College Geometry

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Introduction

> Theory or Practice? But why or? Theory and Practice. This is the Ars Mathematica.
>
> Alfréd Rényi

The word geometry means earth measurement. As far as we know the ancient Egyptians were the first people to do geometry from absolutely practical points of view. The historian Herodotus relates that in 1300 BC "if a man lost any of his land by the annual overflow of the Nile he had to report the loss to Pharao who would then send an overseer to measure the loss and make a proportionate abatement of the tax" [1]. The Greeks were the first to make progress in geometry in the sense that they made it abstract. They introduced the idea of considering idealized points and lines. Using Plato's words the objects of geometric knowledge are eternal. The Greek deductive method gives a kind of answer to the question how to obtain information about this idealized world. It was codified by Euclid around 300 BC in his famous book entitled *Elements* which is a system of conclusions on the bases of unquestionable premisses or *axioms*. The method needs two fundamental concepts to begin working: undefined terms such as points, lines, planes etc. and axioms (sometimes they are referred as premisses or postulates) which are the basic assumptions about the terms of geometry.

The material collected here try to fit the different requirements coming from the different traditional points of view. One of them wants to solve problems in practice, the other wants to develop an abstract theory independently of the empirical world. Although it is hard to realize the equilibrium of different requirements (lecture vs. seminar or theory vs. practice) Alfréd Rényi's Ars Mathematica [2] gives us a perfect starting point: lectures and seminars, theory and practice.

The first chapter is devoted to general computational skills related to numbers, equations, system of equations, functions etc. These tools and the related methods are widely used in mathematics. In chapter 3 we imitate the deductive method by collecting basic facts in geometry. Some of them are axioms in the strict sense of the word such as the axioms of incidence, parallelism, measurement axioms and congruence axiom. We have another collection of facts which are not (or not necessarily) axioms. They are frequently used in geometric argumentations such as the parallel line intersecting theorem or the basic cases of the congruence and the similarity of triangles. In some of these cases the proof is available later on a higher stage of the theory. Chapter 4 is devoted to the investigation of triangles which are the fundamental figures in Euclidean geometry because quadrilaterals (chapter 7) or polygons (chapter 9) are made up of finitely many triangles and most of not polygonal shapes like circles (chapter 10) can be imaged as limits of polygons.

Each chapter includes exercises too. Most of them have a detailed solution. Exercises in separated chapters give an overview about the previous chapter's material. The classical problems (chapter 6 and chapter 13) illustrate how to use geometry in practice. They also have a historical character like the problem of the tunnel (section 6.1) or how far away is the Moon (section 6.3).

Figure 1: Alfréd Rényi (1921-1970).

Part I

Elementary Geometry

Chapter 1

General computational skills

1.1 Numbers

Numbers are one of the most typical objects in mathematics.

1.1.1 Natural numbers

To develop the notion of numbers the starting point is formed by the so-called natural numbers characterized by a set of axioms due to the 19th century Italian mathematician Guiseppe Peano. The Peano axioms define the arithmetical properties of natural numbers, usually represented as a set

$$\mathbf{N} = \{(0), 1, 2, \ldots, n, n+1, \ldots\}$$

The Peano's axioms are formulated as follows.

P1. 1 is a natural number (the set of natural numbers is non-empty).

The naturals are assumed to be closed under a single-valued successor - function S(n)=n+1.

P2. S(n) belongs to **N** for every natural number n.

Peano's original formulation of the axioms used the symbol 1 for the "first" natural number although axiom P1 does not involve any specific properties for the element 1. The number 2 can be defined as 2=S(1) and so on: 3=S(2), 4=S(3), ... The next two axioms define the properties of this representation.

P3. There is no any natural number satisfying S(n)=1.

P4. If S(m)=S(n) then m=n.

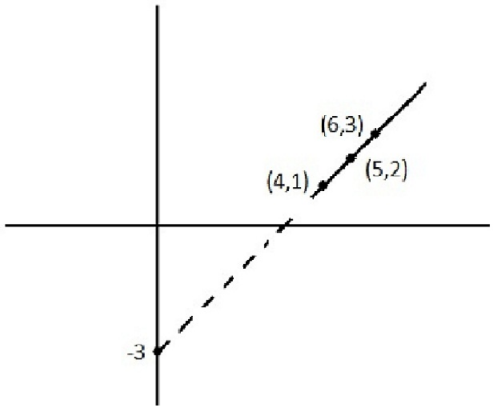

Figure 1.1: Graphical representation of integers.

These axioms imply that the elements 1, 2=S(1), 3=S(2), ... are distinct natural numbers but we need the so-called axiom of induction to provide that this procedure gives all elements of the naturals.

P5. If K is a set such that 1 is in K and for every natural number n, n is in K implies that S(n) is in K then K contains every natural number.

1.1.2 Integers

Equation 5+x=2 has no natural solutions. Let m and n be natural numbers. Equations of the form

$$m + x = n \tag{1.1}$$

without solutions among naturals lead us to new quantities called integers:

$$\mathbf{Z} = \{\ldots, -(n+1), -n, \ldots, -1, 0, 1, \ldots, n, n+1, \ldots\}.$$

Any integer corresponds to a pair (m,n) of naturals by equation 1.1. Two equations are called equivalent if they have exactly the same solutions. If we add the sides of the equations m+x=n and n'=m'+x then m'+n+x=m+n'+x. Therefore

$$m' + n = m + n' \tag{1.2}$$

is a direct consequence of the formal equivalence. The pairs (m,n) and (m',n') satisfying equation 1.2 represent the same integer. In case of (5,2) this new quantity will be written as - 3.

1.1.3 Rationals

Equation 5x=2 has no integer solutions. Let m≠0 and n be integers. Equations of the form

$$mx = n \tag{1.3}$$

without solutions among integers lead us to new quantities called rationals:

$$\mathbf{Q} = \{n/m \mid n,\ m \in \mathbf{Z} \text{ and } m \neq 0\}.$$

Any rational number corresponds to a pair (m,n) of integers by equation 1.3. Two equations are called equivalent if they have exactly the same solutions. If we multiply the sides of the equations mx=n and n'=m'x then m'nx=mn'x. Therefore

$$m'n = mn' \tag{1.4}$$

is a direct consequence of the formal equivalence. The pairs (m,n) and (m',n') satisfying equation 1.4 represent the same rational number. In case of (5,2) this new quantity will be written as 2/5.

1.1.4 Exercises

Excercise 1.1.1 *Calculate the length of the diagonal of a square with side of unit length.*

Hint. Using Pythagorean theorem we have that the diagonal is a number satisfying equation $x^2 = 2$.

Excercise 1.1.2 Prove that $\sqrt{2}$ is not a rational number.

Hint. Suppose in contrary that

$$\sqrt{2} = \frac{n}{m},$$

where n and m are integers. Taking the square of both sides we have that

$$2m^2 = n^2,$$

where the left hand side contains an odd power of 2 in the prime factorization which contradicts to the even power on the right hand side. Therefore the starting hypothesis is false.

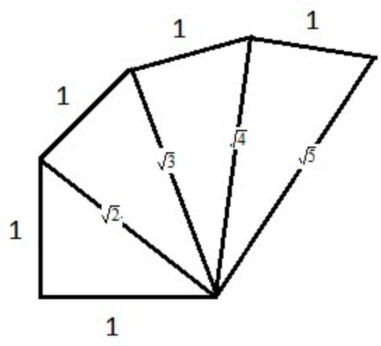

Figure 1.2: The rootspiral.

1.1.5 Irrational numbers

Definition Numbers which can not be written as the ratio of integers are called irrational. The set of real numbers **R** consists of the rational and the irrational numbers.

Irrational numbers can be imaged as limits of sequences of rational numbers; see subsection 1.3.1.

1.1.6 Complex numbers/vectors

To develop the notion of numbers the next level is the complex numbers which can be interpreted as vectors or elements in the Euclidean plane. The algebraic motivation is to provide solutions of the equation $x^2 = -1$.

1.2 Exercises

In what follows we shall use the notation n+1 instead of S(n) for the sake of simplicity.

Excercise 1.2.1 *Using induction prove that*

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \tag{1.5}$$

Solution. We can check directly that if n=1 then

$$1 = \frac{1(1+1)}{2},$$

i.e. equation 1.5 is true. Suppose that n satisfies equation 1.5, i.e.

$$1+2+\ldots+n=\frac{n(n+1)}{2} \quad \text{(inductive hypothesis)}$$

and prove that

$$1+2+\ldots+n+(n+1)=\frac{(n+1)\big((n+1)+1\big)}{2}.$$

Let us start from the left hand side. Using the inductive hypothesis we have that

$$1+2+\ldots+n+(n+1)=\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)\big((n+1)+1\big)}{2}.$$

Therefore n+1 also satisfies equation 1.5. The final conclusion is that the set of natural numbers satisfying equation 1.5 covers **N**.

Remark As we can see induction is a useful general method to prove statements related to naturals. One of its weakness is that we have to guess what to prove.

Excercise 1.2.2 *Prove the so-called Gaussian formula 1.5 without induction.*

Solution. Let

$$s_n=1+2+\ldots+n$$

be the partial sum of the first n natural number. Taking the sum of equations

$$s_n=1+2+\ldots+n$$

and

$$s_n=n+(n-1)+\ldots+1$$

we have that

$$2s_n=n(n+1)$$

and the Gaussian formula follows immediately.

Excercise 1.2.3 *Using induction prove that*

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{1.6}$$

Solution. Follow the steps as above to prove equation 1.6. If n=1 then we can easily check that

$$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6},$$

i.e. equation 1.6 is true. Suppose that n satisfies equation 1.6, i.e.

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

and prove that

$$1^2+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)}{6}.$$

Let us start from the left hand side. Using the inductive hypothesis we have that

$$1^2+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\big(n(2n+1)+6(n+1)\big)}{6} =$$

$$\frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(2n^2+4n+3n+6)}{6} =$$

$$= \frac{(n+1)\big(2n(n+2)+3(n+2)\big)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} =$$

$$= \frac{(n+1)\big((n+1)+1\big)\big(2(n+1)+1\big)}{6}$$

as was to be proved.

Excercise 1.2.4 *Using induction prove that*

$$3 \mid n^3+5n+6. \tag{1.7}$$

Solution. If n=1 then

$$1^3+5\cdot 1+6 = 12$$

and $3 \mid 12$. The expression $(n+1)^3+5(n+1)+6$ can be written into the form

$$(n+1)^3+5(n+1)+6 = (n^3+5n+6)+3n^2+3n+6,$$

where, by the inductive hypothesis, each term can be divided by 3.

Excercise 1.2.5 *Prove that the solutions of the equations*

$$x^2 = 3, \; x^2 = 5 \quad and \quad x^2 = 7$$

are irrationals.

Solution. Let p be an arbitrary prime number and suppose in contrary that

$$\sqrt{p} = \frac{n}{m}$$

where n and m are integers. Taking the square of both sides we have that

$$pm^2 = n^2,$$

where the left hand side contains an odd power of p in the prime factorization which contradicts to the even power on the right hand side. Therefore the starting hypothesis is false.

Excercise 1.2.6 *Prove that the sum and the fraction of rational numbers are rational.*

Excercise 1.2.7 *Is it true or not? The sum of a rational and an irrational number is*

- rational.
- irrational.

Solution. Using the result of the previous exercise the assumption

$$\sqrt{2} + 3 = \text{rational}$$

gives a contradiction. One can easily generalize the argument for the sum of any rational and irrational number. The method is called *indirect proof.*

Excercise 1.2.8 *Find irrational numbers a and b such that*

- a+b is rational,
- a+b is irrational,
- a/b is rational,
- a/b is irrational.

Solution. If

$$a = 1 - \sqrt{2} \text{ and } b = \sqrt{2}$$

then the sum of a+b is obviously rational. Let

$$a = \sqrt{2} \text{ and } b = \sqrt{3}.$$

If

$$a + b = r \text{ then } a = r - b$$

and

$$a^2 = r^2 - 2rb + b^2$$

which means that

$$2 = r^2 - 2r\sqrt{3} + 3,$$

i.e.

$$\sqrt{3} = \frac{r^2 + 1}{2r}.$$

This means that r can not be a rational number. If

$$a = b = \sqrt{2}$$

then its ratio is obviously rational. Finally, if

$$a = \sqrt{2} + 1 \text{ and } b = \sqrt{2} - 1$$

then

$$a/b = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} =$$

$$\frac{3 + 2\sqrt{2}}{1} = 3 + 2\sqrt{2}$$

which is obviously irrational.

Excercise 1.2.9 *Prove that x=2+3i satisfies equation*

$$x^2 - 4x + 13 = 0$$

Solution. Since the imaginary unit is the formal solution of equation $x^2 = -1$ we have that

$$(2 + 3i)^2 - 4(2 + 3i) + 13 = 4 + 12i + 9i^2 - 8 - 12i + 13 =$$

$$4 + 12i - 9 - 8 - 12i + 13 = 0$$

using the principle of permanence: keep all the algebraic rules of calculation with reals.

1.3 Limits

In this section we illustrate how irrational numbers can be interpreted as limits of sequences of rational numbers. Taking the limit is one of the most important operations in mathematics. It is used in the development of the notion of numbers, the theory of length, area and volume of general shapes (curves, surfaces and bodies) and so on. Here we apply only a kind of intuition to create limits without precise definitions.

1.3.1 Approximation of irrational numbers

It can be easily seen that

$$1 < \sqrt{2} < 2.$$

Consider the midpoint

$$q_1 = \frac{1+2}{2} = 3/2$$

of the interval $[1, 2]$. Taking the square of the corresponding sides it can be proved that

$$1 < \sqrt{2} < 3/2$$

and we have a better approximation by the midpoint

$$q_2 := \frac{1 + (3/2)}{2} = 5/4.$$

Taking the square of the corresponding sides again it can be proved that

$$5/4 < \sqrt{2} < 3/2$$

and the midpoint

$$q_3 = \frac{(5/4) + (3/2)}{2} = 11/8$$

is a better approximation of $\sqrt{2}$. The method is similar to looking for a word in a dictionary. The basic steps are

- open the dictionary in a random way (for example open the book in the middle part)

- compare the word we are looking for with the initial letter of the words on the sheet.

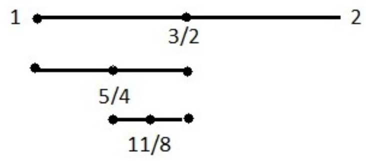

Figure 1.3: Approximation of square root 2.

Every time we bisect the dictionary before running the algorithm again. The process is not exactly the same but we use the same philosophy to solve the problem of approximation of $\sqrt{2}$. The most essential difference is that the method of finding $\sqrt{2}$ is not finite: we always have rational numbers which means that we could not find the exact value of $\sqrt{2}$ among the members of the sequence q_1, q_2, q_3, ... But the errors can be estimated by decreasing values as follows:

$$|\sqrt{2}-q_1| < \text{ the half of the length of the interval } [1,2] = \frac{1}{2} = \frac{1}{2^1}.$$

$$|\sqrt{2}-q_2| < \text{ the half of the length of the interval } [1,3/2] = \frac{1}{4} = \frac{1}{2^2}.$$

In a similar way

$$|\sqrt{2}-q_3| < \text{the half of the length of the interval } [5/4,3/2] = \frac{1}{8} = \frac{1}{2^3}.$$

In general

$$|\sqrt{2}-q_n| < \frac{1}{2^n}.$$

Therefore we can be as close to $\sqrt{2}$ as we want to. In other words the sequence q_1, q_2, q_3, ... tends to $\sqrt{2}$ and this number can be interpreted as the limit of a sequence of rational numbers.

Remark In what follows we present a MAPLE procedure for the approximation of the square root of naturals as we have seen above: let k be a given natural number. We are going to approximate the square root of k by using the basic step n times. The name of the procedure is

$$f := \operatorname{proc}(n,k)$$

At first we should find lower and upper bounds

$$a < \sqrt{k} < b$$

as follows:

$$a := 1;$$

$$b := 1;$$

$$\text{while } a^2 < k \text{ do}$$

$$a := a + 1;$$

$$\text{end do;}$$

$$a := a - 1;$$

This means that if the actual value of the variable a satisfies the inequality $a^2 < k$ then we increase the value of the variable by adding one as far as possible. Finally "a" takes the last value for which the inequality $a^2 < k$ is true. The upper bound is created in a similar way:

$$\text{while } b^2 < k \text{ do}$$

$$b := b + 1;$$

$$\text{end do;}$$

As the next step we give the initial value of a new variable

$$c := \frac{a+b}{2};$$

and we use a "for" loop to take the half of the enclosing intervalls n times:

$$\text{for } i \text{ from } 1\text{to } n \text{ do}$$

$$\text{if } c^2 < k \text{ then}$$

$$a := c;$$

$$c := \frac{a+b}{2};$$

$$\text{else}$$

$$b := c;$$

$$c := \frac{a+b}{2};$$

$$\text{end if;}$$

$$\text{end do;}$$

$$\text{return}(c)$$

$$\text{end proc;}$$

The figure shows how the procedure is working in a standard Maple worksheet environment.

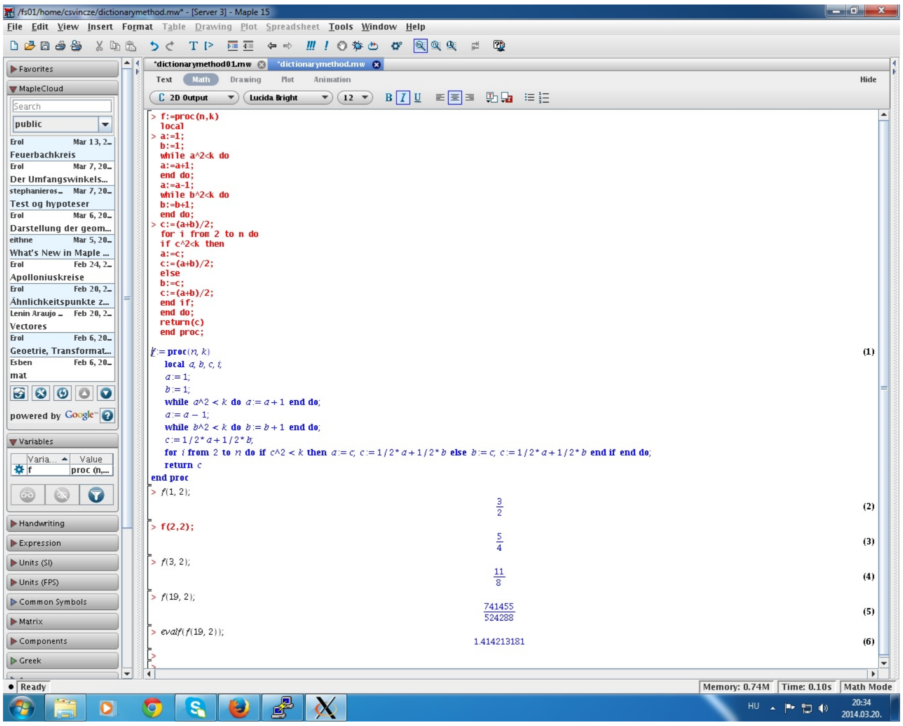

Figure 1.4: A MAPLE procedure.

1.3.2 The problem of the shortest way

One of the most important basic fact in geometry is the so-called triangle inequality

$$AC \leq AB + BC \tag{1.8}$$

to express a more general geometric principle. It says that the shortest way between two points is the straight line. The question is how to derive this principle from inequality 1.8 in general. The first step is the generalization of the triangle inequality. Using a simple induction we can prove polygonal inequalities

$$AC \leq AB_1 + B_1B_2 + B_2C, \quad AC \leq AB_1 + B_1B_2 + B_2B_3 + B_3C \quad \text{and so on.}$$

In general

$$AC \leq AB_1 + B_1B_2 + B_2B_3 + \ldots + B_{n-1}B_n + B_nC \tag{1.9}$$

for any natural number $n \geq 3$. Now image an "arc" from A to C. If the arclength is understood as the limit of lengths of inscribed polygonal chains in some sense then we have that the shortest way between two points is the straight line.

1.3.3 The area of the unit circle

Everybody knows that the area of a circle with radius r is $r^2\pi$. If we have a unit circle then the area is just π. How can we calculate the value of π?

The earliest known textually evidenced approximations of π are from around 1900 BC. They are found in the Egyptian Rhind Papyrus

$$\pi \approx 256/81$$

and on Babylonian tablets

$$\pi \approx 25/8.$$

The Indian text Shatapatha Brahmana gives π as 339/108. Archimedes (287 - 212 BC) was the first to estimate π rigorously. He realized that its magnitude can be bounded from below and above by the area of inscribing and circumscribing regular polygons. For example we can inscribe in the circle a regular hexagon made up of six disjoint equilateral triangles of side 1. The area of each triangle is $3/(4\sqrt{3})$ by Héron's formula, so the area of the hexagon is

$$6\frac{3}{4\sqrt{3}} = \frac{9}{2\sqrt{3}} \approx 2.59808.$$

The area of the circle should be obviously greater than this value. If we circumscribe a regular hexagon around the unit circle then the area can be estimated from above. The area of the circle should be obviously less than the area of the circumscribed regular hexagon of side $2/\sqrt{3}$:

$$\text{the area of the unit circle} \leq 6\frac{1}{\sqrt{3}} \approx 3.46410$$

and so on. Around 480 Zu Chongzhi demonstrated that $\pi \approx 355/113 = 3,1415929$. He also showed that $3,1415926 < \pi < 3,1415927$.

The next major advances in the study of π came with the development of infinite series and subsequently with the discovery of calculus/analysis, which permit the estimation of π to any desired accuracy by considering sufficiently many terms of a relevant series. Around 1400, Madhava of Sangamagrama found the first known such series:

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots .$$

This is known as the Madhava-Leibniz series or Gregory-Leibniz series since it was rediscovered by James Gregory and Gottfried Leibniz in the 17th century. Madhava was able to estimate the value of π correctly to 11 decimal places.

The record was beaten in 1424 by the Persian mathematician, Jamshid al-Kashi by giving an estimation that is correct to 16 decimal digits. The accuracy up to 35 decimal digits was due to the German mathematician Ludolph van Ceulen (1540-1610). Another European contribution to the problem is the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots \tag{1.10}$$

found by Francois Viéte in 1593. Formula 1.10 will be derived in section 10.3 by using inscribed regular n-gons in the unit circle.

1.4 Exercises

Excercise 1.4.1 *Compute the number of steps for the approximation of* $\sqrt{2}$ *with error less than* 10^{-10}.

Solution. We have to solve the inequality

$$\frac{1}{2^n} < \frac{1}{10^{10}} \quad \text{for the unknown natural number } n.$$

Equivalently: $10^{10} < 2^n$. To solve this inequality we use the so-called *logarithm* to have that $10 \log_2 10 < n$. Since $10 < 2^4$ it follows that

$$10 \log_2 10 < 10 \log_2 2^4 = 40$$

steps are enough to approximate $\sqrt{2}$ with error less that 10^{-10} which is just the measure of the unit conversion between meter and Angstrom related to atomic-scale structures.

Excercise 1.4.2 *Find a sequence of rational numbers to approximate* $\sqrt{5}$.

Solution. Using the estimations

$$1 < \sqrt{5} < 3$$

we have

$$q_1 := \frac{1+3}{2} = 2$$

as the first member of the approximating sequence. Since

$$2 < \sqrt{5} < 3$$

it follows that

$$q_2 := \frac{2+3}{2} = \frac{5}{2}.$$

Repeating the basic steps of the dictionary method we have

$$2 < \sqrt{5} < \frac{5}{2} \quad \Rightarrow \quad q_3 := \frac{2+\frac{5}{2}}{2} = \frac{9}{4},$$

$$2 < \sqrt{5} < \frac{9}{4} \quad \Rightarrow \quad q_4 := \frac{2+\frac{9}{4}}{2} = \frac{17}{8}$$

and

$$\frac{17}{8} < \sqrt{5} < \frac{9}{4} \quad \Rightarrow \quad q_5 := \frac{\frac{17}{8}+\frac{9}{4}}{2} = \frac{35}{16},$$

$$\frac{35}{16} < \sqrt{5} < \frac{9}{4} \quad \Rightarrow \quad q_6 = \frac{\frac{35}{16}+\frac{9}{4}}{2} = \frac{71}{32}$$

and so on.

Excercise 1.4.3 *Consider the iterative sequence*

$$q_{n+1} = \sqrt{2+q_n},$$

i.e.

$$q_1 = \sqrt{2},\ q_2 = \sqrt{2+\sqrt{2}},\ q_3 = \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots$$

Prove that

$$q_n \leq 2$$

for any element of the sequence.

Solution. It is clear that the inequality is true for n=1. Using a simple induction

$$q_{n+1}^2 = 2 + q_n \leq 2 + 2 = 4.$$

Excercise 1.4.4 *Find positive integer solutions for the equation*

$$m^2 - m - n = 0.$$

Solution. Using the formula for computing the roots of a quadratic equation

$$m_{12} = \frac{1 \pm \sqrt{1+4n}}{2}.$$

Therefore 1+4n must be an odd square number:

$$1 + 4n = (2k+1)^2,$$

$$1 + 4n = 4k^2 + 4k + 1$$

which means that n must be of the form n=k(k+1), where k is an arbitrary positive integer. In this case the positive root of the equation is m=k+1. For example if k=1 then n=2 and m=2. Further possible solutions are n=12 and m=4 or n=20 and m=5 under the choices of k=3 or k=4.

k	n=k(k+1)	m=k+1
1	2	2
2	6	3
3	12	4
4	20	5
5	30	6

Excercise 1.4.5 *Consider the iterative sequence*

$$q_{n+1} = \sqrt{12 + q_n},$$

i.e.

$$q_1 = \sqrt{12},\ q_2 = \sqrt{12 + \sqrt{12}},\ q_3 = \sqrt{12 + \sqrt{12 + \sqrt{12}}},\ \ldots$$

Prove that

$$q_n \leq 4$$

for any element of the sequence.

Solution. It is clear that the inequality is true for n=1. Using a simple induction

$$q_{n+1}^2 = 12 + q_n \leq 12 + 4 = 16.$$

Excercise 1.4.6 *Consider the iterative sequence*

$$q_{n+1} = \sqrt{20 + q_n},$$

i.e.

$$q_1 = \sqrt{20},\ q_2 = \sqrt{20 + \sqrt{20}},\ q_3 = \sqrt{20 + \sqrt{20 + \sqrt{20}}},\ \ldots$$

Prove that

$$q_n \leq 5$$

for any element of the sequence.

Solution. It is clear that the inequality is true for n=1. Using a simple induction

$$q_{n+1}^2 = 20 + q_n \leq 20 + 5 = 25.$$

Remark The upper bounds in the previous exercises provide that the sequences have finite limits. In case of the sequence

$$q_{n+1} = \sqrt{12 + q_n}$$

we have that the limit must satisfy the equation

$$q_* = \sqrt{12 + q_*}$$

and, consequently, it is just 4.

Excercise 1.4.7 *Find the limit of the sequence*

$$q_{n+1} = \sqrt{2 + q_n}.$$

Solution. As we have seen above the sequence is bounded by 2 from above. This means that we have a finite limit satisfying the equation

$$q_* = \sqrt{2 + q_*}.$$

Therefore

$$0 = q_*^2 - q_* - 2$$

which means that $q_* = 2$ or - 1 but the negative value can be obviously omitted.

Excercise 1.4.8 *Find the limit of the sequence*

$$q_{n+1} = \sqrt{20 + q_n}.$$

Solution. As we have seen above the sequence is bounded by 5 from above. This means that we have a finite limit satisfying the equation

$$q_* = \sqrt{20 + q_*}.$$

Therefore

$$0 = q_*^2 - q_* - 20$$

which means that $q_* = 5$ or - 4 but the negative value can be obviously omitted.

Excercise 1.4.9 *Prove that*

$$a^2 - 1 = (a - 1)(a + 1).$$

Solution. It can be easily derived by direct calculation:

$$(a-1)(a+1) = a^2 + a - a - 1 = a^2 - 1.$$

Excercise 1.4.10 *Prove that*

$$a^3 - 1 = (a-1)(a^2 + a + 1).$$

Solution. It can be easily derived by direct calculation:

$$(a-1)(a^2 + a + 1) = a^3 + a^2 + a - a^2 - a - 1 = a^3 - 1.$$

The formulas involving explicite powers can be given by the help of direct calculations.

Excercise 1.4.11 *Prove that for any natural power*

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \ldots + a + 1).$$

Solution. Let

$$s_{n-1} = 1 + a + \ldots + a^{n-1}$$

be the partial sum of the powers. Then

$$(a-1)s_{n-1} = as_{n-1} - s_{n-1} = a + a^2 + \ldots + a^n - (1 + a + \ldots + a^{n-1}) =$$

$$a^n - 1$$

as was to be proved.

Remark Use the procedure of the induction to prove the statement in Exercise 1.4.11.

Solution.

$$a^{n+1} - 1 = a^{n+1} - a^n + a^n - 1 = a^n(a-1) + \text{ the inductive hypothesis...}$$

Excercise 1.4.12 *Calculate the sum of the series*

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

Hint. Using the previous result with $a = 1/2$ we have that

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}} = \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} \to \frac{-1}{\frac{1}{2} - 1} = 2.$$

Remark We can image the sum of the geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

as taking a 2 units long walk in such a way that each sub - walk takes the half of the distance from the staring point to the end.

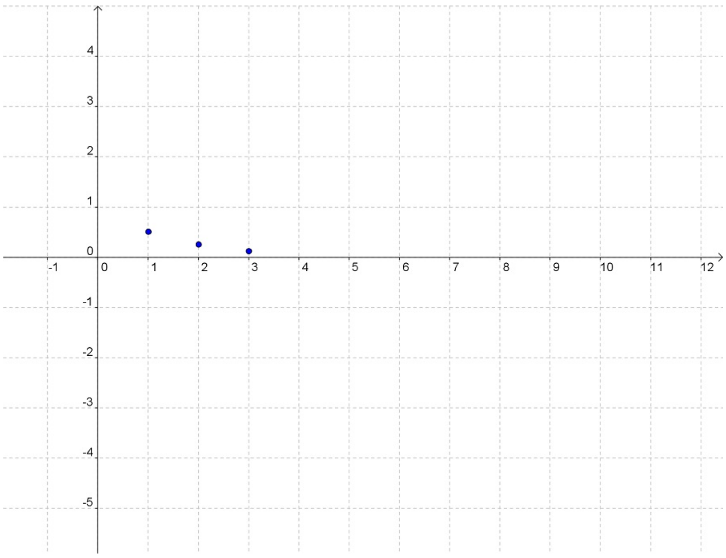

Figure 1.5: Exponentially decreasing tendency.

1.5 Functions

The approximation of square root 2 can be interpreted in the following way.

Step	Value	Bound for the error
1st	$q_1 = 3/2$	$1/2$
2nd	$q_2 = 5/4$	$1/2^2$
3rd	$q_3 = 8/11$	$1/2^3$
	...	...
n - th	q_n	$1/2^n$

Besides the tabular form graphical representation is widely used. Actually this is a direct method to realize relationships and tendencies among data items at a glance.

1.5.1 Exponentials

Exponentials are typical in mathematical modeling of growing without constraints (see eg. cell division, family tree). We also know that each radioactive isotope has its own characteristic decay pattern. Its rate is measured in half - life. The half - life refers to the time it takes for one - half of the atoms of a radioactive material to disintegrate. Half - lives for different radioisotopes can range from a few microsecond to billions of years.

Radioisotope	Half - life
Polonium-215	0.0018 seconds
Bismut-212	60.5 seconds
Barium-139	86 minutes
Sodium-24	15 hours
Cobalt-60	5.26 years
Radium-226	1600 years
Uranium-238	4.5 billion years

1.5.2 Trigonometric functions

Another important type of functions are trigonometric functions; see section 4.5.

1.5.3 Polynomials

Finally we mention polynomial functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

The most important special cases are n=1 (lines) and n=2 (parabolas). Polynomials behave like numbers from some points of view. We can add or multiply them and we can divide two polynomials with each other too.

$$\begin{array}{lcl}
2x^3 + x^2 - 1 & : \ x+1 & = 2x^2 \\
- \ (2x^3 + 2x^2) & & \\ \hline
-x^2 - 1 & : \ x+1 & = -x \\
- \ (-x^2 - x) & & \\ \hline
x - 1 & : \ x+1 & = 1 \\
- \ (x+1) & & \\ \hline
-2 & &
\end{array}$$

Therefore

$$2x^3 + x^2 - 1 = (2x^2 - x + 1)(x + 1) - 2.$$

An important example on a polynomial tendency is the **kinematic law** for the distance travelled during a uniform acceleration starting from rest. It is proportional to the square of the ellapsed time. This is the situation in case of falling bodies investigated by Galileo Galilei. If we are interested in the distance travelled by a falling body as a function of the travelling time it is relatively hard to create an appropriate experimental environment for measuring. It is more reasonable to measure the travelling time as the function of the distance. In other words we are interested in the inverse relationship (inverse function). To create a comfortable experimental situation we can use a slope to ensure a travel during a uniform acceleration starting from rest. A simple scale can be given by using the mid-point technic along the slope. Theoretically we have the formula

$$f(s) = \sqrt{\frac{2s}{a}}$$

to give the travelling time as a function of the distance s along the slope; the constant

$$a = g \sin \alpha$$

is related to the angle of the slope and the gravitational acceleration g. To return to the original problem we need the inverse of the function f. Formally speaking we want to express s in terms of t=f(s):

$$t = \sqrt{\frac{2s}{a}} \quad \Rightarrow \quad \frac{a}{2}t^2 = s$$

and, consequently, the inverse function is working as

$$f^{-1}(t) = \frac{a}{2}t^2$$

on the domain of the non-negative real numbers. Geometrically we change the role of the coordinates x and y in the coordinate plane. Therefore the graphs of a function and its inverse is related by the reflection about the line y=x as we can see in the next figure for the exponential and the logarithmic functions.

1.6 Exercises

Excercise 1.6.1 Suppose that you have 10 grams of Barium - 139. After 86 minutes, half of the atoms in the sample would have decayed into another element called Lanthanum - 139. After one half - life you would have 5 grams Barium - 139 and 5 grams Lanthanum - 139. After another 86 minutes, half of the 5 grams Barium - 139 would decay into Lanthanum - 139 again; you would now have 2.5 grams of Barium - 139 and 7.5 grams Lanthanum - 139. How many time does it take to be Barium - 139 less than 1 gram?

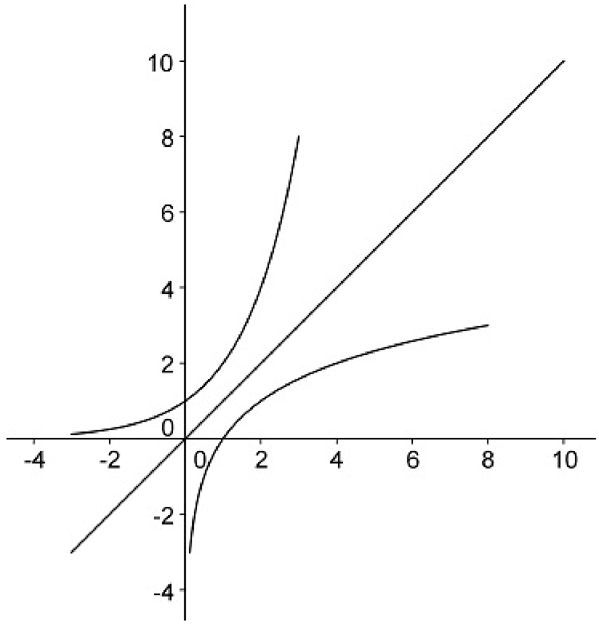

Figure 1.6: The exponential function and its inverse.

Time (minutes)	amount of Barium - 139 (gram)
0	10
86	5
2 × 86	2.5
3 × 86	1.25
4 × 86	0.625
...	...
n × 86	$10/2^n$

Solution. We have to solve the inequality

$$\frac{10}{2^n} < 1 \quad \Rightarrow \quad 10 < 2^n.$$

Therefore $4 \cdot 86$ minutes is enough to be Barium - 139 less than 1 gram.

Excercise 1.6.2 Sketch the functions $f(x) = 2^x$ and $g(x) = \log_2 x$ in a common Cartesian coordinate system.

Excercise 1.6.3 Prove that $\log_2 3$ is irrational.

Solution. Suppose, in contrary that

$$\log_2 3 = \frac{n}{m},$$

where n and m$\neq 0$ are integers. Using that $\log_2 3^m = m \log_2 3$ we have

$$\log_2 3^m = n.$$

By the definition of the logarithm this means that $2^n = 3^m$ which is obviously impossible.

Excercise 1.6.4 Transfer the expression $f(x) = 3x^2 - 5x + 3$ to the canonical form

$$f(x) = a(x - x_0)^2 + y_0$$

and compute the minimum value of the function.

Solution. It can be easily seen that

$$f(x) = 3\left(x^2 - (5/3)x + 1\right) = 3\left((x - (5/6))^2 - (25/36) + 1\right) =$$

$$= 3\left(x - (5/6)\right)^2 + (11/12),$$

i.e. the minimum value is just $y_0 = 11/12$ attained at $x_0 = 5/6$.

Excercise 1.6.5 *Prove the formula*

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the equation

$$ax^2 + bx + c = 0$$

by using the canonical form of a quadratic function.

Hint. Consider the function

$$f(x) = ax^2 + bx + c.$$

Its canonical form is

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

which implies by taking the equation f(x)=0 that

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Therefore

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

provided that the discriminant $D=b^2-4ac$ is non-negative.

Excercise 1.6.6 *Conclude Viéte's formulas*

$$x_1+x_2=-\frac{b}{a}\quad \textit{and}\quad x_1\cdot x_2=\frac{c}{a}.$$

Excercise 1.6.7 *Find the maximum amount of square footage we can enclose in a rectangle using a fence with 128 feet.*

Solution. Let x and y be the sides of a rectangle. To find the maximum of the product xy subject to the equality constrain 2(x+y)=128 consider the function

$$f(x,y)=xy.$$

Substituting y=64 - x we can reduce the number of variables:

$$f(x)=x(64-x)=-x^2+64x=-(x-32)^2+32^2.$$

The maximum area is just 1024 attained at x=32 which is just the case of a square.

Excercise 1.6.8 *The following table shows the average highs of temperature measured on 15th of each month in New York City* [3]. *Using graphical representation find the rule of the average highs. What about the temperature on 30th of October?*

Month	Temperature (Fahrenheit)
February	40
March	50
April	62
May	72
Juny	81
July	85
August	83
September	78
October	66
November	56
December	40

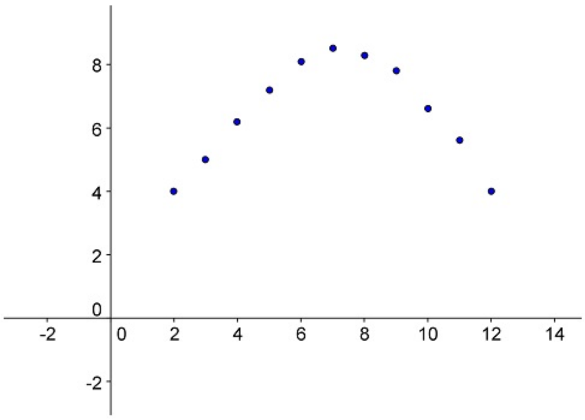

Figure 1.7: The graphical representation of the high temperatures.

Solution. Consider the months as independent variables x=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. For the sake of simplicity we illustrate the corresponding high temperatures as T=4, 5, 6.2, ... and so on. As it can bee seen they form a parabolic arc with canonical form

$$f(x) = a(x-7)^2 + 8.5.$$

To compute the parameter "a" we can use the following substitutions:

$$4 = a(2-7)^2 + 8.5 \quad \Rightarrow \quad a = -0.18,$$

$$5 = a(3-7)^2 + 8.5 \quad \Rightarrow \quad a = -0.21,$$

$$6.2 = a(4-7)^2 + 8.5 \quad \Rightarrow \quad a = -0.25$$

and so on. The following table shows the collection of the possible values of the parameter "a".

x	f(x)=a(x-7)+8.5	a
2 (February)	$4 = a(2-7)^2 + 8.5$	-0.18
3 (March)	$5 = a(3-7)^2 + 8.5$	-0.21
4 (April)	$6.2 = a(4-7)^2 + 8.5$	-0.25
5 (May)	$7.2 = a(5-7)^2 + 8.5$	-0.32
6 (Juny)	$8.1 = a(6-7)^2 + 8.5$	-0.4
7 (July)	$8.5 = a(7-7)^2 + 8.5$	-
8 (August)	$8.3 = a(8-7)^2 + 8.5$	-0.2
9 (September)	$7.8 = a(9-7)^2 + 8.5$	-0.17
October		
November		
December		

Excercise 1.6.9 *Calculate the missing values of the parameter.*

Solution. Using the equations

$$6.6 = a(10-7)^2 + 8.5,$$

$$5.6 = a(11-7)^2 + 8.5,$$

$$4 = a(12-7)^2 + 8.5$$

we have the values a= - 0.21, - 0.18 and - 0.18. Therefore the parameter "a" is about - 0.2. A reasonable model to compute the average high temperature is

$$T(x)/10 = -0.2(x-7)^2 + 8.5.$$

30/October corresponds the value x=10.5. Therefore

$$T(10.5) = -2(10.5-7)^2 + 85 = 60.5$$

Fahrenheit.

Excercise 1.6.10 *Find the inverse of the function*

$$f(x) = 3x - 4.$$

Solution. Express x in terms of y=f(x):

$$y = 3x - 4 \quad \Rightarrow \quad x = \frac{y+4}{3} = \frac{1}{3}y + \frac{4}{3}$$

which means that the inverse function is working as

$$f^{-1}(y) = \frac{1}{3}y + \frac{4}{3}.$$

Excercise 1.6.11 *Find the inverse of the function*

$$f(x) = x^2.$$

Solution. The formal method gives that

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

and, consequently, the inverse function is working as

$$f^{-1}(y) = \sqrt{y}$$

on the domain of the non-negative real numbers.

Excercise 1.6.12 *Find the inverse of the function*

$$f(x) = \frac{3x-4}{x-2}.$$

Solution. The formal method gives that

$$y = \frac{3x-4}{x-2},$$
$$yx - 2y - 3x + 4 = 0,$$
$$x(y-3) - 2y + 4 = 0,$$
$$x = \frac{2y-4}{y-3}.$$

Therefore

$$f^{-1}(y) = \frac{2y-4}{y-3}$$

and the domain of the inverse function does not contain the value y=3.

Excercise 1.6.13 *Find the domains of the functions*

$$f(x) = \frac{2x-1}{x^2-x}, \quad g(x) = \sqrt{5-x} \quad \textit{and} \quad h(x) = \sqrt{(x-3)(5-x)},$$

Solution. The domain of the function f is the set of reals except the roots x=0 or 1 of the denominator. In case of function g we need the set of reals satisfying

$$5 - x \geq 0,$$

i.e. the domain is the set of reals less or equal than 5. Finally we have to solve the inequality

$$(x-3)(5-x) \geq 0.$$

The left hand side is non-negative if and only if

$$x - 3 \geq 0 \text{ and } 5 - x \geq 0$$

or

$$x - 3 \leq 0 \text{ and } 5 - x \leq 0.$$

Therefore

$$3 \leq x \leq 5.$$

Excercise 1.6.14 *Find the domains of the functions*

$$f(x) = \frac{1}{x+3}, \quad g(x) = \sqrt{2x+4} \quad \textit{and} \quad h(x) = \sqrt{(x-2)(x+3)},$$

Solution. The domain of the function f is the set of reals except - 3. For the function g we have

$$2x + 4 \geq 0 \quad \Rightarrow \quad x \geq -2.$$

Finally we have to solve the inequality

$$(x - 2)(x + 3) \geq 0.$$

The left hand side is non-negative if and only if

$$x - 2 \geq 0 \ \text{ and } \ x + 3 \geq 0$$

or

$$x - 2 \leq 0 \ \text{ and } \ x + 3 \leq 0.$$

Therefore

$$x \leq -3 \ \text{ or } \ x \geq 2.$$

Excercise 1.6.15 *Express the numbers*

$$\ln \sqrt{3} \quad \textit{and} \quad \ln \frac{1}{81}$$

in terms of $\ln 3$

Solution. Since

$$\sqrt{3} = 3^{1/2} \ \text{ and } \ \frac{1}{81} = 3^{-4}$$

we have that

$$\ln \sqrt{3} = \frac{1}{2} \ln 3$$

and

$$\ln \frac{1}{81} = -4 \ln 3.$$

Excercise 1.6.16 *Solve the following equations*

$$2^x 3^{x+2} = 54, \quad 3^x 2^{x+2} = 24 \quad \textit{and} \quad \ln\big(x(x-2)\big) = 0.$$

Solution. To solve the first equation observe that

$$9 \cdot 2^x 3^x = 54,$$

$$6^x = 6 \quad \Rightarrow \quad x = 1.$$

In a similar way

$$4 \cdot 2^x 3^x = 24,$$

$$6^x = 6 \quad \Rightarrow \quad x = 1.$$

Finally

$$x(x-2) = 1,$$

$$x^2 - 2x - 1 = 0 \quad \Rightarrow \quad x_{12} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

1.7 Means

In practice estimations are often more important than the exact values of quantities. Lots of numerical values are frequently substituted with only one distinguished quantity as we have seen above in exercise 1.6.8. There are several reasons why to use average (mean, mode, median, ecpectable value etc.) in mathematics. An average is a measure of the middle or typical value of a data set. The general aim is to accumulate the information or to substitute more complicated mathematical objects with relatively simpler ones. In what follows we summarize some theoretical methods to create an average.

- The arithmetic mean of a finite collection of data is
$$A = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$
- In case of nonnegative numbers we can form the so-called geometric mean
$$G = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}.$$
- The harmonic mean of the data set is
$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}.$$

Remark Using Thales theorem we can interpret the arithmetic mean of x=AF and y=FB as the radius of the circumscribed circle of a right triangle with hypothenuse AB. The height is just the geometric mean of x and y. Under the choice x=1 and y=n this gives an alternative method to construct the root of any natural number n by ruler and compass. On the other hand figure 1.7 shows that
$$G \leq A$$
for two variables.

In many situations involving rates and ratios the **harmonic mean** provides the truest average. If a vehicle travels a certain distance d at speed 60 kilometres per hour and then the same distance again at speed 40 kilometres per hour then its average speed is the harmonic mean of 60 and 40, i.e.
$$\frac{2}{\frac{1}{60} + \frac{1}{40}} = 48.$$

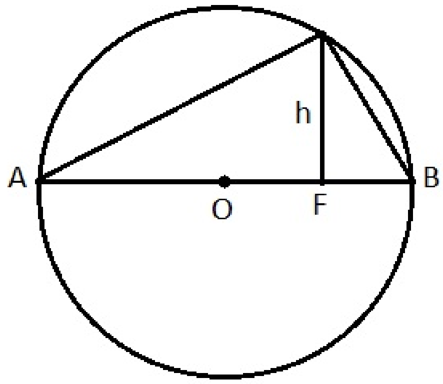

Figure 1.8: Arithmetic vs. geometric means.

In other words the total travel time is the same as if the vehicle had traveled the whole distance at speed 48 kilometres per hour because

$$\frac{d}{t_1} = 60, \quad \frac{d}{t_2} = 40$$

and thus

$$\frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{60} + \frac{d}{40}} = \frac{2}{\frac{1}{60} + \frac{1}{40}}.$$

The same principle can be applied to more than two segments of the motion: if we have a series of sub - trips at different speeds and each sub - trip covers the same distance then the average speed is the harmonic mean of all the sub - trip speeds. After a slight modification we can give the physical interpretation of the **arithmetic mean** too: if a vehicle travels for a certain amount "t" of time at speed 60 and then the same amount of time at speed 40 then the average speed is just the arithmetic mean of 60 and 40, i.e.

$$\frac{60 + 40}{2} = 50.$$

In other words the total distance is the same as if the vehicle had traveled for the whole time at speed 50 kilometres per hour because

$$\frac{s_1}{t} = 60, \quad \frac{s_2}{t} = 40$$

and thus

$$\frac{s_1 + s_2}{2t} = \frac{60 + 40}{2}.$$

1.8 Exercises

Excercise 1.8.1 *Find the arithmetic mean of the possible values of the parameter "a" in exercise 1.6.8.*

Solution.

$$A = -\frac{0.18 + 0.21 + 0.25 + 0.32 + 0.4 + 0.2 + 0.17 + 0.21 + 0.18 + 0.18}{11} =$$

$$-0.2.$$

Excercise 1.8.2 *Prove that for any pair of positive real numbers x and y*

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

.

Solution. At first we prove that for any pair of non-negative numbers x and y

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Taking the square of both sides we have that

$$xy \leq \frac{(x+y)^2}{4}$$

and, consequently,

$$4xy \leq x^2 + 2xy + y^2,$$

$$0 \leq x^2 - 2xy + y^2 = (x-y)^2$$

which is obviously true. If a=1/x and b=1/y then

$$\sqrt{ab} \leq \frac{a+b}{2}$$

which means that

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2}{a+b} \leq \frac{1}{\sqrt{ab}} = \sqrt{xy}$$

as was to be proved.

Excercise 1.8.3 *Suppose that you want to create a rectangular-shaped garden with area 1024 square footage. How many feet in length you need to fence your garden?*

Hint. The problem is to *minimize the perimeter among rectangles with area 1024*. Let x and y be the sides of a rectangle. To find the minimum of the perimeter 2(x+y) subject to the equality constrain xy=1024 introduce the function

$$f(x,y)=2(x+y).$$

Substituting y=1024/x we can reduce the number of variables:

$$f(x)=2\left(x+\frac{1024}{x}\right).$$

The relationship between the arithmetic and the geometric means shows that

$$f(x)=2\left(x+\frac{1024}{x}\right)=4\frac{x+(1024/x)}{2}\geq 4\sqrt{1024}=128$$

and equality happens if and only if x=32. This is the case of the square.

Excercise 1.8.4 *Formulate the physical principle for the arithmetic mean.*

Solution. If we have a series of sub - trips at different speeds and each sub - trip takes the same amount of time then the average speed is the arithmetic mean of all the sub - trip speeds.

1.9 Equations, system of equations

The mathematical formulation of problems often gives a single equation or system of equations (see e.g. coordinate geometry). It is important to isolate relevant information:

> A rectangular box with a base 2 inches by 6 inches is 10 inches tall and holds 12 ounces of breakfast cereal. The manufacturer wants to use a new box with a base 3 inches by 5 inches. How many inches tall should be in order to hold exactly the same volume as the original box?

relevant information	irrelevant information
the base is 2×6	inch
the tall is 10	inch, 12 ounces of breakfast cereal
the new base is 3×5	manufacturer, inch
don't change the volume	-

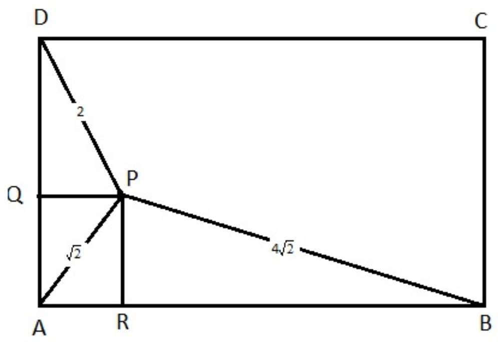

Figure 1.9: Exercise 1.10.1

The only theoretical fact we need to solve the problem is that the volume of a rectangular box is just the product of the area of the base and the tall. Therefore we can write the equation

$$2 \cdot 6 \cdot 10 = 3 \cdot 5 \cdot m,$$

where m denotes the unknown tall (height) of the new box. We have that m=8. Quantities we are looking for may have a more complicated relationship with the given data. Sometimes we should write more than one relationships (together with new auxiliary variables) to compute the missing one.

1.10 Exercises

Excercise 1.10.1 *In rectangle ABCD, side AB is three times longer than BC. The distance of an interior point P from the vertices A, B and D are*

$$PA = \sqrt{2},\ PB = 4\sqrt{2} \ \ \textit{and} \ \ PD = 2,$$

respectively. What is the area of the rectangle.

Hint. Using orthogonal projections of the interior point P to the sides of the rectangle we can use Pythagorean theorem three times:

$$AQ^2 + AR^2 = 2,$$

$$(AD - AQ)^2 + AR^2 = 4,$$

$$(AB - AR)^2 + AQ^2 = (4\sqrt{2})^2 = 32.$$

Since AB=3AD we have three equations for the quantities x=AD, y=AQ and z=AR. Namely

$$y^2 + z^2 = 2,$$
$$(x-y)^2 + z^2 = 4,$$
$$(3x-z)^2 + y^2 = 32.$$

We have that

$$4 = (x-y)^2 + z^2 = x^2 - 2xy + y^2 + z^2 = x^2 - 2xy + 2$$

and

$$32 = (3x-z)^2 + y^2 = 9x^2 - 6xz + z^2 + y^2 = 9x^2 - 6xz + 2.$$

Therefore

$$y = \frac{x^2-2}{2x}, \quad z = \frac{9x^2-30}{6x}$$

and the first equation gives that

$$\frac{(a-2)^2}{4a} + \frac{(9a-30)^2}{36a} = 2,$$

where a=x^2. From here

$$9(a-2)^2 + (9a-30)^2 = 72a,$$
$$90a^2 - 648a + 936 = 0.$$

Finally

$$5a^2 - 36a + 52 = 0$$

which means that

$$a_{12} = \frac{36 \pm \sqrt{256}}{10} \quad \Rightarrow \quad a = 2 \ \text{ or } \ 5.2.$$

If a=2 then we have that

$$x^2 = 2 \quad \Rightarrow \quad y = 0 \ \text{ and } \ z < 0$$

which is impossible. Therefore

$$x^2 = 5.2 \quad \Rightarrow \quad A = 3x^2 = 15.6.$$

Note that there is no need to compute x because the area of the rectangle can be given as $3x^2 = 3a$.

Remark Systems containing quadratic equations are typical in coordinate geometry: the intersection of a line and a circle or the intersection of two circles.

Chapter 2

Exercises

2.1 Exercises

Excercise 2.1.1 *Without calculator find the values of*

$$8^{\frac{2}{3}} \cdot 2^{-2}, \quad 7778^2 - 2223^2, \quad \frac{437^2 - 363^2}{537^2 - 463^2},$$

$$\sqrt{5 - 2\sqrt{6}} + \sqrt{3 - 2\sqrt{2}} + \sqrt{7 - 2\sqrt{12}},$$

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{4}\right) \cdot \ldots \cdot \left(1 + \frac{1}{100}\right).$$

Solution. Using power law identities

$$8^{\frac{2}{3}} \cdot 2^{-2} = \left(8^{\frac{1}{3}}\right)^2 \cdot \frac{1}{2^2} = 2^2 \cdot \frac{1}{2^2} = 1.$$

Secondly

$$7778^2 - 2223^2 = (7778 - 2223)(7778 + 2223) = 5555 \cdot 10001 =$$

$$= 5555(10000 + 1) = 55550000 + 5555 = 55555555.$$

In the same way

$$437^2 - 363^2 = (437 - 363)(437 + 363) = 74 \cdot 800,$$

$$537^2 - 463^2 = (537 - 463)(537 + 463) = 74 \cdot 1000$$

and, consequently,

$$\frac{437^2 - 363^2}{537^2 - 463^2} = 0.8.$$

To compute the exact values of the roots note that

$$(\sqrt{3}-\sqrt{2})^2 = 3-2\sqrt{6}+2 = 5-2\sqrt{6}.$$

Therefore

$$\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}.$$

In a similar way

$$\sqrt{3-2\sqrt{2}} = \sqrt{2}-1,$$
$$\sqrt{7-2\sqrt{12}} = 2-\sqrt{3}.$$

Therefore

$$\sqrt{5-2\sqrt{6}}+\sqrt{3-2\sqrt{2}}+\sqrt{7-2\sqrt{12}} = 1.$$

Observe that

$$1+\frac{1}{n} = \frac{n+1}{n}$$

and thus

$$\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\ldots\cdot\left(1+\frac{1}{100}\right) =$$
$$\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{100}{90}\cdot\frac{101}{100} = \frac{101}{2}.$$

Excercise 2.1.2 *Solve the equations:*

$$3x-\frac{4}{3} = \frac{5}{12}, \quad x^2-x-6=0, \quad x^3+6x^2-4x-24=0,$$

$$\frac{1}{x^2-9}+\frac{1}{x-3} = \frac{48}{(x-3)(x+38)}.$$

Solution. The first equation says that

$$3x = \frac{5}{12}+\frac{4}{3},$$

i.e. 3x=21/12 and thus x=7/12. Secondly

$$x_{12} = \frac{1\pm\sqrt{1+4\cdot 6}}{2} = \frac{1\pm 5}{2}.$$

Using the technic of division of polynomials it can be easily seen that if a polynomial has an integer root m then it must divide the constant term. We are going to guess at least one of the roots of the polynomial by checking the divisors of 24. This results in the root m=2. Using polynomial division again

$$x^3 + 6x^2 - 4x - 24 = (x - 2)(x^2 + 8x + 12).$$

To finish the solution we solve the quadratic equation

$$x^2 + 8x + 12 = 0$$

too. We have

$$x_{12} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 12}}{2} = \frac{-8 \pm 4}{2} = -4 \pm 2.$$

In case of the last equation we use the identity $x^2 - 9 = (x - 3)(x + 3)$ to conclude that

$$\frac{1}{x+3} + 1 = \frac{48}{x+38}$$

which results in a quadratic equation.

Excercise 2.1.3 *Prove that*

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} < 3.$$

Solution. Taking the square of both sides systematically

$$\sqrt{6 + \sqrt{6 + \sqrt{6}}} < 3,$$

$$\sqrt{6 + \sqrt{6}} < 3,$$

$$\sqrt{6} < 3,$$

$$6 < 9$$

which is obviously true.

Excercise 2.1.4 *Which number is the bigger?*

$$297 \cdot 299 \quad or \quad 298^2, \quad 345^2 \quad or \quad 342 \cdot 348, \quad \sqrt{101} - \sqrt{100} \quad or \quad \frac{1}{20}.$$

Solution. Using that

$$297 \cdot 299 = (298 - 1)(298 + 1) = 298^2 - 1$$

it follows that the second number is the bigger one. In a similar way

$$342 \cdot 348 = (345 - 3)(345 + 3) = 345^2 - 3^2$$

and 345^2 is bigger than the product $342 \cdot 348$. Since

$$\sqrt{101} - \sqrt{100} = \left(\sqrt{101} - \sqrt{100}\right) \frac{\sqrt{101} + \sqrt{100}}{\sqrt{101} + \sqrt{100}} = \frac{1}{\sqrt{101} + \sqrt{100}}$$

it is enough to compare the numbers

$$\sqrt{101} + \sqrt{100} \text{ and } 20.$$

Here

$$\sqrt{101} + \sqrt{100} > \sqrt{100} + \sqrt{100} = 20$$

which means that

$$\sqrt{101} - \sqrt{100} = \frac{1}{\sqrt{101} + \sqrt{100}} < \frac{1}{20}.$$

Excercise 2.1.5 *Solve the following systems of equations*

$$3x - 7y = 66$$

$$2x - 9y = -8$$

and

$$x^2 - y = 46$$
$$x^2 y = 147.$$

Solution. In terms of coordinate geometry the solution of the first system of equations gives the common point of two lines. From the first equation we can write y in terms of x as follows

$$y = \frac{3x - 66}{7}.$$

Substituting this expression into the second equation we have that

$$2x - 9\frac{3x - 66}{7} = -8,$$

$$14x - 9(3x - 66) = -56,$$
$$14x - 27x + 594 = -56.$$

Finally

$$x = \frac{650}{13} = 50$$

and, consequently,

$$y = \frac{150 - 66}{7} = 12.$$

To solve the second system of equations it seems to be more convenient to express x^2 from the first equation as follows

$$x^2 = 46 + y.$$

By substitution

$$(46 + y)y = 147,$$
$$y^2 + 46y - 147 = 0$$

which means that

$$y_{12} = \frac{-46 \pm \sqrt{46^2 + 4 \cdot 147}}{2} = \frac{-46 \pm \sqrt{2704}}{2} = \frac{-46 \pm 52}{2} = -23 \pm 26.$$

If $y = 3$ then the corresponding values of x are $x = \pm 7$. If $y = -49$ then there is no any corresponding value of x.

Excercise 2.1.6 *Using induction prove that*

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{2.1}$$

Solution. In case of n=1 the statement is obviously true. Using the inductive hypothesis

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 =$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{2^2} = \frac{(n+1)^2(n^2 + 4(n+1))}{2^2} = \frac{(n+1)^2(n+2)^2}{2^2} =$$
$$\frac{(n+1)^2((n+1)+1)^2}{2^2} = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$$

as was to be proved.

Excercise 2.1.7 *Compute the values of*

$$f(-4),\ f(-3) \quad and \quad f(2)$$

and sketch the graph of the function

$$f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}.$$

Solution. We have

$$f(-4) = -\frac{1}{2}(-4)^2 - (-4) + \frac{3}{2} = \frac{11}{2},$$

$$f(-3) = -\frac{1}{2}(-3)^2 - (-3) + \frac{3}{2} = 0,$$

$$f(2) = -\frac{1}{2}(2)^2 - 2 + \frac{3}{2} = -\frac{5}{2}.$$

To sketch the graph of the function consider the canonical form

$$f(x) = -\frac{1}{2}(x+1)^2 + 2.$$

Therefore the zeros of the function satisfy the equation

$$(x+1)^2 = 4$$

which means that $x_1 = 1$ and $x_2 = -3$. The maximum value is just 2 attained at the arithmetic mean of the zeros:

$$x_{\text{ max }} = \frac{1 + (-3)}{2} = -1.$$

Excercise 2.1.8 *Sketch the graph of the function* $f(x) = x^2 - 8x + 15$.

Solution. To sketch the graph of the function consider the canonical form

$$f(x) = (x-4)^2 - 1.$$

The zeros of the function satisfy the equation

$$(x-4)^2 = 1$$

which means that $x_1 = 3$ and $x_2 = 5$. The minimum value is just - 1 attained at the arithmetic mean of the zeros:

$$x_{\text{ min }} = \frac{3+5}{2} = 4.$$

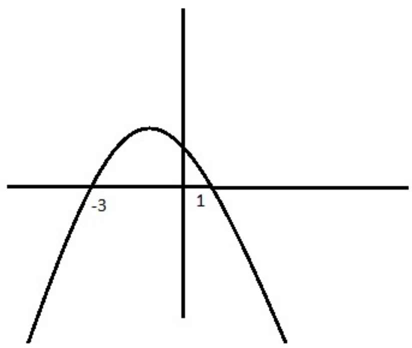

Figure 2.1: Exercise 2.1.7

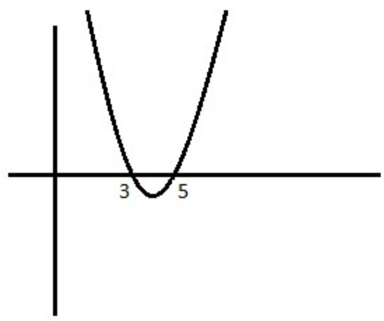

Figure 2.2: Exercise 2.1.8

Excercise 2.1.9 *Find all integer roots of the equation* $2x^3+11x^2-7x-6=0$ *and perform the division*

$$(2x^3+2x-1):(x-1)=$$

Solution. Any integer root must be a divisor of the constant term. Therefore the possible values are

$$\pm 1,\ \pm 2,\ \pm 3, \pm 6.$$

Substituting these values as x we have that the integer roots are x=1 or - 6. Finally

$$\begin{array}{llll}
2x^3+11x^2-7x-6 & : & x-1 & =2x^2 \\
-\ \ (2x^3-2x^2) & & & \\
\hline
13x^2-7x-6 & : & x-1 & =13x \\
-\ \ (13x^2-13x) & & & \\
\hline
6x-6 & : & x-1 & =6 \\
-\ \ (6x-6) & & & \\
\hline
0 & & &
\end{array}$$

Therefore

$$2x^3+x^2-1=(2x^2+13x+6)(x-1).$$

The missing roots are

$$x_{12}=\frac{-13\pm\sqrt{13^2-4\cdot 2\cdot 6}}{4}=\frac{-13\pm 11}{4},$$

i.e. $x_1=-6$ and $x_2=-(1/2)$.

Excercise 2.1.10 *Solve the inequality*

$$x^2-x-6<0.$$

Solution. The standard way of solving quadratic inequalities consists of three steps. At first we determine the roots of the quadratic polynomial if exist:

$$x_{12}=\frac{1\pm\sqrt{1+24}}{2}=\frac{1\pm 5}{2},$$

i.e. $x_1=-2$ and $x_2=3$. Secondly we sketch the graph of the function. Since the coefficient of the term of highest degree is positive the corresponding parabola is open from above (in other words it has a minimum attained at the arithmetic mean of the roots). Finally the solutions are - 2 < x < 3.

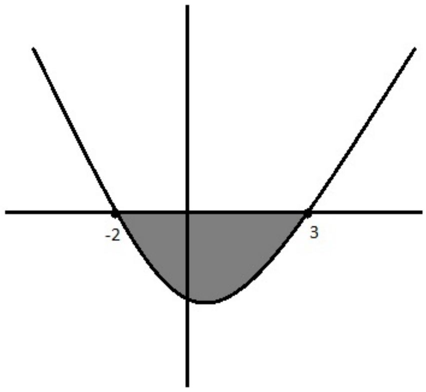

Figure 2.3: Exercise 2.1.10.

Excercise 2.1.11 *Solve the inequality*

$$x^2 - x - 6 > 0.$$

Chapter 3

Basic facts in geometry

Using Plato's words "the objects of geometric knowledge are eternal". The Greek deductive method gives a kind of answer to the question how to obtain information about this idealized world. It was codified by Euclid around 300 BC in his famous book entitled *Elements* which is a system of conclusions on the bases of unquestionable premisses or *axioms*. In terms of a modern language the method needs two fundamental concepts to begin working: undefined terms such as points, lines, planes etc. and axioms (sometimes they are referred as premisses or postulates) which are the basic assumptions about the terms of geometry. Here we present a short review of axioms in Euclidean plane geometry to illustrate its fundamental assumptions, methods and specific points of view.

3.1 The axioms of incidence

The axioms of incidence.

- Through any two distinct points there is exactly one line.

The basic terms (like points, lines etc.) of the axiomatic system are undefined. If we do not know what they mean then there is no point in asking whether or not the axioms are true. Following one of the most expressive examples in [1] suppose that alien beings have landed on Earth by flying saucer and their leader tells you that *through any distinct blurgs there is exactly one phogon*. Unless you know what a blurg and a phogon you will have no way of telling whether or not this statement is true. On the other hand there may be many different interpretations of the undefined terms such as points, lines etc. in an axiomatic system for geometry. An interpretation which makes all the axioms true is called a model for the axiomatic system;

because theorems are all deduced logically from the axioms they will be true in any model as well. To understand the role of models we can consider the classical coordinate geometry as one of the model for the Euclidean plane geometry. Points are interpreted as pairs of real numbers (coordinates) and lines are interpreted as point - sets satisfying equations of special type. In this interpretation the first axiom of incidence can be checked in the following way: consider the points (x_1, y_1) and (x_2, y_2) in the plane; the line passing through these points is just the set of points whose coordinates satisfy the equation

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

provided that the first coordinates of the given points are different. In case of $x_1 = x_2$ the equation of the corresponding line is just x= constant.

Remark As we have seen above points can be interpreted as pairs of real numbers. The lines correspond to more complicated algebraic objects called equations. This is the reason why such a model for the Euclidean geometry is called analytic. It can be easily generalized by admitting more than two coordinates. This results in the geometry of higher dimensional Euclidean spaces. To illustrate what happens note that lines in the space have system of equations of the form

$$\frac{z-z_1}{z_2-z_1}=\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

or

$$z=z_1,\quad \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

in case of $z_1 = z_2$ and so on.

- Any line contains at least two distinct points and we have at least three distinct points which do not lie on the same line.

The statement is labelled as the dimension axiom because it says essentially that lines are one-dimensional and the plane is of dimension two.

Definition Points lying on the same line are called collinear.

3.2 Parallelism

Finally we present the most famous axiom of Euclidean plane geometry which can be expressed in terms of incidence. This is called Euclid's parallel postulate.

Definition Two lines in the plane are parallel if they have no any point in common or they coincide.

- Let l be a line and P be a point in the plane; there is one and only one line e that passes through P and parallel to l.

Theorem 3.2.1 *If l is parallel to e and e is parallel to m then l is parallel to m.*

Proof Suppose that l and m has a point P in common. Since both of the lines are parallel to e we have by the parallel axiom that l=m. Otherwise they are disjoint.

Remark Definitions are shortcut notations from the logical point of view. Theorems are deduced logically from the axioms or other theorems which has been proved.

3.3 Measurement axioms

Another important question is how to measure distance between points in the plane. Like points, lines etc. the absolute distance can also be a new undefined term in our geometry. The main question is not what is the distance but how to measure the distance. The physical instrument to realize distance measurements is a ruler. Its abstract (idealized) version is called the ruler axiom.

- Let l be an arbitrary line in the plane. A ruler for l is a one-to-one correspondence between the points in l and the set of real numbers in such a way that the distance between the points A and B in l is just the absolute value of the difference of the corresponding reals: if A corresponds to the real number a and B corresponds to the real number b then

  $$d(A, B) = |a - b|.$$

  The ruler axiom postulates the existence of such a ruler for any line in the plane.

By the help of a ruler we can use the standard ordering among real numbers to define segments and half - lines. Let A and B be two distinct points in the plane and consider the line l passing through the given points. If a < b then the straight line segment joining A and B is defined as

$$AB := \{\ C \in l \mid a \leq c \leq b\},$$

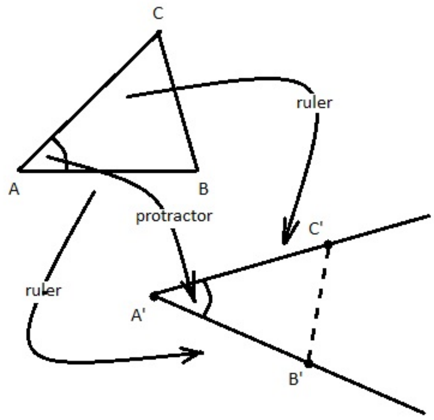

Figure 3.1: Congruence axiom.

where the points correspond to the real numbers a, b and c under a ruler. The half - line starting from A to B is created by cutting the points with coordinates c $<$ a. Segments and half - lines correspond to intervals of the form [a,b] where the starting or the end point can be positioned at plus or minus infinity.

There are several ways of introducing the concept of angle in geometry. Here we consider this concept as a new undefined term governed by its own axioms. Instead of the precise formulation we accept that the protractor axiom formulates the abstract (idealized) version of the physical instrument for measuring angles in the real world.

3.4 Congruence axiom

Using a ruler and a protractor we can compare and copy segments and angles in the plane. The next important question is how to compare and copy triangles.

Let a triangle ABC be given in the plane and consider an arbitrary half - line starting from a point A'. Using a ruler we can copy the segment AB from A' into the given direction. This results in a point B' such that AB=A'B'. Using a protractor and a ruler again we can construct a point C' such that

$$\alpha = \text{ the measure of } \angle CAB = \text{ the measure of } \angle C'A'B' = \alpha'$$

and A'C'=AC. What about the the missing sides BC and B'C', the missing angles γ and γ' or β and β'? Unfortunately we can not know anything about them because nor the axioms of incidence neither the measurement axioms carry any information about the missing data of the triangles. If we want to make them congruent then we have to postulate them to be congruent.

Definition If there is a correspondence between the vertices of two triangles in such a way that all corresponding sides and all corresponding angles are congruent then the triangles are congruent copy of each other.

The congruence axiom allows us to deduce the congruence of triangles under a reduced system of information.

- If there is a correspondence between the vertices of two triangles in such a way that two sides and the angle enclosed by them in one of the triangles are congruent to the corresponding sides and the corresponding angle in the second of the triangles then the triangles are congruent copy of each other.

Sometimes it is referred as side - angle - side - axiom or SAS - axiom.

3.5 Area

Formally speaking [1] area can be considered as a new undefined term in the axiomatic system of geometry. Some obvious requirements can be formulated as follows. Let a polygonal region be defined as the finite union of triangles such that the members of the union have at most common sides or vertices. The area of a bounded polygonal region is a non-negative real number satisfying the following properties:

- (area invariance axiom) The area is invariant under the isometries (chapter 12) of the plane.

- (area addition axiom) The area of the union of two poligonal regions is just the sum of the areas of the regions provided that they have at most common sides or vertices.

- (area normalization axiom) The area of a rectangle of sides a and b is just a·b.

3.6 Basic facts in geometry

In what follows we summarize some further facts which will be frequently used in the forthcoming material. We would like to emphasize that they are not necessarily axioms but we omit the proofs for the sake of simplicity.

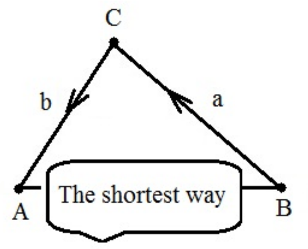

Figure 3.2: Triangle inequalities.

3.6.1 Triangle inequalities

They are special forms of the basic principle in geometry saying that the shortest way between two points is the straight line segment. Consider a triangle with vertices A, B and C. Let us denote the sides opposite to the corresponding vertices by a, b and c. Then

$$a + b > c,\ \ c + a > b \ \text{ and } \ b + c > a.$$

Corollary 3.6.1 *For the sides a, b and c of a triangle*

$$|a - b| < c,\ \ |a - c| < b \ \textit{ and } \ |b - c| < a.$$

Remark If a < b < c then the corollary says that

- the interval [a,b] can be covered by the third side of the triangle,
- the interval [a,c] can be covered by the second side of the triangle,
- the interval [b,c] can be covered by the first side of the triangle.

3.6.2 How to compare triangles I - congruence

The basic cases of congruence of triangles are

- SAS (two sides and the angle enclosed by them), i.e.

  $$a = a',\ b = b' \ \text{ and } \ \gamma = \gamma'$$

  (see congruence axiom).

- ASA (one side and the angles on this side), i.e.

  $$c = c',\ \alpha = \alpha' \ \text{ and } \ \beta = \beta'$$

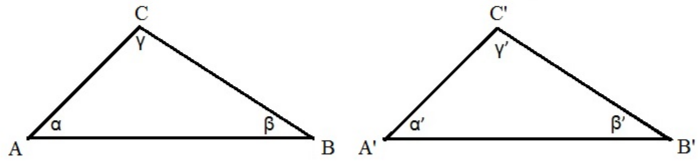

Figure 3.3: Congruent triangles.

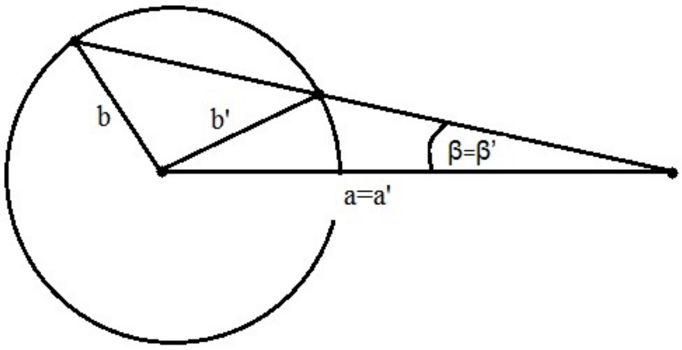

Figure 3.4: The case SsA

- SAA (one side and two angles), i.e.

$$c = c', \ \alpha = \alpha' \ \text{ and } \ \gamma = \gamma'$$

- SSS (all sides), i.e.

$$a = a', \ b = b' \ \text{ and } \ c = c'$$

- SsA (two sides and the angle opposite to the larger one),

$$a = a', \ b = b' \ \text{ and } \ \alpha = \alpha' \ \text{ provided that } \ a > b.$$

Theorem 3.6.2 (The geometric characterization of the perpendicular bisector) *The perpendicular bisector of a segment is the locus of points in the plane having the same distance from each of the endpoints.*

Proof Let AB be a segment with midpoint F and consider the line l through F in such a way that l is perpendicular to the line AB. If X is a point in l then the triangles AFX and BFX are obviously congruent to each other because of

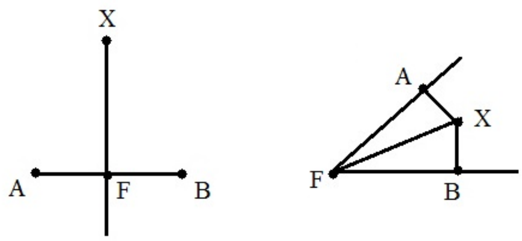

Figure 3.5: Bisectors.

the congruence axiom SAS. Therefore AX=BX. Conversely if AX=BX then the triangles AFX and BFX are congruent because of SSS. Therefore the angles at F are equal and their sum is 180 degree in measure. This means that the line XF is just the perpendicular bisector of the segment.

Excercise 3.6.3 *Formulate the geometric characterization of the bisector of an angle in the plane.*

Hint. Since the triangles FXA and FXB are congruent the bisector is the locus of points in the plane having the same distance from each of the arms of the angle.

Theorem 3.6.4 *The ordering among the sides of a triangle is the same as the ordering among the angles of the triangle.*

3.6.3 Characterization of parallelism

The essential difference between the parallel axiom and the other ones is hidden in the notion of parallelism itself. The parallelism involves the idea of infinity in a rather important way. If we know that two lines are not parallel we still have no idea how far one may have to trace along them before they actually meet. The idea of infinity is always problematic because many errors in mathematics arise from generalizations to the infinite of what is known true for the finite. As one of interesting examples consider a hotel having as many rooms as many natural numbers we have. Is it possible to provide accommodation for one more guest if all of rooms are occupied? The answer is definitely yes because if the guest in room n is moving into room n+1 then room 1 becomes free. In what follows we present a method of checking the parallelism by measuring angles instead of taking an infinite - long walk.

Excercise 3.6.5 *Let e and f be parallel lines in the plane and consider a transversal f. Find the relationships among the inclination angles.*

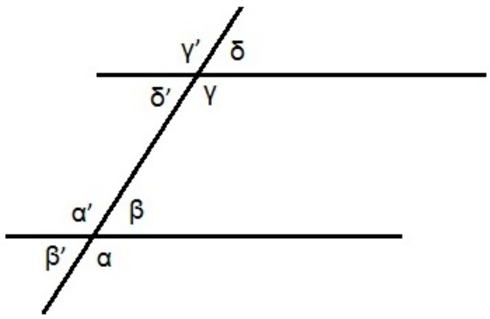

Figure 3.6: Characterization of parallelism.

Theorem 3.6.6 (Characterization of parallelism) *The lines e and f are parallel if and only if one of the following relationships is true for the inclination angles:*

$$\beta = \delta', \quad \beta + \gamma = 180^\circ \quad or \quad \beta = \delta.$$

Excercise 3.6.7 *Prove that the sum of the interior angles of a triangle is 180 degree in measure.*

Hint. Let ABC be a triangle. Taking the line l through the point C in such a way that l is parallel to the side AB, the statement is a direct consequence of the characterization of parallelism.

3.6.4 How to compare triangles II - similarity

Theorem 3.6.8 (Parallel lines intersecting theorem) *Let e and e' be two lines in the plane meeting at the point O. If the lines a and b are parallel to each other such that the line a meets e and e' at the points A and A', the line b meets e and e' at the points B and B' then*

$$OA : OB = OA' : OB'.$$

In the Hungarian educational tradition it is a theorem. It is also possible to consider the statement as an axiom; see Similarity axiom in [1].

Definition Let e and e' be two lines in the plane meeting at the point O. We say that the points O, A, B on the line e correspond to the points O, A', B' on the line e' if they have the same ordering, i.e. the line e separates A' and B' if and only if the line e' separates A and B.

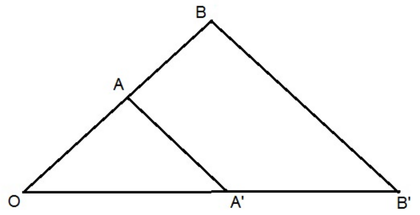

Figure 3.7: Parallel lines intersecting theorem.

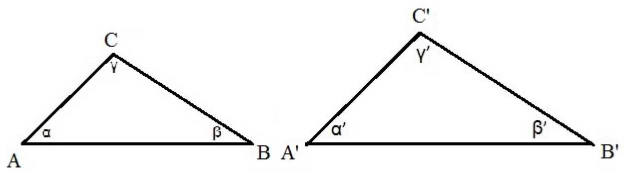

Figure 3.8: Similar triangles.

Theorem 3.6.9 (The converse of the parallel lines intersecting theorem) *Let e and e' be two lines in the plane meeting at the point O. If the line a meets e and e' at the points A and A', the line b meets e and e' at the points B and B' such that O, A, B correspond to O, A', B' and* $OA : OB = OA' : OB'$ *then the lines a and b are parallel.*

The parallel lines intersecting theorem (and its converse) together with the basic cases of the congruence of triangles give automatically the basic cases of similarity.

Definition If there is a correspondence between the vertices of two triangles in such a way that all corresponding angles are congruent and the ratios between the corresponding sides are also equal then the triangles are said to be similar.

The basic cases of similarity of triangles are

- S'AS' (two sides and the angle enclosed by them), i.e.

$$a : a = b : b' \text{ and } \gamma = \gamma'.$$

- AAA (all of angles), i.e.

$$\alpha = \alpha', \ \beta = \beta' \text{ and } \gamma = \gamma',$$

- S'S'S' (all sides), i.e.
$$a : a' = b : b' = c : c'$$

- S's'A (two sides and the angle opposite to the larger one),
$$a : a' = b : b' \text{ and } \alpha = \alpha' \text{ provided that } a > b.$$

Chapter 4

Triangles

4.1 General triangles I

Let us start with the collection of distinguished points, lines and circles related to a triangle

Definition The lines passing through the midpoints of the sides of a triangle are called midlines.

Using the converse of the parallel lines intersecting theorem 3.6.9 it can be easily seen that any midline is parallel to the corresponding side and the line segment between the midpoints is just the half of this side.

Theorem 4.1.1 *The perpendicular bisectors of the sides of a triangle are concurrent at a point which is just the center of the circumscribed circle.*

Proof The statement is a direct consequence of the geometric characterization 3.6.2 of the bisector of a segment.

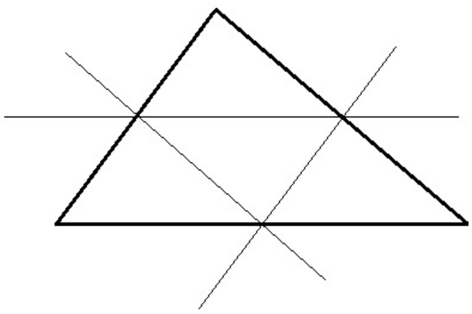

Figure 4.1: Midlines

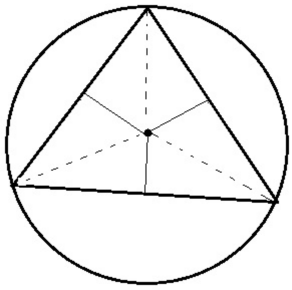

Figure 4.2: Circumcircle

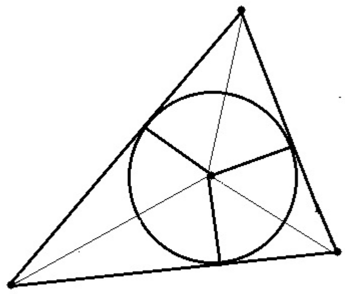

Figure 4.3: Incircle

Theorem 4.1.2 *The bisectors of the interior angles of a triangle are concurrent at a point which is just the center of the inscribed circle.*

Proof The statement is a direct consequence of the geometric characterization of the bisector of an angle.

Theorem 4.1.3 *The altitudes of a triangle are concurrent at a point which is called the orthocenter of the triangle.*

Proof Consider the triangle constituted by the parallel lines to the sides passing through the opposite vertices. The orthocenter of the triangle ABC is the center of the circumscribed circle of A'B'C'.

Definition The medians of a triangle are the straight lines joining the vertices and the midpoints of the opposite sides.

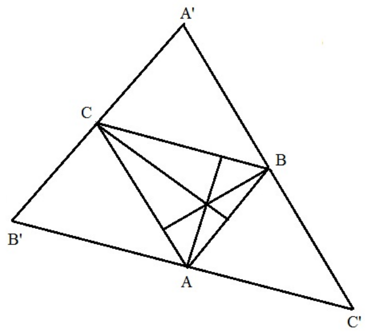

Figure 4.4: Orthocenter - the intersection of the altitudes

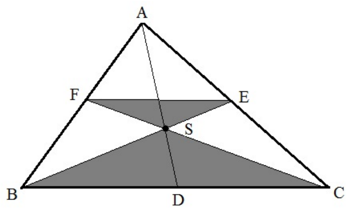

Figure 4.5: Barycenter - the intersection of the medians

Theorem 4.1.4 *The medians are concurrent at a point which is called the barycenter/centroid of the triangle. This point divides the medians in the ratio* $2:1$.

Proof It can easily seen that

- the triangle FSD is similar to the triangle CSA,
- the triangle DSE is similar to the triangle ASB.

The ratio of the similarity is $1:2$. This means that the medians BE and CF intersect AD under the same ratio. Therefore they are concurrent at S.

Remark Each median bisects the area of the triangle.

4.2 The Euler line and the Feuerbach circle

Theorem 4.2.1 *The orthocenter M, the center O of the circumscribed circle and the barycenter S are collinear. The point S divides the segment MO in the ratio 2 : 1. The common line of the points M, O and S is called the Euler line.*

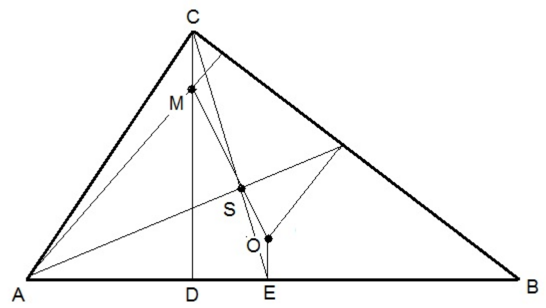

Figure 4.6: Euler-line

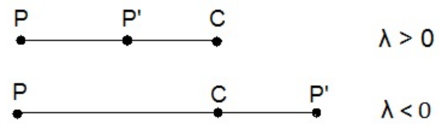

Figure 4.7: Central similarity

Proof The proof is based on the central similarity with respect to the barycenter. A central similarity is a point transformation $P \to P'$ of the plane such that

- there is a distinguished point C which is the only fixpoint (center) of the transformation,
- P, C and P' are collinear,
- there is a real number $\lambda \neq 0$ such that

  $$CP' : CP = |\lambda|.$$

  If $\lambda > 0$ then P and P' are on the same ray emanating from C. In case of $\lambda < 0$ the center separates P and P'.

According to the converse of the parallel lines intersecting theorem any line is parallel to the image under a central similarity. Consider now the central similarity with center S and ratio - 1/2. Then each vertex is transferred into the midpoint of the opposite side and each altitude is transferred into the perpendicular bisector of the corresponding side. This means that M'=O proving the statement.

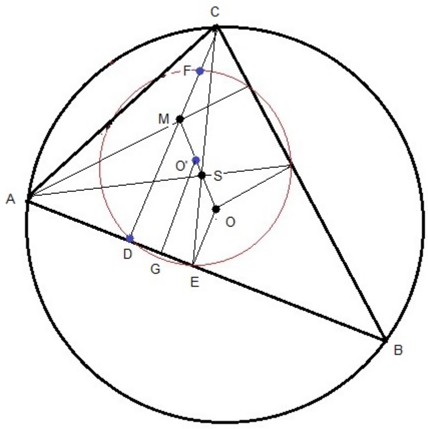

Figure 4.8: Feuerbach-circle

Definition The image of the circumscribed circle under the similarity with center S and ratio - 1/2 is called the Feuerbach circle of the triangle.

Theorem 4.2.2 *The Feuerbach circle passes through nine points:*

- *the midpoints of the sides,*
- *the legs of the altitudes,*
- *the midpoints of the segments joining the orthocenter and the vertices A, B and C.*

Proof The Feuerbach circle passes through the midpoints of the sides because the circumscribed circle passes through the vertices. The radius R' of the Feuerbach circle is just R/2 because of the similarity ratio. Since S divides the segment MO in the ratio 2 : 1 the center O' of the Feuerbach circle is the midpoint of the segment MO. Therefore O'G is the midline of the trapezoid DMOE and G bisects the segment DE. This means that DO'E is an isosceles triangle with

$$O'D = O'E = R/2$$

and the leg point D of the altitude belonging to the side c is on the Feuerbach circle. Finally O'F is a midline in the triangle CMO. Therefore

$$O'F = \frac{1}{2}CO = R/2$$

as was to be stated.

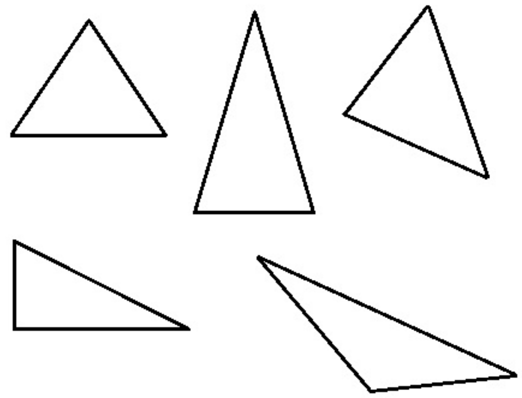

Figure 4.9: Triangles

4.3 Special triangles

Triangles can be classified by angles or sides. In what follows we shall use the basic notations

- A, B and C for the vertices,
- α, β and γ for the angles at the corresponding vertices and
- a, b and c for the opposite sides to the angles α, β and γ, respectively.

The most important cases of special triangles are

- equilateral (regular) triangles: all sides and all angles are equal to each other,
- isosceles triangles: two sides and the opposite angles are equal to each other,
- acute triangles: all angles are less than 90 degree,
- right triangles: one of the angle is 90 degree in measure,
- obtuse triangle: one of the angle is greater than 90 degree

or mixed cases: for example isosceles right triangles. One of the oldest fact in geometry is **Pythagorean theorem** for right triangles.

Theorem 4.3.1 (Pythagoras, 570 BC - 495 BC) *The sum of the squares of the legs is just the square of the hypothenuse:*

$$a^2 + b^2 = c^2.$$

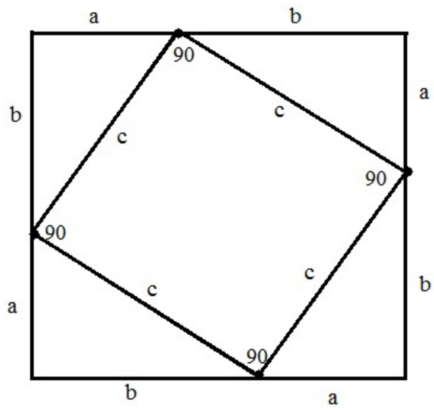

Figure 4.10: Pythagorean theorem.

Proof If we divide a square with sides of length a+b into five parts by the figure then the area can be computed as

$$(a+b)^2 = 4\frac{ab}{2} + c^2.$$

Pythagorean theorem follows immediately by the help of an algebraic calculation.

Remark The meaning of hypothenuse is stretched. The word refers to the ancient method to create right angles by a segmental string in ratio 3 : 4 : 5. Note that

$$3^2 + 4^2 = 5^2.$$

Theorem 4.3.2 (Height theorem) *If m denotes the altitude belonging to the hypothenuse in a right triangle then $m^2 = pq$, where p and q are the lengths of the segments from the vertices to the leg point of the altitude.*

Proof By Pythagorean theorem in the triangles CTB, CTA and ABC

$$p^2 + m^2 = a^2, \quad q^2 + m^2 = b^2, \quad a^2 + b^2 = c^2.$$

Therefore

$$p^2 + q^2 + 2m^2 = a^2 + b^2 = c^2 = (p+q)^2 = p^2 + q^2 + 2pq$$

which means that $m^2 = pq$.

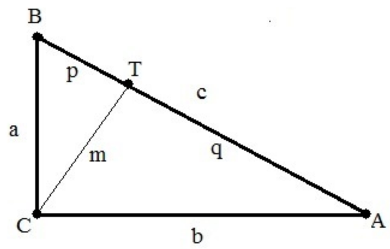

Figure 4.11: Height theorem.

Remark In other words the altitude m is the geometric mean of p and q.

Theorem 4.3.3 (Leg theorems) *$a^2 = cp$ and $b^2 = cq$.*

Proof As above
$$p^2 + m^2 = a^2 \text{ and } q^2 + m^2 = b^2,$$
where m is the geometric mean of p and q. Therefore (for example)
$$a^2 = p^2 + m^2 = p^2 + pq = p(p+q) = pc$$
as was to be stated.

Remark This collection of theorems (Pythagorean, Height and Leg theorems) are often referred as *similarity theorems in right triangles* because there are alternative proofs by using the similar triangles CTB, CTA and ABC.

Theorem 4.3.4 (Thales theorem) *If A, B and C are three different points on the perimeter of a circle such that AB is one of the diagonals then ABC is a right triangle having the angle of measure 90 degree at C.*

Proof Let O be the center of the circle. Since
$$OA = OB = OC = r$$
it follows that AOC and BOC are isosceles triangles. Therefore
$$\angle OAC = \angle OCA = \alpha, \quad \angle OBC = \angle OCB = \beta$$
and, consequently,
$$2(\alpha + \beta) = 180 \quad \Rightarrow \quad \alpha + \beta = 90.$$

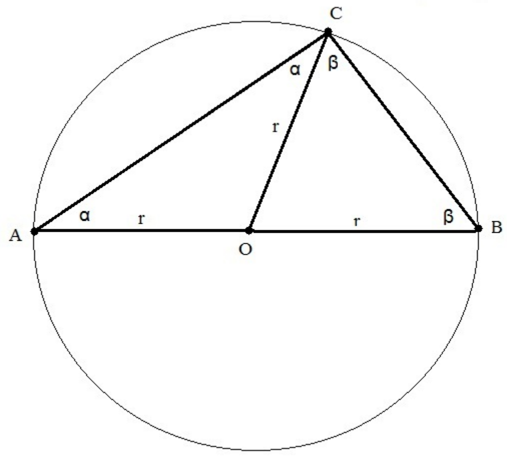

Figure 4.12: Thales theorem.

Remark Thales theorem is actually the special case of a more general observation called inscribed angle theorem: let A, B and C be three different points on the perimeter of a circle with center O and suppose that the angles $\angle AOB$ and $\angle ACB$ are lying on the same arc. Then

$$\angle AOB = 2\angle ACB$$

because

$$\angle AOB = \omega = 2\alpha + 2\beta = 2(\alpha + \beta) = 2\angle ACB.$$

The proof is based on the isosceles triangles AOC and BOC.

4.4 Exercises

Excercise 4.4.1 *Collect the facts we used to prove Pythagorean theorem.*

Solution. The proof of Pythagorean theorem is based on

- the area of squares, right triangles and the basic principles of measuring the area,
- the sum of angles in a (right) triangle is 180 degree,
- algebraic identities.

Excercise 4.4.2 *Prove the height and the leg theorems by using similar triangles. Conclude Pythagorean theorem too.*

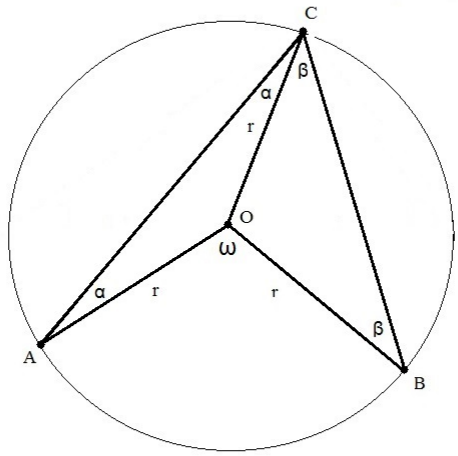

Figure 4.13: Inscribed angle theorem

Excercise 4.4.3 *Find the missing quantities in each row of the following table.*

a	b	c	m	p	q
		12		3	
				4	16
				6	9
6					9
6	8				

Hint. Use Pythagorean, height and leg theorems:

$$a^2 + b^2 = c^2, \quad m^2 = pq, \quad a^2 = cp \text{ and } b^2 = cq.$$

Excercise 4.4.4 *Find the length of the side of a regular triangle inscribed in the unit circle.*

Hint. Using Thales theorem the triangle ABC in the figure has a right angle at the vertex C. Therefore

$$x^2 + 1^2 = 2^2,$$

i.e. $x = \sqrt{3}$.

Excercise 4.4.5 *In a right triangle the length of the longest side AB is* 6. *The leg BC is* 3.

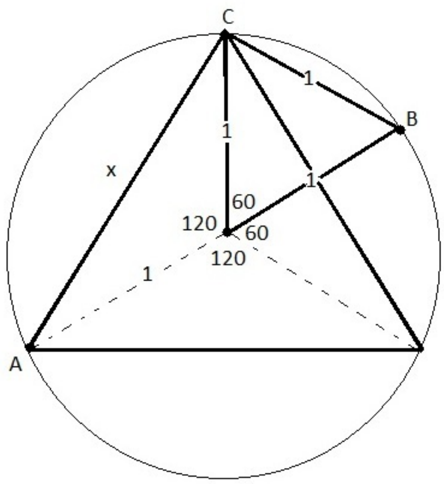

Figure 4.14: Exercise 4.4.4

- *Calculate the missing leg and the area of the triangle.*
- *What is the radius of the inscribed circle?*
- *What are the sine, cosine, tangent and cotangent of the angle at A?*

4.5 Trigonometry

Euclidean geometry is essentially based on triangles. The metric properties of triangles (the length of the sides or the measure of the angles) can be described by elegant formulas. They are very important in practice too (see chapter 6). The word trigonometry directly means the measuring of triangles. Using the basic cases of similarity it can be easily seen that two right triangles with acute angles of the same measure are similar. Therefore the ratios between the legs and the hypothenus are uniquely determined by the angles. This results in the notion of sine, cosine, tangent and cotangent in the following way. Let α be an acute angle, i.e. $0 < \alpha < 90°$. If ABC is a right triangle with legs AC and BC and the angle at the corner A is α then

- the *sine* of α is the ratio between the opposite leg and the hypothenuse: $\sin\alpha = a/c$,
- the cosine of α is the ratio between the adjacent leg and the hypothenuse: $\cos\alpha = b/c$

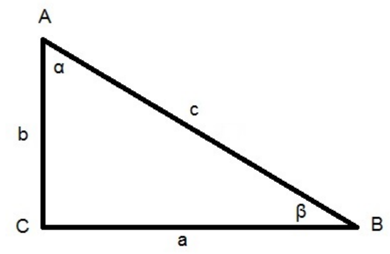

Figure 4.15: Trigonometry in a right triangle

- the tangent of α is the ratio between the opposite and the adjacent leg: $\tan\alpha = a/b$,
- the cotangent of α is the ratio between the adjacent leg and the opposite leg: $\cot\alpha = b/a$.

We can easily conclude that

$$\sin\alpha = \cos(90-\alpha) \quad \text{and} \quad \cos\alpha = \sin(90-\alpha),$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

(trigonometric Pythagorean theorem),

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \ \cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \ \tan\alpha = \frac{1}{\cot\alpha},$$

$$\tan\alpha = \cot(90-\alpha) \quad \text{and} \quad \cot\alpha = \tan(90-\alpha).$$

It is hard to create a geometric configuration to find the sine and cosine (tangent and cotangent) of a given angle in general. The so-called additional rules help us to solve such kind of problems.

Theorem 4.5.1 (Additional rules)

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Special case are

$$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

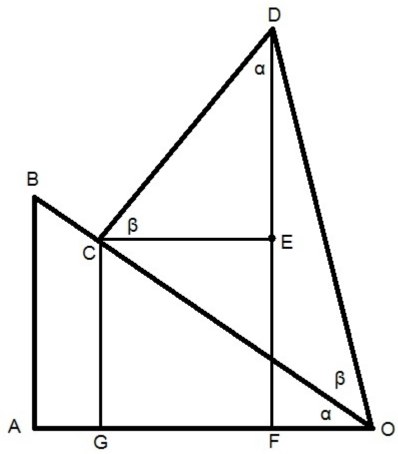

Figure 4.16: Additional rules

Proof Using the notations in the figure we find that

$$\sin(\alpha+\beta) = \frac{DF}{DO}.$$

For the sake of simplicity suppose that DO = BO = 1. Therefore

$$\sin(\alpha+\beta) = DF = DE + EF = DE + CG = CD\cos\alpha + CO\sin\alpha =$$

$$\sin\beta\cos\alpha + \cos\beta\sin\alpha.$$

On the other hand

$$\cos(\alpha+\beta) = OF = OG - GF = OG - CE = CO\cos\alpha - CD\sin\alpha =$$

$$\cos\beta\cos\alpha - \sin\beta\sin\alpha$$

as was to be proved.

The additional rules can be used to extend the notion of sine, cosine, tangent and cotangent. Using the decomposition 90=45+45 we have immediately that

$$\sin 90 = 2\sin 45\cos 45 = 1 \quad \text{and} \quad \cos 90 = \cos^2 45 - \sin^2 45 = 0.$$

The extension in mathematics is usually based on the principle of permanence. This means that we would like to keep all the previous rules (cf. the extension of powers from naturals to rationals). As another example compute sin 105 with the help of decomposition 105=60+45:

$$\sin 105 = \sin(60+45) = \sin 60\cos 45 + \cos 60\sin 45 = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}}.$$

New relationships can be created such as

$$\sin(\alpha + 90) = \sin\alpha\cos 90 + \cos\alpha\sin 90 = \cos\alpha,$$

$$\cos(\alpha + 90) = \cos\alpha\cos 90 - \sin\alpha\sin 90 = -\sin\alpha.$$

Especially the sine is positive in the second quadrant of the plane but the cosine has a minus sign. In a similar way

$$\sin 180 = 2\sin 90\cos 90 = 0 \quad \text{and} \quad \cos 180 = \cos^2 90 - \sin^2 90 = -1.$$

Therefore

$$\sin(\alpha + 180) = -\sin\alpha \quad \text{and} \quad \cos(\alpha + 180) = -\cos\alpha,$$

i.e. both the sine and the cosine are negative in the third quadrant. To investigate the fourth quadrant verify that

$$\cos 270 = 0 \quad \text{and} \quad \sin 270 = -1.$$

We have that the sine is negative in the last quadrant but the cosine keeps its positive sign because of

$$\sin(\alpha + 270) = -\cos\alpha < 0 \quad \text{and} \quad \cos(\alpha + 270) = \sin\alpha > 0$$

for any acute angle α. Finally

$$\sin(\alpha + 360) = \sin\alpha \quad \text{and} \quad \cos(\alpha + 360) = \cos\alpha.$$

The periodicity properties show that the process of extension goes to the end. From now on trigonometric expressions can be considered as functions [6]. The domain of the sine and cosine functions are the set of all angles measured in degree or radian. In mathematics the radian is more typical because it is directly related to the geometric length of the arc along a unit circle (a circle having radius one). The angle belonging to the arc of unit length is 1 radian in measure. The relationship between the degree and the radian is just

$$\frac{\alpha\ (\text{deg})}{360} = \frac{\alpha\ (\text{rad})}{2\pi}.$$

Remark To memorize the signs of trigonometric expressions consider the motion of a point along the unit circle centered at the origin in the Euclidean coordinate plane. The cosine and the sine functions give the first and the second coordinates in terms of the rotational angle. Obviously we have positive coordinates in the first quadrant. After entering in the second quadrant the first coordinate must be negative and so on. For the illustration of the trigonometric functions see figures 4.17 and 4.18.

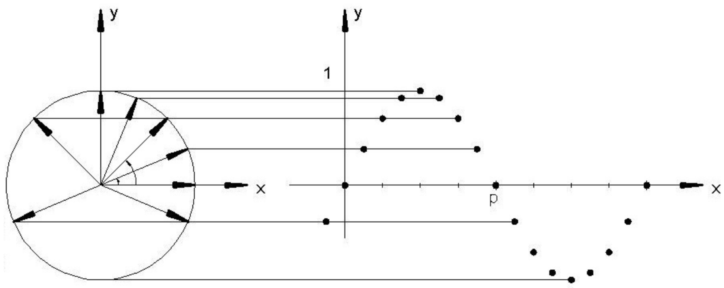

Figure 4.17: The sine function

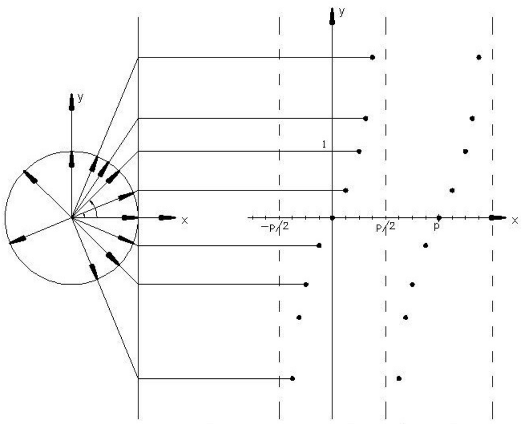

Figure 4.18: The tangent function

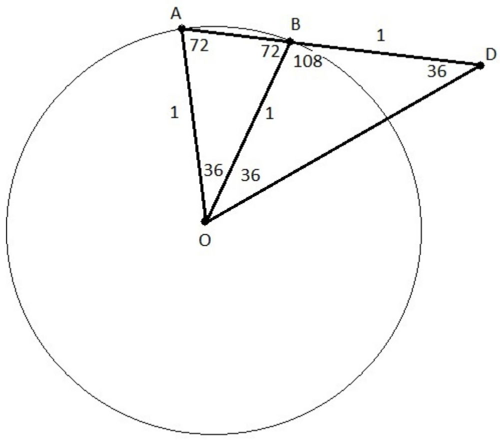

Figure 4.19: Exercise 4.6.2

4.6 Exercises

Excercise 4.6.1 *Compute the exact values of sine, cosine, tangent and cotangent functions for the following angles:*

$$45, \ 30, \ 60.$$

Solution. From an isosceles right triangle we have that

$$\sin 45 = \cos 45 = \frac{1}{\sqrt{2}}.$$

From a regular triangle with sides of unit length we have that

$$\cos 60 = \sin 30 = \frac{1}{2}$$

and

$$\sin 60 = \cos 30 = \frac{\sqrt{3}}{2}.$$

Excercise 4.6.2 *Compute the exact values of sine, cosine, tangent and cotangent functions for the following angles:*

$$72, \ 36, \ 18.$$

Solution. Consider a regular 10–gon inscribed in the unit circle. As the figure shows the triangles OAB and DOA are similar which means that

$$1 : x = (1 + x) : 1,$$

where x denotes the length of the side AB. We have a quadratic equation

$$x^2 + x - 1 = 0.$$

Therefore

$$x = \frac{-1 + \sqrt{5}}{2}.$$

To express (for example) $\cos 72$ consider the perpendicular bisector of the side AB in the triangle OAB. Since the radius is 1 we have that

$$\cos 72 = x/2$$

and, consequently,

$$\sin 72 = \sqrt{1 - (x/2)^2}$$

by the trigonometric Pythagorean theorem. On the other hand

$$\cos 18 = \sin 72 \quad \text{and} \quad \sin 18 = \cos 72.$$

To determine the trigonometric expressions of the angle 36 degree in measure use the perpendicular bisector belonging to the side OD in the triangle OBD. Since the radius of the circle is 1 we have that

$$\cos 36 = (1 + x)/2 \quad \text{and} \quad \sin 36 = \sqrt{1 - (1 + x)^2/4}.$$

Excercise 4.6.3 *Compute the exact values of sine, cosine, tangent and cotangent functions for the following angles:*

$$75, \ 54, \ 22.5.$$

Solution. Using the decompositions

$$75 = 45 + 30, \quad \text{and} \quad 54 = 36 + 18$$

the additional rules give the values of sine, cosine tangent and cotangent. Finally

$$45 = 2 \cdot 22.5$$

and

$$\cos 45 = \cos^2 22.5 - \sin^2 22.5 = 1 - 2\sin^2 22.5$$

because of the trigonometric Pythagorean theorem

$$\cos^2 \alpha + \sin^2 \alpha = 1.$$

Therefore

$$\sin 22.5 = \sqrt{\frac{1 - \cos 45}{2}}$$

and so on.

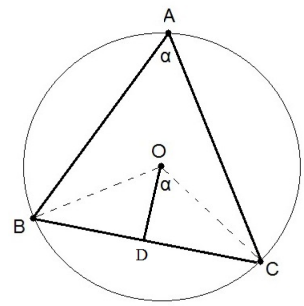

Figure 4.20: Sine rule - acute angles

Excercise 4.6.4 *Express*

$$\cos 3\alpha,\ \sin 3\alpha,\ \cos 4\alpha,\ \sin 4\alpha,\ \ldots$$

in terms of $\sin\alpha$ *and* $\cos\alpha$*.*

Excercise 4.6.5 *Sketch the graph of the cosine function.*

Hint. Use that

$$\cos\alpha = \sin(\alpha + 90).$$

Excercise 4.6.6 *Explain where the name tangent comes from?*

Excercise 4.6.7 *Sketch the graph of the cotangent function.*

Hint. Use that

$$\cot\alpha = \tan(90-\alpha) = -\tan(\alpha - 90).$$

4.7 General triangles II - Sine and Cosine rule

One of the most important applications of the extended sine and cosine functions is to conclude the sine and cosine rules for general triangles.

4.7.1 Sine rule

First of all we investigate the case of acute triangles (all the angles are less than 90 degree in measure). To present the sine rule let us start with the circumscribed circle of the triangle ABC. The center is just the intersection of the perpendicular bisectors of the sides. Since BOC is an isosceles triangle the inscribed angle theorem says that

$$\angle DOC = \alpha,$$

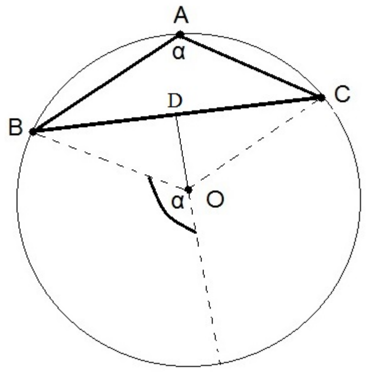

Figure 4.21: Sine rule - an obtuse angle

where D is the midpoint of BC. Therefore

$$\sin\alpha = \frac{a/2}{R} \quad \Rightarrow \quad 2R = \frac{a}{\sin\alpha}.$$

Theorem 4.7.1 (Sine rule)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

Excercise 4.7.2 *Prove the sine rule in case of obtuse triangles.*

Hint. Observe that $\sin(180 - \alpha) = \sin\alpha$.

4.7.2 Cosine rule

The cosine rule is the generalization of Pythagorean theorem. At first we discuss acute triangles again. Using the altitude belonging to the side b we express the square of a in two steps by using Pythagorean theorem. If X is the foot point of the altitude then we can write that

$$AX = c\cos\alpha \quad \text{and} \quad BX = c\sin\alpha.$$

Therefore

$$a^2 = BX^2 + XC^2 = BX^2 + (CA - AX)^2 = c^2\sin^2\alpha + (b - c\cos\alpha)^2 =$$

$$c^2\sin^2\alpha + c^2\cos^2\alpha + b^2 - 2bc\cos\alpha = c^2 + b^2 - 2bc\cos\alpha.$$

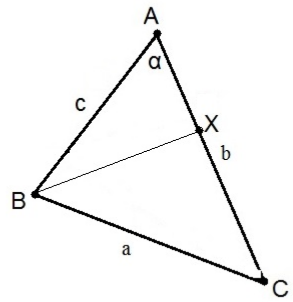

Figure 4.22: Cosine rule - acute angles

Theorem 4.7.3 (Cosine rule)

$$a^2 = c^2 + b^2 - 2bc\cos\alpha.$$

$$b^2 = c^2 + a^2 - 2ac\cos\beta,$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Excercise 4.7.4 *Prove the cosine rule in case of obtuse triangles.*

Hint. Observe that if $\alpha > 90$ then the foot point of the altitude belonging to b is outside from the segment AC. We should use the acute angle $\alpha' = 180-\alpha$ to express AX and BX as above:

$$AX = c\cos\alpha' \quad \text{and} \quad BX = c\sin\alpha'.$$

Therefore

$$a^2 = BX^2 + XC^2 = BX^2 + (CA + AX)^2 = c^2\sin^2\alpha' + (b + c\cos\alpha')^2 =$$

$$c^2 + b^2 + 2bc\cos\alpha' = c^2 + b^2 - 2bc\cos\alpha$$

because of

$$\cos(180-\alpha) = -\cos\alpha.$$

4.7.3 Area of triangles

In what follows we shall use the axioms of measuring area; see section 3.5.
The area of right triangles. Using the area addition axiom we can easily conclude that the area of a right triangle with legs a and b is just ab/2. The

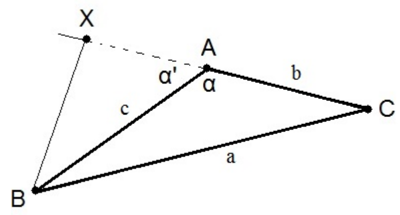

Figure 4.23: Cosine rule - an obtuse angle

altitude belonging to the hypothenuse divides the right triangle into two right triangles. Therefore we have the following formula to compute the area:

$$A = \frac{pm}{2} + \frac{qm}{2} = \frac{(p+q)m}{2} = \frac{cm}{2},$$

where m denotes the altitude (height) belonging to the hypothenuse c. The legs are working as altitudes belonging to each other.
The area of a general triangle can be also computed by the area addition axiom. The basic formulas to compute the area are

$$A = \frac{am_a}{2} = \frac{bm_b}{2} = \frac{cm_c}{2},$$

where m_a, m_b and m_c denote the altitudes belonging to the sides a, b and c, respectively. In practice it is usually hard to measure the altitude (i.e. the distance between a line and a point) in a direct way. Using elementary trigonometry (trigonometry in a right triangle) we can substitute the altitude belonging to a as

$$m_a = b \sin\gamma \quad \text{or} \quad m_a = c \sin\beta.$$

Therefore we have the following trigonometric formulas

$$A = \frac{ab\sin\gamma}{2} = \frac{ac\sin\beta}{2} = \frac{bc\sin\alpha}{2}$$

to compute the area. Another way is given by Héron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s = \frac{a+b+c}{2}$$

is the so-called semiperimeter. The area of a triangle is closely related to the **the radius** r **of the inscribed circle**. Since the bisectors of the interior angles divide the triangle into three parts through the center of the inscribed circle and each of these triangles has altitude r we have that

$$A = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2}$$

and, consequently,

$$r = \frac{A}{s}, \tag{4.1}$$

where s is the semiperimeter.

4.8 Exercises

Excercise 4.8.1 *Two sides of a triangle and the angle enclosed by them are given: 3, 4 and 60 degree in measure.*

- *Find the missing side and angles.*
- *Calculate the area of the triangle.*
- *Calculate the radius of the circumscribed circle of the triangle.*

Hint. See the case SAS.

Excercise 4.8.2 *Three sides of a triangle ABC are given: 6, 8 and 12.*

- *Is it an acute, right or obtuse triangle?*
- *Calculate the area of the triangle.*
- *Calculate the radius of the circumscribed circle of the triangle.*

Hint. See the case SSS. To decide whether ABC is an acute, right or obtuse triangle it is enough to compute the angle opposite to the longest side of length 12:

$$12^2 = 6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cdot \cos\gamma \quad \Rightarrow \quad \cos\gamma = \frac{6^2 + 8^2 - 12^2}{2 \cdot 6 \cdot 8} < 0$$

which means that we have an obtuse angle.

Excercise 4.8.3 *Three sides of a triangle are given: 8, 10 and 12.*

- *Calculate the heights and the area of the triangle.*
- *Calculate the biggest angle of the triangle.*
- *Calculate the radius of the circumscribed circle of the triangle.*

Excercise 4.8.4 *The sides of a triangle are a=5, b=12 and c=13. Calculate the angle opposite to the side c.*

Excercise 4.8.5 *Three sides of a triangle are given: 3, 4 and* $\sqrt{13}$.

- *Find the angles of the triangle.*
- *Calculate the area of the triangle.*
- *Calculate the radius of the circumscribed circle of the triangle.*

Excercise 4.8.6 *Two sides of a triangle are a=8 and b=6, the angle* α *opposite to the side a is 45 degree in measure. Calculate the length of the missing side and find the area of the triangle.*

Hint. See the case SsA.

Excercise 4.8.7 *Two sides of a triangle are a=8 and b=6, the angle* β *opposite to the side b is 45 degree in measure. Calculate the length of the missing side and find the area of the triangle.*

Excercise 4.8.8 *Find the missing quantities in each row of the following table.*

a	b	c	α	β	γ	Area	R	r
12	20				40°			
12				60°	40°			
	20		110°		40°			
	13.4	18.5	110°					
24	25	30						
19	12	9						
8	10	20						
	20	25	60°					
8	10					40		
8	10						5	
			75°	25°	80°			1

Warning. Observe that the cosine rule gives impossible values in case of a=8, b=10 and c=20 (cf. triangle inequalities).

Excercise 4.8.9 *Prove Héron' s formula.*

Hint. Express the cosine of the angle γ from the cosine rule:

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}.$$

Conclude that

$$\sin\gamma = \sqrt{1-\cos^2\gamma} = \sqrt{1-\frac{(a^2+b^2-c^2)^2}{4a^2b^2}}.$$

Use the trigonometric formula to express the area only in terms of the sides of the triangle:

$$A = \frac{ab\sin\gamma}{2} = \frac{ab}{2}\sqrt{1-\frac{(a^2+b^2-c^2)^2}{4a^2b^2}}.$$

Excercise 4.8.10 *Prove that if a polygonal shape has an inscribed circle then the radius can be expressed as the fraction A/s, where A is the area of the polygonal shape and s is the half of its perimeter.*

Chapter 5

Exercises

5.1 Exercises

Excercise 5.1.1 *The radius of the circumscribed circle around a right triangle is 5, one of the legs is 6. What is the area of the triangle?*

Solution. Thales theorem says that the hypothenuse is just $c = 2 \cdot 5 = 10$. Therefore the missing leg must satisfy the equation

$$x^2 + 6^2 = 10^2$$

which means that $x = 8$. The area is

$$A = \frac{6 \cdot 8}{2} = 24.$$

Excercise 5.1.2 *The legs of a right triangle are 6 and 8. How much is the angle of the medians belonging to the legs?*

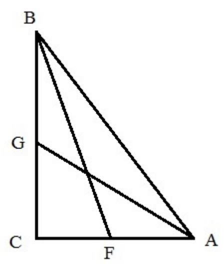

Figure 5.1: Exercise 5.1.2

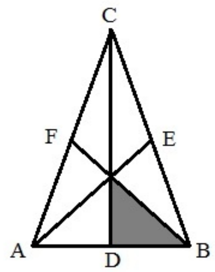

Figure 5.2: Exercise 5.1.3

Solution. Let AC = 6 and BC = 8. At first we compute the lengths of the medians by Pythagorean theorem:

$$3^2 + 8^2 = BF^2 \quad \text{and} \quad 6^2 + 4^2 = AG^2.$$

Therefore BF = $\sqrt{73}$ and AG = $\sqrt{52}$. It is known that the medians intersect each other under the ratio 1 : 2. Therefore we have a triangle constituted by

- the midline FG parallel to the hypothenuse AB=10 (from the Pythagorean theorem)
- (1/3) BF and (1/3) AG.

Using the cosine rule it follows that the angle ω enclosed by the medians satisfies the equation

$$5^2 = \left(\frac{BF}{3}\right)^2 + \left(\frac{AG}{3}\right)^2 - 2\left(\frac{BF}{3}\right)\left(\frac{AG}{3}\right)\cos\omega.$$

Explicitly

$$\cos\omega = \frac{73 + 52 - 225}{2\sqrt{73 \cdot 52}} = -\frac{50}{\sqrt{3796}} \approx -0.811.$$

Therefore $\omega \approx 144.194°$. Usually we consider the acute angle 180 - ω as the angle of medians.

Excercise 5.1.3 *The lengths of the medians of an isosceles triangle are 90, 51 and 51. What is the length of the sides, and the measure of the angles of the triangle?*

Solution. Let AB be the side belonging to the longest median. Since the medians intersect each other by the ratio 1 : 2 we have a right triangle to compute AB/2 because

$$\left(\frac{AB}{2}\right)^2 + 30^2 = 34^2.$$

Therefore AB/2 = 16. Secondly the common length of the missing sides can be computed by Pythagorean theorem again:

$$CB^2 = 16^2 + 90^2 = 8356 \quad \Rightarrow \quad CB = 2\sqrt{2089}.$$

To compute the angles we can use elementary trigonometry in right triangles. For example

$$\tan\frac{\gamma}{2} = \frac{BD}{CD} = \frac{8}{90}$$

and the common measure of the missing angles can be computed as

$$\alpha = \beta = \frac{180-\gamma}{2}.$$

Excercise 5.1.4 *One of the angle of an isosceles triangle is 120 degree, the radius of the inscribed circle is 3. How long are the sides of the triangle?*

Solution. To solve the problem we use the basic cases of similarity. It is clear that an obtuse angle (like 120) can not be repeated inside a triangle which means that the missing angles must be equal to each other. They are 30 degree in measure. Since the angles are given the triangle ABC is determined up to similarity. We can choose one of the side arbitrarily: let (for example) the side AB where the equal angles are lying on is of length 2. The common length x of the missing sides can be determined by the cosine rule

$$2^2 = x^2 + x^2 - 2\cdot x\cdot x\cdot\cos 120$$

i.e. x $= 2/\sqrt{3}$. Now we can compute the radius of the inscribed circle by the formula

$$r = \frac{A}{s},$$

where

$$A = \frac{(2/\sqrt{3})(2/\sqrt{3})\sin 120}{2} = \frac{1}{\sqrt{3}}$$

is the area and

$$s = \frac{AB+AC+CB}{2} = \frac{2+x+x}{2} = 1+x = 1+\frac{2}{\sqrt{3}} \approx 2.1547$$

is the semiperimeter (the half of the perimeter of the triangle). Finally the ratio of the similarity is just r : 3 which means that the real size of the triangle ABC is

$$2 : AB = \frac{r}{3} \quad \text{and} \quad (2/\sqrt{3}) : AC = (2/\sqrt{3}) : BC = \frac{r}{3}.$$

Excercise 5.1.5 *One of the angle of a triangle is 120 degree, one of the sides is just the arithmetic mean of the others. What is the ratio of the sides.*

Solution. Suppose that $a \leq b \leq c$. Then we have to write that

$$b = \frac{a+c}{2},$$

i.e.

$$2 = \frac{a}{b} + \frac{c}{b}.$$

On the other hand c must be opposite to the angle of measure 120 degree. Using the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos 120.$$

Therefore

$$\left(\frac{c}{b}\right)^2 = \left(\frac{a}{b}\right)^2 + 1 + \frac{a}{b}$$

because of cos 120 = - 1/2. We have two equations with two unknown parameters x = a/b and y = c/b :

$$2 = x + y \quad \text{and}$$

$$y^2 = x^2 + 1 + x.$$

Therefore

$$(2-x)^2 = x^2 + 1 + x,$$

$$3 = 5x \quad \Rightarrow \quad x = \frac{3}{5} \quad \text{and} \quad y = \frac{7}{5}.$$

Excercise 5.1.6 *The sides of a triangle have lengths $AC = BC = \sqrt{3}$ and AB=3.*

- *Determine the angles and the area of the triangle.*
- *What is the radius of the inscribed circle.*

Solution. Since it is an isosceles triangle the common measure of the angles lying on the side AB can be easily computed by elementary trigonometry. If D is the midpoint of the segment AB then

$$\cos\alpha = \frac{AD}{AC} = \frac{\sqrt{3}}{2}.$$

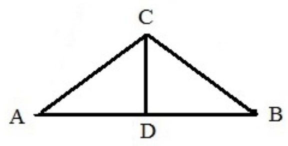

Figure 5.3: Exercise 5.1.6

Therefore $\alpha = \beta = 30$ and $\gamma = 120$. The area is

$$A = \frac{\sqrt{3}\sqrt{3}\sin 120}{2} = \frac{3\sqrt{3}}{4}.$$

To compute the radius of the inscribed circle we need the ratio of the area and the semiperimeter

$$s = \frac{AC + BC + AB}{2} = \frac{2\sqrt{3} + 3}{2}.$$

Finally

$$r = \frac{A}{s} = \frac{3\sqrt{3}}{2(2\sqrt{3} + 3)}.$$

Excercise 5.1.7 *Calculate the length of the sides of an equilateral triangle inscribed in a circle of radius 10. Calculate the area of this triangle and the ratio of the areas of the triangle and the circle.*

Hint. See excercise 4.4.4.

Excercise 5.1.8 *Two sides of a triangle are a=6 and b=3, the angle α opposite to the side a is 60 degree in measure. Calculate the missing side and angles. Find the area of the triangle.*

Solution. The first step is to compute the missing side by the help of the cosine rule:

$$6^2 = 3^2 + c^2 - 2 \cdot 3 \cdot c \cdot \cos 60,$$

$$0 = c^2 - 3c - 27.$$

Therefore

$$c_{12} = \frac{3 \pm \sqrt{9 + 4 \cdot 27}}{2} = \frac{3 \pm \sqrt{117}}{2} = \frac{3 \pm 3\sqrt{13}}{2}.$$

The only possible choice is

$$c = \frac{3+3\sqrt{13}}{2} \approx 6.91.$$

This gives the area immediately by the fomula

$$A = \frac{3 \cdot \frac{3+3\sqrt{13}}{2} \cdot \sin 60}{2} \approx 8.98.$$

One of the missing angle can be computed by the help of the cosine rule again:

$$c^2 = 6^2 + 3^2 - 2 \cdot 6 \cdot 3 \cdot \cos\gamma \quad \Rightarrow \quad \gamma \approx 94.42.$$

Finally $\beta = 180 - \alpha - \gamma$.

Excercise 5.1.9 *The area of a right triangle is 30, the sum of the legs is 17. Calculate the sides of the triangle.*

Solution. Since ab/2=30 and a+b=17

$$60 = ab = a(17-a)$$

which results in a quadratic equation

$$a^2 - 17a + 60 = 0$$

for the unknown length a of one of the legs. We have

$$a_{12} = \frac{17 \pm \sqrt{17^2 - 4 \cdot 60}}{2} = \frac{17 \pm \sqrt{49}}{2} = 12 \text{ or } \; 5.$$

If a = 12 then b = 5 and if a = 5 then b = 12.

Excercise 5.1.10 *Calculate the area of the bright part in the figure.*

Solution. The area is the sum

$$A = 2 \cdot 1 + \frac{4+2}{2}\sqrt{3} + 3\sqrt{3}$$

given by the area of a rectangle, a trapezoid and three equilateral triangles with sides of length 2.

Excercise 5.1.11 *Two sides of a triangle are 8 and 15, its area is 48. How long is the third side?*

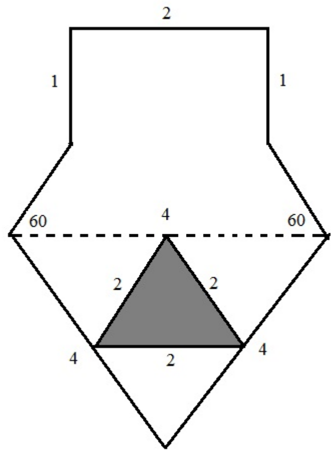

Figure 5.4: Exercise 5.1.10

Solution. Using the trigonometric formula it follows that

$$48 = \frac{8 \cdot 15 \cdot \sin\gamma}{2},$$

i.e. sin γ =0.8. We are going to use the cosine rule to compute the side c opposite to the angle γ. We have that

$$\cos^2\gamma = 1 - \sin^2\gamma = 1 - 0.64 = 0.36$$

and, consequently, cos γ = 0.6 or cos γ = - 0.6 (an acute or an obtuse angle). The cosine rule says that the possible values of the missing side are

$$c_1 = \sqrt{8^2 + 15^2 - 2 \cdot 8 \cdot 15 \cdot 0.6} = 12.04$$

or

$$c_2 = \sqrt{8^2 + 15^2 + 2 \cdot 8 \cdot 15 \cdot 0.6} = 20.8$$

Excercise 5.1.12 *Two sides of a triangle are 8 and 12, the median segment belonging to the third side is 9. What is the area of the triangle?*

Solution. Since

$$8^2 = x^2 + 9^2 - 2 \cdot x \cdot 9 \cdot \cos\omega$$

and

$$12^2 = x^2 + 9^2 - 2 \cdot x \cdot 12 \cdot \cos(180 - \omega)$$

we have that

$$8^2 + 12^2 = 2x^2 + 2 \cdot 9^2$$

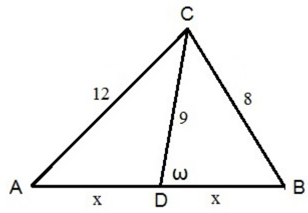

Figure 5.5: Exercise 5.1.12

because of

$$\cos\omega = -\cos(180 - \omega).$$

Therefore $x = \sqrt{23}$. On the other hand

$$\cos\omega = \frac{23 + 81 - 64}{2 \cdot \sqrt{23} \cdot 9} \approx 0.46 > 0,$$

which means that

$$\sin\omega = \sqrt{1 - 0.46^2} \approx 0.88.$$

The area of the triangle ABC is obviously the sum of the areas of triangles ADC and CDB:

$$A = \frac{x \cdot 9 \cdot \sin\omega}{2} + \frac{x \cdot 9 \cdot \sin(180 - \omega)}{2} = 2\frac{x \cdot 9 \cdot \sin\omega}{2} \approx 37.98$$

because of

$$\sin\omega = \sin(180 - \omega).$$

Chapter 6

Classical problems I

"The great book of Nature lies ever open before our eyes and the true philosophy is written in it ... But we cannot read it unless we have first learned the language and the characters in which it is written ... It is written in mathematical language and the characters are triangles, circles and other geometric figures..." (Galileo Galilei)

6.1 The problem of the tunnel

Problem [4]: Due to the increasing population a certain city of ancient Greece found its water supply insufficient, so that water had to be channeled in from source in the nearby mountains. And since, unfortunately, a large hill intervened, there was no alternative to tunneling. Working from both sides of the hill, the tunnelers met in the middle as planned. How did the planners determine the correct direction to ensure that the crews would meet?
Solution. Since the points A (city) and B (source) cannot be connected directly we have to connect them indirectly. Let C be a point from which both A and B are observable. By measuring the distances AC, BC and the angle γ we can easily find the angles α and β by the help of the cosine rule.
Inputs: CA, CB and γ
1. Compute

$$AB = \sqrt{CA^2 + CB^2 - 2 \cdot CA \cdot CB \cdot \cos\gamma}.$$

2. Compute

$$\cos\alpha = \frac{AB^2 + AC^2 - CB^2}{2 \cdot AC \cdot AB} \quad \text{and} \quad \beta = 180 - (\alpha + \gamma).$$

Excercise 6.1.1 *Find the solution if*

$$AC = 2 \text{ Miles}, \quad BC = 3 \text{ Miles} \quad \text{and} \quad \gamma = 53^\circ.$$

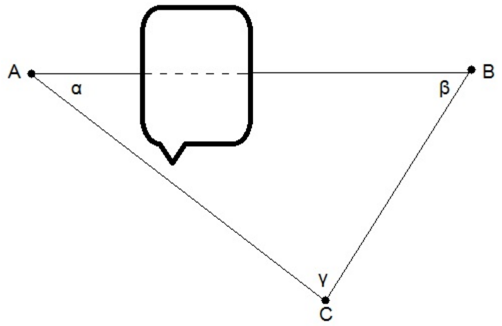

Figure 6.1: The problem of the tunnel - one observer

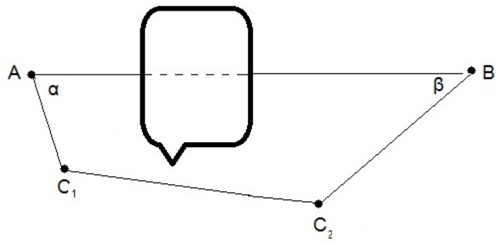

Figure 6.2: Two observers

Excercise 6.1.2 *Can you generalize the method by using more than one observers?*

6.2 How to measure an unreachable distance

In many practical situations the direct measuring of distances is impossible; see for example astronomical measurements or navigation problems. Instead of distances we can measure visibility angles. The following problem is related to the determination of an unreachable distance by measuring visibility angles and a given base line.
Problem: Let the distance of the segment AB be given and suppose that we know

- the visibility angle α of BD from A,
- the visibility angle β of AC from B,
- the visibility angle γ of CD from A,
- the visibility angle δ of CD from B.

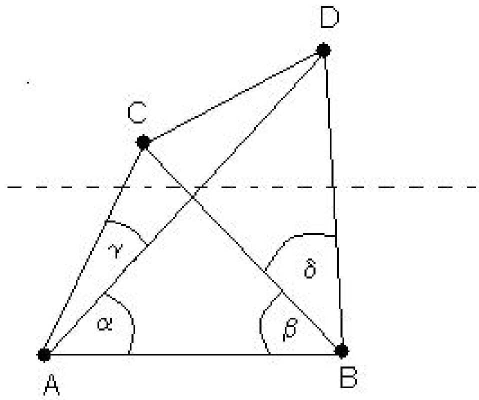

Figure 6.3: Unreachable distance

How can we calculate the distance CD?
Solution: The sine rule in the triangle ABC shows that

$$\frac{AB}{AC} = \frac{\sin(\pi - (\alpha + \beta + \gamma))}{\sin \beta} = \frac{\sin(\alpha + \beta + \gamma)}{\sin \beta}$$

and thus

$$AC = \frac{\sin \beta}{\sin(\alpha + \beta + \gamma)} AB.$$

In a similar way

$$\frac{AD}{AB} = \frac{\sin(\beta + \delta)}{\sin(\pi - (\alpha + \beta + \delta))} = \frac{\sin(\beta + \delta)}{\sin(\alpha + \beta + \delta)}.$$

Therefore

$$AD = \frac{\sin(\beta + \delta)}{\sin(\alpha + \beta + \delta)} AB.$$

Using the cosine rule in the triangle ADC

$$CD^2 = AC^2 + AD^2 - 2 \cdot AC \cdot AD \cdot \cos \gamma.$$

6.3 How far away is the Moon

Problem [4]: How are we to measure the distance of the Moon from the Earth?
Solution. Since the distance between the Earth and the Moon cannot be measured directly it must be measured indirectly. The calculation needs

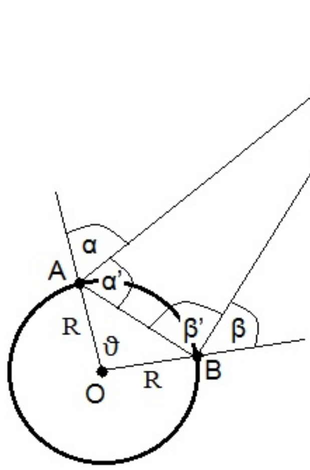

Figure 6.4: How far away is the Moon

accessible distances like the distance between the observers A and B along the perimeter of the Earth. They measure simultaneously the inclination angles of the segments AM and BM to the vertical lines of their positions. If we know the radius of the Earth then we can calculate the distance OM in the following way.
Inputs: the arclength from A to B, α, β and the radius R of the Earth.
1. Compute the central angle θ by the formula

$$\frac{\theta \text{ (degree)}}{360} = \frac{\text{the arclength from A to B}}{2R\pi}.$$

Using that AOB is an isosceles triangle

$$\angle OAB = \angle OBA = \frac{180 - \theta}{2}.$$

2. Compute α' and β' by the formulas

$$\alpha' = 180 - \alpha - \frac{180 - \theta}{2} \quad \text{and} \quad \beta' = 180 - \beta - \frac{180 - \theta}{2}.$$

3. Compute the length of the segment AB by using the cosine rule in the isosceles triangle AOB:

$$AB^2 = 2R^2 - 2R^2 \cos\theta.$$

From now on the triangle AMB is uniquely determined up to congruence because we know one side and the angles lying on this side.

4. Compute AM by using the sine rule in the triangle AMB.
5. Compute OM by using the cosine rule in the triangle OAM.

Remark One obstacle remains; the Moon moves relatively to the Earth. If the observers measure the angles in different times then we are confronted with a quadrilateral instead of a triangle and the method has failed. For triangulation the angles must be measured simultaneously. It is clear that if the observer positions are too close to each other then AM and BM are almost parallel. For accurate measures almost parallel lines must be avoided. But how is the measurer at B to know when the measurer at A is measuring? The ancient Greek's answer to the problem is based on a simple observation. Since both measurers observe the Moon the best is to wait for a signal by the observed object. In other words measurers had to wait for some happening on the Moon visible from Earth. What happening? A lunar eclipse. The eclipse provides four distinct events which are observable simultaneously from A and B:

- the beginning of the Moon's entry to the Earth's shadow,
- the completion of the Moon's entry to the Earth's shadow,
- the beginning of the Moon's emergence from the Earth's shadow,
- the completion of the Moon's emergence from the Earth's shadow.

Chapter 7

Quadrilaterals

In Euclidean plane geometry quadrilaterals mean polygons with four sides and four vertices. Quadrilaterals (or polygons) are tipically built from triangles which may have only common vertices or sides. Especially the quadrilaterals are the union of two triangles having exactly one common side. Sometimes one admits the union of two triangles with exactly one common vertex to be a quadrilateral but these self-intersecting or crossed cases will not be important for us. We restrict ourselves to the case of simple (not self-intersecting) polygons.

7.1 General observations

Theorem 7.1.1 *The sum of the interior angles of a quadrilateral is just 360 degree in measure.*

Corollary 7.1.2 *Any quadrilateral has at most one concave interior angle. Quadrilaterals having concave angles are called concave quadrilaterals. Otherwise the quadrilateral is convex.*

In what follows we summarize some types of quadrilaterals. The most important special class is formed by parallelograms because of their central role in the development of the Euclidean geometry. After declaring the axioms of Euclidean geometry we can prove lots of equivalent characterization for a convex quadrilateral to be a parallelogram. Some of them is crucial to prove the parallel lines intersecting theorem 3.6.8.

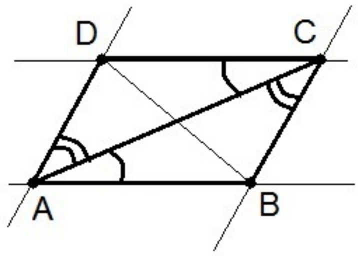

Figure 7.1: Characterization of parallelograms

7.2 Parallelograms

Definition A parallelogram is a quadrilateral with two pairs of parallel sides. The most important special cases are

- squares (all the sides and all the interior angles of the parallelogram are equal),
- rectangle (all the interior angles of the parallelogram are equal),
- rhombus (all the sides of the parallelogram are equal).

Theorem 7.2.1 *The quadrilateral ABCD is a paralellogram if and only if one of the following conditions is satisfied.*

- *The opposite sides are of equal length.*
- *the opposite angles are equal.*
- *One of the pairs of the opposite sides are of equal length and parallel.*
- *It is symmetric with respect to the intersection of the diagonals.*
- *The diagonals bisect each other.*

Proof If ABCD is a parallelogram then ASA implies that any diagonal divides the parallelogram into congruent triangles. Therefore both the opposite sides and the opposite angles are equal. On the other hand the diagonals bisect each other because they divide the parallelogram into four triangles which are pairwise congruent.

The proofs of the converse statements are also based on the cases of congruence of triangles and the characterization of parallelism. If the opposite

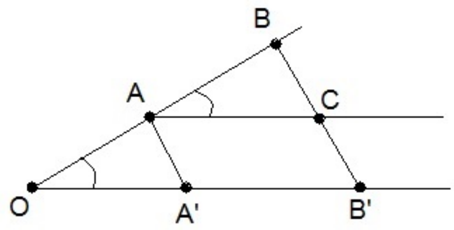

Figure 7.2: Parallel lines intersecting theorem: the first step

sides are of equal length then SSS implies that any diagonal divides the quadrilateral into congruent triangles. Therefore the corresponding angles have the same measure. In the sense of the characterization of parallelism we have that the opposite sides are parallel.

Since the sum of the interior angles is 360 degree in measure the equality of the opposite angles means that the sum of angles lying on the same side is 180 degree. The characterization of parallelism says that the opposite sides are parallel.

If one of the pairs of the opposite sides are of equal length and parallel then the characterization of parallelism and SAS implies that any diagonal divides the quadrilateral into congruent triangles. The proof can be finished as above.

The last two statements are obviously equivalent to each other. Therefore it is enough to discuss one of them. The symmetry with respect to the intersection of the diagonals obviously implies that the opposite sides are parallel.

As an application we prove the parallel lines intersecting theorem 3.6.8
1st step We can conclude that the parallel projections of congruent segments are congruent: if OA=AB then the triangles OAA' and ABC are congruent and AC=A'B'by theorem 7.2.1. Therefore OA:OB=OA':OB'=1:2.
2nd step In case of not necessarily congruent segments OA and AB let n be an arbitrary integer and divide the segment OA into n equal parts by the points

$$X_0 = O,\ X_1,\ \ldots,\ X_n = A.$$

Continue the proccess of copying the segment of length OA/n from A into the direction of B as far as we have

$$k\frac{OA}{n} \leq OB \leq (k+1)\frac{OA}{n}.$$

Using the first step the parallel projections

$$X_0' = O,\ X_1', \dots,\ X_n' = A'$$

gives the divison of OA' into n equal parts. On the other hand

$$k\frac{OA'}{n} \leq OB' \leq (k+1)\frac{OA'}{n}.$$

Therefore

$$\frac{k}{n} \leq \frac{OB}{OA} \leq \frac{k+1}{n} \quad \text{and} \quad \frac{k}{n} \leq \frac{OB'}{OA'} \leq \frac{k+1}{n}$$

which means that

$$\left|\frac{OB}{OA} - \frac{OB'}{OA'}\right| \leq \frac{1}{n}$$

for any integer $n \in \mathbf{N}$. Taking the limit $n \to \infty$ we have that

$$\frac{OB}{OA} = \frac{OB'}{OA'}$$

as was to be stated.

7.3 Special classes of quadrilaterals

Definition A quadrilateral is called trapezoid if it has at least one pair of opposite sides which are parallel. An isosceles trapezoid or symmetric trapezoid have equal base angles in measure.

Definition A quadrilateral is called kite if two pairs of adjacent sides are of equal length.

Excercise 7.3.1 *Prove that in case of a kite the angles between the two pairs of equal sides are equal in measure and the diagonals are perpendicular.*

Solution. From the definition of a kite one of the diagonal divides the kite into congruent triangles by the basic case SSS of the congruence. The perpendicularity follows directly from the geometric characterization of the perpendicular bisector.

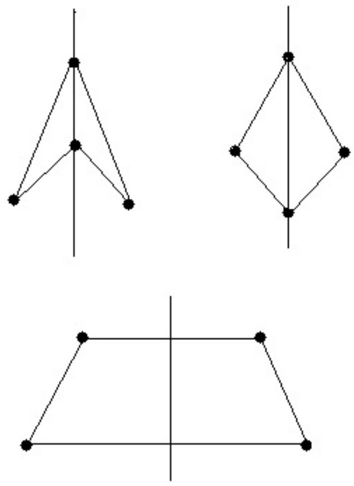

Figure 7.3: Axially symmetric quadrilaterals

7.3.1 Symmetries

Suppose that the quadrilateral ABCD has an axial symmetry, i.e. we have a line such that the quadrilateral is invariant under the reflection about this line. Since any vertex must be transformed into another one we have that

$$k + l = 4,$$

where the number k of the vertices which are not on the axis of symmetry must be even. The possible cases are k=0, 2 or 4:

$$0 + 4 = 4, \quad 2 + 2 = 4 \quad \text{and} \quad 4 + 0 = 4.$$

The case $k = 0$ is obviously impossible. If we have 2 vertices on the axis of symmetry then the quadrilateral must be a convex or concave kite. Otherwise it is a symmetric trapezium.

Definition Rotational symmetry of order n with respect to a particular point means that rotations by angle 360/n does not change the object.

Excercise 7.3.2 *Prove that quadrilaterals with symmetry of order 2 are parallelograms*

Excercise 7.3.3 *Prove that quadrilaterals with symmetry of order 4 are squares.*

7.3.2 Area

The area of a polygonal region can be computed as the sum of the areas of subtriangles. In what follows we consider some special cases with explicit formulas. They are easy consequences of the triangle decomposition. The area of a

- parallelogram is the product of one of the parallel bases and the altitude belonging to this base. The trigonometric version of the formula is

  $$A = ab\sin\alpha.$$

  This follows easily from the division of the parallelogram into congruent triangles by one of the diagonals.

- trapezoid can be computed as

  $$A = \frac{a+c}{2}m,$$

  where a and c are the lengths of the parallel bases and m is the altitude of the trapezoid. One can introduce the mid-line segment for trapezoids on the model of triangles in the same way: the midline of a trapezoid is just the line segment joining the midpoints of the legs. Using the division of the trapezoid into triangles by one of the diagonals it can be easily seen that the length of the midline of a trapezoid is just the arithmetic mean of the lengths of the parallel bases. Another way to conclude the area formula is to put two congruent copies of the trapezoid next to each other in such a way that they form a parallelogram. In terms of geometric transformation it can be realized by a central reflection about the midpoint of one of the legs.

- convex quadrilateral is just

  $$A = \frac{ef\sin\omega}{2},$$

  where e and f are the lengths of the diagonals and ω is the angle enclosed by them.

Excercise 7.3.4 *Prove the area formula of a parallelogram.*

Excercise 7.3.5 *Prove the area formula of a trapezoid.*

Excercise 7.3.6 *Prove the area formula of a kite.*

Theorem 7.3.7 *Let ABCD be a convex quadrilateral. The area can be computed as*

$$A = \frac{AC \cdot BD \cdot \sin\omega}{2},$$

where ω is the angle enclosed by the diagonals AC and BD.

Proof Let E be the point where the diagonals meet at. The triangles AEB, BEC, CED and DEA covers the quadrilateral such that we have only common vertices and edges. Therefore the area can be computed as the sum

$$A = A_{AEB} + A_{BEC} + A_{CED} + A_{DEA}.$$

Since the angles at the common vertex E are alternately ω and 180 - ω we can conclude that

$$A = \frac{AE \cdot EB + EB \cdot EC + EC \cdot ED + ED \cdot EA}{2} \sin\omega =$$

$$\frac{AC \cdot BD \cdot \sin\omega}{2},$$

where ω is the angle enclosed by the diagonals AC and BD.

Excercise 7.3.8 *Let ABCD be a convex quadrilateral. Find the point in the plane to minimize the sum*

$$XA + XB + XC + XD.$$

Solution. By the triangle inequality the point X must be the intersection of the diagonals AC and BD.

Chapter 8

Exercises

8.1 Exercises

Excercise 8.1.1 *Three sides of a symmetrical trapezoid are of length 10. The fourth side has length 20. Calculate the angles and the area of the trapezoid.*

Solution. Using the symmetry we can easily change the trapezoid into a rectangle. Let ABCD be a symmetrical trapezoid having sides of length AB = 20, BC = AD = 10 and CD = 10. The orthogonal projection C'D' of CD onto the longer base AB is of length 10 again. Therefore AD'=5 and BC' = 5 because of the symmetry. From the right triangle AD'D we have that the height is

$$DD' = \sqrt{10^2 - 5^2} = \sqrt{75}.$$

The sides of the rectangle is just a = AB - BC' = 20 - 5 = 15 and b = $\sqrt{75}$. The area is

$$A = 15\sqrt{75} = 75\sqrt{3}.$$

The angles are alternately 60 and 120 degree in measure.

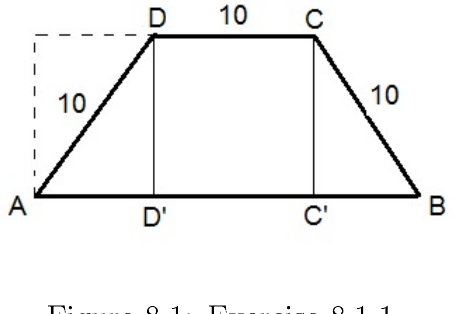

Figure 8.1: Exercise 8.1.1

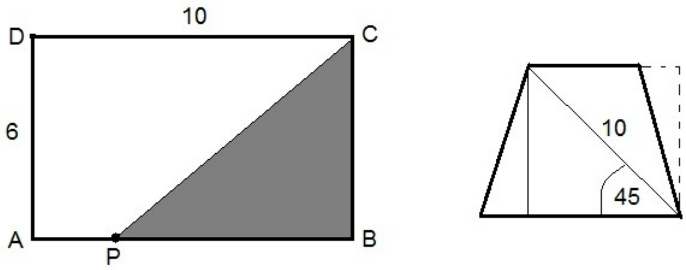

Figure 8.2: Exercises 8.1.2 and 8.1.3

Excercise 8.1.2 *The sides AB and BC of rectangle ABCD are 10 and 6. What is the distance of a point P on the side AB from the vertex D if*

$$AP + PC = 12.$$

Solution. From the right triangle PBC

$$PB^2 + 6^2 = PC^2,$$

i.e.

$$(10 - AP)^2 + 36 = PC^2.$$

Since AP+PC=12

$$(10 - AP)^2 + 36 = (12 - AP)^2$$

and

$$4 \cdot AP = 8 \quad \Rightarrow \quad AP = 2.$$

Finally

$$PD^2 = 6^2 + 2^2 \quad \Rightarrow \quad PD = \sqrt{40}.$$

Excercise 8.1.3 *In a symmetrical trapezoid the inclination angle of the diagonal to the longer parallel base is 45 degree, the length of the diagonal is 10. What is the area of the trapezoid?*

Solution. Using the symmetry we can easily change the trapezoid into a rectangle. Since the diagonal bisects the angles of the rectangle it must be a square. The common length x of the sides can be derived from the Pythagorean theorem

$$x^2 + x^2 = 10^2 \quad \Rightarrow \quad x = \sqrt{50} = 5\sqrt{2}$$

and the area is $x^2 = 50$.

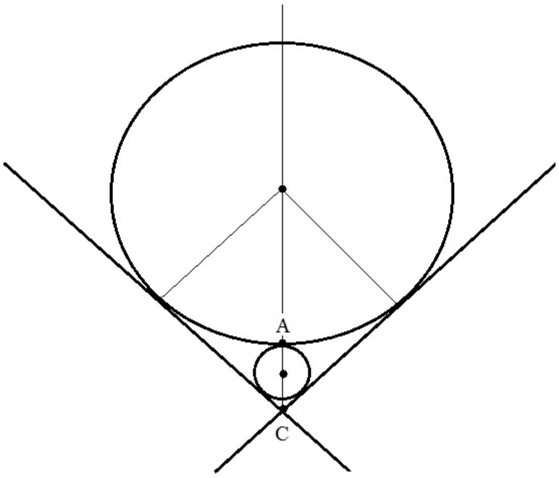

Figure 8.3: Exercise 8.1.4

Excercise 8.1.4 *The side of the square ABCD is 10. Calculate the radius of the circle which passes through the point A, and touches the sides BC and CD.*

Solution. Divide the problem into two parts. At first let us concentrate on the circles touching the sides BC and CD. The center of such a circle must be on the diagonal CA of the square. Let x be the distance of the center from C. Pythagorean theorem says that

$$x^2 = r_x^2 + r_x^2, \quad \text{i.e.} \quad x = r_x\sqrt{2},$$

where r_x is the radius of the circle. It is labelled by the coordinate x. The point A has coordinate $x_A = 10\sqrt{2}$. The circle passes through A if and only if

$$|x - x_A| = r_x, \quad \text{i.e.} \quad x - 10\sqrt{2} = r_x \quad \text{or} \quad 10\sqrt{2} - x = r_x.$$

We have

$$r_x\sqrt{2} - 10\sqrt{2} = r_x \quad \text{or} \quad 10\sqrt{2} - r_x\sqrt{2} = r_x.$$

Therefore

$$r_x = \frac{10\sqrt{2}}{\sqrt{2}-1} \quad \text{or} \quad r_x = \frac{10\sqrt{2}}{\sqrt{2}+1}.$$

Excercise 8.1.5 *In rectangle ABCD side AB is three times longer then BC. The distance of an interior point P from the vertices B, A and D is* $PB = 4\sqrt{2}$, $PA = \sqrt{2}$ *and* $PD = 2$. *What is the area of the rectangle.*

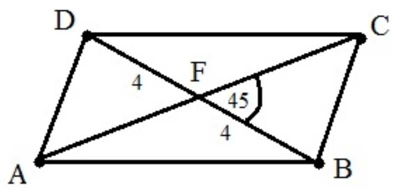

Figure 8.4: Exercise 8.1.6

For the solution see Exercise 1.10.1 in section 1.10, Chapter 1.

Excercise 8.1.6 *The shortest diagonal of a parallelogram has length 8, the angle of the diagonals is 45 degree, and its area is 40. Calculate the perimeter of the parallelogram.*

Solution. The area must be the sum of the areas of triangles AFB, BFC, CFD and DFA. They are pairwise congruent and we also know that the diagonals of a parallelogram bisect each other. If x=AF=FC then

$$40 = 2\frac{x \cdot 4 \cdot \sin 45}{2} + 2\frac{4 \cdot x \cdot \sin(180-45)}{2}.$$

Since sin 45 = sin (180 - 45) it follows that x = $10/\sqrt{2}$. Using the cosine rule in the triangle BFC

$$BC^2 = 4^2 + x^2 - 2 \cdot 4 \cdot x \cdot \cos 45 = 16 + 50 - 40 = 26.$$

In a similar way

$$AB^2 = 4^2 + x^2 - 2 \cdot 4 \cdot x \cdot \cos(180-45) = 4^2 + x^2 + 2 \cdot 4 \cdot x \cdot \cos 45 = 106.$$

Therefore BC≈5.09, AB≈10.29 and the perimeter is P≈30.76.

Excercise 8.1.7 *The length of the mid - line of a symmetric trapezium is 10, the diagonals are perpendicular to each other. What is the area of the trapezium.*

Solution. Because of the symmetry

$$x = DE = CE \text{ and } y = AE = BE.$$

The parallel bases can be computed by the Pythagorean theorem:

$$CD = \sqrt{2}x \text{ and } AB = \sqrt{2}y.$$

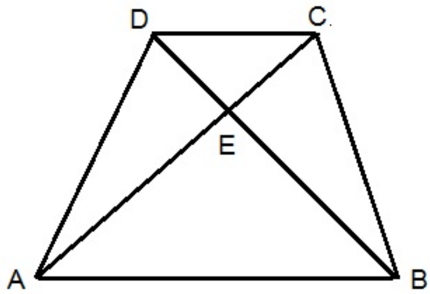

Figure 8.5: Exercise 8.1.7

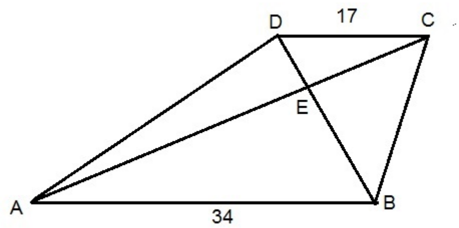

Figure 8.6: Exercise 8.1.8

Since the length of the mid - line is 10 we have that

$$10 = \frac{\sqrt{2}x + \sqrt{2}y}{2}.$$

From here

$$x + y = \frac{20}{\sqrt{2}}$$

and the area is

$$A = \frac{x^2}{2} + \frac{y^2}{2} + 2\frac{xy}{2} = \frac{(x+y)^2}{2} = 100.$$

Excercise 8.1.8 *The diagonals of a trapezium are perpendicular. The lengths of the parallel sides are 17 and 34, one of the legs is* $\sqrt{964}$. *How long is the second leg, what is the area, and the height of the trapezium.*

Solution. The triangles AEB and CED are similar. The ratio of the similarity is just 2=34/17. Therefore

$$AE = 2 \cdot EC \quad \text{and} \quad BE = 2 \cdot DE.$$

Suppose that

$$BC = \sqrt{964}.$$

By Pythagorean theorem in the right triangles BEC and CED:

$$BE^2 + CE^2 = 964 \quad \Rightarrow \quad 4 \cdot DE^2 + CE^2 = 964,$$

$$CE^2 + DE^2 = 17^2.$$

Therefore

$$3 \cdot DE^2 = 675 \quad \Rightarrow \quad DE = 15$$

and CE=8. This means that AE=16 and BE=30. The second leg is

$$AD = \sqrt{AE^2 + DE^2} = \sqrt{481}.$$

The area is

$$A = \frac{AE \cdot BE}{2} + \frac{BE \cdot CE}{2} + \frac{CE \cdot DE}{2} + \frac{DE \cdot AE}{2} = 540.$$

Since

$$540 = \frac{34 + 17}{2} m$$

the height of the trapezium is m = 1080/51 = 360/17.

Excercise 8.1.9 *The parallel bases of a symmetrical trapezoid are 10 and 20. The height is 4.*

- *Calculate the area of the trapezoid.*
- *Calculate the angles of the trapezoid.*

Excercise 8.1.10 *The longest base of a symmetrical trapezoid is 20, the length of the legs is 5, the height is 4.*

- *Calculate the area of the trapezoid.*
- *Calculate the angles of the trapezoid.*

Excercise 8.1.11 *In kite ABCD we know that $AB = BC = 2$ and $CD = DA$. At vertex A the angle is 120 degree, and at D the angle is 60 degree. Calculate the unknown angles, sides and diagonals of the kite and furthermore, the radius of the inscribed circle.*

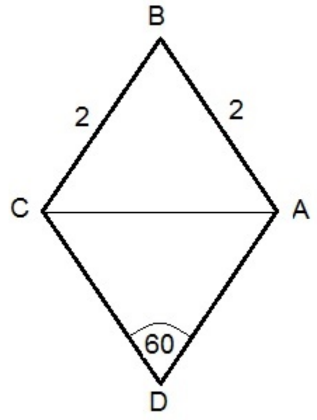

Figure 8.7: Exercise 8.1.11

Solution. Since we have equal adjacent sides it follows that both ACD and ABC are equilateral triangles. Therefore we have a rhombus with sides of length 2. The angles are alternately 60 and 120 degree in measure. Since ABC is an equilateral triangle the diagonal AC is 2 too. To compute the length of the longer diagonal we can use the cosine rule

$$BD^2 = 2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cdot \cos 120 = 12.$$

The radius of the inscribed circle is just

$$r = \frac{A}{s},$$

where the semiperimeter s is 4. To compute A we use the trigonometric formula for the area

$$A = 2 \cdot 2 \cdot \sin 60 = 2\sqrt{3}.$$

Therefore

$$r = \frac{\sqrt{3}}{2}.$$

Excercise 8.1.12 *The perimeter of the rhombus is 40, its area is 96. What are the angles, sides, and diagonals of the rhombus.*

Solution. If a denotes the common length of the sides of the rhombus then $40 = 4a$, i.e. $a = 10$. To compute the area we can write that

$$96 = 10^2 \sin \alpha$$

and, consequently $\sin\alpha = 0.96$. This means that the angles are $\alpha_1 \approx 73.74$ and $\alpha_2 = 180 - \alpha_1 \approx 106.26$. Using the cosine rule systematically the length of the diagonals are

$$d_1 = \sqrt{10^2 + 10^2 - 2 \cdot 10 \cdot 10 \cdot \cos\alpha_1}$$

$$d_2 = \sqrt{10^2 + 10^2 - 2 \cdot 10 \cdot 10 \cdot \cos\alpha_2} = \sqrt{10^2 + 10^2 + 2 \cdot 10 \cdot 10 \cdot \cos\alpha_1}.$$

Since α_1 is an acute angle

$$\cos\alpha_1 = \sqrt{1 - 0.96^2} = 0.28.$$

Therefore

$$d_1 = \sqrt{144} = 12 \ \text{ and } \ d_2 = \sqrt{256} = 16.$$

Excercise 8.1.13 *The length of the two diagonals of a rhombus are given: 6 and 12.*

- *Calculate the area of the rhombus!*
- *Calculate the length of the sides of the rhombus!*
- *Calculate the angles of the rhombus!*

Excercise 8.1.14 *The longer diagonal of a rhombus is given: 12, and one of the angle of the rhombus is 60 degree in measure.*

- *Calculate the area of the rhombus.*
- *Calculate the length of the sides of the rhombus.*

Excercise 8.1.15 *The length of the side of a rhombus is just the geometric mean of the diagonals. What is the ratio of the two diagonals.*

Solution. Let e and f be the lengths of the diagonals. The diagonals of a rhombus are perpendicular to each other because of the geometric characterization of the perpendicular bisector. Therefore

$$a^2 = \frac{e^2}{4} + \frac{f^2}{4}.$$

On the other hand $a^2 = ef$. This means that

$$ef = \frac{e^2}{4} + \frac{f^2}{4}$$

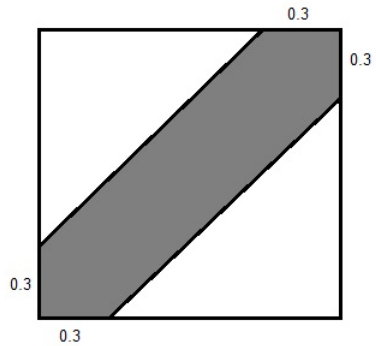

Figure 8.8: Geometric probability

and

$$4 = x + \frac{1}{x},$$

where x = e : f. Therefore

$$0 = x^2 - 4x + 1$$

and

$$x_{12} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

Excercise 8.1.16 *Prove that*

$$2 + \sqrt{3} = \frac{1}{2 - \sqrt{3}}.$$

Excercise 8.1.17 *Two persons are going to meet within one hour. They agree that any of them will wait for the other at most 20 minutes. What is the probability of the meeting.*

Solution. First of all we should find a mathematical model of the problem. Let x and y be the arriving time of persons A and B, respectively. These are randomly chosen from the interval [0,1]. In other words any event correspond to a point P(x,y) of the square with sides of unit length. A and B meet if and only if the absolute value of the difference y - x is less or equal than 0.3 hour=20 minutes. We are going to compute what is the area of the set of points satisfying the inequalities

$$-0.3 \leq y - x \leq 0.3.$$

These points represent successful outcomes. Using the area which is missing:

$$\text{The area of successfull outcomes} = 1 - \frac{(1-0.3)^2}{2} - \frac{(1-0.3)^2}{2} = 0.51.$$

Therefore the probability is

$$P = \frac{\text{the area of successful outcomes}}{\text{the area of all the outcomes}} = 0.51.$$

Chapter 9

Polygons

9.1 Polygons

In general polygons are plane figures bounded by a finite chain of straight line segments. Since they are typically investigated by using a triangle decomposition we agree that any triangle is a polygon.

Definition A simple closed polygon is a finite union of line segments

$$A_1A_2, A_2A_3, \ldots, A_nA_{n+1},$$

where A_1, ..., A_n are distinct points in the plane, $A_1 = A_{n+1}$ and the line segments have no other points in common except their endpoints, each of which lies on two segments.

The boundary of such a shape is a chain of straight line segments. The positions where the chain is broken at are called vertices. The straight line segment between two adjacent vertices is called a side/edge[1] of the polygon. The polygon is called convex if there are no concave interior angles, i.e. all the interior angles are of measure less than 180 degree.

Theorem 9.1.1 *The sum of interior angles of a polygon having n sides is*

$$(n-2)\pi.$$

Proof In case of convex polygons the result follows easily from the triangle decomposition. Otherwise the statement can be proved by induction on the number of concave interior angles.

[1] The name edge is typically used in graph theory.

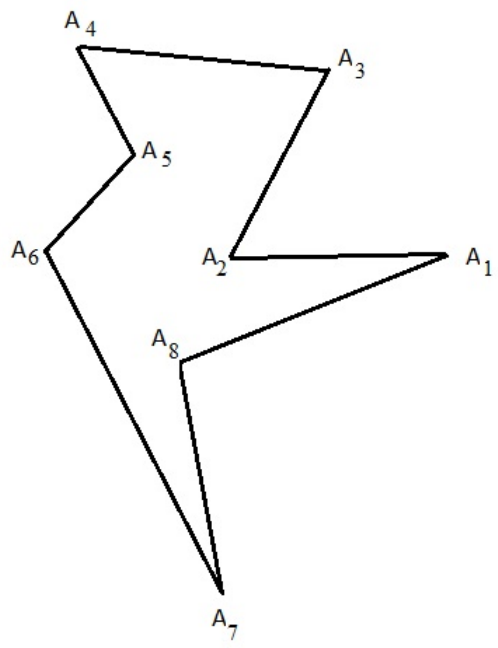

Figure 9.1: A polygon

The sum of the diagonals. Suppose that the polygon has n > 3 vertices. If A is one of them then we have n - 1 vertices left to join with A. These give two sides (adjacent vertices) and n - 3 diagonals. Therefore the sum of the different diagonals of the polygon having n vertices is

$$\frac{n(n-3)}{2}$$

because each diagonal belongs to exactly two vertices.
The area of a polygon can be computed as the sum of the areas of triangles constituting the polygon.

One of the most important special classes of polygons is formed by regular polygons. They are automatically inscribed in a circle in the following way. Let a circle be given and divide the perimeter into n equal parts by the points

$$P_1, P_2, \ldots, P_n,$$

where n is geater or equal than 3. Each of the chords

$$P_1P_2, P_2P_3, \ldots, P_nP_1$$

belong to the central angle 360/n degree in measure. They are the sides of the regular n–gon inscribed in the given circle. The size depends on the radius of the circle.

Chapter 10

Circles

Definition Let a point O in the plane be given. If r is a positive real number then the set of points having distance r from O is called a circle. The point O is the center and r is the radius of the circle. A disk means the set of points having distance at most r from the given point O. The circle is the boundary of the disk with the same center and radius.

The most important problems related to a circle is the problem of tangent lines and the problem of area.

10.1 Tangent lines

Let a line l be given. The definition of the circle suggests us to classify the points of the line by the distance from the center of the circle. At first suppose that l does not pass the center O and consider the line e passing through O such that e is perpendicular to l. Using Pythagorean theorem it can be easily seen that the foot F has the smallest distance from the center among the points of l. Therefore if

- OF=r then the line has exactly one common point with the circle and all the other points are external. In this case we say that the line is tangent to the circle at the point F of tangency.

- OF $<$ r then the line intersect the circle at exactly two points. In this case we speak about a secant line.

- OF $>$ r then l has no points in common with the circle.

The discussion of the lines passing through the center is obvious.

Definition The line l is tangent to a circle if they have exactly one common point and all the other points on the line are external.

Remark Although the condition *all the other points on the line are external* is redundant in case of tangent lines to a circle but not in general as the case of conic sections (ellipse, hyperbola, parabola) shows. To construct tangent lines in general one need taking the limit again. The tangent line is the limit position of chords passing through a given point of the curve.

Excercise 10.1.1 *Find the tangent lines to the parabola given by the graph of the function*

$$f(x) = x^2.$$

Solution. Let $x = 1$ be fixed and consider the chord passing through the points

$$(1, 1) \text{ and } (x, x^2).$$

The slope

$$m(x) = \frac{x^2 - 1}{x - 1}$$

is obviously depend on x. What happens if x tends to 1. Since the division by zero is impossible we have to eliminate the term x - 1. Since

$$m(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$$

the slope at the limit position must be 2. The equation of the tangent line at x=1 is

$$y - f(1) = 2 \cdot (x - 1) \quad \Rightarrow \quad y = 2x - 1.$$

Theorem 10.1.2 *If a line is tangent to a circle then it is perpendicular to the radius drawn to the point of tangency.*

The construction of the **tangent line** to a circle **from a given external point** is based on Thales' theorem. Suppose that P is an external point and F is the point of tangency to a circle with center O. Since PFO is a right triangle the point F must be on the perimeter of the circle drawn from the midpoint of the segment OP with radius OP/2.

Theorem 10.1.3 *Let a circle with center O be given and suppose that P is an external point. The tangent lines from P to the circle can be constructed as follows:*

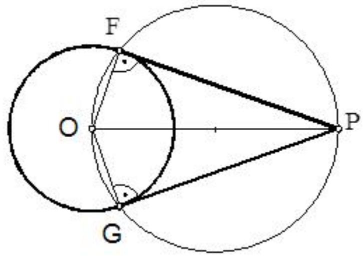

Figure 10.1: Tangent segments from an external point

- *draw a circle with radius $r = OP/2$ around the midpoint of OP,*
- *the circle constructed in the first step meets the given circle at two points F and G,*
- *FP and GP are tangent segments to the given circle.*

To compute the common length of the tangent segments PF and PG we can use Pythagorean theorem:

$$PF^2 + r^2 = OP^2.$$

Corollary 10.1.4 *The tangent segments passing through a given external point are of the same length.*

For two circles there are generally four distinct segments that are tangent to both of them. If the centers are separated then we speak about internal bitangent segments. Otherwise we have external bitangent segments. If the circles

- are outside each other then we have two external and two internal bitangent segments symmetrically about the line of the centers.
- are tangent to each other from outside then we have a common (internal) tangent line at the contact point and two external bitangent segments symmetrically about the line of the centers.
- intersect each other then we have no inner bitangent segments or lines.
- are tangent to each other from inside then we have only a common (external) tangent line at the contact point.

Excercise 10.1.5 *How to construct common bitangent segments to two circles?*

Solution. For the generic cases see figures 10.2 and 10.3.

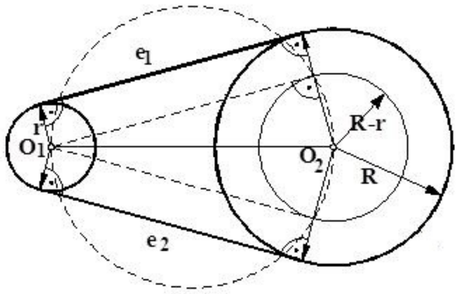

Figure 10.2: External bitangent segments

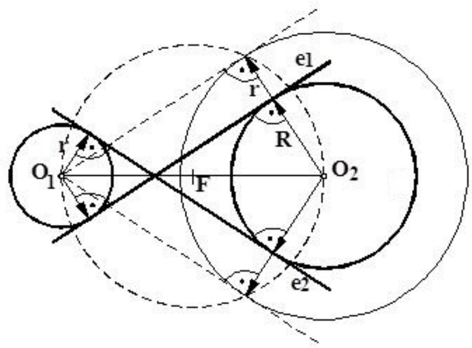

Figure 10.3: Internal bitangent segments

10.2 Tangential and cyclic quadrilaterals

Regular geometric objects can be always imaged together with their inscribed or circumscribed circles. Another type of objects inscribed in a circle are the so - called **cyclic quadrilaterals**. This means that the vertices are lying on the same circle.

Theorem 10.2.1 *The quadrilateral ABCD is a cyclic quadrilateral if and only if the sums of the opposite angles are equal.*

Proof The opposite angles of a cyclic quadrilateral are lying on complement arcs which means that the sum of the corresponding central angles is 360 degree in measure. Therefore the sum of the opposite angles in a cyclic quadrilateral must be 180 degree. Conversely, suppose that for example

$$\angle A = 180 - \angle C$$

and, consequently,

$$\sin \angle A = \sin(180 - \angle C) = \sin \angle C.$$

On the other hand the triangles DAB and BCD have a common side BD. Using the extended sine rule

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

we have that the radius of the circumscribed circles of the triangles DAB and BCD must be the same. The circumscribed circles pass simultaneously through the points B and D. Therefore they are coincide or the (different) centers are situated symmetrically about the line of BD because of the common radius. This is impossible because the angles $\angle A$ and $\angle C$ can not be simultaneously acute (or obtuse) angles.

Definition A quadrilateral is called tangential if it has an inscribed circle which touches all the sides of the quadrilateral.

Theorem 10.2.2 *A convex quadrilateral is tangential if and only if the sum of the opposite sides are equal.*

If we have a tangential quadrilateral then the sides are constituted by tangent line segments to the inscribed circle. If E, F, G and H denote the

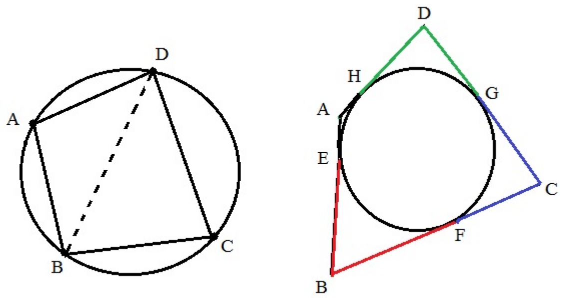

Figure 10.4: Cyclic and tangential quadrilaterals

touching points on the sides AB, BC, CD and DA respectively, then we have that

$$AB + CD = (AE + EB) + (CG + GD) = (AH + BF) + (FC + HD) =$$

$$= (AH + HD) + (BF + FC) = AD + BC$$

because corollary 10.1.4 says that the tangent line segments from an external point to a given circle are of equal length. Therefore the sum of the lengths of the opposite sides are equal. The common value is obviously the half of the perimeter of the quadrilateral. The converse statement fails without the condition of convexity as concave kites show.

10.3 The area of circles

To compute the area of a circle we use an approximation based on inscribed regular n - gons. For the sake of simplicity suppose that the circle has radius one. The vertices $P_1, P_2, \ldots, P_n$ of a regular n - gon inscribed in a circle divides the perimeter into n equal parts. Therefore the area can be computed as the sum of the areas of the congruent triangles

$$P_1OP_2, P_2OP_3, \ldots, P_nOP_1,$$

i.e.

$$\text{the area of a regular n - gon inscribed in the unti circle} = n\frac{\sin\frac{360}{n}}{2}.$$

To simplify the procedure we consider the area

$$A_k = 2^k\frac{\sin\frac{360}{2^k}}{2} = 2^{k-1}\sin\frac{360}{2^k}$$

of 2^k - gons. We are going to express the area

$$A_{k+1} = 2^{k+1}\frac{\sin\frac{360}{2^{k+1}}}{2} = 2^k \sin\frac{360}{2^{k+1}}$$

in the (k+1)th step in terms of A_k. Since

$$\frac{360}{2^k} = 2\frac{360}{2^{k+1}}$$

we have by the additional rules that

$$\cos\frac{360}{2^k} = \cos\left(2\frac{360}{2^{k+1}}\right) = \cos^2\frac{360}{2^{k+1}} - \sin^2\frac{360}{2^{k+1}} = 1 - 2\sin^2\frac{360}{2^{k+1}}$$

because of the trigonometric version of the Pythagorean theorem. Therefore

$$\left(\frac{A_{k+1}}{2^k}\right)^2 = \sin^2\frac{360}{2^{k+1}} = \frac{1-\cos\frac{360}{2^k}}{2} = \frac{1-\sqrt{1-\sin^2\frac{360}{2^k}}}{2} =$$

$$\frac{1-\sqrt{1-\left(\frac{A_k}{2^{k-1}}\right)^2}}{2}$$

and, consequently,

$$A_{k+1} = 2^k\sqrt{\frac{1-\sqrt{1-\left(\frac{A_k}{2^{k-1}}\right)^2}}{2}}$$

We have the following numerical values:

$$A_2 = 2^2\frac{\sin\frac{360}{2^2}}{2} = 2,$$

$A_3 \approx 2.8284$, $A_4 \approx 3.0615$, $A_5 \approx 3.1214$, $A_6 = 3.1365$ and so on.

Theoretically: $A_2 = 2$,

$$A_3 = 2\sqrt{2} = 2\frac{2}{\sqrt{2}}, \quad A_4 = 2\frac{2\cdot 2}{\sqrt{2}\cdot\sqrt{2+\sqrt{2}}},$$

$$A_5 = 2\frac{2\cdot 2\cdot 2}{\sqrt{2}\cdot\sqrt{2+\sqrt{2}}\cdot\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

and so on, see Viéte's formula 1.10 for $2/\pi$.

Chapter 11

Exercises

11.1 Exercises

Excercise 11.1.1 *Let a circle with radius 2 be given. The distance between the point P and the center of the circle is* 4*. Calculate the common length of the tangent segments from P to the given circle and find the length of the shorter arc along the circle between the contact points A and B.*

Solution. The tangent segments have a common length

$$PA = PB = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}.$$

If α is the central angle belonging to the shorter arc between A and B then

$$\sin\frac{\alpha}{2} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \alpha = 120.$$

Therefore

$$\frac{120}{360} = \frac{\text{the arc between A and B}}{2r\pi},$$

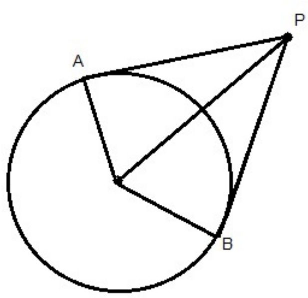

Figure 11.1: Exercise 11.1.1

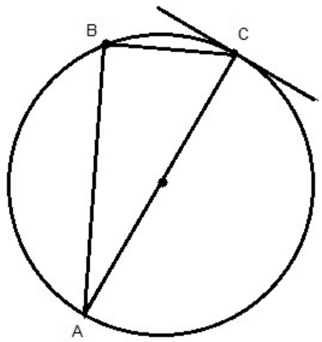

Figure 11.2: Exercise 11.1.2

i.e.

$$\text{the arc between A and B} = \frac{4}{3}\pi.$$

Excercise 11.1.2 *The radius of a circle is 10, the tangent at the point C of the circle has an inclination angle 30 degree to the chord CB. Otherwise AC is the diameter of the circle. Calculate the area and the perimeter of the triangle ABC.*

Solution. Using Thales theorem ABC is a right triangle - the angle of 90 degree in measure is situated at B. The length of the hypothenuse AC is 20. The angle at C is just 60 because the chord BC has an inclination angle 30 degree to the tangent at the point C. The legs are

$$20\cos 60 = 10 \quad \text{and} \quad 20\sin 60 = 10\sqrt{3}.$$

Therefore the area is

$$A = 50\sqrt{3} \quad \text{and} \quad P = 20 + 10 + 10 \cdot \sqrt{3}.$$

Excercise 11.1.3 *Let AB be a diameter of a circle of unit radius. Let C be a point of the tangent to the circle at A for which AC is of length $2\sqrt{3}$ long. Calculate the area of the common part of the triangle ABC and the circle.*

Solution. The angle at B can be easily calculated from the formula

$$\tan\beta = \frac{AC}{AB} = \frac{2\sqrt{3}}{2} \quad \Rightarrow \quad \beta = 60^\circ$$

Therefore the central angle lying on the same arc is of degree 120 in measure. We have that

$$\text{the area of the common part} = \frac{1}{3}r^2\pi + \text{the area of OBD}.$$

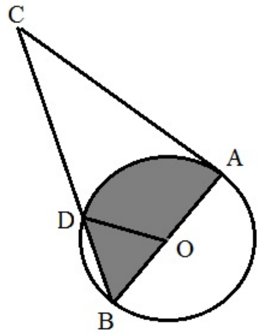

Figure 11.3: Exercise 11.1.3

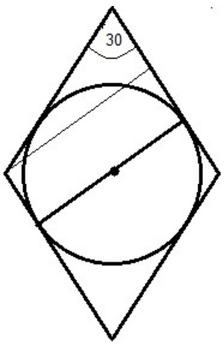

Figure 11.4: Exercise 11.1.4

The area of the triangle OBD can be computed as

$$\frac{r^2 \sin 60}{2},$$

i.e.

$$\text{the area of the common part} = \frac{1}{3}r^2\pi + \frac{\sqrt{3}}{4}r^2.$$

Excercise 11.1.4 *Draw a rhombus around a circle of area 100, so that the rhombus has an angle 30 degree. Calculate the area of the rhombus.*

Solution. The radius of the circle is $10/\sqrt{\pi}$. If a is the common length of the sides of the circumscribed rhombus then

$$\sin 30 = \frac{2r}{a} \quad \Rightarrow \quad a = \frac{40}{\sqrt{\pi}}.$$

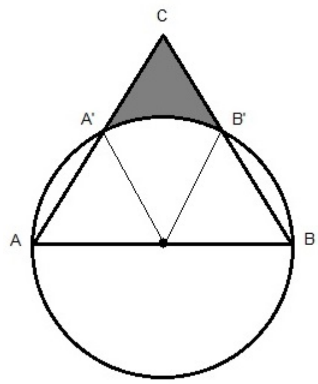

Figure 11.5: Exercise 11.1.5

Therefore the area of the rhombus is

$$A = a^2 \sin 30 = \frac{800}{\pi}.$$

Excercise 11.1.5 *Construct an equilateral triangle above the diameter of a circle with radius r. What is the area of the triangle lying outside the circle.*

Solution. Let AB be the diameter of the circle and consider the common points A' and B' on the perimeter of the circles. Since OAA' and OBB' are equilateral triangles with sides of length r it follows that the area outside from the circle is just

$$\frac{2r \cdot 2r \cdot \sin 60}{2} - 2\frac{r \cdot r \cdot \sin 60}{2} - \frac{1}{6}r^2\pi = r^2\sqrt{3} - r^2\frac{\sqrt{3}}{2} - r^2\frac{\pi}{6}.$$

Excercise 11.1.6 *A circle of unit radius touches the legs of a right angle. What are the radii of the circles which touches the two legs of the right angle and the given circle.*

Solution (cf. exercise 8.1.4). Divide the problem into two parts. At first let us concentrate on the circles touching the legs of a right angle. The center of such a circle must be on the bisector of the angle. Let x be the distance of the center from the vertex. Pythagorean theorem says that

$$x^2 = r_x^2 + r_x^2, \quad \text{i.e.} \quad x = r_x\sqrt{2},$$

where r_x is the radius of the circle. It is labelled by the coordinate x. In case of $r_x = 1$ we have that x=$\sqrt{2}$. Two circles are tangent to each other from outside if and only if the distance of the centers is the sum of the radii:

$$|x - \sqrt{2}| = r_x + 1,$$

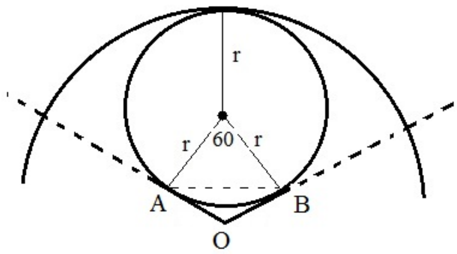

Figure 11.6: Exercise 11.1.7

i.e.

$$x - \sqrt{2} = r_x + 1 \quad \text{or} \quad \sqrt{2} - x = r_x + 1.$$

Therefore

$$r_x = \frac{1+\sqrt{2}}{\sqrt{2}-1} \quad \text{or} \quad r_x = \frac{\sqrt{2}-1}{1+\sqrt{2}}.$$

Excercise 11.1.7 *Draw a circle around the vertex of an angle of 120 degree in measure. Calculate the radius of the circle which touches the given circle inside, and the legs of the angle.*

Solution. Let R be the radius of the circle drawn around the vertex O of an angle of 120 degree in measure. If A and B denote the points of tangency on the legs of the angle then AB=r, where r is the radius of the circle which touches the given circle inside and the legs of the angle. From Pythagorean theorem

$$R - r = \sqrt{r^2 + OA^2},$$

where

$$OA = r \tan 30 \quad \Rightarrow \quad OA^2 = \frac{r^2}{3}.$$

Therefore

$$R = r\left(1 + \frac{2}{\sqrt{3}}\right) \quad \Rightarrow \quad r = \frac{R}{1 + \frac{2}{\sqrt{3}}}.$$

Excercise 11.1.8 *Three sides of a triangle are 13, 14 and 15. What is the radius of the circle whose center lies on the longest side of the triangle and touches the other sides.*

Solution. Consider the radii of the circle which are perpendicular to the sides of lengths 13 and 14, respectively. The area of the triangle can be computed as the sum

$$A = \frac{13r}{2} + \frac{14r}{2}.$$

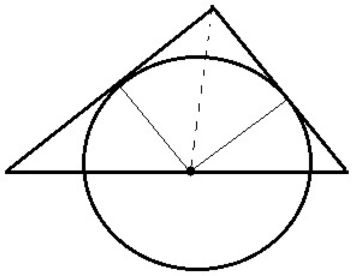

Figure 11.7: Exercise 11.1.8

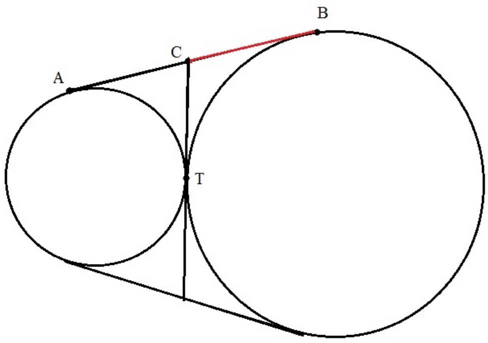

Figure 11.8: Exercise 11.1.9

On the other hand

$$A = \sqrt{42(42-13)(42-14)(42-15)}$$

because of Héron's formula. Finally

$$r = 2\frac{\sqrt{42(42-13)(42-14)(42-15)}}{27}.$$

Excercise 11.1.9 *Let R and r denote the radii of two circles touching each other outside and $R > r$. Calculate the length of the common internal tangent between the common external tangents.*

Solution. Because of the symmetry it is enough to compute the half of the internal common tangent. If T is the point of tangency of the circles it follows that

$$CA = CT = CB.$$

Therefore the length of the internal common tangent between the external common tangents is just AB. On the other hand

$$AB^2 + (R-r)^2 = (R+r)^2$$

and, consequently,

$$AB = 2\sqrt{Rr}$$

Excercise 11.1.10 *The length of the shortest diagonal of a regular 8–gon is given: 10. What is the length of the sides and the area of the polygon.*

Solution. Let $P_1, P_2, \ldots, P_8$ be the vertices of a regular 8–gon inscribed in a circle with center O. The shortest diagonal connecting P_1 and P_3 belongs to the central angle of 90 degree in measure because

$$\angle P_1OP_3 = 2 \cdot \angle P_1OP_2 = 2\frac{360}{8} = 90.$$

Using Pythagorean theorem it follows that

$$r^2 + r^2 = 100 \quad \Rightarrow \quad r = \sqrt{50}.$$

Therefore

$$P_1P_2 = \sqrt{r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos 45} = \sqrt{50 + 50 - 2 \cdot 50 \cdot 50 \cdot \cos 45}$$

and the area is

$$A = 8\frac{r \cdot r \cdot \sin 45}{2}.$$

Excercise 11.1.11 *All sides of a symmetrical trapezoid touch a circle. The parallel bases are 10 and 20.*

- *Calculate the angles of the trapezoid.*
- *Calculate the area of the trapezoid.*

Excercise 11.1.12 *What are the angles of a rhombus if its area is just twice of the area of the inscribed circle?*

Excercise 11.1.13 *The length of the shortest diagonal of a regular 8–gon is given: 20.*

- *What is the length of the sides?*
- *What is the area of the polygon?*

Excercise 11.1.14 *The length of the side of a regular 6–gon is given: 8.*

- *Calculate the angles of the polygon.*

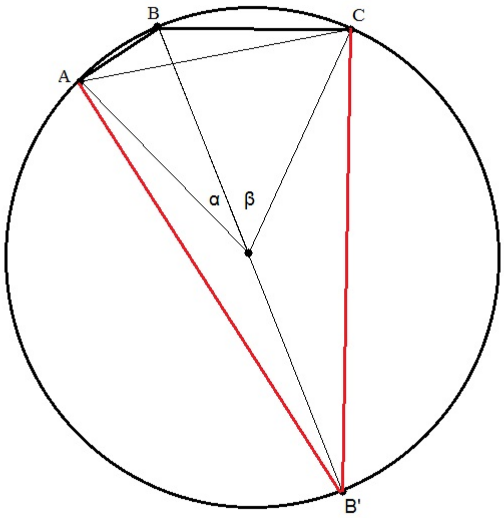

Figure 11.9: Exercise 11.1.16

- *What is the length of the shortest diagonal?*
- *What is the area of the polygon?*

Excercise 11.1.15 *Two circles of radius 5 intersect each other. The distance of the their centers is 8. Calculate the area of the common part of the circles.*

Excercise 11.1.16 *A polygon of 12 sides can be inscribed into a circle. Six of the sides have length $\sqrt{2}$, and the other six sides are equal to $\sqrt{24}$. What is the radius of the circle.*

Solution. Let O be the center of the circle and consider the vertices A, B and C of the polygon such that

$$AB = \sqrt{2} \quad \text{and} \quad BC = \sqrt{24}.$$

If α and β denote the central angles belonging to AB and BC, respectively we have that

$$6\alpha + 6\beta = 360$$

and, consequently

$$\alpha + \beta = 60.$$

Therefore the triangle AOC is equilateral and AC = r. To finish the solution we compute the angle at B in the triangle ABC. Choose the point B' on the circle opposite to B. Then B'ABC form a cyclic quadrilateral and the sum of the measures of the opposite angles must be 180 degree. The inscribed angle theorem says that

$$\angle AB'C = 30$$

and, consequently,

$$\angle ABC = 180 - \angle AB'C = 150.$$

Using the cosine rule we have that

$$AC^2 = 24 + 2 - 2 \cdot 24 \cdot 2 \cdot \cos 150 = r^2.$$

Chapter 12

Geometric transformations

12.1 Isometries

Definition The point transformation $\varphi : P \to P'$ is called an isometry if it preserves the distance between the points:

$$PQ = P'Q'.$$

According to the case SSS of the congruence of triangles any isometry preserves the angles and, by the characterization of parallelism, the parallelism: parallel lines are transformed into parallel lines under any isometry. In what follows we classify the possible cases in terms of the fixpoints.

Theorem 12.1.1 *If an isometry has two fixpoints A and B then for any point X of the line AB we have*

$$X' = X.$$

Proof Since A'=A and B'=B we have that

$$AX = A'X' = AX' \text{ and } BX = B'X' = BX'.$$

Therefore X' must be on

- the circle around A with radius AX,
- the circle around B with radius BX.

Since A, B and X are collinear points these circles are tangent to each other at the uniquely determined point X of tangency: X'=X.

Corollary 12.1.2 *If an isometry has three not collinear fixpoints then it must be the identity.*

Proof Suppose that A, B and C are not collinear fixpoints. Let D be an arbitrary element in the plane and consider the parallel line to BC passing through D. This line intersect both AB and AC at the points F and G, respectively. By theorem 12.1.1 it follows that F'=F, G'=G and D'=D.

The case of two fixpoints gives the identical transformation or the reflection about the line (axis) determined by the fixpoints A and B. In the sense of theorem 12.1.1 for any element of the line AB we have that X'=X. What about the points not in the axis of the reflection? Let Y be one of them. Since

$$YA = Y'A' = Y'A$$

and

$$YB = Y'B' = Y'B$$

it follows by the geometric characterization of the perpendicular bisector that the line AB is just the perpendicular bisector of the segment YY'.
The case of exactly one fixpoint results in the notion of rotation about the uniquely determined fixpoint O.
Translations are typical examples on **isometries** without fixpoints.

Definition Two subsets in the plane are called congruent if there is an isometry which transform them into each other.

According to the **principle of permanence** we should check that in case of two congruent triangles ABC and DEF there is an isometry which maps ABC into DEF. The basic steps of the construction can be formulated as follows:

- If A=D then we use the identical transformation as the first. Otherwise reflect the triangle ABC about the perpendicular bisector of the segment AD. This results in a triangle A'B'C', where A'=D

- If B'=E then we use the identical transformation as the second. Otherwise reflect the triangle A'B'C' about the perpendicular bisector of the segment B'E. This results in a triangle A"B"C", where B"=E. What about A" ? To answer the question we should compare the distances A'B' and A'E:

$$A'B' = AB = DE$$

because the triangles ABC and DEF are congruent. Using the first step

$$DE = A'E \quad \Rightarrow \quad A'B' = A'E.$$

The geometric characterization of the perpendicular bisector implies that A' is lying on the axis of the reflection. Therefore

$$A'' = (A')' = A' = D.$$

- If C"=F then we use the identical transformation as the third. Otherwise reflect the triangle A"B"C" about the perpendicular bisector of the segment C"F. This results in a triangle A"'B"'C"', where C"'=F. What about A"' and B"'?

We are going to prove that

$$A''' = A'' = D \text{ and } B''' = B'' = E.$$

In order to check the first statement we should compare the distances A"C" and A"F:

$$A''C'' = A'C' = AC = DF$$

because the triangles ABC and DEF are congruent. Using the first and the second steps

$$DF = A'F = A''F \quad \Rightarrow \quad A''C'' = A''F.$$

The geometric characterization of the perpendicular bisector implies that A" is lying on the axis of the reflection. Therefore

$$A''' = (A'')' = A'' = D.$$

The proof of the second statement is similar:

$$B''C'' = B'C' = BC = EF$$

because the triangles ABC and DEF are congruent. Using the second step

$$EF = B''F \quad \Rightarrow \quad B''C'' = B''F.$$

The geometric characterization of the perpendicular bisector implies that B" is lying on the axis of the reflection. Therefore

$$B''' = (B'')' = B'' = E.$$

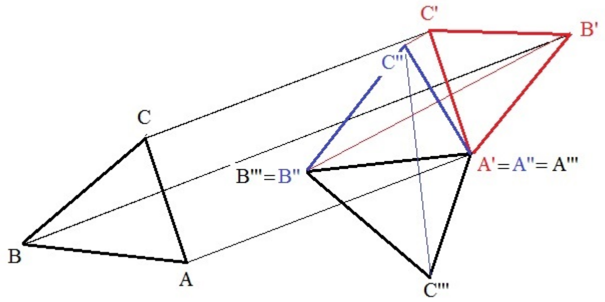

Figure 12.1: The principle of permanence.

Remark The discussion of the principle of permanence shows that for any pair of congruent triangles ABC and DEF there exists an isometry such that

$$A' = D, \; B' = E \; \text{ and } \; C' = F.$$

Corollary 12.1.2 provides the unicity of such an isometry too. Therefore any isometry is uniquely determined by the images of three not collinear points. At the same time any isometry is the product of at most three reflections about lines. This gives a new starting point for the characterization: any isometry is one of the following types

- reflection about a line,
- the product of two reflections about lines,
- the product of three reflections about lines.

Excercise 12.1.3 *Prove that the product of two reflections about lines is a rotation or a translation depending on whether the axes are concurrent or parallel.*

12.2 Similarities

Definition The point transformation $\xi : P \to P'$ is called a similarity if it preserves the ratio of distances between the points:

$$PQ : P'Q' = \lambda,$$

where the positive constant λ is called the similarity ratio.

According to the case S'S'S' of the similarity of triangles any similarity transformation preserves the angles and, by the characterization of parallelism, the parallelism: parallel lines are transformed into parallel lines under any similarity. In what follows we classify the possible cases in terms of the fixpoints.

Remark Isometries are similarities with ratio 1.

Theorem 12.2.1 *If a similarity is not an isometry then it has a uniquely determined fixpoint.*

Proof It is clear that if we have two different fixpoints then the transformation is an isometry. Therefore the number of fixpoints is at most one. For the proof of the existence we can refer to [5], where an elementary ruler construction can be found for finding the fixpoint of a similarity transformation in the plane.

An important subclass of similarities is formed by the central similarities; see the proof of theorem 4.2.1.

Corollary 12.2.2 *Any similarity can be given as the product of a central similarity and an isometry.*

Proof Let ξ be a similarity with ratio λ. If $\lambda = 1$ then we have an isometry. Otherwise the fixpoint C of ξ is uniquely determined in the sense of theorem 12.2.1. Therefore the product of ξ and the central similarity of scaling $1/\lambda$ with respect to C gives an isometry φ.

Remark Since C must be the fixpoint of φ we have the following possible cases: if φ is

- the identity then ξ is a central similarity,
- a reflection about a line passing through C then ξ is a so - called stretch reflection.
- a rotation about C then ξ is a so - called stretch rotation or spiral similarity.

Chapter 13

Classical problems II

Everybody knows the famous geometric principle about the shortest way between two points. In the following problems we can not use the principle in a direct way because the straight line segments are forbidden by some constraints. The indirect way is based on using geometric transformations to create a new configuration for the direct application. To keep all the metric relationships the transformations must be isometries.

13.1 The problem of the bridge

Problem. Suppose that there are two villages A and B on different banks of a river with constant width. We can across the river by a bridge in such a way that it is perpendicular to the banks. Find the best position for the legs of the bridge by minimizing the sum of distances

$$AX + XY + YB,$$

where X and Y denotes the position of the legs of the bridge.

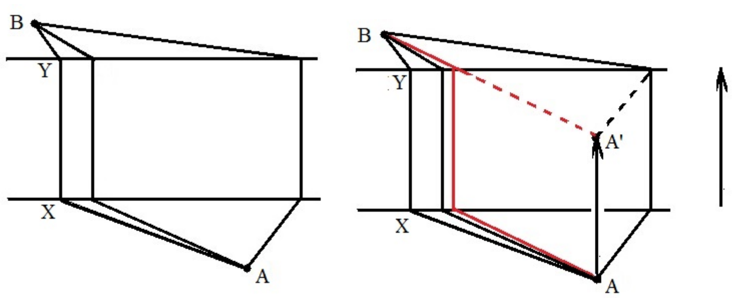

Figure 13.1: The problem of the bridge

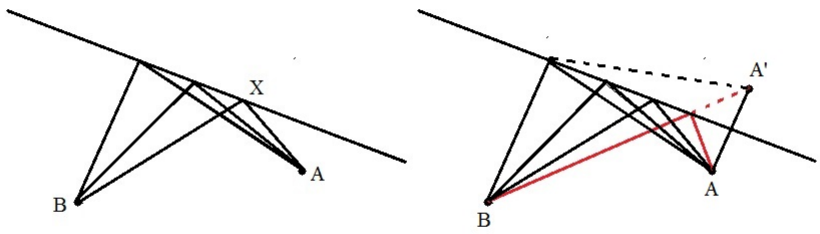

Figure 13.2: The problem of the camel

Solution. Since the river is of constant width, the invariant term XY can be omitted. The translation $X \mapsto X' = Y$ shows that the sub - trips AX and YB correspond to a two - steps long polygonal chain from A' to B. The straight line segment A'B indicates the optimal position for the legs of the bridge.

13.2 The problem of the camel

Problem. Suppose that there are two villages A and B on the same bank of an unswerving river. The distance between them is too large for a camel to walk from A to B without drinking. Find the best position for the camel to have a drink by minimizing the sum of distances

$$AX + XB,$$

where X denotes the position along the river.
Solution. Instead of a reduction by an invariant quantity (see the problem of the bridge) we use an expansion by an invariant quantity to solve the problem of the camel: minimize the sum

$$AA' + A'X + XB,$$

where A' is the image of A under the reflection about the line of the river. The straight line segment A'B indicates the optimal position for the camel to have a drink.

13.3 The Fermat point of a triangle

Problem. Find the point of the triangle ABC which minimizes the sum of distances

$$AX + BX + CX.$$

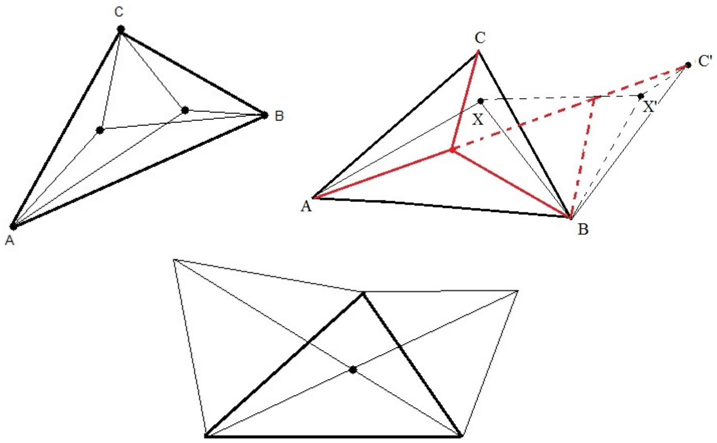

Figure 13.3: The Fermat point of a triangle

Solution. Consider a rotation about B with angle 60 degree into clockwise direction. Since the triangle XBX' is equilateral the sub - trip BX can be substituted by XX'. On the other hand XC=X'C' because rotations are isometries. Therefore every choice of X corresponds to a three - steps - long polygonal chain from A to C'. Since the straight line segment AC' gives the minimal length we have that the minimizer satisfies the conditions

$$\angle AXB = 120^\circ$$

and

$$\angle BXC = \angle BX'C' = 120^\circ.$$

Figure 13.3 shows how to construct the minimizer by using equilateral triangles lying on the sides of the triangle ABC. The method and the argumentation is working as far as all the angles of the triangle ABC is less than 120 degree in measure.

Excercise 13.3.1 *Explain why the method fails in case of an angle of measure greater or equal than 120 degree.*

Remark In case of a triangle having an angle of measure greater or equal than 120 degree the solution is just the vertex where the critical value of the measure is attained or exceeded.

Definition The point of the triangle ABC which minimizes the sum of distances

$$AX + BX + CX$$

is called the Fermat-point of the triangle.

Chapter 14

Longitudes and latitudes

Problem. Find the distance between A and B on the surface of the Earth.
Solution. In geography the longitude and the latitude are used to determine positions on the surface of the Earth. The longitude λ is a rotational angle to specify the east-west position of the point relative to the Greenwich meridian across Royal Observatory, Greenwich. The latitude φ is the inclination angle relative to the plane of the Equator. In what follows we will use the signs $+$ and - instead of north and south or east and west. To simplify the formulas in the calculation we suppose that the radius of the Earth is 1 unit.
First step Pythagorean theorem in the right triangle ABC gives the Euclidean distance between A and B:

$$AB^2 = AC^2 + CB^2.$$

Since AC is the vertical difference between the points,

$$AC^2 = (AA' - BB')^2 = (\sin\varphi_A - \sin\varphi_B)^2.$$

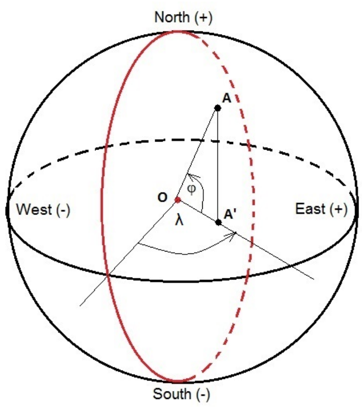

Figure 14.1: Longitudes and latitudes I

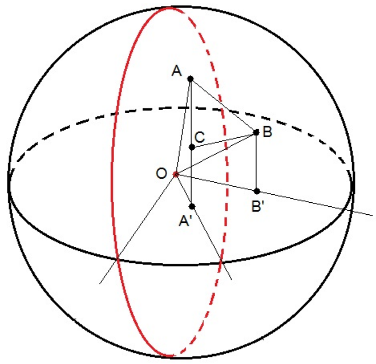

Figure 14.2: Longitudes and the latitudes II

To compute CB consider the projected segment A'B' in the equatorial plane:

$$OA' = \cos\varphi_A,\ OB' = \cos\varphi_B \ \text{ and } \ \angle A'OB' = \lambda_B - \lambda_A.$$

Using the cosine rule we have that

$$A'B'^2 = \cos^2\varphi_A + \cos^2\varphi_B - 2\cos\varphi_A\cos\varphi_B\cos(\lambda_B - \lambda_A) =$$

$$\cos^2\varphi_A + \cos^2\varphi_B - 2\cos\varphi_A\cos\varphi_B(\cos\lambda_B\cos\lambda_A + \sin\lambda_B\sin\lambda_A) =$$

$$\left(\cos\varphi_A\cos\lambda_A - \cos\varphi_B\cos\lambda_B\right)^2 + \left(\cos\varphi_A\sin\lambda_A - \cos\varphi_B\sin\lambda_B\right)^2 = CB^2.$$

Therefore

$$AB^2 = (\cos\varphi_A\cos\lambda_A - \cos\varphi_B\cos\lambda_B)^2 + (\cos\varphi_A\sin\lambda_A - \cos\varphi_B\sin\lambda_B)^2 +$$

$$(\sin\varphi_A - \sin\varphi_B)^2.$$

Second step. Using AB we can compute the central angle ω in the triangle AOB by the cosine rule

$$AB^2 = 2 - 2\cos\omega$$

because the radius of the Earth is chosen as a unit.

Third step. The distance between A and B on the surface of the Earth is just the length of the shorter arc joining A and B along the circle cutted by the plane AOB:

$$\text{the length of the arc from A to B} = \omega \text{ (in radian)}.$$

Remark The distance in kilometers can be expressed from the formula

$$\frac{\text{the length of the arc from A to B}}{R} = \omega,$$

where $R \approx 6378.1$ km.

Excercise 14.0.2 *Find the distances between World cities on the surface of the Earth.*

City	Latitude	Longitude
Aberdeen, Scotland	57 N	2 W
Budapest, Hungary	47 N	19 E
Cairo, Egypt	30 N	31 E
Dakar, Senegal	14 N	17 W
Edinburgh, Scotland	55 N	3 W
Frankfurt, Germany	50 N	8 E
Georgetown, Guyana	6 N	58 W
Hamburg, Germany	53 N	10 E
Irkutsk, Russia	52 N	104 E
Jakarta, Indonesia	6 S	106 E
Kingstone, Jamaica	17 N	76 W
La Paz, Bolivia	16 S	68 W
Madrid, Spain	40 N	3 W
Nagasaki, Japan	32 N N	122 E
Odessa, Ukraine	46 N	30 E
Paris, France	48 N	20 E
Rio de Janeiro, Brasil	22 S	43 W
Sydney, Australia	34 S	151 E
Tananarive, Madagascar	18 S	47 E
Veracruz, Mexico	19 N	96 W
Warsaw, Poland	52 N	21 E
Zürich, Switzerland	47 N	8 E

Part II

Analytical Geometry

Chapter 15

Rectangular Cartesian Coordinates in a Plane

15.1 Coordinates in a plane

Let us draw in the plane two mutually perpendicular intersecting lines Ox and Oy which are termed *coordinate axes* (Fig. 15.1). The point of intersection O of the two axes is called the *origin of coordinates*, or simply the *origin*. It divides each of the axes into two semi-axes. One of the direction of the semi-axes is conventionally called *positive* (indicated by an arrow in the drawing), the other being *negative*.

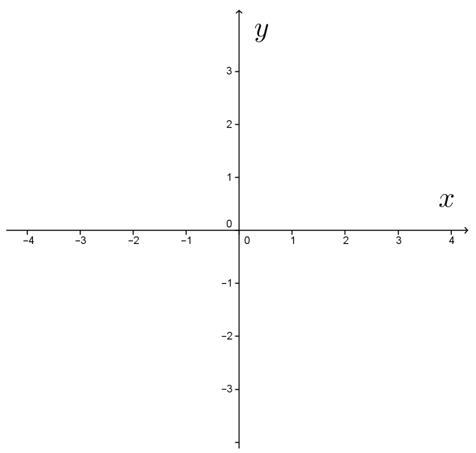

Figure 15.1: The coordinate system

Any point A in a plane is specified by a pair of numbers – called the *rectangular coordinates of the point* A – the *abscissa* (x) and the *ordinate* (y) according to the following rule.

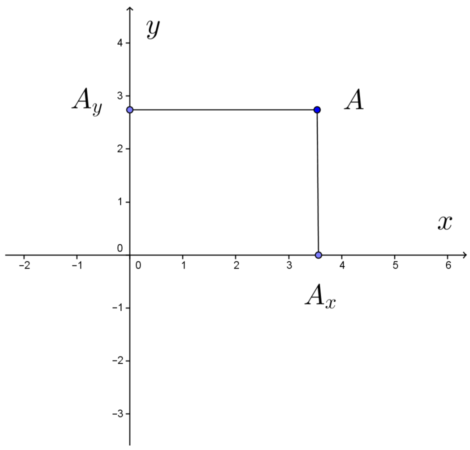

Figure 15.2: Coordinates of a point

Through the point A we draw a straight line parallel to the *axis of ordinates* (Oy) to intersect the *axis of abscissas* (Ox) at some point A_x (Fig. 15.2). The abscissa of the point A should be understood as a number x whose absolute value is equal to the distance from O to A_x which is positive if A_x belong to the positive semi-axis and negative if A_x belongs to the negative semi-axis. If the point A_x coincides with the origin, then we put x equal to zero.

The ordinate (y) of the point A is determined in a similar way.

We shall use following notation: $A(x, y)$ which means that the coordinates of the point A are x (abscissa) and (y) (ordinate).

The coordinate axes separate the plane into four right angles termed the *quadrants* as shown in Fig. 15.3. Within the limits of one quadrant the signs of both coordinates remain unchanged. As we see in the figure, the quadrants are denoted and called the first, second, third, and fourth as counted anticlockwise beginning with the quadrant in which both coordinates are positive.

If a point lies on the *x-axis* (i.e. on the axis of abscissas) then its ordinate y is equal to zero; if a point lies on the *y-axis*, (i.e. on the axis of ordinates), then its abscissa x is zero. The abscissa and ordinate of the origin (i.e. of the point O) are equal zero.

The plane on which the coordinates x and y are introduced by the above method will be called the *xy-plane*. An arbitrary point in this with the coordinates x and y will sometimes be denoted simply (x, y).

For an arbitrary pair of real numbers x and y there exists a unique point A in the xy-plane for which x will be its abscissa and y its ordinate.

Indeed, suppose for definiteness $x > 0$, and $y < 0$. Let us take on the

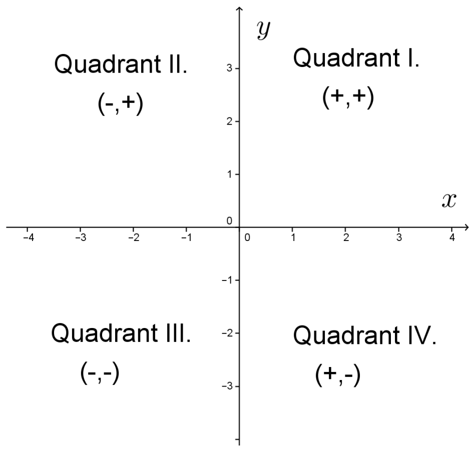

Figure 15.3: Coordinates of a point

positive semi-axis x a point A_x at the distance x from the origin O, and a point A_y on the negative semi-axis y at the distance $|y|$ from O. We then draw through the points A_x and A_y straight lines parallel to the axes y and x, respectively (Fig. 15.4). These lines will intersect at a point A whose abscissa is obviously x, and ordinate is y. In other case ($x < 0$, $y > 0$; $x > 0$, $y > 0$ and $x < 0$, $y < 0$) the proof is analogous.

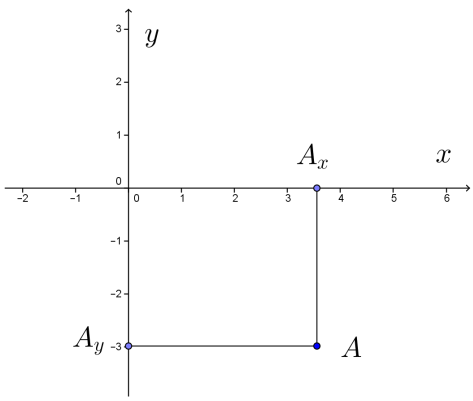

Figure 15.4: Example of coordinates

Let us consider several important cases of analytical representation of domains on the xy-plane with the aid of inequalities. A set of points of the xy-plane for which $x > a$ is a half-plane bounded by a straight line passing through the point $(a, 0)$ parallel to the axis of ordinates (Fig. 15.5). A set of points for which $a < x < b$ represents the intersection (i.e. the common portion) of the half-planes specified by the inequalities $a < x$ and $x < b$. Thus, this set is a band between the straight lines parallel to the y-axis and

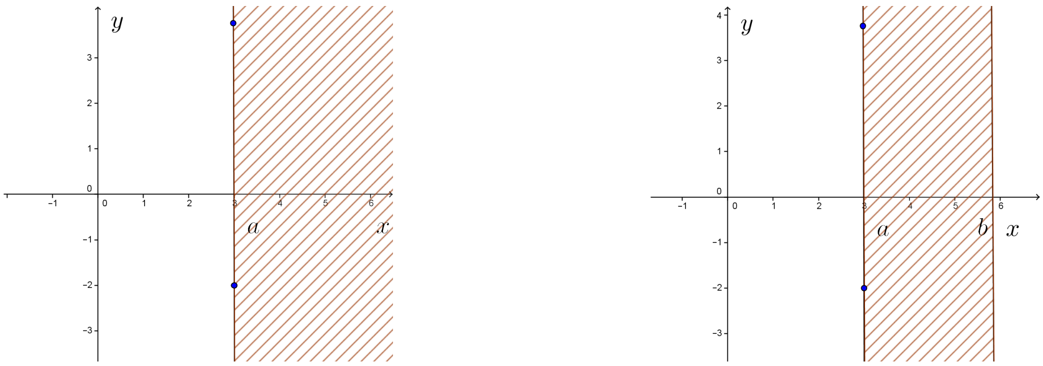

Figure 15.5: Example of a half plane and a strip

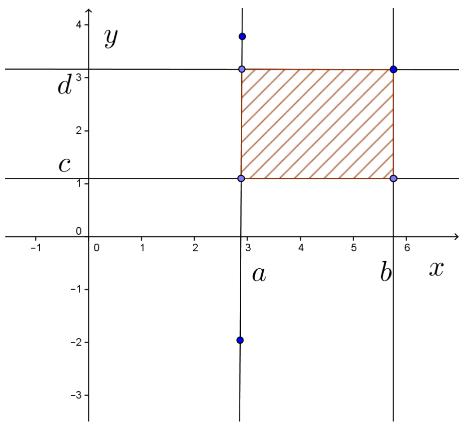

Figure 15.6: Example of a rectangle

passing through the points $(a, 0)$ and $(b, 0)$ (Fig. 15.5). A set of points for which $a < x < b$, $c < y < d$ is a rectangle with vertices at points for which $a < x < b$, $c < y < d$ is a rectangle with vertices at points (a, c) (a, d), (b, c), (b, d). (Fig. 15.6)

In conclusion, let us solve the following problem: *Find the area of a triangle with vertices at points* $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$. Let the triangle be located relative to the coordinate system as is shown in Fig. 15.7. In this position its area is equal to the difference between the area trapezium $B_1A_1A_3B_3$ and the sum of the areas of the trapezia $B_1A_1A_2B_2$ and $B_2A_2A_3B_3$.

The bases of the trapezium $B_1A_1A_3B_3$ are equal to y_1 and y_3, its altitude being equal to $x_3 - x_1$. Therefore, the area of the trapezium

$$S(B_1A_1A_3B_3) = \frac{1}{2}(y_3 + y_1)(x_3 - x_1).$$

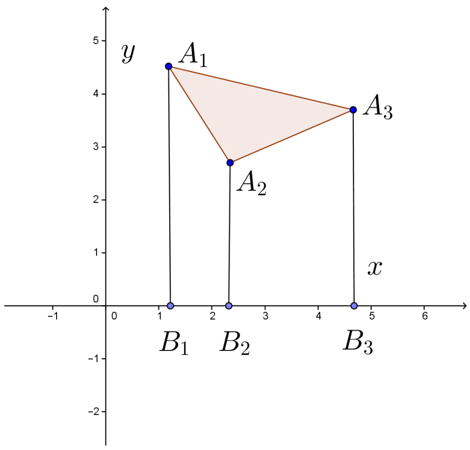

Figure 15.7: Area of a triangle

The areas of two other trapezia are found analogously:

$$S(B_1A_1A_2B_2) = \frac{1}{2}(y_2 + y_1)(x_2 - x_1),$$

$$S(B_2A_2A_3B_3) = \frac{1}{2}(y_3 + y_2)(x_3 - x_2).$$

The area of the triangle $A_1A_2A_3$:

$$\begin{aligned} S(A_1A_2A_3) &= \frac{1}{2}(y_3 + y_1)(x_3 - x_1) \\ &\quad - \frac{1}{2}(y_2 + y_1)(x_2 - x_1) - \frac{1}{2}(y_3 + y_2)(x_3 - x_2) \\ &= \frac{1}{2}(x_2y_3 - y_3x_1 + x_1y_2 - y_2x_3 + x_3y_1 - y_1x_2). \end{aligned}$$

This formula can be rewritten in a convenient form:

$$S(A_1A_2A_3) = \frac{1}{2}\{(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)\}.$$

Though the above formula for computing the area of the triangle has been derived for a particular location of the triangle relative to the coordinate system, it yields a correct result (up to a sign) for any position of the triangle. This will be proved later on (in Section XXXX).

15.2 Exercises

1. What is the location of the points of the xy-plane for which (a) $|x| = a$, (b) $|x| = |y|$?

2. What is the location of the points of the xy-plane for which (a) $|x| < a$, (b) $|x| < a$, $|y| < b$?

3. Find the coordinates of a point symmetrical to the point $A(x, y)$ about the x-axis (y-axis, the origin).

4. Find the coordinates of a point symmetrical to the point $A(x, y)$ about the bisector of the first (second) quadrant.

5. How will the coordinates of the point $A(x, y)$ change if the y-axis is taken for the x-axis, and vice versa?

6. How will the coordinates of the point $A(x, y)$ change if the origin is displaced into the point $A_0(x_0, y_0)$ without changing the directions of the coordinate axes?

7. Find the coordinates of the mid-points of the sides of a square taking its diagonals for the coordinate axes.

8. It is known that three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear. How can one find out which of these points is situated between the other two?

15.3 The distance between points

Let there be given on the xy-plane two points: A_1 with the coordinates x_1, y_1 and A_2 with the coordinates x_2, y_2. *It is required to express the distance between the points A_1 and A_2 in terms of their coordinates.*

Suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. Through the points A_1 and A_2 we draw straight lines parallel to the coordinate axes (Fig. 15.8). The distance between the points A and A_1 is equal to $|y_1 - y_2|$, and the distance between the points A and A_2 is equal to $|x_1 - x_2|$. Applying the Pythagorean theorem to the right-angled triangle A_1AA_2, we get

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2, \qquad (*)$$

Though the formula $(*)$ for determining the distance between points has been derived by us proceeding from the assumption that $x_1 \neq x_2$, $y_1 \neq y_2$, it remains true for other cases as well. Indeed, for $x_1 = x_2$, $y_1 \neq y_2$ d is equal to $|y_1 - y_2|$ (Fig. 15.9). The same result is obtained using the formula $(*)$. For $x_1 \neq x_2$, $y_1 = y_2$ we get a similar result. If $x_1 = x_2$, $y_1 = y_2$ the points A_1 and A_2 coincide and the formula $(*)$ yields $d = 0$.

As an exercise, *let us find the coordinates of the centre of a circle circumscribed about a triangle with the vertices* (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Let (x, y), be the centre of the circumcircle. Since it is equidistant from the vertices of the triangle, we derive the following equations for the required

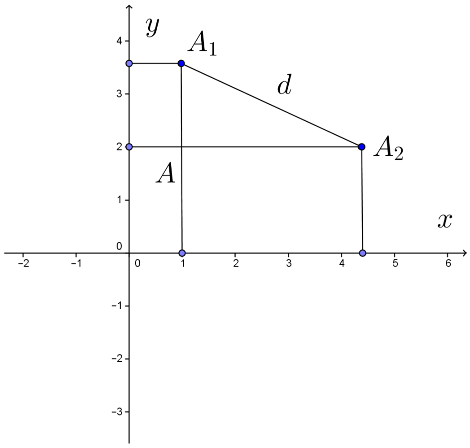

Figure 15.8: Distance of two points

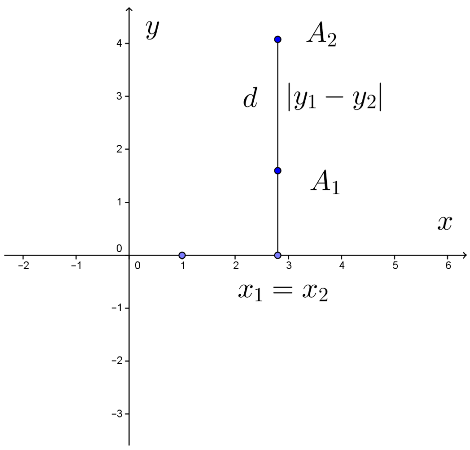

Figure 15.9: Distance of two special points

coordinates of the centre of the circle (x and y). Thus, we have

$$(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2,$$
$$(x-x_1)^2+(y-y_1)^2=(x-x_3)^2+(y-y_3)^2,$$

or after obvious transformations

$$2(x_2-x_1)x+2(y_2-y_1)y=x_2^2+y_2^2-x_1^2-y_1^2,$$
$$2(x_3-x_1)x+2(y_3-y_1)y=x_3^2+y_3^2-x_1^2-y_1^2.$$

Thus, we have a system of two linear equations for determining the unknowns x and y.

15.4 Exercises

1. Find on the x-axis the coordinates of a point equidistant from the two given points $A(x_1, y_1)$, and $B(x_2, y_2)$. Consider the case $A(0, a)$, $B(b, 0)$.

2. Given the coordinates of two vertices A and B of an equilateral triangle ABC. How to find the coordinates of the third vertex? Consider the case $A(0, a)$, $B(a, 0)$.

3. Given the coordinates of two adjacent vertices A and B of a square $ABCD$. How are the coordinates of the remaining vertices found? Consider the case $A(a, 0)$, $B(0, b)$.

4. What condition must be satisfied by the coordinates of the vertices of a triangle ABC so as to obtain a right-angled triangle with a right angle at the vertex C?

5. What condition must be satisfied by the coordinates of the vertices of a triangle ABC so that the angle A exceeds the angle B?

6. A quadrilateral $ABCD$ is specified by the coordinates of its vertices. How to find out whether or not is it inscribed in a circle?

7. Prove that for any real a, a_1, a_2, b, b_1, b_2 there holds the following inequality

$$\sqrt{(a_1-a)^2+(b_1-b)^2}+\sqrt{(a_2-a)^2+(b_2-b)^2}\geq\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}.$$

To what geometrical fact does it correspond?

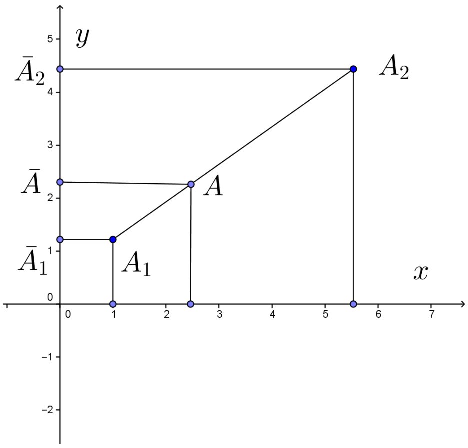

Figure 15.10: Dividing a line segment

15.5 Dividing a line segment in a given ratio

Let there be given two different points on the xy-plane: $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$. *Find the coordinates x and y of the point A which divides the segment A_1A_2 in the ratio $\lambda_1 : \lambda_2$.*

Suppose the segment A_1A_2 is not parallel to the x-axis. Projecting the points A_1, A, A_2 on the y-axis, we have (Fig. 15.10)

$$\frac{A_1A}{AA_2} = \frac{\bar{A}_1\bar{A}}{\bar{A}\bar{A}_2} = \frac{\lambda_1}{\lambda_2}.$$

Since the points $\bar{A}_1$, $\bar{A}_2$, $\bar{A}$ have the same ordinates as the points A_1, A_2, A, respectively, we get

$$\bar{A}_1\bar{A} = |y_1 - y|, \qquad \bar{A}\bar{A}_2 = |y - y_2|.$$

Consequently,

$$\frac{|y_1 - y|}{|y - y_2|} = \frac{\lambda_1}{\lambda_2}.$$

Since the point $\bar{A}$ lines between $\bar{A}_1$ and $\bar{A}_2$, $y_1 - y$ and $y - y_2$ have the same sign.

Therefore

$$\frac{|y_1 - y|}{|y - y_2|} = \frac{y_1 - y}{y - y_2} = \frac{\lambda_1}{\lambda_2}.$$

Whence we find

$$y = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2}. \qquad (*)$$

If the segment A_1A_2 is parallel to the x-axis, then

$$y_1 = y_2 = y.$$

The same result is yielded by the formula $(*)$ which is thus true any positions of the points A_1 and A_2.

The abscissa of the point A is found analogously. For it we get the formula

$$x = \frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2}.$$

We put $\frac{\lambda_1}{\lambda_1+\lambda_2} = t$. Then $\frac{\lambda_2}{\lambda_1+\lambda_2} = 1 - t$.

Consequently, the coordinates of any point C of a segment with the endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ may be represented at follows

$$x = (1-t)x_1 + tx_2, \quad y = (1-t)y_1 + ty_2, \qquad 0 \le t \le 1.$$

Let us find location of points $C(x, y)$ for $t < 0$ and $t > 1$. To do this in case of $t < 0$ we solve our formulas with respect to x_1, y_1. We get

$$x_1 = \frac{1 \cdot x + (-t)x_2}{1-t},$$
$$y_1 = \frac{1 \cdot y + (-t)y_2}{1-t}.$$

Hence, it is clear that the point $A(x_1, y_1)$ is situated on the line segment CB and divides this segment in the ratio $(-t) : 1$. Thus, for $t < 0$ our formulas yield the coordinates of the point lying on the extension of the segment AB beyond the point A. It is proved in a similar way that for $t > 1$ the formulas yield the coordinates of the point located on the extension of the segment AB beyond the point B.

As an exercise, let us prove Ceva's theorem from elementary geometry. It states: *It the sides of a triangle are divided in the ratio $a : b$, $c : a$, $b : c$, taken in order of moving round the triangle* (see Fig. 15.11), *then the segments joining the vertices of the triangle to the points of division of the opposite sides intersect in on one point.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of the triangle and $\bar{A}$, $\bar{B}$, $\bar{C}$ the points of division of the opposite sides (Fig. 15.11). The coordinates of the point $\bar{A}$ are:

$$x = \frac{bx_2 + cx_3}{b+c}, \qquad y = \frac{by_2 + cy_3}{b+c},$$

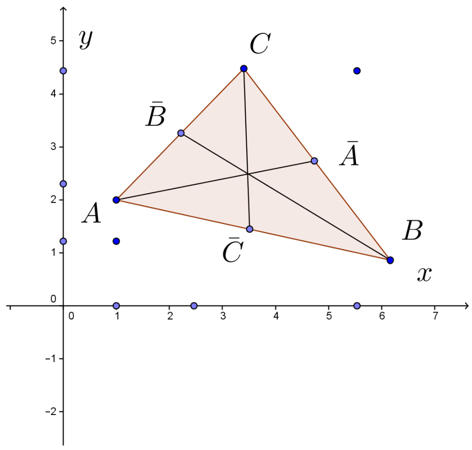

Figure 15.11: Ceva's theorem

Let us divide the line segment $A\bar{A}$ in the ratio $(b+c) : a$. Then the coordinates of the point of division will be

$$x = \frac{ax_1 + bx_2 + cx_3}{a + b + c},$$
$$y = \frac{ay_1 + by_2 + cy_3}{a + b + c}.$$

If the segment $B\bar{B}$ is divided in the ratio $(a+c) : b$, then we get the same coordinates of the point of division. The same coordinates are obtained when dividing the segment $C\bar{C}$ in the ratio $(a + b) : c$. Hence, the segments $A\bar{A}$, $B\bar{B}$, and $C\bar{C}$ have a point in common, which was required to be proved.

Let us note here that the theorems of elementary geometry on intersecting medians, bisectors, and altitudes in the triangle are particular cases of Ceva's theorem.

15.6 Exercises

1. Given the coordinates of three vertices of a parallelogram: (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the fourth vertex and the centroid.

2. Given the coordinates of the vertices of a triangle: (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the point of intersection of the medians.

3. Given the coordinates of the mid-points of the sides of a triangle (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of this vertices.

4. Given a triangle with the vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the vertices a homothetic triangle with the ratio of similarity λ and the centre of similitude at point (x_0, y_0).

5. Point A is said *to divide* the line segment A_1A_2 *externally* in the ratio $\lambda_1 : \lambda_2$ if this point lies on a straight line joining the points A_1 and A_2 outside the segment A_1A_2 and the ratio of its distances from the points A_1 and A_2 is equal to $\lambda_1 : \lambda_2$. Show that the coordinates of the point A are expressed in terms of the coordinates (x_1, y_1), (x_2, y_2) of the points A_1 and A_2 by the formulas

$$x = \frac{\lambda_2 x_1 - \lambda_1 x_2}{\lambda_2 - \lambda_1}, \quad y = \frac{\lambda_2 y_1 - \lambda_1 y_2}{\lambda_2 - \lambda_1}.$$

6. Two line segments are specified by the coordinates of their end-points. How can we find out, without using a drawing, whether the segments intersect or not?

7. *The centre of gravity of two masses* μ_1 and μ_2 situated at points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ is defined as a point A which divides the segment A_1A_2 in the ratio $\mu_2 : \mu_1$.

Thus, its coordinates are:

$$x = \frac{\mu_1 x_1 + \mu_2 x_2}{\mu_1 + \mu_2}, \qquad y = \frac{\mu_1 y_1 + \mu_2 y_2}{\mu_1 + \mu_2}.$$

The centre of gravity of n masses μ_i situated at points A_i is determined by induction. Indeed, if A'_n is the centre of gravity of the first $n-1$ masses, then the centre of gravity of all n masses is determined as the centre of gravity of two masses: μ_n located at point A_n, and $\mu_1 + \cdots + \mu_{n-1}$, situated at point A'_n. We then derive the formulas for the coordinates of the centre of gravity of the masses μ_i situated at points $A_i(x_i, y_i)$:

$$x = \frac{\mu_1 x_1 + \cdots + \mu_n x_n}{\mu_1 + \cdots + \mu_n}, \qquad y = \frac{\mu_1 y_1 + \cdots + \mu_n y_n}{\mu_1 + \cdots + \mu_n}$$

15.7 The equation of a circle

Let there be given a curve on the xy-plane (Fig. 15.12). The equation $\varphi(x, y) = 0$ is called the *equation of a curve in the implicit form* if it is satisfied by the coordinates (x, y) of any point of this curve. Any pair of numbers x, y, satisfying the equation $\varphi(x, y) = 0$ represents the coordinates of a point on the curve. As is obvious, a curve is defined by its equation, therefore we may speak of representing a curve by its equation.

In analytic geometry two problems are often considered: (1) given the geometrical properties of a curve, form its equation: (2) given the equation

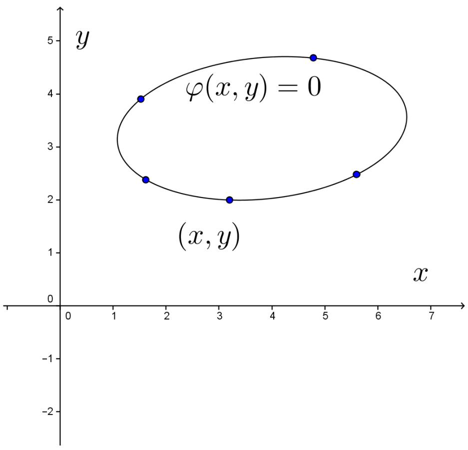

Figure 15.12: Equation of a curve

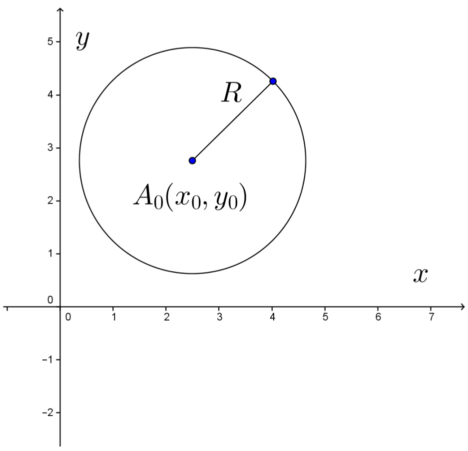

Figure 15.13: Equation of a circle

of a curve, find out its geometrical properties. Let us consider these problems as applied to the circle which is the simplest curve.

Suppose that $A_0(x_0, y_0)$ is an arbitrary point of the xy-plane, and R is any positive number. Let us form the equation of a circle with centre A_0 and radius R (Fig. 15.13).

Let $A(x, y)$ be an arbitrary point of the circle. Its distance from the centre A_0 is equal to R. According to Section 15.3, the square of the distance of the point A from A_0 is equal to $(x - x_0)^2 + (y - y_0)^2$. Thus, the coordinates x, y of any point A of the circle satisfy the equation

$$(x - x_0)^2 + (y - y_0)^2 - R^2 = 0. \qquad (*)$$

Conversely, any point A whose coordinates satisfy the equation $(*)$ belongs to the circle, since its distance from A_0 is equal to R.

In conformity with the above definition, the equation $(*)$ *is an equation of a circle with centre A_0 and radius R.*

We now consider the second problem for the curve given by the equation

$$x^2 + y^2 + 2ax + 2by + c = 0 \qquad (a^2 + b^2 - c > 0).$$

This equation can be rewritten in the following equivalent form:

$$(x + a)^2 + (y + b)^2 - (\sqrt{a^2 + b^2 - c}\,)^2 = 0.$$

Whence it is seen than any point (x, y) of the curve is found at one and the same distance equal to $\sqrt{a^2 + b^2 - c}$ from the point $(-a, -b)$, and, hence, the *curve is a circle with centre $(-a, -b)$ and radius $\sqrt{a^2 + b^2 - c}$.*

Let us consider the following problem as an example illustrating the application of the method of analytic geometry: *Find the locus of points in a plane the ratio of whose distances from two given points A and B is constant and is equal to $k \neq 1$.* (The *locus* is defined as a figure which consists of all the points possessing the given geometrical property. In the case under consideration we speak of a set of all the points in the plane for which the ratio of the distances from the two points A and B is constant).

Suppose that $2a$ is the distance between the points A and B. We then introduce a rectangular Cartesian coordinate system on the plane taking the straight line AB for the x-axis and the midpoint of the segment AB for the origin. Let, for definiteness, the point A be situated on the positive semi-axis x. The coordinates of the point A will then be: $x = a$, $y = 0$, and the coordinates of the point B will be: $x = -a$, $y = 0$. Let (x, y) be an arbitrary point of the locus. The squares of its distances from the points A and B are respectively equal to $(x - a)^2 + y^2$ and $(x + a)^2 + y^2$. The equation of the locus is

$$\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} = k^2,$$

or

$$x^2 + y^2 + \frac{2(k^2 + 1)}{k^2 - 1}ax + a^2 = 0.$$

The locus represents a circle (called Apollonius' circle).

15.8 Exercises

1. What peculiarities in the position of the circle

$$x^2 + y^2 + 2ax + 2by + c = 0 \qquad (a^2 + b^2 - c > 0)$$

relative to the coordinate system take place if

$$\begin{array}{lll} (1)\quad a = 0; & (2)\quad b = 0; & (3)\quad c = 0; \\ (4)\quad a = 0,\ b = 0; & (5)\quad a = 0, c = 0; & (6)\quad b = 0,\ c = 0? \end{array}$$

2. Show that if we substitute in the left-hand member of the equation of a circle the coordinates of any point lying outside the circle, then the square of the length of a tangent drawn from this point to the circle is obtained.

3. The *power of a point* A with reference to a circle is defined as the product of the segments of a secant drawn through the point A taken with plus for outside points and with minus for inside points. Show that the left-hand member of the equation of a circle $x^2 + y^2 + 2ax + 2by + c = 0$ gives the power of this point with reference to a circle when the coordinates of an arbitrary point are substituted in it.

4. Form the equation of the locus of points of the xy-plane the sum of whose distances from two given points $F_1(c, 0)$ and $F_2(-c, 0)$ is constant and is equal to $2a$ (the *ellipse*). Show that the equation is reduced to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$.

5. Form the equation of the locus of points of the xy-plane the difference of whose distances from two given points $F_1(c, 0)$ and $F_2(-c, 0)$ is constant and is equal to $2a$ (the *hyperbola*). Show that the equation is reduced to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$.

6. Form the equation of the locus of points of the xy-plane which are equidistant from the point $F(0, p)$ and the x-axis (the *parabola*).

15.9 The equation of a curve represented by parameters

Suppose a point A moves along a curve, and by the time t its coordinates are: $x = \varphi(t)$ and $y = \varphi(t)$. A system of equations

$$x = \varphi(t), \qquad y = \varphi(t),$$

specifying the coordinates of an arbitrary point on the curve as functions the parameter t is called the *equation of a curve in parametric form.*

The parameter t is not necessarily time, it may be any other quantity characterizing the position of a point on the curve.

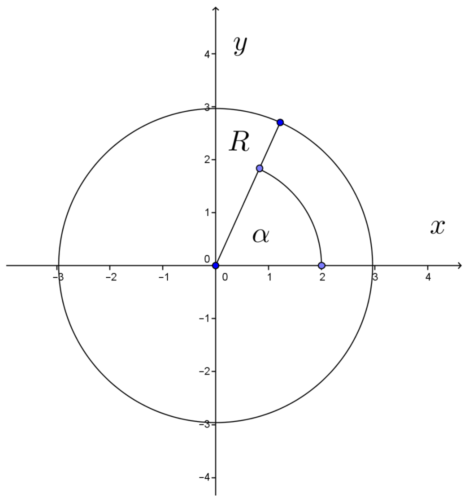

Figure 15.14: Distance of two points

Let us now form the equation of a circle in parametric form.

Suppose the centre of a circle is situated at the origin, and the radius is equal to R. We shall characterize the position of point A on the circle by the angle α formed by the radius OA with the positive semi-axis x (Fig. 15.14). As is obvious, the coordinates of the point A are equal to $R\cos\alpha$, $R\sin\alpha$, and, consequently, the equation of the circle has such a form:

$$x = R\cos\alpha, \qquad y = R\sin\alpha.$$

Having an equation of a curve in parametric form:

$$x = \varphi(t), \qquad y = \varphi(t), \tag{$*$}$$

we can obtain its equation in implicit form:

$$f(x, y) = 0.$$

To this effect it is sufficient to eliminate the parameter t from the equations $(*)$, finding one equation and substituting into the other, or using another method.

For instance, to get the equation of a circle represented by equations in parametric form (i.e. implicitly) it is sufficient to square both equalities and add then termwise. We then obtain the familiar equation $x^2 + y^2 = R^2$.

The elimination of the parameter from the equations of a curve represented parametrically not always yields an equation in implicit form in the sense of the above definition. It many turn out that it is satisfied by the points not belonging to the curve. In this connection let us consider two examples.

Suppose a curve y is given by the equations in parametric form

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \leq t \leq 2\pi.$$

Dividing these equations by a and b, respectively, squaring and adding them termwise, we get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation is obviously satisfied by all the points belonging to the curve y. Conversely, if the point (x, y) satisfies this equation, then there can be found an angle t for which $x/a = \cos t$, $y/b = \sin t$, and, consequently, any point of the plane which satisfies this equation, belongs to the curve y.

Let now a curve y be represented by the following equations

$$x = \cosh t, \quad y = b\sinh t, \qquad -\infty < t + \infty,$$

where

$$\cosh t = (e^t + e^{-t})/2, \qquad \sinh t = (e^t - e^{-t})/2.$$

Dividing these equations by a and b, respectively, and then squaring them and subtracting termwise, we get the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The points of the curve y satisfy this equation. But not any point which satisfies the equation belongs to y. Let us, for instance, consider the point $(-a, 0)$. We see that it satisfies the equation, but does not belong to the curve, since on the curve $ya\cosh t \neq -a$.

Sometimes the equation of a curve represented in implicit form is understood in a wider way. One does not require that any point satisfying the equation, belongs to the curve.

15.10 Exercises

1. Show that the following equations in parametric form

$$x = R\cos t + a, \qquad y = R\sin t + b$$

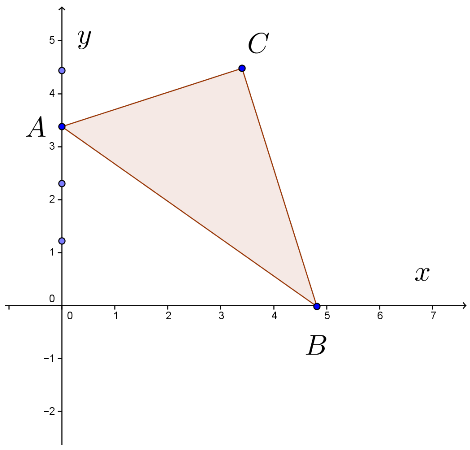

Figure 15.15: Exercise 3

represent a circle of radius R with centre at point (a, b).

2. Form the equation of a curve described by a point on the line segment of length a when the end-points of the segment slide along the coordinate axes (the segment is divided by this point in the ratio $\lambda : \mu$). Take the angle formed by the segment with the x-axis for the parameter. What is the shape of the curve if $\lambda : \mu = 1$?

3. A triangle slides along the coordinate axes with two of its vertices. Form the equation of the curve described by the third vertex (Fig. 15.15).

4. Form the equation of the curve described by a point on a circle of radius R which rolls along the x-axis. For the parameter take the path s covered by the centre of the circle and suppose that at the initial moment $(s = 0)$ point A coincides with the origin.

5. A curve is given by the equation

$$ax^2 + bxy + cy^2 + dx + ey = 0.$$

Show that, by introducing the parameter $t = y/x$, we can obtain the following equations of this curve in parametric form:

$$x = -\frac{d + et}{a + bt + ct^2},$$
$$y = -\frac{dt + et^2}{a + bt + ct^2}.$$

Chapter 16

The Straight Line

16.1 The general equation of a straight line

The straight line is the simplest and most widely used line.

We shall now show that *any straight line has an equation of the form*

$$ax + by + c = 0, \tag{$*$}$$

where a, b, c are constant. And conversely, *if a and b are noth both zero, then there exists a straight line for which $(*)$ is its equation.*

Let $A_1(a_1, b_1)$ and $A_2(a_2, b_2)$ be two different points situated symmetrically about a given straight line (Fig. 16.1). Then any point $A(x, y)$ on this line is equidistant from the points A_1 and A_2. And conversely, any point A which is equidistant from A_1 and A_2 belongs to the straight line. Hence, the equation of a straight line is

$$(x - a_1)^2 + (y - b_1)^2 = (x - a_2)^2 + (y - b_2)^2.$$

Transposing all terms of the equation to the left-hand side, removing the squared parentheses, and carrying out obvious simplifications, we get

$$2(a_2 - a_1)x + 2(b_2 - b_1)y + (a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0.$$

Thus, the first part of the statement is proved.

We now shall prove the second part. Let B_1 and B_2 be two different points of the xy-plane whose coordinates satisfy the equation $(*)$. Suppose

$$a_1 x + b_1 y + c_1 = 0$$

is the equation of the straight line B_1B_2. The system of equations

$$\left.\begin{aligned} ax + by + c &= 0, \\ a_1 x + b_1 y + c_1 &= 0 \end{aligned}\right\} \tag{$**$}$$

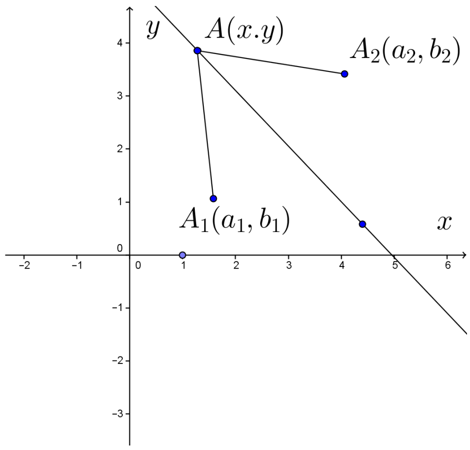

Figure 16.1: Equation of a line

is compatible, it is *a fortiori* satisfied by the coordinates of the point B_1, as well as of B_2.

Since the points B_1, and B_2 are different, they differ in at least one coordinate, say $y_1 \neq y_2$. Multiplying the first equation of $(**)$ by a_1 and the second one by a, and subtracting termwise, we get

$$(ba_1 - ab_1)y + (ca_1 - ac_1) = 0.$$

This equation as a corollary of the equations $(**)$ is satisfied when $y = y_1$ and $y = y_2$. But it is possible only if

$$ba_1 - ab_1 = 0, \qquad ca_1 - ac_1 = 0.$$

Hence it follows that

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1},$$

which means that the equations $(**)$ are equivalent. The second part of the statement is also proved.

As was shown in Section 15.5, the points of a straight line passing through (x_1, y_1) and (x_2, y_2) allow the following representation

$$x = (1-t)x_1 + tx_2, \qquad y = (1-t)y_1 + ty_2.$$

Whence it follows that *any straight line allows a parametric representation by equations of the form*

$$x = at + b, \quad y = ct + d, \qquad -\infty < t < \infty.$$

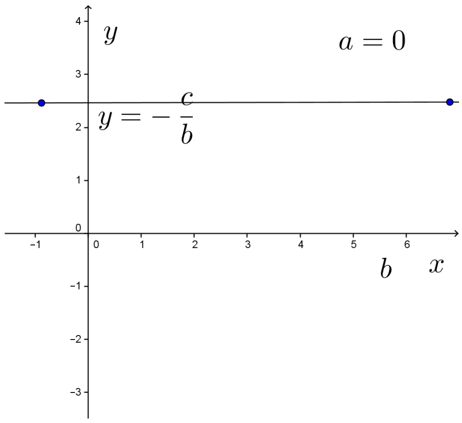

Figure 16.2: x–parallel

Conversely, *any such system of equations may be considered as equations of a straight line in parametric form if a and c are not both equal to zero. This straight line is represented by the equation in implicit form*

$$(x - b)c - (y - d)a = 0.$$

16.2 Particular cases of the equation of a line

Let us find out peculiarities which happen in the location of a straight line relative to the coordinate system if its equation $ax + by + c = 0$ is of a particular form.

1. $a = 0$. In this case the equation of a straight line can be rewritten as follows

$$y = -\frac{c}{b}.$$

Thus, all points belonging to the straight line have one and the same ordinate $(-c/b)$, and, consequently, the *line is parallel to the x-axis* (Fig. 16.2). In particular, if $c = 0$, *then the straight line coincides with the x-axis.*

2. $b = 0$. This case is considered in a similar way. *The straight line is parallel to the y-axis* (Fig. 16.3) *and coincides with it if c is also zero.*

3. $c = 0$. *The straight line passes through the origin*, since the coordinates of the latter $(0, 0)$ satisfy the equation of the straight line (Fig. 16.4).

4. Suppose all the coefficients of the equation of the straight line are non-zero (i.e. the line does not pass through the origin and is not parallel to the coordinate axes). Then, multiplying the equation by $1/c$ and putting

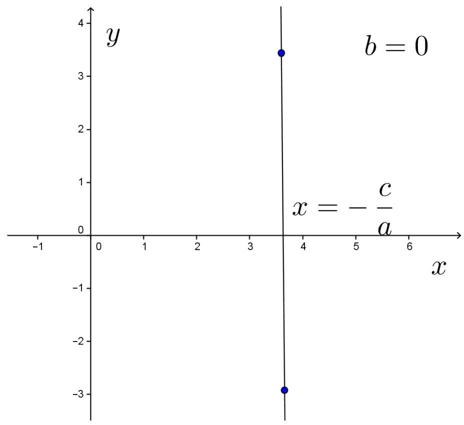

Figure 16.3: y–parallel

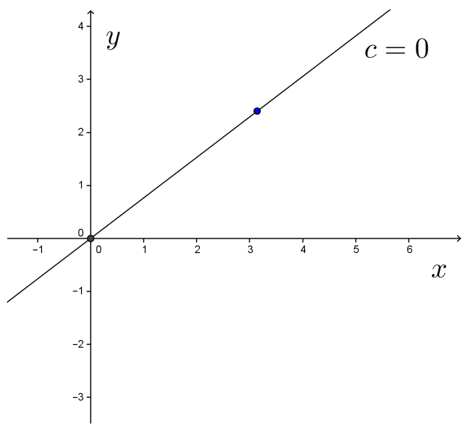

Figure 16.4: A line through the origin

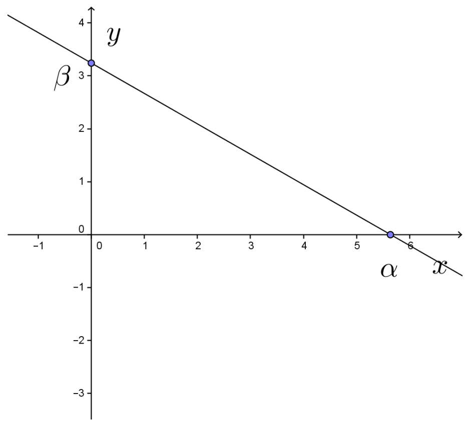

Figure 16.5: x and y intersection

$-c/a = \alpha$, $-c/b = \beta$, we reduce it to the form

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1. \tag{$*$}$$

The coefficients of the equation of a straight line in such a form (which is called the intercept form of the equation of a straight line) have a simple geometrical meaning: *α and β are equal (up to a sign) to the lengths of the line segments intercepted by the straight line on the coordinate axes* (Fig. 16.5). Indeed, the straight line intersects both the x-axis ($y = 0$) at point $(\alpha, 0)$, and the y-axis ($x = 0$) at point $(0, \beta)$.

16.3 Exercises

1. Under what condition does the straight line

$$ax + by + c = 0$$

intersect the positive semi-axis x (the negative semi-axis x)?

2. Under what condition does the straight line

$$ax + by + c = 0$$

not intersect the first quadrant?

3. Show that the straight lines given by the equations

$$ax + by + c = 0, \quad ax - by + c = 0, \qquad b \neq 0,$$

are situated symmetrically about the x-axis.

4. Show that the straight lines specified by the equations

$$ax + by + c = 0; \quad ax + by - c = 0,$$

are arranged symmetrically about the origin.

5. Given a *pencil of lines*

$$ax + by + c + \lambda(a_1x + b_1y + c_1) = 0.$$

Find out for what value of the parameter λ is a line of the pencil parallel to the x-axis (y-axis); for what value of λ does the line pass through the origin?

6. Under what condition does the straight line

$$ax + by + c = 0$$

bound, together with the coordinate axes, an isosceles triangle?

7. Show that the area of the triangle bounded by the straight line

$$ax + by + c = 0 \qquad (a, b.c \neq 0)$$

and the coordinate axes is

$$S = \frac{1}{2}\frac{c^2}{|ab|}.$$

8. Find the tangent lines to the circle

$$x^2 + y^2 + 2ax + 2by = 0,$$

which are parallel to the coordinate axes.

16.4 The angle between two straight lines

When moving along any straight line not parallel to the y-axis x increases in one direction and decreases in the other. The direction in which x increases will be called *positive.*

Suppose we are given two straight lines g_1 and g_2 in the xy-plane which are not parallel to the y-axis. The angle $\theta(g_1, g_2)$ formed by the line g_2 with the line g_1 is defined as an angle, less than π by absolute value, through which the line g_1 must be turned so that the positive direction on it is brought in coincidence with the positive direction on g_2. This angle is considered to be positive if the line g_1 is turned in the same direction in which the positive semi-axis x is turned through the angle $\pi/2$ until it coincides with the positive semi-axis y (Fig. 16.6).

The angle between the straight lines possesses the following obvious properties:

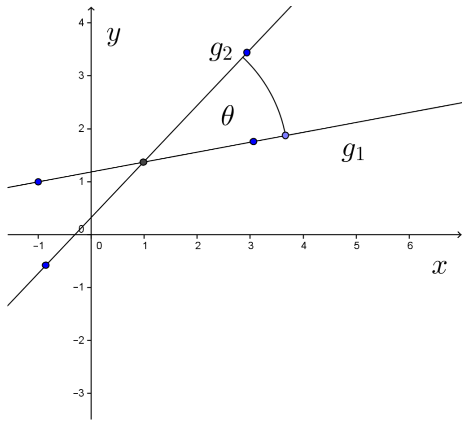

Figure 16.6: Angle of two lines

(1) $\theta(g_1, g_2) = \theta(g_2, g_1)$;

(2) $\theta(g_1, g_2) = 0$ when and only when lines are parallel or coincide;

(3) $\theta(g_3, g_1) = \theta(g_3, g_2) + \theta(g_2, g_1)$.

Let

$$ax + by + c = 0$$

be a straight line not parallel to the y-axis ($b \neq 0$). Multiplying the equation of the is line by $1/b$ and putting $-a/b = k$, $-c/b = \ell$, we reduce it to the form

$$y = kx + \ell. \tag{$*$}$$

The coefficients of the equation of a straight line in this form have a simple geometrical meaning:

k is the tangent of the angle α formed by straight line with the x-axis;

ℓ is the line segment (up to a sign) intercepted by the straight line on the y-axis.

Indeed, let $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ be two points on the straight line (Fig. 16.7). Then

$$\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(kx_2 + \ell) - (kx_1 + \ell)}{x_2 - x_1} = k.$$

The y-axis ($x = 0$) is obviously intersected by the line at point $(0, \ell)$.

Let there be given in the xy-plane two straight lines:

$$y = k_1 x + \ell_1,$$
$$y = k_2 x + \ell_2.$$

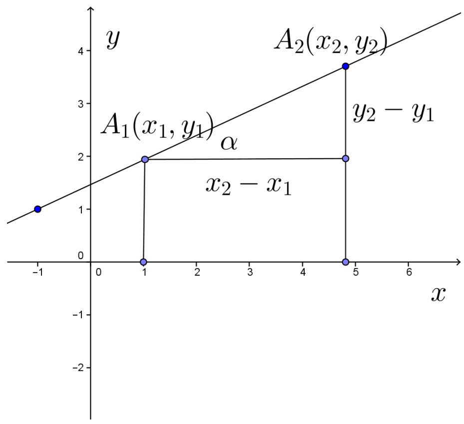

Figure 16.7: The slope of a line

Let us find the angle θ formed by the second line with the first one. Denoting by α_1 and α_2 the angles formed by the straight lines with the x-axis, by virtue of property (3) we get

$$\theta = \alpha_2 - \alpha_1.$$

Since the angular coefficients $k_1 = \tan\alpha_1$, $k_2 = \tan\alpha_2$, we get

$$\tan\theta = \frac{k_2 - k_1}{1 + k_1 k_2}.$$

Whence θ is determined, since $|\theta| < \pi$.

16.5 Exercises

1. Show that the straight lines $ax + by + c = 0$ and $bx - ay + c' = 0$ intersect at right angles.

2. What angle is formed with the x-axis by the straight line

$$y = x\cot\alpha, \quad \text{if } -\frac{\pi}{2} < \alpha 0?$$

3. Form the equations of the sides of a right-angled triangle whose side is equal to 1, taking one of the sides and the altitude for the coordinate axes.

4. Find the interior angles of he triangle bounded by the straight lines $x + 2y = 0$, $2x + y = 0$, and $x + y = 1$.

5. Under what condition for the straight lines $ax+by = 0$ and $a_1x+b_1y = 0$ is the x-axis the bisector of the angles formed by them?

6. Derive the formula $\tan\theta = \frac{c}{a}$ for the angle θ formed by the straight line $x = at + b$, $y = ct + d$ with the x-axis.

7. Find the angle between the straight lines represented by the equations in parametric form:

$$\left.\begin{aligned} x &= a_1t + b_1, \\ y &= a_2t + b_2; \end{aligned}\right\} \qquad \left.\begin{aligned} x &= c_1t + d_1, \\ y &= c_2t + d_2. \end{aligned}\right\}$$

8. Show taht the quadrilateral bounded by the straight lines

$$\pm ax \pm by + c = 0 \qquad (a,\, b,\, c \neq 0),$$

is a rhombus and the coordinate axes are its diagonals.

16.6 The parallelism and perpendicularity of lines

Suppose we have in the xy-plane two straight lines given by the equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0. \end{aligned}$$

Let us find out what condition must be satisfied by the coefficients of the equations of the straight lines for these lines for these lines to be (a) parallel to each other, (b) mutually perpendicular.

Assume that neither of the straight lines is parallel to the y-axis. Then their equations may be written in the form

$$y = k_1x + \ell_1, \qquad y = k_2x + \ell_2,$$

where

$$k_1 = -\frac{a_1}{b_1}, \qquad k_2 = -\frac{a_2}{b_2}.$$

Taking into account the expression for the angle between straight lines, we get the *parallelism condition of two straight lines*:

$$k_1 - k_2 = 0,$$

or

$$a_1b_2 - a_2b_1 = 0. \tag{$*$}$$

The perpendicularity condition of straight lines:

$$1 + k_1 k_2 = 0,$$

or

$$a_1 a_2 + b_1 b_2 = 0. \qquad (*)$$

Thought the conditions $(*)$ and $(**)$ are obtained in the assumption that neither of the straight lines is parallel to the y-axis, they remain true even if this condition is violated.

Let for instance, the first straight line be parallel to the y-axis. This means that, $b_1 = 0$. If the second line is parallel to the first one, then it is also parallel to the y-axis, and, consequently, $b_2 = 0$. The condition $(*)$ is obviously fulfilled. If the second line is perpendicular to the first one, then it is parallel to the x-axis and, consequently, $a_2 = 0$. in this case the condition $(**)$ is obviously fulfilled.

Let us now show that *if the condition* $(*)$ *is fulfilled for the straight lines, then they are either parallel, or coincide.*

Suppose, $b_1 \neq 0$. Then it follows from the condition $(*)$ that $b_2 \neq 0$, since if $b_2 = 0$, then a_2 is also equal to zero which is impossible. In this event the condition $(*)$ may be written in the following way

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}, \quad \text{or} \quad k_1 = k_2,$$

which expresses the equality of the angles formed by the straight lines with the x-axis. Hence, the lines are either parallel, or coincide.

If $b_1 = 0$ (which means that $a_1 \neq 0$), then it follows from $(*)$ that $b_2 = 0$. Thus, both straight lines are parallel to the y-axis and, consequently, they are either parallel to each other, or coincide.

Let us show that the *condition* $(**)$ *is sufficient for the lines to be mutually perpendicular.*

Suppose $b_1 \neq 0$ and $b_2 \neq 0$. Then the condition $(**)$ may be rewritten as follows:

$$1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = 0,$$

or

$$1 + k_1 k_2 = 0.$$

This means that the straight lines form a right angle, i.e. they are mutually perpendicular.

If then $b_1 = 0$ (hence, $a_0 \neq 0$), we get from the condition $(**)$ that $a_2 = 0$. Thus, the first line is parallel to the y-axis, and the second one is parallel to the x-axis which means that they are perpendicular to each other.

The case when $b_2 = 0$ is considered analogously.

16.7 Exercises

1. Show that two straight lines intercepting on the coordinate axes segments of equal lengths are either parallel, or perpendicular to each other.

2. Find the parallelism (perpendicularity) condition of the straight lines represented by the equations in parametric form:

$$\left.\begin{aligned} x &= \alpha_1 t + a_1, \\ y &= \beta_1 t + b_1, \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \alpha_2 t + a_2, \\ y &= \beta_2 t + b_2. \end{aligned}\right\}$$

3. Find the parallelism (perpendicularity) condition for two straight lines one of which is specified by the equation

$$ax + by + c = 0,$$

the other being represented parametrically:

$$x = \alpha t + \beta, \quad y = \gamma t + \delta.$$

4. In a family of straight lines given by the equations

$$a_1 x + b_1 y + c_1 + \lambda(a_2 x + b_2 y + c_2) = 0$$

(λ, parameter of the family) find the line parallel (perpendicular) to the straight line

$$ax + by + c = 0.$$

16.8 Basic problems on the straight line

Let us form the equation of an arbitrary straight line passing through the point $A(x_1, y_1)$.

Suppose

$$ax + by + c = 0 \tag{$*$}$$

is the equation of the required line. Since the line passes through the point A, we get

$$ax_1 + by_1 + c = 0.$$

Expressing c and substituting it in the equation ($*$), we obtain

$$a(x - x_1) + b(y - y_1) = 0.$$

It is obvious that, for any a and b, the straight line given by this equation passes through the point A.

Let us form the equation of the straight line passing through two given points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$.

Since the straight line passes through the point A_1, its equation may be written in the form

$$a(x - x_1) + b(y - y_1) = 0.$$

Since the line passes through the point A_2, we have

$$a(x_2 - x_1) + b(y_2 - y_1) = 0,$$

whence

$$\frac{a}{b} = -\frac{y_2 - y_1}{x_2 - x_1},$$

and the required equation will be

$$\frac{x - x_1}{x_2 - x_1} - \frac{y - y_1}{y_2 - y_1} = 0.$$

Let us now form the equation of a straight line parallel to the line

$$ax + by + c = 0,$$

and passing through the point $A(x_1, y_1)$.

Whatever the value of λ, the equation

$$ax + by + \lambda = 0$$

represents a straight line parallel to the given one. Let us choose λ so that the equation is satisfied for $x = x_1$ and $y = y_1$:

$$ax_1 + by_1 + \lambda = 0.$$

Hence

$$\lambda = -ax_1 - by,$$

and the required equation will be

$$a(x - x_1) + b(y - y_1) = 0.$$

Let us form the equation of a straight line passing through the given point $A(x_1, y_1)$ *and perpendicular to the line*

$$ax + by + c = 0.$$

For any λ the straight line

$$bx - ay + \lambda = 0$$

is perpendicular to the given line. Choosing λ so that the equation is satisfied for $x = x_1$, $y = y_1$ we find the required equation

$$b(x - x_1) - a(y - y_1) = 0.$$

Let us form the equation of a straight line passing through the given point $A(x_1, y_1)$ at an angle α to the x-axis.

The equation of the straight line can be written in the form

$$y = kx + \ell.$$

The coefficients k and ℓ are found from the conditions

$$\tan\alpha = k, \qquad y_1 = kx_1 + \ell.$$

The required equation is

$$y - y_1 = (x - x_1)\tan\alpha.$$

We conclude with the following assertion: *the equation of any straight line passing through the point of intersection of two given straight lines*

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0,$$

can be written in the form

$$\lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0. \tag{$**$}$$

Indeed, for any λ and μ which are not both zero, the equation $(**)$ represents a straight line which passes through the point of intersection of the two given lines, since its coordinates obviously satisfy the equation $(**)$. Further, whatever the point (x_1, y_1) which is different from the point of intersection of the given straight lines, the line $(**)$ passes through the point (x_1, y_1) when

$$\lambda = a_1x_1 + b_2y_2 + c_2, \qquad -\mu = a_1x_1 + b_1y_1 + c_1.$$

Consequently, the straight lines represented by $(**)$ exhaust all the lines passing through the point of intersection of the given straight lines.

16.9 Exercises

1. Form the equation of a straight line parallel (perpendicular) to the straight line

$$ax + by + c = 0,$$

passing through the point of intersection of the straight lines

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0.$$

2. Under what condition are the points (x_1, y_1), (x_2, y_2) situated symmetrically about the straight line

$$ax + by + c = 0?$$

3. Form the equation of a straight line passing through the point (x_0, y_0) and equidistant from the points (x_1, y_1) and (x_2, y_2).

4. Show that three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on a straight line if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Chapter 17

Vectors

17.1 Addition and subtraction of vectors

In geometry, a *vector* is understood as a directed line segment (Fig. 17.1). The direction of a vector is indicated by the arrow. A vector with *initial point* A and *terminal point* B is denoted as $\overrightarrow{AB}$. A vector can also be denoted by a single letter. In printing this letter is given in boldface type ($\mathbf{a}$), in writing it is given with a bar ($\overline{\mathbf{a}}$).

Two vectors are considered to be *equal* if one of them can be obtained from the other by translation (Fig. 17.1). Obviously, if the vector $\mathbf{a}$ is equal to $\mathbf{b}$ is equal to $\mathbf{a}$. If $\mathbf{a}$ is equal to $\mathbf{b}$, and $\mathbf{b}$ is equal to $\mathbf{c}$, then $\mathbf{a}$ is equal to $\mathbf{c}$.

The vectors are said to be *in the same direction (in opposite directions)* If they are parallel, and the terminal points of two vectors equal to them and reduced to a common origin are found on one side of the origin (on different sides of the origin).

The length of the line segment depicting a vector is called the *absolute value* of the vector.

Figure 17.1: Vector representation

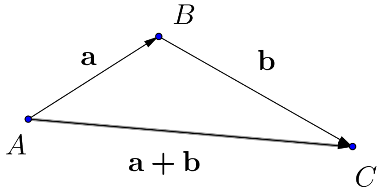

Figure 17.2: Vector addition

A vector of zero length (i.e. whose initial point coincides with the terminus) is termed the *zero vector*.

Vectors may be added or subtracted geometrically, i.e. we may speak of *addition* and *subtraction* of vectors. Namely, the *sum* of two vectors **a** and **b** is a third vector **a**+**b** which is obtained from the vectors **a** and **b** (or vectors equal to them) in the way shown in Fig. 17.2.

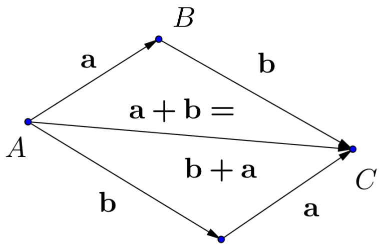

Figure 17.3: Commutativity of vector addition

Vector addition is commutative, i.e. for any vectors **a** and **b** (Fig. 17.3).

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

Vector addition is associative, i.e. if **a**, **b**, **c** are any vectors then

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

This property of addition, as also the preceding one, follows directly from the definition of the operation of addition (Fig. 17.4).

Let us mention here that if the vectors **a** and **b** are parallel, then the vector **a** + **b** (if it is not equal to zero) is parallel to the vectors **a** and **b**, and is in the same direction with the greater (by absolute value) vector. The absolute value of the vector **a** + **b** is equal to the sum of the absolute values of the vectors **a** and **b** if they are in the same direction, and to the difference of the absolute values if the vectors **a** and **b** are in opposite directions.

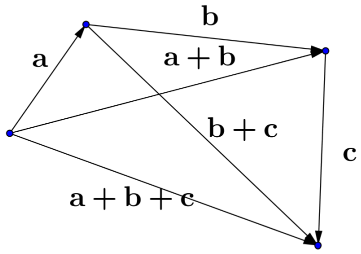

Figure 17.4: Associativity of vector addition

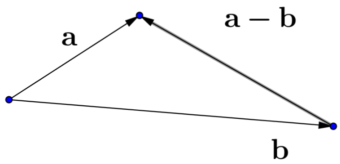

Figure 17.5: Vector subtraction

Subtraction of vectors is defined as the inverse operation of addition. Namely, the *difference* of the vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the vector $\mathbf{a} - \mathbf{b}$ which, together with the vector $\mathbf{b}$, yields the vector $\mathbf{a}$. Geometrically it is obtained from the vectors $\mathbf{a}$ and $\mathbf{b}$ (or vectors equal to them) as is shown in Fig. 17.5.

For any vectors $\mathbf{a}$ *and* $\mathbf{b}$ *we have following inequality*

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(the triangle inequality), geometrycally expressing the fact that in a triangle the sum of its two sides is greater than the third side if the vectors are not parallel. This inequality is obviously valid for any number of vectors:

$$|\mathbf{a} + \mathbf{b} + \cdots + \mathbf{l}| \leq |\mathbf{a}| + |\mathbf{b}| + \cdots + |\mathbf{l}|.$$

17.2 Exercises

1. Show that the sum of n vectors reduced to a common origin at the centre of a regular n-gon and with the terminal points at its vertices is equal to zero.

2. Three vectors have a common origin O and their terminal points are at the vertices of the triangle ABC. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \mathbf{0}$$

if and only if O is the point of intersection of the medians of the triangle.

3. Prove the identity

$$2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2.$$

To what geometrical fact does it correspond if $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors?

4. Show that the sign of equality in the triangle inequality takes place only when both vectors are in the same direction, or at least one of the vectors is equal to zero.

5. If the sum of the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ reduced to a common origin O is equal to zero and these vectors are not coplanar, then whatever is the plane α passing through the point O there can be found vectors $\mathbf{r}_i$ situated on both sides of the plane. Show this.

6. The vector $\mathbf{r}_{mn}$ lies in the xy-plane; its initial point is (x_0, y_0) and the terminus is the point $(m\delta, n\delta)$, where m and n are whole numbers not exceeding M and N by absolute value, respectively. Find the sum of all the vectors $\mathbf{r}_{mn}$ expressing it in terms of the vector with the initial point at $(0, 0)$ and the terminus at the point (x_0, y_0).

7. A finite figure F in the xy-plane has the origin as the centre of symmetry. Show that the sum of the vectors with a common origin and termini at the points whose coordinates are whole numbers of the figure F is equal to zero if and only if the origin of coordinates serves as their common initial point. (It is assumed that the figure F has at least one point whose coordinates are whole numbers.)

8. Express the vectors represented by the diagonals of a parallelepiped in terms of in terms of the vectors represented by its edges.

17.3 Multiplication of a vector by a number

Vectors may also be multiplied by a number. The *product* of the vector $\mathbf{a}$ by the number λ is defined as the vector $\mathbf{a}\lambda = \lambda\mathbf{a}$ the absolute value of which is obtained by multiplying the absolute value of the vector $\mathbf{a}$ by the absolute value of the number λ, i.e. $|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|$, the direction coinciding with the direction of the vector $\mathbf{a}$ or being in the opposite sense depending on whether $\lambda > 0$ or $\lambda < 0$. If $\lambda = 0$ or $\mathbf{a} = \mathbf{0}$, then $\lambda\mathbf{a}$ is considered to be equal to zero vector.

The multiplication of a vector by a number possesses the associative property and two distributive properties. Namely, for any number λ, μ and vectors $\mathbf{a}$, $\mathbf{b}$

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a} \qquad \textit{(associative property)}$$

$$\left.\begin{aligned}(\lambda+\mu)\mathbf{a} &= \lambda\mathbf{a}+\mu\mathbf{a},\\ \lambda(\mathbf{a}+\mathbf{b}) &= \lambda\mathbf{a}+\lambda\mathbf{b}\end{aligned}\right\} \qquad \textit{(distributive properties)}$$

Let us prove these properties.

The absolute values of the vectors $\lambda(\mu\mathbf{a})$ and $(\lambda\mu)\mathbf{a}$ are the same and are equal to $|\lambda|\,|\mu|\,|\mathbf{a}|$. The directions of these vectors either coincide, if λ and μ are of the same sign, or opposite if λ and μ have different signs. Hence, the vectors $\lambda(\mu\mathbf{a})$ and $(\lambda\mu)\mathbf{a}$ are equal by absolute value and are in he same direction, consequently, they are equal. If at least one of the numbers λ, μ or the vector $\mathbf{a}$ is equal to zero, then both vectors are equal to zero and, hence, they are equal to each other. The associative property is thus proved.

We are now going to prove the first distributive property:

$$(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}.$$

The equality is obvious if at least one of the numbers λ, μ or the vector $\mathbf{a}$ is equal to zero. Therefore, we may consider that λ, μ, and $\mathbf{a}$ are non-zero.

If λ and μ are of the same sign, then the vectors $\lambda\mathbf{a}$ and $\mu\mathbf{a}$ are in the same direction. Therefore, the absolute value of the vector $\lambda\mathbf{a}+\mu\mathbf{a}$ is equal to $|\lambda\mathbf{a}|+|\mu\mathbf{a}| = |\lambda|\,|\mathbf{a}|+|\mu|\,|\mathbf{a}| = (|\lambda|+|\mu|)|\mathbf{a}|$. The absolute value of the vector $(\lambda+\mu)\mathbf{a}$ is equal to $|\lambda+\mu|\,|\mathbf{a}| = (|\lambda|+|\mu|)|\mathbf{a}|$. Thus, the absolute values of the vectors $(\lambda+\mu)\mathbf{a}$ and $\lambda\mathbf{a}+\mu\mathbf{a}$ are equal and they are in the same direction. Namely, for $\lambda > 0$, $\mu > 0$ their directions coincide with the direction of $\mathbf{a}$, and if $\lambda < 0$, $\mu < 0$ they are opposite to $\mathbf{a}$. The case when λ and μ have different signs is considered in a similar way.

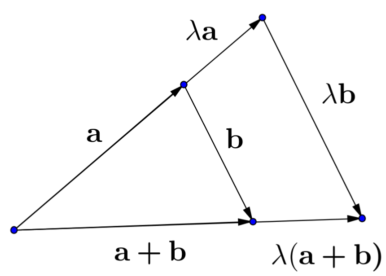

Figure 17.6: Distributive law

Let us prove the second distributive property:

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}.$$

The property is obvious if one of the vectors or the number λ is equal to zero. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{b}$ can be represented in the form $\mathbf{b} = \mu\mathbf{a}$. And the second distributive property follows from the first one. Indeed,

$$\lambda(\mathbf{1}+\mu)\mathbf{a} = \lambda(\mathbf{a}+\mu\mathbf{a}) = \lambda\mathbf{a} + \lambda\mu\mathbf{a}.$$

Hence

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}.$$

Let $\mathbf{a}$ and $\mathbf{b}$ be non-parallel vectors, then for $\lambda > 0$ the vector $\overrightarrow{AB}$ (Fig. 17.6) represents, on the one hand, $\lambda\mathbf{a}+\lambda\mathbf{b}$, and $\lambda\overrightarrow{AC}$ equal to $\lambda(\mathbf{a}+\mathbf{b})$ on the other. If $\lambda < 0$, then both vectors reverse their directions.

17.4 Exercises

1. The vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots$ are called *linearly independent* if there exist no numbers $\lambda_1, \lambda_2, \ldots$, (at least one of which is non-zero) such that

$$\lambda_1\mathbf{r}_1 + \lambda_2\mathbf{r}_2 + \cdots = \mathbf{0}$$

Show that two vectors are linearly independent if and only if they are non-zero and non-parallel.

Show that three vectors are linearly independent when and only when they are non-zero and there is no plane parallel to them.

2. Show that any three vectors lying in one plane are always linearly dependent.

3. Show that if two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in a plane are linearly independent, then any vector $\mathbf{r}$ in this plane is expressed linearly in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\mathbf{r} = \lambda_1\mathbf{r}_1 + \lambda_2\mathbf{r}_2.$$

The numbers λ_1 and λ_2 are defined uniquely.

4. Show that if three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are linearly independent, then any vector $\mathbf{r}$ is uniquely expressed in terms of these vectors in the form

$$\mathbf{r} = \lambda_1\mathbf{r}_1 + \lambda_2\mathbf{r}_2 + \lambda_3\mathbf{r}_3.$$

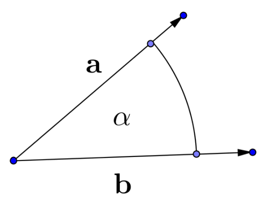

Figure 17.7: Scalar product

17.5 Scalar product of vectors

The *angle* between the vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the angle between the vectors equal to $\mathbf{a}$ and $\mathbf{b}$, respectively, reduced to a common origin (Fig. 17.7). The *scalar product* of a vector $\mathbf{a}$ by a vector $\mathbf{b}$ is defined as the number $\mathbf{ab}$ which is equal to the product of the absolute value of the vectors by the cosine of the angle between them.

The scalar product possesses the following obvious, properties which follow directly from its definition:

(1) $\mathbf{ab} = \mathbf{ba}$;

(2) $\mathbf{a}^2 = \mathbf{aa} = |\mathbf{a}|^2$;

(3) $(\lambda\mathbf{a})\mathbf{b} = \lambda(\mathbf{ab})$;

(4) if $|\mathbf{e}| = \mathbf{1}$, *then* $(\lambda\mathbf{e})(\mu\mathbf{e}) = \lambda\mu$;

(5) *the scalar product of vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is equal to zero if and only if the vectors are mutually perpendicular or one of them is equal to zero.*

The *projection* of a vector $\mathbf{a}$ on a straight line is defined as the vector $\bar{\mathbf{a}}$ whose initial points is the projection of the initial point of the vector $\mathbf{a}$ and whose terminal point is the projection of the terminal point of the vector $\mathbf{a}$.

Obviously, equal vectors have equal projections, the projection of the sum of vectors is equal to the sum of the projections (Fig. 17.8).

The scalar product of a vector $\mathbf{a}$ by a vector $\mathbf{b}$ is equal to the scalar product of the projection of the vector $\mathbf{a}$ onto the straight line containing the vector $\mathbf{b}$ by the vector $\mathbf{b}$. The proof is obvious. It is sufficient to note that $\mathbf{ab}$ and $\bar{\mathbf{a}}\mathbf{b}$ are equal by absolute value and have the same sign.

The scalar product possesses the distributive property. Namely for any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

$$(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}.$$

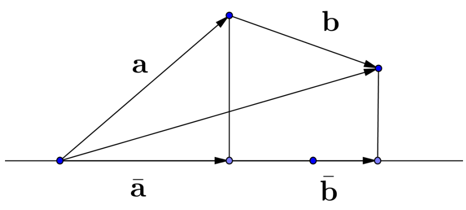

Figure 17.8: Projection of vectors

The statement is obvious if one of the vectors is equal to zero. Let all the vectors be non-zero. Denoting by $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{a}+\mathbf{b}}$ the projections of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ onto the line containing the vector $\mathbf{c}$, we have

$$(\mathbf{a}+\mathbf{b})\mathbf{c} = \overline{(\mathbf{a}+\mathbf{b})}\mathbf{c} = (\overline{\mathbf{a}}+\overline{\mathbf{b}})\mathbf{c},$$

$$\mathbf{ac}+\mathbf{bc} = \overline{\mathbf{a}}\mathbf{c}+\overline{\mathbf{b}}\mathbf{c}.$$

Let $\mathbf{e}$ be a unit vector parallel to $\mathbf{c}$. Then $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$, and $\mathbf{c}$ allow the representations $\overline{\mathbf{a}} = \lambda\mathbf{e}$, $\overline{\mathbf{b}} = \mu\mathbf{e}$, $\mathbf{c} = \nu\mathbf{e}$. We obtain

$$(\overline{\mathbf{a}}+\overline{\mathbf{b}})\mathbf{c} = (\lambda\mathbf{e}+\mu\mathbf{e})\nu\mathbf{e} = (\lambda+\mu)\nu,$$

$$\overline{\mathbf{a}}\mathbf{c}+\overline{\mathbf{b}}\mathbf{c} = \lambda\mathbf{e}\nu\mathbf{e}+\mu\mathbf{e}\nu\mathbf{e} = \lambda\nu+\mu\nu.$$

Whence

$$(\overline{\mathbf{a}}+\overline{\mathbf{b}})\mathbf{c} = \overline{\mathbf{a}}\mathbf{c}+\overline{\mathbf{b}}\mathbf{c}$$

and, hence

$$(\mathbf{a}+\mathbf{b})\mathbf{c} = \mathbf{ac}+\mathbf{bc}.$$

In conclusion we are going to show that *if* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are non-zero vectors which are not parallel to one plane, then from the equalities*

$$\mathbf{ra} = 0, \qquad \mathbf{rb} = 0, \qquad \mathbf{rc} = 0$$

if follows that $\mathbf{r} = \mathbf{0}$.

Indeed, if $\mathbf{r} \neq \mathbf{0}$, then from the above three equalities it follows that the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are perpendicular to $\mathbf{r}$, and therefore parallel to the plane perpendicular to $\mathbf{r}$ which is impossible.

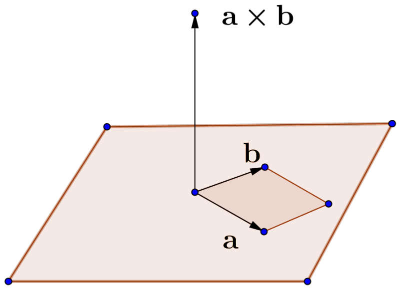

Figure 17.9: Vector product of two vectors

17.6 Exercises

1. Let $A_1, A_2, \dots, A_n$ be the vertices of a regular n-gon. Then $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_nA_1} = 0$. Drive from this that

$$1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{(2n-2)\pi}{n} = 0,$$
$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{(2n-2)\pi}{n} = 0.$$

2. Show that if $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, then $\lambda^2\mathbf{a}^2 + 2\mu\lambda(\mathbf{ab}) + \mu^2\mathbf{b}^2 \geq 0$, the equality to zero taking place only if $\lambda = 0$, and $\mu = 0$.

17.7 The vector product of vectors

The *vector product* of a vector $\mathbf{a}$ by a vector $\mathbf{b}$ is a third vector $\mathbf{a} \times \mathbf{b}$ defined in the following way. If at least one of the vectors $\mathbf{a}$, $\mathbf{b}$ is equal to zero or the vectors are parallel, then $\mathbf{a} \times \mathbf{b} = 0$. in other cases this vector (by its absolute value) is equal to the area of the parallelogram constructed on the vectors $\mathbf{a}$ and $\mathbf{b}$ as sides and is directed perpendicular to the plane containing this parallelogram so that the rotation in the direction from $\mathbf{a}$ to $\mathbf{b}$ and the direction of $\mathbf{a} \times \mathbf{b}$ form a "right-hand screw" (Fig. 17.9).

From the definition of the vector product it directly follows:

(1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$,

(2) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, *where* θ *is the angle formed by the vectors* $\mathbf{a}$ *and* $\mathbf{b}$;

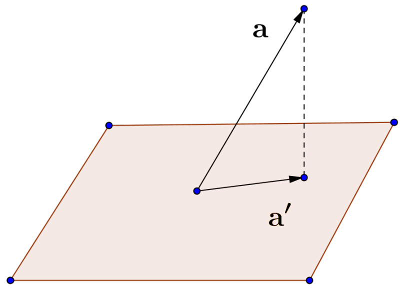

Figure 17.10: Projection on a plane

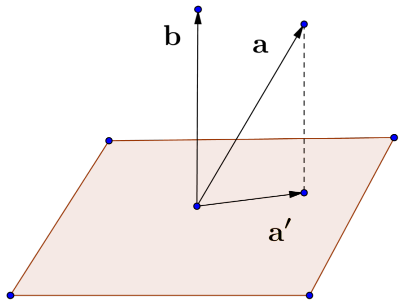

Figure 17.11: Projection on perpendicular plane

(3) $(\lambda\mathbf{a})\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})$.

The *projection* of a vector $\mathbf{a}$ on a plane is defined as the vector $\mathbf{a}'$ whose initial point is the projection of the initial point of the vector $\mathbf{a}$ and whose terminal point is the projection of the terminal point of the vector $\mathbf{a}$. Obviously, equal vectors have equal projections and the projection of the sum of vectors is equal to the sum of the projections (Fig. 17.10).

Suppose we have two vectors $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{a}'$ denote the projection of the vector $\mathbf{a}$ on the plane perpendicular to the vector $\mathbf{b}$ (Fig. 17.11). Then

$$\mathbf{a}\times\mathbf{b}=\mathbf{a}'\times\mathbf{b}.$$

The proof is obvious. It is sufficient to mention that the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}'\times\mathbf{b}$ have equal absolute values and are in the same direction.

The vector product possesses a distributive property, i.e. for any there vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}. \qquad (*)$$

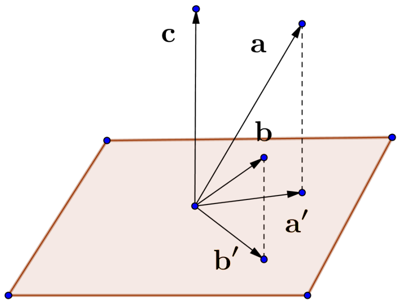

Figure 17.12: Distributive law of vector product

The assertion is obvious if $\mathbf{c} = \mathbf{0}$. It is then obvious that the equality (∗) is sufficient to the for the case $|\mathbf{c}| = \mathbf{1}$, since in the general case it will then follow the above mentioned property (3).

So, let $|\mathbf{c}| = \mathbf{1}$, and let $\mathbf{a}'$ and $\mathbf{b}'$ denote the projections of the vectors $\mathbf{a}$ and $\mathbf{b}$ on the plane perpendicular to the vector $\mathbf{c}$ (Fig. 17.12). Then the vectors $\mathbf{a}' \times \mathbf{c}$, $\mathbf{b}' \times \mathbf{c}$ and $(\mathbf{a}' + \mathbf{b}') \times \mathbf{c}$ are obtained from the vectors $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{a}' + \mathbf{b}'$, respectively, by a rotating through an angle of 90°. Consequently,

$$(\mathbf{a}' + \mathbf{b}') \times \mathbf{c} = \mathbf{a}' \times \mathbf{c} + \mathbf{b}' \times \mathbf{c}.$$

And since

$$\mathbf{a}' \times \mathbf{c} = \mathbf{a} \times \mathbf{c}, \qquad \mathbf{b}' \times \mathbf{c} = \mathbf{b} \times \mathbf{c},$$
$$(\mathbf{a}' + \mathbf{b}') \times \mathbf{c} = (\mathbf{a} + \mathbf{b}) \times \mathbf{c},$$

we get

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c},$$

which was required to be proved.

Let us mention the following simple identity which is true for any vectors $\mathbf{a}$ and $\mathbf{b}$:

$$(\mathbf{a} \times \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{ab})^2.$$

Indeed, if θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then this indentity expresses that

$$(|\mathbf{a}|\,|\mathbf{b}|\sin\theta)^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}|\,|\mathbf{b}|\cos\theta)^2$$

and, consequently, is obvious.

17.8 Exercises

1. If the vectors **a** and **b** are perpendicular to the vector **c**, then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}.$$

Show this.

2. If the vector **b** is perpendicular to **c**, and the vector **a** is parallel to the vector **c**, then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{ac}).$$

Show this.

3. For an arbitrary vector **a** and a vector **b** perpendicular to **c**

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{ac}).$$

Show this.

4. Show that for any three vectors **a**, **b**, **c**

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{ac}) - \mathbf{a}(\mathbf{bc}).$$

5. Find the area of the base of a triangular pyramid whose lateral edges are equal to l, the vertex angles being equal to α, β, γ.

17.9 The triple product of vectors

The *triple (scalar) product* of vectors **a**, **b**, **c** is the number

$$(\mathbf{abc}) = (\mathbf{a} \times \mathbf{b})\mathbf{c}. \qquad (*)$$

Obviously, *the triple product is equal to zero if and only if one of the vectors is equal to zero or all three vectors are parallel to one plane.*

The numerical value of the triple product of non-zero vectors **a**, **b**, **c** which are not parallel to one plane is equal to the volume of the parallelepiped of which the vectors **a**, **b**, **c** are coterminal sides (Fig. 17.13).

Indeed, $\mathbf{a} \times \mathbf{b} = S\mathbf{e}$, where S is the are of the base of the parallelepiped constructed on the vectors **a**, **b**, and **e** is the unit vector perpendicular to the base. Further, **ec** is equal up to a single to the altitude of the parallelepiped dropped onto the mentioned base. Consequently, up to a sign, (**abc**) is equal to the volume of the parallelepiped constructed on the vectors **a**, **b**, and **c**. *The triple product possesses the following property*

$$(\mathbf{abc}) = \mathbf{a}(\mathbf{b} \times \mathbf{c}). \qquad (**)$$

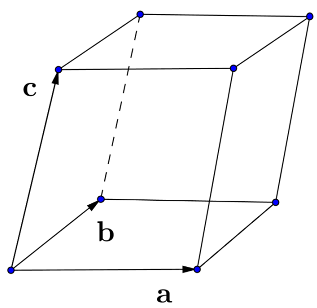

Figure 17.13: Meaning of triple product

It is sufficient to note that the right-hand and the left-hand members are equal by absolute value and have the same sign. From the definition $(*)$ of the triple product and the property $(**)$ it follows that an *interchange of any two factors reverses the sign of the triple product. In particular, the triple product is equal to zero if two factors are equal to each other.*

17.10 Exercises

1. Noting that

$$((\mathbf{a} \times \mathbf{b}) \times \mathbf{c})\mathbf{d} = (\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d}),$$

derive the identity

$$(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{ac} & \mathbf{ad} \\ \mathbf{bc} & \mathbf{bd} \end{vmatrix}.$$

2. With the aid of the identity

$$(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{b}) = (\mathbf{ac})\mathbf{b}^2 - (\mathbf{ab})(\mathbf{bc})$$

derive the formula of spherical trigonometry where α, β, γ are the sides of a triangle on the unit sphere, and B is the angle of this triangle opposite to the side β.

3. Derive the identity

$$(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d}) = \mathbf{b}(\mathbf{acd}) - \mathbf{a}(\mathbf{bcd}).$$

Chapter 18

Rectangular Cartesian Coordinates in Space

18.1 Cartesian coordinates

Let us draw from an arbitrary point O in space three straight lines Ox, Oy, Oz not lying in one plane, and lay off on each of them from the point O three non-zero vectors e_x, e_y, e_z (Fig. 18.1). According to Section 17.6, any vector $\overrightarrow{OA}$ allows a unique representation of the form

$$\overrightarrow{OA} = xe_x + ye_y + ze_z.$$

The *numbers* x, y z are called the *Cartesian coordinates* of a point A.

The straight lines Ox, Oy, Oz are termed the *coordinate axes*: Ox is the x-axis, Oy is the y-axis, and Oz is the z-axis. The planes Oxy, Oyz, Oxz are called the *coordinate planes*: Oxy is the xy-plane, Oyz is the yz-plane, and Oxz is the xz-plane.

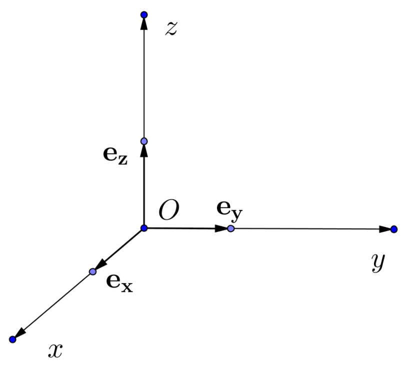

Figure 18.1: Coordinate axes in space

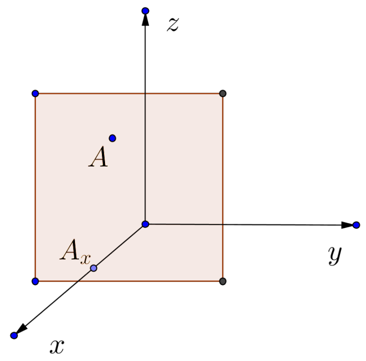

Figure 18.2: Coordinates in space

Each of the coordinate axes is divided by the point O (i.e. by the *origin of coordinates*) into two semi-axes. Those of the semi-axes whose directions coincide with the directions of the vectors e_x, e_y, e_z are said to be *positive*, the others being *negative*. The coordinate system thus obtained is called *right-handed* if $(e_x e_y e_z) > 0$, and *left-handed* if $(e_x e_y e_z) < 0$.

Geometrically the coordinates of the point A are obtained in the following way. We draw through the point A a plane parallel to the yz-plane. It intersects the x-axis at a point A_x (Fig. 18.2). Then the absolute value of the coordinate x of the point A is equal to the length of the line segment OA_x as measured by the unit length $|e_x|$.

It is positive if A_x belongs to the positive semi-axis x, and is negative if A_x belongs to the negative semi-axis x. To make sure of this is sufficient to recall how the coordinates of the vector $\overrightarrow{OA}$ relative to the basis e_x, e_y, e_z are determined. The other two coordinates of the point (y and z) are determined by a similar construction.

If the coordinate axes are mutually perpendicular, and e_x, e_y, e_z are the unit vectors, then the coordinates are called the *rectangular Cartesian coordinates*.

Cartesian coordinates on the plane are introduced in a similar way. Namely, we draw from the point O (i.e. from the origin of coordinates) two arbitrary straight lines Ox and Oy (the coordinate axes) and lay off on each axis (from the point O) a non-zero vector. Thus we obtain the vectors e_x and e_y. The Cartesian coordinates of an arbitrary point A are then determined as the coordinates of the vector $\overrightarrow{OA}$ relative to the basis e_x, e_y.

Obviously, if the coordinate axes are mutually perpendicular, and e_x, e_y are unit vectors, then the coordinates defined in this way coincide with those introduced in Section 15.1 and are called the *rectangular Cartesian* *coordinates.*

Below, as a rule, we shall use the rectangular Cartesian coordinates. If otherwise, each case will be supplied with a special mention.

18.2 Exercises

1. Where are the points in space located if: (a) $x = 0$; (b) $y = 0$; (c) $z = 0$; (d) $x = 0, y = 0$; (e) $y = 0, z = 0$; (f) $z = 0, x = 0$?

2. How many points in space satisfy the following conditions

$$|x| = a, \qquad |y| = b, \qquad |z| = c, \qquad \text{if} \quad abc \neq 0?$$

3. Where are the points in space situated if

$$|x| < a, \qquad |y| < b, \qquad |z| < c?$$

4. Let A be a vertex of a parallelepiped, A_1, A_2, A_3 the vertices adjacent to A, i.e. the end-points of the edges emanating from A. Find the coordinates of all the vertices of the parallelepiped, taking the vertex A for the origin and the vertices A_1, A_2, A_3 for the end-points of the basis vectors.

5. Find the coordinates of the point into which the point (x, y, z) goes when rotated about the straight line joining the point $A_0(a, b, c)$ to the origin through an angle of $\alpha = \pi/2$. The coordinate system is rectangular.

6. Solve Exercises 5 for an arbitrary α.

18.3 Elementary problems of solid analytic geometry

Let there be introduced in space Cartesian coordinates xyz and let $A_1(x_1, y_1, z_1)$ and $A_2(x_2, y_2, z_2)$ be two arbitrary points is pace. *Find the coordinates of the point A which divides the line segment A_1A_2 in the ratio $\lambda_1 : \lambda_2$* (Fig. 18.3).

The vectors $\overrightarrow{A_1A}$ and $\overrightarrow{AA_2}$ are in the same direction, and their absolute values are as $\lambda_1 : \lambda_2$. Consequently,

$$\lambda_2\overrightarrow{A_1A} - \lambda_1\overrightarrow{AA_2} = 0,$$

or

$$\lambda_2(\overrightarrow{OA} - \overrightarrow{OA_1}) - \lambda_2(\overrightarrow{OA_2} - \overrightarrow{OA}) = 0.$$

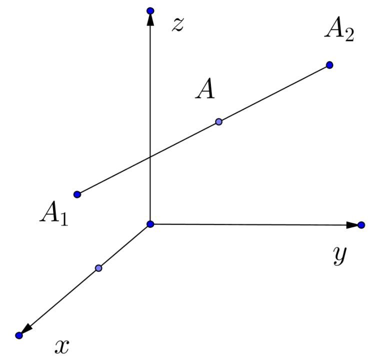

Figure 18.3: Division of a segment in space

Whence

$$\overrightarrow{OA} = \frac{\lambda_2\overrightarrow{OA_1} + \lambda_1\overrightarrow{OA_2}}{\lambda_1 + \lambda_2}.$$

Since the coordinates of the points $A(x, y, z)$ are the same as the coordinates of the vector $\overrightarrow{OA}$, we have

$$x = \frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2},$$

$$y = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2},$$

$$z = \frac{\lambda_2 z_1 + \lambda_1 z_2}{\lambda_1 + \lambda_2}.$$

Let the coordinate system be rectangular. *Express the distance between the points A_1 and A_2 in terms of their coordinates.*

The distance between the points A_1 and A_2 is equal to the absolute value of the vector $\overrightarrow{A_1A_2}$ (Fig. 18.4). We have

$$\overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = e_x(x_2 - x_1) + e_y(y_2 - y_1) + e_z(z_2 - z_1).$$

Whence

$$(A_1A_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Express the area of a triangle in the xy-plane in terms of the coordinates of its vertices: $A_1(x_1, y_1, 0)$, $A_2(x_2, y_2, 0)$, and $A_3(x_3, y_3, 0)$.

The absolute value of the vector $\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}$ is equal to twice the area of the triangle $A_1A_2A_3$;

$$\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3} = e_z \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

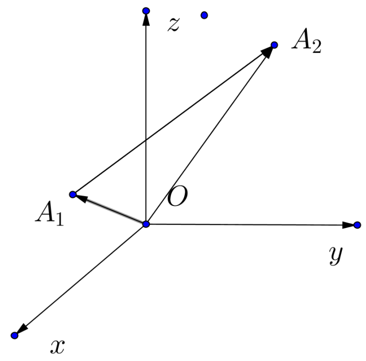

Figure 18.4: The distance of two points

Consequently, the area of the triangle

$$S = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Express the volume of a tetrahedron $A_1A_2A_3A_4$ in terms of the coordinates of its vertices.

The triple scalar product of the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ is equal (up to a sign) to the volume of the parallelepiped constructed on these vectors and, consequently, to six times the volume of the tetrahedron $A_1A_2A_3A_4$. Hence

$$V = \frac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}.$$

18.4 Exercises

1. Find the distance between two points expressed in terms of Cartesian coordinates if the positive semi-axes form pairwise the angles α, β, γ, and e_x, e_y, e_z are unit vectors.

2. Find the centre of a sphere circumsribed about a tetrahedron with the vertices $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, 0, 0)$.

3. Prove that the straight lines joining the mid-points of the opposite edges of a tetrahedron intersect at one point. Express the coordinates of this ponint in terms of the coordinates of the vertices of the tetrahedron.

4. Prove that the straight lines joining the vertices of a tetrahedron to the centroids of the opposite faces intersect at point. Express its coordinates in terms of the coordinates of the vertices of the tetrahedron.

18.5 Equations of a surface and a curve in space

Suppose we have a surface.

The equation

$$f(x, y, z) = 0 \qquad (*)$$

is called the *equation of a surface in implicit form* if the coordinates of any point of the surface satisfy this equation. And conversely, any three numbers x, y, z satisfying the equation $(*)$ represent the coordinates of one of the points of the surface.

The system of equations

$$x = f_1(u, v), \qquad y = f_2(u, v), \qquad z = f_3(u, v), \qquad (**)$$

specifying the coordinates of the points of the surfaces as a function of two parameters (u, v) is called the *parametric equation of a surface.*

Eliminating the parameters u, v from the system $(**)$, we can obtain the implicit equation of a surface.

Form the equation of an arbitrary sphere in the rectangular Cartesian coordinates xyz.

Let (x_0, y_0, z_0) be the centre of the sphere, and R its radius. Each point (x, y, z) of the sphere is located at a distance R from the centre, and, consequently, satisfies the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0. \qquad (* * *)$$

Conversely, any point (x, y, z) satisfying the equation $(***)$ is found at a distance R from (x_0, y_0, z_0) and, consequently, belong to the sphere. According to the definition, the equation $(* * *)$ is the equation of a sphere.

Form the equation of a circular cylinder with the axis Oz and radius R (Fig. 18.5). Let us take the coordinate $z(v)$ and the angle (u) formed by the plane passing through the z-axis and the point (x, y, z) with the xz-plane as the parameters u, v, characterizing the position of the point (x, y, z) on the cylinder. We then get

$$x = R\cos u, \qquad y = R\sin u, \qquad z = v,$$

which is the required *equation of the cylinder in parametric form.*

Squaring the first two equations and adding termwise, we get the *equation of the cylinder in implicit form*:

$$x^2 + y^2 = R^2.$$

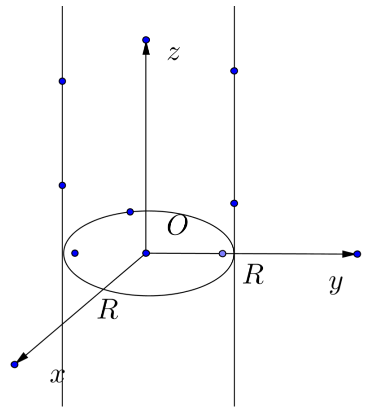

Figure 18.5: Exercise 3

Suppose we have a curve in space. The system of equations

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0$$

is called the *equation of a curve in implicit form* if the coordinates of each point of the curve satisfy both equations. And conversely, any three numbers satisfying both equations represent the coordinates of some point on the curve.

A system of equations

$$x = \varphi_1(t), \qquad y = \varphi_2(t), \qquad z = \varphi_3(t),$$

specifying the coordinates of points of the curve as a function of some parameter (t) is termed the *equation of a curve in parametric form.*

Two surfaces intersect, as a rule, along a curve. Obviously, if the surfaces are specified by equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the curve along which they intersect is represented by a system of equations

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0.$$

Form the equation of an arbitrary circle is space. Any circle can be represented as an intersection of two spheres. Consequently, any circle can be specified by a system of equations

$$\left.\begin{aligned}(x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 - R_1^2 = 0,\\ (x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2 - R_2^2 = 0.\end{aligned}\right\}$$

As a rule, a curve and a surface intersect at separate points. If the surface is specified by the equation $f(x, y, z) = 0$, and the curve by the equations

$f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the points of intersection of the curve and the surface satisfy the following system of equations:

$$f(x, y, z) = 0, \qquad f_1 = (x, y, z) = 0, \qquad f_2(x, y, z) = 0.$$

Solving this system, we find the coordinates of the points of intersection.

Chapter 19

A Plane and a Straight Line

19.1 The equation of a plane

Form the equation of an arbitrary plane in the rectangular Cartesian coordinates xyz.

Let $A_0(x_0, y_0, z_0)$ be a point in a plane and $\mathbf{n}$ a nonzero vector perpendicular to the plane. Then whatever the point of the plane $A(x, y, z)$ is, the vectors $\overrightarrow{A_0A}$ and $\mathbf{n}$ are mutually perpendicular (Fig. 19.1). Hence,

$$\overrightarrow{A_0A} \cdot \mathbf{n} = 0. \tag{$*$}$$

Let a, b, c be the coordinates of the vector $\mathbf{n}$ with respect to the basis e_x e_y, e_z.

Then, since $\overrightarrow{A_0A} = \overrightarrow{OA} - \overrightarrow{OA_0}$, it follows from $(*)$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{$**$}$$

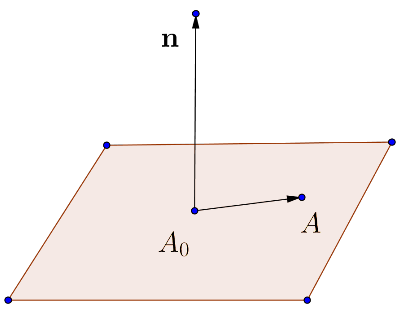

Figure 19.1: Equation of a plane

This is the required equation.

Thus, *the equation of any plane is linear relative to the coordinates* x, y, z.

Since the formulas for transition from one Cartesian system of coordinates to another are linear, we may state that *the equation of a plane is linear in any Cartesian system of coordinates* (but not only in a rectangular one).

Let us now show that *any equation of the form*

$$ax + by + cz + d = 0$$

is the equation of a plane.

Let x_0, y_0, z_0 be a solution of the given equation. Then

$$ax_0 + by_0 + cz_0 + d = 0$$

and the equation may be rewritten in the from

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (***)$$

Let $\mathbf{n}$ be a vector with the coordinates a, b, c with respect to the basis e_x, e_y, e_z, A_0 a point with the coordinates x_0, y_0, z_0 and A a point with the coordinates x, y, z. Then the equation $(***)$ can be written in the equivalent form

$$\overrightarrow{A_0A} \cdot \mathbf{n} = 0.$$

Whence it follows that all points of the plane passing through the point A_0 and perpendicular to the vector $\mathbf{n}$ (and only they) satisfy the given equation and, consequently, it is the equation of this plane.

Let us note that the coefficients of x, y, z in the equation of the plane are the coordinates of the vector perpendicular to the plane relative to the basis e_x e_y, e_z.

19.2 Exercises

1. Form the equation of a plane given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) situated symmetrically about it.

2. Show that the planes

$$ax + by + cz + d_1 = 0,$$
$$ax + by + cz + d_2 = 0, \qquad d_1 \neq d_2,$$

are parallel (do not intersect).

3. What is the locus of points whose coordinates satisfy the equation

$$(ax + by + cz + d)^2 - (\alpha x + \beta y + \gamma z + \delta)^2 = 0?$$

4. Show that the curve represented by the equations

$$f(x, y, z) + a_1 x + b_1 y + c_1 z + d_1 = 0,$$
$$f(x, y, z) + a_2 x + b_2 y + c_2 z + d_1 = 0,$$

is a plane one, i.e. all points of this curve belong to a plane.

5. Show that the three planes specified by the equations

$$ax + by + cz + d = 0,$$
$$\alpha x + \beta y + \gamma z + d = 0,$$
$$\lambda(ax + by + cz) + \mu(alphax + \beta y + \gamma z) + k = 0,$$

have no points in common if $k \neq \lambda d + \mu\delta$.

6. Write the equation of the plane passing through the circle of intersection of the two spheres

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$
$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0.$$

7. Show that inversion transforms a sphere either into a sphere or into a plane.

8. Show that the equation of any plane passing through the line of intersection of the planes

$$ax + by + cz + d = 0,$$
$$\alpha x + \beta y + \gamma z + \delta = 0,$$

can be represented in the from

$$\lambda(ax + by + cz + d) + \mu(\alpha x + \beta y + \gamma z + \delta) = 0.$$

9. Show that the plane passing through the three given points (x_i, y_i, z_i) $(i = 1, 2, 3)$ is specified by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

19.3 Special positions of a plane relative to coordinate system

Let us find out the peculiarities of the of the position of a plane relative to coordinate system which take place when its equation is of this or that particular form.

1. $a = 0$, $b = 0$. Vector $\mathbf{n}$ (perpendicular to the plane) is parallel to the z-axis. The plane is parallel to the xy-plane. In particular, it coincides with the xy-plane if d is also zero.

2. $b = 0$, $c = 0$. The plane is parallel to the yz-plane and coincides with it if $d = 0$.

3. $c = 0$, $a = 0$. The plane is parallel to the xz-plane and coincides with it if $d = 0$.

4. $a = 0$, $b \neq 0$, $c \neq 0$. Vector $\mathbf{n}$ is perpendicular to the x-axis: $e_x\mathbf{n} = 0$. The plane is parallel to the x-axis, in particular, it passes through it if $d = 0$.

5. $a \neq 0$, $b = 0$, $c \neq 0$. The plane is parallel to the y-axis and passes through it if $d = 0$.

6. $a \neq 0$, $b \neq 0$, $c = 0$. The plane is parallel to the z-axis and passes through it if $d = 0$.

7. $d = 0$. The plane passes through the origin (whose coordinates 0, 0, 0 satisfy the equation of the plane).

If all the coefficients are non-zero, then the equation may be divided by $-d$. Then, putting

$$-\frac{d}{a} = \alpha, \qquad -\frac{d}{b} = \beta, \qquad -\frac{d}{c} = \gamma,$$

we get the equation of the plane the following form:

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1. \tag{$*$}$$

The numbers α, β, γ are equal (up to a sign) to the segments intercepted by the plane on the coordinate axes. Indeed, the x-axis ($y = 0$, $z = 0$) is intersected by the plane at point $(\alpha, 0, 0)$, the y-axis at point $(0, \beta, 0)$, and the z-axis at point $(0, 0, \gamma)$. The equation $(*)$ is called the *intercept form of the equation of a plane*.

We conclude with a note that any plane not perpendicular to the xy-plane $(c \neq 0)$ may be specified by an equation of the form

$$z = px + qy + l.$$

19.4 Exercises

1. Find the conditions under which the plane

$$ax + by + cz + d = 0$$

intersects the positive semi-axis $x(y, z)$.

2. Find the volume of the tetrahedron bounded by the coordinate planes and the plane

$$ax + by + cz + d = 0$$

if $abcd \neq 0$.

3. Prove that the points in space for which

$$|x| + |y| + |z| < a,$$

are situated inside an octahedron with centre at the origin and the on the vertices coordinate axes.

4. Given a plane σ by the equation in rectangular Cartesian coordinates

$$ax + by + cz + d = 0.$$

Form the equation of the plane σ' symmetrical to σ about the xy-plane (about the origin O).

5. Given a family of planes depending on a parameter

$$ax + by + cz + d + \lambda(\alpha x + \beta y + \gamma z + \delta) = 0.$$

Find in this family a plane parallel to the z-axis.

7. In the family of planes

$$(a_1 x + b_1 y + c_1 z + d_1) + \lambda(a_2 x + b_2 y + c_2 z + d_2) + \mu(a_3 x + b_3 y + c_3 z + d_3) = 0$$

find the plane parallel to the xy-plane. The parameters of the family are λ and μ.

19.5 The normal form of the equation of a plane

If a point $A(x, y, z)$ belongs to the plane

$$ax + by + cz + d = 0, \tag{$*$}$$

then its coordinates satisfy the equation $(*)$.

Let us find out what geometrical meaning has the expression

$$ax + by + cz + d$$

if the point A does not belong to the plane.

We drop from the point A a perpendicular onto the plane. Let $A_0(x_0, y_0, z_0)$ be the foot of the perpendicular. Since the point A_0 lies on the plane, then

$$ax_0 + by_0 + cz_0 + d = 0.$$

Whence

$$ax + by + cz + d = a(x - x_0) + b(y - y_0) + c(z - z_0) = \mathbf{n} \cdot \overrightarrow{A_0A} = \pm|\mathbf{n}|\delta,$$

where $\mathbf{n}$ is a vector perpendicular to the plane, with the coordinates a, b, c, and δ is the distance of the point A form the plane.

Thus

$$ax + by + cz + d$$

is positive on one side of the plane, and negative on the other, its absolute value being proportional to the distance of the point A from the plane. The proportionality factor

$$\pm|\mathbf{n}| = \sqrt{a^2 + b^2 + c^2}.$$

If in the equation of the plane

$$a^2 + b^2 + c^2 = 1,$$

then

$$ax + by + cz + d,$$

will be equal up to a sign to the distance of the point from the plane. In this case the plane is said to be specified by an *equation in the normal form.*

Obviously, to obtain the normal form of the equation of a plane $(*)$, it is sufficient to divide it by

$$\pm\sqrt{a^2 + b^2 + c^2}.$$

19.6 Exercises

1. The planes specified by the equations in rectangular Cartesian coordinates

$$ax + by + cz + d = 0,$$
$$ax + by + cz + d' = 0$$

where $d \neq d'$, have no points in common, hence, they are parallel. Find the distance between these planes.

2. The plane

$$ax + by + cz + d = 0$$

is parallel to z-axis. Find the distance of the z-axis from this plane.

3. What is the locus of points whose distance to two given planes are in a given ratio?

4. Form the equations of the planes parallel to the plane

$$ax + by + cz + d = 0$$

and located at a distance δ from it.

5. Show that the points in space satisfying the condition

$$|ax + by + cz + d| < \delta^2,$$

are situated between the parallel planes

$$ax + by + cz + d \pm \delta^2 = 0.$$

6. Given are the equations of the planes containing the faces of a tetrahedron and a point M by its coordinates. How to find out whether or not the point M lies inside the tetrahedron?

7. Derive the formulas for transition to a new system of rectangular Cartesian coordinates $x'y'z'$ if the new coordinate plane are specified in the old system by the equations

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

19.7 Relative position of planes

Suppose we two planes

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0,\\ a_2x + b_2y + c_2z + d_2 = 0. \end{aligned}\right\} \qquad (*)$$

Find out under which condition these planes are: (a) *parallel*, (b) *mutually perpendicular*.

Since a_1, b_1, c_1 are the coordinates of vector $\mathbf{n}_1$ perpendicular to the first plane, and a_2, b_2, c_2 are the coordinates of vector $\mathbf{n}_2$ which is perpendicular to the second plane, the planes are parallel if the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ are parallel, i.e. if their coordinates are proportional:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Moreover, this condition is sufficient for parallelism of the planes if they are not coincident.

For the planes $(*)$ to be mutually perpendicular it is necessary and sufficient that the mentioned vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are mutually perpendicular, which for non-zero vectors is equivalent to the condition

$$\mathbf{n}_1\mathbf{n}_2 = 0 \quad \text{or} \quad a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Let the equations $(*)$ specify two arbitrary planes. *Find the angle made by these planes.*

The angle θ between the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is equal to one the angles formed by planes and is readily found. We have

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1|\,|\mathbf{n}_2| \cos\theta.$$

Whence

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

19.8 Equations of the straight line

Any straight line can be specified as an intersection of two planes. Consequently, *any straight line can be specified by the equations*

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0,\\ a_2x + b_2y + c_2z + d_2 = 0. \end{aligned}\right\} \qquad (*)$$

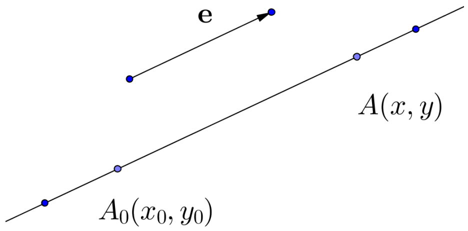

Figure 19.2: Equations of a line i space

the first which represents one plane and the second the other. Conversely, *any compatible system of two such independent equations represents the equations of a straight line.*

Let $A_0(x_0, y_0, z_0)$ be fixed point on a straight line, $A(x, y, z)$ an arbitrary point of the straight line, and $\mathbf{e}(k, l, m)$ a non-zero vector parallel to the straight line (Fig. 19.2). Then the vectors $\overrightarrow{A_0A}$ and $\mathbf{e}$ are parallel and, consequently, their coordinates are proportional, i.e.

$$\frac{x - x_0}{k} = \frac{y - y_0}{l} = \frac{z - z_0}{m}. \qquad (**)$$

This form of the equation of a straight line is called *canonical.* It represents a particular case $(*)$, since it allows an equivalent form

$$\frac{x - x_0}{k} = \frac{y - y_0}{l}, \qquad \frac{y - y_0}{l} = \frac{z - z_0}{m},$$

corresponding to $(*)$.

Suppose a straight line is represented by the equations $(*)$. Let us form its equation in canonical form. For this purpose it is sufficient to find a point A_0 on the straight line and a vector $\mathbf{e}$ parallel to this line.

Any vector $\mathbf{e}(k, l, m)$ parallel to the straight line will be parallel to either of the planes $(*)$, and conversely. Consequently, k, l, m satisfy the equations

$$\left.\begin{aligned} a_1k + b_1l + c_1m = 0,\\ a_2k + b_2l + c_2m = 0.\end{aligned}\right\} \qquad (***)$$

Thus, any solution of the system $(*)$ may be taken as x_0, y_0, z_0 for the canonical equation of the straight line and any solution of $(***)$ as the coefficients k, l, m, for instance

$$k = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad l = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad m = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

From the equation of a straight line in canonical form we can derive its equations in parametric form. Namely, putting the common value of the three ratios of the canonical equation equal to t, we get

$$x = kt + x_0, \quad y = lt + y_0, \quad z = mt + z_0$$

which are the *parametric equations of a straight line.*

Let us find out what are the peculiarities of the position of a straight line relative to the coordinate system if some of the coefficients of the canonical equation are equal to zero.

Since the vector $\mathbf{e}(k, l, m)$ is parallel to the straight line, with $m = 0$ the line is parallel to the xy-plane ($\mathbf{e}\mathbf{e}_x = 0$), with $l = 0$ the line is parallel to the xz-plane, and with $k = 0$ it is parallel to the yz-plane.

If $k = 0$ and $l = 0$, then the straight line is parallel to the z-axis ($\mathbf{e}$ is parallel to $\mathbf{e}_z$; if $l = 0$ and $m = 0$, then it is parallel to the x-axis, and if $k = 0$ and $m = 0$, then the line is parallel to the y-axis.

We conclude with a note that a straight line may be specified by the equations of the form $(*)$ and $(**)$ in Cartesian coordinates in general (and not only in its particular case, i.e. in rectangular Cartesian coordinates).

19.9 Exercises

1. Under what condition does a straight line represented by the equation in canonical form $(**)$ intersect the x-axis (y-axis, z-axis)? Under what condition is it parallel to the plane $xy(yz, zx)$?

2. Show that the locus of points equidistant from three pairwise non-parallel planes is a straight line.

3. Show that the locus of points equidistant from the vertices of a triangle is a straight line. Form its equations given the coordinates of the vertices of the triangle.

4. Show that through each point of the surface

$$z = axy$$

there pass two straight lines entirely lying on the surface.

5. If the straight lines specified by the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0. \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_3x + b_3y + c_3z + d_3 = 0,\\ a_4x + b_4y + c_4z + d_4 = 0. \end{aligned}\right\}$$

intersect, then

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Show this.

19.10 Basic problems of straight lines and planes

Form the equation of an arbitrary plane passing through the point (x_0, y_0, z_0).

Any plane is specified by an equation of the form

$$ax + by + cz + d = 0.$$

Since the point (x_0, y_0, z_0) belongs to the plane, then

$$a_0x + b_0y + c_0z + d_0 = 0.$$

Hence the equation of the required plane is

$$ax + by + cz - (a_0x + b_0y + c_0z) = 0,$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Obviously, for any a, b, c this equation is satisfied by the point (x_0, y_0, z_0).

Form the equation of an arbitrary straight line passing through the point (x_0, y_0, z_0).

The required equation is

$$\frac{x - x_0}{k} = \frac{y - y_0}{l} = \frac{z - z_0}{m}.$$

Indeed, this equation specifies a straight line passing through the point (x_0, y_0, z_0) whose coordinates obviously satisfy the equation. Taking arbitrary (not all equal to zero) values for k, l, m, we obtain a straight line of an arbitrary direction.

Form the equation of a straight line passing through two given points (x', y', z') and (x'', y'', z'').

The equation of the straight line may be written in the form

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m}.$$

Since the second points lies on the line, then

$$\frac{x''-x'}{k}=\frac{y''-y'}{l}=\frac{z''-z'}{m}.$$

This allows us to eliminate k, l, m, and we get the equation

$$\frac{x-x'}{x''-x'}=\frac{y-y'}{y''-y'}=\frac{z-z'}{z''-z'}.$$

Form the equation of a plane passing through three points $A'(x',y',z')$, $A''(x'',y'',z'')$, $A'''(x''',y''',z''')$, not lying on a straight line.

Let $A(x,y,z)$ be an arbitrary point belonging to the required plane. The three vectors

$$\overrightarrow{A'A},\quad \overrightarrow{A'A''},\quad \overrightarrow{A'A'''}$$

lie in one plane. Consequently,

$$(\overrightarrow{A'A},\quad \overrightarrow{A'A''},\quad \overrightarrow{A'A'''})=0,$$

and we get the required equation

$$\begin{vmatrix} x-x' & y-y' & z-z' \\ x''-x' & y''-y' & z''-z' \\ x'''-x' & y'''-y' & z'''-z' \end{vmatrix}=0.$$

Form the equation of a plane passing through a given point (x_0,y_0,z_0) and parallel to the plane

$$ax+by+cz+d=0.$$

The required equation is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

Indeed, this plane passes through the given point and is parallel to the given plane.

Form the equation of a straight line passing through a given point (x_0,y_0,z_0) parallel to a given straight line

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m}.$$

The required equation is

$$\frac{x - x_0}{k} = \frac{y - y_0}{l} = \frac{z - z_0}{m}.$$

A straight line passing through a point (x_0, y_0, z_0) *and perpendicular to a plane*

$$ax + by + cz + d = 0,$$

is specified by the equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{l} = \frac{z - z_0}{c}.$$

A plane perpendicular to a straight line

$$\frac{x - x'}{k} = \frac{y - y'}{l} = \frac{z - z'}{m},$$

passing through a point (x_0, y_0, z_0)*, is specified by the equation*

$$k(x - x_0) + l(y - y_0) + m(z - z_0) = 0.$$

let us form the equation of a plane passing through a point (x_0, y_0, z_0) *and parallel to the straight lines*

$$\frac{x - x'}{k'} = \frac{y - y'}{l'} = \frac{z - z'}{m'},$$
$$\frac{x - x''}{k''} = \frac{y - y''}{l''} = \frac{z - z''}{m''}.$$

Since the vector (k', l', m'), and (k'', l'', m'') are parallel to the plane, their vector product is perpendicular to the plane. Hence the equation is

$$(x - x_0)\begin{vmatrix} l' & m' \\ l'' & m'' \end{vmatrix} + (y - y_0)\begin{vmatrix} m' & k' \\ m'' & k'' \end{vmatrix} + (z - z_0)\begin{vmatrix} k' & l' \\ k'' & l'' \end{vmatrix} = 0,$$

which can be rewritten in a compact form:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ k' & l' & m' \\ k'' & l'' & m'' \end{vmatrix} = 0.$$

Chapter 20

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