§ 3.4

1. $y=\varphi(x)$ yechim chegaralangan bo‘lsin: $\exists m>0\ \forall x\in\mathbb{R}\ |\varphi(x)|\le m$. Agar bu yechimni $[-m',m']\times[-a,a]$ $(m'>m, a-$ ixtiyoriy musbat son) to‘rtburchakda davom ettirsak, yechim to‘rtburchakning yon tomonlaridan chiqib ketadi (yuqori va quyi tomonlariga yetib borolmaydi).

2. Har qanday $y=y(x)$ yechim uchun x nuqtada

$$y(x)=y_0+\int_0^x \frac{ds}{1+s^2+y^{2018}(s)}.$$

Demak,

$$|y(x)|\le |y_0|+\left|\int_0^x \frac{ds}{1+s^2}\right|\le |y_0|+\int_0^\infty \frac{ds}{1+s^2}=\text{const}.$$

3. $x_1,x_2\in(\alpha,\beta)$ nuqtalar uchun $|\varphi(x_1)-\varphi(x_2)|=\left|\int_{x_1}^{x_2} f(s,\varphi(s)ds\right|\le \text{const}\,|x_1-x_2|$ tengsizlikdan kelib chiqib, limitning mavjudligi haqidagi Koshi mezonini qo‘llang va

$$\begin{cases} y'=f(x,y) \\ y(\alpha)=\delta \end{cases} \text{ va } \begin{cases} y'=f(x,y) \\ y(\beta)=\gamma \end{cases}$$

boshlang‘ich masalalarni qarang.

4. Yo‘q. **5.** 3- masalaga qarang.

6. Gronuoll-Bellman tipidagi tengsizlikdan foydalaning (§ 3.2. dagi 6- masalaga qarang).

7. Yechim $[0,b]$ da aniqlangan bo‘lsin. Bu oraliqda $y'>0,\ y(0)=0\ \Rightarrow y(x)>0$. Ravshanki,

$$y(x)=\int_0^x (s^2+y^2(s))ds=\frac{x^3}{3}+\int_0^x y^2(s)ds\le \frac{b^3}{3}+\int_0^x y^2(s)ds.$$

$u(x)=\dfrac{b^3}{3}+\int_0^x y^2(s)ds$ deylik. U holda $y(x)\le u(x)$ va $u'(x)=y^2(x)\le u^2(x)$. Bundan $\dfrac{u'}{u^2}\le 1,\ -\left(\dfrac{1}{u}\right)'\le 1,$ $\dfrac{1}{u(0)}-\dfrac{1}{u(x)}=x,$ $\dfrac{3}{b^3}-\dfrac{1}{u(x)}=x,$ $u(x)=\dfrac{b^3}{3-b^3x}.$ Demak, $y(x)\le u(x)=\dfrac{b^3}{3-b^3x},$ $x\in[0;3/b^3].$ $b=3/b^3\Rightarrow b^4=3\Rightarrow b\approx 1{,}3161.$ Yechim kamida $[0;b]\approx[0;1{,}3161]$ oraliqda aniqlangan. x ni $-x$, y ni $-y$ bilan almashtirib, $y(x)$ yechim kamida $[-b,b]\approx[-1{,}3161;1{,}3161]$ oraliqda aniqlanganligini topamiz.

8. 7- msalaning yechilishidan foydalanib, yechim o‘ngga kamida $[1;b]\approx[1;1{,}1549]$ gacha davom etishini asoslang; aniq hisoblashlar yechim $[1;1{,}25609]$ gacha davom etishini ko‘rsatadi. Yechimni chapga davom ettirish uchun $x=1-t$ almashtirish bajaring.

§ 4.1

1. Tenglamadan y' ni toping. Hosil bo‘lgan differensial tenglamalarni yeching. Yechimning yagonalik xossasi buziladigan nuqtalarni aniqlang. MyaT ni ham qo‘llang.

2. 1- masalaga o‘xshash.

§ 4.4

1. 1) Lagranj tenglamasi.

$$\begin{cases} x(p)=\ln|p|-\arcsin p+c, \\ y(p)=p+\sqrt{1-p^2}, \end{cases} \quad p\in(-1;0) \text{ yoki } p\in(0;1);\ y(x)\equiv 1.$$

2) Lagranj tenglamasi.

$$\begin{cases} x(p)=ce^{-p}-2p+2, \\ y(p)=c(1+p)e^{-p}-p^2+2, \end{cases} \quad p\in(-\infty;+\infty).$$

3) Klero tenglamasi.

$$y=cx+\sqrt{1+c^2}\ ;\ \text{maxsus yechim: } y=\sqrt{1-x^2}.$$

§ 5.1

1. Ushbu

$$\frac{dx}{dt}+\omega^2x=0 \quad (\omega=\text{const}>0)$$

garmonik ossilyator tenglamasining har qanday yechimi $x=c_1\cos\omega t+c_2\sin\omega t$ formuladan c_1,c_2 o‘zgarmaslarning biror tayin qiymatida hosil bo‘lishini ko‘rsating.

2. h balandlikdan v_0 boshlang‘ich tezlik bilan yuqoriga tik otilgan jismning harakat tenglamasini yozing. Mos boshlang‘ich masalani qo‘ying va uni yeching.

1. Soddalik uchun $\omega=1$ deymiz. Ixtiyoriy $y=y(x)$ yechim ($y''(x)+y(x)=0$) ga k`ora ushbu

$$Y_1(x)=y(x)\cos x-y'(x)\sin x,$$
$$Y_2(x)=y(x)\sin x+y'(x)\cos x$$

funksiyalarni tuzaylik. Ularning hosilasi nolga teng:

$$Y_1'(x)=y'(x)\cos x-y(x)\sin x-y''(x)\sin x-y'(x)\cos x=-\big(y(x)+y''(x)\big)\sin x=0,$$
$$Y_2'(x)=\ldots=0.$$

Demak, $Y_1(x)$ va $Y_2(x)$ lar o‘zgarmas:

$$\begin{cases} c_1=y(x)\cos x-y'(x)\sin x \\ c_2=y(x)\sin x+y'(x)\cos x \end{cases} \quad (c_1,c_2-\text{const})$$

Oxirgi sistemani yechib, $y(x)=c_1\cos x+c_2\sin x$ ekanligini topamiz.

2. Ox o‘qni tik yuqoriga yo‘naltirylik $O-$Yer sathida. Jismga (moddiy nuqtaga) faqat og‘irlik kuchi mg ta’sir etadi deb faraz qilamiz. $x=x(t)-$ moddiy nuqtaning t paytagi koordinatasi bo‘lsin. U holda $v=\dfrac{dx}{dt}-$tezlik, $a=\dfrac{dv}{dt}=\dfrac{d^2x}{dt^2}=x''-$tezlznish va Nyutonnig ikkinchi qonuniga ko‘ra $ma=-mg$, $x''=-g$.

Boshlang‘ich shartlar: $x(0)=h, x'(0)=v_0$. Hosil bo‘lgan Koshi masalasining yechimi:

$$x=h+v_0t-\frac{gt^2}{2}.$$

§ 5.2

1. Ushbu

$$\sum_{k=1}^{n}|a_k|/\sqrt{n} \le \sqrt{\sum_{k=1}^{n}a_k^2} \le \sum_{k=1}^{n}|a_k| \ \left(a_k \in \mathbb{R}, k=\overline{1,n}\right)$$

tengsizlikdan (isbotlab) foydalaning.

2. 1^0, 2^0 lar bevosita tekshiriladi. 3^0 Peano teoremasining isbotiga o‘xshah.

§ 5.3

1. $y=c_1, y=2c_1\left(c_1 \pm \sqrt{c_1^2-x-c_2}\right)-x-c_2$. **2.** $y=c_1\exp(-c_2x^3)x^{-1/3}$.

3. $x=te^{-t}+c_1$, $y=(t^2+t+1)e^{-t}+c_2$. Yechilishi:

$$y'=t,\ \frac{dt}{dx}=\frac{e^t}{1-t} \Rightarrow x=te^{-t}+c_1;\ \frac{dy}{dx}=t,\ dy=td(te^{-t}+c_1) \Rightarrow y=(t^2+t+1)e^{-t}+c_2.$$

4. $y=-x+e^{x^2/2}(1+c_1)\int e^{-x^2/2}dx+c_2e^{x^2/2}$.

§ 6.1

1. Yo‘q. Yechimning yagonalik xossasidan foydalaning.

2. Ha.

3. *a*) $2p'+p^2=4q$; *b*) $q'+2pq=0$. y_1 va xy_1ning yechim ekanligidan $2y_1'+p(x)y_1=0 \Rightarrow y_1=\exp\left(-\frac{1}{2}\int p(x)dx\right)$ kelib chiqadi. Bu $y_1$ni tenglamaga qo‘yib, $2p'+p^2=4q$ shartni hosil qilamiz. … **.**

§ 6.2

1. Yo‘q, bunday funksiyalar (nol-funksiya qatnashgani sababli) har doim chiziqli bog‘langan.

4. $\{y_i(x)\}_{i=1}^n$ lar chiziqli erkli va $\det[a_{ij}]\ne 0$ bo‘lsin. Agar $\sum_{i=1}^{n}\lambda_i z_i(x)=0$ bo‘lsa,

$$\sum_{i=1}^{n}\lambda_i z_i(x)=\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n}a_{ij}y_j(x)=\left(\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i a_{ij}\right)y_j(x)=0 \quad \text{va, demak,} \quad \sum_{i=1}^{n}\lambda_i a_{ij}=0,\ j=\overline{1,n},$$

oxirgi tengliklardan $\lambda_1=...=\lambda_n=0$, ya’ni $\{z_i(x)\}_{i=1}^n$ lar ham chiziqli erkli. Agar $\{z_i(x)\}_{i=1}^n$ lar chiziqli erkli bo‘lsa, $\{y_i(x)\}_{i=1}^n$ lar ular orqali chiziqli ifodalanganligi sababli ($\det[a_{ij}]\ne 0$) yuqoridagilarni takrorlab, $\{y_i(x)\}_{i=1}^n$ larning ham chiziqli erkliligini ko‘ramiz.

§ 6.3

1 – **2** mos funksiyalarni chiziqli erkli yechimlar ekanligini isbotlang.

§ 6.4

1. *a*) $y''-\frac{2\varphi'(x)}{\varphi(x)}y'+\left(\frac{2\varphi'^2(x)}{\varphi^2(x)}-\frac{\varphi''(x)}{\varphi(x)}\right)y=0$; *b*) $y''-\left(\frac{2\varphi'(x)}{\varphi(x)}+\frac{\varphi''(x)}{\varphi'(x)}\right)y'+\frac{2\varphi'^2(x)}{\varphi^2(x)}y=0$.

2. Izlanayotgan $y=y(x)$ yechimni Ostrogradskiy-Liuvill formulasiga ko‘ra topish mumkin:

$$\begin{vmatrix} \varphi_1(x) & y \\ \varphi_1'(x) & y' \end{vmatrix} = \exp\left(-\int a_1(x)dx\right), \ \varphi_1(x)y' - \varphi_1'(x)y = \exp\left(-\int a_1(x)dx\right),$$

$$\frac{\varphi_1(x)y' - \varphi_1'(x)y}{\varphi_1^2(x)} = \frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right), \ \left(\frac{y}{\varphi_1(x)}\right)' = \frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right),$$

$$y = \varphi_1(x)\int \frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right)dx .$$

3. $\int_a^{+\infty} a_1(x)dx = +\infty$. Ostrogradskiy-Liuvill formulasidan foydalaning.

§ 6.5

1. $y = \frac{c_1}{x} + \frac{c_2 \ln x}{x} + \frac{x + \mathrm{Si}(x)}{x}$, $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt$.

2. Berilgan differensial tenglamaning ixtiyoriy $x = x(t)$ yechimi

$$x'' + x = -\varphi(t)x \tag{1}$$

tenglikni qanoatlantiradi. Ushbu

$$x'' + x = f(t) \tag{2}$$

tenglama uchun Koshi formulasiga ko‘ra

$$x(t) = A\sin(t-\varphi) + \int_{t_0}^{t} \sin(t-s)f(s)\,ds \ \ (A, \varphi - \mathrm{const}) . \tag{3}$$

(3) formulada $f(t) = -\varphi(t)x(t)$ deb, berilgan (1) tenglamaning yechimi uchun

$$x(t) = A\sin(t-\varphi) - \int_{t_0}^{t} \sin(t-s)\varphi(s)x(s)ds \tag{4}$$

integral tenglamaga kelamiz. $M(t) = \sup\limits_{t_0 \le s \le t} |x(s)|$ deylik. $|x(s)|$ uzluksis funksiya bo‘lgani uchun bu supremum biror $\tau = \tau(t), t_0 \le \tau \le t$, nuqtada erishiladi, ya’ni $M(t) = |x(\tau)|$ bo‘ladi. Demak, (4) integral munosabatdan yetarlicha katta t lar (aniqrog‘i $t > t_0 > c$) uchun quyidagi baholashlarni hosil qilamiz:

$$M(t) = |x(\tau)| \le |A| + M(\tau)\int_{t_0}^{\tau} |\varphi(s)|ds \le |A| + M(t)\int_{t_0}^{t} \frac{c}{s^2}ds \le |A| + cM(t)\left(\frac{1}{t_0} - \frac{1}{t}\right)$$

tengsizlikni hosil qilamiz. Bundan o‘sha t lar uchun

$$M(t) \le \frac{|A|}{1 + \frac{c}{t} - \frac{c}{t_0}}, \text{ ya'ni } M(t) \le \frac{|A|}{1 - \frac{c}{t_0}} \ \ (t > t_0 > c).$$

Demak, $t \to +\infty$ da $|x(t)|$ yuqoridan $\frac{|A|}{1 - \frac{c}{t_0}}$ son bilan chegaralangan.

3. Bevosita tekshiriladi.

§ 7.1

1. Bevosita tekshiriladi. **2.** $k \in \mathbb{N}$ bo‘yicha induksiya qo‘llang.

3. e^z ning haqiqiy va mavhum qismlarini ajrating. **4.** $\mathrm{Re}\,y(x)$ va $\mathrm{Im}\,y(x)$ haqiqiy funksiyalarni qarang.

5. $y'(x)=zy(x)\Leftrightarrow y'(x)e^{-zx}-y(x)ze^{-zx}=0\Leftrightarrow\Leftrightarrow\left(y(x)e^{-zx}\right)'=0\Leftrightarrow y(x)e^{-zx}=c=\text{const}$.

§ 7.2

1. $y=c_1e^x+c_2xe^x+c_3x^2e^x$.

2. $y=c_1e^{2x}+c_2e^x\cos x+c_3e^x\sin x+c_4xe^x\cos x+c_5xe^x\sin x$.

3. $y'''-3y''+3y'-y=0$. Dastlab xarakteristik tenglama tuzing: $(\lambda-1)^3=0,\ \lambda^3-3\lambda^2+3\lambda-1=0$.

4.. $y^{IV}-4y'''+8y''-8y'+4y=0$

5.. $y^{IV}-5y'''+9y''-7y'+2y=0$. Xarakterisnik sonlarni tuzing: $\lambda_1=\lambda_2=\lambda_3=1,\lambda_4=2$.

6. $y=c_1x^2+c_2x+c_3x\ln x$

§ 7.3

1. $y=xe^{2x}+c_1e^{2x}+c_2e^x\cos 2x+c_3e^x\sin 2x$;

§ 8.1

1. $y''^2-4y(y^2+1)y''-16y^2y'+4(y^4-3)y^2=0$.

2. Sistemadagi (1) tenglamani differensiallash natijasida hosil bo‘lgan tenglamadan y' hosilani (2) tenglamadan foydalanib yo‘qoting:

$$x''+3x^2x'-x^3-(x'+3x^2)y-xy^2+2y^3=0 \qquad (3)$$

Endi (1) va (3) tenglamalardan y noma’lumni yo‘qotish uchun (3)ni y ga, (1) ni y va y^2 ko‘paytirib, quyidagi tengliklar sistemasini hosil qiling:

$$x''+3x^2x'-x^3-(x'+3x^2)y-xy^2+2y^3=0 \qquad (3)$$

$$(x''+3x^2x'-x^3)-(x'+3x^2)y^2-xy^3+2y^4=0 \qquad (4)$$

$$x'+x^3-xy-y^2=0 \qquad (1)$$

$$(x'+x^3)y-xy^2-y^3=0 \qquad (5)$$

$$(x'+x^3)y^2-xy^3-y^4=0 \qquad (6)$$

Bu sistemani $1,y,y^2,y^3,y^4$ “noma’lumlar”ga nisbatan chiziqli bir jinsli algebraik tenglamalar sistemasi deb qarab, u notrivial yechimga ega bo‘lgani sababli uning determinantining nolga tengligi shartini yozib, izlangan tenglamani toping:

$$x''^2+(6xx'-7x'-8x^3-2x^2)xx'-x'^3+(9x^2-26x+12)x^2x'-$$
$$-(32x^2-21x-7)x^4x'-4x^9+15x^8+8x^7+x^6=0.$$

§ 8.2

1. Ixtiyoriy tayinlangan $\boldsymbol{z}\in\mathbb{R}^n$ uchun $\varphi(s)=(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{x}+s(\boldsymbol{y}-\boldsymbol{x}))),\quad s\in[0;1]$, bir o‘zgaruvchining funksiyasini qarang. ϕ funksiyaning $[0;1]$ da differensiallanuvchiligini asoslang va bir o‘zgaruvchining haqiqiy funksiyasi uchun Lagranj teoremasiga ko‘ra biror $\theta\in(0;1)$ uchun $|\varphi(1)-\varphi(0)|=|\varphi'(\theta)|\le\sup\limits_{0<\tau<1}|\varphi'(\tau)|$ ekanligidan foydalanib, fikrlashni davom ettiring.

5. $\boldsymbol{x}=0\in\mathbb{R}^n$ da differensiallanuvchi emas.

8. Funksiya $|t|<1$, $|x_1|<1$, $|x_2|<1$ to‘plamda x_1,x_2 bo‘yicha Lipshits shartini qanoatlantirmaydi, $|t|<1$, $\varepsilon<|x_1|<1$, $\varepsilon<|x_2|<1$ $(0<\varepsilon<1)$ to‘plamda esa – qanoatlantiradi.

9. Agar $\mathbf{x}\in E$ bo‘lsa, ixtiyoriy $\mathbf{y}\in E$ uchun berilganga ko‘ra

$$f(\mathbf{x})-f(\mathbf{y})\le|f(\mathbf{x})-f(\mathbf{y})|\le L|\mathbf{x}-\mathbf{y}| \;\Rightarrow\; f(\mathbf{x})\le f(\mathbf{y})+L|\mathbf{x}-\mathbf{y}|$$

va aniq quyi chegara (inf) ta’rifuga ko‘ra

$$f(\mathbf{x})\le\inf_{y\in E}\{f(\boldsymbol{y})+L\,|\,\boldsymbol{x}-\boldsymbol{y}\,|\}=\tilde f(\boldsymbol{x}).$$

Ravshanki, $\mathbf{x}\in E$ bo‘lgani uchun $\tilde f(\boldsymbol{x})=\inf_{y\in E}\{f(\boldsymbol{y})+L\,|\,\boldsymbol{x}-\boldsymbol{y}\,|\}\le f(\boldsymbol{x})$ ($\boldsymbol{y}=\boldsymbol{x}$ olish mumkin).

Demak, $\mathbf{x}\in E$ uchun $\tilde f(\boldsymbol{x})=f(\boldsymbol{x})$. Endi $\tilde f$ ning Lipshits shartini qanoatlantirishini ko‘rsatamiz.

$\mathbf{x}\in\mathbb{R}^n$ va $\mathbf{z}\in\mathbb{R}^n$ bo‘lsin. U holda $L|\mathbf{x}-\mathbf{y}|\le L|\mathbf{z}-\mathbf{y}|+L|\mathbf{x}-\mathbf{z}|$ uchburchak tengsizligiga ko‘ra

$$\tilde f(\boldsymbol{x})=\inf_{y\in E}\{f(\boldsymbol{y})+L\,|\,\boldsymbol{x}-\boldsymbol{y}\,|\}\le\inf_{y\in E}\{f(\boldsymbol{y})+L|\mathbf{z}-\mathbf{y}|+L|\mathbf{x}-\mathbf{z}|\}=\tilde f(\mathbf{z})+L|\mathbf{x}-\mathbf{z}|,$$

ya’ni $\tilde f(\boldsymbol{x})\le\tilde f(\mathbf{z})+L|\mathbf{x}-\mathbf{z}|$. Bu yerda $\mathbf{x}$ va $\mathbf{z}$ ning o‘rinlarini almashtirib, $\tilde f(\mathbf{z})\le\tilde f(\boldsymbol{x})+L|\mathbf{z}-\boldsymbol{x}|$ tengsizlikni ham topamiz. Demak, $\left|\tilde f(\boldsymbol{x})-\tilde f(\mathbf{z})\right|\le L|\boldsymbol{x}-\mathbf{z}|$.

§ 8.3

1. $\mathbf{u}(t)$ va $\boldsymbol{\varphi}(t)$ funksiyalarning $t=a$ nuqtada uzluksizligidan $\mathbf{u}(t)>\boldsymbol{\varphi}(t)$ tengsizlikning biror $t\in[a,a+\delta]$ $(\delta>0)$ ora’iqda bajarilishi kelib chiqadi. Agar bu tengsizlik $[a,b]$ segmentda bajarilmasa, ya’ni $\delta<b-a$ bo‘lsa, u holda shunday eng kichik $c>a+\delta$ son va $j\in[1,n]$ indeks topiladiki, ular uchun

$\mathbf{u}(t)>\boldsymbol{\varphi}(t),\ a\le t<c,\ \mathbf{u}(c)\ge\boldsymbol{\varphi}(c),\ \mathbf{u}^j(c)=\boldsymbol{\varphi}^j(c)$

bo‘ladi. Bu holda shartga ko‘ra

$D^-u^j(c)>f^j(t,\mathbf{u}(c))\ge f^j(t,\boldsymbol{\varphi}(c))=\varphi^{j\prime}(c)$.

Lekin,

$$D^-u^j(c)=\overline{\lim_{h\to0-}}\frac{u^j(c+h)-u^j(c)}{h}\le\overline{\lim_{h\to0-}}\frac{\varphi^j(c+h)-\varphi^j(c)}{h}=\varphi^{j\prime}(c).$$

2. $x'(t)=f(x(t))$ ayniyatni $x'(t)$ga ko‘paytirib, uni hadma-had $t\in[a,b]$ segment bo‘yicha integrallaymiz va $x(a)=x(b)$ ekanligini hisobga olib topamiz:

$$\int_a^b x'^2(t)\,dt=\int_a^b f(x(t))x'(t)\,dt=\int_a^b\frac{dF(x(t))}{dt}dt=F(x(b))-F(x(a))=0,$$

bu yerda $F(x)=\int_0^x f(s)ds$. $x'^2(t)$ uzluksiz va nomanfiy bo‘lgani uchun $\int_a^b x'^2(t)\,dt=0$ tenglikdan $x'(t)\equiv0$, $t\in[a,b]$, ya’ni $x(t)=\text{const}$ ekanligi kelib chiqadi.

Tasdiqning $\mathbf{f}\in C(\mathbb{R}^n,\mathbb{R}^n)$, $n>1$, holida o‘rinli emasligi quyidagi misoldan kelib chiqadi:

$$\begin{cases} x_1' = -x_2 \\ x_2' = x_1 \end{cases}$$

sistema $x_1 = \cos t,\ x_2 = \sin t$ o‘zgarmasdan farqli yechimga ega va bu yechim uchun $x_1(0) = x_1(2\pi),\ x_2(0) = x_2(2\pi)$.

Endi faraz qilaylik, $\mathbf{f} = \mathrm{grad}\varphi$, $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$, bo‘lsin. Bu holda $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ sistemaning $\mathbf{x} = \mathbf{x}(t), t \in [a,b],\ \mathbf{x}(a) = \mathbf{x}(b)$, yechimi uchun quyidagilarga egamiz:

$$\int_a^b \|\mathbf{x}'(t)\|^2 dt = \int_a^b (\mathbf{x}'(t), \mathbf{x}'(t))\, dt = \int_a^b (\mathbf{f}(\mathbf{x}(t)), \mathbf{x}'(t))\, dt = \int_a^b \sum_{j=1}^n \frac{\partial \varphi(\mathbf{x}(t))}{x_j} \cdot x_j'(t)\, dt =$$

$$= \int_a^b \frac{d\varphi(\mathbf{x}(t))}{dt} dt = \varphi(\mathbf{x}(b)) - \varphi(\mathbf{x}(a)) = 0.$$

Demak, $\|\mathbf{x}'(t)\| \equiv 0,\ t \in [a,b]$, ya’ni $\mathbf{x}(t) = \mathrm{const}$.

4. (K) masalaning $t \geq t_0$ da $\boldsymbol{x} = \boldsymbol{x}(t)$ va $\boldsymbol{y} = \boldsymbol{y}(t)$ yechimlari berilgan bo‘lsin:

$$\boldsymbol{x}'(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)),\ \ \boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)),\ \ \boldsymbol{x}(t_0) = \boldsymbol{y}(t_0) = \boldsymbol{x}^0.$$

Ushbu $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t),\ u(t) = |\boldsymbol{z}(t)|$ belgilashlarni kiritaylik. Ravshanki,

$$\boldsymbol{z}(t) \cdot \boldsymbol{z}(t) = |\boldsymbol{z}(t)|^2 = u^2(t)\ ,\ u(t_0) = 0.$$

Quyidagilarga egamiz:

$$2u(t)\frac{du(t)}{dt} = 2\boldsymbol{z}(t) \cdot \frac{d\boldsymbol{z}(t)}{dt} = 2(\boldsymbol{x}(t) - \boldsymbol{y}(t)) \cdot (\boldsymbol{f}(t, \boldsymbol{x}(t)) - \boldsymbol{f}(t, \boldsymbol{y}(t))) \leq$$

$$\leq 2 \cdot |\boldsymbol{x}(t) - \boldsymbol{y}(t)|\, \varphi(|\boldsymbol{x}(t) - \boldsymbol{y}(t)|) = 2u(t) \cdot \varphi(u(t)).$$

Demak,

$$u(t)\frac{du(t)}{dt} \leq u(t)\,\phi(u(t)),\ u(t_0) = 0 \quad (*)$$

Biz $t \geq t_0$ bo‘lganda $u(t) = 0$ bo‘lishini ko‘rsatishimiz kerak. Teskarisini faraz qilaylik, ya’ni biror $t_* > t_0$ nuqtada $u(t_*) > 0$ bo‘lsin. $u(t)$ funksiyaning $[t_0; t_*]$ segmentdagi nollari to‘plamini qaraylik:

$$F = \{ t \in [t_0; t_*] \,|\, u(t) = 0 \}.$$

$F \neq \varnothing$, chunki $t_0 \in F$. $u(t)$ ning uzluksizligidan F ning yopiqligi ravshan. F yuqoridan t_* bilan chegaralangan. Demak, uning aniq yuqori chegarasi mavjud

$$\bar{t} \overset{def}{=} \sup F,\ \ \bar{t} \leq t_*.$$

F yopiq bo‘lgani uchun $\bar{t} \in F$, ya’ni $u(\bar{t}) = 0$ Bundan $\bar{t} < t_*$ ekanligi kelib chiqadi. Endi ravshanki, $(\bar{t}, t_*]$ oraliqda $u(t) > 0$ va (*) dan shu oraliqda

$$\frac{u'(t)}{\varphi(u(t))} \leq 1,\ \ t \in (\bar{t}, t_*].$$

Bu tengsizlikni $[\tau; t_*] \subset (\bar{t}, t_*]$ segmentda integrallaymiz:

$$\int_\tau^{t_*} \frac{du(t)}{\varphi(u(t))} \leq t_* - \tau,\ \text{ya'ni} \int_{u(\tau)}^{u(t_*)} \frac{ds}{\varphi(s)} \leq t_* - \tau,\ \bar{t} < \tau < t_*\ .$$

Oxirgi tengsizlikda $\tau \to \bar{t} + 0$ deb limitga o‘tib,

$$\int_0^{u(t_*)} \frac{ds}{\varphi(s)} \le t_* - \tau < +\infty$$

munosabatni hosil qilamiz. Bu esa berilganga zid. Shunday qilib $u(t) \equiv 0$, ya'ni $\boldsymbol{x}(t) \equiv \boldsymbol{y}(t)$.

5. Yechimning mavjudligi Peano teoremasidan, uning yagonaligi esa yuqoridagi masaladan kelib chiqadi, chunki bu masala shartlari qanoatlanadi. Haqiqatan ham, Koshi Bunyakovskiy tengsizligiga ko‘ra

$$(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\boldsymbol{y})) \le |\boldsymbol{x}-\boldsymbol{y}|\cdot\left|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\boldsymbol{y})\right| \le |\boldsymbol{x}-\boldsymbol{y}|\cdot\varphi\left(|\boldsymbol{x}-\boldsymbol{y}|\right).$$

§ 8.4

1. Yangi $\tau = \ln t$ erkli o‘zgaruvchiga o‘ting. Yechim ko‘rinishidan ravshanki, u $t=0$ nuqtaga davom etmaydi.

3. Teskarisini faraz qilamiz. $(y' > 0,\ y(0) \ge 0 \Rightarrow y -$ o‘suvchi $\Rightarrow y(x) \underset{x\to 2,6-}{\longrightarrow} +\infty$

$$y'(x) = y^2(x) + x^2 \ge y^2 + \varepsilon^2,\ \varepsilon \le x < 2,6\ (0 < \varepsilon < 2,6).$$

$$\frac{dy}{y^2+\varepsilon^2} \ge dx\ \int_\varepsilon^x \Rightarrow \frac{1}{\varepsilon}\left(\operatorname{arctg}\frac{y(x)}{\varepsilon} - \operatorname{arctg}\frac{y(\varepsilon)}{\varepsilon}\right) \ge x - \varepsilon$$

$\operatorname{arctg}\dfrac{y(x)}{\varepsilon} - \operatorname{arctg}\dfrac{y(\varepsilon)}{\varepsilon} \ge \varepsilon(x-\varepsilon)$, bu yerda $x \to 2,6-$ deymiz va

$\operatorname{arctg}\dfrac{y(\varepsilon)}{\varepsilon} \le \dfrac{\pi}{2} - \varepsilon(2,6-\varepsilon)$ ni hosil qilamiz; oxirgi tengsizlikda $\varepsilon = 1,3$ deb ziddiyatga

kelamiz: $\operatorname{arctg}\dfrac{y(1,3)}{1,3} \le \dfrac{\pi}{2} - (1,3)^2 < 0$.

4. Faraz qilaylik, berilgan Koshi masalasining $x = x(t)$ yechimi $[0,b)\ (b < +\infty)$ oraliqda aniqlangan bo‘lsin. Odatdagidek normal sistemaga o‘tamiz:

$$x' = y,\ y' = -g(x) - f(y),\ x(0) = x_0,\ x'(0) = v_0.$$

$x'' + f(x') + g(x) = 0$ tenglikni x' ga ko‘paytirib, uni 0 dan t gacha integrallaymiz. Berilgan $G(x) \ge mx^2$, $y f(y) \ge 0$ shrtlarga ko‘ra ushbu

$$\frac{1}{2}|y(t)|^2 + m|x(t)|^2 \le \frac{1}{2}|v_0|^2 + m|x_0|^2,\ t \in [0,b),$$

tengsizlikni hosil qilamiz. Demak, $x(t)$, $y(t)$ yechim chegaralangan, va shuning uchun u $[0,+\infty)$ gacha davom etadi.

§ 8.5

Teoremani to‘g‘ridan-to‘g‘ri ushbu

$$\boldsymbol{\varphi}(t;\xi') - \boldsymbol{\varphi}(t;\xi'') = \xi' - \xi'' + \int_{t_0}^{t}\left(\boldsymbol{f}(s,\boldsymbol{\varphi}(s;\xi')) - \boldsymbol{f}(s,\boldsymbol{\varphi}(s;\xi''))\right)ds$$

tenglik, Lipshits sharti va Gronuoll tengsizligidan foydalanib isbotlang.

§ 9.1

1- 2. Mos aksiomalarning qanoatlanishini ko‘rsating

§ 9.3

1. Bevosita tekshiriladi. **2.** Shu paragrafdagi teorema isbotiga qarang.

§ 9.4

1. $\Phi'(t) = A(t)\Phi(t)$ bo'lganligi uchun berilganga ko'ra $\frac{d\Phi^T(t)}{dt} = \left(A(t)\Phi(t)\right)^T = \Phi^T(t)A^T(t) = -\Phi^T(t)A(t)$. Demak, $\Psi = \Phi^T(t)$ qiyidagi Koshi masalasining yechimi: $\Psi' = -\Psi A(t), \Psi\big|_{t_0} = \Phi^T(t_0)$. $\Psi = \Phi^{-1}(t)$ ham shu masalaning yechimi, chunki

$$\frac{d\Phi^{-1}(t)}{dt} = -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)A(t)\Phi(t)\Phi^{-1}(t) = -\Phi^{-1}(t)A(t),$$

va $\Phi(t_0)$ ortogonal matritsa bo'lganligi uchun $\Phi^{-1}(t_0) = \Phi^T(t_0)$. Chiziqli normal sisitema uchun yechimning yagonalik xossasiga ko'ra $\Phi^T(t) = \Phi^{-1}(t)$, ya'ni $\Phi(t)$– ortogonal matritsa.

2. $\Phi(t)$ fundamental matritsa

$$\Phi(t) = \Phi(t_0) + \int_{t_0}^{t} A(s)\Phi(s)ds$$

integral tenglamalar sistemasinining yechimidir. Uni ushbu

$$\Phi_0(t) = \Phi(t_0),\ \Phi_k(t) = \Phi(t_0) + \int_{t_0}^{t} A(s)\Phi_{k-1}(s)ds,\ k = 1, 2, 3, \ldots,$$

ketma-ket yaqinlashishlarning limiti sifatida topish mumkin. Ravshanki, simmetrik matritsalar ko'paytmasi va yig'indisi yana simmetrik matritsa bo'ladi. Shuning uchun ketma-ket yaqinlashishlarning barchasi simmetrik matritsalardan iborat. Demak, ularning limiti bo'lmish $\Phi(t)$ ham simmetrik matritsadir.

§ 10.1

1. $\begin{cases} x_1 = e^t(c_1\sin t + c_2\cos t), \\ x_2 = e^t(c_2\sin t - c_1\cos t) \end{cases}$ **2.** $\begin{cases} x = c_1e^{-t} + c_3e^{3t}, \\ y = c_1e^{-t} + c_2e^{2t} + c_3e^{3t}, \\ z = -2c_1e^{-t} + 2c_2e^{2t} + 2c_3e^{3t}. \end{cases}$

3. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} \cos t + \sin t \\ -\cos t \\ -\sin t \end{pmatrix} + c_3e^{2t}\begin{pmatrix} \cos t - \sin t \\ \sin t \\ -\cos t \end{pmatrix}.$

§ 10.2

1. a) $\begin{pmatrix} (1-t)e^{3t} & -te^{3t} \\ te^{3t} & (1+t)e^{3t} \end{pmatrix}$

b) $= \frac{1}{2}\begin{pmatrix} e^{3t}(3\cos t - \sin t) - e^{2t} & e^{3t}(\cos t + 3\sin t) - e^{2t} & e^{3t}(\sin t - 3\cos t) + 3e^{2t} \\ -2e^{3t}\sin t & 2e^{3t}\cos t & 2e^{3t}\sin t \\ e^{3t}(\cos t - \sin t) - e^{2t} & e^{3t}(\cos t + \sin t) - e^{2t} & e^{3t}(\sin t - \cos t) + 3e^{2t} \end{pmatrix}$

2. A matritsaning Jordan kanonik ko'rinishidan foydalaning; yoki $\frac{1}{j!} \geq \frac{1}{k^j}C_k^j$, $j = \overline{0,k}$; $C_k^j = 0$, $j > k$, bo'lgani uchun

$$\left\| e^A - \left(E + \frac{1}{k}A\right)^k \right\| = \left\| \sum_{j=0}^{\infty} \left(\frac{1}{j!} - \frac{1}{k^j} C_k^j \right) A^j \right\| \le \sum_{j=0}^{\infty} \left| \frac{1}{j!} - \frac{1}{k^j} C_k^j \right| \|A\|^j =$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \|A\|^j - \sum_{j=0}^{k} \frac{1}{k^j} C_k^j \|A\|^j = e^{\|A\|} - \left(1 + \frac{1}{k}\|A\|\right)^k \text{ tengsizlikda } k \to \infty \text{ deng.}$$

4. $\mathbf{1^0}$ xossa $A = SBS^{-1} \Rightarrow A^k = SB^kS^{-1}$ va (10.2.4) ta'rifdan kelub chiqadi. $\mathbf{2^0}$. $AB = BA \Rightarrow A^kB = BA^k,\ k \in \mathbb{N}$. Bundan (10.2.4) formulaga ko'ra ixtiyoriy $t \in \mathbb{R}$ uchun $e^{tA}B = Be^{tA}$ ekanligi kelib chiqadi. Endi ravshanki, $\left(e^{tA}e^{tB}\right)' = \left(e^{tA}\right)' e^{tB} + e^{tA}\left(e^{tB}\right)' =$

$$= Ae^{tA}e^{tB} + e^{tA}Be^{tB} = Ae^{tA}e^{tB} + Be^{tA}e^{tB} = (A+B)e^{tA}e^{tB}.$$

Demak, $X = X(t) = e^{tA}e^{tB}$ funksiya $X' = (A+B)X$ sistemaning yechimi va $X(0) = E$. Bu sistemaning shu boshlang'ich shartni qanoatlantiruvchi yechimi, ravshanki, $e^{t(A+B)}$ hamdir, Yechimning yagonalik xossasiga ko'ra har qanday $t \in \mathbb{R}$ uchun $e^{t(A+B)} = e^{tA}e^{tB}$ bo'lishi kerak. Bu tenglikda $t = 1$ deyish kerak. $\mathbf{3^0}$ xossa ikkinchisidan osongina kelib chiqadi: $A(-A) = (-A)A$ bo'lgani uchun $e^Ae^{-A} = e^{-A}e^A = e^{A-A} = e^O = E$ $(O \in \mathbb{M}_{n\times n}(\mathbb{R})-$nol-matritsa), ya'ni e^A matritsa teskarilanuvchi va uning teskarisi e^{-A} matritsadan iborat.

5. a) 4- masaladan foydalaning. b) $J_{\mu,p} = \mu E + N$, bunda

$$E = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{p\times p},\ N == \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{p\times p} \quad \text{va } N^p = N^{p+1} = \ldots = O - \text{ nol-matritsa.}$$

E matritsa har qanday matritsa bilan kommutatsiyalanuvchi bo'lgani uchun Nyuton binomi formulasiga ko'ra $J_{\mu,p}^k = \left(\mu E + N\right)^k = \sum_{j=0}^{k} C_k^j \mu^{k-j} N^j =$

$$= \mu^k + \frac{k}{1!}\mu^{k-1}N + \ldots + \frac{k(k-1)\ldots(k-(p-2))}{(p-1)!}N^{p-1}.$$

Bu yerdagi qo'shish amallarini bajaring.

c) $e^{tJ_{\mu,p}} = e^{\mu t}e^{tN}$. Endi e^{tN} matritsani ta'rifga ko'ra hisoblang.

6. Berilganga ko'ra

$$\left.\begin{aligned} e^{tA+tB} &= e^{tA}e^{tB} \\ e^{tB+tA} &= e^{tB}e^{tA} \end{aligned}\right\} \Rightarrow e^{tA}e^{tB} = e^{tB}e^{tA}, \text{ demak,}$$

$$\left(E + tA + \frac{t^2}{2}A^2 + \ldots\right)\left(E + tB + \frac{t^2}{2}B^2 + \ldots\right) = \left(E + tB + \frac{t^2}{2}B^2 + \ldots\right)\left(E + tA + \frac{t^2}{2}A^2 + \ldots\right).$$

t^2 oldidagi koeffitsientlarni tenglashtiramiz:

$$AB + \frac{1}{2}A^2 + \frac{1}{2}B^2 = BA + \frac{1}{2}B^2 + \frac{1}{2}A^2 \Rightarrow AB = BA.$$

7. $t > 0$. Ta'rifga ko'ra $\left\|e^{At}\right\| \le e^{\|A\|t}$. $E = e^{tA}e^{-tA} \Rightarrow 1 = \left\|e^{tA}e^{-tA}\right\| \le \left\|e^{tA}\right\| \cdot \left\|e^{-tA}\right\| \Rightarrow$ $\left\|e^{-tA}\right\| \ge \left\|e^{tA}\right\|^{-1} = \left\|e^{-tA}\right\|$. Endi A ni $-A$ bilan almashtirish kerak.

8. Aytaylik, $f : (-r, r) \to \mathbb{R}$ funksiya darajali qatorga yoyilsin:

$$f(t)=a_0+a_1t+a_2\frac{t^2}{2!}+\ldots+a_k\frac{t^k}{k!}+\ldots\ ,\ |t|<r,$$

$A\in\mathbb{M}_{n\times n}(\mathbb{R})$ matritsaning barcha $\lambda=\lambda_j$ xos sonlari uchun esa $|\lambda_j|<r$ tengsizlik o‘rinli bo‘lsin. U holda

$$f(A)=a_0+a_1A+\frac{a_2}{2!}A^2+\ldots+\frac{a_k}{k!}A^k+\ldots\ ,$$

matritsaviy qatorning absolyut yaqinlashuvchi ekanligini ko‘rsating.

Ma’lumki, ixtiyoriy $\tilde r, 0<\tilde r<r,$ uchun $\sum_{k=0}^{\infty}|a_k|\frac{\tilde r^k}{k!}<+\infty$. $\|A^k\|$ ni yuqoridan baholaymiz. $\max_j|\lambda_j|=\rho$ deylik. S matritsa A matritsani Jordan ko‘rinishi J ga keltirsin: $A=SJS^{-1}$. Bundan $A^k=SJ^kS^{-1}$, $\|A^k\|\le\|S\|\cdot\|S^{-1}\|\cdot\|J^k\|=c\|J^k\|$ $,k=0,1,2,\ldots;$ bu masalada bitta c harf bilan k ga bog‘liq bo‘lmagan turli o‘zgarmas sonlarni belgilaymiz. J^k matritsaning ko‘rinishidan (5. c)) masala) uning ixtiyoriy elementining moduli $\le c\rho^kn^k$ ekanligini ko‘ramiz. Shunday $\tilde r$, $0<\rho<\tilde r<r$, mavjudki, uning uchun $\|J^k\|\le c\rho^kn^k\le c\tilde r^k$ bo‘ladi. Demak,

$$\left\|\sum_{k=0}^{\infty}\frac{a_k}{k!}A^k\right\|=\sum_{k=0}^{\infty}\frac{|a_k|}{k!}\|A^k\|\le c\sum_{k=0}^{\infty}\frac{|a_k|}{k!}\tilde r^k<+\infty$$

11. $A=[a_{lj}]_{n\times n}, a=\max_{1\le l,j\le n}|a_{lj}|$ deylik. A^2 matritsaning $(l,j)-$ elementi uchun

$|(A^2)_{l,j}|=\left|\sum_{m=1}^{n}a_{lm}a_{mj}\right|\le\sum_{m=1}^{n}|a_{lm}||a_{mj}|\le na^2$. Shunga o‘xshash $|(A^k)_{l,j}|\le n^{k-1}a^k$. $|t|\le\delta$ bo‘lganda

$$\left|(\exp(At))_{lj}\right|=\left|\sum_{k=0}^{\infty}\frac{(A^k)_{lj}}{k!}t^k\right|\le\sum_{k=0}^{\infty}\frac{\left|(A^k)_{lj}\right|}{k!}|t|^k\le$$

$$\le1+\sum_{k=1}^{\infty}\frac{n^{k-1}a^k}{k!}\delta^k=1+\frac{1}{n}\sum_{k=1}^{\infty}\frac{(na\delta)^k}{k!}=1+\frac{1}{n}\left(\sum_{k=0}^{\infty}\frac{(na\delta)^k}{k!}-1\right)=$$

$$=1+\frac{1}{n}\left(\exp(na\delta)-1\right)<+\infty$$

13. Bevosita tekshiriladi.

14. Berilganlarga ko‘ra $X(0)=X(0)X(0)\Rightarrow X(0)=E$. Bundan tashqari

$$X(s+t)-X(t)=X(s)X(t)-X(t)=\left(X(s)-E\right)X(t)=$$
$$=\left(X(s)-X(0)\right)X(t).$$

Demak, $X'(t)=X'(0)X(t)$. Bundan $X(0)=E$ ni hisobga olib, topamiz: $X(t)=\exp\left(tX'(0)\right)$

§ 10.3

4. Ozod hadni vektor koeffitsientli ikkita kvaziko1had yig‘indisi safatida tasvirlang.

§ 11.1

1. Berilganga ko‘ra $f(x)$ funksiya $(-\infty;a),(a;b),(b;+\infty)$ oraliqlarining har birida o‘z ishorasini saqlaydi. Ixtiyoriy $x=x(t)$ yechimni qaraylik. Aytaylik, biror t_0 uchun $x_0=x(t_0)\in(a;b)$ bo‘lsin. U holda t ning o‘zgarish jarayonida bu yechim chekli paytda a ga ham b ga ham yetib bora olmaydi (yechimning yagomalik xossasiga ko‘ra). Demak, u barcha $t\in(-\infty;+\infty)$ larda aniqlangan va $x(t)\in(a;b)$. $x=x(t)$ yechim monoton bo‘lganligi uchun ($f(x)$ funksiya $(a;b)$ da o‘z ishorasini saqlaydi) bu funksiyaning qiymatlar to‘plami, ya’ni fazaviy traektoriya $(a;b)$ intervaldan iborat bo‘ladi.

Endi faraz qilaylik, $x=x(t)$ yechim uchun biror $t=t_0$ da $x_0=x(t_0)\in(-\infty;a)$ bo‘lsin. Aniqlik uchun $f(x_0)<0$ deylik. Demak, $x=x(t)$ kamayuvchi. t kamayishi bilan $x(t)$ ortadi va $\lim\limits_{t\to-\infty}x(t)=a$ bo‘ladi. t ortishi bilan esa $x(t)$ kamayadi va u yo chekli paytda $-\infty$ ga ketib qoladi, yoki $[t_0;+\infty)$ oraliqqacha davom etadi. Oxirgi holda ham $\lim\limits_{t\to-\infty}x(t)=-\infty$ bo‘ladi, chunki aks holda $x(t)$ quyidan biror α son bilan chegaralangan, ya’ni $x(t)\ge\alpha$, $t\in[t_0;+\infty)$, va

$x'(t)=f(x(t))\le\sup\limits_{[\alpha;x_0]}f(x)=\beta<0,\ t\in[t_0;+\infty)$, baholashlardan hosil bo‘luvchi $x(t)\le\beta(t-t_0)+x_0$, $t\in[t_0;+\infty)$, tengsizlikdan yetarlicha katta t lar uchun $x(t)<\alpha$ ziddiyat hosil bo‘lar edi.

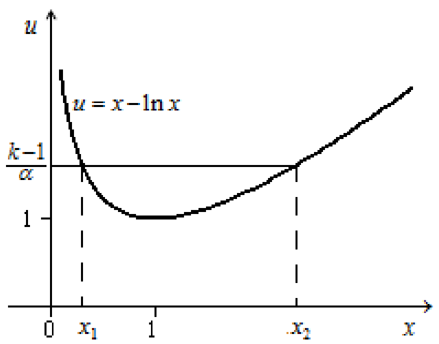
3. $F(x)=\int\limits_0^x\frac{1}{f(s)}ds$ deylik. $F(x+\tau)-F(x)$ funksiyaning o‘zgarmasligini va $F(x(b))-F(x(a))=b-a$ ekanligini ko‘rsating.

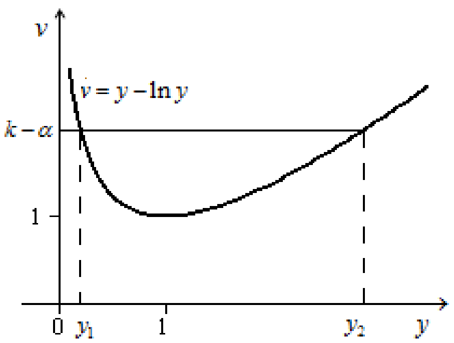
§ 11.2

1. Tenglamani logarifmlaymiz.

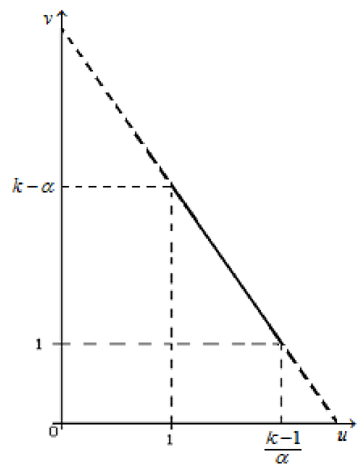
$y-\ln y+\alpha(x-\ln x)=k,k=-\ln c$.

$u=x-\ln x,v=y-\ln y$ desak, $v+\alpha u=k$ munosabatni topamiz (quyidqgi rasmlarga qarang). Yechimlarda $u\ge1$, $v\ge1$ bo‘lishi kerak. $v+\alpha u=k$ munosabatdan $u\ge1$ da $v\le k-\alpha$, $v\ge1$ da esa $u\le(k-1)/\alpha$ bo‘lishini topamiz. Demak, $k\ge1+\alpha$ mavud bo‘lmish traektoriyalar uchun $1\le u\le(k-1)/\alpha$, $1\le v\le k-\alpha$ bo‘ladi.





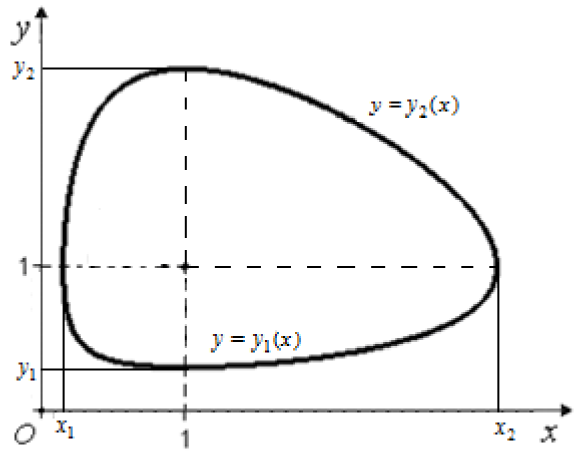
$v \equiv y - \ln y = k - \alpha(x - \ln x)$ tenglamadan y topilishi uchun $k - \alpha(x - \ln x) \geq 1$, ya'ni $u \equiv x - \ln x \leq (k-1)/\alpha$ bo'lishi kerak. Demak, $(k-1)/\alpha \geq 1$, ya'ni $k \geq 1+\alpha$. Shunday qilib, $k < 1+\alpha$ $\left(0 < c < e^{-(1+\alpha)}\right)$ bo'lganda qaralayotgan tenglama bo'sh to'plamni aniqlaydi. $k = 1+\alpha$ da $x = 1$ va $y = 1$, ya'ni muvozanat nuqta hosil bo'ladi. $k > 1+\alpha$ bo'lganda $x - \ln x = (k-1)/\alpha$ tenglama $x_1 < 1$ va $x_2 > 1$ ildizlarga ega bo'lib, $x_1 \leq x \leq x_2$ bo'lganda $y - \ln y = k - \alpha(x - \ln x)$ tenglama $0 < y_1(x) \leq 1$ va $y_2(x) \geq 1$ yechimlarga ega bo'ladi. Bu funksiyalarning hosilasini hisoblaylik.



$y_j(x) - \ln y_j(x) = k - \alpha(x - \ln x),\ j = 1, 2.$ Bundan

$$\frac{dy_j(x)}{dx} = \alpha \frac{(1-x)y_j(x)}{\left(y_j(x)-1\right)x},\ j = 1, 2.$$

Demak, $[x_1, 1]$ oraliqda $y_1(x)$ kamayadi, $y_2(x)$ esa o'sadi; $[1, x_2]$ oraliqda $y_1(x)$ o'sadi, $y_2(x)$ esa kamayadi. $y_1 = \max\limits_{[x_1,x_2]} y_1(x), y_2 = \max\limits_{[x_1,x_1]} y_2(x)$. $k > 1+\alpha$ bo'lgandagi trayektoriya quyidqagi rasmda tasvirlangan.



§ 11.4

1. Muvozanat nuqtalari va ularning tabiatini aniqlang. $\{(1; y) \mid |y| < 1\}$ kesmaning hamda $\{(1; y) \mid -\infty < y\}$, $\{(1; y) \mid y > 1\}$, $\{(x; y) \mid y = 2 - x, x < 1\}$ va $\{(x; y) \mid y = 2 - x, x > 1\}$ nurlarning traektoriya ekanligini asoslang. Tezliklar maydonini va maydon vektorlariga urinuvchi chiziqlarni (traektoriyalarni) quring.

2. Muvozanat nuqtalar to'rtta:

$(1;1)$ – egar, chunki chiziqlashtirilgan sistema matitsasinig xos sonlari $\lambda_1 = \dfrac{-1+\sqrt{17}}{2} > 0$, $\lambda_2 = \dfrac{-1-\sqrt{17}}{2} < 0$;

$(3;3)$ – turg'un tugun, chunki mos matritsaning xos sonlari

$$\lambda_1 = -3 < 0,\ \lambda_2 = -4 < 0;$$

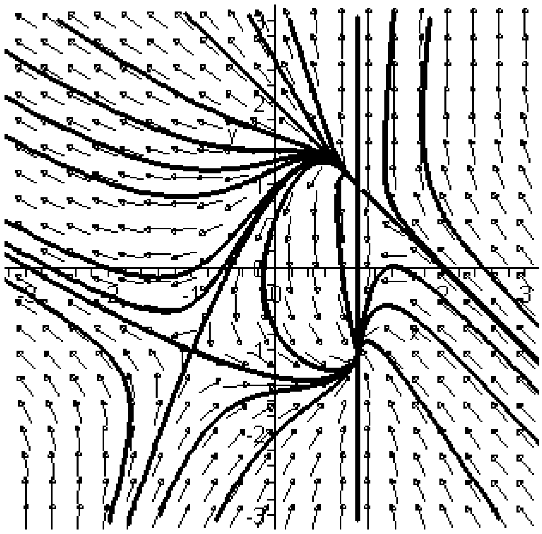
$(\sqrt{3}; -\sqrt{3})$ – noturg'un tugun, chunki mos matritsaning xos sonlari

$$\lambda_1=\frac{2+3\sqrt{3}+\sqrt{12\sqrt{3}-\sqrt{17}}}{2}\approx 4,6>0,\ \lambda_2=\frac{2+3\sqrt{3}-\sqrt{12\sqrt{3}-\sqrt{17}}}{2}\approx 2,6>0;$$

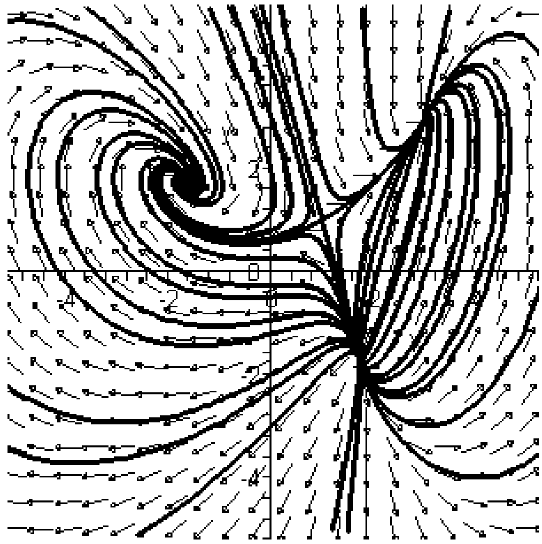
$(-\sqrt{3};\sqrt{3})-$ turg‘un fokus, chunki mos matritsaning xos sonlari

$$\lambda_1=\frac{2-3\sqrt{3}}{2}+i\frac{\sqrt{12\sqrt{3}+\sqrt{17}}}{2}\approx -1,6+i3,1;\ \lambda_2=\frac{2-3\sqrt{3}}{2}-i\frac{\sqrt{12\sqrt{3}+\sqrt{17}}}{2}\approx -1,6-i3,1;$$

$\mathrm{Re}(\lambda_1)=\mathrm{Re}(\lambda_2)<0$



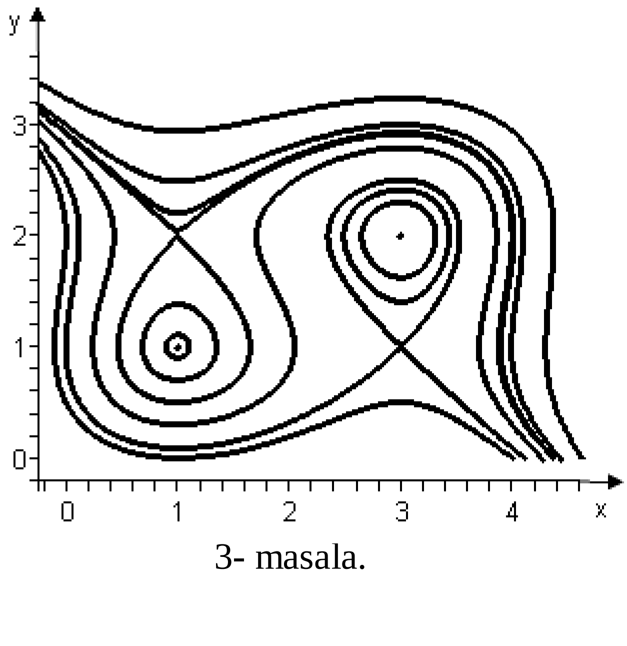
1- masala.



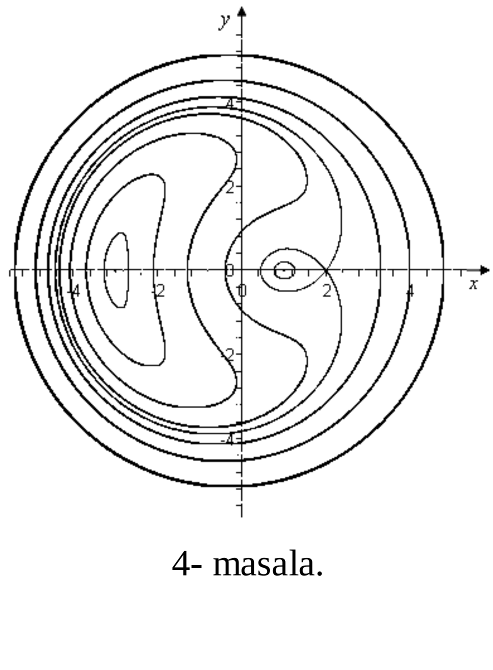
2- masala.

3. To‘rtta muvozanat nuqta bor. Ular: $(1;1)$ va $(3;2)$ - markazlar; $(1;2)$ va $(3;1)$ - egarlar.

4. Muvozanat nuqtalari uchta: $(-3;0),(1;0)-$ markazlar; $(2;0)-$ egar.

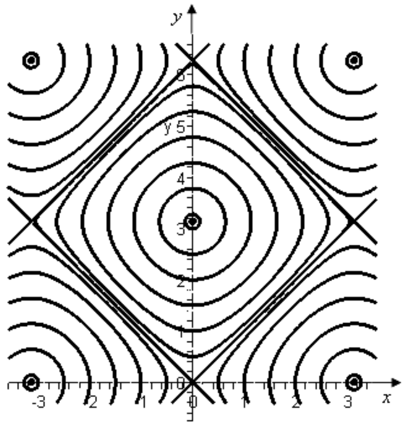


3- masala.



4- masala.

5. Davriy manzarning bir qismi quyidagi rasmda tasvirlangan:



§ 11.5

1. Bu sistema $-p(x')x'$ qarshilik (ishqalanish) kuchi va $-x$ elastik kuch ta'siri ostida moddiy nuqtaning $(m=1)$ harakati tenglamasini ifodalaydi, $x''=-p(x')x'-x$. Moddiy nuqtaning $\upsilon=\upsilon(x,y)=(x^2+y^2)/2$ to'la mexanik energiyasini qarang. $x=x(t), y=y(t)$ harakat mobaynida bu energiya kamayadi:

$$\frac{d\upsilon}{dt}=-p(y)y^2<0, y\neq 0.$$

Sistemaning biror

$$\begin{cases} x=x(t) \\ y=y(t) \end{cases}, 0\le t\le T,$$

T-davriy yecimi mavjud deb hisoblab, bu yechim bo'ylab

$$\int_0^T \frac{d\upsilon}{dt}dt$$

integralni ikki usul bilan hisoblab, ziddiyat hosil qiling.

2. Bendikson-Dyulak teoremasidan foydalaning ($h=be^{-\beta x}$).

3. Bendikson-Dyulak teoremasidan foydalaning ($h=x^k y^l$; k,l larni tanlang).

4. $(x^2+y^2)'\ge 0$ munosabatdan foydalaning.

5. $\frac{dx}{d\tau}$ va $\frac{dy}{d\tau}$ hosilalarning ishoralarini o'rganing va A_7A_6 kesmaning ixtiyoriy $(x;0)$ nutasidan chiqqan yechim τ ning ortishi bilan $(0;0)$ nuqta atrofida aylanib, A_7A_6 ning $(\varphi(x);0)$ nuqtasiga qaytishini ko'rsating. $\varphi(x)$ ning o'suvchi va uzluksiz funksiya ekanligini asoslang. $x=x_{A_7}$ da $\varphi(x)-x>0$, $x=x_{A_6}$ da esa $\varphi(x)-x<0$ bo'lgani uchun $\exists\tilde{x}\in[x_{A_7};x_{A_6}]$ $\varphi(\tilde{x})=\tilde{x}$. Sistemaning $\tilde{x}$ nuqtadan chiqqan traektoriyasi yopiq chiziqdan iborat.

§ 12.1

1. Qisqalik uchun $\boldsymbol{\psi}(t)=\boldsymbol{\psi}(t;t_0,\boldsymbol{x}^0)$ deylik. Ma'lum (12.1.26)

$$\frac{\partial\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0)}{\partial t_0}=-\frac{\partial\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0))}{\partial\boldsymbol{x}^0}\boldsymbol{f}(t_0,\boldsymbol{x}^0)$$

formulaga ko'ra

$$\frac{\partial \boldsymbol{\varphi}(t;s,\boldsymbol{\psi}(s))}{\partial s}=\frac{\partial \boldsymbol{\varphi}(t;s,\boldsymbol{\psi}(s))}{\partial t_0}+\frac{\partial \boldsymbol{\varphi}(t;s,\boldsymbol{\psi}(s))}{\partial \boldsymbol{x}^0}\boldsymbol{\psi}'(s)=\Phi(t;s,\boldsymbol{\psi}(s))\big(\boldsymbol{\psi}'(s)-\boldsymbol{f}(s,\boldsymbol{\psi}(s))\big)=$$
$$=\Phi(t;s,\boldsymbol{\psi}(s))\boldsymbol{r}(s,\boldsymbol{\psi}(s)).$$

Bu tenglikni $s=t_0$ dan $s=t$ gacha integrallab, va $\boldsymbol{\varphi}(t;t,\boldsymbol{\psi}(t))=\boldsymbol{\psi}(t)$ ekanligidan foydalanib, V. A. Alekseev formulasini hosil qiling.

2. Ushbu

$$\frac{\partial \boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0+s(\boldsymbol{y}^0-\boldsymbol{x}^0))}{\partial s}=\Phi\big(t;t_0,\boldsymbol{x}^0+s(\boldsymbol{y}^0-\boldsymbol{x}^0)\big)(\boldsymbol{y}^0-\boldsymbol{x}^0),\ s\in[0;1],$$

ayniyatni integrallang.

3. Bu formulani 1- va 2- masala formulalaridan keltirib chiqaring.

§ 12.2

1. $x=\dfrac{e^t}{1+\mu(e^t-1)}$.

§ 12.3

1. Tenglamalarni hadma-had qo‘shib va ayirib, toping:

$$\begin{cases}(x+y)'=(x+y)^2\\(x-y)'=(x-y)^2\end{cases}$$

Tenglamalarni alohida-alohida yeching ($x+y\neq 0,\ x-y\neq 0$ deb faraz qiling):

$$\begin{cases}x+y=\dfrac{1}{c_1-t}\\[2mm] x-y=\dfrac{1}{c_2-t}\end{cases}\ \Rightarrow\ \begin{cases}x=\dfrac{1}{2}\left(\dfrac{1}{c_1-t}+\dfrac{1}{c_2-t}\right)\\[2mm] y=\dfrac{1}{2}\left(\dfrac{1}{c_1-t}-\dfrac{1}{c_2-t}\right)\end{cases}$$

Yo‘qolgan yechimlarni toping.

2. a). Teskarisini faraz qilaylik: biror B_δ doirada aniqlangan $u(x,y)$ birinchi integral mavjud bo‘lsin. Demak, $u(x,y)\in C^1(B_\delta)$, $u(x,y)\neq \text{const}$ va berilgan sistemaning B_δ da joylashgan har qanday $x=x(t),\ y=y(t)$ yechimi bo‘ylab $u(x(t),y(t))=\text{const}$. Ixtiyoriy $(x_0,y_0)\in B_\delta$ nuqtani olib, berilgan sistemaning $x(t)=x_0e^{-t}$, $y(t)=y_0e^{-t}$ yechimini qaraylik. Ravshanki, ixtiyoriy $t\geq 0$ uchun $(x(t),y(t))=(x_0e^{-t},y_0e^{-t})\in B_\delta$. Demak, $u(x_0e^{-t},y_0e^{-t})=u(x_0,y_0),\ t\geq 0$. Demak, $u(x_0e^{t},y_0e^{t})=u(x_0,y_0)$. Bu ayniyatda $t\to+\infty$ da limitga o‘tamiz. Natijada ixtiyoriy $(x_0,y_0)\in B_\delta$ uchun $u(0,0)=u(x_0,y_0)$ ekanligini hosil qilamiz. Bu esa $u(x,y)\neq\text{const}$ ekanligiga zid.

b). $x>0$ yarim tekislikda aniqlangan birinchi integral osongina topiladi:

$$xy'-yx'=0\ \Rightarrow\ \left(\frac{y}{x}\right)'=0\ \Rightarrow\ \frac{y}{x}=c.$$

3. Sistemaning ikkinchi va uchinchi tenglamalalaridan bitta birinchi integralni topamiz

$zy'+yz'=0\ \Rightarrow\ zy=c_1$ (c_1- ixtiyoriy o‘zgarmas).

Demak, har qanday $x = x(t),\ y = y(t),\ z = z(t)$ yechim bo‘ylab zy ko‘paytma o‘zgarmas. Ikkinchi tenglamani birinchisiga hadma-had bo‘lib, va yechim bo‘ylab $c_1 = yz$ o‘zgarmas ekanligini hisobga olib, ushbu

$$\frac{dy}{dx} = \frac{c_1 x}{1+3y^2}$$

o‘zgaruvchilari ajraladigan tenglamani hosil qilamiz. Bundan

$$c_1 \frac{x^2}{2} = y + y^3 + \frac{c_2}{2} \quad (c_2 - \text{ ixtiyoriy o‘zgarmas}), \text{ ya’ni } x^2 c_1 - 2y - 2y^3 = c_2$$

ekanligi kelib chiqadi. Oxirgi tenglikdan yechim bo‘ylab $c_1 = yz$ bo‘lganligi uchun yana bir birinchi integralni hosil qilamiz:

$$x^2 yz - 2y - 2y^3 = c_2 .$$

Topilgan $u_1 = yz$ va $u_2 = x^2 yz - 2y - 2y^3$ birinchi integrallarni erklilikka tekshiramiz. Buning uchun ushbu

$$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & z & y \\ 2xyz & x^2 z - 2 - 6y^2 & x^2 y \end{pmatrix}$$

Yakobi matritsasini tuzib, uning rangini hisoblaymiz. Agar $y \neq 0$ bo‘lsa, quyidagi ikkinchi tartibli minorning qiymati noldan farqli:

$$\begin{vmatrix} z & y \\ x^2 z - 2 - 6y^2 & x^2 y \end{vmatrix} = y(2+6y^2) \neq 0 ;$$

demak, tuzilgan matritsaning rangi ikkiga teng va $y > 0$ (yoki $y < 0$) sohada topilgan birinchi integrallar erkli.

4. Sistemaning birinchi integrallari:

$$x_1' + x_2' + x_3' = 0 \ \Rightarrow \ x_1 + x_2 + x_3 = c_1 \ \text{(tekislik)}$$

$$x_1 x_1' + x_2 x_2' + x_3 x_3' = 0 \ \Rightarrow \ x_1^2 + x_2^2 + x_3^2 = c_2 \ \text{(sfera)}$$

Demak, traektoriyalar aylanalardan iborat.

§ 13.1.

1. Ta’rifdan ravshan.

2. Davriy sistemaning ixtiyoriy $\boldsymbol{x}(t, t_0, \boldsymbol{x}^0)$ yechimi uchun

$\boldsymbol{x}(t, t_1, \boldsymbol{x}(t_1, t_0, \boldsymbol{x}^0)) = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0),\ \boldsymbol{x}(t \pm mT, t_0 \pm m, \boldsymbol{x}^0) = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0),\ m \in \mathbb{Z}$, ayniyatlarning o‘rinliligi ravshan. Birinchisi yechimning yagonalik xossasidan, ikkinchisi esa sistemaning davriyligidan kelib chiqadi. Ikkinchi ayniyatga ko‘ra tekis yaqinlashishni isbotlashda faqat $t_0 \in [0, T]$ boshlang‘ich paytlar bilan chegaralaninsh mumkin. Endi kerakli $\delta > 0$ sonning mavjudligi yechimning chegaralangan $[0, T]$ segmentda t_0 ga uzluksiz bog‘liqligi to‘g‘risidagi teoremadan kelib chiqadi.

3. Shu paragrafda keltirilgan misolga qarang.

4. Traektoriyalarni eslang.

§ 13.2

1. Yechim formulasi $x(t,t_0,x_0)=x_0\exp\left(\int\limits_{t_0}^{t}a(s)ds\right)$ dan foydalaning.

3. Xarakteristik sonlar $\lambda_1=0,\lambda_{2,3}=\pm i\sqrt{3}$ ($\operatorname{Re}\lambda_j=0$), bir karrali bo‘lgani uchun sistema turg‘un (asimptotik emas) . **4.** Turg‘un emas. Sistema $x=y=z=ce^{2t}$ ko‘rinishdagi yechimlarga ega.

§ 13.3

1. Ha. **2.** Nol-yechim noturg‘un, $v=x^2+y^2$.

3. Nol-yechim turg‘un, $v=x+y-\ln(1+x)-\ln(1+y), x^2+y^2<1$.

4. $v=x^4+2y^2$. **5.** Nol-yechim $k>0$ holida turg‘un, $k<0$ holida esa noturg‘un.

§ 13.4

4. Muvozanat holatlarini ushbu

$$\begin{cases}-\sigma x+\sigma y=0\\ rx-y-xz=0\\ -bz+xy=0\end{cases}$$

sistemadan toping. Ixtiyoriy $r>0$ uchun bu sistema $x=0,y=0,z=0$ yechimga ega. Agar $r>1$bo‘lsa, yana ikkita muvozanat nuqtasi hosil bo‘ladi:

$$x=x_0,y=y_0,z=z_0 \text{ va } x=-x_0,y=-y_0,z=z_0 \;\left(x_0=y_0=\sqrt{b(r-1)}\,,\; z_0=r-1\right).$$

$x=0,y=0,z=0$ muvozanat nuqtani turg‘unlikka tekshiraylik. Bu nuqta atrofida Lorens sistemasining chiziqlashtirilishi

$$\begin{cases}x'=-\sigma x+\sigma y\\ y'=rx-y\\ z'=-bz\end{cases}$$

ko‘rinishga ega. Xarakteristik tenglama

$$\begin{vmatrix}-\sigma-\lambda & \sigma & 0\\ r & -1-\lambda & 0\\ 0 & 0 & -b-\lambda\end{vmatrix}=0.$$

Xarakteristik sonlar

$$\lambda_1=-b,\;\lambda_{2,3}=\frac{1}{2}\left(-1-\sigma\pm\sqrt{(1+\sigma)^2-4\sigma(1-r)}\right).$$

Agar $0<r<1$ bo‘lsa, hamma xususiy sonlar manfiy va, Lorens sistemasining nol-yechimi asimptotik turg‘un.

Agar $r>1$ bo‘lsa, $\lambda_2=\frac{1}{2}\left(-1-\sigma+\sqrt{(1+\sigma)^2-4\sigma(1-r)}\right)$ xarakteristik son musbat va Lorens sistemasining nol-yechimi noturg‘un. Demak, $r=1$ da nol-yechim turg‘unligi almashinadi.

$r=1$ bo‘lganda xarakteristik sonlar $\lambda_1=-b,\;\lambda_2=0,\;\lambda_3=-(1+\sigma)$ va chiziqlashtirilgan sistema Lorens sistemasi nol-yechimining turg‘unligi haqida hech narsa deya olmaydi. Lekin bu holda Lyapunov funksiyasini qurishga harakat qilish mumkin. Kvadratik forma ko‘rinishidagi ushbu

$$v = v(x, y, z) = x^2/\sigma + y^2 + z^2$$

aniq musbat funksiyani qaraylik. Bu funksiyaning Lorens sistemasiga ko‘ra hosilasi

$$\frac{dv}{dt} = 2\frac{x}{\sigma}(-\sigma x + \sigma y) + 2y(rx - y - xz) + 2z(-bz + xy) =$$

$$= -2\left(\left(x - \frac{r+1}{2}y\right)^2 + \left(1 - \frac{(r+1)^2}{4}\right)y^2 + bz^2\right).$$

Agar $r < 1$ bo‘lsa, qurilgan v funksiya Lyapunovning asimptotik turg‘unlik haqidagi teoremasi shartlarini qanoatlantiradi; demak, yana nol-yechim asimptotik turg‘un.

$r = 1$ holini qaraylik. Bu holda v funksiya Lyapunovning turg‘unlik haqidagi teoremasi shartlarini qanoatlantiradi; demak, bu holda nol-yechim turg‘un. $\frac{dv}{dt} = -2\left((x-y)^2 + bz^2\right)$ hosila $x = y, z = 0$ to‘g‘ri chiziqda nolga aylanib, boshqa nuqtalarda qat’iy manfiy. Noldan farqli har qanday yechim bu to‘g‘ri chiziq bilan uchrashgach, undan albatta chiqib ketadi, chunki bunda $z' = -bz + xy = x^2 \neq 0$. Shuning uchun vaqt o‘tishi bilan yechim $v = x^2/\sigma + y^2 + z^2$ funksiyaning $x^2/\sigma + y^2 + z^2 = c$ $(c > 0)$ sath to‘plamlarini (ellipsoidlarni) c ning kamayish yo‘nalishida kesib boradi va koordinatalar boshiga intiladai, ya’ni $r = 1$ holida nol-yechim asimptotik turg‘un hamdir.

Endi $r > 1$ holida $x = x_0, y = y_0, z = z_0$ va $x = -x_0, y = -y_0, z = z_0$ muvozanat nuqtalarni turg‘unlikka tekshiramiz. Buning uchun Lorens sistemasida

$$x = u + x_0, y = v + y_0, z = w + z_0$$

almashtirishni bajaramiz, bunda $u, v, w -$ yangi noma’lum funksiyalar. Natijada

$$\begin{cases} u' = -\sigma u + \sigma v \\ v' = u - v - x_0 w - uw \qquad (\ x_0 = \sqrt{b(r-1)}\ ) \\ w' = x_0 u + x_0 v - bw + uv \end{cases}$$

sistemaga kelamiz. Bu sistemaning birinchi yaqinlashishi uchun xarakteristik tenglama

$$\begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & -x_0 \\ x_0 & x_0 & -b-\lambda \end{vmatrix} = 0.$$

Bu tenglama x_0 ning ishorasi o‘zgarganda o‘zgarmaydi, ya’ni $x = -x_0, y = -y_0, z = z_0$ muvozanat nuqta uchun ham shu xarakteristik tenglama hosil bo‘ladi. Xarakteristik tenglamani

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r-1) = 0 \ (\sigma > 0, b > 0, r > 1).$$

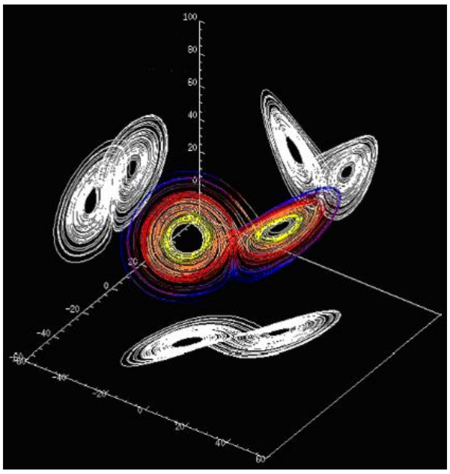
ko‘rinishda yozish mumkin. Bu tenglama ildizlarining haqiqiy qismi manfiy bo‘lishi uchun L’yenar-Shipar mezoniga ko‘ra

$$(\sigma + b + 1)(\sigma + r)b - 2\sigma b(r-1) > 0$$

shart bajarilishi kerak. Oxirgi shart

$$(\sigma - b - 1)r < \sigma(\sigma + b + 3)$$

tengsizlikka teng kuchli. $\sigma = 10,\ b = 8/3$ holini qaraylik. Oxirgi tengsizlikdan $r < r_0$, $r_0 \approx 22,74$, ni topamiz. Demak, $1 < r < r_0$ bo‘lganda qaralayotgan muvozanat nuqtalar asimptotik turg‘un, $r > r_0$ bo‘lganda esa ular notirg‘un (Lorens tekshirgan holda $r = 28 > r_0$ bo‘lgan). $r = r_0$ da bitta manfiy haqiqiy qismli va ikkita sof mavhum xarakteristik sonlar mavjud. Bu kritik holni tekshirmaymiz.



Qaralayotgan sistemaning yechimlarini $\sigma = 10,\ b = 8/3$, $r = 28$ holida Lorens sonli usullar yordamida o‘rgangan. U yechimlarning to‘satdan betartib ravishda to $(\sqrt{72};\sqrt{72};27)$, to $(-\sqrt{72};-\sqrt{72};27)$ noturg‘un muvozanat nuqtalari atrofida burala boshlashini aniqlagan (-rasm). Bunda yechimlarning necha marta bir muvozanat nuqtasi atrofida buralib so‘ngra ikkinichisi atrofiga o‘tib buralishi ham betartib bo‘lgan. Yechimlarning tartibsiz o‘zgarishi boshlang‘ich qiymatga kuchli bog‘liq bo‘lgan. Yechimning bunday betartib tabiati xaos deb ataladi. Xaos nazariyasida nochiziqli dinamik sistemalarning turg‘un bo‘lmagan davrsiz yechimlari tabiati o‘rganiladi.

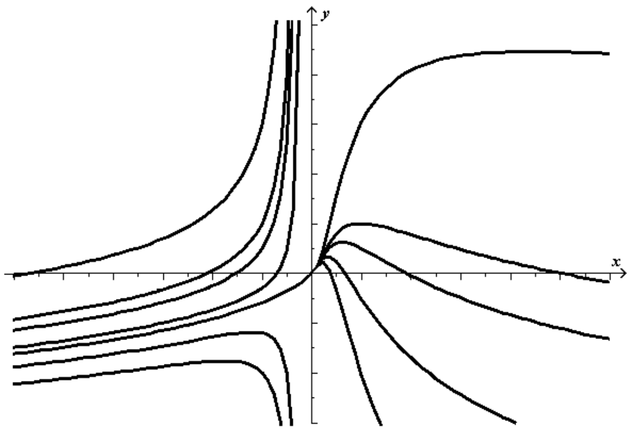
§ 14.2

5. Berilgan tenglamaning yechimlari

$x < 0$ da $y = e^{-1/x}\left(c + \int_{-1}^{1/x} \frac{e^s}{s}\,ds\right)$, $x > 0$ da esa $y = e^{-1/x}\left(c + \int_{1}^{1/x} \frac{e^s}{s}\,ds\right)$ formulalar bilan beriladi. Berilgan boshlang‘ich masalaning yechimi cheksiz ko‘p (J.3- rasm)

$$y = y(x) = \begin{cases} \varphi(x),\ x < 0; \\ 0\ ,\ x = 0; \\ e^{-1/x}\left(c + \int_{1}^{1/x} \frac{e^s}{s}\,ds\right),\ x > 0. \end{cases}, \quad \varphi(x) \overset{def}{=} e^{-1/x}\left(-\int_{-1}^{-\infty} \frac{e^s}{s}\,ds + \int_{-1}^{1/x} \frac{e^s}{s}\,ds\right) = e^{-1/x}\int_{-\infty}^{1/x} \frac{e^s}{s}\,ds\ .$$

$\lim\limits_{x\to 0-} \varphi(x) = 0$, $\lim\limits_{x\to 0+} e^{-1/x}\left(c + \int_{1}^{1/x} \frac{e^s}{s}\,ds\right) = 0$. Analitik yechimi esa mavjud emas.



5- masala uchun rasm.

6. $y = y(x) = c\exp\left(-\frac{1}{x^2}\right)$. $\lim\limits_{x\to 0} y^{(n)}(x) = c\lim\limits_{x\to 0} P\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) = 0$, $n = 0,1,2,\ldots$, $P(t)-$ ko‘phad. Tenglamaning nol nuqtada analitik bo‘lgan yechimi bitta, u ham bo‘lsa $y(x)\equiv 0$.

§ 14.3

2. Dastlab ushbu

$$u'' + \omega^2 u = 0$$

tenglamaning har qanday yechimi $u = A\cos(\omega x + \varphi)$ $(A,\varphi-$ o‘zgarmaslar$)$ ko‘rinishda bo‘lgani uchun uning har qanday notrivial yechimining ixtiyoriy qo‘shni nollari orasidagi masofa π/ω ga teng ekanligini e’tirof etaylik. Endi $y'' + q(x)y = 0$ va $u'' + \omega^2 u = 0$ tenglamalarga Shturm teoremasini qo‘llaymiz.

Faraz qilaylik, $q(x) \le \omega^2$ $(x \in (a,b))$ shart bajarilsin. U holda berilgan tenglamaning har qanday notrivial yechimining qo‘shni $x_0 < x_1$ nollari uchun $x_1 - x_0 \ge \frac{\pi}{\omega}$ bo‘lishi kerak, chunki aks holda $x_1 - x_0 < \frac{\pi}{\omega}$, ya’ni $x_0 < x_1 < x_0 + \frac{\pi}{\omega}$ bo‘lardi va Shturm teoremasiga ko‘ra $u'' + \omega^2 u = 0$ tenglamaning x_0 da nolga aylanuvchi yechimi x_1 dan kichik yoki teng nolga ega bo‘lib, bu nollar orasidagi masofa π/ω dan kichik bo‘lib qolardi.

Endi faraz qilaylik, $q(x) \ge \omega^2$ $(x \in (a,b))$ tengsizlk o‘rinli bo‘lsin. U holda berilgan tenglamaning har qanday notrivial $y = y(x)$ yechimining qo‘shni $x_0 < x_1$ nollari uchun $x_1 - x_0 \le \frac{\pi}{\omega}$ bo‘lishi kerak. Buni isbotlash uchun aksini faraz qiling, $y = y(x)$ va $u'' + \omega^2 u = 0$ tenglamning x_0 va $x_0 + \frac{\pi}{\omega}$ nuqtalarda nolga aylanuvchi yechimiga Shturm teoremasini qo‘llab, ziddiyat hosil qiling.

3. Ushbu

$$y = z(x)\exp\left(-\frac{1}{2}\int \frac{1}{s}\,ds\right) = \frac{z(x)}{\sqrt{x}}$$

almashtirishni bajaring. U holda $z = z(x)$ yangi noma’lum funksiya uchun ushbu

$$z''+\left(1-\frac{\nu^2-1/4}{x^2}\right)z=0$$

tenglamani hosil qilamiz. z oldidagi koeffitsient $\nu>1/2$ bo‘lganda 1 dan kichik, $0\le\nu<1/2$ bo‘lganda esa ≥ 1. Demak, oldingi masalaga ko‘ra Bessel tenglamasi har qanday notrivial yechiming qo‘shni nollari orasidagi masofa $\nu>1/2$ holida π dan katta, $0\le\nu<1/2$ holida esa π dan kichik. Agar $\nu=1/2$ bo‘lsa, bevosita tekshirib ko‘rish mumkinki, Bessel tenglamasining umumiy yechimi elementar funksiyadan iborat va notrivial yechimning qo‘shni nollari orasidagi masofa π ga teng bo‘ladi.

4. Oldingi masalaga qarang.

5. Sturmning taqqoslash teoremasidan foydalaning.

6. 2- masalaga qarang.

§ 15.1

Ma’lumki, (*) tenglamaning umumiy yechimi

$$y=c_1\cos x+c_2\sin x$$

formula bilan beriladi.

1. chegaraviy shartlarni qanoatlantiramiz:

$$c_1\cos 0+c_2\sin 0=0\ ,\ c_1\cos\pi+c_2\sin\pi=0\ .$$

Bundan $c_1=0$, c_2 esa ixtiyoriy ekanligini topamiz. Demak, (*),(1) chegaraviy masalaning yechimlarga cheksiz ko‘p: $y=c\sin x$, $c=\text{const}$.

2. chegaraviy shartlardan ushbu

$$c_1\cos 0+c_2\sin 0=0\ ,\ c_1\cos\pi+c_2\sin\pi=1$$

tengliklarni hosil qilamiz. Bu tengliklarni c_1 ,c_2 larning hech qanday qiymatlarida qanoatlantirib bo‘lmaydi. Demak, (*),(2) chegaraviy masala yechimga ega emas.

3. chegaraviy shartlarning bajarilishi uchun

$$c_1\cos 0+c_2\sin 0=0\ ,\ c_1\cos 1+c_2\sin 1=1$$

bo‘lishi kerak. Bu munosabatlar

$$c_1=0 \text{ va } c_2=2/\sin 1$$

bo‘lgandagina o‘rinli ekanligini topamiz. Demak, (*),(3) chegaraviy masala $y=\dfrac{\sin x}{\sin 1}$ yagona yechimga ega.

§ 15.3

2. $G(x,\xi)=\begin{cases}\ln\dfrac{\xi}{2}\ ,\ 1\le x\le\xi, \\ \ln\dfrac{x}{2}\ ,\ \xi\le x\le 2.\end{cases}$

3. $\lambda_k=-\dfrac{1}{4}-\left(\dfrac{\pi k}{\ln 2}\right)^2$, $y_k=\dfrac{1}{\sqrt{1+x}}\sin\dfrac{\pi k\ln(1+x)}{\ln 2}$, $k\in\mathbb{N}$.

Tenglama yechimini $y=(1+x)^a$ ko‘rinishda izlang, Umumiy yechimni topib, chegaraviy shartlarni yozing. $(1+x)^{i\varphi}=\cos(\varphi\ln(1+x))+i\sin(\varphi\ln(1+x))$ formuladan foydalanib λ_k va y_k larni aniqlang.

4. $y(x)=\lambda\int_0^1 G(x,\xi)d\xi$, $G(x,\xi)=\begin{cases}\frac{1}{2}(e^{-x}-2)e^{-\xi}\ ,\ 0\le x\le\xi,\\ \frac{1}{2}(e^{-\xi}-2)e^{-x}\ ,\ \xi\le x\le 1.\end{cases}$

5. $-\frac{m}{2}\le y(x)\le 0$, $-\frac{m}{3x}\le y'(x)\le\frac{m}{3x}$ **.** Ekvivalent integral tenglamaga o‘ting.

§ 16.1

1. Teskarisini faraz qiling. U holda tenglamaning ixtiyoriy $u\in C^1$ yechimi

$$u=t\cdot g+c(x-t)\quad u=t\cdot g(x-t)+c(x-t)$$

ko‘rinishda ifodalanadi. Bu funksiyaning hech qanday sohada birinchi tartibli uzluksiz xususiy hosilalarga ega bo‘la olmasligini ko‘rsating.

§ 16.2

1. $u=u(t,x)$ yechimni $x=x_0+ct$ xarakteristikada, ya’ni $f(t)=u(t,x_0+ct)$ funksiyani qarang. Tenglamadan $\frac{df}{dt}=f$ munosabatni toping va $f(t)=f(0)e^t$ ekanligidan $u=u(t,x)$ yechim uchun, agar u chegaralangan bo‘lsa, $u(t,x)\equiv 0$ bo‘lishini keltirib chiqaring.

§ 16.3

1. Tenglamani xarakteristikalar bo‘ylab qarang.

2. $v=u_x$ deylik. Berilgan differensial tenglama va boshlang‘ich shartni x bo‘yicha differensiallab, v funksiyaga nisbatan kvazichiziqli tenglama va uning uchun boshlang‘ich masalaga kelamiz:

$$v_t+2vv_x=0, v|_{t=0}=2x\ .$$

Xarakteristik sistema:

$$\frac{dx}{dt}=2v\ ,\frac{dv}{dt}=0\ .$$

Demak,

$$v=\frac{2x}{4t+1}.$$

Bundan $v=u_x$ belgilash va $u|_{t=0}=x^2$ boshlang‘ich shartga ko‘ra berilgan masalaning yechimini topamiz:

$$u=\frac{x^2}{4t+1}.$$

§ 17.3

1. Ha. Koshi-Kovalevskaya teoremasiga ko‘ra. **2.** Koshi-Kovalevskaya teoremasini qo‘llab bo‘lmaydi. Analitik yechimning mavjud emasligini bevosita isbotlang.

§ 17.4

1. Isbotni soddalashtirish maqsadida t o‘rniga $t-t_0$ ni, x_j lar o‘rniga $x_j - x_j^0,\ j=\overline{1,n},$ larni kiritib, boshlang‘ich shartlarni $x_j(0)=0$, $j=\overline{1,n}$, ko‘rinishga keltirish mumkin. Bunda f_j analitik funksiyalar yana analitik funksiyalar bilan almashadi. Shuning uchun umumiylikni cheklamasdan boshlang‘ich nuqtani koordinatalar boshida va f_j funksiyalarni shu nuqtada analitik deb hisoblaymiz. Masalani yanada soddalashtirish maqsadida avtonom sistema uchun Koshi masalasiga o‘tamiz. Buning uchun yangi $\tau=\tau(t)$ noma’lum funksiyani $\tau'=1$, $\tau(0)=0$ masala yechimi (ya’ni $\tau=t$) kabi kiritib, $(\tau,\boldsymbol{x})$ noma’lum funksiyalarga nisbatan quyidagi avtonom sistema uchun Koshi masalasini qarash mumkin:

$$\begin{cases} \tau' = 1, \\ \boldsymbol{x}' = \boldsymbol{f}(\tau,\boldsymbol{x}), \\ \tau(0)=0, \\ \boldsymbol{x}(0)=0 \end{cases}$$

Ravshanki, bu avtonom sistemaning o‘ng tomoni, ya’ni $(1,\boldsymbol{f}(\tau,\boldsymbol{x}))$ vektor-funksiya $(0,0)\in\mathbb{R}^{1+n}$ nuqtada analitik. Shunday qilib, teoremani isbot qilish uchun uni ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \\ \boldsymbol{x}(0)=0 \end{cases} \tag{1}$$

Koshi masalasi holida isbotlash kifoya. Bu yerdagi $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}=0$ nuqta atrofida absolyut yaqinlashuvchi darajali qatorga yoyiladi:

$$f_j(\boldsymbol{x}) = \sum_{\|k\|=0}^{\infty} b_k^{(j)} \boldsymbol{x}^k = \sum_{\|k\|=0}^{\infty} b_k^{(j)} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \ , \ j=\overline{1,n} \quad ;$$

bunda qatorlar koeffitsientlari uchun

$$b_k^{(j)} = \frac{1}{k\,!}\frac{\partial^{\|k\|} f(0)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}},\ k=(k_1,k_2,\ldots k_n)\in\mathbb{Z}_+^n \ ,\ j=\overline{1,n}\ ,$$

formulalar o‘rinli bo‘ladi.

Dastlab qaralayotgan masalaning analitik yechimi yagona bo‘lishini isbotlaylik. Faraz qilaylik, $\boldsymbol{x}=\boldsymbol{x}(t)$ vektor-funksiya (1) masalaning $t=0$ nuqtada analitik yechimi bo‘lsin. Yechimning komponentalari $t=0$ nuqtaning biror atrofida absolyut yaqinlashuvchi darajali qatorga yoyiladi:

$$x_j = \sum_{i=0}^{\infty} c_i^{(j)} t^i \ ,\ c_0^{(j)} = 0 \ .$$

Bu qatorlarning koeffitsientlari

$$c_i^{(j)} = \frac{1}{i!}\frac{d^i x_j(0)}{dt^i},\ i=1,2,\ldots\ ,\ j=\overline{1,n}\ ,$$

formulalar yordamida aniqlanadi. Bu hosilalar qaralayotgan tenglamalardan ketma-ket quyidagicha bir qiymatli hisoblanadi. Ushbu

$$\frac{dx_j(t)}{dt} = f_j(x_1(t),x_2(t),\ldots,x_n(t)),\ \ j=\overline{1,n}\ , \tag{2}$$

tengliklarda $t=0$ deb,

$$c_1^{(j)} = \frac{dx_j(0)}{dt} = f_j(x_1(0), x_2(0), \ldots, x_n(0)) = f_j(0) = b_0^j,\ j = \overline{1,n},$$

bo‘lishini topamiz.

(2) tengliklarni differensiallaymiz:

$$\frac{d^2 x_j(t)}{dt^2} = \sum_{l=1}^{n} \frac{\partial f_j(\boldsymbol{x}(t))}{\partial x_l} \frac{dx_l(t)}{dt}. \tag{3}$$

Bu tengliklarda $t = 0$ deb, $\frac{d^2 x_j(0)}{dt^2}$ qiymatlarni topamiz

$$\frac{d^2 x_j(0)}{dt^2} = \sum_{l=1}^{n} \frac{\partial f_j(0)}{\partial x_l} \frac{dx_l(0)}{dt} = \sum_{l=1}^{n} b_l^{(j)} c_1^l.$$

Shunga o‘xshash (3) tengliklarni differensiallaymiz va hosil bo‘lgan tengliklardan $t = 0$ da $\frac{d^3 x_j(0)}{dt^3}$ qiymatlarni topamiz va h.k. Shunday qilib, $\frac{d^i x_j(0)}{dt^i}$, $i = 1, 2, \ldots$, $j = \overline{1,n}$, hosilalar (demak, $c_i^{(j)}$ koeffitsientlar ham) $b_k^{(j)} = \frac{1}{k!} \frac{\partial^{\|k\|} f(0)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$ koeffitsientlar orqali bir qiymatli aniqlanadi. Bu analitik yechimning yagonaligini isbotlaydi.

Endi analitik yechimning mavjudligini isbotlaylik. Yuqorida hisoblangan $c_i^{(j)}$ larga ko‘ra formal ravishda $x_j = \sum_{i=1}^{\infty} c_i^{(j)} t^i$ qatorlarni tuzaylik. Agar bu qatorning $t = 0$ nuqta atrofida yaqinlashuvchiligini ko‘rsatsak, uning yig‘indisi $c_i^{(j)}$ koeffitsientlarning qurilishiga ko‘ra izlangan yechim bo‘ladi.

Bu qatorning yaqinlashuvchiligini isbotlash uchun majorantlar metodidan foydalanamiz. $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya $\boldsymbol{x} = 0 \in \mathbb{R}^n$ nuqtada analitik bo‘lgani uchun u shu nuqtaning biror yopiq atrofida (masalan, biror $r > 0$ uchun $|x_j| \le r,\ j = \overline{1,n}$, bo‘lganda) absolyut yaqinlashuvchi qatorga yoyiladi:

$$f_j(\boldsymbol{x}) = \sum_{\|k\|=0}^{\infty} b_k^{(j)} \boldsymbol{x}^k,\ |x_j| \le r,\ j = \overline{1,n},$$

Demak, $\sum_{\|k\|=0}^{\infty} |b_k^{(j)}| r^{\|k\|}$ sonli qatorlar yaqinlashuvchi va

$$\exists M > 0 \quad \forall k \in \mathbb{Z}_+^n \quad |b_k^{(j)}| r^{\|k\|} \le M,\ \ j = \overline{1,n}.$$

Bunga ko‘ra ushbu

$$\sum_{\|k\|=0}^{\infty} \frac{M\,\|k\|!}{r^{\|k\|} k!} \boldsymbol{x}^k$$

darajali qator $\sum_{\|k\|=0}^{\infty} b_k^{(j)} \boldsymbol{x}^k = f_j(\boldsymbol{x})$ qatorlar uchun majorant qator bo‘ladi. Majorant qatorning yig‘indisini hisoblash oson:

$$\sum_{\|k\|=0}^{\infty} \frac{M\,\|k\|!}{r^{\|k\|}k!}\boldsymbol{x}^k = \sum_{m=0}^{\infty}\sum_{\|k\|=m}^{\infty} \frac{Mm!}{r^{\|k\|}k!}\boldsymbol{x}^k = \sum_{m=0}^{\infty}\sum_{\|k\|=m}^{\infty} \frac{Mm!}{k!}\left(\frac{\boldsymbol{x}}{r}\right)^k =$$

$$= M\sum_{m=0}^{\infty}\left(\frac{x_1+x_2+\ldots x_n}{r}\right)^m = \frac{M}{1-\dfrac{x_1+x_2+\ldots+x_n}{r}} = \frac{M}{r-(x_1+x_2+\ldots+x_n)} = \Phi(\boldsymbol{x}).$$

Endi qaralayotgan sistemaning o‘ng tomonidagi darajali qatorni (analitik funksiyani) uning majorantasi bilan almashtitib, quyidagi majorant masalani qaraylik:

$$y_j' = \frac{Mr}{r-(y_1+y_2+\ldots+y_n)},\; y_j(0)=0\,,\; j=\overline{1,n}\ .$$

Faraz qilaylik, bu masala anlitik yechimga ega bo‘lsin. Bu yechimning darajali qatorga yoyilmasi $y_j = \sum_{i=1}^{\infty} C_i^{(j)} t^i$ dagi $C_i^{(j)}$ koeffitsientlar musbat bo‘ladi va ular $c_i^{(j)}$ larni topishga o‘xshash topiladi. $C_i^{(j)}$ larni topishda $b_k^{(j)}$ lar o‘rnida uning modulidan kattaroq $\frac{M\,\|k\|!}{r^{\|k\|}k!} \geq |b_k^{(j)}|$ sonlar ishlatiladi va hisoblashlarda musbat songa ko‘paytirish va qo‘shish amallaridan foydalaniladi. Natijada $C_i^{(j)} \geq |c_i^{(j)}|$ hosil bo‘ladi, ya’ni ushbu $\sum_{i=1}^{\infty} C_i^{(j)} t^i = y_j$ darajali qator $\sum_{i=1}^{\infty} c_i^{(j)} t^i = x_j$ darajali qator uchun majorant bo‘ladi. Shuning uchun $\sum_{i=1}^{\infty} C_i^{(j)} t^i = y_j$ bilan birgalikda $\sum_{i=1}^{\infty} c_i^{(j)} t^i = x_j$ ham yaqinlashuvchi bo‘ladi.

Shunday qilib, majorant masalaning $y_j = \sum_{i=1}^{\infty} C_i^{(j)} t^i$, $C_i^{(j)} > 0$, analitik yechimga ega ekanligini ko‘rsatishimiz qoldi.

Majorant masala $y_1 = y_2 = \ldots = y_n = y$ yechimga ega. $y = y(t)$ uchun

$$\frac{dy}{dt} = \frac{Mr}{r-ny},\; y(0)=0$$

masalaga egamiz. U osongina yechiladi:

$$(r-ny)dy = Mrdt,\; y(0)=0;\; ny^2 - 2ry + 2Mrt = 0.$$

Bizga oxirgi kvadrat tenglamaning $t=0$ da nolga aylanuvchi yechimi kerak:

$$y = \frac{r-\sqrt{r^2-2Mnrt}}{n}\ .$$

Bu funksiyaning $t=0$ nuqtada analitik ekanligi ravshan: mos darajali qator koeffitsientlari musbat va u $|t| < \frac{1}{\omega}, \omega = \frac{2Mn}{r},$ intervalda absolyut yaqinlashuvchi. Bu tasdiq binomial qatordan kelib chiqadi:

$$y = \frac{r}{n}(1-(1-\omega t)^{1/2}) = \frac{r}{n}\left(\frac{1}{2}\omega t + \frac{1}{2}\frac{3}{2}\omega^2 t^2 + \ldots\right).$$ ♘

2. Umumiylikni buzmasdan $t_0 = 0$, $x_j^0 = 0$ $(j=\overline{1,n})$ deb hisoblaymiz. Chunki aks holda t o‘rniga $t-t_0$ ni, x_j lar o‘rniga $x_j - x_j^0, j=\overline{1,n},$ larni (mos ravishda) kiritib, boshlang‘ich shartlarni $x_j(0)=0$, $j=\overline{1,n}$, ko‘rinishga keltirish mumkin. Bunda

$a_{ij}(t), g_j(t)$ funksiyalar 0 nuqtada analitik funksiyalar bilan almashinadi. Shunday qilib, biz

$$\begin{cases} x'_j = \sum_{i=1}^{n} a_{ij}(t)x_i + g_j(t)\ ,\ \ j=\overline{1,n}, \\ x_j(0)=0,\ \ j=\overline{1,n}\ . \end{cases} \qquad (*)$$

boshlang‘ich masala uchun teoremani isbotlashimiz kerak. Bizga ushbu

$$a_{ij}(t)=\sum_{l=0}^{\infty} b_l^{(ij)} t^l\ ,\ \ i,j=\overline{1,n}\ ,\ \ g_j(t)=\sum_{l=0}^{\infty} b_l^{(j)} t^l\ ,\ \ j=\overline{1,n}\ ,$$

yoyilmalarning $(-R,R)$ intervalda o‘rinli ekanligi berilgan.

Analitik yechimning yagonaligi yuqoridagi teoremadagidek isbotlanadi. Faraz qilaylik, qaralayotgan masala 0 nuqtada analitik bo‘lgan $x_j(t),\ j=\overline{1,n}$, analitik yechimga ega bo‘lsin. Demak, bu yechim 0 nuqtadaning biror atrofida darajali qatorga yoyiladi:

$$x_j = \sum_{i=0}^{\infty} c_i^{(j)} t^i\ ,\ \ c_0^{(j)}=0\ .$$

Bu yerdagi $c_i^{(j)} = \dfrac{d^i x_j(0)}{dt^i}$ koeffitsientlar (*) tenglamalardagi koeffitsientlar orqali bir qiymatli aniqlanadi. Haqiqatan ham, ularni topish uchun ketma-ket tenglamalarda va undan olingan hosilalarda $t=0$ deyish kerak. Bunda, ravshanki, $c_i^{(j)}$ lar ketma-ket b_l^{ij} va c_l^j lar orqali musbat koeffitsientli ko‘phadlar kabi ifodalanadi.

Endi teoremaning mavjudlik qismini isbotlaymiz. Buning uchun formal ravishda yuqorida hisoblangan $c_i^{(j)}$ larga ko‘ra

$$\sum_{i=0}^{\infty} c_i^{(j)} t^i\ \ (c_0^{(j)}=0)$$

darajali qatorlarni tuzib, ularning $(-R,R)$ intervalda yaqinlashuvchi ekanligini ko‘rsatish kifoya. Ixtiyoriy $t=\tilde{t}\in(-R,R)$, $\tilde{t}\neq 0$, nuqtani olib, $\sum_{i=0}^{\infty} c_i^{(j)}\tilde{t}^i$ sonli qatorlarning yaqinlashuvchi ekanligini isbotlaymiz. Buning uchun yana majorantlar metodidan foydalanamiz.

Berilganga ko‘ra $a_{ij}(t)=\sum_{l=0}^{\infty} b_l^{(ij)} t^l\ ,\ \ i,j=\overline{1,n}\ ,\ \ g_j(t)=\sum_{l=0}^{\infty} b_l^{(j)} t^l\ ,\ \ j=\overline{1,n}\ ,$ qatorlar $t=\rho,\ \rho\in\left(\left|\tilde{t}\right|,R\right)$, nuqtada yaqinlashuvchi. Demak,

$$\exists M>0\ \ \left|b_l^{(ij)}\rho^l\right|\le M\,,\ \left|c_l^{(j)}\rho^l\right|\le M\ ,$$

ya’ni

$$\left|b_l^{(ij)}\right|\le \frac{M}{\rho^l}\,,\ \left|c_l^{(j)}\right|\le \frac{M}{\rho^l}\ \ .$$

Bundan ravshanki, $|t|<\rho$ bo‘lganda

$$\sum_{l=0}^{\infty} \frac{M}{\rho^l} t^l\ \left(=\frac{M\rho}{\rho-t}\right)$$

darajali qator $\sum_{l=0}^{\infty} b_l^{(ij)} t^l \left(= a_{ij}(t)\right),\ i,j=\overline{1,n}$, va $\sum_{l=0}^{\infty} b_l^{(j)} t^l \left(= g_j(t)\right),\ j=\overline{1,n}$, qatorlar uchun majorant qator bo‘ladi. Endi qaralayotgan (*) Koshi masalasi uchun majorant masalani tuzaylik:

$$\begin{cases} y_j' = \sum_{i=1}^{n} \frac{M\rho}{\rho-t} y_i + \frac{M\rho}{\rho-t},\ \ j=\overline{1,n}, \\ y_j(0)=0,\ \ j=\overline{1,n}\ . \end{cases}$$

Bu masalaning analitik yechimi $y_j = \sum_{l=1}^{\infty} C_l^{(j)} t^l, C_l^{(j)} > 0, j=\overline{1,n}$, mavjud bo‘lsa, bu yechim (*) Koshi masalasi uchun majorant qator bo‘ladi. Bu tasdiq yuqoridagi teoremaning isbotidan ravshan. Majorant masalaning analitik yechimini topaylik. t argument o‘rniga τ o‘zgaruvchini ushbu

$$\frac{dt}{\rho-t} = d\tau,\ \tau(0)=0,$$

shartlar orqali kiritaylik. Bu holda

$$\tau = -\ln\left(1-\frac{t}{\rho}\right),\ t=\rho(1-e^{-\tau})$$

bo‘ladi.

Majorant masala $y_1 = y_2 = \ldots = y_n = y$ yechimga ega:

$$\begin{cases} \frac{dy}{d\tau} = M\rho(ny+1), \\ y(0)=0\ \ . \end{cases}$$

Bu yerdagi tenglamada o‘zgaruvchilarni ajratib, masalaning yechimi osongina olamiz:

$$y = \frac{1}{n}\left(e^{M\rho n\cdot\tau} - 1\right).$$

Eski t o‘zgaruvchiga qaytib topamiz:

$$y = \frac{1}{n}\left((1-\frac{t}{\rho})^{-M\rho n} - 1\right).$$

Binomial qator formulasidan bu funksiyaning Makloren qatoriga yoyilmasi $|t|<\rho$ bo‘lganda yaqinlashuvchi va qator koeffitsientlarining musbat ekanligini ko‘ramiz. Demak, bu qator majorant bo‘lgan $\sum_{i=0}^{\infty} c_i^{(j)} t^i\ \ (c_0^{(j)}=0)$ qatorlar ham $|t|<\rho$ bo‘lganda yaqinlashuvchi; xususan, $\sum_{i=0}^{\infty} c_i^{(j)} \tilde{t}^i\ \ (c_0^{(j)}=0)$ sonli qatorlar ham yaqinlashadi.

ILOVA. MAPLE DASTURIDAN FOYDALANISH

Hozirgi kunda MAPLE, MATEMATICA, MATLAB va MATHCAD kabi kompyuter dasturlarlari mavjudki, ular yordamida sof matematikada va amaliy fanlarda uchraydigan ko‘pdan-ko‘p matematik masalalarni analitik (simvolli) va sonli yechish mumkin (Maple inglizcha so‘z, u meypl deb o‘qiladi). Bu dasturlar (sistemalar) foydalanuvchi uchun qulay bo‘lgan interfeyslar (muloqot oynalari)ga ega.

Biz Maple sistemasi yordamida oddiy differensial tenglamalarni tekshirish (ularni yechish, yechim tabiatini o‘rganish va hk.) buyruqlari bilan tanishamiz.

Maple arifmetik hisoblashlarni quvvatli kalkulyator kabi bajaradi, harfli ifodalarni ixchamlaydi, tenglamalarni analitik usullar bilan birgalikda sonli usullar yordamida ham yechadi, tekislikda va fazoda grafiklar quradi.

Maple kompyuterga yuklatilgach, displeyda ishchi varaq (work sheet) paydo bo‘ladi. Ishchi varaqda buyruqlar kiritiladi va natijalar chiqariladi. Maple buyruqlar majmuasi bilan to‘laligicha uning Help (yordam) imkoniyati orqali tanishish mumkin. Buyruq kiritish sohasi (satri) ushbu “ $>$” belgi bilan boshlanadi. Shu belgidan keyin kerakli buyruqni terib, “Enter” tugmasi bosilgach, natija displeyning chiqarish sohasida paydo bo‘ladi.

Mapleda tenglamalarni yechish uchun **solve (yechmoq)** buyrug‘i ishlatiladi. Masalan, ushbu $2x^2-x-3=0$ tenglamani x ga nisbatan yechishni quyidagicha bajarish mumkin

> solve(2*x^2-x-3=0,x);

$$\frac{3}{2},\ -1$$

Ushbu $2x^3-x^2-2a^2x+a^2=0$ tenglamani x ga va a ga nisbatan yechaylik:

> solve(2*x^3-x^2-2*a^2*x+a^2=0,x);

$$\frac{1}{2},\ a,\ -a$$

> solve(2*x^3-x^2-2*a^2*x+a^2=0,a);

$$x,\ -x$$

Qiymatlash operatori **:=** bilan belgilanadi. Masalan, **eq** bilan $2x^3-x^2-2a^2x+a^2=0$ tenglamani belgilab (**eq** ga $2x^3-x^2-2a^2x+a^2=0$ “tenglama qiymatini” berib), yuqoridari ishni quyidagicha bajarish mumkin:

> eq:=2*x^3-x^2-2*a^2*x+a^2=0;

$$eq := 2x^3 - x^2 - 2a^2x + a^2 = 0$$

> **solve(eq,x);**

$$\frac{1}{2}, a, -a$$

> **solve(eq,a);**

$$x, -x$$

Funksiyani aniqlash operaratori yordamida funksiyalarni aniqlash (kiritish) mumkin. Masalan, $f(x) = 2\ln x + \exp(3x) + x^2$ funksiyani aniqlash quyidagicha amalga oshiriladi:

> **f:=x->2*ln(x)+exp(3*x)+x^2;**

$$f := x \mapsto 2\ln(x) + e^{3x} + x^2$$

Aniqlangan bu funksiyaning $f(1)$ va $f(0,5)$ qiymatlarini endi hisoblash oson:

> **f(1);**

$$1 + e^3$$

> **f(0.5);**

$$3.345394709$$

Maple $f(1)$ ni aniq hisobladi. Agar 1 butun sonning o‘rniga 1.0 taqribiy son ko‘rsatsak, taqribiy qiymat hisoblanadi:

> **f(1.0);**

$$21.08553692$$

Aniqlik ko‘rsatilmaganda hisoblashlar 10 ta qiymatli raqam aniqligida bajariladi. Kerak bo‘lsa, qiymatli raqamlar sonini **Digits** buyrug‘i bilan qo‘shimcha ko‘rsatish mumkin (20 ta qiymatli raqam hisoblash):

> **Digits:=20;f(1.0);**

$$Digits := 20$$

$$21.085536923187667741$$

Bu buyruqdan keyingi hisoblashlar endi 20 ta qiymatli raqam bilan bajariladi. Masalan,

> **1/3.0;**

$$0.33333333333333333333$$

Berilgan ko‘rsatmalar, aniqlangan funksiyalar va o‘zgaruvchilarning qiymatlarini bekor qilish uchun **restart** buyrug‘ini berish kerak:

> **restart;**

> **1/3.0;**

$$0.3333333333$$

> **f(1.0);**

$$f(1.0)$$

Bu yerda $f(x)$ funksiya bekor qilingani (yo‘qotilgani) uchun $f(1.0)$ yozuvi chiqarilgan (funksiya yo‘q – qiymat ham yo‘q).

Differensiallash buyrug‘i (amali) **diff** bilan belgilanadi. U mavjud, aniqlangan, ko‘rsatilgan funksiyaning hosilasini hisoblaydi:

> g:=x->exp(3*x)+2*x^2;

$$g := x \mapsto e^{3x} + 2x^2$$

> **diff(g(x),x);**

$$3e^{3x} + 4x$$

> **diff(f(x),x);**

$$f'(x)$$

Ikkinchi tartibli hosilani hisoblash quyidagicha bajariladi:

> **diff(g(x),x,x);**

$$9e^{3x} + 4$$

> **diff(f(x),x,x);**

$$f''(x)$$

Qisqaroq yozuv ham ishlatish mumkin:

> **diff(f(x),x$5);**

$$f^{(5)}(x)$$

Yana misol keltiraylik (ikki o‘zgaruvchining $h(x, y)$ funksiyasi aniqlangan va uning y bo‘yicha uchinchi tartibli hosilasi hisoblangan):

> **h:=(x,y)->3*x^2*y^3-x*y-x;**

$$h := (x, y) \rightarrow 3x^2y^3 - yx - x$$

> **diff(h(x,y),y$3);**

$$18x^2$$

Mapleda **D** differensiallash operatori ham mavjud. **D(f)** yozuv **f** funksiyaning hosilasini anglatadi:

> **D(g); diff(g(x),x);**

$$x \mapsto 3e^{3x} + 4x$$

$$3e^{3x} + 4x$$

> **D(f);**

$$D(f)$$

Ko‘p o‘zgaruvchining funksiyadan hosila olish shunga o‘xshash bajariladi:

> **h(x,y);**

$$3x^2y^3 - yx - x$$

> **D[1,2](h);**

$$(x, y) \mapsto 18xy^2 - 1$$

> **D[1,2](f);**

$$D_{1,2}(f)$$

Quyidagi misollar **D** operatorining mohiyatini ochadi:

> restart;

> D(cos);

$$-\sin$$

> D(ln)

$$z \to \frac{1}{z}$$

> D(ln)(x) = diff(ln(x),x);

$$\frac{1}{x} = \frac{1}{x}$$

> D(D(f));

$$D^{(2)}(f)$$

> (D@@2)(g);

$$D^{(2)}(g)$$

> f := (x,y) -> y^2*sin(x) + x^3*y;

$$f := (x, y) \to y^2 \sin(x) + x^3 y$$

> D[1](f);

$$(x, y) \to y^2 \cos(x) + 3x^2 y$$

> diff(f(x,y), x);

$$y^2 \cos(x) + 3x^2 y$$

> g:=D[2](f);

$$g := (x, y) \to 2y \sin(x) + x^3$$

> g(a,b);

$$2b \sin(a) + a^3$$

> f21:=D[2,1](f);

$$f21 := (x, y) \to 2y \cos(x) + 3x^2$$

> f12:=D[1,2](f);

$$f12 := (x, y) \to 2y \cos(x) + 3x^2$$

Maple da aniqmas va aniq integrallarni hisoblash mumkin. **int** buyrug‘i integral hisoblaydi:

> int(x*ln(x)+x,x);

$$\frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2$$

> int(exp(-x^2),x);

$$\frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)$$

evalf buyrug‘i yordamida aniq integralni sonli hisoblash mumkin. Masalan, $\int_0^1 \frac{\sin x}{x}dx$ integralning qiymati quyidagicha hisoblanadi:

> evalf(int(sin(x)/x,x=0..1));

0.9460830704

Int buyrug‘i inert integralni anglatadi. Bu buyruq integralni hisoblamaydi, yozadi xolos. Masalan,

> Int(x*ln(x)+x,x);

$$\int (x \ln(x) + x)\, dx$$

Tekislikda chizmalar qurish. Ikki o‘lchamli chizmalarni (bir haqiqiy o‘zgaruvchining haqiqiy funksiyasi grafigi, tekislikdagi vektor maydonlar va h.k.) **plot** buyrug‘i quradi. Chizmalar xususiyatlari (fontlar, ranglar, sarlavhalar, turli yozuvlar va h.k.) qo‘shimcha tarzda ko‘rsatiladi. Bu buyruqning ko‘rinishi

plot(f, h, v) yoki **plot**(f, h, v,...).

Bu yerdagi parametrlar:

f – grafigi quriladigan funksiya(lar);

h – gorizontal oraliq;

v – vertikal oraliq (yozilishi shart bo‘l magan argument);

… (nuqtalar) o‘rniga qo‘shimcha ko‘rsatmalar yoziladi. Ular haqida to‘la ma’lumotni **Help** dan **?plot[options]** so‘rovi orqali bilib olish mumkin.

Vertikal oraliq yozilmaganda Maple o‘zi uni avtomatik tarzda tanlaydi.

Oddiy differensial tenglama (ODT) larni Maple yordamida yechishda asosiy qurol, bu **dsolve** buyrug‘idir.

dsolve buyrug‘ining ko‘rinishlari:

dsolve(ODE)

dsolve(ODE, y(x), options)

dsolve({ODE, ICs}, y(x), options)

dsolve({sysODE, ICs}, {funcs}, options)

Bu yerdagi parametrlar

ODE – (ordinary differential equation) oddiy differensial tenglama,

y(x) – x erkli o‘zgaruvchi (argument) ning noma’lum funksiyasi,

ICs – (initial conditions) boshlang‘ich shartlar,

{sysODE} – oddiy differensial tenglamalar sistemasi (to‘plami),

{funcs} – noma’lum funksiyalar to‘plami (sistemasi),

options – yozilishi shartmas (optional) parametrlar (shartmas argumentlar); ular yechiladigan masalaning tipiga bog‘liq.

Buyruq tavsifi.

Erksiz o‘zgaruvchi (noma’lum funksiya)ni ko‘rsatuvchi **options** berilgan ODTda bir necha funksiyalarning hosilalari qatnashgan holda yoziladi, chunki u yozilmaganda Maple tenglamani qaysi noma’lum funksiyaga nisbatan yechish kerakligini tushunmaydi.

Chekli ko‘rinishdagi umumiy yechim oshkor, ya’ni $y(x)=\varphi(x,\_C_1,\_C_2,\ldots,\_C_n)$ yoki oshkormas, ya’ni $\Phi(y(x),x,\_C_1,\_C_2,\ldots,\_C_n)=0$ ko‘rinishda berilishi mumkin; bu yerda $\_C_1,\_C_2,\ldots,\_C_n-$ ixtiyoriy o‘zgarmaslar ($n-$ natural son). Agar mumkin bo‘lsa, yechim oshkor ko‘rinishda chiqariladi.

Birinchi tartibli ODTlarning yechimlari, ayniqsa ular dy/dx ga nisbatan yuqori darajali bo‘lsa, parametrik ko‘rinishda ham berilishi mumkin: [x(_T)=f(_T), y(_T)=g(_T)], bunda _T parametr.

Shartmas argumentlar **(options).**

dsolve buyrug‘iga shartmas argumentlar orqali qo‘shimcha ko‘rsatmalar berish mumkin. Shartmas argumentlarning to‘la tavsifini Help orqali bilib olsa bo‘ladi.

dsolve buyrug‘ida bir dona ODT ko‘rsatilgan holida ko‘p uchraydigan shartmas argunentlar quyida keltirilgan:

‘**implicit'** yechimni oshkormas ko‘rinishda chiqarish.

‘explicit' yechimni oshkor ko‘rinishda chiqarish.

‘parametric' parametrik yechimni topish (faqat birinchi tertibli tenglamalar uchun)

‘**useInt'** inert integralni ishlatish; yechish jarayoni tezlashadi. Yechim topilgach, value buyrug‘i bilan barcha integrallarni hisoblab ko‘rish mumkin.

‘**useint'** yechish jarayonida barcha integrallarni hisolab yurish (agar mumkin bo‘lsa)

‘_mu = int_factor_hint' integrallovchi ko‘paytuvchini ko‘rsatilgan ko‘rinishda izlash va berilgan ODTni yechish

Optional parametrlar (shartmas argumentlar)ning boshqa xususiyatlarini Help dan **dsolve,setup** orqali bilib olish mumkin.

Quyida birinchi tartibli chiziqli differensial tenglama yechilgan:

> deq1:=diff(y(x),x)+y(x)=x;dsolve(deq1,y(x));

$$deq1 := \frac{d}{dx}\,y(x) + y(x) = x$$

$$y(x) = -1 + x + e^{-x}\,\_C1$$

Demak, $y'+y=x$ tenglamaning umumiy yechimi $y=-1+x+ce^{x}$ ($c=\text{const}$).

Ushbu

$$\begin{cases} y' + y = x \\ y(0) = 1 \end{cases}$$

Koshi masalasining yechimi:

> dsolve({deq1,y(0)=1});

$$y(x) = -1 + x + 2\,e^{-x}$$

Differensial tenglamani oshkormas (**implicit)** ko‘rinishda yechishga ham buyruq beriish mumkin.

Bir ikkinchi tartibli nochiziqli tenglamaning yechimi:

> deq4:=x^2*y(x)*diff(y(x),x,x)+3*(y(x)-x*diff(y(x),x))^2=0;

$$deq4 := x^2 y(x) y''(x) + 3\left(y(x) - x y'(x)\right)^2 = 0$$

> dsolve(deq4,implicit);

$$\frac{1}{4}\,\frac{y(x)^4}{x^4} - \frac{\_C1}{x} + \_C2 = 0$$

Ushbu

$$\begin{cases} y'' - 5y' + 6y = x \\ y(0) = a, y'(0) = b \end{cases}$$

Koshi masalasining yechimi:

> deq5:=diff(y(x),x,x)-5*diff(y(x),x)+6*y(x)=x;

$$deq5 := \frac{d^2}{dx^2} y(x) - 5\left(\frac{d}{dx} y(x)\right) + 6\,y(x) = x$$

> Init_con:=y(0)=a, D(y)(0)=b;

$$Init\_con := y(0) = a,\ D(y)(0) = b$$

> dsolve({deq5, Init_con});

$$y(x) = e^{2x}\left(-b - \frac{1}{4} + 3a\right) + e^{3x}\left(-2a + \frac{1}{9} + b\right) + \frac{5}{36} + \frac{1}{6}x$$

Endi ushbu

$$y''' - 3y' + 2y = 0,\ y(0) = 1,\ y'(1) = 1,\ y(1) + y''(0) = 0$$

chegaraviy masalani yechaylik:

> deq6:=diff(y(x),x$3)-3*diff(y(x),x)+2*y(x)=0;

$$deq6 := \frac{d^3}{dx^3} y(x) - 3\left(\frac{d}{dx} y(x)\right) + 2\,y(x) = 0$$

> Bound_con:=y(0)=1, D(y)(1)=1,y(1)+D(D(y))(0)=0;

$$Bound\_con := y(0) = 1, D(y)(1) = 1, y(1) + D^2(y)(0) = 0$$

> dsolve({deq6,Bound_con});

$$y(x) = -\frac{\left(e+2+(e)^2\right)e^{-2x}}{-(e)^2+4\,e\,e^{-2}+8\,e+4\,e^{-2}} - \frac{\left(3\,e\,e^{-2}+5\,e+e^{-2}-3\right)x\,e^{x}}{-(e)^2+4\,e\,e^{-2}+8\,e+4\,e^{-2}} + \frac{\left(4\,e\,e^{-2}+9\,e+4\,e^{-2}+2\right)e^{x}}{-(e)^2+4\,e\,e^{-2}+8\,e+4\,e^{-2}}$$

Quyida bir ODTlar sistemasi yechilgan

> sysdeq := [(D@@2)(x)(t)+2*(D@@2)(y)(t)-x(t)-2*y(t)=0, D(x)(t)+2*D(y)(t)-x(t)+y(t)=0];

$$sysdeq := \left[D^{(2)}(x)(t) + 2\,D^{(2)}(y)(t) - x(t) - 2\,y(t) = 0,\ D(x)(t) + 2\,D(y)(t) - x(t) + y(t) = 0\right]$$

> dsolve(sysdeq);

$$x(t) = \_C1\,e^{t} + \_C2\,e^{-t},\ y(t) = -2\,\_C2\,e^{-t}$$

Endi ODTlar sistemasi uchun Koshi va chegaraviy masalalarni yechaylik:

> sysdeq2:= diff(y(t),t) = x(t)+2*y(t), diff(x(t),t) = 3*x(t)+2*y(t);

$$sysdeq2 := \frac{d}{dt}\,y(t) = x(t) + 2\,y(t),\ \frac{d}{dt}\,x(t) = 3\,x(t) + 2\,y(t)$$

> Boshl_shart:= x(0)=1, y(1)=0;

$$Boshl\_shart := x(0) = 1,\ y(1) = 0$$

> dsolve({sysdeq2, Boshl_shart});

$$\left\{x(t) = \frac{e^4 e^t}{2e+e^4} + \frac{2ee^{4t}}{2e+e^4},\ y(t) = -\frac{e^4 e^t}{2e+e^4} + \frac{2ee^{4t}}{2e+e^4}\right\}$$

> Cheg_shart:=x(0)+y(1)=1,2*x(0)-y(0)=1;

$$Cheg\_shart := x(0) + y(1) = 1,\ 2\,x(0) - y(0) = 1$$

> dsolve({sysdeq2, Cheg_shart});

$$\left\{x(t) = \frac{1}{3}\frac{(-1+e^4)e^t}{e+1+e^4} + \frac{2}{3}\frac{(2+e)e^{4t}}{e+1+e^4},\ y(t) = -\frac{1}{3}\frac{(-1+e^4)e^t}{e+1+e^4} + \frac{1}{3}\frac{(2+e)e^{4t}}{e+1+e^4}\right\}$$

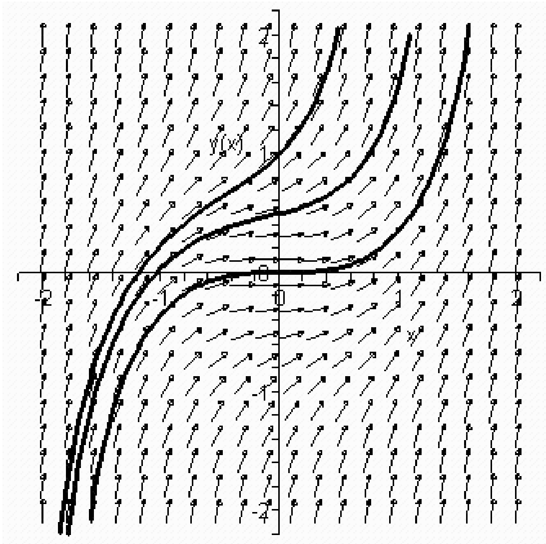
Mapledagi DEtools nomli paket ODTlarni tekshirishda katta imkoniyatlar yaratadi. Bu paket va Maplening boshqa imkonoyatlari bilan internetdagi www.exponenta.ru saytdan tanishish mumkin. Darslikning § 1.2 da $y' = x^3 + y^3$ tenglamaning yo‘nalishlar maydoni va integral chiziqlari ana shu **DEtools** paketdan foydalanib qurilgan.

Differensial tenglamaning yo‘nalishlar maydoni bilan birgalaikda tayin yechim(lar)ining grafigini (integral chiziqlarni) qurish uchun **Deplot** buyrug‘idan foyalanish mumkin. Masalan,

> with(DEtools): deq:=D(y)(x)=x^2+y(x)^2;

$$deq := D(y)(x) = x^2 + y(x)^2$$

> DEplot(deq, y(x), x=-2..2, [[y(0)=0], [y(0)=0.5], [y(0)=1]], y=-2..2, stepsize=.05, arrows=medium, colour=black, linecolour=black, thickness=2);



Endi berilgan sistemaning yo'nalishlar maydoni bilan birgalikda turli boshlang'ich shartlar (**IC**) ga ko'ra mos trayektoriyalarni quramiz. Buning uchun **phaseportrait** buyrug'idan foydalanamiz.

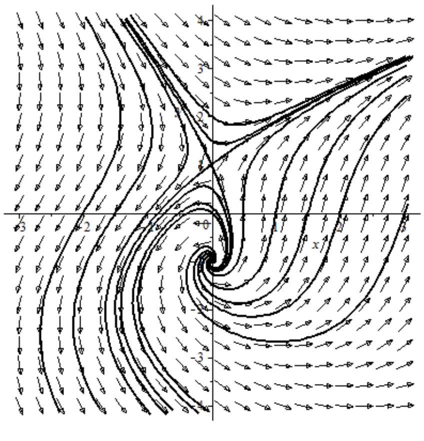
sys:={diff(x(t),t)=2*x(t)+y(t)^2-1,diff(y(t),t)=6*x(t)-y(t)^2+1};

$$sys := \left\{ \frac{d}{dt} x(t) = 2\,x(t) + y(t)^2 - 1,\ \frac{d}{dt} y(t) = 6\,x(t) - y(t)^2 + 1 \right\}$$

> with(DEtools): IC:= [[0,-2,0], [0,-1.5,0], [0,-1,0], [0,-0.5,0], [0,0,0], [0,1,0], [0,0.5,0], [0,1.5,0], [0,2,0], [0,3,0], [0,0,0.5], [0,0,0.99], [0,0,1.01], [0,0,1.5], [0,0,2]];

IC := [[0, -2, 0], [0, -1.5, 0], [0, -1, 0], [0, -0.5, 0], [0, 0, 0], [0, 0.5, 0], [0, 1, 0], [0, 1.5, 0], [0, 2, 0], [0, 3, 0], [0, 0, 0.5], [0, 0, 0.99], [0, 0, 1.01], [0, 0, 1.5], [0, 0, 2]]

> phaseportrait(sys,{x(t),y(t)},-10..10,IC, x=-3..3,y=-4..4, stepsize=0.01, arrows=`SLIM`, thickness=2, linecolor=black);



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Glossariy

(Kitobda uchrashi tartibida)

Oraliq – sonlar o‘qidagi kamida bitta ichki nuqtaga ega bo‘lgan bog‘lanishli (tutash) to‘plam

Uzluksiz diffrensiallanuvchu funksiya – barcha birinchi tartibli (xususiy) hosilalari uzluksiz bo‘lgan funksiya.

Differensial tenglamaning yechimi (oraliqda) shunday funksiyaki, uning tenglamada qatnashgan barcha hosilalari uzluksiz va u differensial tenglamani qanoatlantiradi (ayniyatga aylantiradi)

Differensial tenglamaning umumiy yechimi - differensial tenglamaning berigan sohadagi barcha yechimlarini va faqat ularnigina ifodalovchi funksiya.

Differensial tenglama $F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ ko‘rinishga ega; bunda F funksiya $n+2$ ta erkli haqiqiy o‘zgaruvchining haqiqiy funksiyasi, $x-$ argument, $y(x)-$noma’lum funksiya, $y'(x), \dots, y^{(n)}(x)-$ hosilalar.

Differensial tenglamaning tartibi noma’lum funksiyaning tenglamada qatnashgan hosilalarining eng yuqori tartibi.

Koshi masalasi $n-$ tartibli $F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ differensial tenglama uchun bu – tenglamaning n ta $y(x_0) = y_0, \dfrac{dy(x_0)}{dx} = y_0', \dots, \dfrac{d^{n-1}y(x_0)}{dx^{n-1}} = y_0^{n-1}$ Koshi shartlarini (boshlang‘ich shartlarni) qanoatlantiruvchi yechimini biror $I, x_0 \in I$, oraliqda topish.

Boshlang`ich masala Koshi masalasining boshqacha atalishi.

Differensiallarda yozilgan tenglama $M(x, y)dx + N(x, y)dy = 0$ ko‘rinishga ega.

O‘zgaruvchilari ajraladigan differensial tenglama ushbu $M(x)N(y)dx + P(x)Q(y)dy = 0$ (yoki $y' = f(x)g(y)$) ko‘rinishga ega.

O‘zgaruvchilariga nisbatan bir jinsli differensial tenglama $y' = g(\frac{y}{x})$ ko‘rinishga ega; bunda $g(t)-$ bir haqiqiy o‘zgaruvchi t ning funksiyasi .

Birinchi tartibli chiziqli differensial tenglama $y' + p(x)y = q(x)$ (yoki $a(x)y' + b(x)y = c(x)$, $a(x) \neq 0$) ko‘rinishga ega.

Bernulli tenglamasi $y' = p(x)y + q(x)y^{\alpha}$ $(\alpha \neq 1, \alpha \neq 0)$ ko‘rinishga ega.

Rikkati tenglamasi $y' = a(x)y^2 + b(x)y + c(x)$ ko‘rinishga ega.

To‘la differensialli tenglama $M(x,y)dx + N(x,y)dy = 0$ ko‘rinishda bo‘lib, uning chap tomoni biror $u = u(x,y)$ funksiyaning (to‘la) diffrerensialidan iborat, ya’ni $du(x,y) = M(x,y)dx + N(x,y)dy$.

Integrallovchi ko‘paytuvchi shunday silliq $\mu = \mu(x,y)$ funksiyaki, $M(x,y)dx + N(x,y)dy = 0$ tenglamani unga ko‘paytirilganda bu tenglama to‘la differensialli tenglamaga aylanadi.

MYaT – mavjudlik va yagonalik teoremasi.

Lagranj tenglamasi deb

$$y = x\psi(y') + \chi(y') \ \left(\psi(y') \not\equiv y' \right)$$

ko‘rinishdagi differensial tenglamaga aytiladi.

Klero tenglamasi ushbu

$$y = xy' + \chi(y')$$

ko‘rinshga ega.

Maxsus yechim deb $F(x,y,y') = 0$ differensial tenglamaning shunday $y = \psi(x)$ yechimiga aytiladiki, bu yechim grafigining har qanday nuqtasidan shu integral chiziqqa (grafikka) urinib boshqa bir $y = \varphi(x)$ yechim grafigi ham o‘tadi; bunda $y = \psi(x)$ va $y = \varphi(x)$ yechimlar urinish nuqtasining ixtiyoriy atrofida ustma-ust tushmasliklari kerak.

$\boldsymbol{p}$**– diskriminant chiziq** ($F(x,y,y') = 0$ differensial tenglama uchun) deb ushbu

$$\begin{cases} F(x,y,p) = 0 \\ \dfrac{\partial F(x,y,p)}{\partial p} = 0 \end{cases}$$

sistemani biror p da qanoatlantiruvchi $(x,y) \in \mathbb{R}^2$ nuqtalar to‘plamiga aytiladi.

Lipshits sharti $f(x, y_1, y_2, \ldots, y_n)$ skalyar funksiya uchun $(x, y_1, y_2, \ldots, y_n) \in E \subset \mathbb{R}^{n+1}$ to‘plamda $y_1, y_2, \ldots, y_n$ o‘zgaruvchilarga nisbatan:

shunday $L > 0$ son mavjudki, barcha $(x, y_1', y_2', \ldots, y_n') \in E$ va $(x, y_1'', y_2'', \ldots, y_n'') \in E$ nuqtalar uchun ushbu

$$|f(x,y_1',y_2',\dots,y_n') - f(x,y_1'',y_2'',\dots,y_n'')| \le L(|y_1'-y_1''|+|y_2'-y_2''|+\dots+|y_n'-y_n''|)$$
tengsizlik o‘rinli bo‘ladi.

Avtonom tenglama – erkli o‘zgaruvchi bevosita qatnashmagan differensial tenglama, $F(y,y',\dots,y^{(n)})=0,\ y=y(x)-?$

Noma’lum funksiya va uning hosilalariga nisbatan bir jinsli tenglama

$$F(x,y,y'...,y^{(n)})=0$$

ko‘rinishga ega bo‘lib, bunda tenglamaning chap tomonidagi funksiya $y,y',...,y^{(n)}$ o‘zgaruvchilarni λ ga ko‘paytirilganda u λ ning biror $m-$ darajasiga ko‘payadi, ya’ni

$$F(x,\lambda y,\lambda y',...,\lambda y^{(n)})=\lambda^m F(x,y,y',...,y^{(n)}).$$

Umumlashgan bir jinsli differensial tenglama

$$F(x,y,y'...,y^{(n)})=0$$

ko‘rinishga ega bo‘lib, bunda tenglamaning chap tomoni uchun ushbu

$$F(\lambda x,\lambda^k y,\lambda^{(k-1)}y',...,\lambda^{k-n}y^{(n)})=\lambda^m F(x,y,y',...,y^{(n)})$$

munosabat o‘rinli.

Chap tomoni to‘la hosiladan iborat bo‘lgan tenglama

$$F(x,y,y',...,y^{(n)})=0$$

ko‘rinishga ega bo‘lib, bunda tenglamaning **chap tomoni** $x,y,y',...,y^{(n-1)}$ o‘zgaruvchilarning funksiyasi bo‘lmish biror $\Phi(x,y,y',...,y^{(n-1)})$ ning x bo‘yicha **to‘la hosilasidan iborat** bo‘ladi, ya’ni

$$F(x,y,y',...,y^{(n)})=\frac{d}{dx}\Phi(x,y,y',...,y^{(n-1)})$$

$$\left(F(x,y,y',...,y^{(n)})\overset{def}{=}\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}y'+\frac{\partial\Phi}{\partial y'}y''+...+\frac{\partial\Phi}{\partial y^{(n-1)}}y^{(n)}\right).$$

Yuqori (n-) tartibli chiziqli differensial tenglama

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x)\ \left(a_n(x)\ne 0\right)$$

ko‘rinishga ega.

Chiziqli bir jinsli differensial tenglama

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=0$$

ko‘rinishga ega.

Funksiyalar $\left(y_1(x),\, y_2(x),\, ...,\, y_n(x)\right)$ **ning chiziqli kombinatsiyasi** deb ularni

$\lambda_1, \lambda_2, ..., \lambda_n$ – o'zgarmas sonlarga ko'paytirib qo'shishdan hosil bo'lgan $\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x)$ ifodaga aytiladi; bu yerdagi $\lambda_1, \lambda_2, ..., \lambda_n$ sonlar shu chiziqli kombinatsiyaning koeffitsientlari deb ataladi.

Trivial chiziqli kombinatsiy deb barcha koeffitsientlari nolga teng bo'lgan chiziqli kombinatsiyaga aytiladi. Trivial chiziqli kombinatsiy aynan nolga teng.

Chiziqli bog'langan (I oraliqda) **funksiyalar** deb shunday $y_1(x),\, y_2(x),\, ...,\, y_n(x)$ funksiyalarga aytiladiki, ularning biror notrivial chiziqli kombinatsiyasi **(**I oraliqda) aynan nolga teng bo'ladi:

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x) \equiv 0 \ \ (x \in I),$$

bunda $|\lambda_1| + |\lambda_2| + ... + |\lambda_n| \neq 0$.

Chiziqli erkli funksiyalar deb chiziqli bog'lanmagan funksiyalarga aytiladi.

Vronskian (Vronskiy determinanti) – berilgan $y_1(x),\, y_2(x),\, ...,\, y_n(x) \in C^{n-1}(I)$ funksiyalar uchun tuzilgan ushbu

$$W[y_1(x),\, y_2(x),\, ...,\, y_n(x)] = \begin{vmatrix} y_1(x) & y_2(x) & ... & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

determinant.

Bazis yechimlar

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0 \ \left(a_n(x) \neq 0\right)$ n-tartibli chiziqli bir jinsli differensial tenglama uchun – shu tenglamaning n dona chiziqli erkli yechimlari.

Yechimlarning fundamental sistemasi – bazis yechimlarning boshqacha atalishi.

***n*- tartibli chiziqli differensial operator**

$$L[y] \overset{def}{=} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y \ \ \left(a_n(x) \neq 0\right)$$

ko'rinishga ega.

Ostrogradskiy-Liuvill formulasi – n- tartibli chiziqli bir jinsli differensial tenglama $y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=0$ ning (bosh koeffitsient birga teng) n dona $y=y_1(x),\ y=y_2(x),\ \dots,\ y=y_n(x)$ yechimlarining $W(x)$ vronskiani uchun quyidagi formula

$$W(x)=W(x_0)\exp\left(-\int_{x_0}^{x}a_{n-1}(s)ds\right)$$

Lagranjning ixtiyoriy o‘zgarmaslarni variatsiyalash metodi ushbu $L[y]=g(x)$ n- tartibli chiziqli bir jinsli bo‘lmagan differensial tenglamaning yechimini izlash usuli. Bu usulga ko‘ra mos bir jinsli tenglama $L[y]=0$ ning $y=\sum_{j=1}^{n}c_jy_j(x)$ umumiy yechimi ($y_j(x),\ j=\overline{1,n}$, – bazis yechimlar) dagi ixtiyoriy o‘zgarmaslarni variatsiyalab $L[y]=g(x)$ tenglamaning yechimini $y=\sum_{j=1}^{n}c_j(x)y_j(x)$ ko‘rinishda izlaymiz.

Eyler formulalari quyidagi formulalardir:

$$e^{ix}=\cos x+i\sin x,\ e^{-ix}=\cos x-i\sin x,$$

$$\cos x=\frac{e^{ix}+e^{-ix}}{2},\ \sin x=\frac{e^{ix}-e^{-ix}}{2i}\quad ...$$

Eyler (differensial) tenglamasi deb

$$a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\cdots+xa_1y'+a_0y=0$$

ko‘rinishdagi chiziqli differensial tenglamaga aytiladi.

Xarakteristik tenglama $y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y=0$ chiziqli o‘zgarmas koeffitsientli differensial tenglama uchun ushbu

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0=0$$

algebraik tenglamadan iborat.

Bessel tenglamasi

$$x^2y''+xy'+(x^2-\nu^2)y=0\ \ (\nu=\text{const}\geq 0)$$

ko‘rinishga ega.

Bessel funksiyalari

$J_\nu(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu}$ (ν- tartibli birinchi tur Bessel funksiyasi),

$\Gamma-$ gamma funksiya;

$Y_\nu(x)=\frac{\cos\nu\pi\cdot J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}$ (ν- tartibli ikkinchi tur Bessel funksiyasi yoki Neyman funksiyasi); $\nu=k$ butun son bo‘lganda bu formulani $Y_k(x)=\lim_{\nu\to k}Y_\nu(x)$ deb tushunish kerak.

$J_\nu(x)$ va $Y_\nu(x)$ Bessel funksiyalari Bessel tenglamasining chiziqli erkli yechimlaridir.

Grin funksiyasi $G(x,\xi)$ ushbu

$$\begin{cases} p_2(x)y''+p_1(x)y'+p_0(x)y=g(x), \\ \alpha_1 y'(a)+\alpha_0 y(a)=0\ \left(|\alpha_1|+|\alpha_0|\neq 0\right), \\ \beta_1 y'(b)+\beta_0 y(b)=0\ \left(|\beta_1|+|\beta_0|\neq 0\right) \end{cases}$$

chegaraviy masala uchun quyidagi uchta xossaga ega bo‘lgan funksiyadir:

1. $G(x,\xi)$ funksiya $(x,\xi)\in\left([a,b]\times[a,b]\right)$bo‘lganda aniqlangan va uzluksiz;

2. Tayinlangan $\xi\in[a,b]$ uchun $y(x)=G(x,\xi)$ funksiya x bo‘yicha $x\neq\xi$ nuqtalarda mos bir jinsli $p_2(x)y''+p_1(x)y'+p_0(x)y=0$ tenglamani, $x=a$ va $x=b$ nuqtalarda esa $\alpha_1 G'_x(a,\xi)+\alpha_0 G_x(a,\xi)=0$ va $\beta_1 G'_x(b,\xi)+\beta_0 G_x(b,\xi)=0$ chegaraviy shartlarni qanoatlantiradi;

3. Tayinlangan $\xi\in(a,b)$ uchun $G(x,\xi)$ funksiyaning birinchi tartibli hosilasi $x=\xi$ nuqtada sakrashga ega va

$$\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi+0}-\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi-0}=\frac{1}{p_2(\xi)}.$$

Avtonom (yoki **muxtor) sistema** deb

$$\begin{cases} x_1' = f_1(x_1, x_2, ..., x_n) \\ x_2' = f_2(x_1, x_2, ..., x_n) \\ \dots\dots\dots\dots\dots\dots \\ x_n' = f_n(x_1, x_2, ..., x_n) \end{cases} \quad \text{(vektorli yozuvda } \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})\text{)}$$

ko‘rinishdagi differensial tenglamalar sistemasiga aytiladi; tenglamalarda erkli o‘zgaruvchi bevosita qatnashmagan; bu yerdagi $\boldsymbol{f}(\boldsymbol{x})$ vektor-funksiya aniqlangan soha avtonom sistemaning fazalar fazosi deb ataladi.

Avtonom sistemaning traektoriyasi – shu sistema $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimi grafigining fazalar fazosidagi ortogonal proyeksiyasi.

Avtonom sistema traektoriyalarining turlari:
nuqta (yechimning davri ixtiyoriy son);

o‘z-o‘zini kesmaydigan yopiq chiziq (eng kichik musbat davrli yechim);

o‘z-o‘zini kesmaydigan yopiqmas chiziq (davrsiz yechim).

Normal ko‘rinishdagi differensial tenglamalar sistemasi ushbu

$$\begin{cases} x_1' = f_1(t, x_1, x_2, ..., x_n) \\ x_2' = f_2(t, x_1, x_2, ..., x_n) \\ \dots\dots\dots\dots\dots\dots \\ x_n' = f_n(t, x_1, x_2, ..., x_n) \end{cases} \quad \text{(vektorli yozuvda } \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x})\text{)}$$

ko‘rinishga ega.

Normal sistema - normal ko‘rinishdagi differensial tenglamalar sistemasining qisqacha aytilishi.

Normal sistema yechimi (I oraliqda) deb shu sistemani qanoatlantiruvchi $\boldsymbol{x} = \boldsymbol{\varphi}(t) \in C^1$ vektor-funksiyaga aytiladi.

Chiziqli normal sistema deb

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + ... + a_{1n}(t)x_n + b_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + ... + a_{2n}(t)x_n + b_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + ... + a_{nn}(t)x_n + b_n(t) \end{cases} \quad \text{(vektorli yozuvda}$$

$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{b}(t)$)

ko‘rinishdagi differensial tenglamalar sistemasiga aytiladi.

Chiziqli bir jinsli normal sistema (n- tartibli) ushbu

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + ... + a_{1n}(t)x_n \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + ... + a_{2n}(t)x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + ... + a_{nn}(t)x_n \end{cases}$$

(vektorli yozuvda $\boldsymbol{x}' = A(t)\boldsymbol{x}$)

ko‘rinishga ega.

Bazis yechimlar chiziqli bir jinsli n- tartibli normal sistema $\boldsymbol{x}' = A(t)\boldsymbol{x}$ uchun shu sistemaning n dona chiziqli erkli yechimlaridir.

Fundamental matritsa chiziqli bir jinsli normal sistema uchun bazis yechimlar koordinatalarini ustunlar bo‘ylab yozishdan hosil bo‘lgan matritsa.

Chiziqli o‘zgarmas koeffitsientli bir jinsli normal sistema ushbu

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{cases}$$

(vektorli yozuv: $\boldsymbol{x}' = A\boldsymbol{x}$, $A-$ o‘zgarmas matritsa)

ko‘rinishda bo‘ladi.

Xarakteristik tenglama chiziqli o‘zgarmas koeffitsientli bir jinsli normal sistema $\boldsymbol{x}' = A\boldsymbol{x}$ uchun ushbu $\det(A - \lambda E) = 0$ algebraik tenglama.

Xarakteristik son – xarakteristik tenglama ildizi.

Eksponensial matritsa A kvadrat matritsa uchun quyidagicha aniqlanadi:

$$\exp(A) \overset{def}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!} \; (A^0 = E - \text{birlik matritsa}).$$

Muvozanat nuqtaning turlari (chiziqli hol) tekislikdagi ushbu

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{yoki} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}, \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

chiziqli avtonom sistemaga taaluqli (haqiqiy sohada). Qaralayotgan sistemaning muvozanat nuqtasi $(0;0)$. A matritsaning xos sonlarini λ_1, λ_2 bilan belgilaylik. Bunda,

agar λ_1, λ_2 sonlar kompleks va $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0$ bo‘lsa, muvozanat nuqta **markaz** deb ataladi;

agar λ_1, λ_2 sonlar kompleks va $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$ bo‘lsa, muvozanat nuqta **turg‘un fokus** deb ataladi;

agar λ_1, λ_2 sonlar kompleks va $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > 0$ bo‘lsa, muvozanat nuqta **noturg‘un fokus** deb ataladi;

agar λ_1, λ_2 sonlar haqiqiy va $\lambda_1 \cdot \lambda_2 < 0$ bo‘lsa, muvozanat nuqta **egar** deb ataladi;

agar λ_1, λ_2 sonlar haqiqiy va $\lambda_1 \neq \lambda_2, \lambda_1 < 0, \lambda_2 < 0$ bo‘lsa, muvozanat nuqta **turg‘un tugun** deb ataladi;

agar λ_1, λ_2 sonlar haqiqiy va $\lambda_1 \neq \lambda_2, \text{v}\lambda_1 > 0, \lambda_2 > 0$ bo‘lsa, muvozanat nuqta **noturg‘un tugun** deb ataladi;

agar $\lambda_1 = \lambda_2$ va A matritsa diagonallashtiriluvchi (ya’ni A matritsa ikkita chiziqli erkli xos vektorga ega) bo‘lsa, muvozanat nuqta **dikritik tugun** ($\lambda_1 = \lambda_2 < 0$ holda **turg‘un dikritik tugun,** $\lambda_1 = \lambda_2 > 0$ holda esa **noturg‘un dikritik tugun**) deb ataladi;

agar $\lambda_1 = \lambda_2$ va A matritsa diagonallashtirilmaydigan (ya’ni A matritsa bitta chiziqli erkli xos vektorga ega) bo‘lsa, muvozanat nuqta **aynigan tugun** ($\lambda_1 = \lambda_2 < 0$ holida **turg‘un aynigan tugun,** $\lambda_1 = \lambda_2 > 0$ holda esa **noturg‘un aynigan tugun**) deb ataladi;

Muvozanat nuqtaning turlari (nochiziqli hol) tekislikdagi ushbu

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

nochiziqli avtonom sistemaga taaluqli (haqiqiy sohada). Qaralayotgan sistemaning muvozanat nuqtasining turu shu nuqta atrofida chiziqlashtirilgan sistemaning $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 \neq 0$ holidagi muvozanat nuqtasi turi bilan aniqlanadi.

Chiziqli bog‘langan vektor-funksiyalar. Agar $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$, $t \in I$, vektor-funksiyalarning biror notrivial chiziqli kombinatsiyasi I oraliqda nol-vektorga teng, ya’ni kamida bittasi noldan farqli bo‘lgan $\lambda_1, \lambda_2, \ldots, \lambda_m$ $(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \neq 0)$ sonlar mavjud bo‘lib, ular uchun

$$\lambda_1 \boldsymbol{x}^1(t) + \lambda_2 \boldsymbol{x}^2(t) + \cdots + \lambda_m \boldsymbol{x}^m(t) \equiv 0,\ t \in I,$$

ayniyat o‘rinli bo‘lsa, u holda bu $\boldsymbol{x}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^m(t)$ vektor-funksiyalar I oraliqda **chiziqli bog‘langan** deyiladi.

Chiziqli erkli vektor-funksiyalar deb chiziqli bog‘liq bo‘lmagan vektor-funksiyalarga aytiladi.

Vektor-funksiyalarning vronskiani (Vronskiy determinanti): ushbu

$$\boldsymbol{x}^1(t) = \begin{pmatrix} x_1^1(t) \\ x_2^1(t) \\ \vdots \\ x_n^1(t) \end{pmatrix}, \boldsymbol{x}^2(t) = \begin{pmatrix} x_1^2(t) \\ x_2^2(t) \\ \vdots \\ x_n^2(t) \end{pmatrix}, \ldots, \boldsymbol{x}^n(t) = \begin{pmatrix} x_1^n(t) \\ x_2^n(t) \\ \vdots \\ x_n^n(t) \end{pmatrix}$$

vektor-funksiyalarning vronskiani (Vronskiy determinanti) deb

$$W[\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n] = \begin{vmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix}$$

determinantga aytiladi.

Bazis yechimlar chiziqli bir jinsli normal sistemaning – soni sistema tartibiga teng bo‘lgan chiziqli erkli yechimlari.

Yechimlarning fundamental sistemasi - bazis yechimlarning boshqacha atalishi.

Fundamental matritsa (chiziqli bir jinsli normal sistema uchun) bazis yechimlarning koordinata fynksiyalarini ustunlar bo‘ylab yosilishidan hosil bo‘lgan matritsa.

Liuvill formulasi chiziqli bir jinsli n- tartibli normal sistema $\boldsymbol{x}' = A(t)\boldsymbol{x}$ uchun n dona yechimlarining $W(t)$ vronskiani uchun o‘rinli bo‘lgan ushbu

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} SpA(s)ds\right),\ SpA(s) = \sum_{j=1}^{n} a_{jj}(s)\ \left(A(s) = [a_{pq}(s)]\right),$$

formula.

Lyapunov funksiyalari – normal sistema yechimlarini Lyapunov teoremalariga ko‘ra turg‘unlikka (noturg‘unlikka) tekshirish uchun ishlatiladigan funksiyalar.

Normal sistemaning birinchi integrali. Ushbu $\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x})$ normal sistemaning birinchi integrali deb shunday $u = u(t, \boldsymbol{x}) \not\equiv \text{const}$, $u \in C^1$, funksiyaga aytiladiki, u sistemaning har qanday $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechimida o‘zgarmasga aylanadi: $u(t, \boldsymbol{\varphi}(t)) \equiv \text{const}$.

Xususiy hosilali birinchi tartibli kvazichiziqli tenglama ushbu

$$\sum_{i=1}^{n} a_i(\boldsymbol{x}, u(\boldsymbol{x})) \frac{\partial u}{\partial x_i} = b(\boldsymbol{x}, u(\boldsymbol{x})), \ \boldsymbol{x} = (x_1, x_2, ..., x_n), u = u(\boldsymbol{x}) - ?,$$

ko‘rinishga ega.

Xususiy hosilali birinchi tartibli chiziqli tenglama deb

$$\sum_{i=1}^{n} a_i(\boldsymbol{x}) \frac{\partial u}{\partial x_i} = b(\boldsymbol{x}) u + c(\boldsymbol{x}), \ \boldsymbol{x} = (x_1, x_2, ..., x_n), u = u(\boldsymbol{x}) - ?,$$

ko‘rinishdagi tenglamaga aytiladi.

DILMURADOV NASRUTDIN

ODDIY DIFFERENSIAL TENGLAMALAR

5130100 – ”Matematika” ta’lim yo‘nalishi talabalari uchun darslik

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