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MATEMATIKA 1.2

fanidan
amaliy mashg‘ulotlar uchun

USLUBIY KO‘RSATMALAR

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KIRISH

Matematika fani amaliy mashg‘ulotlaridan uslubiy ko‘rsatmaning o‘ziga xosligi shundan iboratki:

- bir tomondan matematika -matematikaligicha qolib, fundamental fan sifatida matematik tushunchalar, aksiomalar, teoremalarning uzviy bog‘lanishda mantiqiy izchilligida qat’iy bayon qilinishi zarur. Talabalarda mantiqiy, algoritmik, abstrakt fikrlashlarning sintezi bo‘lgan – matematik fikrlashni shakllantirishga xizmat qilishi kerak;

- ikkinchi tomondan konkret sohaning talab va ehtiyojlaridan kelib chiqib, uning o‘ziga xos jihatlarini aks ettirishi lozim. Talabalarning matematikani maqsadli o‘rganishini ta’minlash bilan birga o‘zlashtirishini osonlashtirishi kerak.

Masalaning bu ikki tomoni ma’lum mutanosiblikda shunday uyg‘unlashuvi kerakki, natijada kurs ma’lum sohaning konkret masalalarini yechishga retseplar beruvchi qo‘llanma yoki talabalarda matematika faqat hisoblashlarni o‘rganadigan fan degan tushunchani hosil qilmasligi kerak.

Mana shu prinsipdan kelib chiqib, matematikaga “tabiat haqidagi barcha bilimlarimizni sistemaga soluvchi, tabiat va jamiyatdagi jarayonlarning matematik modellarini o‘rganuvchi fan” deb ta’rif berilgan.

Ushbu uslubiy ko‘rsatma matematikadan yozilgan bo‘lib, ko‘p hollarda matematik tushunchalarni texnikadagi talqini berildi va hayotda uchraydigan masala, misollar keltirildi.

Uslubiy ko‘rsatmada to‘plamlar, ko‘phadlar, kompleks sonlar, matritsa va determinantlar, chiziqli tenglamalar sistemasi, chiziqli fazo elementlari, limitlar nazariyasi, bir o‘zgaruvchili va ko‘p o‘zgaruvchili funksiyalarning differensial hisobi, integral hisob elementlariga doir misol va masalalar berilgan.

Barcha bo‘limlarda qisqa nazariy ma’lumotlar keltirilgan. Qator masalalar yechimlari bilan berilgan nazorat ishlari hamda mustaqil yechish uchun misol va masalalar tavsiya etilgan.

1-amaliy mashg‘ulot. To‘plamlar nazariyasi va matematik mantiq elementlari.

1.To‘plam haqida tushuncha. Bosh va qism to‘plam.

To‘plam tushunchasi matematikaning boshlang‘ich (ta’riflanmaydigan) tushunchalaridan biridir. U chekli yoki cheksiz ko‘p obyektlar (narsalar, buyumlar, shaxslar va h.k.) ni birgalikda bir butun deb qarash natijasida vujudga keladi.

Masalan, O‘zbekistondagi viloyatlar to‘plami; viloyatdagi akademik litseylar to‘plami; butun sonlar to‘plami; to‘g‘ri chiziq kesmasidagi nuqtalar to‘plami; sinfdagi o‘quvchilar to‘plami va hokazo. To‘plamni tashkil etgan obyektlar uning ***elementlari*** deyiladi.

To‘plamlar odatda lotin alifbosining bosh harflari bilan, uning elementlari esa shu alifboning kichik harflari bilan belgilanadi. Masalan, $A = \{a, b, c, d\}$ yozuvi A to‘plam a, b, c, d elementlardan tashkil topganligini bildiradi.

x element X to‘plamga ***tegishli*** ekanligi $x \in X$ ko‘rinishda, ***tegishli emasligi*** esa $x \notin$ ko‘rinishda belgilanadi.

Masalan, barcha natural sonlar to‘plami N va 4, 5, π sonlari uchun $4 \in N$, $5 \in N$, $\pi \in N$ munosabatlar o‘rinli.

1-m i s o 1. $A = \{x|\ x \in N,\ x^2 > 8\}$ to‘plam 2 dan katta bo‘lgan barcha natural sonlardan tuzilgan, ya'ni A = {3,4, 5, 6, 7, 8, 9, ...}. Bu to‘plam - cheksiz to‘plamdir.

Birorta ham elementga ega bo‘lmagan to‘plam ***bo‘sh to‘plam*** deyiladi. Bo‘sh to‘plam Ø orqali belgilanadi. Bo‘sh to‘plam ham chekli to‘plam hisoblanadi.

2-m i s o 1. $x^2 + 3x + 2 = 0$ tenglamaning ildizlari $X = \{-2; -1\}$ chekli to‘plamni tashkil etadi. $x^2 + 3x + 3 = 0$ tenglama esa haqiqiy ildizlarga ega emas, ya'ni uning haqiqiy yechimlar to‘plami $\varnothing$ dir.

Ayni bir xil elementlardan tuzilgan to‘plamlar ***teng to‘plamlar*** deyiladi.

3-miso1. $X=\{x|x \in N, x \le 3\}$ va $Y=\{x|(x-1)(x-2)(x-3)=0\}$ to‘plamlar-ning har biri faqat 1, 2, 3 sonlaridan tuzilgan. Shuning uchun bu to‘plamlar tengdir: $X=Y$.

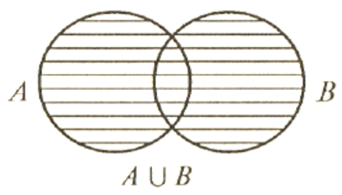
Agar B to‘plamning har bir elementi A to‘plamning ham elementi bo‘lsa, B to‘plam A to‘plamning ***qism-to‘plami*** deyiladi va $B \subset A$ ko‘rinishida belgilanadi. Bunda $Ø \subset A$ va $A \subset A$ hisoblanadi. Bu qism-to‘plamlar ***xosmas qism-to‘plamla r***deyiladi. A to‘plamning qolgan barcha qism-to‘plamlari ***xos qism-to‘plamlar*** deyiladi. Masalan: $N \subset Z \subset Q \subset R$. Agar A={3, 4, 5}, $B=\{x|\ x^2 - 7x + 12 = 0\}$ bo‘lsa, $B \subset A$ bo‘ladi.

2. To‘plamlar ustida amallar. Sonli to‘plamlar.

A va B to‘plamlarning ikkalasida ham mavjud bo‘lgan x elementga shu to‘plamlarning ***umumiy*** elementi deyiladi.

A va B to‘plamlarning ***kesishmasi*** **(yoki *ko‘paytmasi)*** deb, ularning barcha umumiy elementlaridan tuzilgan to‘plamga aytiladi. A va B to‘plamlarning kesishmasi $A\cap B$ ko‘rinishda belgilanadi: $A\cap B=\{x|x\in A$ va $x\in B\}$. 1.1-rasmda Eyler - Benn diagrammasi nomi bilan ataladigan chizmada A va B shakllarning kesishmasi $A\cap B$ ni beradi (chizmada shtrixlab ko‘rsatilgan).

A va B to‘plamlarning ***birlashmasi*** **(yoki *yig‘indisi)*** deb, ularning kamida bittasida mavjud bo‘lgan barcha elementlardan tuzilgan to‘plamga aytiladi. A va B to‘plamlarning birlashmasi $A\cup B$ *k*o‘rinishida belgilanadi: $A\cup B=\{x|\ x\in A$ yoki $x\in B\}$ (1.2- rasm).

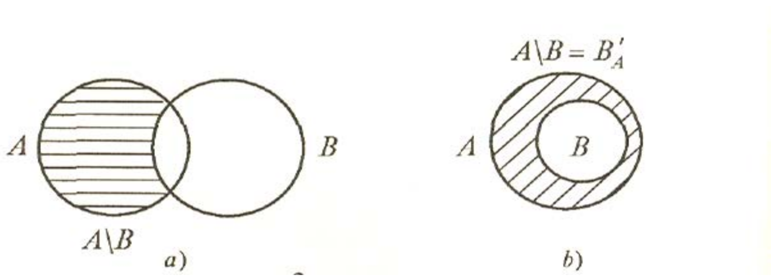


1.1-rasm 1.2-rasm

A va B to‘plamlarning ***ayirmasi*** deb, A ning B da mayjud bo‘lmagan barcha elementlaridan tuzilgan to‘plamga aytiladi. A va B to‘plamlarning ayirmasi $A\backslash B$ ko‘rinishda belgilanadi: $A\backslash B=\{x|x\in A\ va\ x\notin B\}$ (1.3- rasm).

Topshiriq:1.3-*a* rasmda $B\backslash A$ ni ko‘rsating.

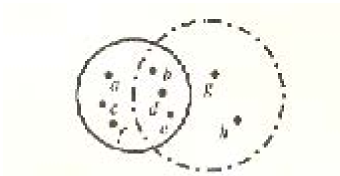
Agar $B\subset A$ bo‘lsa, $A\backslash B$ to‘plam B to‘plamning ***to‘ldiruvchisi*** deyiladi va B' yoki B_A' bilan belgilanadi (1.3- *b* rasm).



1.3-rasm

4-misol. $A=\{a, b, c, d, e, f\}$ va $B=\{b, d, e, g, h\}$ to‘plamlar berilgan. Ularning kesishmasi, birlashmasini topamiz va Eyler - Benn diagrammasida talqin etamiz.

b, d, e elementlari *A* va *B* to‘plamlar uchun umumiy, shunga ko‘ra $A\cap B = \{b, d, e\}$. Bu to‘plamlarning birlashmasi esa $A\cup B = \{a, b, c, d, e, f, h\}$ dan iborat (1.4-rasm).



1.4-rasm

5- misol . Agar $A\subset B$ bo‘lsa, $A\cup B$ =B bo‘lishini isbot qilamiz.

Isbot. $A\subset B$ bo‘lsin.

a) $A\cup B\subset B$ niko‘rsatamiz. $x\in A\cup B$ bo‘lsin. Uholda $x\in A$ yoki $x\in B$ bo‘ladi. Agar $x\in A$ bo‘lsa, $A\subset B$ ekanidan $x\in B$ ekani kelib chiqadi, ikkala holda ham $A\cup B$ ning har qanday elementi *B* ning ham elementidir. Demak, $A\cup B\subset B$;

b) $B\cup A\subset B$ni ko‘rsatamiz. $x\in B$ bo‘lsin. U holda,to‘plamlar birlashmasining ta'rifiga ko‘ra $x\in A\cup B$ bo‘ladi, Demak, *B* ning har qanday elementi $A\cup B$ ning ham elementi bo‘ladi, ya'ni $B\cup A\subset B$. Shunday qilib, $A\cup B\subset B$,$B\cup A\subset B$. Bu esa $B = A\cup B$ ekanini tasdiqlaydi.

To‘plamlar ustida bajariladigan amallarning ***xossalari*** sonlar ustida bajariladigan amallarning xossalariga o‘xshash. Har qanday *X*, Y va *Z* to‘plamlar uchun:

1) $X\cup Y=Y\cup X$;

1) $X\cap Y=Y\cap X$;

2) $(X\cup Y)\cup Z=X\cup(Y\cup Z)=(X\cup Z)\cup Y$;

2) $(Z\cap Y)\cap Z=(X\cap Z)\cap Y=X\cap(Y\cap Z)$;

3) $(X\cup Y)\cap Z=(X\cap Z)\cup(Y\cap Z)$;

3) $(X\cap Y)\cup Z=(X\cup Z)\cap(Y\cup Z)$ tengliklar bajariladi,

Agar qaralayotgan to‘plamlar ayni bir *U* to‘plamning qism-to‘plamlari bo‘lsa, Uto‘plam ***universal*** to‘plam deyiladi.

U universal to‘plam qism-to‘plamlarining kesishmasi, birlashmasi, shuningdek, U to‘plam ixtiyoriy qism-to‘plamining to‘ldiruvchisi ham U ning qism to‘plami bo‘ladi. Biror *X* to‘plamning *U* ga to‘ldiruvchisini $X`_U$ yoki X' shaklida belgilash mumkin. To‘ldirish amalining ayrim ***xossalarmi*** ko‘rsatib o‘tamiz:

1) Ø = U, 2) $U' = Ø$, 3) $(X')' = X$, 4) U dan olingan har qanday *X va* Y to‘plam uchun $(X\cap Y)' = X`\cup Y$ ; $(X\cup Y)` = X`\cap Y`$.

Shuningdek, agar $X\subset Y$ bo‘lsa, $X\cap Y = X$, $X\cup Y = Y$ bo‘ladi. Xususan, $Ø\subset X$ *va* $X\subseteq X$ bo‘lganidan, $Ø\cap X=Ø$, $Ø\cup X=X$, $X\cap X=X$, $X\cup X=X$ bo‘ladi.

3-misol. $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$, $C=\{1, 5, 9\}$ to‘plamlar berilgan. $D = \{1, 2, 3, 4, 5, 9\}$ to‘plam universal to‘plam bo‘ladimi? $E=\{1, 2, 3, 4, 5, 9, 15\}$ va $M=\{1, 3, 4, 5, 9\}$ to‘plamlarchi?

$A\subset D$, $B\subset D$, $C\subset D$ bo‘lgani uchun D to‘plam universal to‘plam bo‘ladi. $D\subset E$ bo‘lgani uchun E to‘plam ham universal to‘plam bo‘ladi. $B\subset M$, $C\subset M$, lekin $A\not\subset M$ bo‘lgani uchun M to‘plam universal to‘plam bo‘la olmaydi.

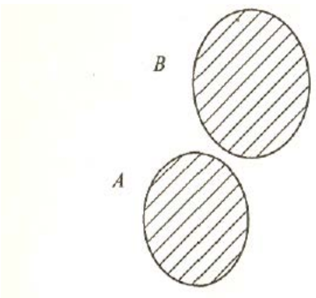
To‘plam elementlarining soni bilan bog‘liq ayrim masalalar.

To‘plamlar nazariyasining muhim qoidalaridan biri — jamlash qoidasidir. Bu qoida kesishmaydigan to‘plamlar birlashmasidagi elementlar sonini topish imkonini beradi

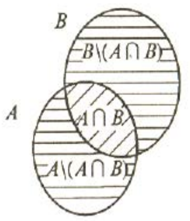
1-teorema (jamlash qoidasi). **Kesishmaydigan A va B chekli to‘plamlarning (1.5 – rasm) birlashmasidagi elementlar soni A va B to‘plamlar elementlari sonlarining yig‘indisiga teng:**

$$n(A\cup B)=n(A)+ n(B)$$

Isbot. $n(A) = k$, $n(B) = m$ bo‘lib, A to‘plam a_1, a_2 ..., a_k elementlardan, B to‘plam esa b_1 b_2..., b_m elementlardan tashkil topgan bo‘lsin.



1.5-rasm



1.6-rasm

Agar A va B to‘plamlar kesishmasa, ularning birlashmasi a_1 a_2 ..., a_k, b_1, b_2, ..., b_m elementlardan tashkil topadi:

$$A\cup B = \{ a_1\ a_2\ ..., a_k, b_1, b_2, ..., b_m \}.$$

Bu to‘plamda k + m ta element mavjud, ya'ni

$$n(A\cup B) = k+m=n(A)+n(B).$$

Xuddi shu kabi, chekli sondagi A, B, ..., F juft-jufti bilan kesishmaydigan to‘plamlar uchun quyidagi tenglik to‘g‘riligini isbotlash mumkin:

$$n(A\cup B\cup ... \cup F) = n(A) + n(B) + ... +n(F).$$

2-teorema. **Ixtiyoriy A va B chekli to‘plamlar uchun ushbu tenglik o‘rinli:**

$$\mathbf{n(A\cup B) = n(A) + n(B) - n(A\cap B).} \quad \mathbf{(1.1)}$$

Isbot. Agar $A\cap B = \varnothing$ bo‘lsa, $n(A\cap B) = 0$ bo‘lib, 1- teoremaga ko‘ra (1.1) tenglik o‘rinli. Agar $A \cap B \neq \varnothing$ bo‘lsa, u holda $A\cup B$ to‘plamni uchta juft-jufti bilan kesishmaydigan to‘plamlarning birlashmasi ko‘rinishida tasvirlash mumkin (1.6- rasm):

$$\boldsymbol{A\cup B = (A\setminus(A\cap B)) \cup (B\setminus(A\cap B))\cup (A\cap B).} \quad \mathbf{(1.2)}$$

$A\setminus(A\cap B)$, $B\setminus(A\cap B\}$ va $A\cap B$ to‘plamlardagi elementlari soni mos ravishda $n(A)$-$n(A\cap B)$, $n(B\}$ -$n(A\cap B)$, $n(A\cap B)$ ga teng.

Jamlash qoidasiga ko‘ra, (1.2) tenglikdan

$n(A\cup B) = (n(A) - n(A\cap B)) + (n(B) - n(A\cap B)) + + n(A\cap B) = n(A) + n(B) - n(A\cap B)$, ya'ni (1.1) tenglik hosil bo‘ladi.

4-M a s a l a . 100 kishidan iborat sayyohlar guruhida 70 kishi ingliz tilini, 45 kishi fransuz tilini, 23 kishi esa ikkala tilni ham biladi. Sayyohlar guruhidagi necha kishi ingliz tilini ham, fransuz tilini ham bilmaydi?

Yechish. Berilgan guruhdagi ingliz tilini biladigan sayyohlar to‘plamini A bilan, fransuz tilini biladigan sayyohlar to‘plamini B bilan belgilaymiz. U holda ham ingliz tilini, ham fransuz tilini biladigan sayyohlar to‘plami $A\cap B$ to‘plamdan, shu ikki tildan hech bo‘lmasa bittasini biladigan sayyohlar to‘plami esa $A\cup B$ to‘plamdan iborat bo‘ladi.

Shartga ko‘ra, n(A)=70, n(B)=45, $n(A\cap B) = 23$. (1) tenglikka ko‘ra,

$n(A\cup B)$=70 + 45 - 23 = 92.

Shunday qilib, 92 kishi ingliz va fransuz tillaridan hech bo‘lmaganda bittasini biladi,

100 - 92 = 8 kishi esa ikkala tilni ham bilmaydi.

Matematik mantiq elementlari.

Matematik mantiq matematikaning bir bo‘limi bo‘lib, unda ,,mulohaza"lar va ular ustidagi mantiqiy amallar o‘rganiladi.

Chinyoki ***yolg‘onligi*** haqida fikr yuritish mumkin bo‘lgan har qanday darak gap ***mulohaza*** deyiladi. Mulohazalar ustida bajariladigan mantiqiy amallar maxsus belgilar yordamida ifodalanadi. Bu belgilar hozirgi zamon matematikasining barcha bo‘limlarida qo‘llaniladi.

Bu belgilar qiyidagilardir:

1) $\Rightarrow$— agar ... bo‘lsa, u holda ... bo‘ladi,

$P\Rightarrow Q$- agar P bo‘lsa, Q bo‘ladi (Pdan Q kelib chiqadi);

2)$\Leftrightarrow$— teng kuchlilik,

$P \Leftrightarrow Q$ - P va Q teng kuchli (P dan Q kelib chiqadi va aksincha);

3)$\vee$— dizyunksiya (,,yoki" amali);

4) $\wedge$ — konyunksiya (,,va" amali);

5) $\forall$— ixtiyoriy, barcha, har qanday;

6) $\exists$— shunday, mavjud;

7) $\nexists$— mavjud emas.

Bu amallarni (belgilarni) qo‘llashga doir misollar keltiramiz.

P={a soni 15 ga bo‘linadi} va *Q= {a* soni 5 ga bo‘linadi} mulohazalari quyidagicha bog‘langan:

P mulohazaning chinligidan *Q* mulohazaning chinligi kelib chiqadi. Mulohazalarning bunday bog‘lanishi ***mantiqiy kelib chiqish*** deyiladi va $\Rightarrow$ belgi yordamida yoziladi: $P\Rightarrow Q$. Bu yerda “a soni 15 ga bo‘linadi” sharti asonining 5 ga bo‘linishi uchun ***yetarlidir***. Shu bilan birga “*a* soni 5 ga bo‘linadi” sharti uning 15 ga bo‘linishi uchun yetarli emas, u ***zaruriy*** shartdir xolos, chunki *a* soni 5 ga bo‘linmasa, uning 15 ga bo‘linishi mumkin emas.

Umuman, *P* mulohazaning chinligidan *Q* mulohazaning chinligi kelib chiqsa $(P\Rightarrow Q)$, Pmulohaza *Q* mulohaza uchun yetarli shart va *Q* mulohaza *P* mulohaza uchun zaruriy shart deyiladi.

Agar $A\Rightarrow B$ va $B\Rightarrow A$ bo‘lsa, *B* mulohaza *A* mulohaza uchun zaruriy va yetarli shartdir. Bu esa quyidagicha yoziladi: $A\Leftrightarrow B$. „$\Leftrightarrow$" — mantiqiy teng kuchlilik belgisidir.

A - ,,*a* soni juft son" mulohazasi bo‘lsin.

B - ,,a^2- juft son" mulohazasi bo‘lsin.

Bu mulohazalar teng kuchli mulohazalar bo‘ladi, ya'ni $A\Leftrightarrow B$.

Boshqacha aytganda, sonning kvadrati juft son bo‘lishi uchun sonning o‘zi juft bo‘lishi zarur va yetarli.

Biror *A* mulohazaning ***inkori*** deb, *A* chin bo‘lganda yolg‘on, *A* yolg‘on bo‘lganda esa chin bo‘ladigan mulohazaga aytiladi va $\overline{A}$ bilan belgilanadi.

A – “yetti - murakkab son”, u holda $\overline{A}$ “yetti -murakkab son emas”. Bu yerda *A* — yolg‘on,

$\overline{A}$ - chin mulohazadir.

A va *B* mulohazalarning ***dizyunksiyasi*** deb, *A* va *B* mulohazalardan kamida bittasi chin bo‘lganda chin bo‘ladigan yangi mulohazaga aytiladi va $A\vee B$ bilan belgilanadi.

Masalan, *A* – “6·4=24”, *B* – “6·4=25” bo‘lsa, $A\vee$B mulohaza “6 • 4 ko‘paytma 24 yoki 25 ga teng”.

A va *B* mulohazalarning konyunksiyasi deb, bu ikkala mulohaza ham chin bo‘lgandagina chin bo‘ladigan yangi mulohazaga aytiladi va $A\wedge B$ bilan belgilanadi.

Masalan, *C* — “13 soni toq va tubdir” mulohazasi quyidagi ikkita mulohazaning konyunksiyasidir. A— “13 soni – toq”, *B* – “13 soni — tub”. Demak, $C=A\wedge B$,

Matematik mulohazalarni yuqoridagi belgilar yordamida ifoda etishga doir misollar keltiramiz.

1-m i s o l. Agar $a > b$ va $b > c$ bo‘lsa, $a > c$ bo‘ladi. $(a > b) \wedge (b > c) \Rightarrow (a > c)$.

2-m i s o l. $a > b$ bo‘lsa, $a + c > b + c$ bo‘ladi. $(a > b) \Rightarrow (a + c > b + c)$.

3-m i s o l. $a = 0$ yoki $b=0$ bo‘lsa, $ab = 0$ bo‘ladi va aksincha, $ab = 0$ bo‘lsa, $a = 0$ yoki $b=0$ bo‘ladi. $(ab = 0) \Leftrightarrow ((a = 0) \wedge (b = 0))$.

4-m i s o l. $a > 0$ va $b > 0$ bo‘lsa, $ab > 0$ bo‘ladi. $(a>0) \wedge (b>0) \Rightarrow (ab>0)$.

5-m i s o l. Ixtiyoriy x haqiqiy son uchun $|x| \geq x$. $\forall x \in R$: $|x| \geq x$.

6-m i s o l. Ixtiyoriy $a \geq 0$ son uchun, shunday $x \in R$ son mavjudki, $x^2 = a$ bo‘ladi, ya'ni $\forall a \geq 0$, $\exists x \in R$: $x^2 = a$.

Mustaqil ishlash uchun topshiriq:

1-misol. $\cup, \cap, \subset$ belgilarda foydalanib, to‘plamlar orasidagi munosabatni yozing:

a) $X_1 = \{-5;6\}, X_2 = \{x / -5 \leq x \leq 6\}$,
$X_3 = \{x / -5 < x < 6\}$

b) $A = \{1;3;5;7;\}, B = \{1;5;7\}$

d) $A = \{x,y,z\}, B = \{y,z,x\}$

2-misol. Agar $A=\{-2;-1;0;1;2;3;4;5\}$, $B=\{3;4;5;6\}$,
$C=\{-3;-2;-1;0;2;3\}$, $M=\{5 \leq x-10 \leq 12 \mid x \in N\}$, $K=\{x+10 \leq 30 \mid x \in N\}$ bo‘lsa, quyidagi to‘plamlar elementlarini ko‘rsatib yozing:

1) $(A \cup B) \cap (C \cup D)$; 2) $(A \cap B \cap C) \cup D$; 3) $(A \cap B) \cup (C \cap D) \cup M$;
4) $(A \cup C) \cap (A \cup B)$; 5) $(B \setminus A) \cup (A \setminus B)$

Nazorat uchun savollar:

1.Bo‘sh va qism to‘plam deb nimaga aytiladi?

2.A va B to‘plamlarning birlashmasi (yoki yig‘indisi) deb nimaga aytiladi?

3. A va B to‘plamlarning kesishmasi (yoki ko‘paytmasi) deb nimaga aytiladi?

4.A va B to‘plamlarning ayirmasi deb nimaga aytiladi?

5.To‘ldiruvchi to‘plam deb nimaga aytiladi?

6. A mulohazaning inkori deb nimaga aytiladi?

7.A va B mulohazalarning dizyunksiyasi deb nimaga aytiladi?

8.A va B mulohazalarning dizyunksiyasi deb nimaga aytiladi?

9. Universal to‘plam deb nimaga aytiladi?

2-amaliy mashg‘ulot. Ko‘phadni ko‘phadga bo‘lish. Bezu teoremasi. Tenglama tushunchasi. Tenglama yechimi.

Bir o‘zgaruvchili $A(x)$ va B(x) ko‘phadlar uchun

$$A(x) = B(x)\cdot Q(x) \qquad (2.1)$$

tenglik o‘rinli bo‘ladigan $Q(x)$ ko‘phad mavjud bo‘lsa, ***A(x) ko‘phad B(x) ko‘phadga bo‘linadi*** (yoki qoldiqsiz bo‘linadi) deyiladi. Bunda A(x) ko‘phad ***bo‘linuvchi***, $B(x)$ ko‘phad ***bo‘luvchi***, $Q(x)$ ko‘phad esa ***bo‘linma*** deyiladi.

1-misol.$x^3-1=(x^2+x+1)(x-1)$ ayniyatdan, $A(x)=x^3-1$ ko‘phadning $B(x) = x^2+x+1$ ko‘phadga (qoldiqsiz) bo‘linishini va bo‘linma $Q(x)=x-1$ ko‘phadga tengligini ko‘ramiz.

Butun sonni butun songa (butun) bo‘lish amali kabi, ko‘phadni ko‘phadga qoldiqsiz bo‘lish amali hamma vaqt ham bajarilavermaydi. Shu sababli ko‘phadni ko‘phadga qoldiqsiz bo‘lishga nisbatan yanada umumiyroq bo‘lgan amal — ko‘phadni ko‘phadga qoldiqli bo‘lish amali kiritiladi.

A(x) ko‘phadni B(x) ko‘phadga qoldiqli bo‘lish deb, uni quyidagicha ko‘rinishda tasvirlashga aytiladi:

$$A(x) = B(x)\cdot Q(x) + R(x). \qquad (2.2)$$

(2.2) tenglikdagi $Q(x)$ va $R(x)$ lar bir o‘zgaruvchili ko‘phadlar bo‘lib, $R(x)$ ko‘phadning darajasi $B(x)$ ko‘phadning darajasidan kichik yoki $R(x) = 0$.

(2.2) tenglikdagi $A(x)$ ko‘phad ***bo‘linuvchi***, $B(x)$ ko‘phad ***bo‘luvchi***, $Q(x)$ ko‘phad ***bo‘linma*** (yoki to‘liqsiz bo‘linma), $R(x)$ ko‘phad esa ***qoldiq*** deyiladi.

Teorema. ***A(x) va B(x) ko‘phadlar haqiqiy koeffitsiyentli va B(x) ≠ 0 bo‘lsin. U holda shunday Q(x) va R(x) ko‘phadlar topiladiki, ular uchun A(x) = B(x) • Q(x) + R(x) tenglik o‘rinli bo‘ladi va bunda R(x) ning darajasi B(x) nikidan kichik yoki R(x) = 0 bo‘ladi hamda Q(x), R(x) ko‘phadlar bir qiymatli aniqlanadi.***

Bu teorema ko‘phadni ko‘phadga bo‘lishning amaliy usulini bermaydi. Ko‘phadni ko‘phadga bo‘lishning amaliy usullari — ***«aniqmas koeffitsiyentlar usuli»*** va ***«burchakli bo‘lish»*** usulini misollarda qaraymiz.

2- misol. $A(x) = x^3 + x + 1$ ko‘phadni $B(x) = x^2 + x + 1$ko‘phadga aniqmas koeffitsiyentlar usuli bilan bo‘lamiz. Yechish. $A(x)$ ko‘phad 3-darajali, $B(x)$ esa 2-darajali ko‘phad bo‘lgani uchun $Q(x)$ ko‘phad 1- darajali ko‘phad bo‘lishi kerak. $A(x)$ ko‘phadni $B(x)$ ko‘phadga bo‘lishdagi qoldiqning darajasi ko‘pi bilan 1 ga teng bo‘ladi. Shu sababli $Q(x)$ ni $Q(x) = ax$

$+ b$ ko‘rinishda, $R(x)$ ni esa ; $R(x)=px + q$ ko‘rinishda izlaymiz. Bu yerdagi a, b, p, q lar topilishi kerak bo‘lgan aniqmas koeffitsiyentlardir.

$A(x)=B(x)\cdot Q(x)+R(x)$ tenglikni $x^3+x+1=(x^2+x+1)\cdot(ax+b)+(px+q)$ ko‘rinishda yozib, uning o‘ng tomonidagi amallarni bajaramiz. Ixchamlashtirishlardan so‘ng, $x^3+x+1=ax^3+(a+b)x^2+(a+b+p)x+(b+q)$ tenglikni hosil qilamiz. Ko‘phadlarning tenglik shartiga ko‘ra,

$$\begin{cases} a = 1 \\ a + b = 0, \\ a + b + p = 1, \\ b + q = 1 \end{cases}$$

sistemaga ega bo‘lamiz. Bundan $a = 1$, $b = -1$, $p=1$, $q=2$ ekanligi aniqlanadi. Demak, $Q(x)=x-1$, $R(x)=x+2$.

3-misol. Ushbu

$$A(x)=\frac{3x^4-10ax^3 22a^2x^2-24a^3x-10a^4}{x^2+22ax-3a^2}$$

ifodadan butun qism ajratamiz. Buning uchun suratdagi ko‘phadni maxrajdagi ko‘phadga bo‘lish lozim. Bo‘lishni «burchakli bo‘lish» usulida bajaramiz:

$$\begin{array}{l|l}
3x^4-10ax^3+22a^2x^2-24a^3x+10a^4 & \underline{x^2-2ax+3a^2} \\
\underline{-3x^4-6ax^3+9a^2x^2} & 3x^2-4ax+5a^2 \\
\quad -4ax^3+13a^2x^2-24a^3x & \\
\quad \underline{-4ax^3+8a^2x^2-12a^3x} & \\
\qquad 5a^2x^2-12a^3x+10a^4 & \\
\qquad \underline{5a^2x^2-10a^3x+15a^4} & \\
\qquad\qquad -2a^3x-5a^4. &
\end{array}$$

Demak, $A(x) = 3x^2 - 4ax + 5a^2 + \dfrac{-2a^3x-5a^4}{x^2-2ax+3a^2}$.

n- darajali A (x) va m- $(m \le n)$ darajali $B(x)$ ikkita ko‘phad berilgan bo‘lib, ularning ***eng katta umumiy bo‘luvchisini*** topish talab qilinsin. Uni topishda ***Yevklid algoritmi*** dan foydalanamiz: oldin $A(x)$ ni $B(x)$ ga bo‘lamiz, so‘ng $B(x)$ ni birinchi $r_1(x)$ qoldiqqa, undan so‘ng $r_1(x)$ ni ikkinchi $r_2(x)$ qoldiqqa bo‘lamiz va hokazo. Bo‘linmalarni q_k orqali belgilaylik, bunda k= 1, 2, 3, ... Quyidagiga ega bo‘lamiz:

$A(x) = B(x)\cdot q_1(x) + r_1(x)$,

$B(x) = r_1(x)\cdot q_2(x) + r_2(x)$,

$r_1(x) = r_2(x)\cdot q_3(x) + r_3(x)$,

……………………….

$r_{n-2}(x) = r_{n-1}(x)\cdot q_n(x) + r_n(x)$,

$r_{n-1}(x) = r_n(x) \cdot q_{n+1}(x)$

Agar A(X) va B(x) lar umumiy bo‘luvchiga ega bo‘lmasa (ya'ni eng katta umumiy bo‘luvchi doimiy son bo‘lsa), ular ***o‘zaro tub ko‘phadlar*** deyiladi.

Tenglamalarning karrali ildizlarini topish kabi masalalarni hal qilishda Yevklid algoritmidan foydalanadilar. Ketma-ket bo‘lishlardan qoladigan qoldiqlarning darajalari (ular natural sonlar) kamayib, bir necha qadamdan so‘ng 0 ga teng bo‘ladi ($r_{n+1}(x) = 0$).

Undan oldingi noldan farqli $r_n(x) \neq 0$ qoldiq A(x) va B(x) ning eng katta umumiy bo‘luvchisi bo‘ladi.

Nazorat uchun savollar:

1.a sonning n-darajasi deb nimaga aytiladi?

2.Birhad deb nimaga aytiladi?

3.Ko‘phad deb nimaga aytiladi?

4.Bir jinsli ko‘phad deb nimaga aytiladi?

5. Simmetrik ko‘phad deb nimaga aytiladi?

6.Paskal uchburchagi qanday hollarda qo‘llaniladi?

7. A(x) ko‘phadni B(x) ko‘phadga qoldiqli bo‘lish deb nimaga aytiladi?

3-amaliy mashg‘ulot: Kompleks sonlarni tasvirlash. Kompleks sonlarning moduli va argumenti. Kompleks sonlarning shakllari. Eyler va Muavr formulalari.

1-misol. $z_1 = l+2i$, $z_2 = 2-i$, $z_3 = 2{,}1$, $z_4 = 2i$, $z_5=0$ kompleks sonlarning haqiqiy va mavhum qismlarini topamiz.

Yechish. Kompleks son haqiqiy va mavhum qismlarining aniqlanishiga ko‘ra, quyidagilarga egamiz:

$Re(z_1)=l$; $Re(z_2) = 2$; $Re(z_3) = 2{,}l$; $Re(z_4) = 0$; $Re(z_s)=0$;

$Im(z_1)=2$; $Im(z_2) = -i$; $Im(z_3) = 0$; $Im(z_4) = 2i$; $Im(z_s) = 0$.

Kompleks sonlar uchun « < », « > » munosabatlari aniqlanmaydi, lekin teng kompleks sonlar tushunchasi kiritiladi. Haqiqiy va mavhum qismlari mos ravishda teng bo‘lgan kompleks sonlar ***teng kompleks sonlar*** deb ataladi. Masalan, z, $=1{,}5 + \frac{4}{5}i$ va $z_2=\frac{3}{2}i + 0{,}8i$ sonlari uchun

$Re(z_1) = Re(z_2) = l{,}5$, $Im(z_1) = Im(z_2) = 0{,}8$. Demak, $z_1 = z_2$

Bir-biridan faqat mavhum qismlarining ishorasi bilan farq qiladigan ikki kompleks son ***o‘zaro qo‘shma kompleks son*****lar** deyiladi. $z=a+ bi$ kompleks songa qo‘shma kompleks son $\bar{z}=a-bi$ ko‘rinishda yoziladi. Masalan, 6+7i va 6-7*i* lar qo‘shma kompleks sonlardir: $\overline{6+7i}$ = 6-7i. Shu kabi $\bar{z}$ soniga qo‘shma son $\bar{\bar{z}}$ = *z* bo‘ladi. Masalan, $\overline{\overline{6+7i}} = \overline{6-7i}$ =6+7i. *a* haqiqiy songa qo‘shma son *a* ning o‘ziga teng:

$\bar{a}=\overline{a+0\cdot i}$ = a - 0·i =*a*. Lekin *bi* mavhum songa qo‘shma son $\overline{bi}$=-*bi* dir. Chunki, $\overline{bi}=\overline{0+bi}$=0-*bi*=-*bi*, $a,b\in R$.. Kompleks sonlar ustida arifmetik amallar quyidagicha aniqlanadi:

$(a + bi) + (c + di) = (a + c) + (b + d)i$; (3.1)

$(a + bi) - (c + di) = (a - c) + (b - d)i$; (3.2)

$(a + bi)\cdot(c + di) = (ac - bd) + (ad + bc)i$ (3.3)

$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+\frac{bc+ad}{c^2+d^2} \quad (3.4)$$

(3.1) va (3.2) tengliklarni bevosita qo‘llash qiyin emas. Kompleks sonlarni ko‘paytirish amalini $i^2 = -1$ ekanligini e'tiborga olib, ko‘phadlarni ko‘paytirish kabi bajarish mumkin.

2-misol. $(2-i)\cdot(\frac{3}{4}+2i)=2\cdot\frac{3}{4}+2\cdot2i-i\cdot\frac{3}{4}-2i^2=\frac{3}{2}+4i-\frac{3}{4}i+2=\frac{7}{4}+\frac{13}{4}i.$

(3.4) formulani eslab qolish va amaliyotda bevosita qo‘llash ancha qiyin. Shu sababli $\frac{a+bi}{c+di}$ ni hisoblash uchun,uning surati va maxrajini c - *di* ga ko‘paytirib, tegishli amallarni bajarish qulaydir.

3-misol $\frac{2-i}{-3+2i}=\frac{(2-i)(-3-2i)}{(-3+2i)(-3-2i)}=\frac{-6-4i+3i-2}{9+6i-6i+4}=\frac{-8-i}{13}=\frac{-8}{13}-\frac{1}{13}i$

Kompleks sonlarni qo‘shish va ko‘paytirish amallari xossalari haqiqiy sonlarnikiga o‘xshash;

1) z + w= w + z; 1') *zw = wz;*

2) *(z + w) + t = z+(w + t);* 2') (zw)t=z(wt);

3) z + 0= z; 3')z·1= z;

4) z(w +t) = zw +zt.

z +w = 0 tenglikni qanoatlantiruvchi *z, w* kompleks sonlari o‘zaro qarama-qarshi sonlar deyiladi. *z* kompleks soniga qarama-qarshi sonni -*z* bilan belgilash qabul qilingan. z = *a* + *bi* kompleks songa ***qarama-qarshi*** bo‘lgan yagona kompleks son mavjud va bu son –*z*=-*a*- *bi* kompleks sonidan iborat.

zw=1 tenglikni qanoatlantiradigan *z* va *w* kompleks sonlari o‘zaro teskari kompleks sonlar deyiladi. z = 0 soniga teskari son mavjud emas. Har qanday $z\neq0$ kompleks songa teskari kompleks son mavjud. Bu son $\frac{1}{z}$- sonidan iborat. *z = a + bi* kompleks songa teskari bo‘lgan $\frac{1}{z}$ sonini topamiz:

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(+bi)(a-bi)} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

4-misol. $\overline{z+w} = \overline{z} + \overline{w}$

Yechish. $z=a+bi, w=c+4i$ bo‘lsin. U holda $\overline{z} = a-bi$, $\overline{w} = c-di$ va $\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = a+c-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}$

5-misol. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

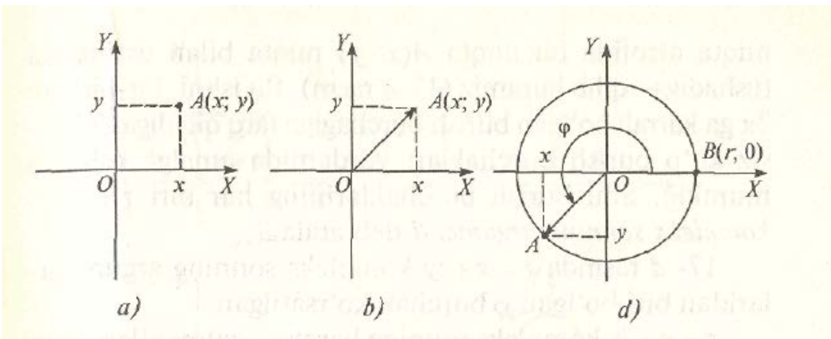
Yechish: $\overline{z \cdot w} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i} = ac-bd-(ad+bc)i.$

$\overline{z} \cdot \overline{w} = (a-bi)(c-di) = ac-bd-(ad+bc)i.$ Natijalar bir xil. Demak, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. Xususan, $z \neq 0$ bo‘lsa, z ga teskari bo‘lgan $\frac{1}{z}$ songa qo‘shma son z ga qo‘shma

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}}$$

N a t i j a. *Kompleks sonning natural ko‘rsatkichli darajasiga qo‘shma son, berilgan songa qo‘shma sonning shu natural ko‘rsatkichti darajasiga teng;* $\overline{z^n} = (\overline{z})^n$.

2. Trigonometrik shakldagi kompleks sonlar va ular ustida amallar.



1.7-rasm

$z = x + yi$ kompleks sonining geometrik tasviri bo‘lgan vektor uning ***radius-vektori*** deyiladi. Har qanday $z = x + yi$ kompleks son yagona radius-vektorga ega, chunki x, y sonlari yagona $A(x; y)$ nuqtani (vektorning oxirini) aniqlaydi. Kompleks son radius-vektorining uzunligi shu ***sonning moduli*** deyiladi. $z=x+yi$ kompleks sonning modulini $|z|$ yoki r bilan belgilaymiz. $|z|$, x, y haqiqiy sonlar quyidagi tenglik bilan bog‘langan: $|z| = \sqrt{x^2+y^2}$. (3.1)

Haqiqatan ham, ikki nuqta orasidagi masofa formulasiga ko‘ra, $|z| = OA = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$ tenglik o‘rinlidir (3.1- *b* rasm).

6- misol. z=$\sqrt{2}$-i$\sqrt{2}$ kompleks sonning modulini toping.

Yechish. $x = \sqrt{2}$, y =-$\sqrt{2}$ bo‘lgani uchun, $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = 2$. -$\overrightarrow{OA}$ vektor z= $x + iy \neq 0$ kompleks sonning radius-vektori bo‘lsin (3.1-brasm). Markazi O(0; 0) nuqtada bo‘lgan r= |z|radiusli aylananing *B(r;* 0) nuqtasini, O nuqta atrofida bu nuqta *A(x; y)* nuqta bilan ustma-ust tushadigan qilib buramiz (3.1-drasm). Bu ishni, bir-biridan 2π ga karrali bo‘lgan burish burchagiga farq qiladigan cheksiz ko‘p burish burchaklari yordamida amalga oshirish mumkin. Shu burish burchaklarining har biri ***z = x + iy kompleks sonning argumenti*** deb ataladi.

3.1- *d* rasmda *z = x + iy* kompleks sonning argument-laridan biri bo‘lgan φ burchak ko‘rsatilgan.

z = x+iy kompleks sonning barcha argumentlari to‘plamini Arg(z) bilan belgilaymiz.

Yuqoridagi mulohazalardan ko‘rinadiki, agar φ$\in$Arg(z) bo‘lsa, u holda ixtiyoriy $k \in Z$ son uchun φ + $2\pi k$ $\in$ Arg(z) bo‘ladi. Shu sababli Arg(z) to‘plamni quyidagicha tasvirlash mumkin:

Arg(z) = {φ + 2πk: |$k \in Z$}.

Burish burchagining kosinusi va sinusi ta'riflaridan ko‘rinadiki, *z = x + yi* kompleks sonning har qanday cp argument! uchun quyidagi munosabatlar o‘rinli:

cosφ=$\frac{x}{|z|}$, sinφ = $\frac{y}{|z|}$

Bu tengliklar asosida, *z = x + yi* kompleks sonini z= |z|(cos(φ + i sinφ) ko‘rinishida yozib olish mumkin. Bunday yozish ***kompleks sonni trigonomelrik shaklda tasvirlash*** deb yuritiladi.

7- m i s o 1. a) 1 + i; b) 3i; d) -1 + i; e) 1 – i sonlarini trigonometrik shaklda ifodalang.

Yechish.

a) $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\varphi = \frac{\pi}{4}$,

1 +i= $\sqrt{2}$(cos$\frac{\pi}{4}$+ isin$\frac{\pi}{4}$)

a) |3i|=3 , $\varphi = \frac{\pi}{2}$, 3i = 3(cos$\frac{\pi}{2}$ +isin$\frac{\pi}{2}$);

Trigonometrik shaklda berilgan kompleks sonlarni ko‘paytirish, bo‘lish, darajaga ko‘tarish.

z^n=(r(cosφ + isinφ))^n*ni hisoblash uchun,* z^n =r^n(cosnφ + isinnφ)^n*tenglikni tuzish va* nφ *argumentni bosh argument bilan almashtirish kerak.* Agar *z* =cosφ+*i*sinφ bo‘lsa, darajaga ko‘tarish formulasi quyidagi ko‘rinishni oladi: $(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$. Bu tenglik ***Muavr formulasi*** deyiladi.

Mustaqil yechish uchun misollar

8-misol. a) 1 + i; b) 3i; d) -1 + i; e) 1 – i sonlarning 7-darajasini toping.

9-misol. a) 1 + i; b) 3i; d) -1 + i; e) 1 – i sonlarning 6-darajali ildizini toping.

Kompleks sondan ildiz chiqarish.

z kompleks sonning n-darajali ildizi deb, $w^n = z$ tenglik bajariladigan har qanday w kompleks songa aytiladi (bu yerda $n \in N$). Agar z=0 bo‘lsa, $w^n = 0$ $(n \in N)$
tenglik w=0 soni uchungina bajariladi.

Agar $z \neq o$ bo‘lsa, $w^n = z$ $(n \in N)$ tenglik w ning n ta har xil kompleks ildizlarga ega bo‘lishini isbotlaymiz.

$z=r(\cos\varphi+i\sin\varphi)\neq 0$ **kompleks soni n ta har xil w_k kompleks ildizlarga ega va bu ildizlar quyidagi formula bilan topiladi:**

$$w_k = \sqrt[n]{r}\left(\cos\frac{\alpha+2\pi k}{n} + i\sin\frac{\alpha+2\pi k}{n}\right), \quad k=0,1,2,\dots,n-1$$

Nazorat uchun savollar:

1.Qanday ifodaga algebraik shakldagi kompleks deyiladi?

2.Kompleks haqiqiy va mavhum qismlari deb nimaga aytiladi?

3. O‘zaro qo‘shma kompleks sonlar deb nimaga aytiladi?

4.Kompleks sonning radius-vektori deb nimaga aytiladi?

5.Kompleks trigonometrik shaklda qanday tasvirlanadi?

6.Kompleks sonlar qanday ko‘paytiriladi,bo‘linadi va darajaga ko‘tariladi?

7.Kompleks sondan qanday ildiz chiqariladi?

4-amaliy mashg‘ulot: Matritsa va ularning ayrim xossalari. Matritsalar va ular ustida amallar

Sonlarning m ta satr va n ta ustundan iborat to‘g‘ri to‘rburchak shaklida tuzilgan jadvali m×n o‘lchamli matritsa deyiladi. U

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad (4.1)$$

ko‘rinishida yoziladi. Bunda a_{ij}- haqiqiy sonlar $(i=\overline{1,m}\ \ j=\overline{1,n})$ va matritsaning elementlari hisoblanib *i* va *j* lar mos ravishda qator va ustun indekslari, $m\times n$ – *A* matritsaning o‘lchami deb ataladi. (4.1) formuladagi *A* matritsaning qisqacha ko‘rinishi quyidagicha yoziladi:

$A=\left\|a_{ij}\right\|,\ (i=\overline{1,m}\ \ j=\overline{1,n})$

Matritsalarni qo‘shish, ayirish, songa ko‘paytirish

1-misol. *A* va *B* matritsalarning yig‘indisini hisoblang.

$$A=\begin{pmatrix}2&3\\6&5\\1&2\end{pmatrix}\quad B=\begin{pmatrix}3&8\\7&2\\4&6\end{pmatrix}$$

Yechish. $A+B=\begin{pmatrix}2&3\\6&5\\1&2\end{pmatrix}+\begin{pmatrix}3&8\\7&2\\4&6\end{pmatrix}=\begin{pmatrix}2+3&3+8\\6+7&5+2\\1+4&2+6\end{pmatrix}=\begin{pmatrix}5&11\\13&7\\5&8\end{pmatrix}$

2-misol. Quyidagi amallarni bajaring. $2\begin{pmatrix}3&-1&4\\2&5&0\end{pmatrix}-3\begin{pmatrix}1&2&3\\2&1&4\end{pmatrix}.$

3-misol. Agar $A=\begin{pmatrix}7&2&1&5\\3-2&4-3\\2&1&1&2\end{pmatrix}$; $B=\begin{pmatrix}5&4&3&0\\2&3-2&1\\1&0&2&4\end{pmatrix}$ bo‘lsa, $\frac{1}{2}B-\frac{5}{2}A$ni hisoblang.

4-misol. Do‘konlar soni ikkita bo‘lsin, u holda tovarlarni keltirishni ikkita satr va uchta ustunli matritsa yordamida ifodalash mumkin. Birinchi satr 1-do‘konga, ikkinchisi 2-do‘konga keltirilgan mahsulotlar miqdori. Tovarlarning ikkita do‘konga birinchi marta olib kelinishi quyidagi $A_1=\begin{pmatrix}10&12&8\\5&20&14\end{pmatrix}$ matritsa bilan, ikkinchi marta olib kelinishi esa, $A_2=\begin{pmatrix}5&8&10\\12&5&10\end{pmatrix}$matritsa bilan berilgan bo‘lsa, keltirilgan ja’mi tovarlar miqdorini aniqlang.

$m\times k$ o‘lchamli *A* matritsaning $k\times n$ o‘lchamli *B* matritsaga ko‘paytmasi deb *mxn* o‘lchamli shunday *C* = *A*·*B* matritsaga aytiladiki, uning c_{ij} elementi *A* matritsaning *i*-satr elementlarini *B* matritsaning *j*-ustinidagi mos elementlariga ko‘paytmalari yig‘indisiga teng, ya’ni

$c_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ik}b_{kj}$

Agar *AB* = *BA* bo‘lsa, u holda *A* va *B* matritsalar o‘rni almashinadigan yoki kommutativ matritsalar deyiladi. Matritsalarning kommutativlik sharti ba’zi hollardagina bajariladi.Masalan:

$$A=\begin{pmatrix}1&0&2\\3&1&0\\0&2&0\end{pmatrix};\quad B=\begin{pmatrix}6&2&6\\10{,}5&6&3\\3&6&3\end{pmatrix}$$ matritsalar uchun

$$A\cdot B=\begin{pmatrix}1&0&2\\3&1&0\\0&2&0\end{pmatrix}\times\begin{pmatrix}6&2&6\\10{,}5&6&3\\3&6&3\end{pmatrix}=\begin{pmatrix}1\cdot6+0\cdot10{,}5+2\cdot3&1\cdot2+0\cdot6+2\cdot6&1\cdot6+0\cdot10{,}5+2\cdot3\\3\cdot6+1\cdot10{,}5+0\cdot3&3\cdot2+1\cdot6+0\cdot6&3\cdot6+1\cdot3+0\cdot3\\0\cdot6+2\cdot10{,}5+0\cdot3&0\cdot2+2\cdot6+0\cdot6&0\cdot6+2\cdot3+0\cdot3\end{pmatrix}=$$

$$=\begin{pmatrix}12&14&12\\28{,}5&12&21\\21&12&6\end{pmatrix},$$

$$B\cdot A=\begin{pmatrix}6&2&6\\10{,}5&6&3\\3&6&3\end{pmatrix}\times\begin{pmatrix}1&0&2\\3&1&0\\0&2&0\end{pmatrix}=\begin{pmatrix}6\cdot1+2\cdot3+6\cdot0&6\cdot0+2\cdot1+6\cdot2&6\cdot2+2\cdot0+6\cdot0\\10{,}5\cdot1+6\cdot3+3\cdot0&10{,}5\cdot0+6\cdot1+3\cdot2&10{,}5\cdot2+6\cdot0+3\cdot0\\3\cdot1+6\cdot3+3\cdot0&3\cdot0+6\cdot1+3\cdot2&3\cdot2+6\cdot0+3\cdot0\end{pmatrix}=$$

$$=\begin{pmatrix}12&14&12\\28{,}5&12&21\\21&12&6\end{pmatrix}$$

AB=BA bo‘lib, *A* va *B* matritsalarning kommutativlik sharti bajarildi.

Matritsalarni ko‘paytirishda quyidagi hollar mavjud:

1) $A\cdot B$ ko‘paytma aniqlanmagan;

2) $A\cdot B$ ko‘paytma aniqlangan lekin $A\cdot B\neq B\cdot A$;

3) shunday A va B matritsalar borki, ular uchun $A\cdot B$ ko‘paytma aniqlangan va $A\cdot B=B\cdot A$ bo‘ladi.

Matritsalarni ko‘paytirish kommutativ emas, lekin assotsiativ ya’ni umumiy holda $A\cdot B\neq B\cdot A$

$A\cdot(B\cdot C)=(A\cdot B)\cdot C$

4) shunday A≠0, B≠0 matritsalar mavjudki A·B=0 bo‘ladi.

$$\begin{bmatrix}3&2&1\\5&5&5\\2&1&0\end{bmatrix}\times\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

7-misol. Matritsalarning ko‘paytmasini aniqlang.

$$A=\begin{pmatrix}2&3&4&-5\\1&2&-3&4\\-1&-2&3&1\end{pmatrix};\quad B=\begin{pmatrix}3&2&1\\4&1&-1\\1&3&2\\2&0&1\end{pmatrix}$$

Yechish

$$A\cdot B=\begin{pmatrix} 2\cdot3+3\cdot4+4\cdot1-5\cdot2 & 2\cdot2+3\cdot1+4\cdot3-5\cdot0 & 2\cdot1-3\cdot1+4\cdot2-5\cdot1 \\ 1\cdot3+2\cdot4-3\cdot1+4\cdot2 & 1\cdot2+2\cdot1-3\cdot3+4\cdot0 & 1\cdot1-2\cdot1-3\cdot2+4\cdot1 \\ -1\cdot3-2\cdot4+3\cdot1+1\cdot2 & -1\cdot2-2\cdot1+3\cdot3+1\cdot0 & -1\cdot1+2\cdot1+3\cdot2+1\cdot1 \end{pmatrix}=$$

$$=\begin{pmatrix} 12 & 19 & 2 \\ 16 & -5 & -3 \\ -6 & 5 & 8 \end{pmatrix}$$

8-misol. $A=\begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$; $B=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ bo‘lsa, $A{\cdot}B$ ni toping.

9-misol. Bozordan 4 hafta davomida xarid qilingan 3 xil mahsulot; go‘sht, guruch, yog‘ miqdori *A* matritsa bilan va ularning narxlari esa *B* matritsa bilan berilgan.

$$A=\begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & 4 \\ 2 & 5 & 6 \\ 3 & 2 & 4 \end{pmatrix}; \qquad B=\begin{pmatrix} 1000 \\ 600 \\ 300 \end{pmatrix}$$

To‘rt hafta davomida bu mahsulotlarni sotib olish uchun sarflanadigan xarajatni aniqlang.

10-misol. Zavoddan yangi ishlab chiqarilgan dvigatellarning 40*%i* qayta ta’mirlashga beriladi, qolgani foydalanishga chiqarib yuboriladi. Statistik ma’lumotlarga qaraganda ta’mirlangan dvigatellarning 65*%i* yana qayta ta’mirlashga qaytariladi va 35*%i* yaxshi ishlab ketadi. Qayta ta’mirlashni talab qilgmagan dvigatellarning 20*%i* 1 oydan keyin qayta ta’mirlashni talab qiladi. Qolgani esa yaxshi ishlab ketadi. 2 oydan keyin yaxshi ishlab ketadigan va qayta ta’mirlash kerak bo‘lgan dvigatellar qismini aniqlang. Masala sharti xuddi shu tarzda davom etsa 3 oydan keyingisini ham aniqlang.

Yechish. Ishlab chiqarilgandan keyin barcha dvigatellarning 0,6 qismi yaxshi ishlaydi, 0,4 qismi esa qayta ta’mirlashni talab qiladi. Bir oydan keyin yaxshi ishlab ketadigan dvigatellar ulushi $0,6\times0,8+0,4\times0,35 = 0,62$ ni, qayta ta’mirlanishi kerak bo‘lgan dvigatellar ulushi esa $0,6\times0,2+0,4\times0,65 = 0,38$ ni tashkil etadi. *t*–holatdagi aniqlikni beruvchi X_t qatorni kiritamiz. $X_t = (x_{1t};x_{2t})$, bunda x_{1t}– *t* - momentdagi yaxshi ishlab ketadigan dvigatellar ulushi. x_{2t} - *t* momentdagi qayta ta’mirlanishi kerak bo‘lgan dvigatellar ulushi. Quyidagi matritsani qaraymiz;

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

bunda a_{ij}– dvigatellar ulushi, i – dvigatellar holati (ishlab ketishi yoki yo‘qligi: 1- yaxshi ishlab ketadi, 2- ta’mirlash kerak), j - bir oydan keyingi holati. Ko‘rinib turibdiki, matritsaning qatoridagi elementlari yig‘indisi 1 ga teng bo‘lishi kerak va barcha elementlar nomanfiy.

$X_0 = (0{,}6 \;\; 0{,}4), \qquad A = \begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}35 & 0{,}65 \end{pmatrix};$

bir oydan keyin

$X_1 = X_0 A = (0{,}6 \;\; 0{,}4)\begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}35 & 0{,}65 \end{pmatrix} = (0{,}62 \; 0{,}38);$

ikki oydan keyin

$X_2 = X_1 \times A = X_0 \times A^2 = (0{,}6 \;\; 0{,}4) \times \begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}35 & 0{,}65 \end{pmatrix} \times \begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}35 & 0{,}65 \end{pmatrix} = = (0{,}6 \;\; 0{,}4) \times \begin{pmatrix} 0{,}71 & 0{,}29 \\ 0{,}5075 & 0{,}4925 \end{pmatrix}$=(0,629 ; 0,371),

$X_3 = X_2 \times A = X_0 A^3 = (0{,}634 \;\; 0{,}366)$ Umumiy holda $X_t = X_0 \times A^t$ formula o‘rinli.

<u>Matritsani transponirlash</u> – A matritsadan satrlari va ustunlari o‘rni almashgan A′ matritsaga o‘tishdir. A′ matritsa A matritsaga nisbatan transponirlangan deyiladi.

Ta’rifdan kelib chiqadiki agar A matritsaning o‘lchami m×n bo‘lsa, u holda transponirlangan matritsaning o‘lchami n×m bo‘ladi.

Masalan:

$A = \begin{bmatrix} 5 & 7 & 9 \\ 10 & 8 & 20 \end{bmatrix}; \quad A' = \begin{bmatrix} 5 & 10 \\ 7 & 8 \\ 9 & 20 \end{bmatrix}$

Transponirlashning xossalari:

1) $(A')' = A$ 3) $(A+B) = A+B$

2) $(\lambda A)' = \lambda A'$ □ 4) $(AB)' = B'A'$

11-misol. Korxona uch turdagi mebel ishlab chiqarib, mahsulotini 4 ta tumanda sotadi.

$B = \begin{pmatrix} 2 & 5 & 1 & 2 \\ 1 & 8 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

matritsada b_{ij} – i – turdagi mebelni j - tumandagi qiymati. Agar $A=\begin{pmatrix}200\\80\\100\end{pmatrix}$ matritsa orqali bir oyda tumanlarga tarqatilgan mebellar miqdori berilgan bo‘lsa, korxonaning har bir tumandan oladigan pul miqdorini aniqlang.

12-misol. Korxona 4 xil xomashyodan foydalanib 3 turdagi mahsulot ishlab chiqaradi. A matritsaning elementlari $a_{ij}\ (i=\overline{1,4};\ j=\overline{1,3})$ orqali j - turdagi mahsulotni ishlab chiqarish uchun sarflanadigan i - xom ashyo miqdori aniqlanadi. B matritsa korxonaning ma’lum bir vaqt oralig‘ida ishlab chiqargan mahsulot miqdorini ifodalaydi.

$$A=\begin{pmatrix}2&5&3\\0&1&8\\1&3&1\\2&2&3\end{pmatrix};\quad B=\begin{pmatrix}100\\80\\110\end{pmatrix}$$

mahsulot ishlab chiqarishga sarflanadigan umumiy xomashyo miqdorini toping.

13-misol. Telefon apparatlarini ta’mirlovchi usta 70% telefonlarni past darajada, 20% o‘rta darajada va 10% to‘liq ta’mirdan chiqardi. Statistik ma’lumotlarga ko‘ra 70% past darajada ta’mirlangan telefonlarni bir yildan keyin qayta 10% past darajada, 60% o‘rta darajada, 30% ni to‘liq ta’mirlashadi. O‘rta darajada ta’mirlangan telefonlarni bir yildan keyin qayta 20% past darajada, 50% o‘rta, 30% ni to‘liq ta’mirlashadi. To‘liq ta’mirlangan telefonlarni bir yildan keyin qayta 60% past darajada, 40% o‘rta darajada ta’mirlashadi. Agar masala sharti shu tarzda davom etsa 1,2,3 – yillardan keyingi har bir darajada ta’mirlangan telefonlar ulushini aniqlang.

14-misol. Ikki turdagi yog‘ mahsuloti uchta magazinda sotiladi. Birinchi va ikkinchi kvartallarda ikki turdagi yog‘ning uchta magazinda sotilish hajmini mos ravishda A va B matritsalar bilan berilgan.

$$A=\begin{pmatrix}20&30\\10&20\\25&20\end{pmatrix}\qquad B\begin{pmatrix}10&15\\14&12\\16&15\end{pmatrix}.$$

1) ikkala kvartal davomida sotilgan mahsulotlar hajmini aniqlang.
2) Ikkinchi kvartalda sotilgan mahsulot hajmining birinchi kvartalda sotilgan mahsulot hajmidan farqini aniqlang.

15-misol. Korxona ikki turdagi xomashyodan foydalanib, 3 xil mahsulot ishlab chiqaradi. A matritsa bilan j- xil mahsulotga i - turdagi xomashyoning ishlatilish hajmi berilgan. B matritsa esa bir kvartalda ishlab chiqarilgan mahsulotlar hajmi. Xom ashyo birligining narxi P matritsa bilan berilgan. Quyidagilarni aniqlang.

1) ishlatilgan jami xomashyo miqdorini aniqlovchi C matritsani;

2) sarflangan jami xomashyoning umumiy narxi;

$a)\ A=\begin{pmatrix}3&2&2\\2&3&4\end{pmatrix};\ B=\begin{pmatrix}1&1\\2&1\\1&1\end{pmatrix};\ P=(6\,;3)\ b)\ A=\begin{pmatrix}2&3&2\\4&3&2\end{pmatrix};\ B=\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix};\ P=(3\,;5)$

16-misol. Zavod tikuv mashinalarini ishlab chiqaradi va ishlab chiqarilgan mashinalar ikki holatda bo‘ladi. 1) yaxshi ishlab ketadigan mashinalar, 2) ta’mirlashni talab qiladigan mashinalar. Ishlab chiqarilgan mashinalarning *P %i* yaxshi ishlab ketadigan va (100-*P*) *%i* qayta ta’mirlashni talab qiladigan mashinalar hisoblanadi. Statistik ma’limotlarga qaraganda yaxshi ishlab ketgan mashinalarning 1 oydan keyin 70*%i* yaxshi ishlaydi va 30*%i* qayta ta’mirlashni talab qiladi. Qayta ta’mirlangan mashinalar esa bir oydan keyin 60*%i* yaxshi ishlab ketadigan va 40*%i* qayta ta’mirlashni talab qiladi. Yana bir oydan keyin bu mashinalarning ishlab ketish holatlari qanday bo‘ladi?

$a)P = 80\,b)P = 50\,c)P = 20$

17-misol. Quyidagi amallarni bajaring:

$$3\begin{pmatrix}3&-1&4\\2&5&0\end{pmatrix}-2\begin{pmatrix}1&2&3\\2&1&4\end{pmatrix}$$

18-misol. *A* va *B* matritsalar berilgan, $A\cdot B=(c_{ij})$ matritsaning c_{32} elementini toping.

$$A=\begin{pmatrix}3&2&4&5\\9&2&-3&4\\-1&-5&3&11\end{pmatrix};\quad B=\begin{pmatrix}3&2\\4&-1\\1&-3\\2&5\end{pmatrix}$$

Mavzu yuzasidan savollar.

1. *Matritsa deb nimaga aytiladi va matritsalar ustida qanday amallar aniqlangan?Uchinchi tartibli determinant deb nimaga aytiladi?*
2. *Kvadrat matritsaning determinanti va xossalri.*
3. *Minor va algebraik to‘ldiruvchi nima?*
4. *Teskari matritsa nima?*
5. *Matritsaning rangi nima?*
6. *Matritsalar ustida qanday elementar almashtirishlarni bilasiz?*

5-amaliy mashg‘ulot. Determinantlar. Matritsa determinanti, Minorlar va algebraik to‘ldiruvchilar. Teskari matritsa.

Ikkinchi tartibli matritsaning determinanti deb quyidagi songa aytiladi:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad (5.1)$$

uchinchi tartibli matritsaning determinanti deb quyidagi songa aytiladi.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{32}a_{23}a_{11} \qquad (5.2)$$

1-misol. Berilgan matritsalarni determinantini hisoblang.

$$a)\ A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \qquad b)\ A = \begin{pmatrix} 5 & 8 \\ 4 & 7 \end{pmatrix} \qquad c)\ A = \begin{pmatrix} 3 & 2 & 8 \\ 4 & 5 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

Yechish.

$$a)\ |A| = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 3\cdot 4 - 2\cdot 5 = 2$$

$$b)\ |A| = \begin{vmatrix} 5 & 8 \\ 4 & 7 \end{vmatrix} = 5\cdot 7 - 4\cdot 8 = 3$$

$$c)\ |A| = \begin{vmatrix} 3 & 2 & 8 \\ 4 & 5 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 3\cdot 5\cdot 4 + 2\cdot 3\cdot 2 + 4\cdot 1\cdot 8 - 8\cdot 5\cdot 2 - 3\cdot 3\cdot 1 - 4\cdot 2\cdot 4 = -17$$

Minor. Algebraik to‘ldiruvchi

Determinant a_{ik} elementining M_{ik} minori deb, bu element turgan qator va ustunni o‘chirish natijasida hosil bo‘lgan determinantga aytiladi. Determinant a_{ik} elementining algebraik to‘ldiruvchisi

$$A_{ik} = (-1)^{i+k} M_{ik} \qquad (5.3)$$

munosabat bilan aniqlanadi.

Har qanday determinant ixtiyoriy satri (ustuni) elementlarining mos algebraik to‘ldiruvchilariga ko‘paytmalarining yig‘indisidan iborat, ya’ni:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} \qquad (5.4)$$

$$\begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} \qquad (5.5)$$

(5.4) va (5.5) tengliklar mos ravishda determinantning *i*-satr va *j*-ustun elementlari bo‘yicha yoyilmasi deyiladi. (5.4) va (5.5) formulalar matritsalarning determinantlarini hisoblash uchun qo‘llaniladi.

Yuqori tartibli matritsaning determinanti va uning xossalari

Kvadrat matritsa uchun shu matritsaning elementlaridan tuzilgan *n* - tartibli determinantni hisoblash mumkin. Bu determinant *det A* yoki $|A|$ orqali belgilanadi:

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \qquad (5.6)$$

Determinantning asosiy xossalari:

1. Agar determinantning barcha satr elementlarini ustun elementlariga (yoki aksincha), almashtirilsa, uning qiymati o‘zgarmaydi $(\det A = \det A')$.

2. Agar determinantning ikki yonma-yon turgan satr (ustun) elementlarini o‘rnini mos ravishda almashtirsak, determinantning qiymati qarama-qarshi ishoraga o‘zgaradi.

3. Agar determinantning biror satr (ustun) elementlari umumiy *k* ko‘paytuvchiga ega bo‘lsa, u holda bu ko‘paytuvchini determinant tashqarisiga chiqarish mumkin.

4. Agar determinantning biror satr (ustun) elementlari mos ravishda boshqa yo‘l (ustun) elementlariga proporsional bo‘lsa, u holda determinant qiymati nolga teng bo‘ladi.

5. Agar determinantning satr (ustun) elementlari ikki ifodaning yig‘indisi ko‘rinishida bo‘lsa, u holda determinant ikki determinant yig‘indisi ko‘rinishida yozish mumkin.

6. Agar determinantning biror ustun (satr) elementlariga boshqa ustun (satr)ning mos elementlarini umumiy ko‘paytuvchi *m* soniga ko‘paytirib qo‘shilsa, uning qiymati o‘zgarmaydi.

2-misol. Berilgan determinantni to‘rtinchi satr elementlari bo‘yicha yoyib hisoblang

$$\begin{vmatrix} 2 & -1 & 3 & 0 \\ -5 & 3 & 1 & -2 \\ -3 & -2 & 0 & 4 \\ 1 & 5 & -4 & 2 \end{vmatrix}$$

Yechish. 1) To‘rtinchi satr elementlari bo‘yicha yoyib yechamiz:

$$|A| = a_{41}A_{41} + a_{42}A_{42} + a_{43}A_{43} + a_{44}A_{44} = -a_{41}M_{41} + a_{42}M_{42} - a_{43}M_{43} + a_{44}M_{44} =$$

$$= -\begin{vmatrix} -1 & 3 & 0 \\ 3 & 1 & -2 \\ -2 & 0 & 4 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 & 0 \\ -5 & 1 & -2 \\ -3 & 0 & 4 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 & 0 \\ -5 & 3 & -2 \\ -3 & -2 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & -2 & 0 \end{vmatrix} = 546$$

2) Uchunchi ustun elementlarini nolga aylantirish usuli bilan hisoblaymiz:

$$\begin{vmatrix} 2 & -1 & 3 & 0 \\ -5 & 3 & 1 & -2 \\ -3 & -2 & 0 & 4 \\ 1 & 5 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 17 & -10 & 0 & 6 \\ -5 & 3 & 1 & -2 \\ -3 & -2 & 0 & 4 \\ -19 & 17 & 0 & -6 \end{vmatrix} = -\begin{vmatrix} 17 & -10 & 6 \\ -3 & -2 & 4 \\ -19 & 17 & -6 \end{vmatrix} = -\begin{vmatrix} 17 & -10 & 6 \\ -3 & -2 & 4 \\ -2 & 7 & 0 \end{vmatrix} =$$

$$= 2\begin{vmatrix} -10 & 6 \\ -2 & 4 \end{vmatrix} + 7\begin{vmatrix} 17 & 6 \\ -3 & 4 \end{vmatrix} = 2(-40+12) + 7(68+18) = 546$$

3-misol. Berilgan determinantlarni hisoblang.

1. $\begin{vmatrix} 4 & -3 & 2 & -4 \\ 0 & 5 & 3 & 2 \\ -3 & 2 & 6 & 1 \\ 13 & 3 & -2 & 0 \end{vmatrix}$ 2. $\begin{vmatrix} 3 & -1 & 0 & 3 \\ 5 & 1 & 4 & -7 \\ 5 & -1 & 0 & 2 \\ 1 & -8 & 5 & 3 \end{vmatrix}$ 3. $\begin{vmatrix} 3 & 1 & 0 & 4 \\ -5 & 1 & 10 & 3 \\ 2 & 3 & -2 & -5 \\ 3 & 2 & 1 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 1 \end{vmatrix}$ 5. $\begin{vmatrix} 2 & 3 & -3 & -4 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & 1 & 0 \\ 2 & 3 & 0 & -5 \end{vmatrix}$ 6. $\begin{vmatrix} 3 & -1 & 4 & 2 \\ 5 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 6 & -2 & 9 & 8 \end{vmatrix}$

7. $\begin{vmatrix} 2 & -2 & 1 & 0 \\ 0 & 2 & 2 & -1 \\ 3 & -2 & 2 & 3 \\ 3 & 2 & 6 & 1 \end{vmatrix}$ 8. $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix}$ 9. $\begin{vmatrix} 5 & 1 & 2 & -3 \\ 4 & -2 & 4 & -3 \\ 2 & 1 & 3 & -2 \\ 4 & 1 & 5 & -4 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$ 11. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}$ 12. $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 3 & 5 \\ 5 & 3 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}$

4-misol. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$ ekanligini isbotlang.

5-misol. $A = \begin{pmatrix} 3 & -1 & 0 & 3 \\ 5 & 1 & 4 & -7 \\ 5 & -1 & 0 & 2 \\ 1 & -8 & 5 & 3 \end{pmatrix}$ bo‘lsa $|A|$ ni hisoblang.

6-misol. Berilgan determinantni uch usul bilan hisoblang:

a) *i*-satr bo‘yicha yoyib;
b) *j*-ustun elementlari bo‘yicha yoyib;
c) Oldin *j* - ustundagi bittadan boshqa elementlarini nolga aylantirib, so‘ngra shu ustun elementlari bo‘yicha yoyib.

$a)\ \begin{vmatrix} 5 & 0 & 1 & -1 \\ 1 & 1 & -3 & 2 \\ 3 & -3 & 4 & 5 \\ 2 & 4 & 0 & 3 \end{vmatrix}$ $\quad b)\ \begin{vmatrix} 3 & 5 & -5 & 0 \\ 4 & 3 & -4 & 2 \\ -1 & 2 & 3 & 5 \\ 5 & -1 & 2 & -3 \end{vmatrix}$

$i = 3, \quad j = 2$ $\qquad i = 4, \quad j = 1$

7-misol. Tenglamani yeching. $\begin{vmatrix} 2 & 3 & x \\ 4 & 2 & 1 \\ 3 & x-1 & 0 \end{vmatrix} = 2$

8-misol. $y = \begin{vmatrix} 1 & x-2 & 2 \\ 3 & x & 4 \\ 2 & 1 & 0 \end{vmatrix}$ to‘gri chiziqning funksiyaning burchak koeffitsiyentini toping va grafigini chizing.

Teskari matritsa. Xosmas matritsa

A kvadrat matritsa uchun $A \cdot A^{-1} = A^{-1} \cdot A = E$ tenglik bajarilsa, u holda A^{-1} matritsa *A* matritsaga teskari matritsa deyiladi.(*E* birlik matritsa).

A kvadrat matritsaning determinanti noldan farqli, ya'ni $|A| \neq 0$ bo‘lsa, u holda *A* matritsa xosmas matritsa deb ataladi.

A kvadrat matritsaning teskari matritsasi mavjud (va yagona) bo‘ladi, faqat va faqat bu matritsa xosmas bo‘lsa.Teskari matritsa quyidagi munosabat yordamida hisoblanadi.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{1n} \\ A_{12} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

bu yerda $|A|$-*A* matritsaning determinanti, A_{ij} esa a_{ij} elmentining algebraik to‘ldiruvchisi.

9-misol. Berilgan matritsaga teskari matritsani toping

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{pmatrix}$$

Yechish. $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{vmatrix} = 5+12+21+10-21+6 = 33;$ va $\begin{matrix} A_{11} = -16, \ A_{12} = 9, \ A_{13} = 31 \\ A_{21} = 9, \ A_{22} = -3, \ A_{23} = -3 \\ A_{31} = 11, \ A_{32} = 0, \ A_{33} = -11 \end{matrix}$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} -16 & 9 & 11 \\ 9 & -3 & 0 \\ 31 & -3 & -11 \end{pmatrix} = \frac{1}{33}\begin{pmatrix} -16 & 9 & 11 \\ 9 & -3 & 0 \\ 31 & -3 & -11 \end{pmatrix} = \begin{pmatrix} -\frac{16}{33} & \frac{3}{11} & \frac{1}{3} \\ \frac{3}{11} & -\frac{1}{11} & 0 \\ \frac{31}{33} & -\frac{1}{11} & -\frac{1}{3} \end{pmatrix}$$

Tekshirib ko‘ramiz:

$$A^{-1}\cdot A=\frac{1}{33}\begin{pmatrix}-16 & 9 & 11\\ 9 & -3 & 0\\ 31 & -3 & -11\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & 1\\ 3 & -5 & 3\\ 2 & 7 & -1\end{pmatrix}=\frac{1}{33}\begin{pmatrix}-16+27+22 & -32-45+77 & -16+27-11\\ 9-9+0 & 18+15+0 & 9-9+0\\ 31-9-22 & 62+15-77 & 31-9+11\end{pmatrix}=$$

$$\frac{1}{33}\begin{pmatrix}33 & 0 & 0\\ 0 & 33 & 0\\ 0 & 0 & 33\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E$$

10-misol. Quyidagi berilgan matritsaga teskari matritsani aniqlang.

$$A=\begin{pmatrix}1 & 5 & 1\\ 3 & 2 & 1\\ 6 & -2 & 1\end{pmatrix}$$

11-misol. Quyidagi matritsali tenglamani yeching:

$$\begin{pmatrix}-3 & 1 & 2\\ 1 & 0 & -1\\ -4 & 3 & 0\end{pmatrix}\cdot X=\begin{pmatrix}-2 & 1 & 2\\ 1 & -1 & 3\\ 1 & -1 & 4\end{pmatrix}$$

12-misol. Quyidagi matritsalarni teskari matritsasini aniqlang.

$$1.\begin{pmatrix}2 & -6 & 2\\ 8 & 4 & -2\\ 2 & 0 & 4\end{pmatrix}\qquad 2.\begin{pmatrix}6 & 0 & 3\\ 0 & 3 & -15\\ 6 & 0 & 0\end{pmatrix}\qquad 3.\begin{pmatrix}4 & 3 & 5\\ 6 & 7 & 1\\ 9 & 1 & 8\end{pmatrix}$$

$$4.\begin{pmatrix}0 & 4 & 8\\ 14 & -4 & -4\\ 4 & 8 & 0\end{pmatrix}\qquad 5.\begin{pmatrix}-1 & -1 & -1\\ -1 & 4 & 7\\ 8 & 1 & -1\end{pmatrix}\qquad 6.\begin{pmatrix}2 & 3 & 4\\ 0 & 4 & 11\\ 7 & -5 & 0\end{pmatrix}$$

$$7.\begin{pmatrix}3 & -5 & 3\\ 1 & 2 & 1\\ 1 & -7 & -2\end{pmatrix}\qquad 8.\begin{pmatrix}1 & -4 & -2\\ 3 & -5 & -6\\ 3 & 1 & 1\end{pmatrix}\qquad 9.\begin{pmatrix}4 & 1 & -3\\ 8 & 3 & -6\\ 1 & 1 & -1\end{pmatrix}$$

13-misol. Quyidagi matritsalarni teskari matritsasini toping

$$a)\begin{pmatrix}1 & 4 & 2 & 3\\ 1 & -2 & 1 & -2\\ 1 & -1 & 1 & 1\\ 0 & -10 & -2 & -5\end{pmatrix}\qquad b)\begin{pmatrix}-2 & 1 & -3 & 1\\ -1 & 1 & -2 & 2\\ 1 & -3 & 1 & -4\\ -3 & 3 & -5 & 5\end{pmatrix}$$

Matritsaning rangi. Elementar almashtirishlar

$m\times n$ o‘lchamli A matritsaning rangi deb, uning noldan farqli minorining eng yuqori tartibiga aytiladi va $rang(A)$ yoki $r(A)$ kabi belgilanadi.

Matritsa ustidagi quyidagi almashtirishlar elementar almashtirishlar deb ataladi:

a) faqat nollardan iborat satrni (ustunni) o‘chirish;

b) ikkita satr (ustun)ning o‘rnini almashtirish;

c) bir satr (ustun)ning barcha elementlarini biror ko‘paytuvchiga ko‘paytirib, boshqa satr (ustun) elementlariga qo‘shish;

d) satr (ustun)ning barcha elementlarini noldan farqli bir xil songa ko‘paytirish;

Elementar almashtirishlar matritsani rangini o‘zgartirmaydi.

e) matritsani transponirlash

14-misol. Berilgan matritsaning rangini toping.

$$A=\begin{pmatrix} 5 & 2 & 1 & 3 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 4 & 8 \\ 3 & -1 & 1 & 2 \end{pmatrix}$$

Yechish.

$$\begin{pmatrix} 5 & 2 & 1 & 3 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 4 & 8 \\ 3 & -1 & 1 & 2 \end{pmatrix} \boxed{-1}\ \boxed{-4}\ \boxed{-1}\ + \quad + \rightarrow \begin{pmatrix} 5 & 2 & 1 & 3 \\ -2 & -3 & 0 & -1 \\ -19 & -6 & 0 & -4 \\ -2 & -3 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 5 & 3 \\ 0 & -3 & -2 & -1 \\ 0 & -6 & -19 & -4 \\ 0 & -3 & -2 & -1 \end{pmatrix} \boxed{-2}\ \boxed{-1}\ + \quad +$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 5 & 3 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & -15 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Demak, $r(A) = 3$.

15-misol. Berilgan matritsalarning rangini aniqlang.

$$A=\begin{pmatrix} 1 & 3 & 1 & 4 & 1 \\ 11 & 3 & 5 & 5 & 2 \\ 5 & 5 & 3 & 17 & 12 \\ 4 & 2 & 2 & 3 & 1 \end{pmatrix}$$

16-misol. Matritsalarning rangini toping.

1. $\begin{pmatrix} 25 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & 3 & 4 \\ -3 & 1 & 10 \end{pmatrix}$ 2. $\begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$

3. $\begin{pmatrix} 5 & 6 & 11 \\ 7 & 8 & 15 \\ 31 & 36 & 67 \end{pmatrix}$ 4. $\begin{pmatrix} 5 & 9 & 15 & 20 & 25 \\ 2 & 3 & 6 & 8 & 10 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

5. $\begin{pmatrix} -3 & 1 & -6 \\ 1 & 0 & 2 \\ 7 & 2 & 14 \end{pmatrix}$ 6. $\begin{pmatrix} 9 & 4 & -1 & 5 \\ 17 & -3 & -8 & 5 \\ 5 & -2 & -3 & 1 \\ -13 & 9 & 10 & -1 \end{pmatrix}$

7. $\begin{pmatrix} 15 & 13 & 21 \\ -7 & 5 & -4 \\ 23 & 31 & 38 \\ 8 & 2-18 & 17 \end{pmatrix}$ 8. $\begin{pmatrix} -9 & 20 & -3 & -6 \\ -5 & 2 & -3 & -4 \\ 8 & 5 & 6 & 7 \end{pmatrix}$

Mavzu yuzasidan savollar.

1. *Determinant deb nimaga aytiladi va determinant ustida qanday amallar aniqlangan? Uchinchi tartibli determinant deb nimaga aytiladi?*
2. *Kvadrat matritsaning determinanti va xossalri.*
3. *Minor va algebraik to'ldiruvchi nima?*
4. *Teskari matritsa nima?*
5. *Matritsaning rangi nima?*
6. *Matritsalar ustida qanday elementar almashtirishlarni bilasiz?*

6-amaliy mashg'ulot. Chiziqli tenglamar sistemasi va uni yechish usullari.

Noma'lumlar soni n ta bo'lgan m ta chiziqli tenglamalar sistamasining umumiy ko'rinishi quyidagicha:

$$\begin{cases} a_{11}x_1 + & a_{12}x_2 + \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + \dots & + & a_{2n}x_n & = & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1}x_1 + & a_{m2}x_2 + \dots & + & a_{mn}x_n & = & b_m \end{cases} \qquad (6.1)$$

bu yerda a_{ij}– noma’lumlar oldidagi koeffitsyentlar; $b_i$lar esa sistemaning – ozod hadlari (i = 1,2,…m, j = 1,2,…n.) deyiladi.(6.1) tenglamalar sistemasida x_1, x_2,..., x_n lar o‘rniga mos ravishda $x_1^0, x_2^0, ..., x_n^0$ o‘zgarmas sonlarni qo‘yish natijasida berilgan tenglamar sistemasi ayniyatlar sistemasiga aylansa, u holda $x_1^0, x_2^0, ..., x_n^0$ lar (6.1) sistemaning yechimi deb ataladi.

Kamida bitta yechimga ega tenglamalar sistemasi birgalikdagi tenglamalar sistemasi deyiladi. Yechimga ega bo‘lmagan tenglamalar sistemasi birgalikda emas deb ataladi.

Agar ikkita tenglamalr sistemasi bir xil yechimga ega bo‘lsa, yoki ikkisi ham yechimga ega bo‘lmasa ular teng kuchli deb ataladi.

Chiziqli tenglamar sistemasini Kramer usulida yechish

Bu holda sistemaning **Kramer usuli**dagi yechimi quyidagicha bo‘ladi:

$x_1 = \frac{\Delta_1}{\Delta}, x_2 = \frac{\Delta_2}{\Delta}, \ldots, x_n = \frac{\Delta_n}{\Delta}$.bunda Δ- sistema matritsasining determinanti, Δ_k- sistema determinantining k- ustunini ozod hadlar ustuni bilan almashtirishdan hosil bo‘lgan determinant.

Agar $\Delta \neq 0$ bo‘lsa sistema yagona yechimga ega bo‘ladi.

Agar $\Delta = 0$ va $\Delta_i = 0 (i = 1, 2, ...n)$ bo‘lsa, berilgan tenglamalar sistemasi cheksiz ko‘p yechimga ega bo‘ladi.

Agar $\Delta = 0$ bo‘lib Δ_i lardan kamida bittasi noldan farqli bo‘lsa, sistema yechimga ega bo‘lmaydi.

1-misol. Berilgan tenglamalar sistemasini Kramer usulida yeching.

$$\begin{cases} 3x_1 + x_2 + 3x_3 = 2 \\ 5x_1 - 2x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 = 1 \end{cases}$$

Yechish.

$$\Delta=\begin{vmatrix}3&1&3\\5&-2&2\\2&2&3\end{vmatrix}=-18+4+30+12-12-15=1\neq 0$$

$$\Delta_1=\begin{vmatrix}2&1&3\\1&-2&2\\1&2&3\end{vmatrix}=-12+2+6+6-8-3=-9$$

$$\Delta_2=\begin{vmatrix}3&2&3\\5&1&2\\2&1&3\end{vmatrix}=9+8+15-6-6-30=-10$$

$$\Delta_3=\begin{vmatrix}3&1&2\\5&-2&1\\2&2&1\end{vmatrix}=-6+2+20+8-6-5=13$$

$$x_1=\frac{\Delta_1}{\Delta}=\frac{-9}{1}=-9 \qquad x_2=\frac{\Delta_2}{\Delta}=-10 \qquad x_3=\frac{\Delta_3}{\Delta}=13.$$

2-misol. Tenglamlar sistemalarini Kramer usulidan foydalanib yeching

1. $\begin{cases}x_1+x_2-2x_3=\dfrac{2}{3}\\ 4x_1+6x_2-14x_3=\dfrac{32}{3}\\ 5x_1+2x_2+2x_3=\dfrac{-32}{3}\end{cases}$

2. $\begin{cases}\dfrac{3}{2}x_1+\dfrac{1}{2}x_2-\dfrac{1}{4}x_3=\dfrac{5}{4}\\ \dfrac{1}{6}x_1+\dfrac{2}{9}x_2+x_3=\dfrac{2}{3}\\ 2x_1-x_2-2x_3=-5\end{cases}$

3. $\begin{cases}3x_1+2x_2+x_3=5\\ 2x_1-x_2+x_3=4\\ x_1+5x_2+2x_3=-1.\end{cases}$

4. $\begin{cases}2x_1-3x_2+x_3=-7\\ x_1+4x_2+2x_3=-1\\ x_1-4x_2-x_3=-3.\end{cases}$

5. $\begin{cases}5x_1+2x_2-3x_3=\dfrac{29}{8}\\ -x_1+3x_2+x_3=-1\\ 2x_1+4x_2-6x_3=\dfrac{3}{2}\end{cases}$

6. $\begin{cases}5x_1+8x_2+x_3=2\\ 3x_1-2x_2+6x_3=-7\\ 2x_1+x_2-x_3=-5.\end{cases}$

7 $\begin{cases}x_1-x_2+x_3=6\\ x_1-2x_2+x_3=9\\ x_1-4x_2-2x_3=4\end{cases}$

8. $\begin{cases}4x_1+2x_2-x_3=1\\ 5x_1+3x_2-2x_3=2\\ 3x_1+2x_2-3x_3=0\end{cases}$

9. $\begin{cases}3x_1+x_2+3x_3=2\\ 5x_1-2x_2+3x_3=1\\ 2x_1+2x_2+3x_3=6\end{cases}$

10. $\begin{cases}5x_1+2x_2+5x_3=4\\ 3x_1+5x_2-3x_3=-1\\ -2x_1-4x_2+3x_3=1\end{cases}$

$$11\begin{cases}3x_1+2x_2+x_3=7\\2x_1+3x_2+x_3=8\\2x_1+x_2+3x_3=-8\end{cases}\quad 12.\begin{cases}4x_1+6x_2+3x_3=-5\\2x_1+6x_2+5x_3=3\\3x_1+6x_2+4x_3=-1\end{cases}$$

$$13.\begin{cases}x_1-2x_2+3x_3=-11\\2x_1+3x_2-4x_3=17\\3x_1-2x_2-5x_3=7\end{cases}\quad 14.\begin{cases}3x_1-3x_2-x_3=-4\\3x_1+5x_2+4x_3=32\\4x_1-2x_2+3x_3=16\end{cases}$$

Chiziqli tenglamalar sistemasini teskari matritsa usulida yechish

(1) tenglamar sistemasini matritsa ko‘rinishda quyidagicha ifodalash mumkin: $A\cdot X=B$ bunda,

$$A=\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}&\dots&a_{2n}\\\dots&\dots&\dots&\dots\\a_{m1}&a_{m2}&\dots&a_{mn}\end{pmatrix},\ X=\begin{pmatrix}x_1\\x_2\\\dots\\x_n\end{pmatrix},\ B=\begin{pmatrix}b_1\\b_2\\\dots\\b_m\end{pmatrix}\qquad(6.2)$$

Chiziqli tenglamalar sistemasida noma’lumlar soni tenglamalar soniga teng ($m=n$) va sistema matritsasi A – xosmas, ya’ni $detA\neq 0$ bo‘lsa, bu sistema yagona yechimga ega bo‘ladi.

Agar tenglamalar sistemasining ($m=n$) matritsasi xosmas, ya’ni $detA \neq 0$ bo‘lsa, u holda sistemaning matritsa ko‘rinishdagi yechimi quyidagicha bo‘ladi:

$$X=A^{-1}\cdot B$$

bunda, A^{-1}- (6.2) munosabatdagi A matritsaning teskari matritsasi, B- esa ozod hadlar matritsasi.

3-misol. Quyidagi tenglamalar sistemasini matritsa usulida yeching.

$$\begin{cases}2x_1+3x_2+4x_3=1\\x_1+2x_2+3x_3=0\\3x_1+4x_2+6x_3=1\end{cases}$$

Yechish.

$$A=\begin{pmatrix}2&3&4\\1&2&3\\3&4&6\end{pmatrix};\quad B=\begin{pmatrix}1\\0\\1\end{pmatrix};\quad X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$|A|=24+27+16-24-24-1=1\neq 0\ \Rightarrow A^{-1}=\frac{1}{\Delta}\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}=\begin{pmatrix}0&-2&1\\3&0&-2\\-2&1&1\end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 0 & -2 \\ -2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

demak, $x_1 = 1$, $x_2 = 1$, $x_3 = -1$.

4-misol. Matritsali tenglamani yeching.

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 5 & -1 & 2 \\ -6 & 4 & 6 \\ -2 & 0 & 7 \end{pmatrix}.$$

Nazorat savollari

1.*Chiziqli tenglamalar sistemasi deb nimaga aytiladi?*
2. Chiziqli tenglamalar sistemasi yechish usullari.
3. Chiziqli tenglamalar sistemasini Kramer usulida yechish.
4.Chiziqli tenglamalar sistemasi matritsalar usuli bilan qanday yechiladi?
5.Kroneker – Kapelli teoremasi qanday ifodalanadi?

7-amaliy mashg'ulot. Chiziqli tenglamalar sistemasi va uni yechishda dasturlar majmuasidan foydalanish.

1-misol. Sistemani Gauss usuli bilan yeching

$$\begin{cases} 2x_1 + x_2 - x_3 + 3x_4 = 20 \\ 5x_1 - x_2 + 2x_3 - x_4 = 17 \\ -3x_1 + 2x_2 - x_3 + 2x_4 = 1 \\ x_1 - x_2 + 4x_3 - 2x_4 = -4 \end{cases}$$

Yechish. Sistemaning kengaytirilgan matritsasini yozib olamiz:
Birinchi qadamda $a_{11} \neq 0$ bo'lishi zarur, lekin $a_{11} = 1$ hisoblashlar uchun qulaydir. Shuning uchun birinchi va to'rtinchi satrlarning o;rnini almashtiramiz

$$\left(\begin{array}{rrrr|r} 2 & 1 & -1 & 3 & 20 \\ 5 & -1 & 2 & -1 & 17 \\ -3 & 2 & -1 & 2 & 1 \\ 1 & -1 & 4 & -2 & -4 \end{array}\right) \leftrightarrow \left(\begin{array}{rrrr|r} 1 & -1 & 4 & -2 & -4 \\ 5 & -1 & 2 & -1 & 17 \\ -3 & 2 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 & 20 \end{array}\right) + \quad \boxed{-5}\ \boxed{3}\ \boxed{-2}$$

1-***qadam.***Birinchi satr elementlarini –5, 3 va -2 ga ko'paytirib, ularni mos ravishda ikkinchi, uchinchi va to'rtinchi satrlarga qo'shamiz, chunki maqsad a_{11} element ostida nollardan iborat " zina " hosil bo'lsin.

2-qadamni o'tkazish uchun, ya'ni matritsada $a_{22} \neq 0$, lekin $a_{22} = 1$ yoki $a_{22} = -1$bo'lgan i qulayroq. Shuning uchun ikkinchi va uchinchi satrlar o'rnini almashtiramiz:

$$\left(\begin{array}{cccc|c} 1 & -1 & 4 & -2 & -4 \\ 0 & 4 & -18 & 9 & 37 \\ 0 & -1 & 11 & -4 & 13 \\ 0 & 3 & -9 & 7 & 28 \end{array}\right) \leftrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 4 & -2 & -4 \\ 0 & -1 & 11 & -4 & 13 \\ 0 & 4 & -18 & 9 & 37 \\ 0 & 3 & -9 & 7 & 28 \end{array}\right) \quad \boxed{4} \quad \boxed{3}$$

2-*qadam*. Ikkinchi satr elementlarini 4 va 3 ga ko‘paytirib mos ravishda uchinchi va to‘rtinchi satr elementlariga qo‘shamiz, natijada a_{22} element tagida ikkinchi ustunda “ zina ” hosil bo‘ladi.

3-*qadam*. Hosil bo‘lgan matritsada $a_{33} = 26 \neq 0$, uchinchi satr elementini $-\frac{24}{26} = -\frac{12}{13}$ ga ko‘paytirib, to‘rtinchi satrga qo‘shamiz. Natijada:

$$-\frac{13}{13} \times \left(\begin{array}{cccc|c} 1 & -1 & 4 & -2 & -4 \\ 0 & -1 & 11 & -4 & -11 \\ 0 & 0 & 26 & -7 & -7 \\ 0 & 0 & 24 & -5 & -5 \end{array}\right) \leftrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 4 & -2 & -4 \\ 0 & -1 & 11 & -4 & -11 \\ 0 & 0 & 26 & -7 & -7 \\ 0 & 0 & 0 & \frac{19}{13} & \frac{19}{13} \end{array}\right).$$

Kengaytirilgan matritsa zinapoya ko‘rinishiga keltirildi. Unga mos keluvchi sistemaning ko‘rinishi quyidagicha:

$$\begin{cases} x_1 - x_2 + 4x_3 - 2x_4 = -4 \\ \quad - x_2 - 11x_3 - 4x_4 = -11 \\ \qquad 26x_3 - 7x_4 = -7 \\ \qquad\qquad \frac{19}{13}x_4 = \frac{19}{13} \end{cases}$$

oxirgi tenglamadan $x_4 = 1$, uchinchidan $x_3 = \frac{-7 + 7x_4}{26} = 0$, ikkinchidan $x_2 = 11 + 11x_3 - 4x_4 = 7$ va birinchidan $x_1 = -4 + x_2 - 4x_3 + 2x_4 = 5$ yechimlarni olamiz. Javob: (5; 7; 0; 1)

2-misol. Berilgan tenglamalar sistemasini Gauss usulida yeching

$$\begin{cases} x_1 + 3x_2 + 4x_3 - 2x_4 = \ \ 2 \\ -3x_1 - 7x_2 - 8x_3 + 2x_4 = -4 \\ 2x_1 - x_2 + 3x_3 \qquad\quad = \ 4 \\ 2x_1 + 4x_2 + 4x_3 \qquad\quad = \ 3 \end{cases}$$

3-misol. Quyidagi berilgan tenglamalar sistemasini Gauss usulida yeching

1. $\begin{cases} 6x_1 + x_2 + 3x_3 = 4 \\ -2x_1 - 5x_2 + 2x_3 = 8 \\ 3x_1 \quad + \quad 2x_3 = 3. \end{cases}$

2. $\begin{cases} x_1 - 3x_2 + 2x_3 = -2 \\ 3x_1 + x_2 - 5x_3 = -4 \\ 4x_1 - 2x_2 - 3x_3 = -11. \end{cases}$

3. $\begin{cases} x_1+x_2+x_3+x_4=10 \\ x_1+x_2-x_3-x_4=-4 \\ x_1-x_2+x_3-x_4=-2 \end{cases}$

4. $\begin{cases} x_1+2x_2+3x_3=8 \\ 2x_1+3x_2-4x_3=5 \\ 3x_1+x_2+x_3=2 \end{cases}$

5. $\begin{cases} x_1+2x_2-x_3+x_4=4 \\ 3x_1-x_2+2x_3-x_4=-1 \\ 2x_1-x_2+3x_3+2x_4=3 \\ 2x_1+3x_2-2x_3+x_4=6 \end{cases}$

6. $\begin{cases} x_1+x_2+3x_3-2x_4=2 \\ 2x_1+2x_2+4x_3-x_4=7 \\ 3x_1+3x_2+5x_3-2x_4=10 \\ 2x_1+2x_2+8x_3-3x_4=13 \end{cases}$

7. $\begin{cases} 2x_1-2x_2+x_3-x_4=-3 \\ x_1+2x_2-x_3+x_4=6 \\ 4x_1-10x_2+5x_3-5x_4=-21 \\ 2x_1-14x_2+7x_3-7x_4=-33 \end{cases}$

8. $\begin{cases} 5x_1+7x_2+4x_3+6x_4=28 \\ 15x_1+30x_2+7x_3+8x_4=97 \\ 9x_1+6x_2+5x_3+8x_4=35 \\ 6x_1+9x_2+3x_3+4x_4=33 \end{cases}$

9. $\begin{cases} x_1-2x_2+x_3-x_4=4 \\ 2x_1+x_2-x_3+2x_4=-2 \\ 3x_1-2x_2-x_3+x_4=0 \\ 2x_1-5x_2+x_3-2x_4=6 \end{cases}$

10. $\begin{cases} 15x_1+2x_2+4x_3-3x_4=19 \\ 3x_1+20x_2+5x_3-2x_4=11 \\ 3x_1+6x_2+2x_3-x_4=8 \\ 9x_1+4x_2+3x_3-2x_4=21 \end{cases}$

11. $\begin{cases} 2x_1+x_2+x_3-2x_4=-5 \\ 13x_1+8x_2+4x_3-3x_4=0 \\ 5x_1+4x_2+2x_3-3x_4=6 \\ 3x_1+2x_2+x_3-x_4=-1 \end{cases}$

12. $\begin{cases} 13x_1-4x_2-x_3-4x_4=-11 \\ 11x_1-2x_2+x_3-2x_4=-7 \\ 5x_1+4x_2+7x_3+4x_4=5 \\ 7x_1+2x_2+5x_3+2x_4=-1 \end{cases}$

13. $\begin{cases} 2x_1+3x_2+x_3+6x_4=-14 \\ x_1-x_2-2x_3+2x_4=-3 \\ 2x_1+x_2+4x_3+2x_4=-2 \\ 3x_1+2x_2+5x_3+4x_4=-5 \end{cases}$

14. $\begin{cases} 3x_1-x_2+3x_3+2x_4=15 \\ 5x_1-3x_2+2x_3+3x_4=18 \\ x_1-3x_2-5x_3-x_4=-12 \\ 7x_1-5x_2+x_3+4x_4=21 \end{cases}$

3-misol. Bir jinsli tenglamalar sistemasining fundamental yechimlar sistemasini toping.

$$\begin{cases} 3x_1+x_2-8x_3+2x_4+x_5=0 \\ 2x_1-2x_2-3x_3-7x_4+2x_5=0 \\ x_1+11x_2-12x_3+34x_4-5x_5=0 \\ x_1-5x_2+2x_3-16x_4+3x_5=0 \end{cases}$$

Yechish. Sistemaning oxirgi tenglamasini birinchi o‘ringa yozamiz, so‘ngra uni zinapoya shakliga keltiramiz:

$$\begin{pmatrix} 1 & -5 & 2 & -16 & 3 \\ 3 & 1 & -8 & 2 & 1 \\ 2 & -2 & -3 & -7 & 2 \\ 1 & 11 & -12 & 34 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & 2 & -16 & 3 \\ 0 & 16 & -14 & 50 & 8 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & 16 & -14 & 50 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & 2 & -16 & 3 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matritsaning rangi r = r(A) = 2. x_1, x_2 o‘zgaruvchilarning bazis minori $\begin{vmatrix} 1 & -5 \\ 0 & 8 \end{vmatrix} \neq 0; x_1$, x_2 ni asosiy o‘zgaruvchilar sifatida tanlab olamiz va ularni asosiy bo‘lmagan x_3, x_4, x_5 lar orqali ifodalaymiz:

$$\begin{cases} x_1 - 5x_2 + 2x_3 - 16x_4 + 3x_5 = 0 \\ \quad 8x_2 - 7x_3 + 25x_4 - 4x_5 = 0 \end{cases}$$

Fundamentall yechimlar sistemasi e_1, e_2, e_3 ni hosil qilish uchun asosiy bo‘lmagan o‘zgaruvchi x_3, x_4, x_5 larni birlik matritsa E_3 satr elementlari bilan almashtiramiz. $x_3 = 1$, $x_4 = 0$, $x_5 = 0$ da sistemaning ko‘rinishi quyidagicha bo‘ladi:

$$\begin{cases} x_1 - 5x_2 + 2 = 0 \\ \quad 8x_2 - 7 = 0 \end{cases}$$

bundan $x_1 = \frac{19}{8}$, $x_2 = \frac{7}{8}$ ya’ni bazis yechimni hosil qilamiz: $e_1 = \left(\frac{19}{8}; \frac{7}{8}; 1, 0, 0\right)$

1) Shunga o‘xshash yana ikkita bazis yechimni topamiz:

$x_3 = 0$; $x_4 = 1$; $x_5 = 0$ da $e_2 = \left(\frac{3}{8}; -\frac{25}{8}; 0, 1, 0\right)$;

$x_3 = 0$; $x_4 = 0$; $x_5 = 1$ da $e_3 = \left(-\frac{1}{2}; \frac{1}{2}; 0, 0, 1\right)$.

Topilgan yechimlar (vektorlar) e_1, e_2, e_3 fundamental sistemani tashkil qiladi. e_1, e_2, e_3 yechimlarning komponentlarini mos ravishda 8, 8, 2 ga ko‘paytirib butun komponentli yechimlar sistemasini hosil qilamiz:
(19; 7; 8; 0; 0), (3; -25; 0; 8; 0), (-1; 1; 0; 0; 2).

Mavzu yuzasidan savollar.

Kramer formulasi qanday ko‘rinishga ega va qachon qo‘llaniladi?

1. *Chiziqli algebraik tenglamalr sistemasi qanday shart bajarilganda yagona yechimga ega bo‘ladi?*
2. *Uchta noma’lumli ikkita tenglamadan iborat chiziqli tenglamalar sistemasi qanday yechiladi?*
3. *Qanday shart bajarilganda n noma’lumli n ta chiziqli bir jinsli tenglamalar sistemasi notrivial yechimga ega?*
4. *Chiziqli tenglamalar sistemasi matritsalar usuli bilan qanday yechiladi?*

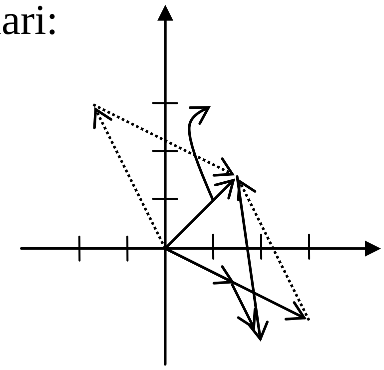
5. Kroneker – Kapelli teoremasi qanday ifodalanadi?

8-amaliy mashg‘ulot. Vektorlar va ularnung ayrim xossalari. Skalyar ko‘paytma. Vektorlarning o‘zaro joylashuvi.

1-misol. Uchta vektor berilgan: $\overline{a}=(1;-1),\ \overline{b}=(1;-2),\ \overline{c}=(1;7).\ \overline{p}=\overline{a}+\overline{b}+\overline{c}$ ni yasang, uning uzunligini toping va $\overline{p}$ vektorni $\overline{a}$ va $\overline{b}$ vektorlar bo‘yicha yoying.

Yechish. $\overline{p}=\overline{a}+\overline{b}+\overline{c}$ vektorni yasash ko‘pburchak qoidasiga asosan rasmda ko‘rsatilgan. U holda vektorni (4.3) formulaga asosan $\overline{p}=\sqrt{3^2+4^2}=5$ $\overline{p}$ vektorni $\overline{a}$ va $\overline{b}$ vektorlar bo‘yicha yoyish, uni quyidagi ko‘rinishda ifodalashdir: $\overline{p}=\alpha\overline{a}+\beta\overline{b},$ bu yerda α va β - haqiqiy sonlar uni aniqlash uchun $(3;\,4)=\alpha(3;\,-1)+\beta(1;\,-2)$ yoki quyidagi $\begin{cases}3=3\alpha+\beta\\4=-\alpha-2\beta\end{cases}$ tenglamalar sistemasini yozib olamiz. Olingan sistemani yechib, $\alpha=2,\ \beta=-3;$ ya’ni $\overline{p}=2\overline{a}-3\overline{b}$ ni olamiz. Demak $\overline{p}$ vektorning $\overline{a}$ va $\overline{b}$ vektorlar bo‘yicha yoyilmasi $2\overline{a}$ va $-3\overline{b}$ vektorlarga qurilgan parallellogram diagonalidan iborat.

Vektorning yo‘naltiruvchi kosinuslari deb $\cos\alpha,\ \cos\beta,\ \cos\gamma$ sonlariga aytiladi, bunda mos ravishda $\alpha,\beta,\gamma-\overline{a}$ vektorning *Ox, Oy, Oz* o‘qlari bilan hosil qilgan burchaklari:



$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}},\qquad \cos\beta=\frac{y}{\sqrt{x^2+y^2+z^2}},\tag{8.1}$$

$$\cos\gamma=\frac{z}{\sqrt{x^2+y^2+z^2}};\ \text{bunda}\ \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\tag{8.2}$$

Ikkita $\overline{a}=(x_1,\,y_1,\,z_1)$va $\overline{b}=(x_2,\,y_2,\,z_2)$ yig‘indisining koordinatalari va $\overline{a}$ vektorning λ songa ko‘paytmasi quyidagi formulalar bo‘yicha aniqlanadi:

$$\overline{a}+\overline{b}=(x_1,\,y_1,\,z_1)+(x_2,\,y_2,\,z_2)=(x_1+x_2,\,y_1+y_2,\,z_1+z_2)\tag{8.3}$$

$$\lambda\overline{a}=\lambda(x_1,\,y_1,\,z_1)=(\lambda x_1,\,\lambda y_1,\,\lambda z_1)\tag{8.4}$$

$\bar{a}$ vektorning l o‘qdagi proyeksiyasi deb $pr_1\bar{a}$

$$pr_l\bar{a} = |\bar{a}| \cdot \cos\varphi \qquad (8.5)$$

songa aytiladi, bu yerda φ $\bar{a}$ vektor va l o‘q orasidagi burchak.

Ikkita $\bar{a}$ va $\bar{b}$ vektorlarning skalyar ko‘paytmasi ($\bar{a}$, $\bar{b}$) deb

$$(\bar{a}, \bar{b}) = \bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cdot \cos\varphi \qquad (8.6)$$

songa aytiladi. $\bar{a} = (x_1, y_1, z_1)$ va $\bar{b} = (x_2, y_2, z_2)$ vektorlarning skalyar ko‘paytmasi:

$$(\bar{a}, \bar{b}) = \bar{a} \cdot \bar{b} = x_1x_2 + y_1y_2 + z_1z_2 \qquad (8.7)$$

formula bilan aniqlanadi.

Vektorning skalyar kvadrati,

$$(\bar{a}, \bar{a}) = \bar{a}^2 = |\bar{a}|^2 = x^2 + y^2 + z^2 \qquad (8.8)$$

yoki

$$|a| = \sqrt{\bar{a}^2} \qquad (8.9)$$

$\bar{a} = (x_1, y_1, z_1)$ va $\bar{b} = (x_2, y_2, z_2)$ vektorlar orasidagi burchak

$$\cos\varphi = \frac{(a,\, b)}{|a||b|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}} \qquad (8.10)$$

formula orqali aniqlahadi.

Ikkita $\bar{a}$ va $\bar{b}$ vektorlar ortogonal deyiladi, agar ularning skalyar ko‘paytmsi nolga teng bo‘lsa, ya’ni $\bar{a} \cdot \bar{b} = 0$.

Ikkita $\bar{a} = (x_1, y_1, z_1)$ va $\bar{b} = (x_2, y_2, z_2)$ vektorlarning kolleniarlik (parallellik) sharti

$$\bar{b} = k\bar{a}, \text{ yoki } \frac{x_2}{x_1} = \frac{y_2}{y_1} = \frac{z_2}{z_1} = \frac{1}{k}; \qquad (8.11)$$

ortogonallik (perpendikulyarlik) sharti

$$\bar{a}\bar{b} = 0 \text{ yoki } x_1x_2 + y_1y_2 + z_1z_2 = 0 \qquad (8.12)$$

1-misol. Berilgan $\bar{a} = (2;\ -1;\ -2)$ va $\bar{b} = (8;\ -4;\ 0)$ vektorlar bo‘yicha quyidagilarni toping:

a) $\bar{c} = 2a$ va $\bar{d} = \bar{b} - \bar{a}$; *b*) $\bar{c}$ va $\bar{d}$ vektorlarning uzunliklarini;

c) $\bar{d}$ vektorning skalyar kvadratini; *d*) $(\bar{c}, \bar{d})$ vektorlarning skalyar ko‘paytmasini

e) $\bar{c}$ va $\bar{d}$ vektorlar orasidagi burchakni

2-misol. Quyidagi $\bar{a}$ va $\bar{b}$ vektorlar berilgan. Berilgan vektorlar modullarini, ularning chiziqli kombinatsiyasi $\bar{c}$ vektor koordinatalarini va uzunligini, $\bar{a}$ va $\bar{b}$ vektorlarning skalyar ko‘paytmasini, ular orasidagi burchak kattaligini, o‘zaro ortogonalligini aniqlang:

$\bar{a} = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\bar{b} = (-1, 2, -2)$, $\bar{c} = 3\sqrt{2}\,\bar{a} - \bar{b}$.

3-misol. $\bar{a} = (2; 1; -1)$ vektorga kolleniar va $(\bar{a}, \bar{b})$ = 3 shartni qanoatlantiruvchi $\bar{b}$ vektorni toping.

4-misol. $a = (5;\ 2;\ 5)$ vektorning $b = (2; -1;\ 2)$ vektor o‘qidagi proyeksiyasini toping.

5-misol.Agar $\bar{a} + 2\bar{b}$ va $5\bar{a} - 4\bar{b}$ vektorlar o‘zaro perpendikulyar bo‘lsa, $\bar{a}$ va $\bar{b}$ birlik vektorlar orasidagi burchakni toping.

6-misol.Quyidagi $b = (8;\ -3;\ -10;\ 10)$ vektorni

$a_1 = (1;\ 0;\ 4;\ 3)$; $a_2 = (-2;\ 3;\ 1; 4)$; $a_3 = (1;\ 1;\ -4;\ 5)$;

$a_4 = (1;\ -2;\ 0;\ 3)$ vektorlar sistemasining chiziqli kombinatsiyasi ko‘rinishida yoyish mumkin yoki yo‘qligini tekshiring.

7-misol a = (5; 1; -2) va *b* = (1; 5; -2) vektorlar berilgan. $3\bar{a} - \bar{b}$ vektorning

a) $3\bar{a} - \bar{b}$ vektorning koordinata o‘qlarida hosil qilgan proyeksiyalarini;

b) uzunligini;

c) yo‘naltiruvchi kosinuslarini toping.

8-misol. Quyidagi $\bar{a}$ va $\bar{b}$ vektorlar berilgan. Berilgan vektorlar modullarini, ularning chiziqli kombinatsiyasi $\bar{c}$ vektor koordinatalarini va uzunligini, $\bar{a}$ va $\bar{b}$ vektorlarning skalyar ko‘paytmasini, ular orasidagi burchak kattaligini, o‘zaro ortogonalligini aniqlang:

a) $\bar{a} = (0, 0, -1, 1)$, $\bar{b} = (1, 1, 1, 1)$ $\bar{c} = 2\bar{a} + \bar{b}$.

b) $\bar{a} = (1, 2, 3)$, $\bar{b} = (-5, 3, 2)$, $\bar{c} = -4\bar{a} + 3\bar{b}$.

9-misol. $\overline{a}=2\overline{m}+4\overline{n}$ va $\overline{b}=\overline{m}-\overline{n}$ vektorlar berilgan, bu yerda $\overline{m}$ *va* $\overline{n}$ birlik vektorlar, ular orasidagi burchak 120°. $\overline{a}$ va $\overline{b}$ vektorlar orasidagi burchakni toping.

10-misol. Tekislikda uchta $\overline{a},\overline{b},\overline{c}$ vektorlar berilgan. Ma'lumki $|\overline{a}|=2,\ |\overline{b}|=3,\ |\overline{c}|=3\ (\overline{a}\,\hat{}\,\overline{b})=60^\circ$,

$(\overline{b}\,\hat{}\,\overline{c})=60^\circ$, $\overline{d}=-\overline{a}+\overline{b}-\overline{c}$ vektorning uzunligini toping.

11-misol. α va β ning qanday qiymatlarida $\overline{a}=-2\overline{i}+3\overline{j}+\beta\overline{k}$ va $\overline{b}=\alpha\overline{i}-6\overline{j}+2\overline{k}$ vektorlar

a) kolleniar, *b*) ortogonal.

12-misol. $\overline{OA}$ vektor *OX*, *OY* va *OZ* o'qlari bilan mos ravishda $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$ burchaklar hosil qiladi. Agar $B(2;\ 2;\ -2\sqrt{2})$ bo'lsa, $\overline{OA}$ va $\overline{OB}$ vektorlarning perpendikulyarligini isbotlang.

13-misol. Uchta $\overline{a}(2;-2),\ \overline{b}(2;-1),\ \overline{c}(2;\ 4)$ vektorlar berilgan. $\overline{p}=2\overline{a}-\overline{b}+\overline{c}$ vektorningkoordinatalarini toping hamda *a* va *b* vektorlar bo'yicha yoying.

14-misol. To'rtta vektor berilgan:

$\overline{a}=(2;\ 1;\ 0),\quad \overline{b}=(1;-1;\ 2),\quad \overline{c}=(2;\ 2;-1),\quad \overline{d}=(3,\ 7,-7)$. $\overline{a}$ vektorni $\overline{b},\ \overline{c},\ \overline{d}$ vektorlar bo'yicha yoying.

15-misol. $\overline{a}=2\overline{i}+3\overline{j}-6\overline{k}$ vektorning uzunligini va yo'naltiruvchi kosinuslarini toping.

16-misol. *m* ning qanday qiymatlarida $\overline{a}=m\overline{i}-3\overline{j}+2\overline{k}$ va $\overline{b}=\overline{i}+2\overline{j}-m\overline{k}$ vektorlar o'zaro perpendikulyar bo'ladi.

17-misol. $\overline{a}=\overline{i}+\overline{j}+2\overline{k}$ vektorning $\overline{b}=\overline{i}-\overline{j}+4\overline{k}$ vektor yo'nalishidagi proyeksiyasini toping.

18-misol. $\overline{a}=3\overline{i}-6\overline{j}-\overline{k}$, $\overline{b}=\overline{i}+4\overline{j}-5\overline{k}$ va $\overline{c}=3\overline{i}+4\overline{j}+2\overline{k}$ vektorlar berilgan. $\overline{a}+\overline{c}$ vektorni $\overline{b}+\overline{c}$ vektorga proyeksiyasini toping.

Mavzu yuzasidan savollar

1. Vektor deb nimaga aytiladi?

2. Vektorning moduli uning koordinatalari orqali qanday ifodalanadi?

3. Vektorlar uchun qanday chiziqli amallar aniqlangan?

4. Vektorlarning yo'naltiruvchi kosinuslari uning koordinatalari orqali qanday ifodalanadi?

5. Vektorlarning kolleniarlik shartlari.
6. Bazis deb nimaga aytiladi?
7. Vektorlarning skalyar ko'paytmasi deb nimaga aytiladi?
8. Ikki vektor orasidagi burchak nima?

9-amaliy mashg'ulot.Vektorlarning vektor ko'paytmasi, aralash ko'paytma, xossalari. Vektorlar algebrasining amaliy qo'llanishi.

$\bar{a}$ vektorning $\bar{b}$ vektorga *vektor ko'paytmasi* deb, $\bar{c}=\bar{a}\times\bar{b}$ ko'rinishda belgilanuvchi va quyidagi shartlarni qanoatlantiruvchi $\bar{c}$ vektorga aytiladi:

1. $\bar{c}$ vektor $\bar{a}$ va $\bar{b}$ vektorlarga perpendikulyar;
2. $\bar{c}$ vektor uchidan qaralganda $\bar{a}$ vektordan $\bar{b}$ vektorga eng qisqa burilish soat mili yo'nalishiga teskari yo'nalishda kuzatiladi ($\bar{a}$, $\bar{b}$, $\bar{c}$ vektorlarning bunday joylashuvini o'ng uchlik deyiladi);
3. $\bar{c}$ vektorning moduli $\bar{a}$ va $\bar{b}$ vektorlarga qurilgan parallellogramning S yuzasini ifodalovchi songa teng, ya'ni $|\bar{c}|=S=|\bar{a}||\bar{b}|\sin\varphi$ (φ - $\bar{a}$ va $\bar{b}$ vektorlar orasidagi burchak).

Vektor ko'paytmasining asosiy xossalari:

1. $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$;
2. $(\lambda\bar{a})\times\bar{b}=\bar{a}\times(\lambda\bar{b})=\lambda(\bar{a}\times\bar{b})$;
3. $\bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times\bar{b}+\overline{a\times}\,\bar{c}$;
4. Agar $\bar{a}=0,$ yoki $\bar{b}=0,$ yoki $\bar{a}\,\|\,\bar{b}$ bo'lsa, u holda $\bar{a}\times\bar{b}=0$. Xususan $\bar{a}\times\bar{a}=0$.

Koordinata o'qlari ortlarining vektor ko'paytmasi:

$$\bar{i}\times\bar{i}=0,\ \bar{j}\times\bar{j}=0,\ \bar{k}\times\bar{k}=0;\qquad \bar{i}\times\bar{j}=\bar{k},\ \bar{j}\times\bar{k}=\bar{i},\ \bar{k}\times\bar{i}=\bar{j}.$$

agar $\bar{a}=a_x\bar{i}+a_y\bar{j}+a_z\bar{k},\ \bar{b}=b_x\bar{i}+b_y\bar{j}+b_z\bar{k}$ bo'lsa, u holda

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

Agar $\bar{a}$ va $\bar{b}$ vektorlar kolleniar bo'lsa, u holda $\frac{a_x}{b_x}=\frac{a_y}{b_y}=\frac{a_z}{b_z}$.

1-misol. $\bar{a}=\bar{i}+3\bar{j}$ va $\bar{b}=3\bar{j}-4\bar{k}$ vektorlarga qurilgan parallelogram yuzasini hisoblang.

Yechish. $\bar{a}$ va $\bar{b}$ vektorlarga qurilgan parallelogrammning S yuzasi shu vektorlarning vektor ko'paytmasining moduliga teng: $S=|\bar{a}\times\bar{b}|$. Vektor ko'paytmani topamiz:

$$\bar{a}\times\bar{b}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 3 & 0 \\ 3 & 0 & -4 \end{vmatrix}=-12\bar{i}+4\bar{j}-9\bar{k}.$$

Demak, $S=\sqrt{(-12)^2+4^2+(-9)^2}=\sqrt{144+16+81}=\sqrt{241}$ kv birlik.

$\bar{a}$, $\bar{b}$, $\bar{c}$ vektorlarning *aralash ko'paytmasi* $(\vec{a}\bar{b}\bar{c})$ deb, $(\bar{a}\times\bar{b})$ vektor ko'paytmaning $\bar{c}$ vektorga skalyar ko'paytmasiga aytiladi. Aralash ko'paytmaning xossalari:

a) $(\bar{a}\times\bar{b})\cdot\bar{c}=\bar{a}\cdot(\bar{b}\times\bar{c})$.

Bu xossadan aralash ko'paytmani $\bar{a}\bar{b}\bar{c}$ ko'rinishda belgilash mumkin ekanligi kelib chiqadi.

b) $\bar{a}\bar{b}\bar{c}=\bar{b}\bar{c}\bar{a}=\bar{c}\bar{a}\bar{b}$, ya'ni ko'paytiriluvchi vektorlar o'rinlari doiraviy almashtirilganda aralash ko'paytma qiymati o'zgarmaydi;

c) $\bar{a}\bar{b}\bar{c}=-\bar{b}\bar{c}\bar{a}$, $\bar{a}\bar{b}\bar{c}=-\bar{c}\bar{b}\bar{a}$, $\bar{a}\bar{b}\bar{c}=-\bar{a}\bar{c}\bar{b}$, ya'ni qo'shni ikkita vektorlarning o'rinlari alamshtirilganda aralash ko'paytma ishorasini o'zgartiradi;

d) agar vektorlardan aqalli bittasi nol vektor yoki $\bar{a}$, $\bar{b}$, $\bar{c}$ vektorlar komplanar bo'lsa, u holda $\bar{a}\bar{b}\bar{c}=0$ bo'ladi.

Agar

$$\begin{aligned} \bar{a} &= a_x\bar{i}+a_y\bar{j}+a_z\bar{k} \\ \bar{b} &= b_x\bar{i}+b_y\bar{j}+a_z\bar{k} \\ \bar{c} &= c_x\bar{i}+c_y\bar{j}+c_z\bar{k} \end{aligned}$$

bo'lsa, u holda

$$(\vec{a}\bar{b}\bar{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Agar $\bar{a}$, $\bar{b}$, $\bar{c}$ vektorlar komplanar bo'lsa, u holda

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0.$$

Aralash ko'paytma ko'paytiriluvchi vektorlarga qurilgan parallelopiped hajmiga ishora aniqligida teng, ya'ni $V=\pm(\bar{a}\bar{b}\bar{c})$.

2-misol. Uchlari $A(1,\ 2,\ 0)$; $B(-1,\ 2,\ 1)$; $C(0;\ -3;\ 2)$ va $D(1,\ 0,\ 1)$ nuqtalarda bo'lgan piramidaning hajmini hisoblang.

3-misol. $\overline{a}$ va $\overline{b}$ vektorlar o‘zaro $\varphi = 45^\circ$li burchak tashkil qilib, $|\overline{a}| = |\overline{b}| = 5$ bo‘lsa, $\overline{p} = \overline{a} - 2\overline{b}$ va $\overline{q} = 3\overline{a} + 2\overline{b}$ vektorlarga qurilgan uchburchak yuzini hisoblang.

4-misol. Uchlari *A*(7, 3, 4), *B*(1, 0, 6), *C*(4, 5, -2) nuqtalardan iborat uchburchak yuzini hisoblang.

5-misol. Piramidaning uchlari berilgan: *A*(2, 3, 1), *B*(4, 1, -2), *C*(6, 3, 7), *D*(-5, -4, 8). Uchburchakning *D* uchidan tushirilganbalandligi uzunligini toping.

6-misol. Uchburchakning uchlari berilgan: *A*(1, -1, 2), *B*(5, -6, 2), *C*(1, 3, -1). Uning *B* uchidan *AC* tomoniga tushirilgan balandligining uzunligini hisoblang.

7-misol. Uchlari *A*(2, -1, 1), *B*(5, 5, 4), *C*(3, 2, -1), *D*(4, 1, 3) nuqtalarda bo‘lgan piramida hajmini hisoblang.

Mavzu yuzasidan savollar

1. *Vektorlarning skalyar ko‘paytmasi deb nimaga aytiladi?*
2. *Ikki vektor orasidagi burchak nima?*
3. *Ikkita vektorning vektor ko‘paytmasi nima?*
4. *Vektorlarning aralash ko‘paytmasi deb nimaga aytiladi?*
5. *Vektorlarning perpendikulyarlik va parallellik shartlari.*
6. *Qanday vektorlar komplanar deb ataladi?*
7. *Vektor va aralash ko‘paytmaning geometrik ma’nosi.*

10-amaliy mashg‘ulot. Tekislikda to‘g‘ri chiziqning umumiy tenglamasi va uning turli xususiy tenglamalari.

Tekislikda to‘g‘ri chiziqning umumiy tenglamasi

$$Ax + By + C = 0 \ \left(A^2 + B^2 \neq 0\right) \qquad (10.1)$$

Burchak koeffitsiyentli tenglamasi

$$y = kx + b, \qquad (10.2)$$

(*k* – burchak koeffitsiyenti, *b* – boshlang‘ich ordinatasi).
Kesmalarga nisbatan tenglamasi

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (10.3)$$

(*a* va *bOx* va *Oy* o‘qlarda ajratgan kesmalar).

M_1(x$_1$; y$_1$) va M_2(x$_2$; y$_2$) nuqtalar orqali o‘tuvchi to‘g‘ri chiziq tenglamasi quyidagicha aniqlanadi:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

Ikkita to‘g‘ri chiziq $y = k_1x+b_1$ va $y = k_2x+b_2$ yoki $A_1x+B_1y+C_1 = 0$ va $A_2x+B_2y+C_2 = 0$ tenglamalar bilan berilgan bo‘lsin, ular orasidagi φ burchak

$$tg\varphi=\frac{k_1-k_2}{1+k_1k_2} \qquad (10.4)$$

yoki

$$\cos\varphi=\frac{\pm(A_1A_2+B_1B_2)}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}$$

formulalar bilan topiladi.

To‘g‘ri chiziqlarning parallellik sharti:

$$k_1 = k_2 \quad \text{yoki} \quad \frac{A_1}{A_2}=\frac{B_1}{B_2} \qquad (10.5)$$

To‘g‘ri chiziqlarning perpendikulyarlik sharti:

$$k_2=-\frac{1}{k_1} \quad \text{yoki} \quad A_1\cdot A_2+B_1\cdot B_2=0 \qquad (10.6)$$

$M_0(x_0; y_0)$ nuqtadan $Ax+By+C = 0$ to‘g‘ri chiziqqacha bo‘lgan masofa

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$$ formula bilan aniqlanadi.

Misollar:

1-misol. $M_1(2, 0)$ va $M_2(3, 4)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

Yechish. $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$ formulaga ko‘ra

$$\frac{x-2}{3-2}=\frac{y-0}{4-0}\Rightarrow\frac{x-2}{1}=\frac{y}{4}$$

2-misol. Berilgan to‘g‘ri chiziqlarni o‘zaro parallel va perpendikulyar bo‘lgan juftliklarga ajrating.

1)$2y + 3x + 5 = 0$ 2)$6y + 9x - 25 = 0$
3)$2y + x + 8 = 0$4) $y - 2x + 10 = 0$

3-misol. $M(0;3)$ nuqtadan o‘tuvchi va $\bar{a} = \{2,1\}$ vektorga parallel tog‘ri chiziq tenglamasini tuzing.

4-misol. $3x + y - 6 = 0$ to‘g‘ri chiziq va $A(-3; 1)$, $B(3; 3)$ nuqtalar orqali o‘tuvchi to‘g‘ri chiziqlar orasidagi burchakni aniqlang.

5-misol. C(1; 1) nuqtadan o‘tuvchi va koordinata burchagidan yuzasi 2 kv. birlik bo‘lgan uchburchak ajratadigan to‘g‘ri chiziq tenglamasini tuzing.

6-misol. Uchlari A(7; 9), B(2; -3), C(3; 6) nuqtalarda bo‘lgan uchburchakning

a) M medianalar kesishish nuqtasi koordinatalarini

b) A uchidan chiqib BC tomonini E nuqtada kesib o‘tuvchi AE bessektrisasi asosi E nuqta koordinatalarini aniqlang.

Yechish. *a*)D nuqta BC tomonni o‘rtasi bo‘lganligi uchun

$$x_D = \frac{x_C + x_B}{2} = \frac{3+2}{2} = \frac{5}{2}$$

$$y_D = \frac{y_C + y_B}{2} = \frac{6-3}{2} = \frac{3}{2} \Rightarrow D\left(\frac{5}{2};\frac{3}{2}\right).y$$

A(7; 9)

M medianalar kesishish nuqta bo‘lganligi uchun bu C(3 ; 6) AD kesmani $\lambda = 2:1$(uchburchak uchidan boshlab hisoblanganda) nisbatda bo‘ladi. Demak M nuqtani koordinatalari quyidagicha aniqlanadi.

$$x_M = \frac{x_A + \lambda x_D}{1+\lambda} = \frac{7 + 2\cdot\frac{5}{2}}{1+2} = 4 \quad y_M = \frac{y_A + \lambda y_D}{1+\lambda} = \frac{9 + 2\cdot\frac{3}{2}}{1+2} = 4$$ B(2; -3)

Demak $M(4;4)$.

b) Ikki nuqta orasidagi masofa formulasiga ko‘ra: $|AC| = \sqrt{(7-3)^2 + (9-6)^2} = 5,\ |AB| = \sqrt{(7-2)^2 + (9+3)^2} = 13$.

AE bissektrisa BC tomonni quyidagicha nisbatda bo‘ladi: $\lambda = \frac{|CE|}{|EB|} = \frac{|AC|}{|AB|} = \frac{5}{13}$

$$\Rightarrow x_E = \frac{x_C + \lambda x_B}{1+\lambda} = \frac{3 + \frac{5}{13}\cdot 2}{1 + \frac{5}{13}} = \frac{49}{18}, \quad y_E = \frac{y_C + \lambda y_B}{1+\lambda} = \frac{6 + \frac{5}{13}\cdot(-3)}{1+\frac{5}{13}} = \frac{7}{2}$$

Demak $E(\frac{49}{18};\ \frac{7}{2})$.

7-misol. Uchlari A(-7; 2); B(5; -3); C(8; 1) nuqtalarda bo‘lgan ABC uchburchakni B uchidan chiqarilgan mediana, balandlik, bissektrisa tenglamalarini tuzing.

8-misol. Quyidagi jadvalda muzqaymoqning narxi va unga mos keluvchi bir kunlik sotilish miqdori berilgan.

P sotilish narxi	100	200	300	400	500
Q sotilish miqdori	900	700	500	300	100

a) $P = f(Q)$ funksiya grafigini chizing.

b) Muzqaymoqqa bo‘lgan talab funksiyasini toping.

9-misol. Ikki turdagi transport vositasi bilan yuk tashish xarajatlari $P = 100 + 4Q$ va $P = 200 + 3Q$ funksiyalar bilan ifodalangan. Bunda Q – yuz kilometrlardagi masofa, P – pul birligidagi transport xarajatlari. Qaysi masofadan boshlab ikkinchi yuk tashish mashinasida birinchisiga qaraganada yuk tashish arzonga tushadi.

10-misol. $5x - y + 10 = 0$ va $8x + 4y + 9 = 0$ to‘g‘ri chiziqlarning kesishish nuqtasidan o‘tuvchi va $x+3y = 0$ to‘g‘ri chiziqqa parallel bo‘lgan to‘g‘ri chiziq tenglamasini tuzing.

11-misol. $3x + 4y - 1 = 0$ va $4x - 3y + 5 = 0$ tenglamalar bilan berilgan ikki to‘g‘ri chiziqlar orasidagi burchakni toping.

12-misol. Parallellogramning ikkita tomonining tenglamalari $x + y + 5 = 0$ va $x - 4y = 0$ bo‘lib, diagonallarining kesishish nuqtasi $O(2;-2)$ bo‘lsa, qolgan tomonlarining tenglamalarini tuzing.

13-misol.$A(-4;\ 1)$ nuqtadan o‘tuvchi va koordinata o‘qlariga parallel bo‘lgan to‘g‘ri chiziqlar tenglamalarini tuzing.

14-misol. Uchlari $A(1;\ -3)$ va $B(4;\ 3)$ nuqtalarda bo‘lgan kesmani uchta teng qismlarga ajrating va bo‘linish nuqtalaning koordinatalarini aniqlang.

15-misol. Agar uchburchak tomonlari o‘rtalri koordinatalari A(-2; 1), B(2; 3), C(4; -1) lar bo‘lsa, uning uchlari koordinatalarini aniqlang.

16 -misol.(3; -1) nuqta orqli o‘tuvchi va Ox o‘qi bilan 45^0 burchak tashkil qiluvchi to‘g‘ri chiziq tenglamasini tuzing.

17-misol. $A(-3; 1)$va $B(3; 3)$ nuqtalar orqali o‘tuvchi to‘g‘ri chiziq va $3x+y-6 = 0$ to‘g‘ri chiziqlar orasidagi burchak bissektrisa tenglamasini tuzing.

18-misol. Uchburchakning $A(-1; 2)$, $B(3; -1)$, $C(0; 4)$ uchlaridan uning qarama – qarshi tomoniga parallell to‘g‘ri chiziq o‘tkazing va shu to‘g‘ri chiziqlarning tenglamalarini tuzing.

19-misol. Uchburchakning uchta uchi koordinatalari A(-1; 3), B(3; -2), C(5; 3) lar berilgan.

a) Uchta tomoni tenglamasi, *b*) *B* uchidan chiqqan medianasi, *c*) *C* uchidan *AB* tomoniga tushirilgan balandlik tenglamalarini tuzing.

20-misol. Uchburchak ikki uchi koordinatalari $A(-2; 1)$, $B(3; -4)$ va balandliklari kesishish nuqtasi $D(5; -1)$ berilgan. Berilgan uchburchakni tomonlari tenglamalarini tuzing.

21-misol. Diagonallari 10 sm va 6 sm, katta diagonali Ox o‘qida, kichigi esa Oy o‘qida joylashgan romb tomonlari tenglamasini tuzing.

22-misol. Tovarni ishlab chiqarish xarajatlari quyidagicha: mahsulot miqdori 100 ta bo‘lganda xarajat 200 p/b., 300 ta bo‘lganda 500 p/b., agar xarajat funksiyasi chiziqli bo‘lsa, 500 ta mahsulot ishlab chiqarishga qancha xarajat sarflanishini aniqlang.

23-misol. Ishlab chiqaruvchiga 60 ta tovardan 300 p/b., 100 ta dan esa 800 p/b. foyda keladi. Agar foyda funksiyasi chiziqli bo‘lsa, u holda 500 ta tovarni sotishdan keladigan foydani toping.

24-misol. Tovarni ikkita magazinda sotishdan keladigan foyda $P = -2+3Q$ va $P = -3+\frac{16}{5}Q$ funksiyalar bilan ifodalanadi. Bunda Q – yuz donada miqdor, P – foyda birligi ming so‘mda. Qaysi miqdordan boshlab ikkinchi magazinda savdo qilish foydali bo‘ladi.

25-misol. Firma tovarning narxi 2000 so‘m bo‘lganda bu tovardan 400 ta, 4000 so‘m bo‘lganda esa 700 ta ishlab chiqaradi. Bu mahsulotga bo‘lgan taklif funksiyasini toping.

26-misol. Gvozdikaga bo‘lgan talab narx 100 so‘m bo‘lganda xarid 2000 dona, 200 so‘m bo‘lganda esa 1500 dona. Gvozdikaga bo‘lgan talab funksiyani toping.

27-misol. B tovarni ishlab chiqarishga sarflanadigan o‘zgaruvchan xarajat quyidagicha:

Q	20	40
VC	500	650

O‘zgarmas xarajat esa 9000 p/b. bo‘lsa, xarajat funksiyasini toping.

28-misol. Ikki turdagi transport vositasi bilan transport vositasi bilan yuk tashish xarajatlari

y = 150+50x va y = 250+25x tenglamalar bilan ifodalanadi. Qaysi masofadan boshalab, ikkinchi turdagi transport vositasiga ketgan xarajatlar birinchisiga nisbatan kam bo‘ladi.

29-misol. Ishlab chiqarish halmi y ni mehnat unumdorligi x ga bog‘liqligi chiziqli va$x$ = 3 da$y$ = 185, x = 5 da y = 305 bo‘lsa, ishlab chiqarish tenglamasini toping. x = 20 da ishlab chiqarish hajmini aniqlang.

Mavzu yuzasidan savollar.

1. *Tekislikdagi analitik geometriyaning sodda masalalarini ko‘rsating va sanab o‘ting.*
2. *Tekislikdagi to‘g‘ri chiziq tenglamalarini yozing.*
3. *Nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa.*
4. *Parallell to‘g‘ri chiziqlar orasidagi masofa.*
5. *Tekislikdagi 2 ta to‘g‘ri chiziq orasidagi burchak.*
6. *Tekislikdagi 2 ta to‘g‘ri chiziqning parallellik va perpendikulyarlik shartlari.*
7. *Tekislikdagi to‘g‘ri chiziqning burchak koeffitsiyentini aniqlash formulalari.*
8. *Tekislikda to‘g‘ri chiziqlar dastasining tenglamasini yozing.*

11-amaliy mashg‘ulot. Fazoda tekislikning umumiy tenglamasi va uning turli xususiy ko‘rinishlari.

Berilgan $M_0(x_0, y_0, z_0)$ nuqta orqali o‘tuvchi va n=(A, B, C) vektorga perpendikulyar tekislik tenglamasi

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \qquad (11.1)$$

Kesmalarga nisbatan tenglamasi esa

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (11.2)$$

(a, b, c mos ravishda Ox, Oy, Oz o‘qlaridan ajratilgan kesmalar);

Tekislikning umumiy tenglamasi

$$Ax + By + Cz + D = 0 \qquad (11.3)$$

$A(x_0,\ y_0,\ z_0)$ nuqtadan $Ax + By + Cz + D = 0$ tekislikkacha bo‘lgan masofa

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

Ikkita tekislik $A_1x + B_1y + C_1z + D_1 = 0$ va $A_2x + B_2y + C_2z + D_2 = 0$ berilgan bo‘lsin. Ikkita tekislik orasidagi burchak kosinusi φ quyidagi munosabatdan topiladi:

$$\cos\varphi=\pm\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}} \qquad (11.4)$$

Ikkita tekislikning parallellik sharti:

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \qquad (11.5)$$

Tekisliklarning perpendikulyarlik sharti:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0 \qquad (11.6)$$

Fazoda to‘g‘ri chiziq tenglamasi:

Ikkita tekislikning kesishish chizig‘i sifatida:

$$\begin{cases} A_1x_1+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases} \qquad (11.7)$$

Berilgan $M(x_1,\ y_1, z_1)$ nuqta orqali o‘tuvchi va yo‘naltiruvchi vektori $S=(m,\ n,\ p)$ bo‘lgan.

To‘g‘ri chiziqning kanonik tenglamasi

$$\frac{x-x_1}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p} \qquad (11.8)$$

to‘g‘ri chiziqning parametrik tenglamasi

$$\begin{aligned} x&=x_1+mt, \\ y&=y_1+nt, \\ z&=z_1+pt \end{aligned} \qquad (11.9)$$

Berilgan ikki nuqta $M_1(x_1,\ y_1,\ z_1)$ va $M_2(x_2,\ y_2,\ z_2)$ orqali o‘tuvchi to‘g‘ri chiziq tenglamasi.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \qquad (11.10)$$

Ikkita to‘g‘ri chiziqning yo‘naltiruvchi vektorlari $s_1(m_1, n_1, p_1)$ va $s_2(m_2, n_2, p_2)$ berilgan bo‘lsin. Ikkita to‘g‘ri chiziq orasidagi burchak quyidagi munosabatdan topiladi:

$$\cos\varphi = \pm\frac{m_1m_2 + n_1n_2 + p_1p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2}\sqrt{m_2^2 + n_2^2 + p_2^2}} \qquad (11.11)$$

Fazoda ikkita to‘g‘ri chiziqning parallellik sharti:

$$\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2} \qquad (11.12)$$

Fazoda ikkita to‘g‘ri chiziqning perpendikulyarlik sharti:

$$m_1m_2 + n_1n_2 + p_1p_2 = 0 \qquad (11.13)$$

$\frac{x - x_1}{m} = \frac{y - y_1}{n} = \frac{z - z_1}{p}$ to‘g‘ri chiziq va $Ax + By + Cz + D = 0$ tekislik berilgan bo‘lsin.

To‘g‘ri chiziq va tekislik orasidagi burchak φ quyidagi munosabatdan aniqlanadi:

$$\sin\varphi = \frac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2}\sqrt{m^2 + n^2 + p^2}} \qquad (11.14)$$

To‘g‘ri chiziq va tekislikning parallellik sharti:

$$Am + Bn + Cp = 0 \qquad (11.15)$$

To‘g‘ri chiziq va tekislikning perpendikulyarlik sharti:

$$\frac{A}{m} = \frac{B}{n} = \frac{C}{p} \qquad (11.16)$$

1-misol. *a*) $M(1; -2; 3)$ nuqtadan o‘tuvchi va $n = (3; -4; 5)$ vektorga perpendikulyar,

b) $M(1; -2; 3)$ nuqtadan o‘tuvchi va $3x - 4y + 5z + 6 = 0$ tekislikka parallel bo‘lgan tekisliklarning tenglamasini tuzing.

Yechish. *a*)(1) formulaga ko‘ra $A = 3$, $B = -4$, $C = 5$ va $x_0 = 1$, $y_0 = -2$, $z_0 = 3$

$=> 3(x - 1) - 4(y + 2) + 5(z - 3) = 0$

$3x - 4y + 5z - 26 = 0$

b) $Ax + By + Cz + D = 0$ tenglama bilan berilgan tekislik $M(1; -2; 3)$ nuqtadan o‘tsin va $3x - 4y + 5z + 6 = 0$ tekislikka parallel bo‘lsin. U holda (11.8) formulaga asosan

$\frac{A}{3} = \frac{B}{-4} = \frac{C}{5} \Rightarrow A = \frac{3C}{5}$ $B = -\frac{4C}{5}$; $\frac{5D}{C} = 4$ bo‘lsin $=> 3x - 4y + 5z + 4 = 0$

M nuqta shu tekislikka tegishli ekanligidan $3\cdot 1 - 4\cdot(-2) + 5\cdot 3 + 4 = 0$ $=> 4 = -26$

demak $3x - 4y + 5z - 26 = 0$

2-misol. $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z+1}{-1}$ to‘g‘ri chiziq va $M(2; 0; 1)$ nuqtadan o‘tuvchi tekislik tenglamasini tuzing.

3-misol. *a*) *Ox* o‘qi va $A(1; -1; 3)$ nuqta orqali o‘tuvchi tekislik tenglamasini tuzing.

b) *Oy* o‘qi va $B(2; 1; -1)$ nuqta orqali o‘tuvchi tekislik tenglamasini tuzing.

4-misol. $M_0(2; -3; 1)$ nuqta orqali o‘tuvchi va $x - 4y + 5z + 1 = 0$ tekislikka parallel tekislik tenglamasini tuzing.

5-misol. $M_1(2; -15; 1)$ va $M_2(3; 1; 2)$ nuqtalar orqali o‘tuvchi va $3x - y - 4z = 0$ tekislikka perpendikulyar tekislik tenglamasini tuzing.

6-misol. $M_1(2;-1;-1)$ va $M_2(3; 3; -1)$ nuqtalar orqali o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

7-misol. $A(1; 2; 1)$ nuqtaning $\frac{x+2}{3}=\frac{y}{-1}=\frac{z-1}{2}$ to‘g‘ri chiziqdagi proyeksiyasini toping.

8-misol. $M(4; -4; 2)$ nuqta orqali o‘tuvchi va *xOz* tekisligiga parallell tekislik tenglamasini tuzing.

9-misol. *Ox* va *Oy* o‘qlaridan $a = 1$, $b = -1$ kesma ajratuvchi va $A(2; 3; 4)$ nuqta orqali o‘tuvchi tekislik tenglamasini tuzing.

Mavzu yuzasidan savollar.

1. *Tekislik tenglamasi $Ax + By + Cx + D = 0$ bo‘lsa, u koordinatalar sistemasiga nisbatan quyidagu hollarda qanday joylashadi?*
   a) $D = 0$; b) $A = 0$; v) $A = 0, B = 0$; g) $A = 0, B = 0, D = 0$; d) $A = 0, D = 0$.
2. *Tekislik tenglamasini yozing.*
3. *Nuqtadan tekislikgacha bo‘lgan masofa.*
4. *Ikkita parallell tekislik orasidagi masofa.*
5. *Tekisliklarning parallellik va perpendikulyarlik shartlari.*
6. *Ikki tekislik orasidagi burchak.*
7. *To‘g‘ri chziq va tekislikning kesilish nuqtasi qanday topiladi?*

12-amaliy mashg‘ulot. Tekislikda 2-tartibli chiziqlar. Aylana, ellips, giperbola va parabola tenglamalari.

Ikkinchi tartibli egri chiziqning umumiy tenglamasi:

$$Ax^2 + Bxy + Cx + Dy^2 + Ey + F = 0 \ \left(A^2 + B^2 \neq 0\right) \qquad (12.1)$$

1-misol. Quyidagi tenglama bilan berilgan aylana markazining koordinatalari va radiusini toping.

$$x^2 + y^2 - 8x + 6y - 11 = 0$$

Yechish. Hadlarni guruhlab, to‘la kvadrat ajratamiz. $x^2 - 8x + 16 - 16 + y^2 + 6y + 9 - 9 - 11 = 0$ yoki $(x-4)^2 + (y+3)^2 = 36$. Bundan aylana markazi $C(4; -3)$ va radiusi $R = 6$.

2-misol. Markazi (0; 3) nuqtada bo‘lgan (3; 7) nuqtadan o‘tuvchi aylana radiusini toping.

3-misol. 3 Radiusi $\sqrt{13}$ ga teng hamda (1; 0) va (0; -1) nuqtalardan o‘tuvchi aylana tenglamasini tuzing.

$$\left(x-3\right)^2 + \left(y+3\right)^2 = 3\left(x+2\right)^2 + \left(y-2\right)^2 = 13$$

4-misol. Uchta $A(-4; 1)$, $B(2; 7)$, $C(8; 1)$ nuqtalardan o‘tuvchi aylana tenglamasini tuzing.

5-misol. Markazi $A(4; 7)$ nuqtada bo‘lgan va $3x-4y+1 = 0$ to‘g‘ri chiziqqa urinadigan aylana tenglamasini yozing.

6-misol. $9x^2 + 25y^2 = 225$ ellips berilgan. Uning yarim o‘qlari, fokuslari koordinatalarini, ekssentrisitetini, direktrisalari tenglamalarini toping.

7-misol. O‘z harakati davomida $x = 9$ to‘g‘ri chiziqqa nisbatan $A(1; 0)$ nuqtaga uch marta yaqinroq bo‘lgan nuqtalarning trayektoriyasini aniqlang.

8-misol. Agar $2x-5y-30 = 0$ to‘g‘ri chiziq $\frac{x^2}{75} + \frac{y^2}{24} = 1$ ellipsga urinib o‘tishi ma’lum bo‘lsa, shu urinish nuqtaning koordinatalarini toping.

9-misol. $16x^2 - 9y^2 = 144$ giperbola berilgan. Uning yarim o‘qini, fokuslari koordinatalarini, ekssentrisitetini, dirktrisasi va assimptotalari tenglamasini toping.

Yechish. Berilgan giperbolaning kanonik ko‘rinishdagi tenglamasini yozib olamiz.

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \ \Rightarrow a^2 = 9,\ b^2 = 16 \ \Rightarrow \text{o‘qlari } a = 3,\ b = 4$$

$c^2 = a^2 + b^2 = 25$ $c = \pm 5$. Fokuslari: $F_1(5;\ 0)$ va $F_2(-5;\ 0)$ ekssentrisiteti $e = \frac{c}{b} = \frac{5}{3}$.

Direktrisasi $y = \pm\frac{b^2}{c} = \pm\frac{9}{5}$, asimptotalari tenglamalari

$y = \pm\frac{b}{a}x = \pm\frac{4}{3}x$

10-misol. Fokuslari abssissa o‘qida, koordinata boshiga nisbatan simmetrik va uchlari orasidagi masofa $2c=20$, asimptota tenglamalari $y = \pm\frac{3}{4}x$ bo‘lgan giperbola tenglamasini tuzing.

11-misol. Tenglamasi $\frac{x^2}{8} + \frac{y^2}{5} = 1$ bo‘lgan ellips berilgan. Uchlari ellipsning fokuslaridan, fokuslari esa uning uchlarida bo‘lgan giperbola tenglamasini tuzing.

12-misol. Agar parabolaning uchi koordinatalar boshida bo‘lib, u $A(2;\ 4)$ nuqtadan o‘tsa va Ox o‘qiga nisbatan simmetrik bo‘lsa, uning tenglamasini tuzing va fokusini toping.

Yechish. Ox o‘qiga simmetrik va $O(0;\ 0)$ nuqtadan o‘tganligi uchun

$y^2 = 2px \;=>\; 4^2 = 4p,\;\; p = 4 \;=>\; y^2 = 8x$ va $F = \left(\frac{p}{2};0\right) \;=>\; F\ (2;\ 0)$

13-misol. Uchi koordinata boshida bo‘lgan parabola $A(2;\ 4)$ nuqta orqali o‘tadi va Ox o‘qiga nisbatan simmetrik. Parabolaning tenglamasi, fokuslari va direktrisalarini toping.

14-misol. Berilgan $A(4;\ 4)$ nuqta va $x^2 + y^2 + 4x - 4y = 0$ aylana bilan $y = -x$ to‘g‘ri chiziqning kesishish nuqtasi orqali o‘tuvchi aylana tenglamasini yozing.

15-misol. Koordinata boshidan $x^2 + y^2 - 8x - 4y + 16 = 0$ aylanaga o‘tkazilgan urinma tenglamasini yozing.

16-misol. Quyidagi aylanalarning markazlari va radiuslarini toping.

$a)\ x^2 + y^2 - 6x + 4y - 23 = 0$ $\qquad$ $b)\ x^2 + y^2 + 5x - 7y + 2{,}5 = 0$

$c)\ x^2 + y^2 + 7y = 0$ $\qquad$ $d)\ x^2 + y^2 - 2x + 8y - 19 = 0$

17-misol. $A(-3;\ 0)$, $B(3;\ 6)$ nuqtalar berilgan. Diametri AB kesmadan iborat bo‘lgan aylana tenglamasini yozing.

18-misol. Koordinata boshidan va $x+y+a = 0$ to‘g‘ri chiziqning $x^2 + y^2 = a^2$ aylana bilan urinish nuqtalari orqali o‘tuvchi aylana tenglamasini yozing.

19-misol. Berilgan A(1; -2), B(0; -1) va C(-3; 0) nuqtalar orqali o‘tuvchi aylanaga koordinata boshidan o‘tkazilgan urinma tenglamasini yozing.

20-misol. $x^2+y^2-4x+6y-5=0$ aylananing Ox o‘qi bilan kesishish nuqtalariga o‘tkazilgan radiuslari orasidagi burchakni toping.

21-misol. A(3; 0) nuqta $x^2+y^2-4x+2y+1=0$ aylanani ichida yotishini ko‘rsating va A nuqtada teng ikkiga bo‘linadigan vatar tenglamasini yozing.

(*Ko‘rsatma: izlanayotgan vatar OA ga perpendikulyar*, *bunda O – aylananing markazi*.)

22-misol. $3x^2 + 3y^2 - 6x + 8y = 0$ tenglama bilan berilgan aylana radiusini va markazini aniqlang.

23-misol. (3; 1) va B(-1; 3) nuqtalardan o‘tuvchi va markazi

$3x – y – 2 = 0$ to‘g‘ri chiziqda yotuvchi aylana tenglamasini tuzing.

24-misol. Yarim o‘qi 5, ekssentrisiteti $\frac{12}{13}$ ga teng bo‘lgan ellipsning kanonik tenglamasini yozing.

25-misol. Yer ellips bo‘yicha harakatlanadi va uning fokuslaridan birida quyosh joylashgan. Yerdan quyoshgacha bo‘lgan eng qisqa masofa taxminan 147,5 million kilometr, eng katta masofa esa 152,5 million kilometr. Yer orbitasining katta yarim o‘qi va ekssentrisitetini toping.

26-misol. Koordinata o‘qlariga nisbatan simmetrik ellips M(2; $\sqrt{3}$) va B(0; 2) nuqtalar orqali o‘tadi. Uning tenglamasini yozing va M nuqtadan fokuslarigacha bo‘lgan masofani toping.

27-misol. Koordinata o‘qlariga nisbatan simmetrik, fokuslari Ox o‘qida joylashgan, M(2; $\sqrt{21}$) nuqta orqali o‘tuvchi ellipesning ekssentrisiteti $\varepsilon=\frac{3}{4}$. Ellips tenglamasini yozing va uning fokal radius vektorini aniqlang. (*Ko‘rsatma: f okal radius vektorlar, ya’ni M*(x, y) *nuqtadan f ikuslargacha bo‘lgan masof alar* $r_1 = a – ex$, $r_2 = a + ex$ *f ormulalar bilan topiladi*)

28-misol. Fokal radiuslarini yig‘indisi $2\sqrt{5}$, fokuslari F_1(-2; 0),

F_2(2; 0) nuqtalarda bo‘lgan ellips tenglamasini yozing.

29-misol. Fokuslari orasidagi masofa katta va kichik yarim o‘qlari orasidagi masofaga teng bo‘lgan ellipsning ekssentrisitetini toping.

30-misol. $x^2+4y^2 = 4$ ellipsga uchlaridan biri katta yarim o‘qning oxiri bilan ustma – ust tushadigan to‘g‘ri burchakli uchburchak chizilgan. Uning qolgan ikki uchining koordinatalarini toping. (*Ko‘rsatma: tomonlaridan biri* $k = tg30^0$ *og‘ma tenglamasi yozilib, ellips bilan kesishish nuqtasi topiladi.*)

31-misol. $9x^2+25y^2 = 225$ ellipsda, o‘ng fokusigacha bo‘lgan masofa chap fokusigacha bo‘lgan masofadan to‘rt marta uzun bo‘lgan nuqtani toping.

32-misol. $x^2+y^2 = 36$ aylananing barcha ordinatalarini uch marta qisqartirishdan hosil bo‘lgan egrichiziqning tenglamsini yozing.

33-misol. O‘z harakati davomida $A(0; 1)$ nuqtagacha bo‘lgan masofa y-4 = 0 to‘g‘ri chiziqqacha bo‘lgan masofadan ikki marta qisqa bo‘lgan M nuqtaning traektoriyasini aniqlang.

34-misol. $x^2-4y^2=16$ giperbolani yasang, asimptotalarini toping. fokuslari, ekssentrisiteti, asimptotalari orasidagi burchakni toping.

35-misol. $x^2-4y^2=16$ giperbolada ordinatasi 1 bo‘lgan M nuqta olingan. Undan fokuslargacha bo‘lgan masofani toping.

36-misol. Giperbolaning kanonik tenglamasini yozing: *a*) fokuslari orasidagi masofa 10, uchlari orasidagi masofa esa 8 ga teng. *b*) haqiqiy o‘q $a = 2\sqrt{5}$, ekssentrisiteti esa $e = \sqrt{1{,}2}$.

37-misol. Uchlari $\frac{x^2}{25}+\frac{y^2}{9}=1$ ellepsning fokuslarida, fokuslari esa uning uchlarida bo‘lgan giperbola tenglamasini yozing.

38-misol. Berilgan $M_1\left(2\sqrt{7}; -3\right); M_2\left(-7; -6\sqrt{2}\right)$ nuqtalar orqali o‘tuvchi koordinata o‘qlariga nisbatan simmetrik giperbola tenglamasini yozing.

39-misol. Asimptotasi $y=\pm\frac{3}{5}x$ va $M\left(10; -3\sqrt{3}\right)$ nuqtadan o‘tuvchi giperbola tenglamasini yozing.

40-misol. F (0; 2) nuqtadan va y = 4 to‘g‘ri chiziqdan baravar uzoqlikda joylashgan nuqtalar geometrik o‘rnining tenglamasini yozing.

41-misol. *a*) $y^2=4x$; *b*) $y^2=-4x$; *c*) $x^2=4y$; *d*) $x^2=-4y$ tenglamalr bilan berilgan parabolani chizing, fokuslari, direktrisasi tenglamasini yozing.

42-misol. $y^2=4x$ parabolada, fokal radiusi 4 bo‘lgan nuqtani toping.

43-misol. Agar parabola $x+y = 0$ to‘g‘ri chiziq va $x^2+y^2+4y = 0$ aylananing kesishish nuqtalari orqali o‘tsa hamda Oy o‘qiga nisbatan simmetrik bo‘lsa, uning kanonik tenglamasi va direktrisasini yozing.

44-misol. $A(0; 0)$, $B(-1; 2)$ nuqtalar orqali o‘tuvchi va Ox o‘qiga nisbatan simmetrik bo‘lgan parabola tenglamasini yozing.

45**-misol.** $A(0; 0)$, $B(2; 4)$ nuqtalar orqali o‘tuvchi va Oy o‘qiga nisbatan simmetrik bo‘lgan parabola tenglamasini yozing.

46-misol. Diametri 80 m va chuqurligi 10 m bo‘lgan parabola shaklidagi chuqurlik qazilgan. Bu chuqurlikning quyi nuqtasidan markaz bo‘yicha qanday masofada parabolaning fokusi joylashgan.

Nazorat savollari

1. *Ikkinchi tartibli egri chiziqlarning umumiy ko'rinishini yozing .*
2. *Aylana kanonik tenglamasini ko'rsating.*
3. *Ellips kanonik tenglamasini ko'rsating.*
4. *Parabola kanonik tenglamasini ko'rsating.*

13-amaliy mashg'ulot. Analitik geometriyaning amaliy masalalarga tatbiqi.

Faraz qilaylik, $M_1(x_1;y_1)$, $M_2(x_2;y_2)$, $M_3(x_3;y_3)$ - uchburchak uchlari. U holda yuza quyidagi formula bilan hisoblanadi:

$$\pm S = \frac{1}{2}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} \qquad (13.1)$$

Uchburchak yuzasi

$$\pm S = \frac{1}{2}\left(\left(y_2 - y_3\right)\left(x_1 - x_3\right) - \left(y_1 - y_3\right)\left(x_2 - x_3\right)\right) = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

Agar M_3 koordinata boshi bilan ustma-ust tushsa, u holda $x_3 = y_3 = 0$ va

$$\pm S = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Agar uchta nuqta bir to'g'ri chiziqda yotsa, u holda uchburchakning yuzi nolga teng va biz bundan uch nuqatning bir to'g'ri chiziqda yotish shartini hosil qilamiz:

$$\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = 0 \qquad (13.2)$$

1-misol. $M_1(0;\ 2)$; $M_2(2;\ 6)$; $M_3(1;\ 4)$ nuqtalar bir to'g'ri chizqda yotishini ko'rsating.

Yechish.

$$\begin{vmatrix} 0-1 & 2-4 \\ 2-1 & 6-4 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix} = -2-(-2) = 0$$

2-misol. Uchlari $M_1(3;-2)$; $M_2(-4;0)$; $M_3(2;5)$ nuqtalarda bo‘lgan uchburchak yuzasini hisoblang.

3-misol. Quyidagi nuqtalar bir to‘g‘ri chiziqda yotishini tekshiring.

Variant	M_1	M_2	M_3
1.	(0;5)	(1;3)	(2;1)
2.	(1;5)	(-2;-1)	(3;9)
3.	(-1;-9)	(2;6)	(3;11)
4.	(-1;2)	(2;11)	(3;14)
5.	(0;5)	(-1;1)	(2;13)
6.	(0;-2)	(2;0)	(3;1)
7.	(2;5)	(1;3)	(-2;-3)
8.	(0;5)	(1;8)	(-1;2)
9.	(0;7)	(-1;9)	(2;3)
10.	(2;5)	(7;2)	(-1;3)
11.	(1;2)	(-1;14)	(-2;20)
12.	(3;5)	(-2;5)	(4;4)
13.	(0;1)	(1;10)	(-1;8)
14.	(6;3)	(2;4)	(6;5)
15.	(1;8)	(0;5)	(2;11)

Variant	M_1	M_2	M_3
16.	(9;1)	(1;2)	(2;1)
17.	(-1;5)	(0;3)	(2;-1)
18.	(-3;2)	(0;2)	(1;5)
19.	(2;8)	(-2;2)	(4;11)
20.	(1;0)	(0;8)	(-1;3)
21.	(-1;-3)	(-2;-1)	(0;-5)
22.	(0,5;4)	(-2;-4)	(4;0,5)
23.	(0;1)	(-1;5)	(-3;10)
24.	(1;-3)	(2;-8)	(0;2)
25.	(2;12)	(-1;-12)	(0;-4)
26.	(0;5)	(1;12)	(2;19)
27.	(1;3)	(-1;-9)	(3;15)
28.	(0;1)	(1;3)	(-1;-1)
29.	(0;3)	(1;8)	(-2;-7)
30.	(1;5)	(0;-2)	(-2;-6)

4-misol. Uchlari quyidagi nuqtalarda bo‘lgan ucburchak yuzini hisoblang.

Variant	M_1	M_2	M_3
1.	(2;5)	(1;2)	(3;1)
2.	(3;5)	(-2;-1)	(3;10)
3.	(-10;-5)	(0;3)	(4;1)
4.	(-1;2)	(2;5)	(3;10)
5.	(0;2)	(3;1)	(3;4)
6.	(0;2)	(2;0)	(4;2)
7.	(2;5)	(3;3)	(-2;-3)
8.	(1;5)	(2;8)	(-1;3)
9	(2;7)	(0;4)	(2;3)
10.	(3;5)	(7;2)	(-1;4)
11.	(3;2)	(-1;10)	(-2;12)
12.	(4;5)	(-2;3)	(4;4)
13.	(3;1)	(2;5)	(-1;4)
14.	(6;4)	(2;5)	(6;3)
15.	(1;0)	(0;5)	(2;11)

Variant	M_1	M_2	M_3
16.	(1;6)	(3;5)	(2;4)
17.	(0;7)	(4;8)	(3;9)
18.	(1;8)	(3;6)	(3;4)
19.	(3;2)	(2;11)	(3;20)
20.	(3;5)	(-1;1)	(2;3)
21.	(2;-2)	(3;1)	(4;2)
22.	(5;5)	(1;2)	(-2;-2)
23.	(5;5)	(9;8)	(3;0)
24.	(-7;7)	(5;2)	(2;-2)
25.	(2;5)	(-2;2)	(-8;-4)
26.	(3;2)	(-1;10)	(-2;11)
27.	(3;4)	(5;6)	(7;8)
28.	(0;1)	(1;2)	(-1;5)
29.	(5;3)	(3;4)	(7;5)
30.	(1;8)	(10;5)	(2;15)

Mavzu yuzasidan savollar.

1. *Tekislikdagi 2 ta to‘g‘ri chiziqning parallellik va perpendikulyarlik shartlari.*
2. *Tekislikdagi to‘g‘ri chiziqning burchak koefisentini aniqlash formulalari.*
3. *Tekislikda to‘g‘ri chiziqlar dastasining tenglamasini yozing.*
4. *Ikki nuqta orqali o‘tuvchi to‘g‘ri chiziq tenglamasidan foydalanib talab va taklif funksiyasinini toping.*
5. *Tekislik tenglamasi $Ax + By + Cx + D = 0$ bo‘lsa, u koordinatalar sistemasiga nisbatan quyidagu haollarda qanday joylashadi?*
   a) $D = 0$; b) $A = 0$; v) $A = 0, B = 0$; g) $A = 0, B = 0, D = 0$;

d) $A = 0, D = 0$.

6. *Tekislik tenglamasini yozing.*
7. *Nuqtadan tekislikgacha bo‘lgan masofa qanday topiladi?*
8. *Ikkita parallell tekislik orasidagi masofa.*
9. *Tekisliklarning parallellik va perpendikulyarlik shartlari.*

14-amaliy mashg‘ulot. Funksiya haqida tushuncha. Funksiyaning berilish usullari. Elementar funksiyalar. Parametrik, oshkormas va transendent ko‘rinishidagi funksiyalar.

1. Funksiya va argument tushunchasi.

Amaliyotda vaqt, harorat, bosim, kuch, tezlik, yuz, hajm va hokazo miqdorlar (kattaliklar) bilan ish ko‘rishga, ular orasidagi bog‘lanishlarning xususiyatlarini o‘rganishga to‘g‘ri keladi. Bunga ko‘plab misollarni fizika, geometriya, biologiya va boshqa fanlar beradi. Jism o‘tgan S masofaning t vaqtga, aylana C uzunligining R radiusga bog‘liq ravishda o‘zgarishi bunga oddiy misol.

Ta’rif: Agar x o‘zgaruvchi miqdor X sonli to‘plamdan qabul qila oladigan har bir qiymatga biror f qoida bo‘yicha y o‘zgaruvchi miqdorning Y sonli to‘plamdagi aniq bir qiymati mos kelsa, y o‘zgaruvchi x o‘zgaruvchining sonli funksiyasi deb ataladi.

Y o‘zgaruvchining x o‘zgaruvchiga bog‘liq ekanligini ta'kidlash maqsadida uni erksiz o‘zgaruvchi yoki funksiya, x o‘zgaruvchini esa erkli o‘zgaruvchi yoki argument deb ataymiz. y o‘zgaruvchi x o‘zgaruvchining funksiyasi ekanligi $y = f(x)$ ko‘rinishda belgilanadi.

Ta’rif: Argument x ning X to‘plamdan qabul qila oladigan barcha qiymatlar to‘plami f funksiyaning aniqlanish sohasi deyiladi va D(f) orqali belgilanadi. $\{f(x) \mid x \in D(f)\}$ to‘plam f funksiyaning qiymatlar sohasi (to‘plami) deb ataladi va E(f) orqali belgilanadi.

Ta'rif: Ixtiyoriy $x \in D(f)$ qiymatda funksiya faqat y = b (o'zgarmas miqdor — constanta), $b \in R$ qiymatga ega bo'lsa, unga X to'plamda berilgan doimiy funksiya deyiladi.

Masalan, koordinatalar sistemasida Ox o'qqa parallel to'g'ri chiziqni ifodalovchi y = 3 funksiya $D(f) = \{-\infty < x < +\infty\}$ da doimiydir.

1-misol. Agar $y=x^2$ funksiya R to'plamda berilgan bo'lsa, u holda D(f)=R va $E(f)=R_+ U\{0\}$ bo'ladi.

2-misol. $y=x^2$ funksiya D(f)=[-3;4] da berilgan bo'lsin. Bu funksiyaning qiymatlar sohasi E(f)=[0;16] dan iborat.

2. Funksiyani bo'laklab ajratib berish.

Ta'rif: Aniqlanish sohasining turli qismlarida turli hil qoida bilan berilgan funksiyaning bo'laklarga ajratib berilgan funksiya (yoki bo'lakli berilgan funksiya) deb ataymiz.

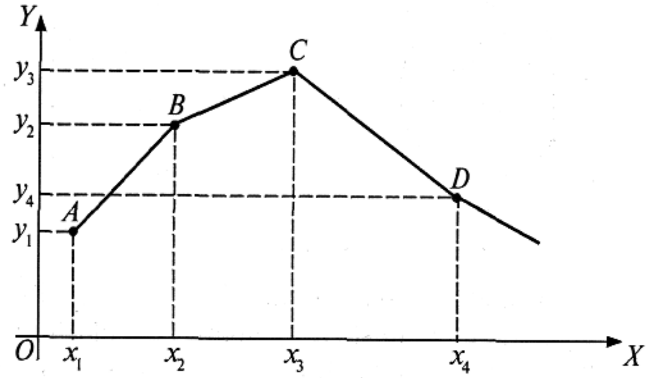
3-misol. Jism harakatni boshlab, dastlabki t_1 vaqt davomida tekis tezlanuvchan (a_1 tezlanish bilan), so'ng t_2 vaqt davomida tekis sekinlanuvchan ($-a_2$ tezlanish bilan) harakat qilgan. Uning v harakat tezligini t ning funksiyasi sifatida ifodalaymiz.

Yechish. 1) Jismning harakat boshidagi tezligi $u_0=0$, jism t_1 vaqt davomida tekis tezlanuvchan harakat qilgan:

$v=v_0+a_1t=a_1t,\ 0\le t\le t_1$

2) t_1 vaqt momentidagi tezligi v_l= o, $t_2$vaqt davomida tekis sekinlanuvchan harakat qilgan: $v = v_l-a_2t=a_lt_l- a_2t,\ t_l\le t\le t_l + t_2$ Shunday qilib,

$$v = \begin{cases} a_1t, & 0 \le t \le t_1, \\ a_1t_1 - a_2t, & t_1 \le t \le t_1 + t_2. \end{cases}$$



14.1- rasm.

4-miso1. Koordinatalar tekisligida f(x) funksiya ABCD siniq chiziq ko‘rinishida tasvirlangan. (14.1-rasm). Uning tugunlari A(x_1;y_1,), B(x_2; y_2), C(x_3; y_3), D(x_4; y_4) nuqtalarda yotadi. Funksiyaning ifodasini yozing.

Yechish. Ikki nuqta ustidan o‘tuvchi to‘g‘ri chiziq tenglamasidan foydalanamiz. AB bo‘g‘in uchun:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \qquad \text{yoki} \qquad y=y_1+\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$

Qolgan bo‘g‘inlarm'ng tenglamalari ham shu kabi aniqlanadi. Natijada funksiya ifodasi quyidagi ko‘rinishga ega bo‘ladi:

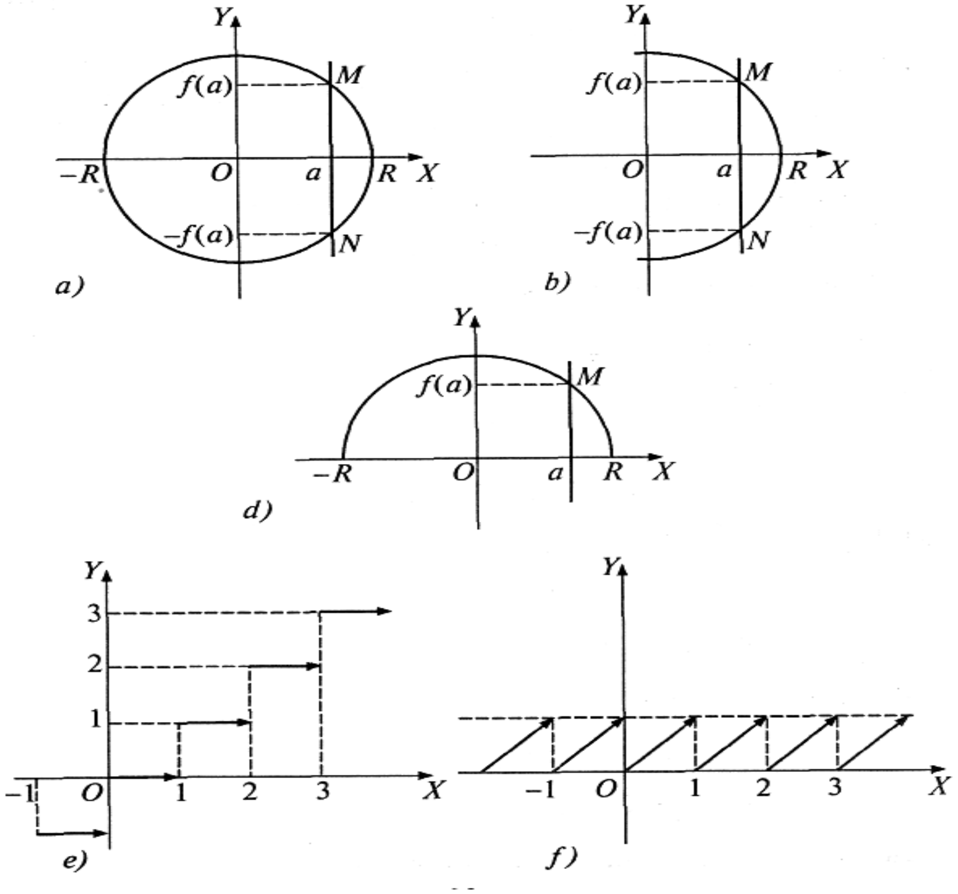
$$f(x)=\begin{cases} y_1+\frac{y_2-y_1}{x_2-x_1}(x-x_1),\ x_1\le x\le x_2,\\ y_2+\frac{y_3-y_2}{x_3-x_2}(x-x_2),\ x_2\le x\le x_3,\\ y_3+\frac{y_4-y_3}{x_4-x_3}(x-x_3),\ x_3\le x\le x_4.\end{cases}$$

3.Funksiya grafigini nuqtalar bo‘yicha yasash

Biror X sonli oraliqda berilgan y=f(x) sonli funksiya grafigi F ni «nuqtalar usuli» bilan yasash uchun X oraliqdan argumentning bir necha $x_{1,}x_2$, …,x_n qiymati tanlanadi, funksiyaning ularga mos f(x_1), ..., f(x_n) qiymatlari hisoblanadi, koordinatalar tekisligida M(x_l ;f(x_2)), ..., M(x_n; f(x_n)) nuqtalar belgilanadi va bu nuqtalar ustidan silliq chiziq o‘tkaziladi. Bu chiziq f(x) funksiya grafigini taqriban ifodalaydi.

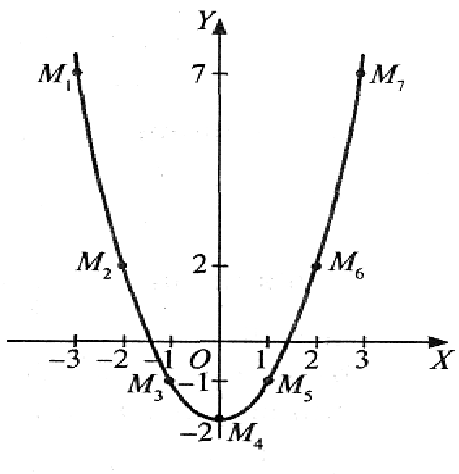
Agar ordinata o‘qiga parallel bo‘lgan har qanday to‘g‘ri chiziq F chiziqni ko‘pi bilan bitta nuqtada kessa, u holda F chiziq biror f(x) funksiyaning grafigi bo‘ladi. Shunga ko‘ra $x^2+y^2=R^2$ aylana hech qanday funksiyaning grafigi emas, chunki Oy o‘qiga parallel bo‘lgan x= a to‘g‘ri chiziq (2-a rasm) bu aylanani bittadan ortiq (aynan ikkita M va N) nuqtalarda kesadi, demak, x=a qiymatga y ning ikki qiymati to‘g‘ri keladi, ya'ni $y=\pm\sqrt{R^2-x^2}$, R≤ x≤ R.

Shu kabi $y=\pm\sqrt{R^2-x^2}$, 0≤ x≤R va $y=\pm\sqrt{R^2-x^2}$, - R≤ x≤0 yarim aylanalar ham funksiya grafigi Lekin $y=\pm\sqrt{R^2-x^2}$, -R≤ x≤ R yarim aylana shu ifodali funksiyaning grafigi (14.2- b, d rasm) (Γ — yunoncha gamma, bosh harf).



(14.2-rasm).

Funksiya grafigi uzilishga ega bo‘lishi mumkin. y = [x] va y = {x} funksiyalar grafiklari uzilishlidir (14.2- e,f rasm). Bu yerda Y=[x] — x ning butun qismi va y = {x} — x ning kasr qismi. Ulardan birinchisi pog‘onasimon joylashgan birlikkesmalardan, ikkinchisi esa y=x+n ($n<x<n+1$ $n\in Z$) to‘g‘ri chiziqlardan iborat.



(14.3-rasm).

5-misol. $y = x^2-2$ funksiya grafigi eskizini chizamiz, bunda $-3 \leq x \leq 3$ (eskiz homaki, asbob yordamisiz chizilgan chizma).

Yechish. Argumentning x = -3;-2;-l; 0; 1; 2; 3 qiymatlarini tanlaylik. f(x) qiymatlarini hisoblaymiz:

f(-3)=f(3) = 9-2 = 7, f(-2) =f(2) = 4 - 2 = 2,

f(-l)=f(l)=-l, f(0) = -2.

Koordinatalar tekisligida $M_1(-3; 7)$, $M_2(-2; 2)$,
$M_3(-l; -1)$, $M_4(0; -2)$, $M_5(l; -1)$,
$M_6(2; 2)$, $M_7(3; 7)$ nuqtalarni belgilab, ularni qo‘l bilan tutashtirib silliq chiziq chizamiz .(14.3-rasm).

Mashqlar:

1.Quyidagi funksiyalar grafiklarini «nuqtalar bo‘yicha» yasang:

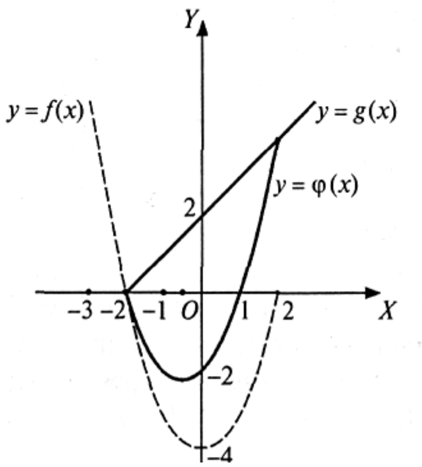
a) $y = x^2 + 2$; b) $y = x^3-1$; d) $y = x^4-!$;
h) $y=|x-2|$; i) $j = |x^2-l|$; j) $y = |x|-|x- 1|$;
k) $y = x-|xl$; l) $y=lx-|x + 2||$; m) $y = 3x^2-4x$.

4. Funksiyalar ustida amallar.

D(f) to‘plamda berilgan f(x) va D(g) to‘plamda berilgan g(x) funksiyalarning yig‘indisi deb $D(\varphi) = D(f) \cap D(g)$ to‘plamda berilgan yangi $\varphi(x) =f(x) + g(x)$funksiyaga aytiladi.

6-misol. $f=x^2-4$, $-3\le x\le 2$ va $g=x+2$, $-2\le x\le 3$ funksiyalar yig‘indisi

$\varphi(x)=(x^2 - 4) + (x + 2)$, $-2 \le x \le 2$ funksiyadan iborat. Uning grafigini chizishda f va g funksiyalar mos ordinatalarini qo‘shishdan foydalanish mumkin (4- rasm).



(14.4-rasm).

f(x) va g(x) funksiyalarning ko‘paytmasi $D(\varphi)=D(f)\cap D(g)$ to‘plamda berilgan $\varphi(x) =f(x)-g(x)$ funksiyadan iborat.

$\frac{1}{g(x)}$ funksiya D(g) to‘plamning g(x)≠0 bo‘lgan barcha sonlarida aniqlangan. $f(x)\cdot\frac{1}{f(x)}$ funksiya f va g funksiyalar bo‘linmasi deb ataladi.Uni $\frac{f}{g}$ orqali belgilaymiz.

7-misol. f(x) = (x- 1)^3 - 3 berilgan. $\frac{4}{f^2-3}$ funksiya ifodasi $\frac{4}{((x-1)^3-3)^2-3}$ ko‘rinishida yoziladi: $\frac{4}{f^2-3}=\frac{4}{((x-1)^3-3)^2-3}$.

f va g sonli funksiyalar berilgan va E(f) ⊂ D(g} bo‘lsin. f va g funksiyalar kompozitsiyasi deb D(f) da berilgan va har qaysi x∈D(f) songa g(f(x)) sonni mos qo‘yuvchi yangi F(x) funksiyaga aytiladi (lot. compositio — tuzish). F funksiya gof orqali ham belgilanadi: (g°f)(x)=g(f(x)). Kompozitsiya ifodasini tuzish uchun g(x) dagi X o‘rniga f funksiya ifodasi qo‘yiladi.

8-misol. f(x) = x^3 - 2 va g(x) = $\frac{1}{x}$ funksiyalarning g°f va f°g kompozitsiyalarini tuzing.

Yechish. l) g°f =g(f(x))=$\frac{1}{x^3-2}$; 2) f°g=f(g(x))=($\frac{1}{x}$)^3-2.

Mashqlar:

1. f(x) = $\frac{x-1}{x+1}$ bo‘lsa, f$\left(\frac{1}{x^2}\right)$ ni toping.
2. f(x) = $\sqrt{x^3-1}$ bo‘lsa, f($\sqrt[3]{x^2+1}$) ni toping.
3. f(x)=$\frac{x^2}{\sqrt{1+x^2}}$ bo‘lsa, f(tgx) ni toping.
4. f($\frac{3x-1}{x+2}$)=$\frac{x+1}{x-1}$ bo‘lsa, f(x) ni toping.

Nazorat savollari:

1. *Funksiya deb nimaga aytiladi?*
2. *Erkli va erksiz o‘zgaruvchi deganda nimani tushunasiz?*
3. *Funksiyaning aniqlanish va qiymatlar sohasi deb nimaga aytiladi?*
4. *Bo‘laklarga ajratib berilgan funksiya deb qanday funksiyaga aytiladi?*
5. *Koordinatalar sistemasida nuqtaning o‘rni qanday belgilanadi?*

15-amaliy mashg‘ulot. Sonli ketma-ketlik. Ketma-ketlik limiti.

Agar har bir natural n songa biror qoida yoki qonun asosida bitta a_n son mos qo‘yilgan bo‘lsa, u holda $\{a_n\}$ sonli ketma – ketlik deyiladi:

$$a_1, a_2, \ldots, a_n, \ldots. \qquad (15.1)$$

Boshqacha qilib aytganda sonli ketma–ketlik n - natural argumentning funksiyasidir: $a_n = f(n)$.

Masalan: $\left\{\frac{1}{n}\right\}$; $\left\{\frac{1}{2^n}\right\}$; $\{-1+(-1)^n\}$, yoki

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots;$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots;$

$0, 2, 0, 2, \ldots.$

1-misol. $\left\{\frac{n}{n+1}\right\}$ - yaqinlashuvchiligini tekshiring.

Yechish. $\lim\limits_{n\to\infty}\frac{n}{n+1} = \lim\limits_{n\to\infty}\frac{n}{n(1+\frac{1}{n})} = \lim\limits_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$ demak yaqinlashuvchi.

2-misol. $\{a_n\} = (-1)^n$, yoki -1, 1, -1, 1, … limitga ega emasligini ko‘rsating.

Yechish. Haqiqatan, limit sifatida qanday sonni tasavvur qilmaylik 1 yoki – 1, ε<0,5 da, $|x_n - a| < \varepsilon$ tengsizlik qanoatlantirilmaydi. Bu ketma-ketlikning barcha toq nomerlar – 1, juftlari 1 ga teng.

3-misol. Ketma-ketlikning limitini toping.

$$\lim_{n\to\infty}\frac{3n^2+2n+4}{4n^2+n-3}$$

Yechish. Kasrning surat va maxrajini n^2 ga bo‘lib, bo‘linma va yig‘indining limiti qoidalaridan foydalanamiz.

$$\frac{\lim\limits_{n\to\infty}(3+2/n+4/n^2)}{\lim\limits_{n\to\infty}(4+1/n-3/n^2)} = \frac{\lim\limits_{n\to\infty}3+\lim\limits_{n\to\infty}(2/n)+\lim\limits_{n\to\infty}(4/n^2)}{\lim\limits_{n\to\infty}4+\lim\limits_{n\to\infty}(1/n)-\lim\limits_{n\to\infty}(3/n^2)} = \frac{3+0+0}{4+0-0} = \frac{3}{4}.$$

4-misol. Ketma-ketlikning limitini toping.

$$x_n = \frac{\sqrt{n}}{n+1}.$$

5-misol. Ketma-ketmalikning n→∞ dagi limitini hisoblang.

$$x_n = \sqrt{n+1} - \sqrt{n}$$

6-misol. Quyidagi limitlarni hisoblang

$a)\ \lim\limits_{n\to\infty}\frac{(n+2)^3+(n-2)^3}{n^4+2n^2-1}$ $b)\ \lim\limits_{n\to\infty}\frac{3n^2-4n+8}{4n^2+5n-9}$

$c)\ \lim\limits_{n\to\infty}\frac{n!+(n+2)!}{(n-1)!+(n+2)!}$ $d)\ \lim\limits_{n\to\infty}\frac{3+6+9+...+3n}{n^2+4}$

$e)\ \lim\limits_{n\to\infty}\frac{2^n+7^n}{2^n-7^{n-1}}$ $f)\ \lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{n+2}-\sqrt{n-3}\right)$

7-misol. Quyidagi limitlarni hisoblang

$a)\ \lim\limits_{n\to\infty}\frac{(n+1)^3-(n+1)^2}{(n-1)^2-(n+1)^3}$ $b)\ \lim\limits_{n\to\infty}\frac{(3-n)^3}{(n+1)^2-(n+1)^3}$

$c)\ \lim\limits_{n\to\infty}\frac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!}$ $d)\ \lim\limits_{n\to\infty}\frac{(n+4)!-(n+2)!}{(n+3)!}$

$e)\ \lim\limits_{n\to\infty}\frac{3^n-2^n}{3^{n-1}+2^n}$ $f)\ \lim\limits_{n\to\infty}\sqrt{n^3+8}\left(\sqrt{n^3+2}-\sqrt{n^3-3}\right)$

Umuman $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty-\infty$, $0.\infty$, 0^0, ∞^0, 1^∞, aniqmasliklar mavjud va ularni ochishni misollarda ko‘rsatamiz.

8-misol: $\lim_{x\to2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)$ limitni toping.

Yechish: agar x o‘rniga 2 ni qo‘ysak, $\infty-\infty$, ko‘rinishidagi aniqmaslik hosil bo‘ladi. Bu aniqmaslikni ochish uchun qavs ichidagi ifodani umumiy maxrajga keltiramiz. Natijada $\lim\limits_{x\to2}\left(-\frac{2-x}{x^2-4}\right)$, ya’ni $\frac{0}{0}$ ko‘rinishdagi aniqmaslik hosil bo‘ladi. Agar $x-2\neq0$ deb kasr qisqartirilsa, berilgan limit quyidagiga teng bo‘ladi:

$$\lim_{x\to2}\left(-\frac{1}{x+2}\right)=-\frac{1}{4}$$

9-misol: $\lim\limits_{x\to+\infty}\frac{3x^2-x^2+5}{3x^3+x^2-x}$ limitni toping.

10-misol.

$$\lim_{n\to\infty}\frac{(3n+2)^{100}}{(3n-1)^{98}(n+2)^2}$$

Yechish. $\lim\limits_{n\to\infty}\frac{3^{100}n^{100}\left(1+\frac{2}{3n}\right)^{100}}{3^{98}n^{100}\left(1-\frac{1}{3n}\right)^{98}\left(1+\frac{2}{n}\right)^2}=\lim\limits_{n\to\infty}9\frac{\left(1+\frac{2}{3n}\right)^{100}}{\left(1-\frac{1}{3n}\right)^{98}\left(1+\frac{2}{n}\right)^2}=9.$

11-misol. $\lim\limits_{n\to\infty}\dfrac{7x^4+2x^3-1}{3x^2-2x^4+x}$

12-misol. $\lim\limits_{x\to 64}\dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$

13- misol Limitni hisoblang

a. $\lim\limits_{x\to 1}\dfrac{x^2-3x+2}{x^2+x-2}$ b. $\lim\limits_{x\to -2}\dfrac{x^2-x-6}{x^2+6x+8}$

c. $\lim\limits_{x\to 2}\dfrac{x^2-2x}{x^2-4x+4}$ d. $\lim\limits_{x\to 4}\dfrac{x^2-5x+4}{x^2-16}$

e. $\lim\limits_{x\to 1}\dfrac{x^2-1}{\sqrt{x}-1}$ f. $\lim\limits_{x\to -3}\dfrac{x^3+27}{x^2-9}$

j. $\lim\limits_{x\to 7}\dfrac{x^2-49}{\sqrt{x}-\sqrt{7}}$ h. $\lim\limits_{x\to 1}\dfrac{\sqrt{x-2}\left(\sqrt{x}-\sqrt{2}\right)}{x^2-4}$

i. $\lim\limits_{x\to 1}\dfrac{x^3-2x^2-5x+6}{x^3-7x+6}$ g. $\lim\limits_{x\to 2}\dfrac{\sqrt{x-1}-1}{x-2}$

k. $\lim\limits_{x\to -1}\dfrac{x^2-1}{\sqrt[3]{x}+1}$ l. $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-2}{\sqrt{x}-\sqrt{2}}$

Ajoyib limitlar

$\lim\limits_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha}=1$ – *birinchi ajoyib limit;*

$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=\lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e$ – *ikkinchi ajoyib limit.*

Bundan tashqari quyidagi umumiy holdagi formulalarni keltirib o‘tamiz:

1. $\lim\limits_{x\to a}\left(1+\dfrac{1}{f(x)}\right)^{f(x)}=e,$ bunda $x\to a$ bo‘lganda $f(x)\to\infty$.

2. $\lim\limits_{x\to a}(1+\varphi(x))^{\frac{1}{\varphi(x)}}=e,$ bunda $x\to a$ bo‘lganda $\varphi(x)\to 0$.

3. $\lim\limits_{x\to\infty}\left(1+\dfrac{k}{x}\right)^{mx}=e^{km};\quad \lim\limits_{x\to 0}\sqrt[x/m]{1+kx}=\lim\limits_{x\to 0}(1+kx)^{\frac{m}{x}}=e^{km}$

14-misol. Berilgan limitlarni hisoblang.

$$a)\ \lim_{x\to0}\frac{\sin 7x}{x};\qquad b)\ \lim_{x\to0}\frac{\sin Ax}{\sin Bx};\qquad c)\ \lim_{\alpha\to0}\frac{\sin^n\alpha}{\sin\alpha^m};$$

$$d)\ \lim_{x\to\infty}\frac{2x-\sin x}{x+3\cos x};\qquad e)\ \lim_{x\to0}\frac{tgx}{x};\qquad f)\ \lim_{x\to0}\frac{1-\cos x}{x^2}$$

Yechish. $a)\ \lim_{x\to0}\frac{\sin 7x}{x}=7\lim_{x\to0}\frac{\sin 7x}{7x}=7\cdot1=7.$

$$b)\ \lim_{x\to0}\frac{\sin Ax}{\sin Bx}=\lim_{x\to0}\frac{Ax\sin Ax}{Ax}\cdot\frac{Bx}{\sin Bx}\cdot\frac{1}{Bx}=\frac{A}{B}\lim_{x\to0}\frac{Ax\sin Ax}{Ax}\cdot\lim_{x\to0}\frac{Bx}{\sin Bx}=\frac{A}{B}\cdot1\cdot1=\frac{A}{B}.$$

$$c)\ \lim_{\alpha\to0}\frac{\sin^n\alpha}{\sin\alpha^m}=\lim_{\alpha\to0}\left(\frac{\sin\alpha}{\alpha}\right)^n\cdot\alpha^n\cdot\left(\frac{\alpha^m}{\sin\alpha^m}\right)\cdot\frac{1}{\alpha^m}=\lim_{\alpha\to0}\frac{\alpha^n}{\alpha^m}\cdot1=\lim_{\alpha\to0}\alpha^{n-m}=\begin{cases}0,\ agar\ \ n>m,\\ 1,\ agar\ n=m,;\\ \infty,\ agar\ n<m.\end{cases}$$

$$d)\ \lim_{x\to\infty}\frac{2x-\sin x}{x+3\cos x}=\lim_{x\to\infty}\frac{2-\frac{1}{x}\sin x}{1+\frac{3}{x}\cos x}=\frac{2-0}{1+0}=2.$$

$$e)\ \lim_{x\to0}\frac{tgx}{x}=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=1\cdot1=1.$$

$$f)\ \lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{(x/2)}\right)^2=\frac{1}{2}\cdot1^2=\frac{1}{2}.$$

15-misol. limitlarni hisoblang

a. $\lim_{x\to\infty}\left(\frac{2x+1}{2x-3}\right)^{4x-1}$.　　**b.** $\lim_{x\to\infty}\left(\frac{2x^2-3}{2x^2+1}\right)^{-3x^2}$.

c. $\lim_{x\to0}(\cos x)^{\frac{1}{x^2}}$　　**d.** $\lim_{x\to\infty}\left(\frac{x+5}{x+2}\right)^{x}$　　**e.** $\lim_{x\to\infty}\left(\frac{2x+1}{2x+5}\right)^{7x}$

f. $\lim_{x\to\infty}\left(\frac{4x^2+2}{4x^2-1}\right)^{5x^2}$　　**g.** $\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2-4}\right)^{3x}$

h. $\lim_{x\to\infty}\left(\frac{x^2+x-1}{x^2-2x+5}\right)^{-2x}$　　**i.** $\lim_{x\to\infty}\left(\frac{5x^3-2}{5x^3+1}\right)^{-6x^3}$

j. $\lim_{x\to\infty}\left(\frac{2x^3-3x^2+x+1}{2x^3-3x^2-2x+3}\right)^{5x^2}$

k. $\lim\limits_{x\to\infty}\left(\dfrac{7x^{10}-3}{7x^{10}+2}\right)^{-2x^{10}}$

16-amaliy mashg‘ulot. Funksiyaning limiti. Funksiyaning cheksizlikdagi limiti. Bir tomonlama limitlar. Funksiyaning uzluksizligi

***f*(*x*) funksiya x_0 nuqtada uzluksiz deyiladi, agar u quyidagi uchta shartni qanoatlantirsa:**

a) x_0 nuqtada aniqlangan (ya’ni $f(x_0)$mavjud);

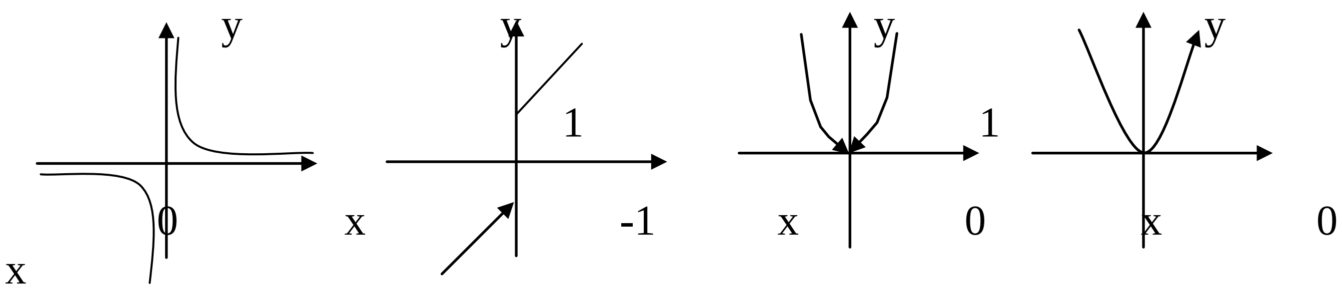
b) $x\to x_0+0$, $x\to x_0-0$ chekli limitlarga ega;

c) bu limitlar funksiyaning x_0 nuqtadagi qiymatiga teng, ya’ni:

$$\lim_{x\to x_0+0} f(x)=\lim_{x\to x_0-0} f(x)=\lim_{x\to x_0} f(x)=f(x_0)$$

1-misol. Quyidagi funksiyalarni uzluksizlikka tekshiring:

a) $y=\dfrac{1}{x}$; b) $y=\begin{cases} x+1, & agar\ \ x\ge 0, \\ x-1, & agar\ \ x<0 \end{cases}$; c) $\begin{cases} x^2, & agar\ \ x\ne 0, \\ 1, & agar\ \ x=0 \end{cases}$; d) $y=x^2$.



a) b) c) d)

16.1-rasm

Yechish. a) Berilgan $y=\dfrac{1}{x}$ funksiya (16.1 *a* – rasmga qarang) $x = 0$ nuqtada uzilishga ega, chunki uzluksizlikning birinchi sharti buzilgan – $f(0)$ mavjud emas.

b) $y=\begin{cases} x+1, & agar\ \ x\ge 0, \\ x-1, & agar\ \ x<0 \end{cases}$ funksiya (16.1 *b* – rasmga qarang) $x = 0$ nuqtada uzilishga ega, chunki uzluksizlikning birinchi sharti bajarilgan, $f(0)$ mavjud ($f(0) = 1$), lekin uchinchi shart buziladi. (bu yerda funksiyaning bir tomonlama limitlari mavjud chapdan $\lim\limits_{x\to 0-0} f(x) = -1$, o‘ngdan $\lim\limits_{x\to 0+0} f(x) = 1$, lekin ular teng emas).

c) $y = \begin{cases} x^2, & agar \ \ x \neq 0, \\ 1, & agar \ \ x = 0 \end{cases}$ funksiya (c – rasmga qarang) $x = 0$ nuqtada uzilishga ega uzluksizlikning ikkita sharti bajariladi, ya’ni $f(0)$ aniqlangan ($f(0) = 1$) va $\lim\limits_{x\to 0} f(x) = 0$ chekli limit mavjud, lekin uchinchi asosiy shart buzilgan: $\lim\limits_{x\to 0} f(x) \neq f(0)$.

$y = x^2$ funksiya (16.1 d – rasmga qarang) $x = 0$ nuqtada uzluksiz, chunki uzluksizlikning uchchala sharti bajariladi.

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

Funksiyaning uzilishi va uning turlari

f(x) funksiya uchunuzluksizlik shartlaridan aqalli bittasi bajarilmasa, bu funksiya x nuqtada uzilishga ega deyiladi.

Agar $f(x)$ funksiya berilgan x_0 nuqtada uzluksiz bo‘lmasa, bu uzilishga ega deyiladi.

Uzilish turlari quyidagicha:

I – tur uzilish – funksiyaning chap va o‘ng chekli limitlari mavjud, lekin ular teng emas. (16.1. b rasm) misol

II – tur uzilish – bir tomonlama chap va o‘ng limitlardan biri cheksiz yoki mavjud emas. (16.1 a rasm) misol)

I – tur uzilishga bartaraf qilinadigan uzilish deyiladi, bunda $x \to x_0$ da funksiyaning limiti mavjud, lekin funksiyaning x_0 nuqtadagi qiymatiga teng emas. (1. c) misol)

2-misol. $y=f(x)$ funksiyani $x=1$ nuqtada uzluksizlikka tekshiring. Uzluksizlikka ega bo‘lgan holda $x=1$ nuqtadagi xarakterini aniqlang.

$a)\ y(x) = \dfrac{(x-1)^3}{x-1}$; $b)\ y(x) = \dfrac{x}{x-1}$; $c)\ y(x) = x - 1$;

$d)\ y(x) = \begin{cases} x-1, & agar \ \ x \geq 1 \\ x+1, & agar \ \ x < 1 \end{cases}$

Yechish. $a)\ y(x) = \dfrac{x^3 - 3x^2 + 3x - 1}{x-1}$ funksiya $x=1$da aniqlanmagan. Demak bu nuqtada uzilishga ega.

Funksiya limitini hisoblaymiz:

$$\lim_{x\to 1}\frac{x^3-3x^2+3x-1}{x-1}=\lim_{x\to 1}(x-1)^2=\lim_{x\to 1}(1-1)^2=0$$, ya’ni chekli limit mavjud, demak x=1 bartara*f* qilinadigan 1-tur uzilish .

Funksiyani x=1 nuqtada aniqlanishini to‘ldirib, ya’ni f(1)=0 deb faraz qilib,

$$F(x)=\begin{cases}\dfrac{x^3-3x^2+3x-1}{x-1}, & x\neq 1 \ da\\ 0, & x=1 \ da\end{cases}$$

funksiyani hosil qilamiz. Bu funksiya x=1 nuqtada uzluksiz.

b) $y(x)=\frac{x}{x-1}$ funksiya x=1 nuqtada aniqlanmagan va x=1 nuqtada uzilishga ega chunki

$$\lim_{x\to 1+0}\frac{x}{x-1}=+\infty, \quad \text{va} \quad \lim_{x\to 1-0}\frac{x}{x-1}=-\infty \qquad \text{(16.3-rasm)}$$

Bir tomonlama limitlar (bitta limit mavjud bo‘lsa ham yetarli edi) cheksiz bo‘lgani uchun x=1 2-tartibli uzilish nuqtasi.

c) $y(x)=x$-1 funksiya x=1da aniqlangan, $\lim_{x\to 1+0}(x-1)=0,\quad \lim_{x\to 1-0}(x-1)=0,\quad y(1)=0,$ ya’ni

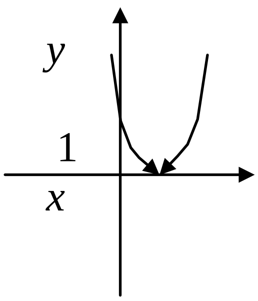
$$\lim_{x\to 1+0}y(x)=\lim_{x\to 1+0}y(x)=y(1)=0$$

demak funksiya x=1 nuqtada uzluksiz. (16.4 – rasm)

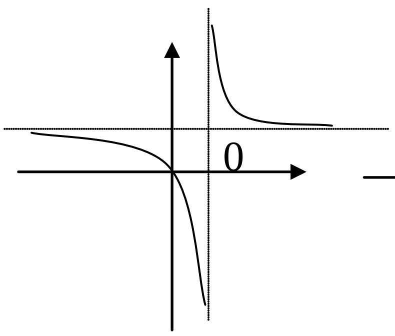
d) $y(x)=\begin{cases}x-1, \ agar \ x\geq 1\\ x+1, \ agar \ x<1\end{cases}$ funksiya x=1 da aniqlangan y(1)=0,

$$\lim_{x\to 1-0}y(x)=\lim_{x\to 1-0}(x+1)=2,\quad \lim_{x\to 1+0}y(x)=\lim_{x\to 1+0}(x-1)=0,\quad \lim_{x\to 1-0}y(x)\neq\lim_{x\to 1+0}y(x)$$

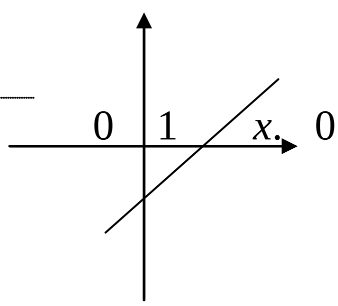
ga ega bo‘lamiz, shunday qilib, x=1 nuqtada funksiya bartara*f* qilinadigan uzilishga ega (16.5 – rasm).



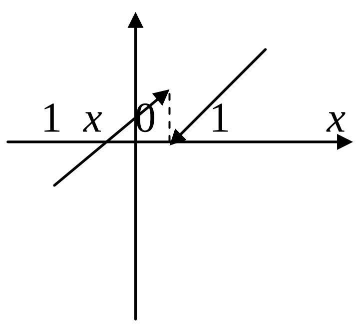
16. 2-rasm



16. 3-rasm



16. 4-rasm



16. 5-rasm

3-misol. Funksiyani uzluksizlikka tekshiring, uzilish nuqtalarini aniqlang.

4-misol. $y = \frac{x}{x-3}$ funksiyaning uzilish nuqtasini toping va uzilish turini aniqlang.

5-misol. Agar $f(x) = signx = \begin{cases} -1, & agar \quad x < 0; \\ 0, & agar \quad x = 0; \\ 1, & agar \quad x > 0. \end{cases}$ bo‘lsa, $f(x)$ ning uzilish nuqtasini va uzilish turini aniqlang.

6-misol. $f(x) = e^{\frac{1}{x}}$ funksiyani uzilish nuqtasi va turlari bo‘yicha tekshiring.

Uzluksiz funksiyalarning asosiy xossalari

Agar funksiya qaralayotgan oraliqning hamma nuqtasida uzluksiz bo‘lsa, u holda funksiya shu oraliqda uzluksiz deyiladi. Elementar funksiyalarning hammasi o‘zlarining aniqlanish sohalarida uzluksizdir.

1. Agar $f(x)$ va $\varphi(x)$ funksiyalar x_0 nuqtada uzluksiz bo‘lsa, u holda ularning yig‘indisi, ko‘paytmasi, bo‘linmasi (maxraj noldan farqli bo‘lganda) shu nuqtada uzluksiz bo‘ladi.

2. Agar $y = f(u)$ funksiya $u_0 = \varphi(x_0)$ nuqtada, uzluksiz bo‘lsa $u = \varphi(x)$ funksiya esa x_0 nuqtada uzluksiz bo‘lsa, u holda $y = f[\varphi(x)]$ murakkab funksiya x_0 nuqtada uzluksiz bo‘ladi.

3. Agar funksiya biror oraliqning har bir nuqtasida uzluksiz bo‘lsa, u shu oraliqda uzluksiz deyiladi. Barcha elementar funksiyalar o‘zining aniqlanish sohasida uzluksizdir.

Bolsano Koshining teoremalari

Bo‘lsano Koshining 1-teoremasi

Agar $f(x)$ funksiya $[a; b]$ segmentda aniqlangan, uzluksiz bo‘lib, segmentning chetki nuqtalarida har xil ishorali qiymatlarga ega bo‘lsa, u holda shunday c $(a < c < b)$ nuqta topiladiki, u nuqtada funksiya nolga aylanadi:

$f(c) = 0.$

Bo‘lsano Koshining 2-teoremasi

Agar $f\ (x)$ funksiya $[a;\ b]$ segmentda aniqlangan va uzluksiz bo‘lib, uning chetki nuqtalarida $f(a)=A$, $f(b)=B$ qiymatlarga ega va $A\neq B$ bo‘lsa, A va B sonlari orasida har qanday C son olinganda ham a bilan b orasida shunday c nuqta topiladiki, bunda $f(c)=C$ bo‘ladi.

Veyershtrasning 1- teoremasi

Agar $f(x)$ funksiya $[a;\ b]$ segmentda aniqlangan va uzluksiz bo‘lsa, funksiya shu segmentda chegaralangan bo‘ladi.

Veyershtrasning 2 - teoremasi

Agar $f(x)$ funksiya $[a;\ b]$ segmentda aniqlangan va uzluksiz bo‘lsa, funksiya shu segmentda o‘zining aniq yuqori hamda quyi chegaralariga erishadi.

7-misol. $f(x)=\frac{|x|-x}{2x^2}$ funksiyani uzluksizligini tekshiring.

Yechish. $|x|=\begin{cases} x, & agar\ \ x\geq 0\ bo'lsa, \\ -x & agar\ \ x<0\ bo'lsa. \end{cases}$

Bundan ko‘rinadiki,

$$f(x)=\begin{cases} 0, & agar\ \ x>0\ bo'lsa, \\ -\frac{1}{x} & agar\ \ x<0\ bo'lsa. \end{cases}$$

$x=0$ nuqtada funksiya aniqlanmagan bolib, $\lim\limits_{x\to+0} f(x)=0\ va\ \lim\limits_{x\to-0} f(x)=+\infty$ munosabat o‘rinli. Demak, $x=0$ nuqta $f(x)$funksiya uchun ikkinchi tur uzilish nuqtasi.

8-misol. Quyidagi funksiyaning uzluksizligini tekshiring va grafigini chizing.

$$f(x)=\frac{1}{\sin x^2}$$

9-misol. Berilgan funksiya a ning qanday qiymatida uzluksiz bo‘ladi.

$$f(x)=\begin{cases} x\cdot ctgx,\ agar\ x\neq 0\ va\ |x|<\frac{\pi}{2} bo'lsa, \\ a\ ,agar\ \ x=0\ bo'lsa. \end{cases}$$

10-misol. Funksiyaning uzilish nuqtasini toping

a. $f(x)=4^{\frac{2}{5-x}}$ **b.** $f(x)=6^{\frac{1}{x+3}}$

c. $f(x)=3^{\frac{2}{1-x}}$ **d.** $f(x)=\begin{cases}3x+4, & agar \ \ x\le 1 \ bo'lsa,\\ x^2-2, & agar \ \ -1<x<2 \ bo'lsa,\\ x, & agar \ \ x\ge 2 \ \ bo'lsa.\end{cases}$

e. $f(x)=\begin{cases}x+1, & agar \ x\le 0 \ bo'lsa,\\ (x+1)^2, & agar \ 0<x\le 2 \ bo'lsa,\\ 4-x, & agar \ x>2 \ bo'lsa.\end{cases}$

11- **misol.** Berigan funksiyalarni uzilish nuqtasini va turini aniqlang

a. $f(x)=\begin{cases}x-2, & agar \ x<0 \ bo'lsa,\\ 2, & agar \ x=0 \ bo'lsa,\\ x^2-2, & agar \ x>0 \ bo'lsa.\end{cases}$ **b.** $f(x)=\begin{cases}x-2, & agar \ x<0 \ bo'lsa,\\ -2, & agar \ x=0 \ bo'lsa,\\ -x-2, & agar \ x>0 \ bo'lsa.\end{cases}$

c. $f(x)=\dfrac{x^2+2}{x-2}$ **d.** $f(x)=\dfrac{x-2}{x^2+2}$

e. $f(x)=\sin\dfrac{1}{x}$ **f.** $f(x)=\dfrac{1}{x^2+x}$

g. $f(x)=\begin{cases}x-2, & agar \ \ x<2 \ bo'lsa,\\ x+2, & agar \ \ x\ge 2 \ bo'lsa.\end{cases}$ **h.** $f(x)=\begin{cases}x-3, & agar \ \ x<0 \ bo'lsa,\\ x+1, & agar \ \ 0\le x\le 4 \ bo'lsa,\\ 3+\sqrt{x}, & agar \ x>4 \ \ bo'lsa.\end{cases}$

12-misol. Funksiyani uzluksizlikka tekshiring va grafigini chizing

a. $y=\dfrac{3}{x-4}.$ **b.** $y=|x|.$

c. $a) \ y=-\dfrac{5}{x};$ $b) \ y=tgx.$ **d.** $a) \ y=x-|x|;$ $b) \ y=3-\dfrac{|x|}{x}.$

e. $a) \ y=3^{\frac{1}{x-3}};$ $b) \ y=1-3^{\frac{1}{x}}.$ **f.** $a) \ y=2^{\frac{1}{x-3}};$ $b) \ y=5-4^{\frac{1}{x^2}}.$

13-misol. Funksiya uzilish nuqtalarini toping. Funksiyaning uzilish nuqtasi atrofidagi shaklini chizing

1. $f(x)=2^{\frac{1}{x-5}}$

2. $f(x)=7^{\frac{1}{5-x}}$

3. $f(x)=3^{\frac{1}{x-4}}$

4. $f(x)=8^{\frac{3}{4-x}}$

5. $f(x)=4^{\frac{1}{x-3}}$

6. $f(x)=6^{\frac{1}{2-x}}$

7. $f(x)=5^{\frac{3}{x-8}}$

8. $f(x)=5^{\frac{4}{3-x}}$

9. $f(x)=4^{\frac{2}{x-6}}$

10. $f(x)=4^{\frac{2}{5-x}}$

14-misol. Berilgan funksiyani uzilish nuqtalarini toping. Ularning grafigini chizing

1. $f(x)=\begin{cases} 2 & agar\ \ x\geq 2 \\ 2^x & agar\ \ 0\leq x<2 \\ x+1 & agar\ \ x<0 \end{cases}$

2. $f(x)=\begin{cases} x-1 & agar\ \ x\geq 1 \\ x^2 & agar\ 0\leq x<1 \\ x^3 & agar\ \ x<0 \end{cases}$

3. $f(x)=\begin{cases} \cos x & agar\ \ x\leq 0 \\ x+1 & agar\ 0<x\leq 2 \\ 2 & agar\ \ x>2 \end{cases}$

4. $f(x)=\begin{cases} 2x & agar\ \ x\geq 0 \\ -x^2 & agar -1\leq x<0 \\ -x & agar\ \ x<-1 \end{cases}$

5. $f(x)=\begin{cases} x-3 & agar\ \ x\geq 3 \\ x^3 & agar\ \ 0\leq x<3 \\ x & agar\ \ x<0 \end{cases}$

6. $f(x)=\begin{cases} x^2 & agar\ \ 0<x\leq 2 \\ x-1 & agar\ \ x\leq 0 \\ x+2 & agar\ \ x>2 \end{cases}$

7. $f(x)=\begin{cases} \sin x & agar\ \ x\geq 0 \\ x^2 & agar\ \ -1\leq x<0 \\ x & agar\ \ x<-1 \end{cases}$

8. $f(x)=\begin{cases} 1-x & agar\ \ x\geq 1 \\ \log_x x^2 & agar\ \ -1<x<1 \\ x+3 & agar\ \ x\leq -1 \end{cases}$

9. $f(x)=\begin{cases} x & agar\ \ x\geq 0 \\ (x+3)^2 & agar\ -2<x<0 \\ x+3 & agar\ \ x\leq -2 \end{cases}$

10. $f(x)=\begin{cases} x^2 & agar\ \ x\leq 1 \\ \lg x & agar\ \ 1<x<10 \\ 11-x & agar\ \ x>10 \end{cases}$

11. $f(x)=\begin{cases} x-1 & agar\ \ x\geq 1 \\ x^2 & agar\ 0\leq x\leq 1 \\ -x & agar\ \ x<0 \end{cases}$

12. $f(x)=\begin{cases} x^2+1 & agar\ \ 0<x<1 \\ 1 & agar\ \ x\geq 1 \\ x+1 & agar\ \ x\leq 0 \end{cases}$

15-misol. Funksiya uzilish nuqtasini toping. Funksiyaning uzilish nuqtasi atrofidagi shaklini chizing

1. $f(x) = 4^{\frac{2}{x-3}}$	2. $f(x) = 6^{\frac{1}{3-x}}$
3. $f(x) = 5^{\frac{2}{x+2}}$	4. $f(x) = 3^{\frac{1}{x-6}}$
5. $f(x) = 2^{\frac{2}{x-4}}$	6. $f(x) = 7^{\frac{3}{x-4}}$
7. $f(x) = 8^{\frac{1}{4-x}}$	8. $f(x) = 9^{\frac{1}{x-2}}$
9. $f(x) = 6^{\frac{2}{3-x}}$	10. $f(x) = 5^{\frac{3}{x+3}}$
11. $f(x) = 3^{\frac{4}{5-x}}$	12. $f(x) = 4^{\frac{3}{5+x}}$

Mavzu yuzasidan savollar

1. Funksiya uzluksizligining ta'rifi.
2. Uzluksiz funksiyaning xossalari.
3. Elementar funksiyalarning uzluksizligi.
4. Funksiyaning uzilishi, uzilish turlari.
5. Bolsano Koshining teoremalari.
6. Veyershtras teoremasi.

17-amaliy mashg'ulot. Cheksiz katta va cheksiz kichik miqdorlar. Ajoyib limitlar.

Agar $\alpha(x)$ va $\beta(x)$ $x \to x_0$ holda cheksiz kichik funksiyalar bo'lib,
$\lim\limits_{x \to x_0} \frac{\alpha(x)}{\beta(x)} = 1$

bo'lsa, u holda ular ekvivalent deyiladi va $x \to x_0$ da $\alpha(x)$ ~ $\beta(x)$ kabi belgilanadi. Masalan, $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1,$ shu sababli $x \to 0$ da $\sin x$ ~ x. Shunga o'xshash $x \to 0$ da quyidagi cheksiz kichik funksiyalar ekvivalentdir:

sin$x$ ~ tg$x$~ $\arcsin x$ ~ arctg$x$ ~ ln(1+x) ~ e^x-1 ~ x, 1-cos$x$ ~ $\frac{x^2}{2}$, e^x ~ 1+x, a^x ~1+ $x$ln$a$, $(1+x)^m$ ~ 1+mx, $\sqrt[m]{1+x} = (1+x)^{\frac{1}{m}}$ ~ $1 + \frac{x}{m}$, $\log_a^{(1+x)} = \frac{\ln(1+x)}{\ln a}$ ~ $\frac{x}{\ln a}$.

1-misol. Limitlarni toping. 1) $\lim\limits_{x \to 0} \frac{\sin^2 3x}{\ln^2(1+2x)}$, 2) $\lim\limits_{x \to 0} \frac{1-\cos^{\mu} x}{x^2}$.

Yechish. 1) $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{\ln^2(1+2x)}=\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{\ln(1+2x)}\right)^2=\lim\limits_{x\to 0}\left(\dfrac{3x}{2x}\right)^2=\dfrac{9}{4}.$

2) $\lim\limits_{x\to 0}\dfrac{1-\cos^{\mu} x}{x^2}=\lim\limits_{x\to 0}\dfrac{1-\left(1-0{,}5x^2\right)^{\mu}}{x^2}=\lim\limits_{x\to 0}\dfrac{1-1+0{,}5\mu x^2}{x^2}=0{,}5\mu.$

2-misol. Limitlarni hisoblang

1. $\lim\limits_{x\to\infty}\dfrac{3x^2+4x-6}{2x^3-7x^2+2}$
2. $\lim\limits_{x\to\infty}\dfrac{x^4-4x+1}{2x^4+3x^2+x}$
3. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+1}{x^2+2x+3}$
4. $\lim\limits_{x\to\infty}\dfrac{3x^3+2x-5}{x^4+5x^2+1}$
5. $\lim\limits_{x\to\infty}\dfrac{2x^4-3x^2+x}{4x^3+3x-5}$
6. $\lim\limits_{x\to\infty}\dfrac{5+3x-4x^2}{2x^2-x+4}$
7. $\lim\limits_{x\to\infty}\dfrac{6x^2-2x+1}{2x^4+x^2+5}$
8. $\lim\limits_{x\to\infty}\dfrac{3-2x+4x^2}{6+5x-3x^2}$
9. $\lim\limits_{x\to\infty}\dfrac{5x^3-3x^2+4}{6x^3+x-5}$
10. $\lim\limits_{x\to\infty}\dfrac{x^4+3x^2-x}{x^2-x^3+3x^4}$
11. $\lim\limits_{x\to\infty}\dfrac{14x^2+3x}{7x^2+2x-8}$
12. $\lim\limits_{x\to\infty}\dfrac{6x^4-4x^2+5}{2x^3-3x^2+1}$
13. $\lim\limits_{x\to\infty}\dfrac{x^6-3x^2+7}{3x^5+4x^3-2}$
14. $\lim\limits_{x\to\infty}\dfrac{3x^2-2x+5}{2x^3-3x^2+1}$

3-misol. Limitni hisoblang

1. $\lim\limits_{x\to 4}\dfrac{\sqrt[3]{x^2-16}}{\sqrt{x+12}-\sqrt{3x+4}}$
2. $\lim\limits_{x\to}\dfrac{\sqrt{4x-3}-\sqrt{5x-6}}{\sqrt[3]{x^2-9}}$
3. $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{2-3x}-\sqrt[6]{6-x}}{\sqrt[3]{8+x^3}}$
4. $\lim\limits_{x\to 5}\dfrac{\sqrt{4x+5}-\sqrt{6x-5}}{\sqrt{4+x}-\sqrt{2x-1}}$
5. $\lim\limits_{x\to 8}\dfrac{\sqrt[3]{x}-2}{\sqrt{2x+9}-\sqrt{3x+1}}$
6. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{\sqrt[3]{8+x}-\sqrt[3]{8-x}}$
7. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{5+x}-\sqrt[3]{5-x}}{\sqrt[3]{x^2+x^4}}$
8. $\lim\limits_{x\to\frac{2}{3}}\dfrac{\sqrt{\frac{1}{3}+x}-\sqrt{2x-\frac{1}{3}}}{\sqrt[3]{3x-2}}$
9. $\lim\limits_{x\to 4}\dfrac{\sqrt[3]{x-3}-\sqrt[3]{2x-7}}{\sqrt{1+2x}-3}$
10. $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{6+x}-\sqrt[3]{10+3x}}{\sqrt{2-x}-2}$

11. $\lim\limits_{x\to 1}\dfrac{4x^2-3x-1}{\sqrt{3+2x}-\sqrt{x+4}}$ 12. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{\sqrt{5x+1}-4}$

13. $\lim\limits_{x\to -1}\dfrac{\sqrt{3+x}-\sqrt{5+3x}}{4x^2+3x-1}$ 14. $\lim\limits_{x\to 5}\dfrac{\sqrt{2x+1}-\sqrt{x+6}}{x^2-8x+15}$

4-misol. Hisoblang

1. $\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x+3}\right)^{3x+2}$ 2. $\lim\limits_{x\to 1}(3x-2)^{\frac{5x}{x-1}}$

3. $\lim\limits_{x\to -\infty}(4x-1)(\ln(x+2)-\ln(x-1))$ 4. $\lim\limits_{x\to\infty}\left(\dfrac{5x-1}{5x+2}\right)^{2x-1}$

5. $\lim\limits_{x\to 1}(2x-1)^{\frac{3x}{x-1}}$ 6. $\lim\limits_{x\to\infty}(x+2)(\ln(2x+3)-\ln(2x-1))$

7. $\lim\limits_{x\to\infty}\left(\dfrac{2x+5}{2x+1}\right)^{5x-3}$ 8. $\lim\limits_{x\to 1}(3-2x)^{\frac{x}{1-x}}$

9. $\lim\limits_{x\to +\infty}(2x+3)(\ln(x+2)-\ln(x))$ 10. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{2x-5}$

11. $\lim\limits_{x\to 2}(2x-3)^{\frac{3x}{x-2}}$ 12. $\lim\limits_{x\to +\infty}(3x+5)(\ln(x+5)-\ln(x))$

13. $\lim\limits_{x\to\infty}\left(\dfrac{2-3x}{5-3x}\right)^{4x+3}$ 14. $\lim\limits_{x\to -1}(2-x)^{\frac{2x}{1-x}}$

5-misol. Limitlarni hisoblang

1. $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{4x^2}$ 2. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^3 x}{3x^2}$

3. $\lim\limits_{x\to 0}\dfrac{\cos 5x-\cos x}{2x^2}$ 4. $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{4x}$

5. $\lim\limits_{x\to 0}\dfrac{1-\cos^2 x}{xtgx}$ 6. $\lim\limits_{x\to 0}\dfrac{tg2x-\sin 2x}{x^2}$

7. $\lim\limits_{x\to 0}\dfrac{5x}{\sin x+\sin 7x}$ 8. $\lim\limits_{x\to 0}\dfrac{\cos^2 x-\cos^2 2x}{x^2}$

9. $\lim\limits_{x\to 0}\dfrac{tgx}{3\sin 3x}$ 10. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{\sin 5x}$

11. $\lim\limits_{x\to 0}\dfrac{\sin^2 2x-\sin^2 x}{3x^2}$ 12. $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos^3 6x}{4x^2}$

13. $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{4x^2}$ 14. $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{3x^2}$

Mavzu yuzasidan savollar.

1. *Funksiya ta'rifi va misollar.*
2. *Funksiyaning berilish usullari.*
3. *Qanday funksiyalar elementar funksiyalar deyiladi?*
4. *Ketma-ketlik limitning ta'rifi.*
5. *Ajoyib limitlar.*
   a) Birinchi ajoyib limit.
   b) Ikkinchi ajoyib limit.
7. *Cheksiz kichik va cheksiz katta miqdorlar nima?*
6. *Aniqmasliklarni ochish*

18-amaliy mashg'ulot. Funksiyaning nuqtadagi hosilasi. Hosilaning geometrik, mexanik, iqtisodiy, kimyoviy va boshqa talqinlari. Hosila olishning asosiy qoidalari.

Aytaylik $y=f(x)$ funksiya biror X sohada aniqlangan b'olib, $x_0\in X$ va $x_0+\Delta x\in X$ bo'lsin.

Ta'rif. Agar $\Delta x\to 0$ da $\dfrac{\Delta y}{\Delta x}=\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

nisbatning limiti mavjud va chekli bo'lsa, bu limit $y=f(x)$ funksiyaning x_0 nuqtadagi hosilasi deyiladi. Hosilaning belgilanishi:

$y', f'(x_0), \dfrac{dy}{dx}, \dfrac{df}{dx}$.

Demak, ta'rifga ko'ra

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}. \qquad (18.1)$$

Agar $y=f(x)$ funksiya x_0 nuqtada hosilaga ega bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada differensiallanuvchi deyiladi, hosilani topish jarayoni differensiallash deyiladi.

1-misol. $y=x^2$ funksiyaning hosilasini hisoblang.

Yechish. Avval x ga Δx orttirma beramiz va funksiya orttirmasi Δy ni topib (8.1) - formulaga qo'yamiz:

$$\Delta y = f(x_0+\Delta x)-f(x_0)=\left(x+\Delta x\right)^2-x^2=x^2+2x\Delta x+\Delta x^2-x^2=2x\Delta x+\Delta x^2=\Delta x\left(2x+\Delta x\right)$$

(18.1) formulaga ko‘ra:

$$f^{'}(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

2-misol. $y=\frac{1}{x}$ funksiyaning hosilasini hisoblang

3-misol. $y=sinx$ funksiyaning hosilasini toping.

4-misol. $y=\frac{2x+3}{2x+1}$ funksiyaning hosilasini hosilaning ta’rifidan foydalanib hisoblang

5-misol. $y=|x|$ funksiya hosilasini hisoblang

Yechish. $x = 0$ nuqtada argumentga Δx orttirma beramiz, u holda funksiya Δy orttirma oladi:

$$\Delta y=|\Delta x|=\begin{cases}-\Delta x, & agar\ \ \Delta x<0\ \ \ bo'lsa,\\ \Delta x, & agar\ \Delta x>0\ \ \ bo'lsa.\end{cases}$$

$$\frac{\Delta y}{\Delta x}=\begin{cases}-1, & agar\ \ \Delta x<0\ \ \ bo'lsa,\\ 1, & agar\ \Delta x>0\ \ \ bo'lsa.\end{cases}$$

Ko‘rinib turibdiki, $\Delta x = 0$ nuqtada $y=|x|$ funksiya hosilaga ega emas, chunki $\frac{\Delta y}{\Delta x}$ nisbatning $\Delta x\to 0$ dagi limiti mavjud emas.

6-misol. $y=x^2+3$ funksiya grafigiga $M(1;4)$ nuqtadan o‘tkazilgan urinmaning burchak koeffitsiyentini toping.

Yechish. $k=tg\varphi=f'(x_0)=2x_0=2\cdot 1=2$

7-misol. $y=x^2+3x+4$ funksiya grafigiga M (-1; 2) nuqtadan o‘tkazilgan urinma tenglamasini tuzing.

8-misol. $y=2x^3+1$ funksiya grafigining $M(1;\ 3)$ nuqtasiga o‘tkazilgan urinmaning burchak koeffitsiyentini toping.

9-misol. $y=\sin x+\cos x$ funksiyaning $x_0=\frac{\pi}{2}$ nuqtasidan o‘tkazilgan urinmaning tenglamasini tuzing.

10-misol. $y=\ln x$ ($x>0$) funksiyaning $x_0=1$ nuqtasidan o‘tkazilgan urinma Ox o‘qning musbat yo‘nalishi bilan qanday burchak hosil qiladi.

11-misol. $S=2t^2+t$ (m) qonuniyat bilan harakatlanayotgan moddiy nuqtaning $t=3$ (sek) dagi oniy tezligini toping.

Yechish. $\upsilon(t)=\lim\limits_{\Delta t\to 0}\frac{\Delta S}{\Delta t}=\lim\limits_{\Delta t\to 0}\frac{S(t+\Delta t)-S(t)}{\Delta t}=\lim\limits_{\Delta t\to 0}\frac{2(t+\Delta t)^2+(t+\Delta t)-2t^2-t}{\Delta t}=4t+1$

Demak, (3) = 4·3 + 1 = 13 m/sek, ***v*** **=** 13 m/sek.

12-misol. $S=t^3-t^2-t+1$ (m) qonuniyat bo‘yicha harakatlanayotgan moddiy nuqta qancha vaqtdan keyin to‘xtaydi.

13-misol. $y=(3x+1)(\frac{x}{2}+3)$ funksiyaning hosilasini hisoblang

Yechish. $f(x)=3x+1$, $g(x)=(\frac{x}{2}+3)$

$y`=f`(x)\,g(x)+g`(x)\,f(x)=(3x+1)`(\frac{x}{2}+3)+(\frac{x}{2}+3)`(3x+1)=3(\frac{x}{2}+3)+$

$+\frac{1}{2}(3x+1)=3x+9\frac{1}{2}$

14-misol. $y=\frac{8x+1}{2-3x}$ funksiyaning hosilasini hisoblang

Mavzu yuzasidan savollar.

1. *Funksiya nuqtadagi hosilasi ta’rifi va misollar keltiring.*
2. *Funksiyaning berilish usullarini ayting.*
3. *Funksiya hosilasining geometriyaga tatbiqi haqida gapiring.*
4. *Funksiya hosilasining mehanikaga tatbiqi haqida gapiring.*
5. *Funksiya hosilasining iqtisodiyotga tatbiqihaqida gapiring.*

19-amaliy mashg‘ulot. Murakkab va teskari funksiyalarning hosilasi.

$y=f(x)$ funksiya (a, b) oraliqda berilgan va unga teskari $x=\varphi(y)$ funksiya mavjud bo‘lsin.

Agar $y=f(x)$ funksiya x_0 ($x_0\in(a,b)$) nuqtada $f`(x_0)$ hosilaga ega bo‘lib $f`(x_0)\neq 0$ bo‘lsa, u holda bu funksiyaga teskari bo‘lgan $x=\varphi(y)$ funksiya $y_0=f(x_0)$ nuqtada hosilaga ega va

$$\varphi'(y_0)=\frac{1}{f'(x_0)} \qquad (19.1)$$

tenglik o‘rinli bo‘ladi.

1-misol. $y = arsinx$ ($-1 \le x \le 1$) funksiyani hosilasini aniqlang.

Yechish. $y=arcsinx$ funksiya $x=siny$ funksiyaga nisbatan teskari funksiyadir.

(19.1) - formuladan $f'(y_0)=\frac{1}{\varphi'(x_0)}$ formulani olamiz. Bunda $f(x)=arcsinx$, $\varphi(y)=\sin y$. Demak,

$$(\arcsin x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

2-misol. y = arccos$x$ funksiyaning hosilasini hisoblang

3-misol. y = arctg$x$ funksiyaning hosilasini hisoblang

4-misol. y = arcctg$x$ funksiyaning hosilasini hisoblang

Murakkab funksiyaning hosilasi

$y = \varphi(x)$ funksiya X to‘plamda, $u = f(y)$ funksiya esa Y ($Y = \{\varphi(x);\ x \in X\}$) to‘plamda aniqlangan bo‘lib, $u = f(\varphi(x))$ murakkab funksiya berilgan bo‘lsin. Agar $y = \varphi(x)$ funksiya x_0 nuqtada $\varphi`(x_0)$ hosilaga ega bo‘lib, $u=f(y)$ funksiya esa $y_0$ ($y_0 = \varphi(x_0)$) nuqtada $f`(y_0)$ hosilaga ega bo‘lsa, u holda $u = f(\varphi(x))$ murakkab funksiya ham x_0 nuqtada hosilaga ega bo‘ladi va quyidagi formula bilan hisoblanadi;

$$\left(f(\varphi(x))\right)'_{x=x_0} = f'(y_o)\cdot\varphi'(x_0) = f'(\varphi(x_0))\cdot\varphi'(x_0) \qquad (19.2)$$

Funksiyalarning hosilasini hisoblang:

5-misol. $y = cos^3x$

Yechish. $u=cosx$ deb belgilashni kiritsak, $y=u^3$ hosil bo‘ladi. (19.2) – formulaga ko‘ra:

$$y'=(u^3)'=3u^2\cdot u',\quad u'=(\cos x)'=-\sin x,\ y'=-3\sin x\cdot\cos^2 x.$$

b. $y = (1+\sin^3 2x)^4$

c. $y = tg^2(e^{-x})$. **d.** $y=\frac{(1+x)^2\sqrt[3]{(1-x)^2}}{(x^2+4)^4 e^{-\sin x}}.$

i. $y = xarctg^3 5x + \ln tgx$. **f.** $y = arc\sec x$

g. $y = x^3\cdot e^{x^2}\cdot\sin 2x.$

Oshkormas funksiya hosilasi

Ikkita x va y o‘zgaruvchilar orasidagi bog‘lanish *F(x,y)=0* tenglama ko‘rinishida berilgan bo‘lsin. F(x,y)=0 oshkormas funksiyani oshkor ko‘rinishga keltirmasdan hosilasini topish qoidasini ko‘rsatamiz. y ni x ning funksiyasi deb F(x,y)=0 tenglamaning ikkala qismini differensiallash, so‘ngra hosil qilingan tenglamani hosilasini topish kerak. Buni quyidagi misolda ko‘rsatamiz.

6-misol.: $x^4+y^4-3xy=0$ oshkormas funksiyaning hosilasini hisoblang

***Yechish*:** y ni x ning funksiyasi deb belgilangan tenglamasining ikkala qismini differensiallaymiz $4x^3+4y^3y'-3y-3xy'=0$ bundan esa $y'=\left(4x^3-3y\right)/\left(3x-4y^3\right)$ ni topamiz.

Funksiyalarning hosilasini hisoblang.

7-misol. a. $x\cdot e^y+y\cdot e^x=xy$.

Yechish: $\left(x\cdot e^y+y\cdot e^x\right)'=\left(xy\right)'$, $e^y+xe^yy'+y'\cdot e^x+y\cdot e^x=y+xy'$,

$y'\cdot\left(xe^y+e^x-x\right)=-e^y-y\cdot e^x+y$, bundan $y'=\dfrac{y-\left(e^y+ye^x\right)}{e^x+x(e^y-1)}$.

b-misol. $\ln(x^2+y^2)=2arctg\dfrac{y}{x}$,

c. $y=x^x$

8-misol. a. $y=\ x^3+2x^2+2$ funksiyaning differensialini hisoblang

Yechish. $dy=y'dx=(x^3+2x^2+2)'dx=(3x^2+4x)dx.$

b. $y = e^{2x} + cox2x$ funksiya differensialini toping.

c. $y=\ln(e^{2x}+x+1)$ funksiyaning differensialini toping.

d. $\sqrt[3]{28}$ miqdorni taqribiy hisoblang.

Parametrik ko‘rinishda berilgan funksiyaning hosilasi

Faraz qilaylik y funksiyaning x argumentga bo‘g‘liqligi $\begin{cases} x=x(t) \\ y=(t) \end{cases}$ tenglamalar bilan parametrik shakldа berilgan bo‘lsin. Masalan:

1) $x=x_0+l\cdot t,\ \ y=y_0+m\cdot t$ – to‘g‘ri chiziqning parametrik tenglamasi.

2) $x=x_0+R\cdot\cos t,\ \ y=y_0+r\cdot\sin t$ – aylananing parametrik tenglamsi. $\left(x-x_0\right)$ va $\left(y-y_0\right)$ larni kvadratga ko‘tarib, qo‘shish natijasida (haqiqatan ham $\left(x-x_0\right)^2+\left(y-y_0\right)^2=R^2$– markazi $M_0(x_0,\ y_0)$nuqtada bo‘lgan aylana tenglamasini hosil qilamiz).

3) Xuddi shuningdek, $x = a \cdot \cos t$, $y = b \cdot \sin t$ – ellipsning parametrik tenglamasi, $\frac{x}{a}$ va $\frac{y}{b}$ ni kvadratga ko‘tarib qo‘shib, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamani hosil qilamiz.

Parametr t ga Δt orttirma beramiz, mos ravishda Δx va Δy orttirmalar hosil bo‘ladi. U holda y dan x bo‘yicha hosila:

$$y'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}} = \frac{y'_t}{x'_t} \quad yoki \quad \frac{dy}{dx} = \frac{y'_t(t)}{x'_t(t)}.$$

9-misol. $x = e^{2t}\cos^2 t,\ \ y = e^{2t}\sin^2 t$; y'_x hosilani hisoblang

Yechish. $x'_t = 2e^{2t}\cos^2 t + e^{2t}(-2\cos t \cdot \sin t) = 2e^{2t}(\cos t - \sin t)\cos t,$

$y'_t = 2e^{2t}\sin^2 t + e^{2t} \cdot 2\cos t \cdot \sin t = 2e^{2t}(\cos t + \sin t)\sin t,$

$$y'_x = \frac{y'_t}{x'_t} = tgt \cdot \frac{\sin t + \cos t}{\cos t - \sin t} = tgt \cdot \frac{1+tgt}{1-tgt} = tgt \cdot tg\left(t + \frac{\pi}{4}\right), \quad \left(t \neq \frac{\pi}{4} + \pi k,\ t \neq \frac{\pi}{2} + \pi k\right).$$

10-misol. $x = a(t - \sin t),\ \ y = a(1 - \cos t)$.; $y'_x = ?$

11-misol. Hosilasini hisoblang.

a. $x = a\cos t,\quad y = a\sin x$; **b** $x = t^2,\ \ y = \frac{t^3}{3} - t$;

c $x = e^{2t},\quad y = e^{3t}$.

12-misol. Quyida berilgan funksiyalar hosilalarini differensiallash qoidalari va formulalardan foydalanib hisoblang

Variant	Funksiya	Variant	Funksiya
1	$y = x^2 \sin 2x$	8	$y = arctg\sqrt{1+e^{-x^2}}$
2	$y = e^{4x} tg 2x$	9	$y = (\sin 2x)^{\cos 4x}$
3	$y = \sqrt[3]{x^3 + \sin^3 x}$	10	$y = (x^2+1)^{ctg 2x}$

4	$y=\sqrt{x^2+1}\cdot ct^2 3x$	11	$y=\dfrac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}$
5	$y=3^{-\cos^3 3x}$	12	$y=\dfrac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}}$
6	$y=e^{-\arcsin\sqrt{x}}$	13	$y=x^2\sqrt{1-x^2}$
7	$y=\left(3x^3-ctg^4 x\right)^3$	14	$y=\dfrac{\sqrt{4x+1}}{x^2}$

Mavzu yuzasidan savollar

1. Hosilaning ta'rifi: geometrik ma'nosi, iqtisodiy ma'nosi, mexanik ma'nosi.

2. Quyidagi tasdiqlardan qaysi biri to'g'ri

a) agar funksiya biror nuqtada uzluksiz bo'lsa, u holda shu nuqtada funksiya differensiallanuvchi bo'ladi.

b) agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u holda shu nuqtada funksiya uzluksiz bo'ladi.

3. Elementar funksiyalarning hosilalari jadvalini yozing.

4. Murakkab, teskari, oshkormas ko'rinishdagi, parametrik ko'rinishdagi funksiyalarning xosilalari.

5. Funksiya hosilasining geometriyaga tatbiqi.

6. Murakkab funksiyaning hosilasini topish usuli.

7. Oshkormas funksiyaning hosilasini topish usuli.

8. Funksiyaning differensiali deb nimaga aytiladi?

20-amaliy mashg'ulot. Yuqori tartibli hosilalar va differensiallari.

Birinchi tartibli hosiladan olingan hosila ikkinchi tartibli hosilasi deyiladi va $\frac{d^2y}{dx^2}$ belgilarning biri bilan belgilanadi.

Ikkinchi tartibli hosilaning hosilasiga uchinchi tartibli hosila deyiladi va $\frac{d^3y}{dx^3}$ belgilarning belgilarning biri bilan belgilanadi.
Umuman, $y=f(x)$ funksiyaning n-tartibli hosilasi deb, uning $(n-1)$ - tartibli hosilasining hosilasiga aytiladi va $y^{(n)}$, $f^{(n)}$,$\frac{d^ny}{dx^n}$ belgilarning biri bilan belgilanadi.

$y = f(x)$ funksiya differensiali dy ning differensiali berilgan funksiyaning ikkinchi tartibli differensiali deb ataladi va d^2y yo‘ki $d^2f(x)$ kabi belgilanadi:

$$d^2y = d(dy) \text{yoki} \quad d^2f(x) = d(df(x)) \qquad (20.1)$$

Funksiyaning differensialini uning hosilasi orqali ifodalovchi

$dy = y`dx$ formulaga ko‘ra:
$d^2y = d(dy) = d(y`dx) = dxd(y`) = dx(y`)`dx = y”dx^2$

Demak, funkisyaning ikkinchi tartibli differensiali funksiyaning ikkinchi tartibli hosilasining argument differensiali kvadrat ko‘paytmasiga teng.

Umumiy holda funksiyaning n – tartibli differensiali uning $(n-1)$ – tartibli differensialining differensialidan iboratdir:

$$d^ny = d(d^{n-1}y), \; d^nf(x) = d(d^{n-1}f(x)) \qquad (20.2)$$

1-misol. $y = e^x + x^2$ funksiyaning uchinchi tartibli hosilasini toping.
Yechish. (20.2) – formulaga ko‘ra $n = 3$ da $y^{(3)} = (y^{(2)})`$

$y” = (y`)` = ((e^x + 2x)`)` = (e^x + 2x)` = e^x + 2$

$y^{(3)} = (e^x + 2)` = e^x+ 0 = e^x$

Differensial hisobining asosiy teoremalari

1. *Ferma teoremasi.* $f(x)$ funksiya quyidagi shartlarni qanoatlantirsin:

a) $[a, b]$ kesmada uzluksiz;

b) (a, b) oraliqda differensiallanuvchi;

c) kesmaning oxirlarida $f(a) = f(b)$teng qiymatlar qabul qilsa, u holda, aqalli bitta shunday $x = c$ $(a < c < b)$ nuqta mavjudki, unda $f`(c)=0$ bo‘ladi.

2. *Roll teoremasi.* Agar$f(x)$ funksiya quyidagi shartlarni qanoatlantirsa:

a) $[a, b]$ kesmada uzluksiz;

b) (a, b) oraliqda differensiallanuvchi bo‘lsa, u holda berilgan oraliqda aqalli bitta$x = c$ $(a < c < b)$ nuqta mavjudki,

$f(b) - f(a) = f'(c)(b - a)$ bo‘ladi.

3. *Lagranj teoremasi.* $y=f(x)$ *va* $y=g(x)$ quyidagi shartlarni qanoatlantirsin;

a) $[a, b]$ kesmada uzluksiz;

b) (a, b) oraliqda differensialanuvchi;

c) $g'(x) \neq 0, \quad x \in [a,b]$

u holda bu oraliqda aqalli bitta shunday $x = c$ $(a < c < b)$ nuqta mavjudki,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

bo‘ladi.

4. *Koshi teoremasi.* Agar $f(x)$ funksiya x oraliqning ichki nuqtasi x_0 da o‘zining eng katta (eng kichik) qiymatiga erishsa, hamda shu x_0 nuqtada chekli hosilaga ega bo‘lsa, u holda funksiya hosilasining x_0 nuqtadagi qiymati nolga teng bo‘ladi, ya’ni: $f`(x) = 0$

Lopital qoidasi

$f(x)$ va $g(x)$ funksiyalar (a, b) oraliqda aniqlangan bo‘lsin.

Agar: $\lim\limits_{x\to a} f(x) = 0$ $\lim\limits_{x\to a} g(x) = 0$ bo‘lsa, u holda $x \to a$ da $\frac{f(x)}{g(x)}$ nisbat $(\frac{0}{0})$ korinishdagi aniqmaslik bo‘ladi.

Agar $\lim\limits_{x\to a} f(x) = \infty$, $\lim\limits_{x\to a} g(x) = \infty$ bo‘lsa, u holda $x \to a$ da $\frac{f(x)}{g(x)}$ nisbat $\left(\frac{\infty}{\infty}\right)$ ko‘rinishdagi aniqmaslik bo‘ladi.

Agar berilgan funksiyalar hosilaga ega bo‘lsa, yuqoridagi aniqmasliklarni ochish mumkin. Bunda Lopital qoidasidan foydalaniladi.

$f(x)$ va $g(x)$ funksiya $x = a$ nuqta atrofida mavjud va differensiallanuvchi bolib,

1) $\lim\limits_{x\to a} f(x) = g(a) = 0,$ yoki ∞.

2) $\lim\limits_{x\to a} g(x) = g(a) = 0,$ yoki ∞ va $g‘(x) \neq 0$ bo‘lsa, u holda

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad (20.3)$$

tenglik o‘rinli bo‘ladi.

2-misol. $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}$ ni hisoblang

Yechish: $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}\lim\frac{0}{0}$ tipidagi aniqmaslik bolib, Lopital qoidasiga ko‘ra:

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{(e^x-1)'}{(\sin x)'}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1.$$

3-misol. $\lim\limits_{x\to 0}\frac{x-\sin x}{x^2}$ limitni hisoblang.

$\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}$ ni hisoblang

Yechish: $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}\lim\frac{0}{0}$ tipidagi aniqmaslik bolib, Lopital qoidasiga ko‘ra:

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{(e^x-1)'}{(\sin x)'}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1.$$

4-misol. $\lim\limits_{x\to 0}\frac{x-\sin x}{x^2}$ limitni hisoblang.

$\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}$ ni hisoblang

Yechish: $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}\lim\frac{0}{0}$ tipidagi aniqmaslik bolib, Lopital qoidasiga ko‘ra:

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{(e^x-1)'}{(\sin x)'}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1.$$

5-**misol.** Lopital qoidasidan foydalanib quyidagi limitlarni hisoblang hisoblang

a. $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\ln(x+1)}$ **b.** $\lim\limits_{x\to 0}\frac{x-\sin x}{x^2}$ **c.** $\lim\limits_{x\to 1}\frac{tg\frac{\pi x}{2}}{\ln(1-x)}$

d. $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ **e.** $\lim\limits_{x\to 0}\frac{e^{3x}-1}{\sin 3x}$

6-misol. Lopital qoidasidan foydalanib quyidagi limitlarni hisoblang

Variant	Funksiya	Variant	Funksiya

1	$\lim\limits_{x\to1}\dfrac{x^3-7x^2+4x+2}{x^3-5x+4}$	8	$\lim\limits_{x\to0}(\sin x)^x$
2	$\lim\limits_{x\to0}\dfrac{tgx-\sin x}{x-\sin x}$	9	$\lim\limits_{x\to\pi}\dfrac{1+\cos x}{x-\pi}$
3	$\lim\limits_{x\to0}\dfrac{e^{5x}-1}{\sin 3x}$	10	$\lim\limits_{x\to-1}\dfrac{x^3+3x^2-2}{x^3-4x^2+5}$
4	$\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$	11	$\lim\limits_{x\to0}\dfrac{e^{2x}-e^{\frac{x}{2}}}{\ln(1+x)}$
5	$\lim\limits_{x\to0}tgx\cdot\ln x$	12	$\lim\limits_{x\to\infty}\dfrac{\pi-2arctgx}{e^{\frac{3}{x}}-1}$
6	$\lim\limits_{x\to0}(tgx)^{\sin x}$	13	$\lim\limits_{x\to\infty}\dfrac{e^x}{x^2}$
7	$\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x$	14	$\lim\limits_{x\to\infty}\dfrac{\ln x}{x^3}$

Mavzu yuzasidan savollar

1. Elementar funksiyalarning hosilalari jadvalini yozing.
2. Murakkab, teskari, oshkormas ko‘rinishdagi, parametrik ko‘rinishdagi funksiyalarning xosilalari.
3. Funksiyaning differensiali deb nimaga aytiladi?
4. O‘rta qiymat haqidagi teoremalarni ayting:
- *a) Ferma teoremasi.*
- *b) Roll teoremasi.*
- *c) Lagranj teoremasi.*
- *d) Koshi teoremasi.*
- *e) Lopital teoremasi.*

21-amaliy mashg‘ulot. Lagranj formasida qoldiq hadli Teylor formulasi. Elementar funksiyalarni Teylor va Makloren qatoriga yoyish.

Teylor teoremasi: Agar $y = f(x)$ funksiya $x = a$ nuqtada aniqlangan va uning biror atrofida $(n+1)$ – tartibgacha hosilaga ega bo‘lsa, u holda shunday $x=\xi$ nuqta mavjudki unda Teylor formulasi o‘rinli bo‘ladi:

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+...+\frac{f^n(a)}{n!}(x-a)^n+$$
$$+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad (21.1)$$

ξ nuqta x va a orasida yotadi, ya'ni $\xi=a+\theta(x-a)$ va $0<\theta<1$. Teylor formulasidagi oxirgi qo'shiluvchi Lagranj shaklidagi qoldiq had deyiladi. a = 0 da Teylor formulasi Makloren formulasi deyiladi:

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...+\frac{f^n(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$
$$0<\theta<1 \qquad (21.2)$$

1-misol. $f(x)=\frac{1}{x-2}$ funksiyani x-1 ning darajasi ko'rinishida ifodalang.

Yechish. Funksiya a=1 nuqtada aniqlangan va bu nuqtaning atrofida barcha hosilalari mavjud. Bu nuqtada funksiyaning qiymatini va beshinchi tartibgacha bo'lgan hosilasini hisoblaymiz.

$$f(1)=-1,\ f'(1)=\left(\frac{-1}{(x-2)^2}\right)_{x=1}=-1,\quad f''(1)=\left(\frac{-1\cdot 2}{(x-2)^3}\right)_{x=1}=-2,\ ....,\ f^5(1)=$$
$$\left(\frac{-1\cdot 2\cdot 3\cdot 4\cdot 5}{(x-2)^6}\right)_{x=1}=-120$$

U holda Teylor formulasiga ko'ra x-1 ga nisbatan ko'phadni hosil qilamiz:

$$\frac{1}{x-2}=-1-(x-1)-\frac{2}{2!}(x-1)^2-\frac{6}{3!}(x-1)^3-\frac{24}{4!}(x-1)^4-\frac{120}{5!}(x-1)^5+R_6=-1-(x-1)-$$
$$(x-1)^2-(x-1)^3-(x-1)^4-(x-1)^5+R_6,\ bunda\ \ R_6=\left(\frac{6!}{6!(x-2)^6}\right)_{x=\xi}(x-1)^6=$$
$$=\frac{(x-1)^6}{(\xi-1)^6}\ va\ 1<\xi<x.$$

2-misol. Berilgan funksiyalarning qator yoyilmasini tuzing.

Variant	$f(x)$ funksiya va x_0 nuqta	Variant	$f(x)$ funksiya va x_0 nuqta
1	$f(x)=x^3+3x^2-6x+4, x_0 = 1.$	8	$f(x)=e^{2x-4}+2\ln x,\ x_0 = 2.$

2	$f(x) = 2x^2+5x+2,\ x_0 = -2.$	9	$f(x) = x - \cos x,\ \ x_0 = \frac{\pi}{4}.$
3	$f(x) = x^4+3x^2-5x-7, x_0=-1.$	10	$f(x) = x - \sin x,\ \ x_0 = \frac{\pi}{4}.$
4	$f(x) = x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 6, x_0= 4.$	11	$f(x) = x^2 + \ln(x-1),\ x_0 = 2.$
5	$f(x) = 2^{-x} + 2^{-2x},\ x_0 = 2.$	12	$f(x) = x^2 + 3x + 2,\ x_0 = 0.$
6	$f(x) = 3^{-x} + 3^x, x_0 = 2.$	13	$f(x) = \ln x,\ \ x_0 = e.$
7	$f(x) = \frac{x+3}{3-x},\ \ x_0 = 4.$	14	$f(x) = \log_4^{(3x-4)},\ x_0 = 2.$

Mavzu yuzasidan savollar

1. *Elementar funksiyalar Teylor qator yoyilmasi qanday topiladi?*
2. *Hosilaning iqtisodiyotda qo'llanilishi.*
3. *O'suvchi va kamayuvchi funksiyalar ta'riflari.*
4. *Funksiyaning ekstremumi nima va u qanday topiladi.*
5. *Ekstremumning zaruriy shartlari.*
6. *Ekstremumning yeterli shartlari.*
7. *Funksiyaning qavariq va botiqliligi, bukilish nuqtalari qanday topiladi?*
8. *Funksiyaning kesmadagi eng katta va eng kichik qiymatlari qanday topiladi?*
9. *Funksiya qanday sxema bilan tekshirilib, grafigi chiziladi?*

22-amaliy mashg'ulot. Funksiyaning monotonlik sharti. Funksiya ekstremumi. Ekstremumning zaruriy va yetarlilik shartlari. Funksiyaning kesmadagi eng katta va eng kichik qiymatlarini topish.

Agar x_0 ning etarlicha kichik atrofidagi barcha nuqtalar uchun $f(x_0)> f(x)$ bo'lsa, $y = f(x)$ funksiya uchun $x = x_0$ lokal maksimum nuqta deb ataladi.

Agar x_0 ning yetarlicha kichik atrofidagi barcha nuqtalar uchun $f(x_0)< f(x)$bo'lsa,
$x = x_0$lokal minimum nuqtasi deb ataladi. Funksiyaning maksimum va minimumi uning ekstremal qiymatlari yoki ekstremumlari deyiladi.

Differensiallanuvchi funksiyaning ekstremum nuqtasidagi hosilasi nolga teng bo‘ladi.

Funksiyaning *ekstremumlarini* topish.

1) $y` = f`(x)$ topiladi.

2) $f`(x)=0$ tenglama yechilib, statsionar nuqtalarning $x_1;\ x_2;\ ...,\ x_k$ abssissalari topiladi.

3) $f``(x) = 0$ tenglama yechimlari $x_1$ , $x_2$ bilan $f``(x)$ funksiya ishoralari aniqlanadi va $f``(x_1) > 0$ da $x = x_1$ minimum, $f``(x_2) < 0$ da esa $x = x_2$ maksimum nuqta bo‘ladi.

1-misol. $y = \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} + 1$ funksiyani hosila yordamida tekshiring.

Yechish. $f'(x) = x^3 - 4x^2 + 3x, \quad f''(x) = 3x^2 - 8x + 3$

1) $f`(x)= 0 \Rightarrow x(x-1)(x-3) = 0 \Rightarrow x_1=0,\ x_2=1,\ x_3=3$

2) $f``(0) = 3 > 0,\ f``(1)= -2 < 0,\ f``(3)= 6 > 0$ demak $x_2=1$ maksimum nuqta.

3) $f``(x)=0 \Rightarrow 3x^2-8x+3=0 \Rightarrow x_1 = \frac{4+\sqrt{7}}{3},\ x_2 = \frac{4-\sqrt{7}}{3}.$

$\forall x \in \left(-\infty; \frac{4-\sqrt{7}}{3}\right)$ *uchun* $f''(x) > 0$ $\forall x \in \left(\frac{4-\sqrt{7}}{3}; \frac{4+\sqrt{7}}{3}\right)$ *uchun* $f''(x) < 0$

$\forall x \in \left(\frac{4+\sqrt{7}}{3}; +\infty\right)$ *uchun* $f''(x) > 0$

Demak funksiya garafigi $\left(-\infty; \frac{4-\sqrt{7}}{3}\right)$ va $\left(\frac{4+\sqrt{7}}{3}; +\infty\right)$ oraliqlarda botiq, $\left(\frac{4-\sqrt{7}}{3}; \frac{4+\sqrt{7}}{3}\right)$ oraliqda esa qavariq bo‘ladi.

2-misol. Funksiyalarni monotonlik oraliqlarini toping.

a. $y = x^3 + 2x - 3$ **b.** $y = x\left(1 + \sqrt{x}\right)$

c. $y = 2 - sinx,\ 0 \le x \le 2\pi.$

5-misol. Funksiyaning ekstremumlarini toping.

a. $y = \frac{x^2 - 4x}{x - 2}$ **b.** $y = \frac{\ln x}{x}$

c. $y = x\sqrt{4 - x^2}$ **d.** $y = x - cosx$

6-misol. Berilgan funksiyalarning ekstremumlarini toping.

Variant	Funksiya	Variant	Funksiya
1	$y=3\left(\frac{x^4}{2}-x^2\right)$	8	$y=x^4+8x^3+16x^2$
2	$y=x^3-9x^2+24x-15$	9	$y=x^4+\frac{1}{3}x^3-\frac{1}{4}x^4$
3	$y=x^5-\frac{5}{3}x^3$	10	$y=\frac{1}{10}\left(2x^3-6x^2-18x+15\right)$
4	$y=2x^3+3x^2-12x-5$	11	$y=1-x^2-\frac{x^4}{8}$
5	$y=(x-3)^2(x-2)$	12	$y=-4x+x^3$
6	$y=\frac{x-1}{x^2-2x}$	13	$y=\frac{e^{x-1}}{x-1}$
7	$y=\frac{2-4x^2}{1-4x^2}$	14	$y=(x-2)e^{3-x}$

Mavzu yuzasidan savollar

1. O‘suvchi va kamayuvchi funksiyalar ta’riflari.
2. Funksiyaning ekstremumi nima va u qanday topiladi.
3. Ekstremumning zaruriy shartlari.
4. Ekstremumning yetarli shartlari.
5. Funksiyaning qavariq va botiqliligi, bukilish nuqtalari qanday topiladi?
6. Funksiyaning kesmadagi eng katta va eng kichik qiymatlari qanday topiladi?
7. Funksiya qanday sxema bilan tekshirilib, grafigi chiziladi?

23-amaliy mashg‘ulot. Funksiya grafigining qavariqligi, botiqligi va burilish nuqtalari. Asimptotalari. Funksiyani tekshirishning umumiy sxemasi.

To‘g‘ri chiziq *y=f(x)* funksiyaning grafigiga asimptota deyiladi, agar *(x,f(x))* nuqtadan bu to‘g‘ri chiziqqacha bo‘lgan masofa, grafigning nuqtasi koordinata boshidan cheksiz uzoqlashganda nolga intilsa.

Funksiya grafigining vertikal, gorizontal va og‘ma asimptotalari bo‘ladi.

Agar chap yoki o‘ng $\lim_{x\to x_o\pm 0} f(x)=\pm\infty$ limitlardan hech bo‘lmaganda bittasi $\pm\infty$ ga teng bo‘lsa $x=x_0$ to‘g‘ri chiziq vertikal asimptota deyiladi.

Agar uzilish nuqtasi yoki aniqlanish sohasining chegaraviy nuqtasi bo‘lsa $x=x_0$ to‘g‘ri chiziq *y=f(x)* funksiyaning vertikal asimptotasi bo‘lishi mumkin.

Agar $\lim_{x\to\infty} f(x)=b$ bo‘lsa y=b to‘g‘ri chiziq gorizontal asimptota deyiladi.

Agar $\lim_{x\to\infty}\frac{f(x)}{x}=k\neq 0$ va $\lim_{x\to\infty}[f(x)-kx]=b$ bo‘lsa, u holda $y=kx+b$ to‘g‘ri chiziq $y=f(x)$ funksiya grafigiga og‘ma asimptota bo‘ladi.

Funksiyani tekshirishning umumiy sxemasi

1) funksiyaning aniqlanish sohasini topish;

2) funksiyani toq yoki juftligini tekshirish;

3) vertikal asimptotalarni topish;

4) funksiyani cheksizlikda tekshirish; gorizontal va og‘ma asimptotalarni topish;

5) funksiyaning ekstremumlari va monoton oraliqlarini topish;

6) funksiyaning qavariqlik, botiqlik oraliqlari, bukilish nuqtalarini topish;

7) funksiya grafigining koordinata o‘qlari bilan kesishish nuqtalarini topish;

Funksiyani tekshirish grafigini chizish bilan bir vaqtda olib boriladi.

Funksiya hosila yordamida monotonlikka, ekstremumga va grafigining qavariq hamda botiqligi tekshiriladi.

$f'(x-h)$	$f'(x+h)$	Kritik nuqta haqida
+	-	Max

-	+	Min
+	+	Ekstremum yo‘q, funksiya o‘suvchi
-	-	Ekstremum yo‘q, funksiya kamayuvchi.

1-misol. $y = 2x^3 - 9x^2 + 12x - 2$ funksiyaning ektremumlarini toping.

Yechish.

1) $y` = 6x^2 - 18x + 12$

2) $y` = 6x^2 - 18x + 12 = 0$ $6(x^2 - 3x + 2) = 0 \Rightarrow 6(x-1)(x-2) = 0$, $x_1 = 1$, $x_2 = 2$

3) *a*) $x = x_1 - 1$ nuqtaning atrofida $y`$ ning ishorasi qanday o‘zgarishini tekshiramiz.

$h = -0,2$ bo‘lsin.

$y` = f`(0,8) = 6(0,8 - 1)(0,8 - 2) > 0.$

$y` = f`(1,2) = 6(1,2 - 1)(1,2 - 2) < 0.$

Hosilaning ishorasi (+) dan (-) ga o‘zgaryapti, demak $x = 1$ kritik nuqta maksimumdir.

b) $x_2 = 2$da $f`(1,8) < 0$ va $f`(2,2) > 0$

Demak $x = 2$ min nuqta.

2-misol. Quyidagi funksiyalarning hosilasini hisoblang

a. $y = \frac{\sqrt{6x+1}}{x^2}$ **b.** $y = \sqrt[4]{1+\sin^2 x}$

c. $y = ctg^3 x - 3ctgx + 3x$ **d.** $y = \sin^4 x + \cos^4 x$

e. $y = \frac{(x-3)^9}{\sqrt{(x-1)^5 (x-4)^{11}}}$ **f.** $y = x^{\frac{1}{\ln x}}$

3-misol. Quyidagi berilgan funksiyalarni ikkinchi tartibli hosilalarini hisoblang

a. $y = \sin^2 x$ **b.** $y = -\frac{3}{x^2 - 5}$

c. $y = tgx$ **d.** $y = ctgx$

e. $y = \sqrt{1+x^2} \cdot arctgx$. **f.** $y = arctgx$

4-misol. Berilgan funksiyaning birinchi va ikkinchi tartibli differensalini hisoblang

a. $y = arctgx$ **b.** $y = tgx$

c. $y=\sin 2x+\cos x$ **d.** $y=\ln x^2$

e. $y=e^{x^2-2x}+tg2x.$

5-misol. Funksiyaning qavariq va botiqlik oraliqlarini toping.

a. $y=\dfrac{3-x^2}{x+2}$ **b.** $y = 3x^4 + 4x^3 + 1$

c. $y=\dfrac{x^4}{4}-2x^2-\dfrac{9}{4}$ **d.** $y=\ln\dfrac{x}{x+5}-1$

e. $y=x\ln x$ **f.** $y=x-\ln x$

6-misol. Berilgan funksiyalarning hosilasini hisoblang va abssissasi x_0 nuqtadan oʻtkazilgan urinma tenglamasini tuzing.

Variant	$f(x)$ funksiya va x_0 nuqta	Variant	$f(x)$ funksiya va x_0 nuqta
1	$f(x) = x^3+3x^2-6x+4$, $x_0 = 1.$	8	$f(x)=e^{2x-4}+2\ln x$, $x_0 = 2.$
2	$f(x) = 2x^2+5x+2$, $x_0 = -2.$	9	$f(x)=x-\cos x$, $x_0=\frac{\pi}{4}.$
3	$f(x) =x^4+3x^2-5x-7$, $x_0 = -1.$	10	$f(x)=x-\sin x$, $x_0=\frac{\pi}{4}.$
4	$f(x)=x^{\frac{3}{2}}+2x^{\frac{1}{2}}-6$, $x_0 = 4.$	11	$f(x)=x^2+\ln(x-1)$, $x_0 =2.$
5	$f(x)=2^{-x}+2^{-2x}$, $x_0 = 2.$	12	$f(x)=x^2+3x+2$, $x_0 = 0.$
6	$f(x)=3^{-x}+3^{x}$, $x_0 = 2.$	13	$f(x) = \ln x$, $x_0 = e.$
7	$f(x)=\dfrac{x+3}{3-x}$, $x_0 = 4.$	14	$f(x)=\log_4^{(3x-4)}$, $x_0 = 2.$

Mavzu yuzasidan savollar

1. Funksiyaning qavariq va botiqliligi, bukilish nuqtalari qanday topiladi?

2. Funksiyaning kesmadagi eng katta va eng kichik qiymatlari qanday topiladi?

3. Funksiya qanday sxema bilan tekshirilib, grafigi chiziladi?

24-amaliy mashg‘ulot. Ko‘p o‘zgaruvchili funksiyalar, uning aniqlanish va o‘zgarish sohasi, limiti, uzluksizligi va xususiy hosilalari.

Agar biror D to‘plamning har bir (x, y) haqiqiy sonlar juftligi biror qoida bilan Z to‘plamdagi yagona z haqiqiy songa mos qo‘yilgan bo‘lsa, u holda D to‘plamda ikki o‘zgaruvchining funksiyasi z aniqlangan deyiladi va quyidagi ko‘rnishlarda belglanadi:

$z = f(x, y)$, $z = Z(x, y)$, $z = F(x, y)$, va h.k.

Bu yerda x va y erkli o‘zgaruvchilar yoki argumentlar, z esa erksiz o‘zgaruvchi yoki funksiya deb ataladi. D - to‘plam bu funksiyaning aniqlanish sohasi deyiladi. Z - to‘plam funksiyaningo‘zgarish sohasi deyladi.

$z=f(x, y)$ funksiyaning argumentlarining tayin $x = x_o$ va $y = y_0$ qiymatlarida qabul qiladigan z_0 xususiy qiymatini topish quyidagicha yoziladi :

$z_0=z\Big|_{\substack{x=x_0\\ y=y_0}}$ yoki $z_0=f(x_0, y_0)$.

Geometrik nuqtai nazardan $z=f(x, y)$ funksiyaning O_{xyz} to‘g‘ri burchakli koordinatalar sistemasidagi tasviri (funksiyaning grafigi) biror sirtdan (nuqtalar to‘plamidan) iborat.

1-misol. $z = \sqrt{4 + 4x + 2y - x^2 - y^2}$ funksiyaning aniqlanish sohasini toping.

Yechish.

$$z = \sqrt{4 + 4x + 2y - x^2 - y^2} = \sqrt{9 - (4 - 4x + x^2) - (1 - 2y + y^2)} = \sqrt{9 - (x-2)^2 - (y-1)^2}$$

Demak $9-(x-2)^2-(y-1)^2 \geq 0$, ya’ni $(x-2)^2+ (y-1)^2 \leq 9$ shartda berilgan funksiya haqiqiy qiymatlar qabul qiladi. Demak, funksiyaning aniqlanish sohasi markazi (2; 1) nuqtada, radiusi 3 ga teng bo‘lgan doiradan, o‘zgarish sohasi esa $[0, 3]$ sohadan iborat.

Istalgan chekli sondagi o‘zgaruvchining funksiyasi ham yuqoridagidek aniqlanadi n o‘zgaruvchili funksiyasining aniqlanish sohasi

n ta haqiqiy sonning $(x_1, x_2, \ldots, x_n)$ sistemasidan tuzilgan D to‘plamdan iborat bo‘ladi, n ta o‘zgaruvchining funksiyasi quyidagicha belgilanadi.
$y = f(x_1, x_2, \ldots, x_n)$, $y = y(x_1, x_2, \ldots, x_n)$.

2-misol. Funksiyalarning aniqlanish sohasini toping va grafigini chizing:

a) $z = x^2 + y^2$; *b*) $z = \sqrt{R^2 - x^2 - y^2}$

3-misol. Quyida berilgan funksiyalarni aniqlanish sohasini toping va garafigini chizing.

a. $z = x + y$ **b.** $z = \frac{1}{\sqrt{9 - x^2 - y^2}}$

c. $z = \sqrt{xy}$ **d.** $z = y\sqrt{x}$

e. $z = \frac{1}{\sqrt{x^2 + y^2 - 4}}$ **f.** $z = \arcsin(x + y)$

j. $z = \ln(x^2 + y^2 + z^2 - 1)$ **h.** $u = \sqrt{x + y + z}$

Ko‘p o‘zgaruvchili funksiya limiti

Agar ikki o‘zgaruvchining $z = f(x, y) = f(P)$ funksiyasi $P_0 = P(x_0; y_0)$ nuqtaning biror atrofida aniqlangan bo‘lsa va $\varepsilon > 0$ sonuchun shunday $\delta > 0$ son topilsaki $d(P;\ P_0) < \delta$ (d – ikki nuqta orasidagi masofa) tengsizlikni qanoatlantiruvchi barcha $P(x,y)$ nuqtalar uchun

$|f(x; y) - A| < \varepsilon$

tengsizlik bajarilsa, u holda A o‘zgarmas son $z=F(x,y)$ funksiyaning $P_0(x_0,y_0)$nuqtadagi limiti deyiladi. A sonining $z = f(x,y)$ funksiyaning $P(x,y) \to P(x_0,y_0)$ dagi limiti bo‘lishi quyidagicha yoziladi

$$\lim_{p \to p_0} y = A \text{ yoki } \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A$$

Uch va undan ortiq o‘zgaruvchining limiti ta’rifi shunga o‘xshash kiritiladi.

Agar bir necha o‘zgaruvchi funksiyasining limiti nolga teng bo‘lsa, u holda u cheksiz kichik deb ataladi. Bir o‘zgaruvchining funksiyasi uchun limitlar haqidagi barcha asosiy teoremalar bir necha o‘zgaruvchining funksiyasi uchun ham o‘rinlidir.

4-misol. Quyidagi limitni hisoblang.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\ln(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}$$

Yechish. x va y nuqtalar orasidagi masofa $p=\sqrt{x^2+y^2}$ ($p=\sqrt{x^2+y^2}$ deb belgilash kiritamiz). $x\to 0$, $y\to 0$ dan $\rho\to 0$ ekanligi kelib chiqadi.

Demak, $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\ln(1-x^2-y^2)}{\sqrt{x^2+y^2}}=\lim\limits_{p\to 0}\dfrac{\ln(1-p^2)}{p}=\lim\limits_{p\to 0}\dfrac{(\ln(1-p^2))'}{p'}=\lim\limits_{p\to 0}\dfrac{\frac{1}{1-p^2}\cdot(-2p)}{1}=0$

Limitga ega bo‘lgan ikki o‘zgaruvchili funksiyalarning bir necha xossalarini keltirib o‘tamiz:

1) Agar $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$ limit mavjud va chekli bo‘lsa, u holda $f\,(x, y)$ funksiya (x_0,y_0) nuqtaning yetarlicha kichik atrofda chegalangan bo‘ladi.

2) Agar $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} g(x,y)=B$ limitlar mavjud bo‘lsa, u holda $f(x,\ y)\pm g(x,\ y)$ funksiyaning ham limiti mavjud va $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}[f(x,y)\pm g(x,y)]=A\pm B$. bo‘ladi.

3) Agar $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} g(x,y)=B$ limitlar mavjud bo‘lsa, u holda $f(x,\ y)\cdot g(x,\ y)$ funksiyaning ham limiti mavjud va $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}[f(x,y)\cdot g(x,y)]=A\cdot B$ bo‘ladi.

4) Agar $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} g(x,y)=B$ limitlar mavjud bo‘lib, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} g(x,y)\neq 0$ bo‘lsa, u holda $\dfrac{f(x,y)}{g(x,y)}$ funksiya ham limtga ega va $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}\dfrac{f(x,y)}{g(x,y)}=\dfrac{A}{B}$ bo‘ladi.

Ikki o‘zgaruvchili funksiyaning limitini hisoblash bir o‘zgaruvchili funksiyani limitini hisoblashga qaraganda ancha murakkab. Buni sababi to‘g‘ri chiziqda faqat ikkita yo‘nalish bor, argument limit nuqtaga faqat ikki tarafdan o‘ng va chapdan intiladi. Tekislikda bunday yo‘nalishlar cheksiz ko‘p va funksiyaning limiti turli yo‘nalishlar bo‘yicha usma – ust tushmasligi mumkin.

5-misol $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{2xy}{x^2+y^2}$ limit mavjud emasligini isbotlang.

6-misol. Limitlarni hisoblang

$a)\lim\limits_{\substack{x\to 0\\ y\to a}}\dfrac{\sin xy}{y},\quad b)\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\left(x^2+y^2\right)e^{-(x+y)},\quad c)\lim\limits_{\substack{x\to\infty\\ y\to a}}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}},\quad d)\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\sin\dfrac{\pi x}{2x+y},$

$e)\ \lim\limits_{\substack{x\to 1\\ y\to 0}}\dfrac{\ln(x+e^{-3x})}{\sqrt{x^2+y^2}}$.

Yechish.

$a)\ \lim\limits_{\substack{x\to 0\\ y\to a}}\dfrac{\sin xy}{x}=\lim\limits_{\substack{x\to 0\\ y\to a}}\dfrac{xy}{x}=\lim\limits_{y\to a}y=a,\quad b)\ \lim\limits_{\substack{x\to\infty\\ y\to\infty}}\left(x^2+y^2\right)e^{-(x+y)}=\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{\left(x^2+y^2\right)}{e^{(x+y)}}=0,$

$c)\ \lim\limits_{\substack{x\to\infty\\ y\to a}}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}=\lim\limits_{\substack{x\to\infty\\ y\to a}}e^{\frac{x}{x+y}}=\lim\limits_{y\to a}e^1=e,\qquad d)\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\sin\dfrac{\pi x}{2x+y}=\lim\limits_{y\to\infty}\dfrac{\pi}{2}=1,$

$e)\ \lim\limits_{\substack{x\to 1\\ y\to 0}}\dfrac{\ln(x+e^{-3x})}{\sqrt{x^2+y^2}}=\dfrac{\ln(1+e^0)}{\sqrt{1+0}}=\ln 2$.

7-misol. Limitlarni hisoblang

a. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(xy\sqrt{1+xy})$

b. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2y^3}{x^3+y^3}$

c. $\lim\limits_{\substack{x\to 1\\ y\to -1}}\dfrac{\sin(x+y)}{x+y}$

d. $\lim\limits_{\substack{x\to 0\\ y\to 0}}[xy\cdot\ln xy]$

e. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x+2y)e^{\frac{1}{x}}$

f. $\lim\limits_{\substack{x\to 1\\ y\to 1}}\dfrac{x-y}{x^3-y^2}$

g. $\lim\limits_{\substack{x\to\infty\\ y\to +0}}\dfrac{x^y}{1+x^y}$

h. $\lim\limits_{\substack{x\to 0\\ y\to\infty}}\dfrac{1}{xy}tg\dfrac{xy}{1+xy}$

i. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{x^2+y^2}{x^2+y^4}$

j. $\lim\limits_{\substack{x\to 1\\ y\to 0}}\log_x(x+y)$

Funksiyaning uzluksizligi

$z = f(x, y)$ funksiya M to‘plamda berilgan bo‘lib (x_0, y_0) nuqta shu M to‘plamning limit nuqtasi bo‘lsin.

Agar

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0) \qquad (24.1)$$

bo‘lsa, u holda $f(x, y)$ funksiya (x_0, y_0) nuqtada uzluksiz deb ataladi.

Agar ixtiyoriy $E>0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $d\{(x, y),(x_0, y_0)\}< \delta$ tengsizlikni qanoatlantiruvchi barcha $(x, y)\in M$ nuqtalar uchun $|f(x,y)-f(x_0,y_0|<E$ bo‘lsa, u holda $f(x, y)$ funksiya (x_0, y_0) nuqtada uzluksiz deb ataladi.

Agar f (x, y) funksiya M to‘plamanng har bir nuqtasida uzluksiz bo‘lsa, u holda funksiya shu M to‘plamda uzluksiz deyiladi.

Agar argument orttirmalari Δx va Δy nolga intilganda funksiyaning to‘liq orttirmasiΔf (x_0, y_0) = $f\left(x_0+\Delta x, y_0+\Delta y\right)-f\left(x_0, y_0\right)$ ham nolga intilsa, ya’ni $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(\Delta f(x_0, y_0)=0$ bo‘lsa, u holda f (x, y) funksiya (x_0, y_0) nuqtada uzluksiz deb ataladi.

8-misol. $f(x, y) = x^2 + y^2$ funksiyaning uzluksizligini tekshiring.

Yechish. $(x_0; y_0)$ nuqtani hamda $(x_0 + \Delta x; y_0 + \Delta y)$ ni olib, funksiyaning to‘liq orttirmasini hisoblaymiz :

$\Delta f(x_0, y_0) = f(x_0 + \Delta x_0; y_0 + \Delta y_0) - f(x_0, y_0) = (x_0 + \Delta x_0)^2 + (y_0 + \Delta y_0)^2 - x_0^2 - y_0^2 =$

$= 2\Delta x\cdot x + \Delta x^2 + 2\Delta y\cdot y + \Delta y^2 = \Delta x(2x + \Delta x) + \Delta y(2y + \Delta y)$.

bundan esa $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(\Delta f(x_0, y_0)=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(\Delta x(2x+\Delta x)+\Delta y(2y+\Delta y)=0$ ekanligini topamiz. Yuqoridagi tariflarga ko‘ra f (x, y) funksiya (x_0, y_0) nuqtada uzluksiz.

Ikki o‘zgaruvchili uzluksiz funksiyalarning bazi xossalari:

f (x, y) va $g(x, y)$ funksiyalar M to‘plamda berilgan bo‘lib, $(x_0; y_0)\in M$ bo‘lsin:

1)Agar f (x, y) va $g(x, y)$ funksiyalarning har biri $(x_0; y_0)$ nuqtada uzluksiz bo‘lsa, u holda $f(x, y) \pm g(x, y)$ funksiya ham shu $(x_0; y_0)$ nuqtada uzluksiz bo‘ladi.

2)Agar f (x, y) va $g(x, y)$ funksiyalarning har biri $(x_0; y_0)$ nuqtada uzluksiz bo‘lsa, u holda $f(x)\cdot g(x)$ funksiya ham $(x_0; y_0)$ nuqtada uzluksiz bo‘ladi.

3)Agar f (x, y) va $g(x, y)$ funksiyalarning har biri $(x_0; y_0)$ nuqtada uzluksiz bo‘lib, $g(x_0; y_0)\neq 0$ bo‘lsa, u holda $\frac{f(x)}{g(x)}$ funksiya ham shu $(x_0;y_0)$ nuqtada uzluksiz bo‘ladi.

4)Agar f (x, y) funksiya chegaralangan yopiq M to‘plamda uzluksiz bo‘lsa, u holda funksiya shu to‘plamda chegaralangan bo‘ladi.

9-misol. Funksiyani $M_0(1; 1)$ nuqtada uzluksizlikka tekshiring. $z = x^5y + 10x^2\sqrt{y}$.

10-misol. Funksiyani (0; 0) nuqtada uzlusizlikka tekshiring. $z=\frac{x+y}{x-y}$.

Xususiy hosilalar

$z = f(x, y)$ funksiya M to‘plamda berilgan bo‘lsin. Bu M to‘plamda $(x_0; y_0)$ va $(x_0+\Delta x; y_0)$ nuqtalarni olib, bu nuqtalardagi funksiya qiymatlari ayirmasini hisoblaymiz:
$f(x_0+\Delta x; y_0) - f(x_0; y_0)$

Bu ayirma $f(x, y)$ funksiyaning $(x_0; y_0)$ nuqtadagi x o‘zgaruvchi bo‘yicha xususiy orttirmasi deyiladi va $\Delta_x f(x_0; y_0)$ kabi belgilanadi: $\Delta_x f(x_0; y_0) = f(x_0+\Delta x; y_0) - f(x_0; y_0)$.

Xuddi shunga o‘xshash $\Delta_y f(x_0; y_0) = f(x_0, y_0+\Delta y) - f(x_0; y_0)$ ayirma $f(x, y)$ funksiyaning $(x_0; y_0)$ nuqtadagi y argument bo‘yicha xususiy orttirmasi deyiladi.

Agar $\Delta x\to 0$da

$$\frac{\Delta_x f(x_0, y_0)}{\Delta x}$$

nisbatning limiti mavjud va chekli bo‘lsa, bu limit $f(x, y)$ funksiyaning $(x_0; y_0)$ nuqtadagi x argument bo‘yicha xususiy hosilasi deb ataladi va $\partial f(x_0; y_0)/(\partial x)$ yoki $f'_x(x_0; y_0)$ qisqacha qilib $(\partial f)/(\partial x)$ yoki $f'x$ kabi belgilanadi

$$\frac{\partial f(x_0, y_0)}{\partial x}=f'_x(x_0, y_0)=\lim_{\Delta x\to 0}\frac{\Delta_x f(x_0, y_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0)-f(x_0, y_0)}{\Delta x}$$

Xuddi shunga o‘xshash $\Delta y\to 0$ da

$$\frac{\Delta_y f(x_0; y_0)}{\Delta y}$$

nisbatning limiti mavjud va chekli bo‘lsa, bu limit $f(x, y)$ funksiyaning $(x_0; y_0)$ nuqtasidagi y argument bo‘yicha xususiy hosilasi deb ataladi va qisqacha qilib $(\partial f)/(\partial y)$ yoki f_y kabi belgilanadi

$$\frac{\partial f(x_0, y_0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x_0, y_0+\Delta y)-f(x_0, y_0)}{\Delta y} \qquad (24.2)$$

11-misol. Funksiyaning xususiy hosilalarini hisoblang $f(x, y)=x^y$

Yechish. $(\partial f)/(\partial x)=\left(x^y\right)'_x=y\cdot x^{y-1}$, $(\partial f)/(\partial y)=\left(x^y\right)'_y=x^y\ln x$.

12-misol. Funksiyalarning $\frac{\partial z}{\partial x}$ *va* $\frac{\partial z}{\partial y}$ xususiy hosilalarini toping. $z = x^2 + 3x\sqrt{y} - y + \frac{y^2}{x}.$

13-misol. $z = xy \cdot e^{x^2 - y^2}$.

14-misol. Funksiyaning birinchi tartibli xususiy hosilalarini hisoblang.

a) $u = 2x^2y + 3x^3y^2 + xyz^5$; *b*) $u = z^{xy}$; *c*) $u = x^y + y^x$.

15-misol. Funksiyalarning xususiy hosilalarini toping.

1. $z = 2x^2 - xy^2 + 3x^2y - 2y^3 + 3x - 4y + 1$ 2. $u = yx^3 + xz^2 + y^2z$

3. $u = s^3\cos 4t$ 4. $z = \frac{x^2}{y} + \frac{y}{x^2}$

5. $z = \ln\left(x^2 + y^2\right)$ 6. $z = \frac{xy}{x+y}$

7. $u = e^{xyz}\left(x^2 + y^2 + z^2\right)$ 8. $u = e^{\frac{x}{y}} + e^{-\frac{z}{y}}$

9. $z = x\sqrt{y} + \frac{y}{\sqrt[3]{x}}$ 10. $z = xye^{x+2y}$

11. $z = \ln(x + \ln y)$ 12. $z = e^{3x^2 + 2y^2 - xy}$

13. $u = e^{\frac{z}{xy}} tg\left(\frac{x}{y+z}\right)$ 14. $z = \arcsin\sqrt{xy}$

15. $z = e^{x-y}(2x - 1)$ 16. $z = \sin\left(x + \sqrt{y}\right)$

17. $z = xe^y + x^y$ 18. $z = \ln\sqrt{x + y^2}$

19. $z = \ln\left(\sqrt{x} + \sqrt{y}\right)$ 20. $z = x^{\sqrt{y}}$

Ko‘p o‘zgaruvchili funksiya differensiali

$z = f(x, y)$ funksiya M to‘plamda berilgan bo‘lsin. Bu M to‘plamda $(x_0; y_0)$ va $(x_0+\Delta x;\ y_0+\Delta y)$ nuqtalarni olib, funksiyaning to‘liq orttirmasini aniqlaymiz:

$$\Delta f(x_0, y_0) = f(x_0+\Delta x_0; y_0 + \Delta y_0) - f(x_0, y_0) \qquad (24.3)$$

Agar $f(x, y)$ funksiyaning (x_0, y_0) nuqtadagi orttirmasi

$$\Delta f(x_0; y_0) = A\Delta x + B\Delta y + \alpha\Delta x + \beta\Delta y \qquad (24.4)$$

ko‘rinishda ifodalansa, u holda funksiya (x_0, y_0) nuqtada differensiallanuvchi deb ataladi, bunda A, B - o‘zgarmaslar, α va β esa Δx va Δy ga bog‘liq, hamda $\Delta x \to 0$, $\Delta y \to 0$ da $\alpha \to 0, \beta \to 0$ bo‘ladi.

Yoqoridagi (24.4) formulada $A = f'_x(x_0;\ y_0),\qquad B = f'_y(x_0;\ y_0)$

$A = f_x(x_0; y_0)$, $B = f_y(x_0; y_0)$.

Agar $f\ (x_0; y_0)$ funksiya $(x_0; y_0)$ nuqtada $f_x(x_0; y_0)$, $f_y(x_0; y_0)$ xususiy hosilalarga ega bo‘lib, bu xususiy hosilalar $(x_0; y_0)$ nuqtadada uzluksiz bo‘lsa, u holda $f\ (x, y)$ funksiya $(x_0; y_0)$ nuqtada differensiallanuvchi bo‘ladi. $Z = f\ (x; y)$ funksiya $(x_0; y_0)$ nuqtada differensiallanuvchi bo‘lsa

$$\Delta f\,(x_0, y_0) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y + \alpha\cdot\Delta x + \beta\cdot\Delta y \qquad (24.5)$$

tenglik o‘rinli bo‘ladi. Bu ifodadagi $\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y$ yig‘indi $f\ (x; y)$ funksiyaning $(x_0; y_0)$ nuqtadagi differensiali deb ataladi va y $df\ (x_0; y_0)$ yoki dz kabi belgilanadi:

$$df\,(x_0; y_0) = dz = \frac{\partial f\,(x_0, y_0)}{\partial x}\Delta x + \frac{\partial f\,(x_0, y_0)}{\partial y}\cdot\Delta y \qquad (24.6).$$

Agar $\Delta x = dx$, $\Delta y = dy$ bo‘lishini e`tiborga olsak, u holda

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\cdot dy \qquad (24.7)$$

16-misol. $z = x^2y - xy^2$ funksiyaning differensialini toping.

Yechish. $\frac{\partial z}{\partial x} = \left(x^2y - xy^2\right)'_x = 2xy - y^2$, $\quad \frac{\partial z}{\partial y} = \left(x^2y - xy^2\right)'_y = x^2 - 2xy$.

(24.7) formulaga ko‘ra $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$

17-misol. Funksiyaning differensialini toping.

$a)$ $z = e^{x^2+y^2}\cdot\sin\left(x^2y^2\right)$ $\qquad b)$ $u = arctg\left(xyz\right)$

Mavzu yuzasidan savollar.

1. Ko‘p o‘zgaruvchili funksiya deb nimaga aytiladi?

2. Ko‘p o‘zgaruvchili funksiya limiti, uzluksizligi haqida gapiring.

3. Ko‘p o‘zgaruvchi funksiyaning birinchi tartibli xususiy hosilasi haqida gapiring.

4. Ko‘p o‘zgaruvchi funksiyaning to‘la differensiali haqida gapiring.

5. Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli hosilalari qanday hisoblanadi?

25-amaliy mashg‘ulot. Ko‘p o‘zgaruvchini funksiyaning yuqori tartibli hosila va differensiali. Ko‘p o‘zgaruvchili funksiya ekstremumi.

$z = f(x,\ y)$ funksiya M to‘plamda berilgan va u $(x_0;\ y_0)$ nuqtada differensiallanuvchi bo‘lsin.

$z = f(x,\ y)$ funksiyaning xususiy xosilalari $f'_x,\ f'_y$ dan olingan $\frac{\partial f'_x}{\partial x},\ \frac{\partial f'_x}{\partial y}\ \frac{\partial f'_y}{\partial x},\ \frac{\partial f'_y}{\partial y}$ xususiy hosilalarga berilgan funksiyaning ikkinchi tartibli xususiy hosilasi deyiladi va

$f''_{x^2},\ f''_{xy},\ f''_{y^2},\ f''_{y^2}$ yoki $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y\,\partial y},\ \frac{\partial^2 f}{\partial y^2}$

kabi belgilanadi.

Demak: $f''_{x^2} = \frac{\partial^2 f}{\partial x^2} = (f'_x(x,y))'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right);\ f''_{xy} = \frac{\partial^2 f}{\partial x \cdot \partial y} = (f'_x(x,y))'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$f''_{yx} = \frac{\partial^2 f}{\partial y \cdot \partial x} = (f'_y(x,y))'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right);\ f''_{y^2} = \frac{\partial^2 f}{\partial y^2} = (f'_y(x,y))'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

Yuqorida ko‘rsatib o‘tilgan $\frac{\partial^2 f}{\partial x \partial y}$ va $\frac{\partial^2 f}{\partial y \partial x}$ xususiy hosilalar aralash hosilalar deb ataladi. Bu aralash hosilalar ($x;y$) nuqtada uzluksiz bo‘lsa, bir-biriga teng bo‘ladi.

Xuddu shuningdek $z=f(x,\ y)$ funksiyaning uchinchi, to‘rtinchi va hokazo tartibli xususiy hosilalar aniqlanadi.

1-misol. $f\ (x,\ y) = x^3+y^3$ funksiyaning ikkinchi tartibli xususiy hosilalari hisoblansin

Yechish.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^3\right) = 3x^2,\quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^3\right) = 3y^2$$

Demak,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2) = 6x,\quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3y^2) = 6y,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2) = 0.$$

2-misol. Ikki o‘zgaruvchili $z = \ln(1 + x + 2y)$ funksiyaning xususiy hosilalarini hisoblang

3-misol. Funksiyaning ikkinchi tartibli xususiy hosilasilalarini hisoblang $y = xy \cdot \ln\frac{x}{y}$

4-misol. Agar $z = \sin x \cdot tgy$ bo‘lsa, u holda $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ ekanligini isbotlang.

5-misol. Berilgan funksiyalarning ikkinchi tartibli hosilalarini hisoblang

a. $z = 3x^2 + 2xy^2 - 4xy + x^2 y - y^3$ **b.** $u = e^{xyz}$

c. $u = \sin\left(\frac{xy}{2}\right)$ **d.** $z = \arcsin(x + y)$

e. $z = \ln tg\frac{x + y}{x - y}$ **f.** $z = x \sin xy + y \cos xy$

g. $z = x^2 \ln(x + y)$ **h.** $z = x^2 \sin\sqrt{y}$

Ko‘p o‘zgaruvchili funksiyalarning lokal ekstremumlari

Agar $z = f(x, y)$ funksiyaning $P=(x_0;\ y_0)$ nuqtadagi qiymati uning shu nuqtaning biror atrofiga tegishli ixtiyoriy $P(x, y)$ qiymatidan katta (kichik) bo‘lsa $P_0(x_0; y_0)$ nuqta maksimum (minimum) nuqta deyiladi.

Lokal maksimum va minimum nuqtalar lokal ekstremum nuqtalar deyiladi. Bunda $z = f(x, y)$ funksiya $P_0(\ x_0;\ y_0)$ nuqtada lokal ekstremumga erishadi.

Teorema: Agar $z = f(x, y)$ differensiallanuvchi funksiya $P_0(x_0,\ y_0)$ nuqtada lokal ekstremumga erishsa, u holda uning bu nuqtadagi birinchi tartibli xususiy hosilalari nolga teng bo‘ladi:

$$\frac{\partial z(P_0)}{\partial x} = 0; \quad \frac{\partial z(P_0)}{\partial y} = 0.$$

Birinchi tartibli xususiy hosilalari nolga teng (yoki mavjud bo‘lmagan nuqtalar), kiritik nuqtalar deyiladi. Ularni ekstremumga tekshirish, ikki o‘zgaruvchili funksiya ekstremumi mavjudligining yetarlilik shartlari yordamida tekshiriladi.

$P_0(x_0,\ y_0)$ nuqta $z = f(x, y)$ funksiyaning statsionar nuqtasi bo‘lsin. $P_0(x_0,\ y_0)$ nuqtadagi ikkinchi tartibli xususiy hosilalari

$\frac{\partial^2 z(P_0)}{\partial x^2} = A$ $\frac{\partial^2 z(P_0)}{\partial x \cdot \partial y} = B$ $\frac{\partial^2 z(P_0)}{\partial y^2} = C$ uchun $\Delta = \begin{vmatrix} A, B \\ B, C \end{vmatrix} = AC - B^2$ determinant tekshiriladi. $z = f(x, y)$ funksiyaning $P_0(x_0, y_0)$ statsionar nuqtada ekstremumining yetarlilik sharti quyidagicha ifodalanadi:

1) $\Delta > 0$ - ekstremum mavjud bo‘lib, bunda agar $A > 0$ (yoki $A = 0$ da $C > 0$) bo‘lsa $P_0(x_0, y_0)$ nuqtada funksiya lokal minimumga, agar $A<0$ (yoki $A=0$ da $C<0$) bo‘lsa lokal maksimumga ega bo‘ladi.

2) $\Delta < 0$ - lokal ekstremum yo‘q;

3) $\Delta = 0$ - qo‘shimcha tekshirishlarni talab qiladi.

6-misol. Berilgan $u = xy + \frac{50}{x} + \frac{20}{y}, \ x \neq 0, \ y \neq 0$ funksiyaning ekstremumlarini toping.

Yechish. Kritik nuqtani topish uchun birinchi tartibli xususiy hosilalarini nolga tenglashtiramiz:

$$\frac{\partial u}{\partial x} = y - \frac{50}{x^2} = 0, \ \frac{\partial u}{\partial y} = x - \frac{20}{y^2} = 0.$$

Bu sistemani yechib $x^2 y = 50, \ xy^2 = 20$, $x = 5$; $y = 2$ ekanligini topamiz. Shunday qilib $M_0(5, 2)$ kritik nuqta.

Ikkinchi tartibli hosilalarni va ularni $M_0(5, 2)$ nuqtadagi qiymatlarini aniqlaymiz: $u''_{xx} = \frac{100}{x^3}, \ u''_{xx}(5, 2) = \frac{100}{125} = \frac{4}{5}; \ u''_{xy} = 1, \ u''_{xy}(5, 2) = 1$; $u''_{yy} = \frac{40}{y^3}$,

$$u''_{yy}(5, 2) = \frac{40}{8} = 5$$

$A = \frac{4}{5} > 0$ va $\Delta = AC - B^2 = \frac{4}{5} \cdot 5 - 1^2 = 3 > 0$ dan funksiya $M_0(5, 2)$ nuqtada minimumga ega. $u_{\min} = u(5, 2) = 5 \cdot 2 + \frac{50}{5} + \frac{20}{2} = 30.$

Mavzu yuzasidan savollar.

1. *Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli hosilalari qanday hisoblanadi?*
2. *Ikki o‘zgaruvchili funksiyaning ekstremumlari qanday hisoblanadi?*
3. *Ko‘p o‘zgaruvchili funksiya ekstremumlarining zarurriy va yetarli shartlari.*
4. *Ikki o‘zgaruvchili funksiyaning eng katta va eng kichik qiymatlari qanday topiladi?*

5. Ikki o‘zgaruvchili funksiyaning shartli ekstremumlari qanday topiladi?

26-amaliy mashg‘ulot. Ko‘p o‘zgaruvchili funksiya ekstremumi.

Shartli ekstremumlar

$z = f(x, y)$ funksiyaning shartli ekstremumi deb bu funksiyaning x va y o‘zgaruvchilarning bog‘lash tenglamasi deb ataluvchi $\varphi(x, y) = 0$ tenglama bilan bog‘langanlik shartida erishadigan ekstremumga aytiladi.

Nuqtalarni bog‘lovchi tenglamalar sistemasi: $G=\{(x,y)|Y_i(x,y)i = 1,2,\dots m\}$ ni qanoatlantiradigan G sohada aniqlangan va differensialanuvchi $z = f(x, y)$ funksiyani qaraymiz. Bu sohada shunday $M_0(x_0; y_0)$ nuqtani topish kerakki $\forall$ M(x,y) $\in$ G uchun $f(M_0) \geq$ f(M) shart bajarilishi kerak. Bunday masalalar $z = f(x, y)$ funksiyanig shartli eksremumini topish masalasi deyiladi.

Shartli ekstremumni topish uchun Lagranjning noma’lum koeffitsentlar usulini keltiramiz

$L(x,y,\lambda_i) = f(x,y) + \sum_{i=1}^{m} \lambda_i Y_i(x,y)$

Lagranj funksiyasi ekstremumini zaruruiy shari quyidagi ko‘rinishga ega:

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$$

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \sum_{i=1}^{m} \frac{\partial Y_i}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \sum_{i=1}^{m} \frac{\partial Y_i}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda_i} = Y_i(x,y) = 0 \end{cases} \qquad \begin{cases} L_x = 0 \\ L_y = 0 \\ \varphi(x) = 0 \end{cases}$$

i=1,2,….m

Bu m+2 noma’lumli tenglamadan iborat sistemadan $x,y,\lambda_i (i=1,2,\dots m)$ noma’lumlarning topiladi. λ_i sonlar Lagranj koeffitsienlari deyiladi.

1-misol. $z = xy$ funksiyaning x va y lar $2x+y-3=0$ tenglama bilan bog‘langanlik sharti ostidagi ekstremumini toping.

2-misol. Funksiyani shartli ekstremumlarini toping.

a. $z = x^2 – xy + y^2 + 9x – 6y + 20$ **b.** $z = xy^2 – xy + xy^3 (x > 0;\ y > 0)$
c. $z = 3x^2 – x^3 + 3y^2 + 4y$ **d.** $z = y\sqrt{x} - y^2 - x + 6y$

e. $z = e^{\frac{x}{2}}(x + y^2)$. **f.** $z = 4 - \sqrt[3]{x^2 + y^2}$

3-**misol.** Birinchi tartibli xususiy hosilalar topilsin.
a) $z = 5x^2 – 13xy + 6y^2$; *b*) $z = 4x^3 + 2x^2y^5 – 8y^3$;
c) $z = 7w^3 + 6wx + 4x^2 – 8xy - 3y^2$; *d*) $z = 2w^4 + 7wxy – 3x^2 + 4y^3$.
4-misol. *a*) $z = (5x + 3y)(12x – 7y)$; *b*) $z = (4x^2 – 3y)(2x + 9y^3)$;
c) $z = (4w - 3x + 7y)(7w^2 + 11x^4 – 3y^5)$; *d*) $z = 13x/ (9x – 4y)$.
5-misol. *a*) $z = \dfrac{4w + 7x + 2y}{3w - 2x + 3y}$ *b*) $z = (5x – 7y)^3$;
c) $z = 4 e^{5xy}$; *d*) $z = \ln|4x + 7y|$
6-misol. *a*) $z = \dfrac{(8x + 7y)^4}{2x - 3y}$; *b*) $z = \dfrac{(3x + 4y)(7x - 8y)}{5x - 2y}$;
c) $z = (12x-5y)^3(6x-7y)$; *d*) $z = (2x+11y)/ (5x+4y)$;

7-misol. *a*) $z = 7x^3e^{5xy}$; *b*) $z = 6xy/e^{5x+2}$;

12-misol. Funksiyaning ekstrimumlarini aniqlang
a. $f(x, y) = x^2 + 3xy + 2{,}5y^2 – 5x – 6y + 1{,}5$
b. $f(x, y) = x^2 + 3xy + 2{,}5y^2 – 5x – 8y + 3{,}5$
c. $f(x, y) = -x^2 + 2xy – 1{,}5y^2 + 0{,}5y + 5$
d. $f(x, y) = -x^2 + 2xy – 1{,}5y^2 – 2x + 5y + 0{,}5$
e. $f(x, y) = 3x^2 + 2xy + 0{,}25y^2 – 10x + 3y + 4$
f. $f(x, y) = -2x^2 + 2xy – 1{,}5y^2 + 75x – 12{,}5y + 9375$.
13-misol. Ikkinchi tartibli xususiy hosilalarni toping
a) $z = x^2y^3 + 5x$; *b*) $z = (3xy – 4y^2)(x - 4)$;
c) $z = 5x/y + 3y/x$; *d*) $z = (x + y)/(x – y)$;
e) $z = \ln[(x^2 – y^2)/(x^2 + y^2)]$; *f*) $z = e^{x^2 - 2xy + y^2}$
14-misol. Ekstremal nuqtalarni aniqlang
a) $z = 60x + 34y – 4xy – 6x^2 – 3y^2 + 30$; *b*) $z = 6x^3 + 6y^3 – 12xy$;
c) $z = 5x^3 + 3x^2 + 6xy – 2y^2 – 2.5$;
d) $z = -6x^2 + 8xy – 2y^2 + 5x + 3y + 17$.

Mavzu yuzasidan savollar.

1. *Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli hosilalari qanday hisoblanadi?*

2. Ikki o‘zgaruvchili funksiyaning ekstremumlari qanday hisoblanadi?

3. Ko‘p o‘zgaruvchili funksiya ekstremumlarining zarurriy va yetarli shartlari.

4. Ikki o‘zgaruvchili funksiyaning eng katta va eng kichik qiymatlari qanday topiladi?

5. Ikki o‘zgaruvchili funksiyaning shartli ekstremumlari qanday topiladi?

27-amaliy mashg‘ulot. Boshlang‘ich funksiya. Aniqmas integral va uning xossalari. Aniqmas integralda bevosita, o‘zgaruvchilarni almashtirish va bo‘laklab integrallash usullari. Asosiy integrallash jadvali.

Berilgan f(x) funksiya (a,b) intervalda aniqlangan bo‘lsin. Agar *F (x)=f(x)* (bunda x$\in (a,b)$) tenglik o‘rinli bo‘lsa, *F(x)* funksiya *f(x)* funksiyaning *(a,b)* intervaldagi boshlang‘ichi deyiladi. Berilgan *f(x)*funksiyaning ixtiyoriy ikkita boshlang‘ich funksiyasi bir-biridan o‘zgarmas songa farq qiladi.

f(x) funksiyaning *F(x)+c* (bunda *c*- o‘zgarmas son) boshlang‘ich funksiyalar to‘plami *f(x)* funksiyaning aniqmas integrali deyiladi va $\int f(x)dx = F(x) + C$ ko‘rinishida ifodalanadi

Aniqmas integral xossalari

1. $(\int f(x)dx)' = f(x), \int f'(x)dx = f(x) + C$, bu yerda *C* – ixtiyoriy o‘zgarmas son.

2. $\int Af(x)dx = A\int f(x)dx$, bu yerda A – o‘zgarmas son.

3. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

4. Agar $\int f(x)dx = F(x) + C$ va x = $y(t)$ differensiallanuvchi funksiya bo‘lsa, u holda $\int f(y(t))dy(t) = F(y(t)) + C$.

Xususan, $\int f(at+b)dt = \frac{1}{a}F(at+b) + C, (a \neq 0)$.

Integrallashning asosiy usullari

1. **Bevosita integrallash.** Bunda integral ostidagi ifoda elementar almashtirishlar bilan jadvalga keltiriladi. So‘ngra integral xossalaridan foydalanib, boshlang‘ich funksiya topiladi.

1-misol. Integralni hisoblang: $I_1 = \int \frac{42ax\sqrt{x} - 5bx^2 + 14x + 20}{x^2}dx$.

Yechish. Darajaning va aniqmas integralning xossalaridan foydalanib:

$$I_1 = 42a\int x^{-\frac{1}{2}}dx - 5b\int dx + 14\int x^{-1}dx + 20\int x^{-2}dx = 84a\sqrt{x} - 5bx + 14\ln|x| - \frac{20}{x} + C.$$

2-misol. Integralni hisoblang:

a. $I_2 = \int \frac{(1+x)^2}{x(1+x^2)}dx$.**b.** $I_3 = \int (a^m b^n)^x dx = \frac{(a^m b^n)^x}{\ln(a^m b^n)} + C$

c. $I = \int \frac{\cos 2x}{\sin^2 x \cos^2 x}dx.$ **d** $I = \int \frac{x^4}{x^2+1}dx.$

e. $\int \frac{x+1}{\sqrt{x^3}}dx$.

2.O‘rniga qo‘yish usuli. Erkli o‘zgaruvchi x ni boshqa ixtiyoriy x ga bog‘liq differensiallanuvchi funksiya bilan almashtirish mumkin.
Integrallarni hisoblashda quyidagi qoidalarni hisobga olish foydalidir:

1) Agar $\int f(x)dx = F(x) + C,$ u holda $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.

Masalan:

$a)$ $\int \sin x dx = -\cos x + C,$ bo‘lgani uchun $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C;$

$b)$ $\int \frac{dx}{x} = \ln|x| + C,$ bo‘lgani uchun $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$.

2) Agar integral ostidagi ifodani $f(x)\cdot f'(x)$ yoki $f'(x) : f(x)$ ko‘rinishida ifodalash mumkin bo‘lsa, u holda $f'(x)dx = df(x)$ekanligidan quyidagilar kelib chiqadi:

$$\int f(x)f'(x)dx = \int f(x)df(x) = \frac{1}{2}f^2(x) + C \ ; \int \frac{f'(x)}{f(x)}dx = \int \frac{df(x)}{f(x)} = \ln|f(x)| + C$$

3) $\int [f'(x)\varphi(x) + f(x)\varphi'(x)]' dx = \int (f\cdot\varphi)' dx = \int d(f\cdot\varphi) = f(x)\cdot\varphi(x) + C$.

4) $\int x^3 f(x^2)dx = \frac{1}{2}\int x^2 f(x^2)dx^2 = \frac{1}{2}\int t\cdot f(t)dt, \quad bunda \ t = x^2.$

5) $\int \frac{f'(x)dx}{a^2 + f^2(x)} = \int \frac{df(x)}{a^2 + f^2(x)} = \frac{1}{a}arctg\frac{f(x)}{a} + C.$

6) $\int \frac{f'(x)dx}{a^2 - f^2(x)} = \frac{1}{2a}\ln\left|\frac{a+f(x)}{a-f(x)}\right| + C.$

7) $\int \frac{f'(x)dx}{\sqrt{a^2 - f^2(x)}} = \int \frac{df(x)}{\sqrt{a^2 - f^2(x)}} = \arcsin\frac{f(x)}{a} + C.$

8) $\int \frac{f'(x)dx}{\sqrt{f^2(x) \pm a^2}} = \int \frac{df(x)}{\sqrt{f^2(x) \pm a^2}} = \ln\left| f(x) - \sqrt{f^2(x) \pm a^2}\right| + C.$

3-misol. Integralni hisoblang:

$a) \int \frac{arctgx}{1+x^2}dx \qquad b) \int \frac{a\sin x - b\sin x}{a\sin x + b\sin x}dx \quad c) \int \frac{\sin x}{\cos^7 x}dx$

Yechish. $a) \int \frac{arctgx}{1+x^2}dx = \int (arctgx)' arctgx\, dx = \int arctgx\, d(arctgx) = \frac{1}{2}arctg^2 x + C$

$b) \int \frac{a\cos x - b\sin x}{a\sin x + b\cos x}dx = \int \frac{(a\cos x + b\sin x)'}{a\sin x + b\cos x}dx = \int \frac{d(a\cos x + b\sin x)}{a\sin x + b\cos x} = \ln|a\sin x + b\cos x| + C.$

$c) \int \frac{\sin x}{\cos^7 x}dx = -\int \frac{(\cos x)'}{\cos^7 x}dx = -\int \frac{d\cos x}{\cos^7 x} = \frac{1}{6\cos^6 x} + C.$

4-misol. Integralni hisoblang:

a. $\int \left(\frac{\ln(1+x)}{1+x^2} + \frac{arctgx}{1+x}\right)dx$ **b.** $I = \int \cos 9x \cdot dx$.

c. $I = \int e^{-x^2} xdx.$ **d.** $I = \int \frac{xdx}{\sqrt{10 - x^2}}.$

5-misol. Integralni hisoblang:

$a) I = \int \frac{dx}{ax+b} \qquad b) I = \int \frac{dx}{\left(x + \sqrt{x^2 - 1}\right)^2}.$

3. O‘zgaruvchilarni almashtirish usuli. Agar aniqmas integral jadval ko‘rinishida bo‘lmasa u holda ba’zan o‘rniga qo‘yish usuliga murojat qilinadi. $\int f(x)dx$ ni topish kerak bo‘lsa $x = \varphi(t)$ almashtirish bajariladi.

$\varphi(t)$ funksiya uzluksiz, uzluksiz hosilaga ega va teskari funksiyasi mavjud. $dx = \varphi'(t)dt$ bo‘lgani uchun $\int f(x)dx = \int f(\varphi(t)) \cdot \varphi'(t)dt.$ Qanday qilib foydali almashtirishni olish mumkin degan savolga umumiy javob berish mumkin emas. Foydali almashtirishni topish mashqlar bilan o‘zlashtiriladi. Aniqmas integralda o‘zgaruvchilarni almashtirish usuliga qator misollar keltiramiz.

6-misol. Integralni hisoblang: $I=\int\frac{dx}{\sqrt{x}-\sqrt[3]{x}}$.

Yechish. Ildiz ostidagi ifodalardan ozod bo‘lish uchun $x=t^6$ belgilashni kiritamiz. $dx=d(t^6)=(t^6)'dt=6t^5\cdot dt$ ekanligidan

$$I=\int\frac{6t^5dt}{t^3-t^2}=6\int\frac{t^3dt}{t-1}=6\int\frac{t^3-1+1}{t-1}dt=6\int\frac{t^3-1}{t-1}+$$

$$+\int\frac{dt}{t-1}=6\int(t^2+t+1)dt+6\ln|t-1|=2t^3+3t^2+6t+6\ln|t-1|=2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C.$$

7-misol. Integralni hisoblang: $I=\int\frac{\sqrt{10+x^2}}{x}dx.$

Ko‘rsatma: $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ ko‘rinisdagi funksiyalarni integrallashda quyidagi belgilashlar kiritiladi:

1) $\sqrt{a^2-x^2}$ funksiya uchraganda $x=a\cdot\sin t,$ belgilanib $\sqrt{a^2-x^2}=a\cdot\cos t$ topiladi. $dx=a\cdot\cos t\cdot dt,\ t=\arcsin(x/a).$

Bunda $-a<x<a,\ -\frac{\pi}{2}<t<\frac{\pi}{2}.$

2) $\sqrt{a^2+x^2}$ ko‘rinishdagi funksiyalar uchraganda $x=atgt$ belgilanadi. Bundan $\sqrt{a^2+x^2}=\sqrt{a^2(1+tg^2t)}=\frac{a}{\cos t},\ dx=\frac{a}{\cos^2 t}dt,\ t=arctg\frac{x}{a},\ -\frac{\pi}{2}<t<\frac{\pi}{2}.$

3) $\sqrt{x^2-a^2}$ ko‘rinishdagi funksiyalar uchraganda $x=\frac{a}{\cos t}$ kabi belgilanadi. Bundan $\sqrt{x^2-a^2}=atgt,\ dx=\frac{a\sin t}{\cos^2 t}dt,$ $t=\arccos\frac{a}{x},\ -\frac{\pi}{2}<t<\frac{\pi}{2}.$

8-misol. Integralni hisoblang: $I=\int\frac{dx}{\sqrt{(a^2-x^2)^3}}.$

Yechish $x=a\sin t$ belgilashni kiritamiz. Bundan esa $dx=a\cos t\cdot dt\ (0<t<\pi/2)$ va

$$I=\int\frac{dx}{\sqrt{(a^2-x^2)^3}}=\int\frac{a\cos t\cdot dt}{\left(\sqrt{a^2-a^2\sin^2 t}\right)^3}=\int\frac{a\cos t\cdot dt}{\sqrt{a^6\cos^6 t}}=\frac{1}{a^2}\int\frac{dt}{\cos^2 t}=\frac{tgt}{a^2}+C.$$

Yuqoridagi belgilashga ko‘ra $\sin t=\frac{x}{a},\ \cos t=\sqrt{1-\left(\frac{x}{a}\right)^2}=\frac{\sqrt{a^2-x^2}}{a},\ tgt=\frac{x}{\sqrt{a^2-x^2}},$ bo‘lib, integralning oxirgi ko‘rinishi $I=\frac{x}{a^2\sqrt{a^2-x^2}}+C.$

9-misol. Integralni hisoblang:

a. $I=\int\frac{\sqrt{4+x^2}}{x^4}dx.$ **b .** $I=\int\frac{dx}{x^2\sqrt{x^2-9}}\quad (x>3).$

c. $\int\frac{e^x dx}{\sqrt{1+e^x}}$.**d.** $\int\frac{(1-3x)dx}{3+2x^2}$.

e $\int\frac{2x+3}{2x+1}dx$. **f** $\int\frac{x^2+1}{x-1}dx$.

g. $\int\frac{xdx}{x^2-5}$. **h** $\int\frac{xdx}{(x+1)^2}$ **i** $\int\frac{x^2dx}{1+x^6}$.

j $\int\frac{xdx}{\sqrt{x^2+1}}$. **k** $\int\frac{\sqrt{x}+\ln x}{x}dx$. **l** $\int\frac{dx}{\sqrt{7+8x^2}}$.

m $\int\frac{dx}{\sqrt{7-5x^2}}$. **n** $\int\frac{xdx}{2x^2+3}$. **o.** $\int\frac{xdx}{\sqrt{a^4-x^4}}$.

p. $\int\frac{x^3dx}{1+x^8}$. **q.** $\int\frac{x^2dx}{\sqrt{x^6-1}}$. **r.** $\int\sqrt{\frac{\arcsin x}{1-x^2}}dx$.

s. $\int\frac{arctg\,\frac{x}{2}}{4+x^2}dx$. **t.** $\int e^{-(x^2+1)}xdx$. **u.** $\int\frac{e^{1/x}}{x^2}dx$.

v. $\int\frac{5^{\sqrt{x}}}{\sqrt{x}}dx$. **w.** $\int\frac{e^xdx}{e^x-1}$. **x.** $\int\cos\sqrt{x}\,\frac{dx}{\sqrt{x}}$.

y. $\int\sin(\lg x)\frac{dx}{x}$. **z.** $\int\frac{xdx}{\cos^2 x^2}$.

4. Bo‘laklab integrallash. Agar $u(x)$ va $v(x)$ – differensiallanuvchi funksiyalar bo‘lsa, u holda ular ko‘paytmasining differensiali $d(uv)=udv+vdu.$ Bu ifodaning ikkala tomonini ibtegrallab $\int d(uv)=\int udv+\int vdu\quad yoki\quad \int udv=uv-\int vdu$ formulani olamiz.

Bo‘laklab integrallash usuli har xil sinfdagi funksiyalar ko‘paytmalarini integrallashda foydalaniladi:

$\int P_n(x)e^{\alpha \cdot x}dx,\quad \int P_n(x)\cos ax\,dx,\quad \int P_n(x)\sin ax\,dx,\quad \int P_n(x)arctgx\,dx,\quad \int P_n(x)\arcsin x\,dx,$

$\int P_n(x)\arccos x\,dx,\quad \int P_n(x)\ln x\,dx.$

Dastlabki uchta integralda u uchun $P_n(x)$ ko‘phad qabul qilinadi, oxirgi to‘rtta integralda uchun esa mos ravishda arctg$x$, arcsin$x$, arccos$x$, ln$x$ lar qabul qilindi. Ba’zi hollarda bo‘laklab integrallash formulasini bir necha marta qo‘llash zarur bo‘ladi.

10-misol. a. Integralni hisoblang: $\int x\cdot e^{-5x}dx.$

Yechish. $u = x$ va $dv = e^{-5x}dx$ deb olamiz, u holda

$$\int xe^{-5x}dx = \left\{\begin{matrix} u=x,\ du=dx \\ dv=e^{-5x}dx, v=\int e^{-5x}dx=-\frac{1}{5}e^{-5x} \end{matrix}\right\} = -\frac{x}{5}e^{-5x}dx-\frac{1}{25}e^{-5x}+C.$$

v ni topishda integrallash doimiysini har doim nolga teng deb hisoblash mumkin.

b. $\int arctgx\,dx$ **c.** $\int (x^2+1)\cos x\cdot dx$

d. $\int e^{\alpha x}\cos\beta x\,dx$.**e.** $I=\int x^n\ln x\,dx,\ n\neq -1.$

f. $\int x\sin xdx$.

11-misol. Integrallarni hisoblang:

a. $\int e^x\cos xdx$. **b.** $\int \ln xdx$. **c.** $\int x\ln(x-1)dx$.

d. $\int (5x+6)\cos 2xdx$. **e.** $\int xarctgxdx$. **j.** $\int xe^{2x}dx$.

h. $\int e^x\sin xdx$ **i.** $\int \frac{xdx}{\sin^2 x}$. **j.** $\int \frac{xdx}{\cos^2 x}$.

5. Kvadrat uchhadni o‘z ichiga olgan ba’zi funksiyalarni integrallash. Quyidagi integrallarni qaraymiz:

$$I_1=\int\frac{dx}{ax^2+bx+c},\quad I_2=\int\frac{Ax+B}{ax^2+bx+c}dx,\quad I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}},\quad I_4=\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx.$$

Bu integrallarni hisoblash uchun maxrajdagi uchhaddan to‘la kvadrat ajratamiz.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+\frac{b}{2a}x+\frac{c}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)=a\left(t^2\pm m^2\right),$$

bu yerda $t = x + \frac{b}{2a},\ \frac{c}{a} - \frac{b^2}{4a^2} = \pm m^2$.

12-misol. Integralni hisoblang $I_1 = \int \frac{dx}{4x^2 + 4x + 5}$.

Yechish. To‘la kvadrat ko‘rinishga keltirib olamiz:

$$4x^2 + 4x + 5 = 4\left(x^2 + \frac{1}{2}\cdot 2\cdot x + \frac{5}{4} + \frac{1}{4} - \frac{1}{4}\right) = 4\left[\left(x + \frac{1}{2}\right)^2 + 1\right] = 4\left(t^2 + 1\right) bunda\ t = x + \frac{1}{2},\ dx = dt.$$

Demak $I_1 = \frac{1}{4}\int \frac{dt}{t^2 + 1} = \frac{1}{4} arctgt + C = \frac{1}{4} arctg\frac{2x+1}{2} + C.$

13-misol. Integralni toping:

a. $I_2 = \int \frac{dx}{\sqrt{2 + 3x - 2x^2}}$. b. $\int \frac{3x - 1}{4x^2 - 4x + 17}dx.$

Mavzu yuzasidan savollar

1. *Funksiyaning boshlang‘ichi deb va aniqmas integral deb nimaga aytiladi?*
2. *Aniqmas integralning asosiy xossalari.*
3. *Funksiyani integrallashning asosiy usullarini sanab o‘ting.*
4. *Noto‘g‘ri kasr qanday integrallanadi?*
5. *Bo‘lakalab integrallash formulasi.*
6. *Qanday funksiyalarni integrallashda aniqmas koeffitsiyentlar usulidan foydalaniladi?*
7. *Sodda trigonometrik funksiyalarni integrallash usullari.*
8. *[a; b] kesmada f(x) funksiyani aniq integrallash deb nimaga aytiladi.*
9. *Aniq integralning geometrik ma’nosini.*
10. *Aniq integralning asosiy xossalari.*
11. *Nyuton Leybnits formulasini keltirib chiqaring.*
12. *Aniq integralni hisoblashning asosiy usullari.*

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