O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

ISLOM KARIMOV NOMIDAGI

TOSHKENT DAVLAT TEXNIKA UNIVERSITETI

# Oliy matematika

# Kompleks o‘zgaruvchili funksiyalar bo‘limidan uslubiy qo‘llanma

TOSHKENT-2020

**Tuzuvchilar: Yuldoshev A.Y., Yusupov A.I., Abdikayimova G.A.**

Oliy matematika. Kompleks o‘zgaruvchili funksiyalar bo‘limidan uslubiy qo‘llanma. Toshkent, ToshDTU, 2020, 76b.

Ushbu uslubiy qo‘llanma ishlab chiqarish texnik sohasi bilim sohasi ta’lim yo‘nalishlari talabalariga mo‘ljallab tayyorlangan bo‘lib, «Oliy matematika» faninig maxsus bo‘limlaridan biri bo‘lgan «Kompleks o‘zgaruvchili funksiyalar» bo‘limi bo‘yicha tayyorlangan.

Uslubiy qo‘llanmada har bir mavzu bo‘yicha qisqacha nazariy tushunchalar, ularga doir namunaviy mashqlar bajarib ko‘rsatilgan, mustaqil bajarish uchun mashqlar berilgan. Talabalar o‘z bilimlarini tekshirib ko‘rishlari uchun mavzuga oid nazariy savollar keltirilgan. Uslubiy qo‘llanmadan talabalar, magistrantlar va yosh o‘qituvchilar ham foydalanishlari mumkin.

*Islom Karimov nomidagi Toshkent davlat texnika universiteti ilmiy –uslubiy kengash qaroriga muvofiq chop etildi.*

**Taqrizchilar:**

1) Shodimetov X.M.- f.-m.f.d., ToshTYMI “Informatika va kompyuter grafikasi” kafedrasi professori

2) Abdukarimov A.-f.-m.f.n., ToshDTU “Oliy matematika” kafedrasi dotsenti

# KIRISH

Mazkur uslubiy qo‘llanma O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi tomonidan 2018-yil 26-avgustda ro‘yxatga olinib tasdiqlangan. “Ishlab chiqarish texnik soha” bilim sohasi ta’lim yo‘nalishlari uchun “Oliy matematika” faninig o‘quv dasturi asosida amaliy mashg‘ulotlar uchun tayyorlangan. Uslubiy qo‘llanma fan dasturining “Kompleks sonlar va ular ustida amallar. Kompleks o‘zgaruvchili funksiyalar” 19- moduli bo‘yicha bayon qilingan bo‘lib, quyidagi kichik modullarni o‘z ichiga olgan. Kompleks sonlar va ular ustida amallar. Kompleks o‘zgaruvchili funksiya, ularning limiti, uzluksizligi va hosilasi. Koshi-Riman shartlari. Kompleks o‘zgaruvchili funksiyaning integrali. To‘g‘ri va maxsus nuqtalar. Chegirmalar nazariyasi va uning tatbiqlari.

Uslubiy qo‘llanma amaliy mashg‘ulot darslariga mo‘ljallanib tayyorlangani uchun, har bir katta mavzu kichik mavzularga ajratilgan va dastlab qisqacha nazariy ma’lumotlar keltirilgan. So‘ngra bir nechta tipik mashqlar bajarilib ko‘rsatilgan. Mustaqil bajarish uchun mashqlar berilgan. Talabalar nazariy bilimlarini mustahkamlash maqsadida mavzu bo‘yicha nazorat savollari keltirilgan. Kompleks sonlardan o‘zbek tilida adabiyotlar yo‘q.

Uslubiy qo‘llanmadan talabalar, magistrantlar va yosh o‘qituvchilar ham foydalanishlari mumkin.

## 1. Kompleks sonlar va ular ustida amallar

### 1.1. Kompleks sonning algebraik ko‘rinishi

Kompleks son tushunchasi yuqori darajali algebraik tenglamalarni yechishda yuzaga keladi. Ushbu

$$ax^2 + bx + c = 0$$

kvadrat tenglamani $D = b^2 - 4ac < 0$ bo‘lsa, u holda

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

dan ikkita kompleks ildiz kelib chiqadi.

Masalan: $x^2 + 1 = 0 \;=> x_{1,2} = \pm\sqrt{-1} = \pm i$, bu yerda $i=\sqrt{-1}$

Ushbu $z=a+ib$ , ko‘rinishdagi sonlar kompleks sonlar deyiladi. Bu yerda $a$ va $b$ lar haqiqiy sonlardir, bunda $a$ son $z$ ning haqiqiy qismi, $b$ son esa mavhum qismi deyiladi va $Rez=a$, $Imz=b$ ko‘rinishida yoziladi. $i$ –mavhum birlik. Ma’lumki, $i^2 = -1\,, i^3 = -i, i^4 = 1, i^5 = i, \ldots.$ umumiy holda

$$i^{4k} = 1\,, i^{4k+1} = i, i^{4k+2} = -1\,, i^{4k+3} = -i$$

Masalan: $z=\frac{3}{4} + \frac{5}{6}i$ bo‘lsa $Rez=\frac{3}{4}$ ; $Imz=\frac{5}{6}$.

### 1.2. Kompleks sonlar ustida amallar

Agar $\alpha=a+ib$ , $\beta=c+id$ kompleks sonlar o‘zaro teng bo‘lishi uchun, yani $\alpha=\beta$ bo‘lishi uchun $a=c$ va $b=d$ tenglik bajarilishi kerak.

***Qo‘shish va ayirish***

$\alpha=a+ib$ , $\beta=c+id$ kompleks sonlar berilgan bo‘lsin, u holda

$\alpha\pm\beta=(a+ib)\pm(c+id)=(a\pm c)+(b\pm d)i$ kabi topiladi.

**Misollar**

1. $(5-4i) \pm (3+2i) = (5\pm3) + (-4\pm2)i = \begin{cases} 8-2i \\ 2-6i \end{cases}$; ko‘rinishga keladi.

2. $(7+8i)-(5-4i)=(7-5)+(8+4)i=2+12i$.

$3.\ (\frac{6}{5} - \frac{3}{4}i) - (\frac{3}{2} + \frac{3}{5}i) = (\frac{6}{5} - \frac{3}{2}) + (-\frac{3}{4} - \frac{3}{2})i = -\frac{3}{10} - \frac{9}{8}i.$

4. $z_1 = 2+3i, z_2 = -4+7i, z_3 = 5-11i$ bo‘lsa $z_1 - z_2 + z_3$ ni toping.

$z_1 - z_2 + z_3 = (2-(-4)+5)+(3-7+(-11))i = 11-15i$ .

***Ko‘paytirish va bo‘lish***

$\alpha = a+ib$ , $\beta = c+id$ kompleks sonlar berilgan bo‘lsin, u holda ularni qo‘paytmasi

$\alpha \cdot \beta = (a+ib)(c+id) = ac+iad+ibc+i \cdot ibd = (ac-bd)+(ad+bc)i$ kabi topiladi.

Agar $\alpha = a + ib$, $\bar{\alpha} = a - ib$ berilgan bo‘lsa, ular o‘zaro qo‘shma kompleks sonlar deyiladi, hamda $\alpha + \bar{\alpha} = 2a \ va \ \alpha \cdot \bar{\alpha} = a^2 + b^2$ bo‘ladi.

$\alpha = a+ib$ , $\beta = c+id$ kompleks sonlarni bo‘lish uchun

$\frac{\alpha}{\beta} = \frac{a+ib}{c+id}$ – kasrni surat va maxrajini $\beta = c - id$ ga ko‘paytiramiz u holda ularni bo‘linmasi

$$\frac{\alpha}{\beta} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

kabi topiladi.

**Misollar**

$1.\ (3-4i)(2+3i) = (6+12) + (9-8)i = 18 + i;$

2. $\frac{\sqrt{3}+i}{3-4i} = \frac{(\sqrt{3}+i)(3+4i)}{(3-4i)(3+4i)} = \frac{(3\sqrt{3}-4)+(3+4\sqrt{3})i}{9+16}$ ;

3. $(\frac{1}{2} - \frac{\sqrt{3}}{2}i)^2 = \frac{1}{4} - 2\frac{\sqrt{3}}{4}i - \frac{3}{4} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i;$

4. Ushbu tenglamani yeching.

$$(1+3i)x + (4-2i)y = 2-5i$$

Yechilishi: Dastlab qavslarni ochib tenglamani chap tomonida haqiqiy va mavhum qismlarini ajratamiz

$x + 3xi + 4y - 2yi = 2 - 5i$

$(x+4y) + (3x-2y)i = 2-5i$

va

$$\begin{cases} x + 4y = 2 \\ 3x - 2y = -5 \end{cases}$$

sistemani hosil qilamiz, bu sistemani yechib,

$$\begin{cases} 7x = -8 \\ -14y = -11 \end{cases} \quad x = -\frac{8}{7}, \quad y = \frac{11}{14}.$$

larni topamiz.

5. Quyidagi kompleks sonni hisoblang, (n€N).

$$\frac{\alpha}{\beta} = \frac{(1+i)^n}{(1-i)^{n-2}}$$

Yechilishi:

$$\frac{\alpha}{\beta} = \frac{(1+i)^n}{(1-i)^n(1-i)^{-2}} = \frac{(1+i)^n}{(1-i)^n}(1-i)^2 = \frac{((1+i)^2)^n}{((1-i)(1+i))^n}(1-2i+i^2) = \frac{(1+2i+i^2)^n}{2^n}(-2i) = \frac{2^n i^n}{2^n}(-2i) = -2i^{n+1}$$

.

**Mustaqil yechish uchun misollar**

Quyidagilarni hisoblang:

1) z= (5+2$i$)-(8-3$i$);
2) z= (7+4$i$)+(15-4$i$);
3) z= (14-8$i$)-(8-$i$);
4) z= (9-8$i$)-(-5-2$i$);
5) z= (-7+3$i$)+(6+4$i$) ;
6) z= $(1-4i)(5-6i)$ ;
7) z= $(2+4i)(2-3i)$;
8) z= $(7-5i)(2+7i)$;
9) z= $(9-i)(2-i)$;
10) z= $(4+4i)(9-3i)$;
11) $z = \frac{5+i}{2+4i}$;
12) $z = \frac{2\sqrt{3}-i}{1+i}$;
13) $z = \frac{5+i}{7-2i}$;

14) $z = \frac{\sqrt{3}-\sqrt{3}i}{\sqrt{3}-i}$;

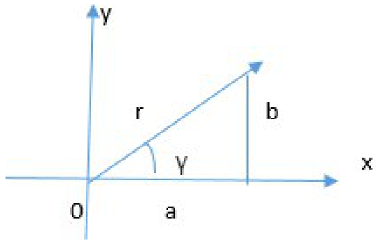
15) $z = \frac{8+2i}{\sqrt{2}-2i}$ .

## 1.3. Kompleks sonning trigonometrik shakli

$\alpha = a + ib$ kompleks sonning ikki xil geometrik ma'nosi bor.

a) XOY tekislikda $(a, b)$ nuqtani tasvirlaydi;

b) XOY tekislikda (0,0) nuqta bilan $(a, b)$ nuqtani tutashtiruvchi vektorni tasvirlaydi(1-chizma). Bu tekislik kompleks tekisligi deyiladi va Z ko'rinishida belgilanadi.




*1-chizma*

1- chizmadan $a = r\cos\varphi,\ b = r\sin\varphi,\ r = \sqrt{a^2 + b^2}$

$$\operatorname{tg}\varphi = \frac{b}{a} \ => \ \varphi = arctg\frac{b}{a}$$

$$r = |\alpha| = |a + ib| = \sqrt{a^2 + b^2} \quad va \quad \varphi = arctg\frac{b}{a}$$

$$0 \le r < +\infty \quad va \quad 0 \le \varphi < 2\pi$$

agar $(a;b)$ nukta 1-chorakda bo'lsa, $\varphi = arctg\frac{b}{a}$;

2-chorakda bo'lsa, $\varphi = \pi - arctg\frac{b}{a}$;

3-chorakda bo'lsa , $\varphi = \pi + arctg\frac{b}{a}$;

4-chorakda bo'lsa , $\varphi = 2\pi - arctg\frac{b}{a}$ ko'rinishida bo'ladi.

U holda $\alpha = a + ib = r(\cos\varphi + i\sin\varphi)$ ko‘rinishiga ega bo‘lib, u kompleks sonning trigonometrik shakli deyiladi.

Eylerning $e^{ix} = \cos x + i\sin x$
formulasidan foydalanib $\alpha = re^{i\varphi}$ ko‘rsatkichli shakl ko‘rinishda yozish mumkin.

**Misollar**

1. Ushbu $\alpha = 1 - \sqrt{3}i$ kompleks sonni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

Yechilishi: $a = 1;\ b = -\sqrt{3}$ bo‘lganligi sababli

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2.$$

$\text{tg}\,\varphi = \frac{b}{a} = -\sqrt{3}$ bo‘lib, $1 - \sqrt{3}i$ ga tegishli vektor tekislikning to‘rtinchi choragida yotganligi uchun

$\varphi = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ bo‘ladi.

Demak, $\alpha = 1 - \sqrt{3}i = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2e^{i\frac{7\pi}{4}}$.

2. Ushbu $\alpha = \sqrt{2} + \sqrt{2}i$ kompleks sonni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

Yechilishi: $a = \sqrt{2}; b = \sqrt{2};\ r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2,\ \text{tg}\varphi = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$\sqrt{2} + \sqrt{2}i$ ga tegishli vektor birinchi chorakda yotganligi uchun

$\varphi = arctg = \frac{\pi}{4}$ bo‘ladi.

Demak, $\alpha = \sqrt{2} + \sqrt{2}i = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 2e^{i\frac{\pi}{4}}$.

3. $\alpha = -1 - \sqrt{3}i$ kompleks sonni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

Yechilishi: $a=-1,\ b=-\sqrt{3}\ ,\ r=2,\ tg\,\varphi=\frac{-\sqrt{3}}{-1}=\sqrt{3}$ bo‘ladi, $-1-\sqrt{3}i$ ga tegishli vektor uchinchi chorakda joylashganligi sababli

$$\varphi=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$$

$$\alpha=-1-\sqrt{3}i=2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=2e^{i\frac{4\pi}{3}}.$$

4. $\alpha=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ kompleks sonni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

Yechilishi: $a=-\frac{1}{2}; b=\frac{\sqrt{3}}{2}, r=1, \text{tg}\,\varphi=-\sqrt{3}$ vektor ikkinchi chorakda, shu sababli $\varphi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

$$\alpha=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+\sin\frac{2\pi}{3}i=e^{i\frac{2\pi}{3}}.$$

5. z = $1+i$ kompleks sonni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

Yechilishi: $a=1, b=1,$ r=$\sqrt{a^2+b^2}=\sqrt{2}$, tg$\varphi=\frac{b}{a}=1, \varphi=\frac{\pi}{4}$ chunki (1,1) nuqta birinchi chorakda yotadi , demak

$1+\mathrm{i}=\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ yoki umumiy ko‘rinishda

$$1+\mathrm{i}=\sqrt{2}\left(\cos\left(\frac{\pi}{4}+2k\pi\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+2k\pi\right)\right)\ ;\ k=0;\pm1;\pm2;$$

**Mustaqil yechish uchun misollar**

Quyidagi kompleks sonlarni trigonometrik shaklga keltiring va ko‘rsatkichli shaklda yozing.

1) $\alpha=-i;$
2) $\alpha=-2i;$
3) $\alpha=-\sqrt{2}+i\sqrt{2};$
4) $\alpha=\frac{1}{2}+\frac{\sqrt{3}}{2}i;$
5) $\alpha=\frac{1}{2}-\frac{\sqrt{3}}{2}i;$

6) $\alpha = -cos\varphi - isin\varphi$;

7) $\alpha = \sin\varphi + i(1-\cos\varphi)$;

8) $\alpha = 1 - cos\varphi + isin\varphi$;

9) $\alpha = \frac{1+i}{1-i}$;

10) $\alpha = -1 + i$;

11) $\alpha = \frac{1+i\sqrt{3}}{1-i}$;

12) $\alpha = -2 + 2i$;

13) $\alpha = -\sqrt{3} + i$;

14) $\alpha = sin\,48^{\circ} + icos48^{\circ}$;

15) $\alpha = cos111^{\circ} - isin\,111^{\circ}$.

## 1.4. Trigonometrik shaklda berilgan kompleks sonlarni ko‘paytirish va darajaga ko‘tarish

Kompleks sonlar $\alpha = r_1(\cos\varphi_1 + i\,sin\,\varphi_1), \beta = r_2(\cos\varphi_2 + i\,sin\,\varphi_2)$ kabi trigonometrik shaklda berilgan bo‘lsin.

Trigonometrik shaklda berilgan kompleks sonlarni o‘zaro ko‘paytirish va darajaga ko‘tarish quyidagicha bajariladi:

$$\alpha \cdot \beta = r_1 \cdot r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

Agar $\alpha = \beta$ bo‘lsa

$$\alpha^2 = (r(\cos\varphi + i\sin\varphi))^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

Agar $\alpha_1 = r_1(\cos\varphi_1 + i\sin\varphi_1), \alpha_2 = r_2(\cos\varphi_2 + i\sin\varphi_2), \ldots,$

$\alpha_n = r_n(\cos\varphi_n + i\sin\varphi_n)$ bo‘lsa,

$\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n =$
$r_1 \cdot r_2 \cdot \ldots \cdot r_n(\cos(\varphi_1 + \varphi_2 + \ldots + \varphi_n) + +i\sin(\varphi_{1+}\varphi_2 + \ldots + \varphi_n))$

Agar $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ bo‘lsa

$\alpha^n = (\,r(\cos\varphi + i\sin\varphi))^n = r^n(\cos n\varphi + i\sin n\varphi)$

$r = 1$ bo‘lsa,

$$(\cos\varphi + i\sin\varphi))^n = \cos n\varphi + i\sin n\varphi \qquad (1)$$

hosil bo‘ladi.

Bu tenglik *Muavr formulasi* deb ataladi. Buning chap tomonidagi qavslarni Nyuton binomi bo‘yicha ochib, so‘ngra tenglikning ikki tomonidagi haqiqiy qismlarni o‘zaro tenglashtirish natijasida trigonometriyaga doir turli formulalarni keltirib chiqarish mumkin. Misol uchun (1) da *n*=2 bo‘lsin, u holda

$$(\cos\varphi + i\sin\varphi))^2 = cos^2\varphi + 2i\cos\varphi\sin\varphi - sin^2\varphi = cos2\varphi + isin2\varphi\ ,$$

bu yerdan

$$cos2\varphi = cos^2\varphi - sin^2\varphi \text{ va } sin2\varphi = 2\sin\varphi\cos\varphi)).$$

Agar algebraik shaklda berilgan kompleks sonni darajaga ko‘tarish talab qilinsa, dastlab uni trigonometrik shaklga keltirib olish maqsadga muvofiqdir.

**Misollar**

1. $\alpha = (1+\sqrt{3}i)(\sqrt{3}-i)$ ni hisoblang.

Yechilishi. Dastlab $1+\sqrt{3}i$ va $\sqrt{3}-i$ kompleks sonlarni trigonometrik shaklga keltirib olamiz.

$$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}), \qquad \sqrt{3}-i = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}).$$

U $h$olda

$$(1+\sqrt{3}i)(\sqrt{3}-i) = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}) =$$

$$= 4(\cos\tfrac{13\pi}{6} + i\sin\tfrac{13\pi}{6}).$$

2. $\alpha = (1+i)^{24}$ ni hisoblang.

Yechilishi:

$$1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$$\alpha = (\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right))^{24} = 2^{12}(\cos 6\pi + i\sin 6\pi).$$

3. $\alpha = \left(\frac{\sqrt{3}+i}{1-i}\right)^{20}$ ni hisoblang.

Yechilishi: Dastlab $\sqrt{3}+i$ va $1-i$ larni trigonometrik shaklga keltirib olamiz.

$$\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right),\;\; 1-i=\sqrt{2}(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})$$

$$\left(\frac{\sqrt{3}-i}{1-i}\right)^{20}=\left(\frac{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}{\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}}\right)^{20}=$$

$$=2^{10}\left(\frac{(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})(\cos\frac{7\pi}{4}-i\sin\frac{7\pi}{4})}{cos^2\frac{7\pi}{4}+sin^2\frac{7\pi}{4}}\right)^{20}$$

$$=2^{10}\left(\cos\frac{19\pi}{12}-i\sin\frac{19\pi}{12}\right)^{20}\quad=2^{10}(\cos\frac{95\pi}{3}-i\sin\frac{95\pi}{3}).$$

Ikkinchi holda $1-i=\sqrt{2}(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)$ deb olsa ham bo‘ladi, u holda

$$1-i==\sqrt{2}(\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right))$$

$$\left(\frac{\sqrt{3}-i}{1-i}\right)^{20}=\left(\frac{\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})}{\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}}\right)^{20}=2^{10}\left(\frac{(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}{cos^2\frac{\pi}{4}+sin^2\frac{\pi}{4}}\right)^{20}\quad=$$

$$=2^{10}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)^{20}$$

.

4. $\sqrt[3]{1}$ ni barcha qiymatlari topilsin.

Yechilishi: Trigonometrik ko‘rinishda tasvirlaymiz.

$$z=1+0\cdot i\,;\;\; |z|=1;\, r=\sqrt{1^2+0^2}=1\,;\; \varphi=\text{arctg}\frac{0}{1}=0;$$

$$z=1(\cos 0+i\sin 0)=\cos 0+i\sin 0\,;$$

$$\sqrt[3]{1}=\sqrt[3]{\cos 0+i\sin 0}=\cos\frac{2\pi k}{3}+i\cdot\sin\frac{2\pi k}{3};\quad k=0{,}1{,}2\,...\;\;.$$

$k=0$ da $W_1=\cos 0+i\sin 0=1;$

$k=1$ , da $W_1=\cos 0+i\sin 0=1;$

$k = 2$, da $W_3 = \cos\frac{4\pi}{3} + i \cdot \sin\frac{4\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i.$

**Mustaqil yechish uchun misollar**

Quyidagi kompleks sonlarni darajaga ko‘taring:

1) $\alpha = (1 + i)^{25};$
2) $\alpha = \left(\frac{1+i\sqrt{3}}{1-i}\right)^{20};$
3) $\alpha = \frac{(1+i\sqrt{3})^{15}}{(1+i)^{10}};$
4) $\alpha = (2 - 2i)^7;$
5) $\alpha = (\sqrt{3} - i)^6;$
6) $\alpha = \left(\frac{1-i}{1+i}\right)^8;$
7) $\alpha = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3;$
8) $\alpha = \left(-2 + 2\sqrt{3}\, i\right)^{10};$
9) $\alpha = (\sin\alpha - i\cos\alpha)^{18}, \quad (\pi < \alpha < \frac{3}{2}\pi);$
10) $\alpha = (-4\sqrt{3} + 4i)^2;$
11) $\alpha = (-2 + 2i)^4;$
12) $\alpha = (\sqrt{3} - i)^{10};$
13) $\alpha = (\sqrt{2} - \sqrt{6}i)^3;$
14) $(3\,(cos11^{\circ} + i\sin 11^{\circ}))^3;$
15) $(\sqrt{2}\,(cos9^{\circ} + i\sin 9^{\circ}))^5;$
16) $\sqrt[3]{-8}$ ni barcha qiymatlari topilsin.

Quyidagi ifodalarni hisoblang:

1) $\frac{(1-i\sqrt{3})^6}{1+i\sqrt{3})^4} + (1 + i)(3 - i);$
2) $\frac{i(9-3i)(3-8i)}{(3-5i)-i(4-5i)};$
3) $\frac{-3+5i+2i(3+7i)}{(2-3i)(7-2i)};$
4) $\frac{(-2+2i)^5}{(-1+i)^3} + 2i - 5;$
5) $(2 + 3i)(3-2i) + (2\text{-}3i)(3\text{+}2i);$

6) $\frac{4+i}{2-i} + \frac{5-3i}{3+i}$;

7) $\frac{2}{2-i} + \frac{2}{2+i}$ ;

8) $\frac{1+i}{1-i} + \frac{1-i}{1+i}$;

9) $\frac{2+i}{1+i} + \frac{3-i}{i}$ ;

10) $\frac{5i+2(1-7i)}{(3-2i)}$ ;

11) $(5-3i)(2-2i) - (2-4i)$;

12) $(2+4i)(4-5i) + (1-i)$.

Quyidagi kompleks sonlardan ildiz chiqaring:

1) $\sqrt[5]{1}$ ;

2) $\sqrt{-1}$;

3) $\sqrt[8]{-i}$ ;

4) $\sqrt[7]{-1+i}$ ;

5) $\sqrt[9]{-1-i}$ ;

6) $\sqrt{2-2\sqrt{3}i}$ ;

7) $\sqrt[3]{\sqrt{3}-i}$;

8) $\sqrt[10]{\frac{(3\sqrt{3}+4)-i\,4\sqrt{3}-3)}{3-4i}}$;

9) $\sqrt[8]{-4}$;

10) $\sqrt{2-2\sqrt{3}\,i}$ .

### 1.5. Kompleks sonning logarifmi

Elementar matematikada faqat musbat sonlarning logarifmlarini tekshirish bilan chegaralaniladi, xolos. Kompleks argumentli funksiyalar nazariyasidan ma’lumki, noldan boshqa har kanday kompleks sonning, shu jumladan, manfiy sonning ham logarifmi mavjud.

Berilgan $z = a + ib$ kompleks sonning logarifmini topish uchun uni dastlab trigonometrik shaklga keltirib olamiz:

$$z = a + ib = r(cos\varphi + isin\varphi), \qquad 0 < r < \infty, \qquad 0 \leq \varphi < 2\pi;$$

$$r = |z| = \sqrt{a^2 + b^2}, \varphi = argz, \ \ tg\varphi = \frac{a}{b} \ .$$

U holda, $Lnz = \ln(re^{i\varphi}) = \ln r + i\varphi \ln e = \ln r + i\varphi$, ya'ni

$$Ln\, z = \ln r + i\varphi + 2k\pi i, \quad k = 0; \pm 1; \pm 2; \pm 3; \dots$$

$$Ln\, z = \ln r + i(\varphi + 2k\pi), \quad k = 0; \pm 1; \pm 2; \pm 3; \dots$$

Bundan ko'rinadiki, bitta z sonining cheksiz ko'p logarifmi bor ekan. Agar $k$ =0 bo'lsa, u holda logarifmning bosh qiymati hosil bo'ladi va u quyidagicha yoziladi:

$$Ln\, z = \ln r + i\varphi$$

**Misollar**

1. $z = 1$ sonining logarifmini toping.

Yechilishi: Ma'lumki, $a = -1, \ b = 0, r = |z| = |-1| = 1, \varphi = \pi$

$z = -1 = cos\pi + isin\pi,$

$lnr = ln1 = 0,$

Shu sababli, $\text{Ln}(-1) = \ \pi i + 2k\pi i = (2k + 1)\pi i, k = 0 \pm 1; \pm 2; \dots$

$$\text{Lnz} = ln1 + i\pi = i\pi$$

$$\text{Lnz} = i\pi + 2k\pi i = i\pi(1 + 2k), \ \ k = 0; \ \pm 1; \ \pm 2; \ \dots$$

Bundan ko'rinadiki, (-1) ning logarifmlari mavhum sonlardan iborat bo'lib, bosh qiymati $i\pi$ ga teng ekan.

2. $z = 1 - i$ sonining logarifmini toping.

Yechilishi: Ma'lumki, $a = 1, \ b = -1, \ r = |z| = \sqrt{2}, \varphi = \frac{\pi}{4}$,

$$z = 1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right],$$

Shu sababli, $\text{Ln}(1 - i) = \frac{1}{2}ln2 + \left(2k - \frac{1}{4}\right)\pi i.$

3. $z=\sqrt{3}+i$ soning logarifmini toping.

Yechilishi: Ma'lumki,

$a=\sqrt{3}, \qquad b=1, \qquad r=|z|=2\,,\varphi=\frac{\pi}{6}$

$$z=\sqrt{3}+i=2\left[\cos\left(-\frac{\pi}{6}\right)+isin\left(-\frac{\pi}{6}\right)\right],$$ demak

$\mathrm{Ln}\left(\sqrt{3}+i\right)=ln2+\left(2k+\frac{1}{6}\right)\pi i.$

**Mustaqil yechish uchun misollar**

Berilgan kompleks sonning logarifmini toping:

1) $z=-i;$

2) $z=-1+i;$

3) $z=-2-2i;$

4) $z=1-i\sqrt{3};$

5) $z=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2};$

6) $z=3+7i;$

7) $z=\sqrt{3}-i;$

8) $z=1+i;$

9) $z=\sqrt{3}+i;$

10) $z=-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}.$

## 1.6. Kompleks sonning kompleks darajasi

Bu yerda kompleks sonni ixtiyoriy kompleks darajaga ko'tarish masalasi ko'riladi.

$\mathrm{W}=z^2;\ z=x+iy\neq 0, \qquad \alpha=a+ib.$

$z=x+iy\neq 0$ ni quyidagicha yozish mumkin:

$z = e^{lnz}$

Buning ikkala tomonini $\alpha$ darajaga ko‘tarib, so‘ngra o‘rniga

$$\text{Ln } z = \ln r + i(\varphi + 2k\pi), \quad k = 0; \pm1; \pm2; \pm3; \ldots$$

formuladan qiymatini qo‘yamiz, u holda

$W = z^{\alpha} = e^{\alpha Lnz} = e^{\alpha(\ln r + i(\varphi + 2k\pi))}$ hosil bo‘ladi yoki ixchamlashtirsak:

$W = z^{\alpha} = e^{\alpha(lnr+i\varphi)} e^{2k\pi i}$ , $\quad k = 0; \pm1; \pm2; \ldots$

Agar $k=0$ desak, darajaning bosh qiymati

$$W_0 = e^{\alpha(lnr+i\varphi)}$$

hosil bo‘ladi. Shunday qilib umumiy formula

$$W = W_0 e^{2k\pi i} \qquad (*)$$

dan iborat bo‘lib, bu yerda $z = x + iy = r(cos\varphi + isin\varphi)$.

Demak, $z = x + iy$ sonni $\alpha$ darajaga ko‘tarish uchun dastlab $z$ ni trigonometrik shaklga keltirib, $r$ bilan $\varphi$ ni aniqlab olish lozim. (*) da $k$ qatnashganligi uchun umumiy daraja cheksiz ko‘p qiymatga ega. Boshqacha aytganda istalgan $z$ sonning $\alpha$ darajasi cheksiz ko‘p sonlar bo‘lishi mumkin.

**Misollar**

1. $W = (-1 + i)^i$ ni hisoblang

Yechilishi: $z = -1 + i = \sqrt{2}\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)$, bu yerda

$r = \sqrt{2}, \ \ \varphi = \frac{3\pi}{4}$.

Demak, $W_0 = e^{i(ln\sqrt{2}+i\varphi)} = e^{-\frac{3\pi}{4}} \cdot e^{iln\sqrt{2}}$,

$W = (-1 + i)^i = e^{-\frac{3\pi}{4}} \cdot e^{iln\sqrt{2}} \cdot e^{2k\pi i \cdot i} = e^{-\left(2k+\frac{3}{4}\right)\pi} \cdot e^{iln\sqrt{2}} =$

$= e^{-\left(2k+\frac{3}{4}\right)\pi}(\cos(\,ln\sqrt{2}) + isin(ln\sqrt{2})), k = 0, \pm1, \pm2, \ldots.$

2. $W = (1 + \sqrt{3}i)^{-i}$ darajani hisoblang.

Yechilishi: $z = 1 + \sqrt{3}i = 2\left(cos\frac{4\pi}{3} + isin\frac{4\pi}{3}\right)$ bu yerda

$r = 2,\ \varphi = \frac{4\pi}{3}.$

$$W_0 = e^{-i(ln2+\frac{4\pi}{3}i)} = e^{\frac{4\pi}{3}} \cdot e^{-iln2},$$

U holda

$$W = (1+\sqrt{3}i)^{-i} = e^{\frac{4\pi}{3}} \cdot e^{-iln2} \cdot e^{2(-k\pi i\cdot i)} == e^{\left(2k+\frac{4}{3}\right)\pi}\big(\cos(ln2) + isin(ln2)\big).$$

**Mustaqil yechish uchun misollar**

Hisoblang:

1. $W = (-1)^i$ ;
2. $W = (-i)^{i+1}$ ;
3. $W = (\sqrt{3}+i)^i$;
4. $W = (i)^i$;
5. $W = (\sqrt{3}-i)^{i+1}$;
6. $W = (\sqrt{3}+\sqrt{3}i)^{-i}$;
7. $W = (3+5i)^{-i}$;
8. $W = (-1+i)^{1-i}$;
9. $W = (1-i)^{i-1}$;
10. $W = (-1+\sqrt{3}i)^{i-1}$ ;
11. $W = (i)^{-i}$;
12. $W = (-1-i)^i$;
13. $W = (1+i)^{-i-1}$;
14. $W = (-\sqrt{3}-\sqrt{3}i)^{-i}$.

**Nazorat savollari**

1. Kompleks sonning algebraik ko‘rinishi.
2. Algebraik ko‘rinishdagi kompleks sonlar ustida amallar.
3. Kompleks sonning moduli va argumenti.
4. Kompleks sonning trigonometrik va ko‘rsatkichli shakli.
5. Trigonometrik shakldagi sonni darajaga ko‘tarish.
6. Trigonometrik sondan ildiz chiqarish.

## 2. Kompleks o‘zgaruvchili funksiyaning limiti, uzluksizligi va hosilasi. Koshi-Riman shartlari

### 2.1. Kompleks o‘zgaruvchili funksiyalar

Agar $G$ kompleks sonlar to‘plamidan olingan har bir $x+iy$ kompleks songa biror qonun bo‘yicha aniq bir $W=u+iv$ kompleks son mos kelsa, u holda $G$ to‘plamda $W$ kompleks o‘zgaruvchili funksiya berilgan deyiladi va

$\mathrm{W}=f(z)=u(x,y)+iv(x,y)$ kabi yoziladi.

**Misollar**

1. $f(z)=z^2+z$ funksiya berilgan. Bu funksiyaning ba’zi $f(1+i),\quad f(2-i),\ f(i)$ lardagi xususiy qiymatlarini toping.

Yechilishi: $f(1+i)=(1+i)^2+(1+i)=1+2i-1+1+i=1+3i.$

$$f(2-i)=(2-i)^2+(2-i)=4-4i-1+2-i=5-5i=5(1-i).$$

$$f(i)=i^2+i=-1+i.$$

2. $\mathrm{W}=x^2-iy^2$ berilgan funksiyaning ba’zi $f(1+i), f(5-3i)$ xususiy qiymatlarini toping:

Yechilishi: $f(1+i)=1^2-i1^2=1-i. f(5-3i)=25-9i.$

3. $\mathrm{W}=3z^2+2z$ funksiyaning haqiqiy va mavhum qismlarini toping.

Yechilishi:

$$\mathrm{W}=f(x+iy)=3(x+iy)^2+2(x+iy)=3(x^2+2xyi-y^2)+2x+2iy=3x^2++2x-3y^2+(6xy+2y)i$$

demak, funksiyaning haqiqiy va mavhum qismlari:

$$u(x,y)=3x^2-3y^2+2x\,,\ v(x,y)=6xy+2y=2y(3x+1).$$

4. $\mathrm{W}=\frac{1}{\bar{z}}\quad(\bar{z}\neq 0)$ funksiyaning haqiqiy va mavhum qismlarini toping.

Yechilishi: $\bar{z}=x-iy$ ni qo‘yib, maxrajini komplekslikdan ozod qilamiz:

$$\mathrm{W}=\frac{1}{x-iy}=\frac{x+iy}{(x-iy)(x+iy)}=\frac{x+iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\,\frac{y}{x^2+y^2}\quad .$$

Demak, $(x,y)=\frac{x}{x^2+y^2}$, $v(x,y)=\frac{y}{x^2+y^2}$.

**Mustaqil yechish uchun misollar**

1. $f(z)=z^3$ funksiya berilgan. Ushbu xususiy qiymatlarni toping:

$a)\ f(2+3i);\quad$ b) $f(-1-i)\ ;\quad v)\ f(a+bi).$

2. $f(z)=\frac{\bar{z}}{z}$ funksiya berilgan. Ushbu xususiy qiymatlarni toping:

$a)\ f(1+i);\quad$ b) $f(-i)\ ;\quad v)\ f(3+4i).$

3. $\mathrm{W}=\frac{4+i\bar{z}}{3-z}$ funksiyaning haqiqiy va mavhum qismlarini toping.

4. $\mathrm{W}=iz^2+2\bar{z}^3$ funksiyaning haqiqiy va mavhum qismlarini toping.

### 2.2.Kompleks o‘zgaruvchili funksiyaning aniqlanish sohasi

Berilgan har qanday

$$W = u(x, y) + iv(x, y) = f(z)$$

Funksiyaning aniqlanish sohasi $G_1$-ga ega. Malumki, agar chegara ham sohaga tegishli bo‘lsa, u yopiq soha deyiladi, u $\overline{G_1}$ ko‘rinishida yoziladi. Masala yechishda soha berilgan bo‘ladi yoki funksiyaning o‘ziga qarab sohani aniqlash talab qilinadi.

**Misollar**

1. $|z - \alpha| < R$ tengsizlik qanday sohani bildiradi, bu yerda

$z = x + iy$ va $\alpha = a + ib$

Yechilishi:

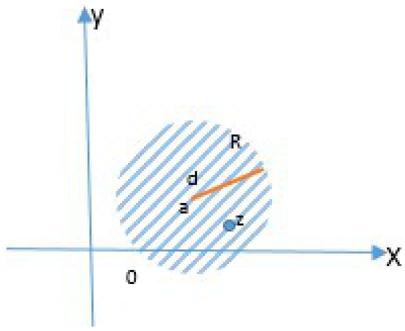
$z - \alpha = x + iy - a - ib = (x - a) + i(y - b)$

$|z - \alpha| = |(x - a) + i|y - b| = \sqrt{(x-a)^2 + (y-b)^2} < R$ bundan

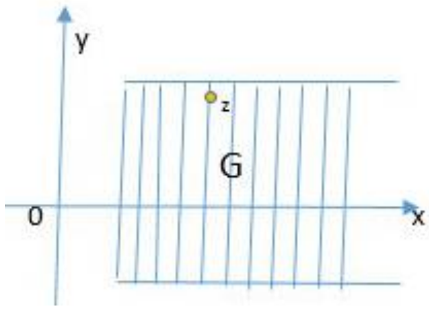
$(x - a)^2 + (y - b)^2 < R^2$ kelib chiqadi.

Bu markazi ($a;b$) nuqtada joylashgan, $R$ radiusli aylananing ichki nuqtalar to‘plamidan, yani doiradan iborat. (2-chizma)

Agar $\alpha = 0,\ R = 1$ bo‘lsa, uni birlik doira deyiladi va misolni berilishidan |z|<1 bo‘ladi.




*2-chizma*




*3-chizma*

2. Ushbu $Rez \geq 1$

tengsizlikni qanoatlantiruvchi nuqtalar qanday to‘plamni aniqlaydi?

Yechilishi: Ma’lumki,

$$z = x + iy,\ \ Rez = Re(x+iy) = x \geq 1.$$

Bu esa $x$=1 to‘g‘ri chiziq va uning o‘ng tomonida yotuvchi nuqtalar to‘plamidir. (3- chizma).

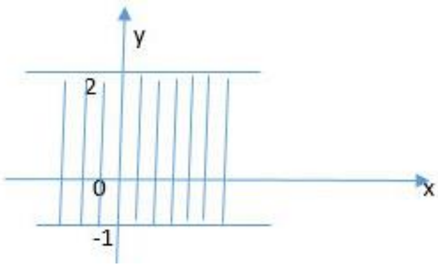
3. Ushbu

$$-1 \leq Imz \leq 2$$

tengsizlikni qanoatlantiruvchi nuqtalar to‘plami qanday sohani tashkil etadi?

Yechilishi:

2-misolga o‘xshash $-1 \leq y \leq 2, y = -1$ va $y = 2$ to‘g‘ri chiziqlar hamda ular orasida yotuvchi nuqtalar to‘plamini tashkil etadi.




*4-chizma*

**Mustaqil yechish uchun misollar**

Tengsizliklar qanday sohalarni aniqlaydi:

1) $|z^2 - 1| \geq a^2, a > 0$;
2) $4 \leq |z-1| + |z+1| \leq 8$;
3) $\frac{1}{4} < Re\left(\frac{1}{\bar{z}}\right) + Im\left(\frac{1}{\bar{z}}\right) < \frac{1}{2}$;
4) $\mathrm{Re}z \leq -1$;
5) $-1 < Rez < 1$;
6) $1 \leq Imz \leq 2$;
7) $Imz > 1$;
8) $1 \leq |z+2+i| \leq 2$;
9) $|z-1| < |z-i|$;
10) $|z-a| < |1-az|$, a –haqiqiy son bo‘lib $a \neq 1$;
11) $Im\left(\mathrm{Im}\frac{1}{z}\right) < \frac{1}{2}$;
12) $\left|\frac{z-3}{z-2}\right| \geq 1$.

## 2.3. Ba’zi egri chiziq tenglamalarini kompleks o‘zgaruvchi orqali ifodasi

Tekislikdagi egri chiziq turli tenglamalar orqali ifoda qilinishi mumkin.

1. Agar tekislikdagi egri chiziq dekart koordinatalari sistemasidagi

$$y = f(x) \quad (a \leq x \leq b)$$

tenglamasi bilan berilgan bo‘lsa , uni kompleks o‘zgaruvchi orqali tenglamasi quyidagicha amalga oshiriladi .

$$z = x+iy \text{ va } \bar{z} = x-iy$$

lardan foydalanib,

$$z+\bar{z} = (x+iy)+(x-iy) = 2x; \quad z-\bar{z} = (x+iy)-(x-iy) = 2iy$$

$$x = \frac{z+\bar{z}}{2} \quad \text{va} \quad y = \frac{z-\bar{z}}{2i} \qquad (2.1)$$

tengliklarni topamiz, bu qiymatlarni (1) ga qo‘yib,

$$\frac{z-\bar{z}}{2} = f\left(\frac{z+\bar{z}}{2}\right) \qquad (2.2)$$

ni hosil qilamiz.

2. Agar tekislikdagi egri chiziqning tenglamasi $x=x(t)$, $y=y(t)$ $(t_0 \leq t \leq T)$ parametrik tenglamalari orqali berilgan bo‘lsa, uni kompleks o‘zgaruvchi orqali

*z=x+iy=x(t)+iy(t)=z(t),*

yani *z=z(t)* $(t_0 \le t \le T$ ko'rinishda yozish mumkin.

**Misollar**

1. Ellipsning tenglamasini kompleks ko'rinishga keltirilsin.

a) Ellipsning dekart koordinatalari sistemasidagi tenglamasi berilgan

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Bu tenglamani kompleks ko'rinishiga keltiring.

Yechilishi: (2.1) formulaga asosan

$$\frac{(z+\bar{z})^2}{4a^2} + \frac{(z-\bar{z})^2}{-4b^2} = 1$$

2. Giperbolaning dekart koordinatalari sistemasidagi tenglamasini kompleks ko'rinishga keltirilsin .

Yechilishi: (2.1) formuladan foydalanamiz

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Longrightarrow \frac{(z+\bar{z})^2}{4a^2} + \frac{(z-\bar{z})^2}{4b^2} = 1$$

Agar ellips *x=acost, y=bsint,*

$(\ 0 \le t \le 2\pi\ )$, parametrik tenglamalari orqali berilgan bo'lsa, uning kompleks shaklidagi tenglamasi

$z = acost\ + ibsint$ ko'rinishda bo'ladi.

agar *a=b=r* bo'lsa,

$z = r(\ cost + isint) = re^{it}$ $(\ 0 \le t \le 2\pi\ )$ ko'rinishida yozish mumkin.

Buni esa $\frac{(z+\bar{z})^2}{4a^2} + \frac{(z-\bar{z})^2}{-4b^2} = 1$ formulaga qo'yamiz:

$$[r(\cos t + i \sin t) + r(\cos t - i \sin t)]^2 - [r(\cos t + i \sin t) - r(\cos t - i \sin t)]^2$$
$$= 4r^2 \cos^2 t + 4r^2 \sin^2 t = 4r^2(\cos^2 t + \sin^2 t) = 4r^2.$$

$(z+\bar{z})^2$- $(z-\bar{z})^2$=$(2r)^2$ hosil bo'ladi

3. Ushbu $Rez^2$=$a^2$ tenglamani qanoatlantiruvchi nuqtalar to'plami qanday chiziqni aniqlaydi.

Yechilishi: $z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$$Rez^2 = x^2 - y^2 = a^2;$$

Bu giperbola tenglamasidir.

4. Ushbu |*z-y*|=|*z+2*| tenglama qanday chiziqni ifodalaydi?

Yechilishi: Kompleks sonning modulini aniqlaymiz

$$|z-i\ | = |x+iy-i| = |x+i(y-1)| = \sqrt{x^2 + (y-1)^2},$$

$$|z+2| = |x+\ iy\ +2\ | = \sqrt{(x+2)^2 + y^2},$$

bularni berilgan tenglamaga qo'yamiz:

$$(\sqrt{x^2 + (y-1)^2})^2 = (\sqrt{y^2 + (x+2)^2})^2$$

$x^2 + (y-1)^2 = y^2 + (x+2)^2$; $x^2 + y^2 - 2u + 1 = y^2 + x^2 + 4x + 4$;
$4x + 2u + 3 = 0$

To'g'ri chiziq tenglamasi kelib chiqadi. Bu chiziq

$z_0 = -2,\ z_1 = i$ nuqtalarni tutashtiruvchi kesmaning o‘rtasidan o‘tadigan perpendikulyar to‘g‘ri chiziq ekanligini ko‘rsatish mumkin.

5. To‘g‘ri chiziqning ushbu

$$Ax + By + S = 0$$

tenglamasini kompleks ko‘rinishiga keltiring.

Yechilishi (2) formuladan foydalanib,

$A\frac{z+\bar{z}}{2} + B\frac{z-\bar{z}}{2i} + C = 0 \Rightarrow (A - iB)z + (A + iB)\bar{z} + 2S = 0$ agar

$A + iB = \alpha$, $A - iB = \bar{\alpha}$ desak, u holda $\alpha\bar{z} + \bar{\alpha}z + 2S = 0$ bo‘ladi.

**Mustaqil yechish uchun misollar**

Quyidagi tenglamalarning har biri qanday chiziqni ifodalashini aniqlang:

1) $Re\left(\frac{1}{z}\right) = 1;$
2) $J_m(\overline{z^2 - z}) = 2 - Imz;$
3) $z^2 + \bar{z}^2 = 1$
4) $|z-i| - |z+2| = 2;$
5) $|z| - Rez = 12;$
6) $Re(z^2 - \bar{z}) = 9$
7) $Re(1+z) = |z|;$
8) $z = t + \frac{1}{t} + t;$
9) $z = i + \frac{1}{i};$
10) $2z\bar{z} + (2+i)z + (2-i)\bar{z} = 2;$
11) $Arg(z-i) = \frac{\pi}{4};$
12) $|z-3|Imz = 6.$

## 2.4. Kompleks o‘zgaruvchili funksiyalar limiti va uzluksizligi

Agar aniq o‘zgarmas $A$ kompleks son va ixtiyoriy kichik musbat $\varepsilon$ son uchun shunday $\delta(\varepsilon) > 0$ sonni topish mumkin bo‘lib, $|z - z_0| < \delta$ tengsizlikni qanoatlantiradigan barcha $z_0$ dan farqli z lar uchun $|f(z) - A| < \varepsilon$ tengsizlik bajarilsa, $A$ son $f(z)$ funksiyaning z o‘zgaruvchili $z_0$ (qo‘zg‘almas) nuqtaga intilgandagi limiti deb ataladi va

$$\lim_{z \to z_0} f(z) = A$$

ko‘rinishida yoziladi.

$\lim_{z \to z_0} f(z) = f(z_0)$ bo‘lsa, $W = f(z)$ funksiya $z_0$ nuqtada uzluksiz deyiladi.

**Misollar**

1. $W = z^2$ funksiyaning uzluksizlik sohasi topilsin.

Yechilishi: $z_0 = x_0 + iy_0$ ixtiyoriy o‘zgarmas son uchun funksiya orttirmasi

$$\Delta W = (z_0 + \Delta z)^2 - z_0{}^2 = z_0{}^2 + 2\Delta z z_0 + \Delta z^2 - z_0{}^2 = 2\Delta z z_0 + \Delta z^2;$$

Orttirmaning limitini hisoblaymiz.

$$\lim_{\Delta z \to 0} \Delta W = \lim_{\Delta z \to 0} (2z_0 \Delta z + \Delta z^2) = 0$$

demak, $W = z^2$ funksiya $z_0$ nuqtada uzluksiz. $z_0$ nuqta tekislikning ixtiyoriy nuqtasi bo‘lgani uchun funksiya butun tekislikda uzluksiz.

2. $W = \frac{1}{z}$ funksiyaning uzluksizligi tekshirilsin.

Yechilishi:

$$\Delta W = \frac{1}{z_0 + \Delta z} - \frac{1}{z_0} = \frac{z_0 - z_0 - \Delta z}{z_0(z_0 + \Delta z)} = \frac{-\Delta z}{z_0(z_0 + \Delta z)}; \qquad (z_0 \neq 0)$$

bo‘lsa, oxirgi ifodaning limiti nolga intiladi.

$$\lim_{\Delta z \to 0} \Delta W = \lim_{\Delta z \to 0} \frac{-\Delta z}{z_0(z_0 + \Delta z)} = -\frac{1}{z_0} \lim_{\Delta z \to 0} \frac{\Delta z}{z_0 + \Delta z} = -\frac{1}{z_0} \cdot 0 = 0.$$

demak, $W = \frac{1}{z}$ funksiya tekislikning $z_0 = 0$ nuqtasidan boshqa barcha nuqtalarida uzluksiz.

3. $W = z^2$ funksiyaning uzluksizlik sohasi topilsin.

Yechilishi: $z_0 = x_0 + iy_0$ ixtiyoriy o‘zgarmas son uchun

$$\Delta w = (z_0 + \Delta z)^2 - z_0^2 = 2z_0 \Delta z + \Delta z^2;$$

$$\lim_{\Delta z \to 0} \Delta w = \lim_{\Delta z \to 0} \left(2z_0 \cdot \Delta z + \Delta z^2\right) = 0.$$

Demak, $W = z^2$ funksiya $z_0$ nuqtada uzluksiz. $z_0$ nuqta tekislikning ixtiyoriy nuqtasi bo‘lganligi uchun u butun tekislikda uzluksiz.

4. $W = \bar{z} = x - iy$ bo‘lsa, $\Delta W = \Delta x - i\Delta y$ ,

$$\lim_{\Delta z \to 0} \Delta w = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x - i\Delta y) = 0.$$

Demak, bu funksiya ham tekislikning hamma chekli nuqtalarida uzluksiz.

5. $W = \frac{1}{z}$.

$$\Delta w = \frac{1}{z_0 + \Delta z} - \frac{1}{z_0} = \frac{-\Delta z}{z_0(z_0 + \Delta z)}.$$

Agar $z_0 \neq 0$ bo‘lsa, o‘ng tomonning limiti nolga intiladi, ya’ni

$$\lim_{\Delta z \to 0} \Delta w = -\frac{1}{z_0} \lim_{\Delta z \to 0} \frac{\Delta z}{z_0 + \Delta z} = 0.$$

Demak, $W = \frac{1}{z}$ funksiya tekislikning $z_0 = 0$ nuqtasidan boshqa barcha nuqtalarida uzluksiz.

**Mustaqil yechish uchun misollar**

1) $W = 2z^3$ funksiyaning uzluksizligi ko‘rsatilsin.
2) $W = \frac{1}{z^2+1}$ funksiyaning uzluksizlik sohasi ko‘rsatilsin.
3) $f(z) = \frac{1}{z-1}$ funksiya birlik doira ichida uzluksiz bo‘ladimi?
4) $W = 3xy - 5(x + y^2)i$ uzluksizlikka tekshirilsin.

## 2.5. Kompleks o‘zgaruvchili funksiyaning hosilasi

Agar $\Delta z$ har qanday yo‘l bilan nolga intilganda ham $\frac{\Delta w}{\Delta z}$ nisbat faqat birgina aniq limitga intilsa, o‘sha limit $f(z)$ funksiyaning $z_0$ nuqtadagi hosilasi deyiladi va $w'$, $f'(z_0), \frac{dw}{dz}, \frac{df}{dz}$ belgilashlarning birortasi orqali belgilanadi. Ya’ni

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \qquad (2.3)$$

bunda $z = z_0 + \Delta z$ nuqta ham $G$ sohada yotadi.

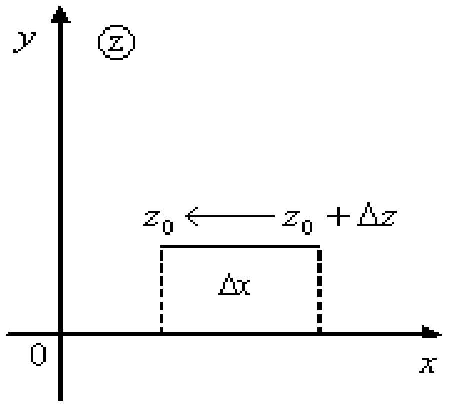
$$w = f(z) = u(x,y) + iv(x,y), \ \Delta w = \Delta u + i\Delta v, \ \ \Delta z = \Delta x + i\Delta y$$

tengsizliklarga asoslanib (1) tenglikni

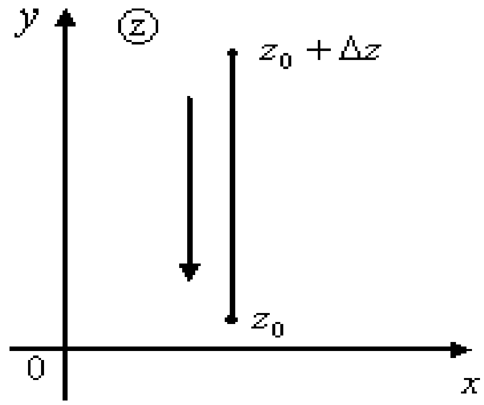
$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \qquad (2.4)$$

ko‘rinishda yozish ham mumkin.

Agar $w = f(z)$ funksiya $z_0$ nuqtada hosilaga ega bo‘lsa, uni shu nuqtada differensiallanuvchi (yoki monogen) funksiya deyiladi.




*5- chizma.*




*6- chizma.*

**Differensiallash qoidasi**. Ko‘rdikki, agar $w=f(z)$ ning $G$ sohaga tegishli biror *z* nuqtada hosilaga ega bo‘lsa, u holda Dalamber-Eyler sharti bajariladi. Ana shu shartlardan foydalanib hosilani topish uchun quyidagi to‘rtta formulalarga ega bo‘lamiz:

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}. \qquad (2.5)$$

Lekin funksiya $w=f(z)$ ko‘rinishida berilganda, ya’ni uning haqiqiy va mavhum qismi ajratilmagan bo‘lsa, uning hosilasini topish uchun (2.5) formulalardan foydalanish noqulaylik tug‘diradi.

Kompleks o‘zgaruvchili funksiyaning yuqorida keltirilgan hosilasining ta’rifi haqiqiy o‘zgaruvchili funksiyaning hosilasiga berilgan ta’rifdan mutlaqo farq qilmaydi. Shu sababli $w=f(z)$ funksiyadan hosila olishda haqiqiy o‘zgaruvchili funksiyalarning hosilalarini topish uchun chiqarilgan hosilalar jadvalidan hamda differensiallash qoidalaridan to‘liq foydalanish mumkin.

**Hosilalar jadvali**

1. $C'=0,$ (S-ixtiyoriy son)
2. $z'=\frac{dz}{dz}=1;$
3. $\left(\frac{1}{z}\right)'=-\frac{1}{z},\ z\neq -1;$
4. $(z^n)'=nz^{n-1};$
5. $(z^{-n})'=-\frac{n}{z^{n+1}},\ z\neq 0;$
6. $(\sqrt{z})'=\frac{1}{2\sqrt{z}};$
7. $(e^z)'=e^z;$
8. $(e^{mz})'=me^{mz};$
9. $(lnz)'=\frac{1}{z}\ \ z\neq 0;$
10. $(sinz)'=cosz;$
11. $(cosz)'=-sinz;$

12. $(tgz)' = \frac{1}{cos^2 z}$;

13. $(ctgz)' = -\frac{1}{sin^2 z}$;

14. $(shz)' = chz$ ;

15. $(chz)' = shz$;

16. $(thz)' = \frac{1}{ch^2 z}$;

17. $(cthz)' = -\frac{1}{sh^2 z}$ va hokazo.

Agar funksiya $w=u+iv$ ko‘rinishida berilgan bo‘lsa, undan hosila olish uchun to‘rtta funksiyaning biridan foydalanishga to‘g‘ri keladi.

$$\mathcal{W}'=f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \qquad (2.6)$$

Berilgan funksiyaning biror $z\epsilon G_1$ nuqtada hosilaga ega bo‘lishi uchun quyidagi Koshi-Riman

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2.7)$$

sharti bajarilishi zarur va yetarlidir. Bu shartlar Koshi-Riman ba’zan Dalamber- Eyler shartlari deyiladi.

**Misollar**

1. $\mathcal{W}=z^2$ funksiyaning hosilasini toping.

Yechilishi : Jadvaldan foydalanib $\mathcal{W}' = (z^2)' = 2z$ ni topamiz.

Buning to‘g‘riligini Koshi-Riman shartidan foydalanib tekshirib ko‘raylik

$\mathcal{W}=z^2=(x+iy)^2 = x^2 - y^2 + 2ixy; \ u = x^2 - y^2; \ v = 2xy;$

$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = -2y; \ \frac{\partial v}{\partial x}=2u, \ \frac{\partial v}{\partial y} = 2x$. Bulardan esa

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \ , \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$$

Ya’ni (2.7) shartlar bajariladi, demak hosila mavjud. Endi o‘sha hosilani quyidagicha topamiz:

$$\mathcal{W}' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}=2x+i2y = 2z.$$

2. Ushbu $W = 3x^2 - 5y^2 + i2xy$ funksiyaning hosilasini toping

Yechilishi: Dastlab (2.7) shartlarni tekshirib ko‘ramiz:

$$u = 3x^2 - 5y^2; \ v = 2xy; \frac{\partial u}{\partial x} = 6x; \ \frac{\partial u}{\partial y} = -10y,$$

$$\frac{\partial v}{\partial y} = 2y; \ \frac{\partial v}{\partial x} = 2x; \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \quad 6x = 2x \ \ => 4x = 0 \ ,x = 0;$$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \ dan \ \ -10y = -2y => 8y = 0 \ => y = 0$ demak funksiya (0;0) nuqtada hosilaga ega ekan.

Shu hosilani topamiz :

$$W' = f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Big|_{z=0} = (6x + i2y)|_{z=0}$$

3. Ushbu $W = 5xy - 6y + 9y + i(x^2 - y^2)$ funksiyaning hosilasini toping.

Yechilishi: $u = 5xy - 6x + 9y, \quad v = x^2 - y^2;$

$$\frac{\partial u}{\partial x} = 5y - 6, \frac{\partial u}{\partial y} = 5x + 9, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2y;$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ dan } 5y - 6 = 2y => \ 7y = 6 => y = \frac{6}{7};$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \text{ dan } \ 5x + 9 = -2x => 7x = -9 => x = \frac{-9}{7}$$

Demak, $z_0 = \frac{-9}{7} + i\frac{6}{7}$ nuqtadagi hosila mavjud bo‘lib u quyidagilardan iborat:

$$W' = f'\left(\frac{-9}{7} + i\frac{6}{7}\right) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} = (5y - 6 + i2x)_{z_0} = 5\cdot\frac{6}{7} + 2i\left(\frac{-9}{7}\right) =$$

$$= \frac{30 - 18i}{7}.$$

4. $W = x - iy$ funksiyaning hosilasini toping.

Yechilishi: Bu funksiya $z = x + iy$ ga qo‘shmadir.

Berilgan misolda $u = x, \ v = -y$.

$\frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial y} = -1$ larni topamiz. Bunda ushbu $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ shart buziladi , chunki

1= -1 bo‘lib qoldi.

Demak , $W = x - iy = z$ funksiya hech qanday nuqtada hosilaga ega emas.

Lekin funksiya uzluksiz. Odatda funksiyaning uzluksizligi quyidagidan ma’lum bo‘ladi.

$$\lim_{\Delta z \to 0} \Delta W = 0$$

Berilgan misolda

$$\Delta W = \Delta x - i\Delta y, \quad \lim_{\Delta z \to 0}(\Delta x - i\Delta y) = 0,$$

Shunday qilib ,funksiya uzluksiz bo‘lsa ham hosilaga ega bo‘lmay qolishi mumkin, lekin aksincha, funksiya hosilaga ega bo‘lsa , u shu nuqtada uzluksiz bo‘ladi .

5. $f(z) = z \cdot \overline{z}$ funksiyaning hosilasini toping.

Yechilishi : $f(z) = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2,$

$$u = x^2 + y^2, \qquad v = 0, \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0.$$

Koshi-Riman shartlariga muvofiq

$2x = 0 => x = 0; \ 2y = 0 => y = 0; z = 0$

$$f'(0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Bigg|_{z=0} = (2x + i \cdot 0)|_{x=0} = 0$$

Demak, berilgan funksiya faqat birgina $z = 0$ nuqtada hosilaga ega bo‘lib, boshqa nuqtalarda hosilaga ega emas.

**Quyidagi kompleks argumentli funksiyalarning hosilalarini toping**

1 . $W = \sin(2e^z)$;
2. $W = e^{\cosh z}$;
3. $W = ze^{-z}$;
4. $W = \frac{e^z}{z}$;
5. $W = \frac{z\cos z}{1+z^2}$;
6. $W = \frac{e^z+1}{e^z-1}$;
7. $W = \frac{1}{\operatorname{tg} z + \cot z}$
8 . $W = (e^z + e^{-z})^{-2}$;
9 . $W = \cos 3(1 - z^2)$;
10 . $W = \frac{\operatorname{tg} z}{\sin z}$;
11. $W = \frac{z^3+z^2}{1-z}$.

## 2.6.Analitik funksiyalar

**1-ta’rif**. Agar bir qiymatli $w = f(z)$ funksiya $G$ sohaning barcha nuqtalarida differensiallanuvchi ya’ni hosilaga ega bo‘lsa, u funksiya o‘sha sohada *analitik* (*golomorf* yoki *regulyar*) deyiladi.

**2-ta’rif**. Agar $w = f(z)$ funksiya $z_0$ nuqtada va uning biror atrofida ham differensiallanuvchi bo‘lsa, u funksiya shu nuqtada *analitik* deyiladi.

Agar $w = f(z)$ funksiya faqat $z_0$ nuqtada hosilaga ega bo‘lib, lekin uning atrofida hosilasi mavjud bo‘lmasa, u holda funksiya $z_0$ nuqtada *monogen* bo‘ladi. Demak, funksiyaning $z_0$ nuqtada monogen bo‘lishidan uning shu nuqtada analitik bo‘lishi kelib chiqmaydi.

Agar noma’lum analitik funksiyaning haqiqiy yoki mavhum qismi ma’lum bo‘lsa , u holda analitik funksiyaning o‘zini topish mumkin . Birinchi usul $F(z)$ analitik bo‘lsa , u holda $f(z)$ analitik funksiya uchun boshlang’ich funksiya ya’ni $F'(z) = f(z)$ bo‘lsa $F'(z) = \int f(z)\,dz + c$ ko‘rinishida yozish mumkin.

Agar $W = f(z) = u(x, y) + iv(x, y)$ bo‘lsa $W_{\bar{z}} = 0$ va

$$W_z' = f'(z) = u_x - -iu_y = v_y + iv_x = u_x + iv_x = u_y + iv_y \quad , \text{ unda}$$

$$f(z) = \int_{z_0}^{z} (u_x - iu_y) dz = \int_{z_0}^{z} (v_y - iv_x) dz.$$

Shunday qilib, berilgan $u(x, y)$ yoki $v(x, y)$ ga asosan $f(z) -$ ni o‘zini topish mumkin.

**Misollar**

1. $w = e^z$ ko‘rsatkichli funksiyaning analitiklik sohasini toping.

Yechilishi: $e^z = e^x(\cos y + i \sin y)$;

$$u(x, y) = e^x \cos y; \ \frac{\partial u}{\partial x} = e^x \cos y; \ \frac{\partial u}{\partial y} = -e^x \sin y,$$

$$v(x, y) = e^x \sin y; \ \frac{\partial v}{\partial x} = e^x \sin y; \ \frac{\partial v}{\partial y} = e^x \cos y.$$

Bu xususiy hosilalar uzluksiz va

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y.$$

Dalamber-Eyler shartlari bajariladi. Demak, $e^z$ ko‘rsatkichli funksiya butun kompleks tekislikda analitik (uzluksiz ham). $e^z$ ko‘rsatkichli funksiya maxsus nuqtalarga ega emas.

2. $w = \sin z$ funksiyaning analitiklik sohasi topilsin.

Yechilishi: $\sin z$ funksiyaning haqiqiy va mavhum qismlari

$$u(x, y) = chy \sin x, \qquad v(x, y) = shy \cos x$$

ekanligi ko‘rsatilgan edi.

$$\frac{\partial u}{\partial x} = chy\cos x, \quad \frac{\partial u}{\partial y} = shy\sin x, \quad \frac{\partial v}{\partial x} = -shy\sin x, \quad \frac{\partial v}{\partial y} = chy\cos x$$

va

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = chy \cos x; \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -shx \sin x.$$

Barcha xususiy hosilalar uzluksiz va Dalamber-Eyler shartlari bajariladi. Demak $w = \sin z$ funksiya butun kompleks tekisligida analitik (uzluksiz ham).

3. $z = x^2 + y^2 + ixy^2$ funksiyaning analitiklik sohasini toping.

Yechilishi: $u = x^2 + y^2, \ v = xy^2$;

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = 2y, \ \frac{\partial v}{\partial x} = y^2, \ \frac{\partial v}{\partial y} = 2xy.$$

Bulardan ko‘rinib turibdiki, Dalamber-Eyler shartlarning bajarilishi uchun $x = 0, \ y = 0$ bo‘lishi kerak. Demak funksiya $(0; 0)$ nuqtadan boshqa nuqtalarda hosilaga ega emas, ya’ni berilgan funksiya butun kompleks tekisligida analitik emas. Bu funksiyaning butun kompleks tekisligida uzluksizligini ko‘rsatish mumkin. Shunday qilib butun kompleks tekisligida uzluksiz funksiya uchun shu tekislikning barcha nuqtalari maxsus nuqta bo‘lar ekan.

4. $w = z^2$ funksiyaning analitiklik sohasini toping.

Yechilishi: $\frac{dw}{dz}=2z$ va demak, $w=z^2$ funksiya kompleks tekisligining barcha nuqtalarida analitik.

5. $f(z)=z^3$ funksiyaning analitikligini tekshiring.

Yechilishi:

$$f'(z)=3z^2;\ f'\left(\frac{1}{i}\right)=3\left(\frac{1}{i}\right)^2=-3;\quad f'(2-3i)=3(2-3i)^2,\ldots,$$

Demak har qanday chegaralangan sohada bu funksiya analitik ekan .

6. $f(z)=\frac{1}{z+i}$ funksiyaning analitikligini tekshiring.

Yechilishi : $f'(z)=\left(\frac{1}{z+i}\right)'=-\frac{1}{(z+i)^2}$ , $f'(i)=-\frac{1}{(i+i)^2}=\frac{1}{4}$ , ...

Demak, $z=-i$ nuqtadan boshqa barcha nuqtalarda funksiya analitik bo‘lib, $-i$ maxsus nuqtadir.

7. $u(x,y)=\frac{x}{x^2+y^2}$ $va$ $f(\pi)=\frac{i}{\pi}$,

Berilgan bo‘lib funksiyaning o‘zini topish talab qilinadi.

Yechilishi:

$$u'_x=\frac{y^2-x^2}{(x^2+y^2)^2}\ ;\quad u'_y=\frac{-2yx}{(x^2+y^2)^2}.$$

$$f'(z)=\frac{y^2-x^2}{(x^2+y^2)^2}-i\frac{2yx}{(x^2+y^2)^2}=\frac{y^2+2ixy-x^2}{(x^2+y^2)^2}=\frac{(y-ix)^2}{(x^2+y^2)^2}=$$

$$=\frac{i(x-iy)}{(x^2+y^2)^2}=\frac{(i\bar{z})^2}{(z\bar{z})^2}=-\frac{z^{-2}}{z^2\bar{z}^2}=-\frac{1}{z^2}$$

U holda $f(z)=-\int\frac{dz}{z^2}+c=\frac{1}{z}$ +c

$$f(\pi)=\frac{1}{\pi}+c=\frac{1}{\pi}\ .\ =>\ c=0$$

demak, $f(z)=\frac{1}{z}$.

8. Analitik funksiya $f(z)$ ning mavhum qismi

$$v(x,y)=arctg\frac{y}{x}\quad (x>0),\quad f(1)=0$$

berilgan . Funksiyaning o‘zini toping.

Yechilishi. $\frac{\partial v}{\partial x}=-\frac{y}{x^2+y^2}$ , . $\frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}$

$$f'(z)=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=\frac{x-iy}{x^2+y^2}=\frac{\bar{z}}{z\cdot\bar{z}}=\frac{1}{z}.$$

$f(z)=\int\frac{1}{z}dz+C=lnz+C. f(1)=C=0$ dan

$$f(z)=lnz.$$

9. Analitik funksiya $f(z)$ ning mavhum qismi berilgan:

$$v(x,y)=-2\sin 2x\cdot sh2x+y,\quad f(0)=2$$

Funksiyaning o‘zini toping.

Yechilishi: $\frac{\partial v}{\partial x} = -4cos2xsh2y;$

$$\frac{\partial v}{\partial y} = -4sin2ch2y + 1$$

$$f'(z) = -4sin2xch2y + 1 - i4cos2xsh2y = -4(sin2(x+iy)) + 1$$

$$= -4sin2z + 1.$$

$$f(z) = -4\int sin2\,zdz + \int dz + C = \frac{4cos2z}{2} + z + C.$$

$$f(0) = 2 + C = 2 \qquad => C=0.$$

Demak, $f(z) = 2cos2z+z.$

**Mustaqil yechish uchun mashqlar**

1. $cosz;\ z^2;\ e^{-z^2}$ funksiyalarning butun kompleks tekisligida analitikligi isbotlansin.

2. Funksiya hosilasining ta'rifidan foydalanib $(e^z)' = e^z;\ (\sin z)' = \cos z,$ $(cosz)' = -sinz,\ (Lnz)' = \frac{1}{z}$ tengliklarning to'g'riligi isbotlansin.

3. Mavhum qismi $v = \frac{y}{x^2+y^2}$ va o'zi $w(2) = 0$ shartni qanoatlantiruvchi funksiya topilsin.

4. Haqiqiy qismi $u = x^2 - y^2 + xy$ va o'zi $w(0) = 0$ shartni qanoatlantiruvchi funksiya topilsin.

5. $w = \ln r + i\varphi\ \ (-\pi < \varphi < \pi)$ funksiyaning, bunda $z = r(\cos\varphi + i\sin\varphi)$, nol nuqtadan boshqa hamma nuqtalarda analitikligi isbotlansin.

6. $f(z) = \frac{1}{z-1}$ funksiya birlik doira ichida uzluksiz bo'ladimi?

7. $w = 3xy - 5(x + y^2)i$ uzluksizlikka va analitiklikka tekshirilsin.

8. $w = \frac{1}{z}$ funksiyaning analitiklik sohasi topilsin.

9. $w = (1-2xy) + i(x^2 - y^2)$ funksiyaning analitiklik sohasi topilsin.

10. $w = x^3 - 2xy^2 + i(3x^2y - y^3)$ funksining analitiklik sohasini toping.

11. $w = 5 - 2xy - i(3y^2 + x^2)$ funksiyaning maxsus nuqtalarini toping.

### 2.7. Garmonik funksiyalar

Ushbu $$\Delta\omega = \frac{\partial^2\omega}{\partial x^2} + i\frac{\partial^2\omega}{\partial y^2} = 0 \quad (2.8)$$

Laplas tenglamasini qanoatlantiradigan har qanday ikki argumentli $\omega = \omega(x, y)$ funksiya garmonik funksiya deyiladi.

**Misollar**

1. Funksiyaning haqiqiy qismi berilgan: $u = x^2 + 2x - y^2$

Funksiyaning garmonikligini tekshiring.

Yechilishi: *u* dan ikki marta xususiy hosila olib, (2.8) Laplas tenglamasiga qo‘yib ko‘ramiz:

$$\frac{\partial u}{\partial x} = 2x + 2, \quad \frac{\partial^2 u}{\partial x^2} = 2\,; \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2;$$

shunga asosan

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0.$$

Demak, berilgan funksiya garmonik ekan.

2. Funksiyaning garmonikligini tekshiring:

$$u = ln(x^2 + y^2)$$

Yechilishi: $\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$ , $\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

shularga asosan $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Funksiya garmonik ekan.

**Ikkinchi usul:**

Noma’lum analitik funksiyaning haqiqiy yoki mavhum qismi berilgan bo‘lsa, uning o‘zini ikkinchi usul bilan topish ham mumkin. Uning uchun Koshi – Riman shartlaridan foydalanishga to‘g‘ri keladi.

3. Funksiyaning haqiqiy qismi va qo‘shimcha shart berilgan.

$u(x,y) = 2e^x cosy, \; f(0) = 2$

$W = f(z)$ analitik funksiyani ikkinchi usul bilan toping.

Yechilishi. Koshi-Riman shartlari

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

dan foydalanib, $v(x,y)$ funksiyani topish mumkin.

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2e^x cosy$, bundan $y$ bo‘yicha integral olsak

$v = 2e^x \int cosydy + \varphi(x) = 2e^x siny + \varphi(x)$

tenglikning ikki tomonidan $x$ bo‘yicha xususiy hosila olib,

(1) shartlarning ikkinchisidan foydalanamiz, u holda

$\frac{\partial v}{\partial x} = 2e^x siny + \varphi'(x) = -\frac{\partial u}{\partial y} = -(2e^x cosy)' = 2e^x siny$

$\varphi'(x) = 0 \;\; => \;\; \varphi(x) = C$ kelib chiqadi. Demak,

$v = 2e^x siny + C$

Bularga asosan

$W = f(z) = u(x,y) + iv(x,y) = 2e^x cosy + i2e^x siny +$

$+ iC = 2e^x(cosy + i\,siny) + iC = 2e^x \cdot e^{iy} + iC = 2e^{x+iy} + iC =$

$= 2e^z + iC.$

Endi boshlang‘ich shartdan foydalanib C ni topamiz, ya’ni

$f(0) = 2, \Rightarrow 2e^0 + iS=2 => S = 0$

Shunday qilib $W = f(z) = 2e^z$ .

**Mustaqil yechish uchun misollar**

Haqiqiy yoki mavhum qismi berilgan analitik funksiyalarni tiklang:

1. $u = x^2 - y^2 + 2y,\ f(i) = 2i - 1;$
2. $v = 2(chx \cdot siny - xy), f(0)=0;$
3. $v = -2sin2xsh2y + y,\ f(0) = 2;$
4. $v = 2cosx\, chy - x^2 + y^2,\ f(0) = 2;$
5. $u = x^2 - y^2 + 5x + y - \frac{y}{x^2 + y^2}$ ;
6. $v = 3 + x^2 - y^2 - \frac{y}{2(x^2 + y^2)}$ ;
7. $v = e^x(xcosy - ysiny) + 2sinx \cdot shy + x^3 - 3x^2y + y;$
8. $v = ln(x^2 + y^2) + x - 2y;$
9. $u = x^2 - y^2 + xy$ ;

**Nazorat savollari**

1. Limit nuqta nima?
2. Ketma-ketlikning limitini ta’riflang.
3. Funksiyaning nuqtadagi limitini ta’riflang.
4. Limitning xossalarini ayting.
5. Funksiyaning nuqtada va sohada uzluksizligini ta’riflang.
6. Tekis uzluksizlikka ta’rif bering.
7. Uzluksiz funksiya qanday xossalarga ega?
8. Funksiyaning hosilasiga ta’rif bering.
9. Monogen funksiyaning ta’rifini ayting.
10. Dalamber-Eyler shartlarini ayting.
11. Funksiya differensiallanuvchi bo‘lishining zaruriy va yetarlilik shartlarini ayting.
12. Differensiallash qoidalarini ayting.
13. Analitik funksiyaga ta’rif bering.
14. Differensiallanuvchi va uzluksiz funksiyalar orasida qanday munosabat mavjud?
15. To‘g‘ri va maxsus nuqtalarga ta’rif bering.

## 3. Kompleks o‘zgaruvchili funksiyaning integrali

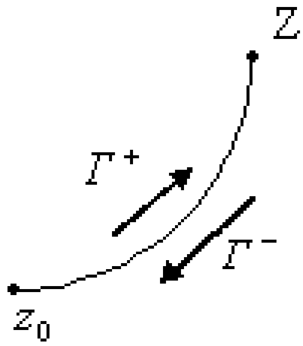
### 3.1. Integralning ta’rifi

Kompleks $Z$ tekislikdagi biror $\tilde{A}$ sohada uzluksiz

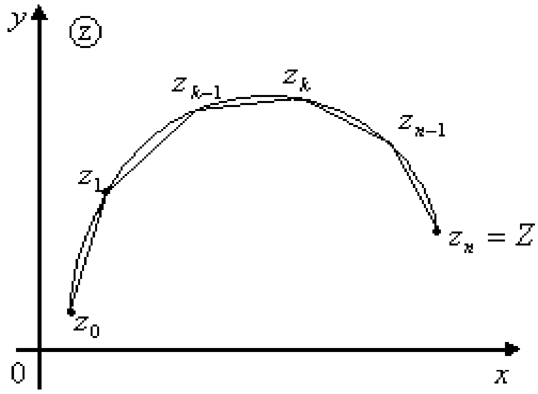
$$W = f(z) = u(x, y) + iv(x, y) \tag{3.1}$$

funksiya berilgan bo‘lsin. $\Gamma$ silliq chiziq $G$ sohaning ichida yotuvchi ixtiyoriy egri chiziq bo‘lib, $z = z(t)\ (\alpha \le t \le \beta)$ uning tenglamasi, $z_0$ boshlang‘ich nuqtasi, z oxirgi nuqtasi bo‘lsin, ya’ni $z_0 = z(\alpha), z = z(\beta)$.

$\tilde{A}$ chiziqda ikki yo‘nalishni aniqlash mumkin: bulardan bittasi *t* parametrning o‘sishiga mos keluvchi musbat yo‘nalish $\tilde{A}$ yoki $\tilde{A}^+$ ikkinchisi esa bunga qarama–qarshi yo‘nalish–manfiy yo‘nalish $\tilde{A}$ yoki $\tilde{A}^-$ (1-chizma).




*7-chizma.*




*8-chizma.*

$\tilde{A}$ chiziqni

$$z_0, z_1, z_2, \ldots, z_k, \ldots, z_{n-1}, z_n = Z, \tag{3.2}$$

nuqtalar yordamida *n* ta ixtiyoriy yoychalarga bo‘lib yoychalarning har birida bittadan ixtiyoriy nuqtalarni olib ularni

$$\eta_1, \eta_2, \ldots, \eta_k, \ldots, \eta_n, \ldots$$

orqali belgilaymiz (8-chizma). *f(z)* funksiyaning bu nuqtalardagi qiymatlari $f(\eta_1), f(\eta_2), \ldots, f(\eta_k), \ldots f(\eta_n), \ldots$

larini hisoblab ularni

$$\Delta z_1 = z_1 - z_0,\ \Delta z_2 = z_2 - z_1,\ \ldots, \Delta z_k = z_k - z_{k-1},\ \ldots, \Delta z_n = z_{n-1} - z_n$$

mos ayirmalarga ko‘paytirib quyidagi *integral yig‘indini* tuzamiz:

$$S_n = f(\eta_1)\Delta z_1 + f(\eta_2)\Delta z_2 + \cdots + f(\eta_k)\Delta z_k + \cdots + f(\eta_n)\Delta z_n = \sum_{k=1}^{n} f(\eta_k)\Delta z_k \tag{3.3}$$

(3.2) nuqtalarni ketma-ket kesmalar orqali tutashtirib $\Gamma$ egri chiziqqa ichki chizilgan siniq chiziqni hosil qilamiz. Bu siniq chiziq bo‘g‘inlarining uzunliklari $|\Delta z_1|, |\Delta z_2| \ldots, |\Delta z_k|, \ldots, |\Delta z_n|$ lardan iborat bo‘ladi. Bularning eng kattasini $\lambda$ orqali belgilaymiz.

**1-ta’rif.** Agar $\lambda \to 0$ da (4) integral yig‘indi $z_1, z_2, \ldots, z_n$ va $\eta_1, \eta_2, \ldots, \eta_n$ nuqtalarning tanlashiga bog‘liq bo‘lmagan aniq chekli limitga intilsa, bu limit *f(z)* funksiyadan G chiziq bo‘yicha olingan *integral* deyiladi va

$$\int_G f(z)dz \qquad (3.4)$$

kabi yoziladi. Demak ta'rifga binoan:

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\eta_k)\Delta z_k = \int_{\Gamma} f(z)\,dz .$$

G chiziq *integrallash yo'li* yoki *konturi* deyiladi.

(1) ni hisobga olsak

$$\int_\Gamma f(z)\ dz = \int_\Gamma (u+iv)\,(dx+idy) \qquad (3.5)$$

ga ega bo'lamiz. (3.5) tenglikning o'ng tomoni haqiqiy argumentli funksiyalardan olingan egri chiziqli integrallardan iborat.

Agar chiziq yopik bo'lsa

$$\oint_G f(z)\,dz \qquad (3.6)$$

$$\int_G f(z)dz = \int_G (u-iv)(dx-idy) = \int_G udx - vdu + +i\int_G vdx + udy \qquad (3.7)$$

integralni hisoblashning turli usullari mavjud. Agar G chiziqning tenglamasi dekart koordinatalar sistemasida berilgan bo'lsa:

$$y = f(x), \quad a \le x \le b \qquad (3.8)$$

(3.7) dagi $y$ va $dy$ o'rniga (3.8) dan qiymatlar qo'yilib aniq integralga aylantiriladi. Agar G chiziqning

$$x = x(t),\ u = u(t) \quad t_0 \le t \le T \qquad (3.9)$$

parametrik tenglamalari, ya'ni $z = z(t)$ berilgan bo'lsa uni (3.6) –ga qo'yib aniq integral hosil qilinadi.

Agar G chiziq markazi $\alpha = a + ib$ nuqtada joylashgan aylanadan iborat bo'lsa, integralni topish uchun ushbu aylana tenglamasidan foydalanamiz:

$$z - \alpha = re^{i\varphi} \qquad (3.10)$$

Agar G chiziq $\alpha$ nuqtadan chiquvchi to'g'ri chiziq nurdan iborat bo'lsa ham (6) dan foydalanishimiz mumkin bo'ladi, bunda $\varphi = const$ bo'ladi ( $0 \le z < \infty$).

**Misollar**

1. Ushbu integralni hisoblang: $I = \int_G (1 + i - z\cdot\bar{z})dz$ ,

bu yerda G chiziq $z_0$=0, $z = 1 + i$ nuqtalardan o'tgan $y = x^2$ paraboladan iborat.

Yechilishi: $y = x^2, x = 0, x = 1, dy = 2xdx; z = x + iy,\ \bar{z} = x - iy$

$I = \int_0^1 (1 + i - (x+iy)(x-iy)(dx+idy) =$

$$= \int_0^1 (1 + i - (x + ix^2)9(x + +ix^2))\cdot(1+2xi)dx =$$

$= \int_0^1 [1 + i - x^2 + x^3 i - ix^3 - x^4](1+2x)dx=$

$$= \int_0^1 [(1 - -2x)dx - (1 + 2y)dy] + i[(1 + 2y)dx + (1 - 2x)dy].$$

Natijada $I = 2 + \frac{4}{3}i$

2. Ushbu integralni hisoblang: $I = \int_0^1 zsinzdz$

Yechilishi: $I$- ni bo‘laklab integrallaymiz:

$$\int zsinzdz = -zcosz + \int coszdz = -zcosz + sinz$$

$$I = (-zcosz + sinz)| = \left(-z\left(\frac{e^{iz} + e^{-iz}}{2}\right) + \left(\frac{e^{iz} - e^{-iz}}{2}\right)\right) = -\frac{1}{2}.$$

3. Ushbu integralni hisoblang: $I = \int_G e^{|z|^2} Rezdz$

bu yerda G chiziq $z_0 = 0,\ z = 1 + i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasidan iborat.

Yechilishi: Malumki, $z = x + iy, \quad dz = dx + idy,$

$|z|^2 = (x + iy)^2 = x^2 + y^2,$

$Re\, z = x,$

$z_0 = x_0 + iy_0,\ => x_0 = 0,\ z = 1 + i = x_1 + +iy_1 => x_1 \quad x\epsilon[0;1]$

G – chiziqning tenglamasi $y = x$ bo‘lgani uchun $dy = dx$; $e^{|z|^2}$

$Rezdz = e^{x^2+y^2}x(1 + i)dx = e^{2x^2}(1 + i)\frac{d(2x^2)}{4}$

$I = \frac{1}{4}\int_0^1 e^{2x^2} d(2x^2)(1 + i) = \frac{1+i}{4}e^{2x^2}| = \frac{1}{4}(1 + i)(e^2 - 1).$

4. Ushbu integralni hisoblang: $I = \oint_T \ln zdz,$

Bu yerda G |z|=1 aylana bo‘ylab $z_0 = -1$ nuqtadan chiquvchi soat strelkasiga teskari yo‘nalishli chiziq.

Yechilishi: $z = e^{i\varphi} \qquad lnz = lne^{i\varphi} = i\varphi,\ dz = ie^{i\varphi}d\varphi,$

$Lnzdz = -\varphi e^{i\varphi}d\varphi, \quad 0 \leq \varphi \leq 2\pi.$

U holda bo‘laklab integrallash natijasida quyidagiga ega bo‘lamiz:

$$I = -\int_0^{2\pi} \varphi e^{i\varphi}d\varphi = 2\pi i.$$

5. Ushbu integralni hisoblang:

$$I = -\oint_G \frac{d\alpha}{\alpha - (1+i)},$$

Bu yerda G chiziq $|\alpha - (1 + i)|$ =1 aylanadan iborat.

Yechilishi: Berilgan aylananing parametrik tenglamalarini yozib olamiz:

$x - 1 = cos\varphi, \qquad y - 1 = sin\varphi,\ x = 1 + cos\varphi,\ y = 1 + sin\varphi;$

$z = (1 + cos\varphi) + i(1 + sin\varphi) = (1 + i) + (cos\varphi + isin\varphi) =$

$= 1 + i + e^{i\varphi}; \qquad 0 \leq \varphi \leq 2\pi;$

$d\alpha = ie^{i\varphi}\, d\varphi;$

$$I = \oint_G \frac{d\alpha}{\alpha - (1+i)} =$$

$$= \oint_G \frac{ie^{i\varphi}d\varphi}{1+i+e^{i\varphi}-1-i} = \int_G \frac{ie^{i\varphi}}{e^{i\varphi}} = \oint_G id\varphi = i\oint_0^{2\pi} d\varphi = 2\pi i.$$

## 3.2. Ko‘p qiymatli funksiyalar va ularni integrallash

Agar $z = x + iy$ ga bitta qiymat berganda $W = f(\alpha)$ funksiya ham birgina qiymatni qabul qilsa, funksiya bir qiymatli, aks holda ko‘p qiymatli deyiladi.

Bu holda funksiyaning har bir qiymati o‘sha funksiyaning tarmog‘i deyiladi.

**Misollar**

1. Ushbu integralni hisoblang: $I = \int_G \frac{dz}{\sqrt{z}}$

bu yerda G chiziq $|z| = 1$ aylananing yuqori qismi

$w = \frac{1}{\sqrt{z}}$ funksiyaning shunday tarmog‘i olinsinki, natijada $\sqrt{1} = -1$ bo‘lsin.

Yechilishi. Bu misolni quyidagi usullar bilan yechish mumkin:

a) $z = r(\cos\varphi + i\sin\varphi), r = |z|, \varphi = \arg z;$

$\alpha_k = \sqrt{z} = \left(\cos\frac{\varphi+2\pi k}{2} + i\sin\frac{\varphi+2\pi k}{2}\right), k = 0,1,2,...$ bu yerda $r = 1$

$\alpha_0 = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2};$ $\alpha_1 = \cos\left(\frac{\varphi}{2} + \pi\right) + i\sin\left(\frac{\varphi}{2}+\pi\right) = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2};$

Berilgan shartga ko‘ra $\left(\sqrt{1} = -1\right)$ $\alpha_1$ ildiz olishga majburmiz, chunki $z = 1$, bo‘lganda

$\varphi = 0$ bo‘lib $\alpha_1 = -1$ bo‘ladi.

Nyuton-Leybnis formulasiga muvofiq

$I = \int_{+1}^{-1} z^{-1/2} dz = 2\sqrt{z}\,| = 2\left(\sqrt{-1} - \sqrt{+1}\right)$, Ma’lumki

$z = -1$ bo‘lganda $\alpha = \pi$ bo‘lib, $\alpha_1$ ga muvofiq

$$\sqrt{-1} = -\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = -i.$$

Demak, $I = 2(-i+1) = 2(1-i)$.

b) $z = re^{i\varphi}$, $r = |z| = 1$, $z = e^{i\varphi}$

$$\sqrt{z} = \sqrt{e^{i\varphi}} = \sqrt{\cos\varphi + i\sin\varphi} = \cos\frac{\varphi + 2k\pi}{2} + i\sin\frac{\varphi + 2k\pi}{2} = dk,$$

$k = 0,1;...$

$$\alpha_0 = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2} = e^{i^1/2},$$

$\alpha_1 = cos\frac{\varphi + 2k\pi}{2} + \, isin\frac{\varphi + 2k\pi}{2} = e^{i\left(\frac{\varphi}{2}+\pi\right)}$

Endi $\sqrt{1} = -1$ shartga ko‘ra $\alpha_1$ olamiz, chunki $z = 1$ bo‘lganda
$\varphi = 0$ bo‘lib,

$\alpha = e^{\pi i} = cos\pi + isin\pi = -1;$

$dz = ie^{i\varphi}d\varphi.$

Aylananing ustki yarmiga $0 \le \varphi \le \pi$.

Shu sababli berilgan integralni aniq integralga aylantiramiz:

$$I = i\int_0^{\pi} \frac{e^{i\varphi}d\varphi}{i\left(\frac{\varphi}{2}+\pi\right)} = i\int_0^{\pi} e^{i\left(\frac{\varphi}{2}-\pi\right)} d\left(i\left(\frac{\varphi}{2}-\pi\right)\right) = 2e^{i\left(\frac{\varphi}{2}-\pi\right)}\Big| =$$

$$= 2\left(e^{-\frac{\pi}{2}i} - e^{-\pi i}\right) = 2(1-i).$$

Demak, $I = 2(1-i)$.

**Mustaqil yechish uchun misollar**

1. $\int_G (x^2 + iy^2)dz$, G chiziq $z_0 = 1 + i$ va $z = 2 + 3i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq.
2. $\int_G zdz$, bu yerda G chiziq $z_0 = i$, $z = 1 + i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq.
3. $\oint_G \bar{z}dz$, bu yerda G chiziq x=cost, y=sint $(0 \le t \le 2\pi)$ aylanadan iborat.
4. $\oint_G \frac{dz}{z-4}$, bu yerda G chiziq $x = 3\text{cost}$, $y = 2\text{sint}$ ellipsdan iborat.
5. $\oint_G z^2dz$, bu yerda G chiziq $z_0 = 1, z = i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasi.
6. $\oint_G \frac{dz}{z}$, bu yerda G chiziq $x = cost$, $y = sint$ aylanadan iborat.
7. $\int_G (2z + 1)dz$ bu yerda G chiziq $z_0 = 1 + i$, $z = -1 - i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasi.
8. $\int e^z\,dz$, bu yerda G-chiziq $z_0 = a\,, z = 1 + i$ nuqtalar o‘tadigan $y = x^2$ parabolaning qismidir.
9. $\int \cos z\,dz$, bu yerda G-chiziq $z_0 = \frac{\pi}{2}\,, z = 1 + i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasi.
10. $\int_{1+i}^{2i}(z^3 - z)\,dz.$
11. $\int_0^i z\cos z\,dz.$
12. $\int_0^i z\sin z\,dz.$
13. $\int_G^{1+i} Re(\sin z)\cos z\,dz,$ bu yerda G chiziq $|Imz| \le 1$ $Rez = \frac{\pi}{4}$.

14. $I=\int_G \frac{dz}{\sqrt[4]{z^3}}$ ni hisoblang, bu yerda G chiziq $|z|=1$aylananing ustki yarmi, $W=\frac{1}{\sqrt[4]{z^3}}$ funksiyaning to‘rtta tarmog‘idan shunday bittasini tanlab olinsinki natijada $\sqrt[4]{1}=1$ bo‘lsin.

15. $\int_r zIm(z^2)dz, bu\ yerda\ |Imz|\le 1\ Rez=1.$

16. $\int_{-i}^{i} ze^{z^2}dz.$

17. $\int_0^1 \frac{dt}{1+it}, t-$ haqiqiy o‘zgaruvchi.

### 3.3. Koshi teoremalari haqida

Ba’zi integrallarning javobini Koshi teoremasiga asoslanib aytish mumkin . Bunday integrallarni hisoblab o‘tirish shart emas.

**1-teorema.** Agar bir bog‘lamli G sohada f(z) funksiya analitik bo‘lsa, u holda G sohada yotuvchi har bir $G$ yopiq kontur bo‘ylab f(z) funksiyadan olingan integral nolga teng bo‘ladi :

$$\oint_G (z)dz=0$$

**2-teorema.** Agar ko‘p bog‘lamli yopiq G sohada *f(z)* funksiya analitik bo‘lsa, u holda funksiyadan tashqi kontur bo‘ylab olingan integral ichki konturlar bo‘ylab olingan interallar yig‘indisiga teng bo‘lib, bunda barcha konturlar bo‘ylab yo‘nalish bir xilda olinadi

$$\oint_G f(z)dz=\oint_{G_1} f(z)dz+\oint_{G_2} f(z)dz+\cdots+\oint_{G_n} f(z)dz$$

### 3.4. Koshining integral formulasi

Koshining integral formulasi:

$$f(a)=\frac{1}{2\pi i}\oint\frac{f(z)dz}{z-a} \qquad (3.11)$$

Bu yerda $a$ nuqta G soha ichida , $z$ nuqta esa $G$ chegarada yotadi va $f(z)$ G da analitikdir

(1) dan quyidagi formulani keltirib chiqarish mumkin.

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_G\frac{f(z)dz}{(z-a)^{n+1}}(n=1,2,3,\dots\dots\dots) \qquad (3.12)$$

(3.11) va (3.12) formulalardan

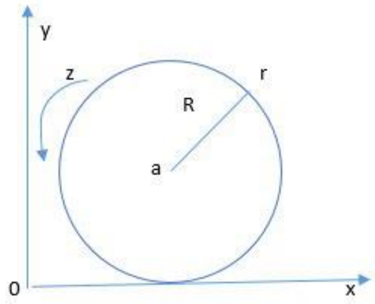
$$\oint\frac{f(z)dz}{z-a}=2\pi i f(a) \qquad (3.13)$$

$$\oint_G\frac{f(z)\,dz}{(z-a)^{n+1}}=\frac{2\pi i}{n!}f^{(n)}(a) \qquad (3.14)$$

larni yozish mumkin.

**Misollar**

1. $\oint \frac{dz}{z-a}$ ni hisoblang, bu yerda G-chiziq markazi a nuqtaga joylashgan aylana.




*5-chizma*

Yechilishi: Bu misolda $f(z) = 1$, bo‘lgani uchun (3.13) dan

$$\oint_G \frac{dz}{z-a} = 2\pi i$$

Agar $a = 0$ bo‘lsa, u holda

$$\oint_G \frac{dz}{z} = 2\pi i \ .$$

2. Ushbu $\oint_G \frac{\sin z dz}{z+i}$ integralni hisoblang, bu yerda G chiziq $|z+i| = R$ aylanadan iborat bo‘lib, $a = -i$ nuqta uning ichida.

Yechilishi:

$f(z) = \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), f(a) = f(-i) = \frac{1}{2i}(e - e^{-1}) = \frac{1}{i}\operatorname{sh} 1;$ shu sababli (3)-formulaga muvofiq $I = 2\pi i f(a) = 2\pi i \operatorname{sh} 1$.

3. $I = \oint_G \frac{dz}{z^2+9}$ ni hisoblang, bu yerda G chiziq $|z-2i| = 2$ aylanadan iborat.

Yechilishi:

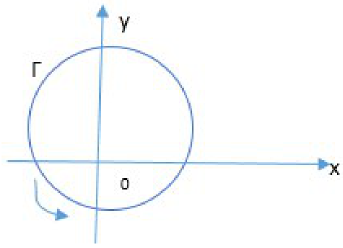
$z^2 + 9 = z^2 - i^2 3^2 = z^2 - (3i)^2 = (z-3i)(z+3i)$

Bu yerda ikkita nuqta bor bo‘lib, shulardan $a = 3i$ berilgan aylana ichidadir. Shu sababli berilgan integralni quyidagicha yozib olamiz:

$$I = \oint_G \frac{\frac{dz}{(z+3i)}}{(z-3i)}, \qquad f(z) = \frac{1}{(z+3i)}; f(a) = f(3i) = \frac{1}{3i+3i} = \frac{1}{6i}.$$

Demak, (3.13) asosan $I = 2\pi i f(a) = \frac{\pi}{3}$.

4. $\oint_G \frac{e^z dz}{(z+2)^4}$ ni hisoblang, bu yerda G chiziq $a = -2$ nuqtani o‘z ichiga olgan har qanday yopiq kontur (6-chizma).




*6 - chizma*

Yechilishi: $f(z)=e^z; n=3;\ f'''(z)=e^z; a=-2; f'''(-2)=e^{-2};$

$I=\frac{2\pi i}{n!}f'''(a)=\frac{2\pi i}{3!}e^{-2}=\frac{\pi i}{3e^2}.$

5. $I=\oint\frac{dz}{(z-1)^3(z+1)^3}-$ ni hisoblang, bu yerda G chiziq $|z-1|=R$ aylanadan iborat bo‘lib, $R<2$.

Yechilishi: Maxrajdagi ikkita 1 va -1 nuqtadan birinchisi aylana ichida yotgani uchun $I=\oint_G\frac{\frac{dz}{(z+1)^3}}{(z-1)^3}$

ko‘rinishda yozib olamiz.

$$f(z)=\frac{1}{(z+1)^3}=(z+1)^{-3};\ n=2;\ f'(z)=-3(z+1)^{-4};\ f''(z)$$

$$=-12(z+1)^{-5};\ f''(1)=-12(2)^{-5}=\frac{3}{8};$$

(3.14) ga asosan : $I=\frac{2\pi i}{n!}f^{(n)}(a)=\pi i\cdot\frac{3}{8}=\frac{3}{8}\pi i.$

6. $I=\oint\frac{dz}{z^2+1}-$. ni hisoblang.

Yechilishi: $I-$ ni (3.13) yoki (3.14) larning biriga keltiramiz:

$z^2+1=z^2-i^2=(z-i)(z+i);$

Bu yerda $i$ va $-i$ nuqtalar $|z|=2$ aylana ichida bo‘lganlari uchun integralni shunday ikkiga ajratamizki natijada aylana ichida o‘sha nuqtalardan faqat bittasi joylashgan bo‘lsin. Uning uchun quyidagicha ish ko‘ramiz.

$$\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{A}{z-i}+\frac{B}{z+i};\quad 1=A(z+i)+B(z-i);$$

$$\Rightarrow\quad A=\frac{1}{2i};\quad B=-\frac{1}{2i};\quad \frac{1}{z^2+1}=\frac{1}{2i}\cdot\frac{1}{z-i}-\frac{1}{2i}\cdot\frac{1}{z+i};$$

U holda

$$I=I_1+I_2=\frac{1}{2i}\oint\limits_{|z|=2}\frac{dz}{z-i}-\frac{1}{2i}\oint\limits_{|z|=2}\frac{dz}{z+i},$$

bularning ikkalasida ham $f(z)\equiv 1$ bo‘lib, birinchisida $a=i$, ikkinchisida $a=-i$.

Demak,

$$I=I_1+I_2=\frac{1}{2i}\cdot 2\pi i-\frac{1}{2i}\cdot 2\pi i=\pi-\pi=0.$$

7. $I = \oint_{|z+1|=1} \frac{dz}{(z+1)(z-1)^3}$ ni hisoblang.

Yechilishi: $I = \oint_{|z+1|=1} \frac{\frac{dz}{(z-i)^3}}{(z+i)}$ , chunki $a = -1$ nuqta $|z+1| = 1$ nuqta aylana ichida, demak,

$f(z) = \frac{1}{(z-i)^3}$ , $f(a) = f(-1) = \frac{1}{(-2)^3} = -\frac{1}{8}$,

$(3.13)$ formulaga asosan

$$I = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{8}.$$

8. $I = \oint_G \frac{e^z\,dz}{z\cdot(1-z)^3}$ni hisoblang, bu yerda $G$ chiziq $|z| < \frac{3}{2}$ doiraning chegarasidir.

Doira ichida ikkita $a = 0$, $a = 1$nuqtalar joylashadi. Shu sababli bu integralni bir nechta integralga ajratamiz uning uchun matematik analiz kursidagi ushbu qoidadan foydalanamiz

$\frac{1}{z(1-z)^3} = \frac{A}{z} + \frac{B}{(1-z)^3} + \frac{C}{(1-z)^2} + \frac{D}{1-z}$,

$1 = A(1-z)^3 + Bz + Cz(1-z) + Dz(1-z)^2$, bu yerda $z$ ga $0, \pm 1, \pm 2$ qiymatlar berilsa $A = B = C = D = 1$ hosil bo‘ladi. Demak,

$I = I_1 + I_2 + I_3 + I_4 = \oint_G \frac{e^z\,dz}{z} + \oint_G \frac{e^z\,dz}{(1-z)^3} + \oint_G \frac{e^z\,dz}{(1-z)^2} + \oint_G \frac{e^z\,dz}{(1-z)}$,

bu yerda $f(z) = e^z$, $f'(z) = e^z, f''(z) = e^z$,

$f(0) = e^0 = 1$, $\quad f'(1) = e$, $\quad f''(1) = e$,

(3) va (4) formulalarga asosan :

$I = 2\pi i \cdot 1 - \frac{2\pi i}{2!} e + 2\pi i \cdot e - 2\pi i \cdot e \ = (2-e)\pi i.$

Shunday qilib, $|z| < \frac{3}{2}$bo‘lganda $I = (2-e)\pi i$ bo‘lar ekan.

**Mustaqil yechish uchun misollar**

1. $\oint\limits_{|z-1|=\frac{1}{2}} \frac{e^{\frac{1}{z}}dz}{(1-z)^2}$; 2. $\oint\limits_{|z|=1} \frac{e^z \cos\pi z\,dz}{z^2+2z}$; 3. $\oint\limits_{|z-2|=2} \frac{\cosh z\,dz}{z^4-1}$;

4. $\oint\limits_{|z|=2} \frac{dz}{z^2+1}$; 5. $\oint\limits_{|z|=2} \frac{e^z dz}{z^2-1}$; 6. $\oint\limits_{|z|<\frac{1}{2}} \frac{e^z dz}{z(i-z)^3}$;

7. $\oint\limits_{|z-2|=3} \frac{e^{z^2} dz}{z^2-6z}$; 8. $\oint\limits_{|z-1|=1} \frac{\sin\pi z\,dz}{(z^2-1)^2}$; 9. $\oint\limits_{|z|=2} \frac{\cosh z\,dz}{(z+1)^3\cdot(z-1)}$;

10. $\oint\limits_{|z|=3} \frac{\cos(z+\pi i)\,dz}{z(e^z+2)}$; 11. $\oint\limits_{|z|=5} \frac{dz}{z^2+16}$; 12. $\oint\limits_{|z|=4} \frac{dz}{(z^2+9)\cdot(z+9)}$;

13. $\oint\limits_{|z|=1} \frac{\sin(z-1)\pi\,dz}{z^2-2z+2}$; 14. $\oint\limits_{|z|=2} \frac{\cosh iz\,dz}{z^2+4z+3}$; 15. $\oint\limits_{|z|=1} \frac{\tan z\,dz}{z\cdot e^{\frac{1}{z+i}}}$;

16. $\oint\limits_{G} \frac{\sinh(z+1)\,dz}{z^2+1}$;

Bu yerda $G$ astroidadan iborat $x^{\frac{2}{3}}+y^{\frac{2}{3}}=3^{\frac{2}{3}}$;

17. $\oint\limits_{|z|=2} \frac{\sin z\cdot\sin(z-1)\,dz}{z^2-z}$; 18. $\oint\limits_{|z|=1} \frac{\cos z\,dz}{z}$;

19. $\oint\limits_{|z-1|=1} \frac{\sin\frac{\pi z}{4}\,dz}{(z-1)^2(z-3)}$; 20. $\oint\limits_{|z|=2} \frac{z\sinh z\,dz}{(z^2-1)^2}$;

21. $\oint\limits_{|z-2|=3} \frac{\cosh e^{i\pi z}\,dz}{z^3-4z^2}$; 22. $\oint\limits_{|z|=\frac{1}{2}} \frac{\cos\frac{\pi}{z+1}\,dz}{z^3}$;

23. $\oint\limits_{|z|=\frac{1}{2}} \frac{1-\sin z\,dz}{z^2}$; 24. $\oint\limits_{|z-1|=\frac{1}{2}} \frac{e^{iz}\,dz}{(z^2-1)^2}$;

25. $\oint\limits_{|z|=1} \frac{\sinh^2 z\,dz}{z^3}$; 26. $\oint\limits_{|z-3|=6} \frac{z\,dz}{(z-2)^3(z-4)}$;

27. $\oint\limits_{|z-2|=1} \frac{e^{\frac{1}{z}}\,dz}{(z^2+4)^2}$; 28. $\oint\limits_{|z|=2} \frac{dz}{(z-a)^n\cdot(z-b)}$;

$(|a|<z<|b|), n=1,2,3,\ldots$

29. $\oint\limits_{|z-i|=1} \frac{\cos z\,dz}{(z-i)^3}$; 30. $\oint\limits_{|z|=2} \frac{1-\cos z\,dz}{(z+1)^2}$.

## 3.5. Yuqori tartibli hosila va analitik funksiyaning istalgan tartibli hosilalarining mavjudligi

**Teorema**. Koshi turidagi integral bilan aniqlanadigan $\Phi(z)$ funksiya $\Gamma$ chiziqda yotmaydigan, har qanday chekli $z$ nuqtada analitik bo‘ladi. Shunday nuqtalarda funksiya barcha yuqori tartibli hosilalarga ega bo‘lib, ular

$$\Phi^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{\varphi(\eta)d\eta}{(\eta - z)^{n+1}} \tag{3.15}$$

formula orqali topiladi.

Ko‘pincha (3.15) ning quyidagi xususiy holi ko‘proq ishlatiladi. $\overline{G}$ yopiq chiziq, $f(z)$ esa $\overline{G}$ sohada analitik bo‘lsin. U holda (1) formula ushbu

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\eta)d\eta}{\eta - z}$$

Koshi formulasini

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(\eta)d\eta}{(\eta - z)^{n+1}} \tag{3.16}$$

analitik funksiyaning yuqori tartibli hosilalarini hisoblash formulasini beradi. $n = 0, 1, 2, 3, ...$ deb olsak (3.16) dan

$$f(z), f'(z)\, f''(z), ..., f^{(n)}(z), ...$$

hosilalar kelib chiqadi.

Shunday qilib biz ushbu teoremani isbotladik.

**Teorema**. Yopiq $\overline{G}$ sohada analitik bo‘lgan har qanday $f(z)$ funksiya shu sohada *istalgan tartibli hosilalarga* ega bo‘lib, ular

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(\eta)d\eta}{(\eta - z)^{n+1}}$$

formula bilan ifodalanadi, bunda $\Gamma$ kontur $\overline{G}$ sohaning chegarasi.

### Misollar

**1.** $I = \oint_{C}\frac{z^2 dz}{z - 2i}$ integral hisoblansin, bunda $C$ markazi nol nuqtada bo‘lib radiusi $R = 3$ ga teng aylana.

Yechilishi: $f(z) = z^2$ funksiya $C$ aylana bilan chegaralangan yopiq doirada analitik funksiya, $a = 2i$ nuqta shu doira ichida yotgani uchun Koshining

$$\oint_{C}\frac{f(z)dz}{z - a} = 2\pi i f(a)$$

formulasidan foydalanamiz

$$I=\oint_C \frac{z^2dz}{z-2i}=2\pi i f(2i)=2\pi i(2i)^2=-8\pi i\,.$$

2. $I=\oint_C \frac{dz}{z^2+9}$ integral hisoblansin, bunda $C$-aylana bo‘lib, uning radiusi $R=2$ va markazi $2i$ nuqtada joylashgan.

Yechilishi: $z^2+9=z^2-(3i)^2=(z+3i)(z-3i)$ bo‘lgani uchun

$$I=\oint_C \frac{dz}{(z+3i)(z-3i)}=\oint_C \frac{\frac{1}{z+3i}}{z-3i}dz\,.$$

Bundan: $f(z)=\frac{1}{z+3i}$ va $a=3i$ bo‘lib, u $C$ aylana ichida yotadi,

(-3$i$) esa doira tashqarisida yotadi. Demak, $f(z)$ funksiya $C$ bilan chegaralangan doira ichida analitikdir. U holda Koshi formulasiga asosan:

$$I=2\pi i\cdot f(3i)=2\pi i\cdot\frac{1}{3i+3i}=\frac{\pi}{3}\,.$$

Demak, $I=\oint_C \frac{dz}{z^2+9}=\frac{\pi}{3}\,.$

3. $\oint_\Gamma \frac{\sin z\,dz}{\left(z-\frac{\pi}{2}\right)^9}$ integral hisoblansin, bu yerdagi $\Gamma$-chiziq $\frac{\pi}{2}$ nuqtani o‘z ichiga olgan sodda yopiq chiziq.

Yechilishi: (2) formuladagi $\eta$ ni $z$ ga va $z$ ni $\alpha$ ga almashtirib, u yerdagi integralni topsak

$$\oint_\Gamma \frac{f(z)dz}{(z-\alpha)^{n+1}}=\frac{2\pi i}{n!}f^{(n)}(\alpha)$$

tenglikka ega bo‘lamiz. Bunga $f(z)=\sin z$, $\alpha=\frac{\pi}{2}$, $n=8$ qiymatlarni qo‘ysak

$$\oint_\Gamma \frac{\sin z\,dz}{\left(z-\frac{\pi}{2}\right)^9}=\frac{2\pi i}{8!}(\sin z)^{(8)}_{z=\frac{\pi}{2}}=\frac{2\pi i}{8!}\sin\frac{\pi}{2}=\frac{2\pi i}{8!}$$

bo‘ladi.

**Mustaqil yechish uchun mashqlar**

$\Gamma$ egri chiziq yopiq silliq yoki silliq bo‘laklardan iborat bo‘lganda quyidagi integrallar hisoblansin.

1. a) $\oint\limits_{\Gamma} \sin z^2 dz$; b) $\oint\limits_{\Gamma} \cos z^2 dz$; c) $\oint\limits_{\Gamma} e^{-z^2} dz$.

2. $\oint\limits_{\Gamma} \frac{(z^2-1)^n}{z-x} dz$ integral hisoblansin.

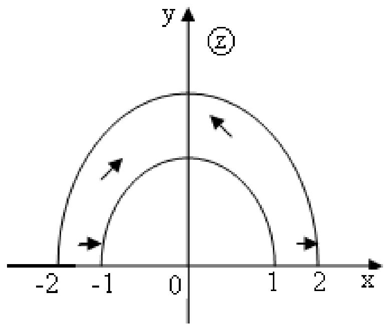
3. $\oint\limits_{\Gamma} \frac{dz}{1+z^2}$ integralning:

a) $\Gamma$ kontur $|z-i|=1$ aylana, b) $\Gamma$ kontur $|z+i|=1$ aylana, c) $\Gamma$ kontur $|z|=2$ aylana bo‘lgandagi qiymati hisoblansin.

4. $\int\limits_{|z|=a} |dz|$, $\int\limits_{|z|=a} dz$ hisoblansin.

5. Ushbu $\oint\limits_{\Gamma} \frac{dz}{z^2+4}$ integral Koshi formulasiga asosan hisoblansin, bunda $\Gamma$: a) $-2i$ $2i$ va $+2i$ nuqtalarni o‘z ichiga olgan kontur; b) $-2i$ nuqtani o‘z ichiga olgan kontur; c) $2i$ nuqtani o‘z ichiga olgan kontur.

6. Ushbu $\oint\limits_{\Gamma} |z|\, z\, dz$ integral 7-chizmada ko‘rsatilgan yarim halqa konturi bo‘yicha hisoblansin.




*7-chizma.*

7. Ushbu $\int\limits_{\Gamma} \frac{dz}{\sqrt{z}}$ integral hisoblansin, bunda $\Gamma$ $|z|=1$ aylananing yarmi, $\sqrt{1}=1$, $y \geq 0$.

8. $\Gamma$ egri chiziq $S$ yuzga ega sohani chegaralovchi yopiq egri chiziq bo‘lsin. U holda $\oint\limits_{\Gamma} x dz = iS$ tenglik isbotlansin.

9. Ushbu $\int\limits_{\Gamma} y dz$ integral quyidagi yo‘llar bo‘yicha hisoblansin:

a) $z = 2+i$ nuqtaning radius vektori bo‘ylab;

b) $|z|=1$ yarim aylana bo‘ylab, $0 \leq \arg z \leq \pi$ (yo‘lning boshlang‘ich nuqtasi $z=1$ da);

c) $|z-\alpha|=R$ aylana bo‘ylab.

10. Ushbu $\oint\limits_{\Gamma} Lnz dz$ integral hisoblansin, bunda $\Gamma$-birlik aylana va $Lni = \frac{\pi i}{2}$.

11. Ushbu $\oint\limits_{\Gamma} \frac{dz}{z^2+9}$ integral hisoblansin, bunda. $\Gamma$-radiusi $R=2$ ga teng va markazi $(-2i)$ nuqtaga joylashgan aylana.

12. Ushbu $\oint\limits_{\Gamma} \frac{dz}{(z^2+9)^2}$ integral hisoblansin, bunda $\Gamma$-radiusi $R=2$ ga teng va markazi $2i$ nuqtaga joylashgan aylana.

13. Bundan oldingi misoldagi $\Gamma$ aylana markazi $(-2i)$ nuqtaga joylashagan bo‘lsa, integralning qiymati nimaga teng bo‘ladi?

14. Ushbu $\oint\limits_{\Gamma} \frac{e^z dz}{(z+2)^4}$ integral hisoblansin, bundagi $\Gamma$-chiziq, $(-2)$ nuqtani bir marta o‘rab olgan sodda yopiq chiziq.

15. Ushbu $\oint\limits_{\Gamma} \frac{dz}{(z-1)^3(z+1)^3}$ integral hisoblansin, bunda $\Gamma$-aylana, uning markazi $1$ nuqtada va radiusi $1<R<2$.

16. Bundan oldingi misolda $1<R<2$ va aylana markazi $(-1)$ nuqtaga joylashgan bo‘lsa, integralning qiymati nimaga teng bo‘lada?

17. Ushbu integral hisoblansin: $\oint\limits_{\Gamma} \frac{z}{\bar{z}} dz$, bu yerdagi $\Gamma$-chiziq 7-chizmada ko‘rsatilgan yarim halqadan iborat.

**Nazorat savollari**

1. Integralga ta’rif bering.
2. Integralning xossalarini ayting.
3. Integral qanday hisoblanadi.
4. Koshining teoremalarini ayting.
5. Boshlang‘ich funksiyaga ta’rif bering.
6. Aniqmas integralga ta’rif bering.
7. Nyuton-Leybnis formulasini yozing.
8. Koshi formulasini yozing.
9. Koshi integralini yozing.
10. Uzluksiz funksiyaning moduli o‘zining eng katta va eng kichik qiymatlariga qayerda erishishi mumkin?
11. Qanaqa integral Koshi turidagi integral deyiladi?
12. Koshi turidagi integralning hosilalari qanday topiladi?
13. Analitik funksiyaning istalgan tartibli hosilalari qanday topiladi?

## 4. Kompleks hadli qatorlar

### 4.1. Kompleks sonli qatorlar

Kompleks sonlarning ushbu cheksiz ketma-ketligi berilgan bo‘lsin.

$z_1, z_2, \dots, z_n, \dots$ bunda $z_n = x_n + iy_n$, $n = 1,2,\dots$ (4.1)

vositasida

$$z_1 + z_2 + z_3 + \cdots + z_n + \cdots \quad (4.2)$$

sonli qator tuziladi. Haqiqiy sonlardan tuzilgan qatorga o‘xshash qatorni yaqinlashuvchi yoki uzoqlashuvchi ekanligini alomatlar yordamida tekshirish mumkin

$$|z_1| + |z_2| + \cdots + |z_n| + \cdots \quad (4.3)$$

berilgan bo‘lsin

1. **Dalamber alomati.**

Agar limit $\lim_{n\to\infty} \frac{|z_n|}{|z_{n-1}|} = \lambda$ bo‘lib, (4.4)

$a)$ $\lambda < 1$ bo‘lsa. (4.3) qator yaqinlashuvchi bo‘ladi;

$b)$ $\lambda > 1$ bo‘lsa. (4.3) qator uzoqlashuvchi bo‘ladi.

2. **Koshi alomati.**

Agar limit $\lim\limits_{n\to\infty} \sqrt[n]{|z_n|} = \lambda$ bo‘lib, (4.5)

$a)$ $\lambda < 1$ bo‘lsa. (4.3) qator yaqinlashuvchi bo‘ladi;

$b)$ $\lambda > 1$ bo‘lsa. (4.3) qator uzoqlashuvchi bo‘ladi.

**Misollar**

1. $\sum_{n=1}^{\infty} \frac{cosin}{2^n}$ qatorni tekshiring.

Yechilishi. Dalamber alomatidan foydalanamiz.

$$\frac{C_n}{C_{n-1}} = \frac{\cos in}{2^n} : \frac{\cos i(n-1)}{2^{n-1}} = \frac{1}{2} \cdot \frac{\cos in}{\cos i(n-1)}$$

Ma’lumki,

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right),$$

$$\cos in = \frac{1}{2}(e^{-n} + e^{n}), \cos i(n-1) = \frac{1}{2}\left(e^{-(n-1)} + e^{n-1}\right),$$

$$\frac{C_n}{C_{n-1}} = \frac{1}{2} \cdot \frac{e^{-n} + e^{n}}{e^{-n} \cdot e + e^{n} \cdot e^{-1}} = \frac{1}{2} \cdot \frac{e^{2n} + 1}{e + e^{-1} \cdot e^{2n}} = \frac{1}{2} \cdot \frac{1 + \frac{1}{e^{2n}}}{\frac{e}{e^{2n}} + e^{-1}}, bundan$$

$$\lambda = \lim_{n\to\infty} \left|\frac{C_n}{C_{n-1}}\right| = \frac{1}{2e^{-1}} = \frac{e}{2} > 1 \; demak\; qator\; uzoqlashuvchi$$

2. $\sum_{n=1}^{\infty} \frac{\cos in^2}{5^{n^2}}$ qatorni tekshiring.

Yechilishi. Koshi alomati yordamida tekshiramiz:

$$\cos in^2 = \frac{1}{2}\left(e^{-n^2} + e^{n^2}\right) = \frac{e^{2n^2} + 1}{2e^{n^2}};$$

$$\sqrt[n]{|C_n|} = \sqrt[n]{\frac{\cos in^2}{5^{n^2}}} = \frac{\sqrt[n]{1 + e^{2n^2}}}{5^n e^n \sqrt[n]{2}} \approx \frac{\sqrt[n]{e^{2n^2}}}{5^n e^n \sqrt[n]{2}} = \left(\frac{e}{5}\right)^n \cdot \frac{1}{\sqrt[n]{2}}, \quad \frac{e}{5} < 1, \lim_{n\to\infty} \left(\frac{e}{5}\right)^n = 0;$$

$$\sqrt[n]{2} = \omega, \qquad 2 = \omega^n, \;\; \omega > i, \qquad \frac{1}{\sqrt[n]{2}} < 1$$

Bularga asosan

$$\lambda = \lim_{n\to\infty} \sqrt[n]{|C_n|} = 0 < 1.$$

Demak, qator yaqinlashuvchi ekan.

**Mustaqil yechish uchun misollar**

Quyidagi kompleks hadli qatorlarning yaqinlashuvchi yoki uzoqlashuvchi ekanligini aniqlang.

1. $\sum_{n=1}^{\infty} \frac{n\,\sin in}{3^n}$.  2. $\sum_{n=1}^{\infty} \frac{e^{i2n}}{n\sqrt{n}}$.  3. $\sum_{n=1}^{\infty} \frac{e^{i\frac{\pi}{n}}}{\sqrt{n}}$.

4. $\sum_{n=1}^{\infty} \frac{(1+i)^n}{2^{\frac{n}{2}} \cos in}$  5. $\sum_{n=1}^{\infty} \frac{\mathrm{Sh}\, i\sqrt{n}}{\sin in}$  6. $\sum_{n=1}^{\infty} \frac{e^{in}}{n^2}$.

7. $\sum_{n=1}^{\infty} \frac{\ln n}{\mathrm{Sh}\, in}$.  8. $\sum_{n=1}^{\infty} \frac{\mathrm{ch}\, \frac{\pi i}{n}}{n^{\ln n}}$  9. $\sum_{n=1}^{\infty} \frac{n}{\mathrm{tg}\, in\pi}$

10. $\sum_{n=1}^{\infty} \frac{1}{[n+(2n-1)i]^2}$. 11. $\sum_{n=1}^{\infty} \frac{n(2+i)^n}{2^n}$.  12. $\sum_{n=1}^{\infty} [\frac{n(2-i)+1}{n(3-2i)-3i}]^n$

## 4.2. Darajali qatorlar

Kompleks sohadagi darajali qatorning umumiy ko‘rinishi quyidagicha:

$$c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots = \sum_{n=0}^{\infty} c_n z^n \qquad (4.6)$$

(4.6) qatorning yaqinlashish radiusi va doirasi quyidagicha aniqlanadi:

a) Yaqinlashish doirasini Dalamber alomati yordamida topish:

$$\overline{\lim_{n\to\infty}} \left|\frac{c_n z^n}{c_{n-1} z^{n-1}}\right| = \overline{\lim_{n\to\infty}} \left|\frac{c_n}{c_{n-1}}\right| \cdot |z| < 1.$$

$Agar\ L = \overline{\lim_{n\to\infty}} \left|\frac{c_n}{c_{n-1}}\right|,\ \frac{1}{L} = R$, deb belgilasak izlanayotgan doira $|z| < R$ bo‘ladi.

b) Yaqinlashish doirasini Koshi alomati yordamida topish:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n z^n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|} \cdot |z| < 1$$

Agar $\overline{\lim_{n\to\infty}} \sqrt[n]{|C_n|}$, $\frac{1}{L} = R$ yaqinlashish doirasi $|z| < R$ bo‘ladi.

**Misollar**

1. $\sum_{k=1}^{n}\left(\frac{z}{in}\right)^{n}$ qatorning yaqinlashish doirasi va radiusini toping.

Yechilishi: $C_n=\left(\frac{1}{in}\right)^{n}$ , $\overline{\lim_{n\to\infty}}\sqrt[n]{|C_n|}=\overline{\lim_{n\to\infty}}\frac{1}{|in|}=\overline{\lim_{n\to\infty}}\frac{1}{n}=0=L$ , $\frac{1}{L}=R\ \Rightarrow$ $R=+\infty$ ,

Demak, qator butun kompleks tekislikda yaqinlashadi.

2. $\sum_{n=1}^{\infty} ch\frac{iz^n}{n}$ qatorning yaqinlashish doirasi va radiusini toping.

Yechilishi: Dalamber alomatidan foydalanamiz:

$$ch\,z=\frac{e^{z}+e^{-z}}{2};\,C_n=\ \mathrm{ch}\frac{i}{n}=\frac{1}{2}\left(e^{\frac{i}{n}}+e^{-\frac{i}{n}}\right)=\cos\frac{1}{n}.$$

$C_{n-1}=\mathrm{ch}\frac{i}{n-1}=\cos\frac{1}{n-1}$; $\lim_{n\to\infty}\left|\frac{C_n}{C_{n-1}}\right|=\lim_{n\to\infty}\frac{\cos\frac{1}{n}}{\cos\frac{1}{n-1}}=\frac{\cos 0}{\cos 0}=1$, demak, $R=1$.

3. $\sum_{n=0}^{\infty}\cos inz^n$ qatorning yaqinlashish doirasi va radiusini toping.

Yechilishi: Dalamber alomatidan foydalanamiz:

$C_n=\cos in=ch\,n=\frac{e^{n}+e^{-n}}{2}$, $\frac{C_{n+1}}{C_n}=\frac{ch\,(in)}{chn}=\frac{chn\cdot ch1+sh1\cdot shn}{chn}$ $=ch1+sh1\cdot thn.$

$$\lim_{n\to\infty}thn=\lim_{n\to\infty}\frac{e^{n}-e^{-n}}{e^{n}+e^{-n}}=\lim_{n\to\infty}\frac{1-e^{-2n}}{1+e^{-2n}}=1.$$

$$\lim_{n\to\infty}e^{-2n}=\lim_{n\to\infty}\frac{1}{e^{2n}}=0$$

Demak, L=$\overline{\lim_{n\to\infty}}\left|\frac{c_{n+1}}{c_n}\right|$ $=\overline{\lim_{n\to\infty}}(ch1+sh1\cdot thn)=ch1+sh1=\frac{e^{1}+e^{-1}}{2}+\frac{e^{1}-e^{-1}}{2}=e.$

$L=e$, $R=\frac{1}{L}=\frac{1}{e}$ ya'ni berilgan qatorning yaqinlashish radiusi $R=e^{-1}$.

3. $\sum_{n=1}^{\infty}(1+i)^{n}z^{n}$ qatorning yaqinlashish doirasi va radiusini toping.

Yechilishi: Koshi alomatidan foydalanamiz:

$$c_n=(1+i)^{n};1+i=\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4}),\ |1+i|=\sqrt{2};$$

$$|C_n|=|1+i|^{n}=(\sqrt{2})^{n};\ \sqrt[n]{|C_n|}=\sqrt{2};$$

$L = \overline{\lim_{n\to\infty}} \sqrt[n]{|C_n|} = \sqrt{2}$; $\quad R = \frac{1}{L} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ demak qatorning yaqinlashish doirasi $|Z| < \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ .

5-misol. $\sum_{n=0}^{\infty} (n+i)\, z^n$ qatorni yaqinlashish doirasi va radiusini toping.

Yechilishi: Koshi alomati bo‘yicha

$$|C_n| = |n+1| = \sqrt{n^2+1};\ \sqrt[n]{|C_n|} = (n^2+1)^{\frac{1}{2n}} = \omega;\ \lim_{n\to\infty} \omega = ?$$

$$\ln\omega = \frac{1}{2n}\ln(n^2+1);\ \lim_{n\to\infty}(\ln\omega) = \lim_{n\to\infty}\frac{\ln(n^2+1)}{2n} = \lim_{n\to\infty}\frac{(\ln(n+1))'}{(2n)'}$$
$$= \lim_{n\to\infty}\frac{\frac{2n}{n^2+1}}{2} = 0;$$

Demak $\lim_{n\to\infty}(\ln\omega) = 0;\ \lim_{n\to\infty}\omega = 1;\ L = \lim_{n\to\infty}\sqrt[n]{|C_n|} = 1,\ R = 1.$

Shunday qilib qatorning yaqinlashish sohasi $|Z| < 1$.

6-misol. $\sum_{n=0}^{\infty} \frac{z^n}{\sin^n n}$ qatorni tekshiring.

Yechilishi: Bu misol uchun ham Koshi-Adamar formulasiga asosan

$C_n = \frac{1}{\sin^n n}$, $\sqrt[n]{|C_n|} = \frac{1}{|\sin n|}$ , ma’lumki,

$$L = \overline{\lim_{n\to\infty}}\frac{1}{|\sin n|} = \frac{1}{\lim_{n\to\infty}|\sin n|} = \infty,\ \ R = \frac{1}{L} = 0.$$

Demak, berilgan qator faqat bitta $|z| = 0$ nuqtada yaqinlashib, qolgan barcha nuqtalarda uzoqlashuvchi:

$$\sqrt[n]{|C_n|} = \frac{1}{\lim_{n\to\infty} n|\sin n|} =- \frac{1}{\lim_{n\to\infty}\left|\frac{e^{in}-e^{-in}}{2i}\right|}$$

$$\lim_{n\to\infty}\sqrt[n]{|C_n|} = \frac{1}{\lim_{n\to\infty}|sinn|} = \frac{1}{\lim_{n\to\infty}|e^{in}|\left|\frac{1-e^{-2in}}{2i}\right|}$$
$$= \frac{2}{\lim_{n\to\infty}|1-e^{-2in}|}\cdot\lim_{n\to\infty}\left|1-e^{-2in}\right| = 0;$$

$$\lim_{n\to\infty}\sqrt[n]{|C_n|}=\frac{1}{\lim\limits_{n\to\infty}|sinn|}=\frac{1}{\lim\limits_{n\to\infty}|e^{in}|\left|\frac{1-e^{-2in}}{2i}\right|}=$$

$$=\frac{2}{\lim_{n\to\infty}|1-e^{-2in}|}\cdot\lim_{n\to\infty}\left|1-e^{-2in}\right|=0;$$

Shuning uchun ham $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|C_n|}=\infty.$

U holda $\overline{\lim\limits_{n\to\infty}}\frac{1}{|\cos n|}=\overline{\lim\limits_{n\to\infty}}\frac{1}{\sin\left(n+\frac{\pi}{2}\right)}=+\infty.$

7-misol. $\sum_{n=1}^{\infty}(\frac{z}{1-i})^n$ qatorni tekshiring.

Yechilishi: Koshi alomati bo‘yicha :

$\sqrt[n]{|c_n|}=\frac{1}{|1-i|}=\frac{1}{\sqrt{2}};\quad L=\lim_{n\to\infty}\sqrt[n]{|c_n|}=\frac{1}{\sqrt{2}}, R=\frac{1}{L}=\sqrt{2}.$

Demak, $|z|<\sqrt{2}.$

8. $\sum_{n=1}^{\infty}\frac{n^n}{n!}z^n$ qatorni tekshiring.

Yechilishi: Dalamber alomatidan foydalanamiz

$\left|\frac{c_{n+1}}{c_n}\right|=\frac{(n+1)^{n+1}}{(n+1)!}:\frac{n^n}{n!}=\left(\frac{n+1}{n}\right)^n=(1+\frac{1}{n})^n,$

$L=\lim_{n\to\infty}(1+\frac{1}{n})^n=e;\ R=\frac{1}{L}=\frac{1}{e}$. Demak $|z|<\frac{1}{e}$ .

**Mustaqil yechish uchun misollar**

Quyidagi darajali qatorlarning yaqinlashish radiuslari va yaqinlashish doiralarini toping.

1. $\sum\limits_{n=1}^{\infty}(n+a^n)\,z^n.$ 2. $\sum\limits_{n=0}^{\infty}(\mathrm{n}-i)^n\,z^n.$ 3. $\sum\limits_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}z^n.$

4. $\sum\limits_{n=1}^{\infty}z^{n!}.$ 5. $\sum\limits_{n=0}^{\infty}(1+i)^n\,z^n.$ 6. $\sum\limits_{n=1}^{\infty}n^{\alpha}\,z^n, \alpha\in R.$

7. $\sum_{n=1}^{\infty} \frac{z^n}{n}$. 8. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. 9. $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

10. $\sum_{n=0}^{\infty} z^{n!}\, 2^n$. 11. $\sum_{n=1}^{\infty} \frac{z^n n}{2^n}$. 12. $\sum_{n=0}^{\infty} z^{2^n}$.

13. $\sum_{n=0}^{\infty} \cos in\, z^n$. 14. $\sum_{n=0}^{\infty} \frac{z^n}{(n!)}$. 15. $\sum_{n=0}^{\infty} [\ln(n+2)]^k z^n$.

16. $\sum_{n=0}^{\infty} e^{-\sqrt{n}} z^n$. 17. $\sum_{n=1}^{\infty} n^k z^n$. 18. $\sum_{n=1}^{\infty} \frac{(kn)!\, z^n}{n!\,(n+1)!\ldots(n+1-k)!}$

19. $\sum_{n=1}^{\infty} \frac{z^n (2+i)^n n}{2^n}$ 20. $\sum_{n=1}^{\infty} \frac{(z-2i)^n}{n 3^n}$. 21. $\sum_{n=0}^{\infty} i^n z^n$.

22. $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}$. 23. $\sum_{n=1}^{\infty} \cos^n \frac{\pi i}{\sqrt{n}} z^n$. 24. $\sum_{n=0}^{\infty} \frac{z^n}{\mathrm{sh}(1+i\,n)}$.

### 4.3. Teylor va Makloren qatorlari

Agar $f(z)$ funksiya $a$ nuqtaning biror atrofida analitik (hosilaga ega) bo‘lsa, uni$(z-a)$ ga nisbatan musbat darajali qatorga yoyish mumkin, u holda qator quyidagicha Teylor qatori deb ataladi va quyidagicha bo‘ladi.

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n. \qquad (4.7)$$

Agar $a = 0$ deb faraz qilinsa ,(1)dan ushbu Makloren qatori kelib chiqadi.

$$f(z) = f(a) + \frac{f'(a)}{1!} z + \frac{f''(a)}{2!} z^2 + \ldots + \frac{f^{(n)}(a)}{n!} z^n + \ldots =$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} z^n \qquad (4.8)$$

Bunda $f(z)$ funksiya $|z-a| < R$ doirada analitik.

**Mustaqil yechish uchun misollar**

Quyidagi funksiyalarni Makloren qatoriga yoying.

1. $f(z) = \frac{1}{az+b}, \ b \neq 0.$
2. $f(z) = chz.$
3. $f(z) = \mathrm{sh}\, z.$
4. $f(z) = \mathrm{ch}^2 z.$
5. $f(z) = \ln\frac{1-z}{1+z}.$
6. $f(z) = \ln(z^2 - 3z + 2).$
7. $f(z) = \frac{z(z+a)}{(a-z)^2}, |z| < a, a \neq 0.$
8. $f(z) = \cos\sqrt{z}\,.$
9. $f(z) = \mathrm{sh}^2\frac{z}{2}.$
10. $f(z) = \ln(2 - z)\,.$

Quyidagi $funksiyalarni$ Teylor qatoriga $z = a$ nuqta bo‘yicha yoying.

11. $f(z) = \mathrm{ch}(1 - z)$ funksiyani $z - \left(1 - \frac{\pi i}{2}\right)$
12. $f(z) = \cos z$ funksiyani nisbatan.
13. $f(z) = \frac{1}{3-2z}$ funksiyani $z - 3$ ga nisbatan .
14. $f(z) = \frac{z^2-5}{z^2-4z+3}$ funksiyani $z - 2$ ga nisbatan.
15. $f(z) = \frac{z^2-5}{(z+1)(2-z)}$ funksiyani $z - 1$ ga nisbatan .

## 4.4. Funksiyaning nollari

Faraz qilaylik , $\omega = f(z)$ -funksiya biror a nuqtada analitik, ya’ni hosilaga ega bo‘lsin (bu yerda $z = x + iy, a$ kompleks son, $\omega = u(x,y) + iv(x,y)$ kompleks o‘zgaruvchili funksiyadir). Agar o‘sha a son $f(z)$ funksiyani nolga aylantirsa , ya’ni $f(\mathrm{a}) = 0$ bo‘lsa , a son $f(z)$ ning *noli* deyiladi . Agar

$$f(a) = 0, f'(a) = 0, f''(a) = 0, \dots., \ f^{(n-1)}(a) = 0\,, f^{(n)}(a) \neq 0$$

bo‘lsa , u holda a son $f(z)$ funksiyaning *n-tartibli* yoki *n-karrali noli* deyiladi.

Agar n=1 bo‘lsa , a son *oddiy nol* deyiladi. Demak ta’rifga muvofiq a nuqtada analitik bo‘lgan $f(z)$ funksiya uchun a nuqta *n-karrali* nol bo‘lishi uchun shu nuqtaning biror atrofida ushbu

$$f(z) = (z-a)^n \varphi(z)$$

tenglik bajarilib $\varphi(z)$ funksiya a nuqtada analitik va $\varphi(a) \neq 0$ bo‘lishi zarur va yetarlidir. Chunki $z = a$ da $f(z)$ analitik funksiya bo‘lganligi uchun u shu nuqta atrofida $(z-a)$ ga nisbatan darajali qatorga yoyiladi.

**Misollar**

Quyidagi funksiyalarning nollari va ularning karraliklarini aniqlang.

1. $f(z) = z - i$.

Yechilishi: $f(i) = i - i = 0,\ a = i\ ,\ f'(z) = 1\ ,\ f''(z) \neq 0$

Demak, $z = a = i$ son funksiyaning oddiy nolidir.

2. $f(z) = z^2 + 1$.

Yechilishi: $f(\pm i) = (\pm i)^2 + 1 = -1 + 1 = 0;\ \ z_1 = i, z_1 = -i$.

$f'(z) = 2z\ , f'(\pm i) = 2(\pm i) = \pm 2i \neq 0$.

Demak, ikkala ildiz ham oddiy noldir.

3. $f(z) = 1 + chz$.

Yechilishi: Quyidagi tenglamani yechamiz:

$1 + chz = 0, chz = \frac{1}{2}(e^z + e^{-z}) = -1;\ e^{2z} + 2e^z + 1 = 0;$

$(e^z + 1)^2 = 0, e^z = -1, z = Ln(-1) = ln1 + \pi i + 2n\pi i = (2n+1)\pi i,$

$\text{n} = 0, \pm 1, \pm 2, \dots;\ f'(z) = shz,\ f''(z) = chz.$

Tekshirish kursatadiki,

$f'((2n+1)\pi i) = 0, f''((2n+1)\pi i) \neq 0$. Demak, $(2n+1)\pi i$ ikki karrali nollardir.

**Mustaqil yechish uchun misollar**

Quyidagi funksiyalarning nollari va ularning tartiblari aniqlansin.

1. $f(z) = 1 + cosz;$
2. $f(z) = 1 - e^z;$
3. $f(z) = \frac{z^3}{z - sinz};$
4. $f(z) = (z^2 + 1)shz;$
5. $f(z) = cosz^3;$
6. $f(z) = \frac{(1 - shz)^2}{z};$
7. $f(z) = \frac{z^3}{1 + z - e^z};$
8. $f(z) = \frac{(1 - cos2z)^2}{z - shz}$.

## 4.5. Manfiy darajali qatorlar

$z-a$ bo‘yicha manfiy darajali qatorlarni quyidagi ko‘rinishida yozish mumkin:

$$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\cdots+\frac{b_n}{(z-a)^n}+\cdots=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n} \qquad (4.9)$$

Uning yaqinlashish sohasini turli usullar bilan topish mumkin, jumladan Dalamber alomati bo‘yicha ham topish mumkin.

$$\overline{\lim_{n\to\infty}}\left|\frac{b_n}{(z-a)^n}:\frac{b_{n-1}}{(z-a)^{n-1}}\right|=\overline{\lim_{n\to\infty}}\left|\frac{b_n}{b_{n-1}}\right|\cdot\frac{1}{|z-a|}<1.$$

$$\text{Ya'ni } \overline{\lim_{n\to\infty}}\left|\frac{b_n}{b_{n-1}}\right|=r\ ,\ |z-a|>r. \qquad (4.10)$$

Demak, manfiy darajali (1) qatorning yaqinlashish sohasi $|z-a|>r$ doira tashqarisidan iborat ekan. Agar $b_n=c_{-n}$ deb (1) ni quyidagicha yozish mumkin.

$$\ldots+c_{-n}(z-a)^{-n}+c_{-(k-1)}(z-a)^{-(k-1)}+\cdots+c_{-2}(z-a)^{-2}+c_{-1}(z-a)^{-1}$$

$$=\sum_{n=-\infty}^{-1}c_n(z-a)^n \ (3)$$

Agar $a=0$ bo‘lsa $(3)$ quyidagi qator kelib chiqadi .

$$\sum_{n=-\infty}^{-1}c_n z^n \ (4)$$

U holda (2) doira tashqarisi $|z|>r$ ko‘rinishiga ega bo‘ladi .

**Misollar**

Quyidagi manfiy darajali qatorlarning yaqinlashish sohalarini toping

1. $\sum_{n=1}^{\infty}e^n(iz)^{-n}$

Yechilishi: $c_{-n}=e^n(i)^{-n}=\frac{e^n}{i^n},\ c_{-(n-1)}=\frac{e^{n-1}}{(i)^{n-1}},$

$r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{e}{|i|}=e,\ r=e$, demak qatorning yaqinlashish sohasi $|z|>e$ doira tashqarisidan iborat ekan.

2. $\sum_{n=1}^{\infty}\frac{z^{-n}}{\cos in}$.

Yechilishi: $C_{-n}=\frac{1}{\cos in}=\frac{1}{\operatorname{ch} n},C_{-(n-1)}=\frac{1}{\operatorname{ch}(n-1)},$

$r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{ch(n-1)}{chn}=\lim_{n\to\infty}\frac{e^{n-1}+e^{-(n-1)}}{e^n+e^{-n}}=\lim_{n\to\infty}\frac{e^{-1}+e\cdot e^{-2n}}{1+e^{-2n}}=e^{-1}=\frac{1}{e}$

Chunki $\lim_{n\to\infty}e^{-2n}=\lim_{n\to\infty}\frac{1}{e^{2n}}=0,|z|>e^{-1}=\frac{1}{e}$.

3. $\sum_{n=1}^{\infty}\frac{n\cdot(2)^{-n}}{(z-2-i)^n}$

Yechilishi:

$z-2-i=z-(i+2),\ a=2+i,\ C_{-n}=n\cdot 2^{-n},\ C_{-(n-1)}=(n-1)\cdot 2^{-(n-1)},$
$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right| =\frac{1}{2},\ \ |z-2-i|>\frac{1}{2}.$

Quyidagi funksiyalarni manfiy darajali qatorlarga yoying.

4 . $f(z)=\frac{1}{z-b}$ funksiyani $\frac{b}{z}$ ga nisbatan manfiy darajali qatorga yoying.

Yechilishi: Ma'lumki , $|q|<1$ bo'lsa,

$a+aq+aq^2+\cdots+aq^{n-1}+\cdots=\frac{a}{1-q}$, bo'ladi shunga asosan

$$\frac{1}{z-b}=\frac{1}{z\left(1-\frac{b}{z}\right)}=\frac{1}{z}\left(1+\frac{b}{z}+\left(\frac{b}{z}\right)^2+\left(\frac{b}{z}\right)^3+\cdots+\left(\frac{b}{z}\right)^n+\cdots\right)=$$

$$=\sum_{n=0}^{\infty}\frac{b^n}{z^{n+1}},\ \left|\frac{b}{z}\right|<1 => |z|>|b|$$

5. $f(z)=\frac{1}{(z-b)^2}$ funksiyani $\frac{b}{z}$ ga nisbatan manfiy darajali qatorga yoying.

Yechilishi: Oldingi misoldagi yaqinlashish sohasida z bo'yicha hosila olish kifoya:

$$\left(\frac{1}{z-b}\right)'=\sum_{n=0}^{\infty}b^n(z^{-k-1})',\qquad \text{bundan}\ \frac{1}{(z-b)^2}=\sum_{n=0}^{\infty}\frac{(n+1)b^n}{z^{n+2}}\quad ,|z|>b.$$

6. $f(z)=\frac{z^2}{z^2+b^2}$ funksiyani $\frac{b}{z}$ ga nisbatan manfiy darajali qatorga yoying.

Yechilishi:

$$\frac{z^2}{z^2+b^2}=\frac{z^2+b^2-b^2}{z^2+b^2}=1-\frac{b^2}{z^2+b^2}=\frac{\left(\frac{b}{z}\right)^2}{1+\left(\frac{b}{z}\right)^2}$$

$$=1-\left(\frac{b}{z}\right)^2-\left(\frac{b}{z}\right)^4-\left(\frac{b}{z}\right)^6+\cdots=\sum_{n=0}^{\infty}(-1)^n(\frac{b}{z})^{2n}.$$

**Mustaqil yechish uchun misollar**

Quyidagi qatorlarning yaqinlashish sohasini aniqlang:

1. $\sum_{n=1}^{\infty}\frac{1}{(1-i)^n z^n}$. 2. $\sum_{n=1}^{\infty}\frac{1}{4^n(z+1)^n}$. 3. $\sum_{n=1}^{\infty}\frac{(z+1-i)^{-n}}{n+i}$. 4. $\sum_{n=1}^{\infty}\frac{(\sqrt{2}+i\sqrt{2})^n}{z^n}$.

5. $\sum_{n=-\infty}^{-1} b^{-2(n+1)} z^{2n}$. 6. $\sum_{n=-\infty}^{-1} (b-a)^{-(n+1)} (z-a)^n$.

Quyidagi funksiyalarni manfiy darajali qatorga yoying:

7 .$f(z)=\frac{1}{2z-5}$ funksiyani $z$ ga nisbatan manfiy darajali qatorga yoying.

8 .$f(z)=z^2\cos\frac{1}{z}$ funksiyani $z=0$ atrofida qatorga yoying.

9. $f(z)=\frac{1}{z^2-a^2}$ funksiyani z=0 nuqta atrofida qatorga yoying.

10. $f(z)=\frac{1}{z-a}$ funksiyani $(z-b)$ ga nisbatan qatorga yoying.

Quyidagi qatorlarning yaqinlashish doiralari aniqlansin:

11. $\sum_{n=0}^{\infty}\frac{z^n}{n}$; 12. $\sum_{n=0}^{\infty}\frac{(z-1)^n}{(1-i)^n}$.

Quyidagi darajali qatorlarning yaqinlashishi radiuslarini toping.

13. $\sum_{n=1}^{\infty}\frac{z^n}{n!}$. 14. $\sum_{n=1}^{\infty} n^n z^n$.

15. $\sum_{n=1}^{\infty}\frac{n}{2^n}z^n$. 16. $\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$.

17. $\sum_{n=0}^{\infty} z^{n!}$. 18. $\sum_{n=0}^{\infty} z^{2n}$

Quyidagi funksiyalar darajali qatorlarga yoyilsin va yaqinlashish radiusi topilsin.

19. $chz$. 20. $\sin^2 z$.

21. $\frac{1}{az+b}$ $(b\neq 0)$. 22. $\ln\frac{1+z}{1-z}$.

## 4.6. Loran qatori

Loran darajali qatorlarning umumiy ko‘rinishi

$$\dots+\frac{C_{-n}}{(z-a)^n}+\frac{C_{-(n-1)}}{(z-a)^{n-1}}+\frac{C_{-(n-2)}}{(z-a)^{n-2}}+\dots+\frac{C_{-2}}{(z-a)^2}+\frac{C_{-1}}{z-a}+C_0$$
$$+C_1(z-a)+C_2(z-a)^2+C_3(z-a)^3+\dots+C_n(z-a)^n+\dots=$$
$$=\sum_{n=-\infty}^{\infty} C_n(z-a)^n \qquad (4.12)$$

bo‘lib, u ikki qismdan iborat:

1) To‘g‘ri qismi:

$$P = C_0 + C_1(z-a) + C_2(z-a)^2 + C_3(z-a)^3 + \cdots + C_n(z-a)^n + \cdots \quad (4.13)$$

Ma'lumki, bu qatorning yaqinlashish sohasi

$|z-a| < R$ doiradan iborat;

2) Bosh qismi:

$$Q = \cdots + C_{-n}(z-a)^{-n} + C_{-(n-1)}(z-a)^{-(n-1)} + C_{-(n-2)}(z-a)^{-(n-2)} + \cdots + C_{-2}(z-a)^{-2} + C_{-1}(z-a)^{-1} = \sum_{n=-\infty}^{-1} C_n(z-a)^n \quad (4.14)$$

Buning yaqinlashish sohasi $|z-a| > r$, ya'ni doirada tashqaridan iborat.

(4.12) Qatorning yaqinlashish sohasi markazi $a$ nuqtada bo'lgan r va R radiusli aylanalar orasidagi halqadan iborat bo'ladi $r < |z-a| < R$

Bu yerda uch xil bo'lishi mumkin:

1. Agar $r < R$ bo'lsa, halqa mavjud bo'lib, Loran qatori usha halqa ichida absolyut yaqinlashadi.

2. Agar $r = R$ bo'lsa, halqa fakat aylanadan iborat bo'lib, Loran qatori uning ba'zi nuqtalardagina yaqinlashuvchi bo'lishi mumkin.

3. Agar $r > R$ bo'lsa, hech qanday halqa yo'q, shu sababli Loran qatori uzoqlashuvchi bo'ladi.

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, Loran qatorining $C_n$ koeffitsiyenti ushbu formula orqali ifodalanadi.

$$C_n = \frac{1}{2\pi i}\oint_g \frac{f(z)dz}{(z-a)^{n+1}}, \; n = 0, \pm1, \pm2, \pm3, \ldots \quad (4.15)$$

**Misollar**

Quyidagi Loran qatorlarining yaqinlashish sohalarini aniqlang:

1. $\sum_{n=1}^{\infty}\left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$

Yechilishi: Dastlab qatorning to'g'ri qismini tekshiraylik:

$$P = \sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n; \; c_n = \frac{1}{4^n}, \qquad c_{n-1} = \frac{1}{4^{n-1}},$$

$$L = \lim_{n\to\infty}\left|\frac{C_n}{C_{n-1}}\right| = \lim_{n\to\infty}\frac{4^{n-1}}{4^n} = \frac{1}{4}, \qquad L = \frac{1}{4}, R = \frac{1}{L} = 4, |z| < 4\,.$$

Endi qatorning bosh qismini tekshiramiz:

$$Q = \sum_{n=1}^{\infty}\frac{2^n}{z^n} = \sum_{n=-\infty}^{-1} z^n 2^{-n}, C_n = 2^{+n}, C_{-(n-1)} = 2^{+(n-1)} = 2^{n-1}$$

$$r = \lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right| = \lim_{n\to\infty}\frac{2^n}{2^{n-1}} = 2, |z| > 2$$

Shunday qilib izlanayotgan soha $2 < |z-a| < 4$ halqadan iborat ekan.

2. $\sum_{n=1}^{\infty}\frac{\sin in}{(z-i)^n} + \sum_{n=0}^{\infty}\frac{(z-i)^n}{n!}$.

Yechilishi: Qatorning to'g'ri qismi:

$$P=\sum_{n=0}^{\infty}\frac{(z-i)^n}{n!}, a=i, C_n=\frac{1}{n!}, C_{n-1}=\frac{1}{(n-1)!};$$

$$L=\lim_{n\to\infty}\left|\frac{C_n}{C_{(n-1)}}\right|=\lim_{n\to\infty}\frac{(n-1)!}{n!}=\lim_{n\to\infty}\frac{1}{n}=0, \quad R=\frac{1}{L}=+\infty, \quad |z-i|<+\infty$$

Qatorning bosh qismi:

$$Q=\sum_{n=1}^{\infty}\frac{\sin in}{(z-i)^n}; \quad C_{-n}=\sin in=\frac{(e^{-n}-e^{n})}{2i}, \quad C_{-(n-1)}=\frac{\left(e^{-(n-1)}-e^{n-1}\right)}{2i}$$

$$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{e^{-2n}-1}{e\cdot e^{-2n}-e^{-1}}=\frac{1}{e^{-1}}=e, \quad r=e, \quad |z-i|>e$$

Demak, soha $e<|z-i|<+\infty$.

3. $\sum_{n=1}^{\infty}\frac{2^n-1}{(z+1)^n}+\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+i)^n}$

Yechilishi: Qatorning to‘g‘ri qismi:

$$P=\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+i)^n}, \qquad a=-1, C_n=\frac{1}{(n+i)^n}; \sqrt[n]{|C_n|}=\sqrt[n]{\frac{1}{|n+i|^n}}=\frac{1}{\sqrt{1+n^2}};$$

$$L=\lim_{n\to\infty}\sqrt[n]{|C_n|}=0, \qquad R=\frac{1}{L}=\infty, \qquad |z+1|<\infty$$

Qatorning bosh qismi : $Q=\sum_{n=1}^{\infty}\frac{2^n-1}{(z+1)^n}, C_{-n}=2^n-1,$

$$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{2^n-1}{2^{n-1}-1}=\lim_{n\to\infty}\frac{2-\frac{1}{2^{n-1}}}{1-\frac{1}{2^{n-1}}}=2, \qquad |z+1|>2.$$

Demak, $2<|z+1|<\infty$, ya’ni markazi $a=-1$ nuqtadan iborat bo‘lgan doira tashqarisi.

4 . $\sum_{n=1}^{\infty}\frac{a^n}{z^n}+\sum_{n=0}^{\infty}\frac{z^n}{b^n}, \quad b\neq 0.$

Yechilishi : Qatorning to‘g‘ri qismi:

$P=\sum_{n=0}^{\infty}\frac{z^n}{b^n}, \left|\frac{C_n}{C_{(n-1)}}\right|=\left|\frac{1}{b^n}:\frac{1}{b^{n-1}}\right|=\frac{1}{|b|}$, L=$\frac{1}{|b|}, R=|b|, |z|=|b|$

Qatorning bosh kismi :

$$Q=\sum_{n=1}^{\infty}\frac{a^n}{z^n}; \qquad \left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\left|\frac{a^n}{a^{n-1}}\right|=|a|, \quad r=|a|, \qquad |z|>|a|.$$

Demak $|a|<|b|$ bo‘lsa, yaqinlashish halqasi $|a|<|z|<|b|$

Agar $|a|>|b|$ bo‘lsa, qator uzoqlashuvchidir.

5. $\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}-\frac{1}{2(z-i)}$

Yechilishi :

Bu masalaning bosh qismi birgina haddan iborat bo‘lgani uchun uning faqat to‘g‘ri qismining yaqinlashish sohasini tekshiramiz.

$$a = i; \quad C_n = \frac{1}{4}\left|\frac{1}{(2i)^n}\right| = \frac{1}{4}\cdot\frac{1}{2^n}, \qquad |i| = 1;$$

$$\left|\frac{C_n}{C_{n-1}}\right|, \quad L = \frac{1}{2}, \quad R = \frac{1}{L} = 2, \quad 0 < |z - i| < 2$$

6. $f(z) = z^4 \cdot cos\frac{1}{z}$ funksiyani z=0 nuqta atrofida Loran qatoriga yoying.

Yechilishi. Ma’lumki,

$$cos\frac{1}{z} = 1 - \frac{1}{2!}\cdot(\frac{1}{z})^2 + \frac{1}{4!}\cdot\left(\frac{1}{z}\right)^4 - \frac{1}{6!}\cdot(\frac{1}{z})^6 + \ \dots,$$

Shusababli

$$z^4 cos\frac{1}{z} = z^4 - \frac{1}{2!}z^2 + \frac{1}{4!} - \frac{1}{6!}\cdot\frac{1}{z^2} + \frac{1}{8!}\cdot\frac{1}{z^4} - \ \dots., \quad R = +\infty$$

7. $f(z) = \frac{sin^2 z}{z}$ ni z=0 nuqta atrofida Loran qatoriga yoying.

Yechilishi. Ma’lumki,

$$sin^2 z = \frac{1 - \cos 2z}{2} = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \ \dots.\right] =$$

$$= \left[\frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - \frac{(2z)^8}{8!} + \ \dots.\right]\frac{1}{2}.$$

Shunga asosan

$$\frac{sin^2 z}{z} = \left[\frac{2^2 z}{2!} - \frac{2^4 z^3}{4!} + \frac{2^6 z^5}{6!} - \frac{2^8 z^7}{8!} + \ \dots.\right]\frac{1}{2}; \quad R = +\infty.$$

8. $f(z) = \frac{1}{(z-2)(z-3)}$ funksiyani 2<z<3 halqada Loran qatorga yoying.

Yechilishi. Buning uchun berilgan funksiyani eng sodda kasrlarga ajratib olamiz;

$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$; $\quad 1 = A(z-3) + B(z-2), \ z = 2 \ A = -1,$ z=3 da V=1.

Berilgan xalqaga binoan ish ko‘ramiz: a) $|z| > 2$, ya’ni $\frac{2}{|z|} < 1$ bo‘lishi uchun quyidagi ko‘rinishda yozamiz:

$$-\frac{1}{z-2} = -\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} = -\frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right] = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n};$$

6) $|z|<3$ ya’ni $\left|\frac{z}{3}\right| < 1$ bo‘lishi uchun quyidagi ko‘rinishda yozamiz:

$$\frac{1}{z-3} = -\frac{1}{3-z} = -\frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Demak, $\frac{1}{(z-2)(z-3)} = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$.

9. $f(z) = \frac{2}{z^2-1}$ Funksiyani $1 < |z+2| < 3$ xalqada Loran qatoriga yoying.

Yechilishi. $\frac{2}{z^2-1}=\frac{2}{(z-1)(z+1)}=\frac{A}{z-1}+\frac{B}{z+1}$; $A(z+1)+B(z-1)=2; z=1$ bo‘lganda A=1, Z= -1 bo‘lganda V =-1 bo‘ladi, u holda

$$a)\quad \frac{1}{z-1}=\frac{1}{(z+2)-3}=-\frac{1}{3}\cdot\frac{1}{1-\frac{z+2}{3}}=-\frac{1}{3}\sum_{n=0}^{\infty}(\frac{z+2}{3})^4\ ;$$

$$b)\quad -\frac{1}{z+1}=-\frac{1}{(z+2)-1}=-\frac{1}{(z+2)\left(1-\frac{1}{z+2}\right)}=\frac{-\frac{1}{z+2}}{1-\frac{1}{z+2}}=$$

$$=-\frac{1}{z+2}[1+\frac{1}{(z+2)}+\frac{1}{(z+2)^2}+\ \dots=-\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}$$

Demak, $\frac{2}{z^2-1}=\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}-\frac{1}{3}\sum_{n=0}^{\infty}(\frac{z+2}{3})^n$.

**Mustaqil yechish uchun misollar**

Quyidagi Loran qatorining yaqinlashish sohalarini aniqlang

$$1)\sum_{n=1}^{\infty}\frac{(3+4i)^n}{(z+2i)^n}+\sum_{n=0}^{\infty}\frac{(z+2i)^n}{6^n};$$

$$2)\sum_{n=1}^{\infty}\frac{1}{n^n(z-2+i)^n}+\sum_{n=0}^{\infty}(1+in)(z-2+i)^n;$$

$$3)\sum_{n=1}^{\infty}\frac{1}{z^n}+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}};$$

$$4)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4\cdot z^n}+\sum_{n=1}^{\infty}\frac{z^n}{n\cdot 2^n};$$

$$5)\ -\frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{9}\cdot(z-1)^n+\sum_{n=0}^{\infty}\frac{3n+5}{9\cdot 2^{n+2}}\cdot(z-1)^n;$$

$$6)\sum_{n=-\infty}^{-1}(n+2)\cdot i^{n+1}(z-i)^n.$$

Quyidagi funksiyalarni Loran qatoriga yoying

7) $f(z)=\frac{2z+3}{z^2+3z+2}$ funksiyani $1<|z|<2$ halqada.

8) $f(z)=\frac{z^2-z+3}{z^2-3z+2}$ funksiyani $1<|z|<2$ halqada.

9) $f(z)=\frac{1}{z^2+2z-8}$ funksiyani $1<|z+2|<4$ halqada.

10) $f(z)=\frac{2z-3}{z^2-3z+2}$ funksiyani $1<|z-2|<1$ halqada.

Quyidagi funksiyalar ko‘rsatilgan nuqtalarning atrofida Loran qatoriga yoyilsin.

11. $\frac{1}{z-2}$, $z=0$ va $z=\infty$.

12. $\frac{1}{z(1-z)}$, $z=0, z=1, z=\infty$.

13. $\frac{1}{(z^2+1)^2}$, $z=i$ va $z=\infty$.

14. $z^2 e^{\frac{1}{z}}$, $z=0$, $z=\infty$,

15. $\frac{1}{(z-a)(z-b)}$ $(0<|a|<|b|)$ $z=0$, $z=a$, $z=\infty$ va $|a|<|z|<|b|$ halqada.

## 5. Maxsus nuqtalar

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, $\omega = f(z)$ funksiyalar $z_0$ nuqtada hosilaga ega bo'lsa, bu nuqta *to'g'ri nuqta* deyiladi. Agar shu nuqtada analitiklik shartlari bajarilmasa, bu nuqta funksiyaning *maxsus nuqtasi* deyiladi. Masalan, $\omega = \frac{(z-1)}{(z+i)}$ funksiya uchun $z=-i$ мaxsus nuqtadir chunki bu nuqtada hosila mavjud emas.

### 5.1. Ajralgan maxsus nuqtalar

**1-ta'rif**. Agar $f(z)$ funksiya $a$ nuqtada analitik bo'lsa u holda $a$ nuqta $f(z)$ ning *to'g'ri nuqtasi* deyiladi.

**2-ta'rif**. $f(z)$ funksiyaning to'g'ri bo'lmagan nuqtasi *maxsus nuqtasi* deyiladi.

**3-tarif**. Agar $f(z)$ funksiya $a$ nuqtaning biror $0<|z-a|<R$ atrofida analitik bo'lib, $a$ nuqtaning o'zida analitik bo'lmasa, u holda $a$ nuqta $f(z)$ funksiyaning *ajralgan (yakkalangan) maxsus nuqtasi* deyiladi.

Ajralgan maxsus nuqtalar uch turga, ya'ni: *qutilib bo'ladigan* (yoki *chetlashtiriladigan) maxsus nuqtalar, qutblar* va *muhim maxsus nuqtalarga* bo'linadi.

**4-ta'rif**. Agar a) $\lim\limits_{z\to a} f(z) = A$ bo'lib, $A$ aniq chekli son bo'lsa, u holda $a$ nuqta $f(z)$ funksiyaning *qutilib bo'ladigan* (yoki *chetlashtiriladigan*) *maxsus nuqtasi*.

b) agar $\lim\limits_{z\to a} f(z) = \infty$ bo'lsa, u holda $a$ nuqta $f(z)$ funksiyaning *qutbi*,

c) Agar $\lim\limits_{z\to a} f(z)$ mavud bo'lmasa $a$ nuqta $f(z)$ ning *muhim maxsus nuqtasi* deyiladi.

$a$ nuqta $f(z)$ funksiyaning qutilib bo'ladigan maxsus nuqtasi bo'lishi uchun shu funksiyaning $a$ nuqta atrofidagi Loran qatoriga yoyilmasi bosh qismga ega bo'lmasligi, ya'ni

$$f(z) = C_0 + C_1(z-a) + C_2(z-a)^2 + \ldots + C_n(z-a)^n + \ldots$$

ko'rinishga ega bo'lishi zarur va yetarlidir.

**Misollar**

1. $f(z)=\frac{1}{z}$ ning hosilasi $f'(z)=-\frac{1}{z^2}$ bo‘lib $z=0$ dan boshqa har qanday $z$ nuqta to‘g‘ri nuqta $z=0$ esa maxsus nuqta.

2. $f(z)=\frac{1}{z^2+1}$ funksiyaning maxsus nuqtalari $i,-i$ lardan iborat. Bulardan farqli har qanday $z$ to‘g‘ri nuqtadir.

3. $\frac{1}{z}$ ning ajralgan maxsus nuqtasi $z=0$ dan iborat, chunki bu nuqtada funksiya hosilaga ega bo‘lmay, uning har qanday $0<|z|<R$ atrofida hosila mavjud.

4. $\frac{1}{z+i}$ ning ajralgan maxsus nuqtasi $z=-i$ dan iborat.

5. $\frac{\sin z}{z}$ funksiya $z=0$ nuqtaga aniq emas. $z\neq 0$ bo‘lganda

$$\frac{\sin z}{z}=\frac{1}{z}.\sin z=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+...,\qquad \lim_{z\to a}f(z)=\lim_{z\to a}\frac{\sin z}{z}=1.$$

Demak $z=0$ nuqta $\frac{\sin z}{z}$ uchun qutilib bo‘ladigan maxsus nuqta ekan. Shu sababli $f(0)=1$ deb qabul qilsak u holda $f(z)$ funksiya $z=0$ nuqtada analitik bo‘ladi.

$f(z)$ funksiyaning ajralgan maxsus $a$ nuqtasi qutb bo‘lishi uchun $f(z)$ funksiyaning $a$ nuqtasi atrofida Loran qatori bosh qismi *hadlarining soni chekli* bo‘lishi, ya’ni

$$f(z)=\frac{C_{-k}}{(z-a)^k}+...+\frac{C_{-1}}{z-a}+C_0+C_1(z-a)+C_2(z-a)^2+...+C_n(z-a)^n+...$$

ko‘rinishga ega bo‘lishi zarur va yetarlidir.

$f(z)$ funksiyaning ajralgan maxsus $a$ nuqtasi muhim maxsus nuqtadan iborat bo‘lishi uchun shu funksiyaning $a$ nuqta atrofidagi Loran qatorining bosh qismi *cheksiz ko‘p* hadlarga ega bo‘lishi, ya’ni

$$f(z)=\sum_{n=-\infty}^{\infty}C_n(z-a)^n$$

ko‘rinishga ega bo‘lishi zarur va yetarlidir.

6. $f(z)=e^{\frac{1}{z}}$ funksiyaning ajralgan maxsus nuqtasi $z=0$ ekanligi ayon.

$$e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+...\frac{1}{n!z^n}+...$$

yoyilmadan $z=0$ berilgan funksiyaning muhim maxsus nuqtasi ekanligi kelib chiqadi.

Muhim maxsus nuqtaning har qanday kichik atrofida $f(z)$ funksiyaning qiymatlari oldindan berilgan har qanday *chekli* yoki *cheksiz songa istalgancha yaqinlashadi*.

7. $f(z)=\frac{(z+1)(z-2)}{(z+2)^3(z^2+4)}$ funksiyaning qutblari aniqlansin.

Yechilishi: Berilgan funksiya kasr-ratsional funksiya. Shuning uchun kasrning maxrajini nollari funksiyaning maxsus nuqtalari bo‘ladi. $z_1 = -2$ uchinchi tartibli nol, $z_2 = 2i,\ z_3 = -2i$ oddiy nollar. Demak, shular $f(z)$ funksiyaning qutblari bo‘lib, $z_1 = -2$ uchinchi tartibli, qolgan nollari oddiy qutblardir.

8. $f(z) = \dfrac{z^2}{1-\cos z}$. Bu funksiya $z = 0$ nuqtada aniqlangan emas. Ma’lumki,

$$1-\cos z = 1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots\right) = \frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\ldots,$$

U holda: $f(z) = \dfrac{1}{\dfrac{1}{2!}-\dfrac{z^2}{4!}+\dfrac{z^6}{6!}-\ldots}$

Bundan: $\lim\limits_{z\to 0} f(z) = 2$.

Demak, $z = 0$ nuqta $f(z)$ ning qutilib bo‘ladigan maxsus nuqtasidir. Agar $f(0) = 2$ deb qabul qilinsa, $z = 0$ nuqtada berilgan funksiya analitik bo‘ladi.

Funksiyaning boshqa maxsus nuqtalarini topish uchun kasrning maxrajini nolga tenglashtiramiz: $1-\cos z = 0$ yoki $\cos z = 1$ bundan $z = 2k\pi$ ($k$ -butun son) nuqtalar hosil bo‘lib, berilgan funksiyaning ikkinchi tartibli qutblaridir, chunki $z = 2k\pi$ lar

$$\frac{1}{f(z)} = \frac{1-\cos z}{z^2} = 2\left(\frac{\sin\frac{z}{2}}{z}\right)^2$$

funksiyaning nollari.

9. $f(z) = \dfrac{z}{e^z - 1}$ bu funksiya $z = 0$ nuqtada aniq emas, chunki $f(0) = \dfrac{0}{0}$. Buni aniqlash uchun $e^Z$ ni qatorga yoyamiz:

$$e^z - 1 = \left(1+z+\frac{z^2}{2!}+\ldots\right)-1 = z+\frac{z^2}{2!}+\ldots,\qquad f(z) = \frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots}.$$

Bundan $\lim\limits_{z\to 0} f(z) = 1$ ya’ni, $z = 0$ qutilib bo‘ladigan maxsus nuqta. Endi boshqa maxsus nuqtalarni ham $e^Z - 1 = 0$ yoki $e^Z = 1$ tenglikdan aniqlaymiz. Bizga ma’lumki $e^{2k\pi i} = \cos 2k\pi + i\sin 2k\pi = 1,$ bunda $k$ -butun son. Demak, $z = 2k\pi$ nuqtalar berilgan funksiyaning oddiy qutblari.

**Mustaqil yechish uchun misollar**

Quyidagi funksiyalarining maxsus nuqtalari va ularning turlari aniqlansin:

1. $\dfrac{1}{z(1-z^2)}$.

2. $\dfrac{z^4}{1+z^4}$.

3. $\dfrac{z^5}{(1-z)^2}$.

4. $\dfrac{1}{z\left(z^2+4\right)^2}$.

5. $\dfrac{e^z}{1+z^2}$.

6. $\frac{1+z^2}{e^Z}$.

7. $\dfrac{1}{e^z-1}-\dfrac{1}{z}$.

8. $ze^{\frac{1}{z}}$.

## 5.2 Funksiyalarning ajralgan maxsus nuqta atrofidagi chegirmasi (qoldig‘i)

Ma’lumki agar $f(z)$ funksiya $a$ nuqtani o‘z ichiga oluvchi biror $G$ sohada analitik bo‘lsa, u holda $a$ nuqtani o‘rab olgan $G$ ichida yotgan yopiq $\Gamma$ kontur bo‘ylab olingan integral, Koshi teoremasiga muvofiq nolga teng bo‘ladi:

$$\oint_{\Gamma} f(z)dz = 0 .$$

$a$ nuqta $f(z)$ ning ajralgan maxsus nuqtasi bo‘lganda shu integralning qiymatini topish talab etiladi. $f(z)$ ni $0<|z-a|<R$ halqada Loran qatoriga yoyamiz:

$$f(z)=C_0+C_1(z-a)+C_2(z-a)^2+\ldots+C_n(z-a)^n+\ldots+\frac{C_{-1}}{z-a}+\frac{C_{-2}}{(z-a)^2}+\ldots+\frac{C_{-n}}{(z-a)^n}+\ldots..$$

.

Halqa ichidagi $a$ nuqtani o‘rab oluvchi yopiq silliq $\tilde{A}$ chiziqni olamiz. Yuqoridagi qator $\tilde{A}$ chiziqda tekis yaqinlashuvchi bo‘lgani uchun uni hadlab integrallash mumkin. U holda

$$\oint_{\Gamma}(z-a)^k dz = \begin{cases} 0, \; agar \quad k \neq -1 \; o'lsa \\ 2\pi i, \; agar \; k=-1 \; o'lsa \end{cases}$$

ekanini hisobga olsak

$$\oint_{\Gamma} f(z)dz = 2\pi i C_{-1} \qquad (5.1)$$

bo‘ladi

**5-ta’rif**. Agar $f(z)$ funksiya $0<|z-a|<R$ halqada analitik bo‘lsa, shu funksiyaning ajralgan maxsus $a$ nuqtaga nisbatan *chegirmasi (qoldig‘i)* deb

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz$$

integralning qiymatiga aytiladi va

$$res\, f(a) = \frac{1}{2\pi i}\oint_{\Gamma} f(z)dz$$

ko‘rinishda yoziladi. (5.1) ga e’tibor bersak, $C_{-1}=\dfrac{1}{2\pi i}\oint_{\Gamma} f(z)dz = res\, f(a)$ ekanini ko‘ramiz.

Shunday qilib $f(z)$ funksiyaning ajralgan maxsus $a$ nuqtaga *nisbatan chegirmasi* deb shu funksiyaning $a$ atrofidagi Loran qatori $(z-a)^{-1}$ hadining $C_{-1}$ koeffitsiyentini qabul qilish mumkin ekan.

Agar $a$ nuqta $f(z)$ ning to‘g‘ri nuqtasi yoki qutulib bo‘ladigan maxsus nuqtasi bo‘lsa, u holda Loran qatorining bosh qismi bo‘lmaydi, ya’ni $C_{-1}=0,\ C_{-2}=0,...$ va

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz = res\, f(a) = C_{-1} = 0$$

bo‘ladi.

**6-ta’rif**. Cheksiz uzoqlashgan $z=\infty$ nuqtaning biror atrofida bir qiymatli va analitik bo‘lgan $f(z)$ funksiyaning $z=\infty$ ga nisbatan chegirmasi deb $\frac{1}{2\pi i}\oint_{\Gamma^-} f(z)dz$ integralga aytiladi, bunda $\Gamma^-$ yetarli darajada katta radiusga ega bo‘lgan aylana bo‘lib uning yo‘nalishi soat milining yo‘nalishi bo‘yicha olinadi.

**Koshi teoremasi**. Agar $f(z)$ funksiya $\Gamma$ chiziq bilan chegaralangan $\overline{G}$ yopiq sohaning ajralgan maxsus $a_1, a_2, ..., a_k$ nuqtalardan boshqa hamma nuqtalarida analitik bo‘lsa, u holda $f(z)$ funksiyadan $\Gamma$ bo‘ylab olingan integralning qiymati $\Gamma$ ichidagi barcha maxsus $a_k$ nuqtalarga nisbatan funksiya *chegirmalari yig‘indisining* $2\pi i$ ga ko‘paytirilganiga teng:

$$\oint_{\Gamma} f(z)dz = 2\pi i\sum_{n=1}^{k} res f(a_n)$$

Bu teoremaga muvofiq $f(z)$ funksiyadan yopiq $\Gamma$ kontur bo‘ylab olingan integralni hisoblash, $\Gamma$ ichidagi ajralgan maxsus nuqtalarga nisbatan barcha *chegirmalarni* hisoblashga keltirildi.

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz = -res\, f(\infty)$$

ekanligini hisobga olib oxirgi tenglikni

$$res f(a_1) + res f(a_2) + ... + res f(a_k) + res\, f(\infty) = 0$$

ko‘rinishda yozish mumkin.

### 5.3 Qutbga nisbatan funksiyaning chegirmasini topish

1. Agar $a$ nuqta $f(z)$ funksiyaning oddiy qutbi bo‘lsa, $a$ atrofida $f(z)$ ning Loran qatori

$$f(z) = \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + C_2(z-a)^2 + ...$$

ko‘rinishda bo‘lar edi. Buning ikki tomonini $z-a$ ga ko‘paytirib limitga o‘tilsa,

$$\lim_{z\to a}[(z-a)f(z)] = \lim_{z\to a}\left[C_{-1} + C_0(z-a) + C_1(z-a)^2 + ...\right] = C_{-1}$$

bo‘ladi. Bundan

$$res f(a) = C_{-1} = \lim_{z\to a}[(z-a)f(z)] \tag{5.2}$$

chegirmani hisoblash formulasiga ega bo‘lamiz.

Masalan, $f(z)=\dfrac{z^3}{z+2}$ funksiya $z=-2$ oddiy qutbga ega bo‘lgani uchun

$$resf(-2)=C_{-1}=\lim_{z\to-2}\left[(z+2)\cdot\frac{z^3}{z+2}\right]=(-2)^3=-8.$$

2. Agar $a$ nuqta $f(z)$ funksiyaning $k$ -tartibli qutbi bo‘lsa, uning bu nuqtaga nisbatan Loran qatori

$$f(z)=\frac{C_{-k}}{(z-a)^k}+...+\frac{C_{-1}}{z-a}+C_0+C_1(z-a)+C_2(z-a)^2+...$$

ko‘rinishda bo‘lib, $C_{-1}$ koeffitsiyentni topmoq uchun tenglikning ikki tomonini $(z-a)^k$ ko‘paytirib $(k-1)$ marta hosila olamiz:

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]=\frac{d^{k-1}}{dz^{k-1}}\left[C_{-k}+...+C_{-1}(z-a)^{k-1}+C_0(z-a)^k+C_1(z-a)^{k+1}+...\right]=(k-1)!C_{-1}+k!(z-a)+...$$

.

Endi $z\to a$ da limitga o‘tsak

$$C_{-1}=resf(a)=\frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right] \qquad (5.3)$$

$k$-tartibli qutbga nisbatan chegirmani hisoblash formulasi kelib chiqadi.

**Misollar**

1. $f(z)=\dfrac{1}{(z^2+1)^3}$ funksiyaning *qutblariga* nisbatan *chegirmalari* topilsin.

Yechilishi: $z_1=i$ va $z_2=-i$ funksiyaning uchinchi tartibli qutblaridir. (5.2) ga binoan:

$$resf(i)=\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z^2+1)^3}\right]=\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z-i)^3(z+i)^3}\right].=\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z+i)^{-3}\right]=$$

$$=\frac{1}{2!}\lim_{z\to i}(-3)(-4)(z+i)^{-5}=\frac{6}{(-2i)^5}=\frac{-3i}{16},$$

$$resf(-i)=\lim_{z\to-i}\frac{d^2}{dz^2}\left[(z+i)^3\frac{1}{(z^2+1)^3}\right]=\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-i)^{-3}\right]=\frac{1}{2!}\lim_{z\to i}(-3)(-4)(z-i)^{-5}=\frac{6}{(-2i)^5}=\frac{3i}{16}.$$

2. Ushbu $\displaystyle\int_0^{2\pi}\frac{dx}{a+\cos x}$ ($a$ -haqiqiy son va $a>1$) integral hisoblansin.

Yechilishi: $z=e^{ix}$ almashtirish olamiz. U holda $x$ 0 dan $2\pi$ gacha o‘zgarganda kompleks o‘zgaruvchi $z$ birlik doira chizadi.

$$\cos x=\frac{e^{ix}+e^{-ix}}{2}=\frac{z+z^{-1}}{2}=\frac{z^2+1}{2z}, \qquad dz=ie^{ix}dx,\ dx=\frac{dz}{iz}$$

ekanligini hisobga olsak:

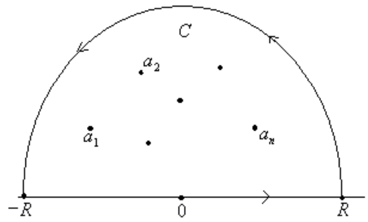
$$\int_0^{2\pi}\frac{dx}{a+\cos x}=\oint_c\frac{\frac{1}{iz}dz}{a+\frac{z^2+1}{2z}}=\frac{2}{i}\oint_c\frac{dz}{z^2+2az+1}=\frac{2}{i}2\pi i\sum res\left(\frac{1}{z^2+2az+1}\right).$$

$\frac{1}{z^2+2az+1}$ funksiyaning maxsus nuqtalari $z_{1,2}=-a\pm\sqrt{a^2-1}$ oddiy qutblar bo‘ladi. $a>1$ bo‘lgani uchun bu qutblardan faqat $z_1=-a+\sqrt{a^2-1}$ birlik aylana ichida yotadi. Shu nuqtadagi chegirmani hisoblaymiz. (1) formulaga binoan:

$$resf(z)=\lim_{z\to z_1}(z-z_1)\cdot\frac{1}{z^2+2az+1}=\lim_{z\to z_1}\frac{1}{z_1-z_2}=\frac{1}{z_1-z_2}=\frac{1}{-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}}=\frac{1}{2\sqrt{a^2-1}}$$

Demak, $\int_0^{2\pi}\frac{dx}{a+\cos x}=\frac{2}{i}\cdot 2\pi i\cdot\frac{1}{2\sqrt{a^2-1}}=\frac{2\pi}{\sqrt{a^2-1}}.$

Endi ayrim integrallarni chegirmalardan foydalanib hisoblash usuli bilan tanishamiz. $f(z)$ funksiya haqiqiy o‘qdan yuqorida yotuvchi chekli sondagi $a_1,a_2,...,a_n$ maxsus nuqtalardan tashqari barcha yuqori yarim tekislikda va haqiqiy o‘qda analitik funksiya bo‘lsin. Shu bilan birga




*12-chizma*

cheksiz uzoqlashgan nuqta $f(z)$ funksiya uchun kamida ikkichi tartibli nol bo‘lsin. U holda:

$$\int_{-\infty}^{+\infty}f(x)dx=2\pi i\sum resf(z) \qquad (5.4)$$

Haqiqatan, koordinatalar boshini markaz qilib, yuqori yarim tekislikda yetarli katta $R$ radiusli shunday $C$ yarim aylana chizamizki, $a_1,a_2,...,a_n$ maxsus nuqtalar shu yarim aylana ichida yotsin. Natijada chegarasi $C$ dan va haqiqiy o‘qini $(-R,R)$ kesmasidan tashkil topgan yarim doira hosil bo‘ladi (12-chizma).

Chegirmalar haqidagi asosiy teoremaga ko‘ra:

$$\oint_{\tilde{A}}f(z)dz=\int_{-\infty}^{+\infty}f(x)dx+\int_C f(z)dz=2\pi i\sum resf(z).$$

$$\lim_{R\to\infty}\int_C f(z)dz=0 \qquad (5.5)$$

ekanini ko‘rsatamiz. $f(z)$ funksiyani cheksiz uzoqlashgan nuqtaning atrofida Loran qatoriga yoyamiz:

$$f(z)=\frac{C_{-2}}{z^2}+\frac{C_{-3}}{z^3}+....=\frac{1}{z^2}\varphi(z)$$

$$\varphi(z)=C_{-2}+\frac{C_{-3}}{z}+\frac{C_{-4}}{z^2}+....$$

Funksiya $z\to\infty$ da $C_{-2}$ ga intiladi, ya’ni cheksiz uzoqlashgan nuqtaning atrofida chegaralangan, jumladan $S$ yarim aylanada ham chegaralangan; ya’ni $|\varphi(z)|\le M$

Demak

$$\left|\int\limits_{\tilde{n}} f(z)dz\right| = \left|\int\limits_{c} \frac{\varphi(z)}{z^2}dz\right| \le \frac{M}{R^2}\cdot \pi R = \frac{M\pi}{R} \text{ va } \lim_{R\to\infty}\int\limits_{C} f(z)dz = 0$$

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \int_{-\infty}^{+\infty} f(x)dx.$$

3. $\int\limits_{-\infty}^{+\infty}\frac{\cos xdx}{(x^2+1)(x^2+9)}$ integral hisoblansin.

Yechilishi:

$$I = R\cdot 2\pi i\sum res\frac{e^{iz}}{(z^2+1)(z^2+9)}$$

bundagi yig‘indi funksiyaning tekisligini yuqori yarmida yotgan qutblarga nisbatan chegirmalari yig‘indisidir. Bu qutblar bir karrali $z_1 = i$ va $z_2 = 3i$ nuqtalardir.

(5.2) formulaga ko‘ra:

$$resf(+i) = \lim_{z\to i}\left[(z-i)\cdot\frac{e^{iZ}}{(z^2+1)(z^2+9)}\right] = \lim_{z\to i}\frac{e^{iZ}}{(z+1)(z^2+9)} = \frac{e^{-1}}{2i\cdot 8} = \frac{e^{-1}}{16i},$$

$$resf(3i) = \lim_{z\to 3i}\left[(z-3i)\frac{e^{iz}}{(z^2+1)(z-3i)(z+3i)}\right] = \frac{e^{-3}}{-8\cdot 6i} = -\frac{e^{-3}}{48i}.$$

Demak, $2\pi\sum resf(z) = \pi\left(\frac{1}{8e}-\frac{1}{24e^3}\right) = \frac{\pi}{24e^3} = \frac{\pi}{24e^3}(3e^2-1).$

Shunday qilib, $\int\limits_{-\infty}^{+\infty}\frac{e^{ix}dx}{(x^2+1)(x^2+9)} = \int\limits_{-\infty}^{+\infty}\frac{\cos x + i\sin x}{(x^2+1)(x^2+9)}dx = \frac{\pi}{24e^3}(3e^2-1).$

Tenglikni har ikkala tomonidagi ifodalarning haqiqiy qismlarini tenglashtirib

$$\int\limits_{-\infty}^{+\infty}\frac{\cos x}{(x^2+1)(x^2+9)}dx = \frac{\pi}{24e^3}(3e^2-1)$$

ga ega bo‘lamiz.

**Mustaqil yechish uchun misollar**

Quyidagi funksiyalarning ajratilgan maxsus nuqtaga hamda cheksiz uzoqlashgan nuqtalarga nisbatan chegirmalari topilsin.

1. $f(z) = \frac{1}{z^3 - z^5}.$
2. $f(z) = \frac{z^2}{(z^2+1)^2}$
3. $f(z) = \frac{1}{z(1-z^2)}.$
4. $f(z) = \frac{z^2+z-1}{z^2(z-1)}.$
5. $f(z) = \frac{e^z}{z^2(z^2+9)}.$
6. $f(z) = \frac{1}{\sin z}.$

7. $f(z)=\cos\frac{1}{z-2}$.

**Ko‘rsatma**. Barcha maxsus nuqtalar (cheksiz uzoqlashgan nuqtani ham qo‘shib hisoblanganda) ga nisbatan chegirmalarning yig‘indisi nolga tengligidan foydalanilsin.

8. $\frac{1}{2\pi i}\oint_{\Gamma}\sin\frac{1}{z}dz,$ bunda $\Gamma-|z|=r$ aylana.

9. $\int_0^{2\pi}\frac{dx}{(\cos x+2)^2}$.

10. $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}(a>0,b>0)$.

11. $\int_0^{2\pi}\frac{\sin^2 tdt}{a+b\cos t}$ $(a>b>0)$.

12. $\int_0^{\pi}ctg(x-a)dx$ $(a=\alpha+i\beta,\beta\neq 0)$.

13. $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^3}$.

14. $\int_{-\infty}^{+\infty}\frac{x^2dx}{(x^2+a^2)^2}$ $(a>0)$.

15. $\int_0^{\infty}\frac{\cos x}{x^2+9}dx.$

16. $\int_0^{\infty}\frac{\sin xdx}{x(x^2+1)^2}$.

**Nazorat savollari**

1. Golomorf funksiyaga ta’rif bering.
2. $e^z,\sin z,\cos z$ funksiyalarini Teylor qatoriga yoying.
3. Funksiyaning nollarini ta’riflang.
4. Funksiyaning Loran qatori deb qanaqa qatorga aytiladi?
5. Funksiya Loran qatoriga qanday qilib yoyiladi?
6. Loran qatorining koeffitsiyentlari qanday topiladi?
7. To‘g‘ri va maxsus nuqtalarga ta’rif bering.
8. Ajralgan maxsus nuqta deb qanaqa nuqtaga aytiladi?
9. Qutulib bo‘ladigan, qutb va muhim maxsus nuqtalarga ta’rif bering.
10. Funksiya muhim maxsus nuqta atrofida o‘zini qanday tutadi?
11. Funksiyaning chegirmasi (qoldig‘i) deb nimaga aytiladi?
12. Qutulib bo‘ladigan maxsus nuqtaga nisbatan chegirma nimaga teng?
13. Chegirmalar uchun Koshi teoremasini isbotlang.
14. Oddiy qutbga nisbatan chegirmani topish formulasini yozing.
15. $k$ -tartibli qutbga nisbatan chegirmani topish formulasini yozing.

ADABIYOTLAR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## MUNDARIJA

### 1. Kompleks sonlar va ular ustida amallar



### 2. Kompleks o‘zgaruvchili funksiya, ularning limiti, uzluksizligi va hosilasi. Koshi-Riman shartlari



### 3. Kompleks o‘zgaruvchili funksiyaning integrali

## 4. Kompleks hadli qatorlar



## 5. Maxsus nuqtalar. Chegirmalar nazariyasi va uning tatbiqlari



Muharrir: Sidikova K.A.