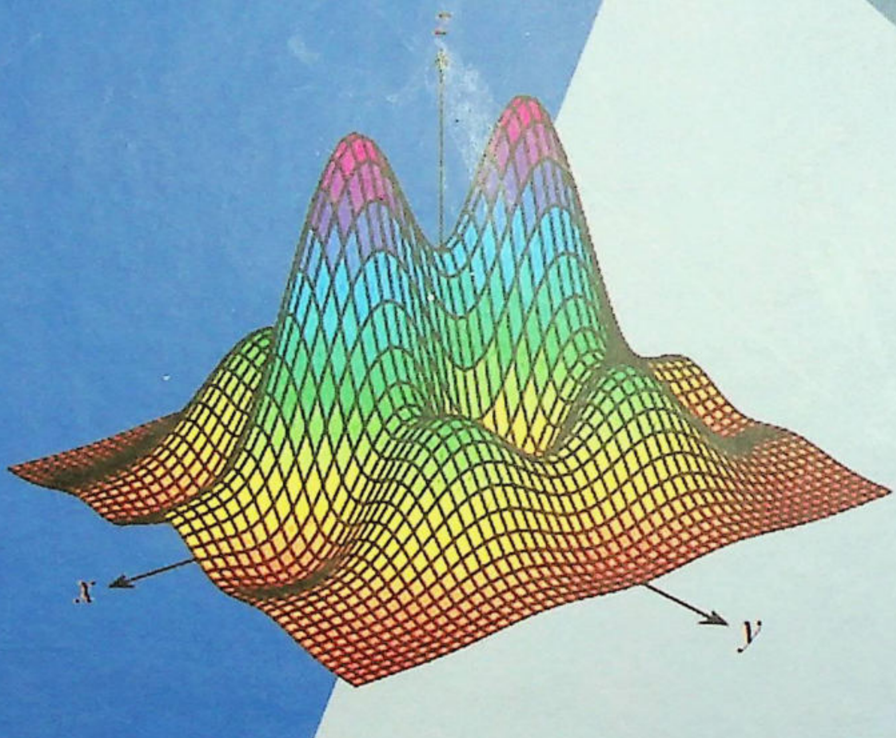
OLIY MATEMATIKA



1-QISM

O‘ZBEKISTON RESPUBLIKASI OLIY VA
O‘RTA MAXSUS TA’LIM VAZIRLIGI

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OLIY MATEMATIKA

1-QISM

O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi tomonidan 300 000- Ishlab-chiqarish texnik soha, 400000-Qishloq va suv xo‘jaligi, 200000-Ijtimoy soha, iqtisod va huquq hamda barcha tabiiy va muhandis-texnik sohalarining bakalavriat ta’lim yo‘nalishlari talabalari uchun darslik sifatida tavsiya etligan

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Ushbu darslik oliy o‘quv yurtlarining tabiiy va muhandis-texnik sohalarining barcha ta’lim yo‘nalishlarida oliy matematikani o‘rganuvchi talabalar uchun mo‘ljallab yozilgan. Kitobning birinchi qismida chiziqli va vektor algebrasi elementlari, tekislikda va fazoda analitik geometriya, haqiqiy va komleks sonlar, funksiyaning limiti va uzluksizligi, funksiyaning hosilasi va differensiali, hosilaning tadbiqlari, aniqmas, aniq va xosmas integrallar boblarini qamrab olgan.

Darslik sodda, ayni paytda matematik qatiyat va izchillikda bayon qilingan bo‘lib, amaliy mashg‘ulot mavzulariga mos tipik misol va masalalar batafsil yechib ko‘rsatilgan hamda mustaqil ishlash uchun yetarli miqdorda misol va masalalar javoblari bilan keltirilgan. Modul va kredit tizimida ta’lim olayotgan talabalarga auditoriyada va auditoriyadan tashqarida bajarishi zarur bo‘lgan mustaqil ishlar hamda ularni bajarish bo‘yicha tavsiyalar berilgan.

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So'z boshi

Buyuk vatandoshimiz Muhammad ibn Muso al-Xorazmiy o'zining mashhur "Aljabr va almuqobala hisobi haqida qisqacha kitob"ini yozib algebra faniga asos solgan bo'lsa, yana bir buyuk alloma Abu Rayhon Beruniy asarlarida geometriya, arifmetika, algebra, sonlar nazariyasi va trigonometriya tushunchalari ma'lum tartib bilan ta'riflangan. U "Al-Qonun al-Ma'sudiy" asarida yassi va sferik trigonometriyani bayon qilgan.U yulduzlar hukmiga oid ilm - matematika fanlarning eng yaxshi hosilasidir deb yozib qoldirgan. Buyuk astranom va matematik Mirzo Ulug'bekning fan va madaniyat tarixida so'nmas iz qoldirgan ilmiy merosi uning "Ziji Ko'ragoniy" asaridir. Bu asar sayyoralar, Quyosh va Oy harakatini talqin qilish, yulduzlar katalogi va unda qo'llanilgan matematik usullari bo'yicha o'rta asrlardagi astranomik asarlarning eng mukammali bo'lganligi uchun, avvalo, u musulmon mamlakatlari olimlarini keyinchalik Yevropa olimlarining diqqatini jalb qilgan. Akademik T.A.Qori Niyoziy o'tgan asrning 30-yillaridan boshlab yozgan asarlarini 1967-1970 yillarda 8 jilddan iborat tanlangan asarlar shaklida nashr yettirgan. Uning birinchi to'rt jildi oliy matematikaning analitik geometriya, matematik tahlil va differensial tenglamalar bo'limlarining o'zbek tilidagi birinchi darsliklaridir. Ular yarim asr davomida respublikamiz oliy o'quv yurtlarida oliy matematikadan asosiy darslik vazifasini o'tab keldilar.Oliy matematikaning deyarli barcha bo'limlarini qamrab olgan texnika oliy o'quv yurtlari uchun o'tgan asrimizning 80-yillarida Yo.U.Soatov tomonidan yozilgan darslik ham oliy matematika ta'limida muhim ahamiyatga ega bo'ldi.

Mazkur darslik "Oliy matematika", 1-qismi bo'lib, unda oliy matematikaning chiziqli va vektor algebrasi elementlari, tekislikda va fazoda analitik geometriya. haqiqiy va komleks sonlar, funksiyaning limiti va uzluksizligi, funksiyaning hosilasi va differensiali, hosilaning tadbiqlari, aniqmas, aniq va xosmas integrallar mavzulari bayon qilingan.

Darslikning har bir boblarida mavzularning nazariy qismi, uni mustahkamlash uchun o'z-o'zini tekshirish savollari, amaliy mashg'ulotlar uchun misol va masalalar, mustaqil bajarish uchun yozma ishlar materiallari berilgan.

Darslikning 1-, 2-, 3- qismlari o'ziga xos jihatlari sifatida quyidagilarni ta'kidlash mumkin:

1) Mustaqil ta'lim bo'yicha ko'rsatmalarda muayyan mavzuni nazariy va amaliy jihatdan o'zlashtirish uchun bo'limlar kesimida ro'yxatda keltirilgan adabiyotlarning nomerlari va zaruriy sahifalari ko'rsatilgan.

2) Amaliy mashg'ulot mavzulariga mos tipik misol va masalalar batafsil yechib ko'rsatilgan hamda mustaqil ishlash uchun yetarli miqdorda misol va masalalar javoblari bilan keltirilgan va ulardan masalalar to'plami sifatida foydalanish mumkin.

3) Yozma nazorat ishlari uchun keltirilgan topshiriqlar har bir talabaga har bir mavzudan bittadan olib mustaqil bajarishga mo'ljallangan bo'lib, ular boblar bo'yicha yozma ish shaklida belgilangan grafik bo'yicha davriy ravishda bajarilishi va baholanib borilishi ko'zda tutilgan.

Masofaviy ta'lim shaklida samarali foydalanib kelinayotgan *moodule* va boshqa tizimlariga kiritish uchun didaktik materiallar tanlash imkoniyati mavjud. Kredit tizimida talabalar bilimini o'lchab baholash uchun nazorat o'lchov materiallarini tanlab olish mumkin.

Mazkur kitob, asosan tabiiy, muhandis-texnik fanlar sohasidagi bakalavriat talabalari uchun mo'ljallangan. Uch xil o'quv shakli (kunduzgi, kechki, sirtqi) uchun amaliy mashg'ulotlar jarayonlari va nazorat turlarini (darsxona topshiriqlari, mustaqil va nazorat ishlari, namunaviy hisob topshiriqlari, laboratoriya ishlari va hokazolar) tashkil qilishga kerakli bo'lgan tushunchalar, formulalar, qoidalar va usullar hamda ularning mohiyati ko'p miqdordagi misollar yordamida tushuntirilgan.

Mualliflar darslikning qo'lyozma va elektron variantlarini o'qib, uning sifatini yanada oshirish borasidagi fikr va mulohazalari uchun prof. A.B.Hasanov va dots. H.Qurbonovlarga hamda matnlarni kompyuterga kiritishga yordam berganliklari uchun D.Boymurodov va R. Eshbekovlarga o'zlarining minnatdorchiliklarini bildiradilar.

Kitobda matematik belgilardan keng foydalanish bilan birga tasdiqlar, teoremalar isbotining tugaganligi esa, ■ belgi orqali belgilangan. Hamda, kitobning boshidan oxirigacha mavzular, ta'riflar, teoremalar kerakli formulalar bandlarga ajratilgan va ular tartib bilan nomerlangan.

O'quv darslik haqidagi fikr mulohazalar, undagi mavjud kamchiliklar bo'yicha takliflarni mualliflar mamnuniyat bilan qabul qiladilar.

Mualliflar

Matematik belgilar

T	Matematik belgilar	Matematik belgilarning ishlatilish mazmuni
1.	$\in$	Tegishli belgisi, a element A to'plamning elementi bo'lsa, $a \in A$ yoziladi.
2.	$\notin$	Tegishli emaslik belgisi, b element B to'plamning elementi bo'lmasa, $b \notin B$ kabi ifodalanadi.
3.	$\subset$	Qism belgisi. A to'plam B to'plamning qismi bo'lsa, uni $A \subset B$ kabi yoziladi.
4.	$\forall$	Umumiylik belgisi. «Har qanday», «Ixtiyoriy», «barchasi uchun» so'zlari va so'z birikmalari o'rnida ishlatiladi.
5.	$\exists$	Mavjudlik belgisi. «Mavjudki», «Topiladiki», o'rnida ishlatiladi.
6.	$\Rightarrow$	«Implikasiya belgisi» bo'lsa, ….. bo'ladi «…, kelib chiqadi» ma'nosida ishlatiladi.
7.	$\Leftrightarrow$	Ekvivalent belgisi

I BO'LIM. CHIZIQLI ALGEBRA

1-bob. CHIZIQLI ALGEBRA ELEMENTLARI

1.1-§. Matritsalar

1.1. Matritsa tushunchasi

1.1.1–ta'rif. m ta satr n ta ustundan iborat ushbu

$$A=\begin{bmatrix} a_{11}\ a_{12}\ \dots\ a_{1n} \\ a_{21}\ a_{22}\ \dots\ a_{2n} \\ \dots\dots\dots\dots \\ a_{m1}\ \ a_{m2}\dots\ a_{mn} \end{bmatrix}$$

jadval to'g'ri burchakli $m\times n$ *tartibli (o'lchamli) matritsa* deyiladi.

Matritsaning tuzuvchi $a_{ij}(i=1,2,...,m,\ \ j=1,2,...,n)$ sonlar uning *elementlari* deyiladi. Matritsada satrlar soni ustunlar sonidan kichik, unga teng yoki katta, ya'ni $m<n,\ m=n,\ m>n$ bo'lishi mumkin.Umumiy holda matritsaning elementlari, odatda, pastiga ikkita indeks qo'yilib, bitta kichik lotin harfi bilan yoziladi. Indekslardan birinchisi satr tartibini, ikkinchisi esa ustun tartibini bildiradi. Matritsalar uchun quyidagi ko'rinishdagi belgilashlar ham ishlatiladi:

$$A=\begin{Vmatrix} a_{11}a_{12}...a_{1n} \\ \dots\dots\dots\dots \\ a_{m1}a_{m2}...a_{mn} \end{Vmatrix} \text{ yoki } A=\begin{pmatrix} a_{11}a_{12}...a_{1n} \\ \dots\dots\dots\dots \\ a_{m1}a_{m2}...a_{mn} \end{pmatrix},\ A=(a_{ij}).$$

Agar matritsaning satrlari soni ustunlar soniga teng (ya'ni $m=n$) bo'lsa, matritsani *kvadrat matritsa* deyiladi. Bunday matritsa (n - tartibli) matritsa deb yuritiladi.

Ushbu

$$D=\begin{bmatrix} d_{n}\ 0...\ \ 0 \\ 0\ \ d_{22}...0 \\ \dots\dots\dots\dots \\ 0\ \ 0...\ \ d_{nn} \end{bmatrix} \qquad (1.1.2)$$

ko'rinishdagi kvadrat matritsa *diagonal matritsa* deyiladi va qichqacha quyidagicha yoziladi $D=\{d_{11}d_{22},...,d_{nn}\}$ yoki $D=\{d_{ii}\}$.

Agar (1.1.2) diagonal matritsa $d_{ii}=1(i=1,2,...,n)$ bo'lsa, bu matritsa birlik matritsa deyiladi va E harfi orqali belgilanadi, ya'ni

$$E=\begin{bmatrix} 1\ \ 0...0 \\ 0\ \ 1...0 \\ \dots\dots\dots \\ 0\ \ 0...1 \end{bmatrix}$$

Agar matritsaning barcha elementlari nollardan iborat bo'lsa, u *nol matritsa* deyiladi va E^0 orqali belgilanadi, ya'ni

$$E^0=\begin{bmatrix} 0\ \ 0...0 \\ 0\ \ 0...0 \\ \dots\dots\dots \\ 0\ \ 0...0 \end{bmatrix}$$

Agar m ta satrli va n ta ustunli ikkita A va B matritsadan birining hamma elementlari ikkinchisining hamma mos elementlariga teng, ya'ni $a_{ij}=b_{ij}$ bo'lsa, bu *matritsalar teng* deyiladi va $A=B$ kabi yoziladi.

Agar bir matritsaning kamida bitta elementi ikkinchisining mos elementiga teng bo'lmasa, bu matritsalar *teng emas* deyiladi va $A\neq B$ ko'rinishda yoziladi. Matritsalar uchun kichik va katta tushunchalari ma'noga ega emas.

1. 2. Matritsalar ustida chiziqli amallar

1.2.1–ta'rif. Bir xil o'lchamli ikkita $A=(a_{ij})$ va $B=(b_{ij})$($i=1,2,..m, j=1,2,...,n$) matritsalarning *yig'indisi* deb, shunday $C=(c_{ij})$ matritsaga aytiladiki, bu matritsaning elementlari A va B matritsalarning mos elementlari yig'indilariga $c_{ij}=a_{ij}+b_{ij}, \forall i,j,\ \ i=1,2,...,m, j=1,2,...,n$ teng bo'ladi va $C=A+B$ deb belgilanadi.

Ta'rif bo'yicha

$$C=A+B=\begin{pmatrix} a_{11}+b_{11}\ \ a_{12}+b_{12}...a_{1n}+b_{1n} \\ a_{21}+b_{21}\ \ a_{22}+b_{22}...a_{2n}+b_{2n} \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{m1}+b_{m1}\ \ a_{m2}+b_{m2}...a_{mn}+b_{mn} \end{pmatrix}$$

Matritsalar yig'indisi tarifidan uning quyidagi xossalari kelib chiqadi:
1^0. $A+(B+C)=(A+B)+C$. 2^0. $A+B=B+A$.

3^0. $A+E^0=A$ (bunda $E^0=(0)$, A,B,C –berilgan bir xil tartibli kvadrat matritsalar).

Matritsalarning ayirmasi ularning yig'indisiga o'xshash ta'riflanadi va

$$C=A-B=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \ldots\ldots a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} \ldots a_{2n}-b_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} \ldots a_{mn}-b_{mn} \end{pmatrix}$$

ko'rinishda yoziladi. Agar matritsalarning tartibi bir xil bo'lmasa, unday matritsalarda qo'shish va ayrish amallari kiritilmagan.

1.2.3–ta'rif. $A=(a_{ij})$ matritsani $\alpha \neq 0$ songa ko'paytmasi deb, A matritsaning hamma elementlarini shu α songa ko'paytirishdan hosil bo'lgan matritsaga aytiladi va αA ko'rinishda yoziladi.

Ta'rifga ko'ra,

$$A\alpha=\alpha A=\begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \ldots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \ldots & \alpha a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha a_{m1} & \alpha a_{m2} & \ldots & \alpha a_{mn} \end{pmatrix}.$$

Matritsani songa ko'paytirish ta'rifidan quyidagi xossalar kelib chiqadi:
1^0. $1\cdot A=A\cdot 1=A$. 2^0. $A\cdot 0=0\cdot A=E^0$.
3^0. $\alpha(\beta)A=\beta(\alpha A)=(\alpha\beta)A$. 4^0. $(\alpha+\beta)A=\alpha A+\beta A$.
5^0. $\alpha(A+B)=\alpha A+\alpha B$.
Bu yerda A va B – bir xil tartibli kvadrat matritsalar, α,β - haqiqiy sonlar.
Misollar:

1) $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ va $B=\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ berilgan bo'lsa, u holda

$$A+B=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}+\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}=\begin{pmatrix} 1+2 & 3-2 \\ 4+3 & 5+4 \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ 7 & 9 \end{pmatrix} \text{ bo'ladi.}$$

2) $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ va $\alpha=2$ berilgan bo'lsa, u holda

$$A\alpha=\alpha A=2\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}=\begin{pmatrix} 2\cdot 2 & 3\cdot 2 \\ 4\cdot 2 & 5\cdot 2 \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix}.$$

1.3. Matritsalarni ko'paytirish. Tartiblari mos ravishda $m\times n$ va $p\times q$ bo'lgan

$$A=\begin{pmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2}\ldots a_{mn} \end{pmatrix} \text{ va } B=\begin{pmatrix} b_{11} & b_{12}\ldots b_{1q} \\ b_{21} & b_{22}\ldots b_{2q} \\ \ldots\ldots & \ldots\ldots \\ b_{p1} & b_{p2}\ldots b_{pq} \end{pmatrix}$$

to'g'ri burchakli matritsalar berilgan bo'lsin.

Agar A matritsaning ustunlari soni n berilgan B matritsaning satrlari soni p ga teng bo'lsa, u holda bu matritsalar uchun ko'paytirish amalini aniqlash mumkin.

1.3.1–ta'rif. Berilgan $m\times n$ o'lchamli A va $n\times p$ o'lchamli B matritsalarning ko'paytmasi AB deb, shunday $m\times p$ o'lchamli

$$C=A\cdot B=\begin{pmatrix} c_{11} & c_{12}\ldots c_{1p} \\ c_{21} & c_{22}\ldots c_{2p} \\ \ldots\ldots & \ldots\ldots \\ c_{m1} & c_{m2}\ldots c_{mp} \end{pmatrix}$$

matritsaga aytiladi, C matritsaning elementlari

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+c_{in}b_{nj}=\sum_{k=1}^{n}a_{ik}b_{kj}, \quad i=1,2,\ldots,m \quad j=1,2,\ldots,n$$

formulalar bilan aniqlanadi.

Agar A va B lar n- tartibli kvadrat matritsalar bo'lsa, ularning $C=AB$ ko'paytmasi ham n- tartibli kvadrat matritsa bo'ladi.

Qoida. Ikkita matritsani ko'paytirishdan hosil bo'lgan matritsaning i- satri va j - ustunida turuvchi c_{ij} elementni topish uchun birinchi matritsaning i - satrida turuvchi elementlarni ikinchi matritsaning j - ustunida turuvchi elementlarga mos ravishda ko'paytirib qo'shish kerak.

Misol. Quyidagicha

$$A=\begin{bmatrix}3 & 2 & 8 & 1\\ 1 & -4 & 0 & 3\end{bmatrix} \text{ va } B=\begin{bmatrix}2 & -1\\ 1 & -3\\ 0 & 1\\ 3 & 1\end{bmatrix}$$

matritsalar ko'paytmasini topamiz.

$$C=AB=\begin{bmatrix}3\cdot2+\ 2\cdot1+\ \ 8\cdot0+1\cdot3 & 3\cdot(-1)+2\cdot(-3)+8\cdot1+1\cdot1\\ 1\cdot2+(-4)\cdot1+0\cdot0+3\cdot3 & 1\cdot(-1)+(-4)\cdot(-3)+0\cdot1+3\cdot1\end{bmatrix}=\begin{bmatrix}11 & 0\\ 7 & 14\end{bmatrix}.$$

Matritsalarning ko'paytmasi quyidagi xossalarga ega:

1⁰. $A(BC)=(AB)\cdot C$. **2⁰**. $\alpha(AB)=(\alpha A)\cdot B=A(\alpha B),\ \ \alpha\neq 0$.
3⁰. $(A+B)C=AC+BC$. **4⁰**. $C(A+B)=CA+CB$.
5⁰. $A^m=AA...A$. **6⁰**. $AE=EA=A$. **7⁰**. $OA=AO=E^0$.

Bunda A,B,C matritsalar, α -haqiqiy son.

Ikki matritsaning ko'paytmasi uchun kommutativlik (o'rin almashtirish) xossasi umuman aytganda, o'rinli emas, ya'ni

$$AB\neq BA.$$

Misol. Agar $A=\begin{pmatrix}5 & 1\\ 2 & 4\end{pmatrix}$ va $B=\begin{pmatrix}2 & 4\\ 1 & -2\end{pmatrix}$ bo'lsa, u holda $AB=\begin{pmatrix}11 & 18\\ 8 & 0\end{pmatrix}$, $BA=\begin{pmatrix}18 & 18\\ 1 & 6\end{pmatrix}$ bo'ladi.

Shunday qilib, $AB\neq BA$.

Satr va ustun matritsalarni ko'paytirish.

1. Istalgan m o'lchamli ustun matritsani istalgan n o'lchamli satr matritsaga ko'paytirish mumkin, natijada $m\times n$ o'lchamli matritsa hosil bo'ladi.

2. n o'lchamli satr matritsani n o'lchamli ustun matritsaga ko'paytirish mumkin, natijada 1×1 o'lchamli matritsa hosil bo'ladi.

1.4. Transponirlangan matritsa. Elementar almashtirishlar.

Ushbu

$$A=\begin{bmatrix}a_{11} & a_{12}...a_{1n}\\ a_{21} & a_{22}...a_{2n}\\ \dots\dots & \dots\dots\\ a_{m1} & a_{m2}...a_{mn}\end{bmatrix}$$

matritsa berilgan bo'lsin.

1.4.1–ta'rif. A matritsaning satrlari bilan ustunlarining o'rinlarini nomerlarini o'zgartirmasdan almashtirishga *transponirlash* deyiladi va quyidagicha belgilanadi

$$A^T=\begin{bmatrix}a_{11} & a_{21}...a_{m1}\\ a_{12} & a_{22}...a_{m2}\\ \dots\dots & \dots\dots\\ a_{1n} & a_{2n}...a_{mn}\end{bmatrix}.$$

Masalan, ushbu $A=\begin{bmatrix}1 & 2 & -7\\ 4 & 0 & 9\end{bmatrix}$ matritsani transponirlashdan $A^T=\begin{bmatrix}1 & 4\\ 2 & 0\\ -7 & 9\end{bmatrix}$ matritsa hosil bo'ladi.

Transponirlash amali quyidagi xossalarga ega:

1⁰ . $(A^T)^T=A$. **2⁰** . $(A+B)^T=A^T+B^T$. **3⁰** . $(\alpha A)^T=\alpha A^T$,

bunda α - haqiqiy son, A va B lar $m\times n$ o'lchamli matritsalardir.

Agar A kvadrat matritsa uchun $A=A^T$ ya'ni $\forall ij$ lar uchun $a_{ij}=a_{ji}$ tenglik o'rinli bo'lsa, u holda A *simmetrik matritsa* deyiladi. Masalan, ushbu

$$A=\begin{pmatrix}1 & 2 & -3\\ 2 & 4 & 5\\ -3 & 5 & 7\end{pmatrix}$$

simmetrik matritsa bo'ladi.

Agar $A=-A^T$ $\forall ij$ lar uchun $a_{ij}=-a_{ji}$ tenglik o'rinli bo'lsa, u holda A matritsa antisimmetrik matritsa deyiladi.

Masalan, ushbu

$$A=\begin{pmatrix}0 & -2 & -3\\ 2 & 0 & -8\\ 3 & 8 & 0\end{pmatrix}$$

antisimmmetrik matritsa bo'ladi.

Eslatma. Har qanda A matritsani simmetrik va antisimmetrik matritsalar yig'indisi ko'rinishda tasvirlash mumkin:

$$a_{ij}=\frac{1}{2}(a_{ij}+a_{ij})+\frac{1}{2}(a_{ij}-a_{ij}),\quad i=1,2,...m>j=1,2,...n.$$

1.4.2–ta'rif. Matritsada elementar almashtirishlar deb, quyidagi amallarni tushunamiz:

1. Transponirlashni;
2. Istalgan ikki satr (ikki ustun) ni o'zaro almashtirishni;
3. Istalgan satr (ustun) ning elementlarini noldan farqli har qanday m songa ko'paytirishni;
4. Biror satri(ustuni)ning elementlarini istalgan m songa ko'paytirib, boshqa satri (ustuni) ning mos elementlariga qo'shishni.

Agar B matritsa A matritsaning satrlari (yoki ustunlari) ni bir necha marta ketma-ket elementar almashtirishlar yordamida hosil qilingan bo'lsa, u holda A matritsa B matritsaga *ekvivalent* deyiladi va $A\approx B$ ko'rinishda yoziladi.

Masalan, ushbu $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $B=\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}$ matritsalar ekvivalentdir. Haqiqatan ham, A matritsaning 2- satrni 1- satrga qo'shamiz,

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\approx\begin{pmatrix}4 & 6\\ 3 & 4\end{pmatrix}\approx$$

2- satrni (-1) ga ko'paytirib, 1- satrni qo'shamiz;

$$\approx\begin{pmatrix}4 & 6\\ 3 & 4\end{pmatrix}\approx\begin{pmatrix}4 & 6\\ 1 & 2\end{pmatrix}\approx\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}=B$$

2- satrni (-1) ga ko'paytirib, 1- satrni qo'shamiz natijada B matritsa hosil bo'ladi.

1.2-§. Determinantlar va ularning xossalari

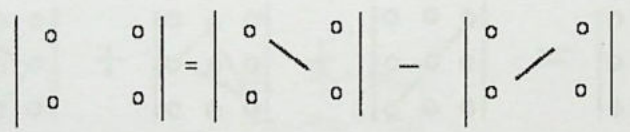
1.5. Ikkinchi, uchinchi va yuqori tartibli determinantlarning ta'riflari. n-tartibli kvadratik A matritsaga mos determinanti deb ataluvchi va $\det A$ (yoki $|A|,\Delta$) kabi belgilanuvchi sonni quyidagicha ta'riflaymiz.

1.5.1-ta'rif. Berilgan ixtiyoriy $a(a\in R)$ haqiqiy son, birinchi taribli determinat deyiladi va $\det A:=\Delta:=|A|:=a$ kabi yoziladi.

1.5.2-ta'rif. Ushbu $a_{11}a_{22}-a_{12}a_{21}$ ifodaga (songa) *ikkinchi tartibli determinant* deyiladi va u

$$n=2,\ \det A:=\Delta:=|A|:=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

kabi yoziladi. Bunda $a_{11},a_{12},a_{21},a_{22}$ larga determinantning *elementlari* bo'lib, a_{11},a_{22} elementlar determinantning bosh diagonalini, a_{12},a_{21} elementlar esa determinantning yordamchi diagonalini tashkil etadi. Ikkinchi tartibli determinant bosh diagonal elementlari ko'paytmasi bilan yordamchi diagonal elementlari ko'paytmasining ayirmasiga teng:



1.5.3-misol. Determinantlarni hisoblang:

$$1)\ \Delta=\begin{vmatrix}2 & 4\\ 5 & 6\end{vmatrix};\quad 2)\ \Delta_1=\begin{vmatrix}x & 2x\\ 3 & 6\end{vmatrix}.$$

Yechilishi. Determinantlarni ta'rif (sxema) asosida topamiz:

1) $\Delta=2\cdot 6-5\cdot 4=-8$;
2) $\Delta_1=x\cdot 6-3\cdot 2x=0$.

1.5.4-ta'rif. Ushbu

$$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}$$

ifodaga uchinchi tartibli determinant deyiladi va u

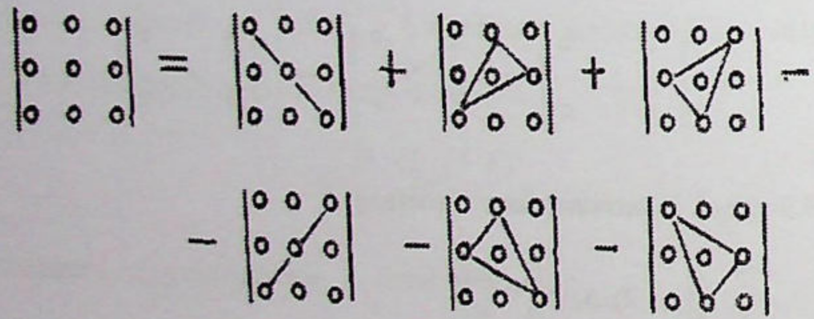
$$n=3,\ \det A:=\Delta:=|A|:=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33} \quad (1.5.5)$$

kabi yoziladi.

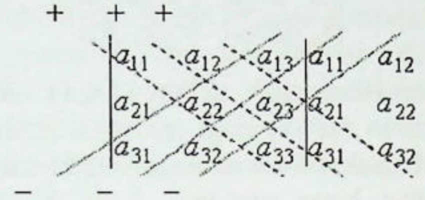
Bu ifoda uchburchaklar qoidasi (*Sarryus qoidasi*) bo'yicha topiladi. Uni quyidagi jadvallar orqali tasvirlash mumkin bo'lib, bir xil ishora bilan bitta ko'paytmada ishtirok etuvchi elementlar kesmalar bilan birlashtirilib ko'rsatilgandir:

yoki



Uchinchi tartibli determinantni hisoblashning yana bir usuli quyidagicha. Determinantning o'ng tomonidagi birinchi ustunlarni yozamiz va asosiy diagonal elementlari va ikkita parallel diagonalning elementlari ko'paytmalarini plyus belgisi bilan yig'amiz. Keyin ikkilamchi diagonal

elementlari va ikkita parallel diagonalning elementlari ko'paytmalarini minus belgisi bilan qo'shamiz.



1.5.6-misol. Ushbu

$$A=\begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & -3 \\ 5 & 0 & 4 \end{pmatrix}$$

matritsaning determinantini uchburchak qoidasi bo'yicha hisoblang.

Yechilishi. Determinantni ta'rif (sxema) asosida topamiz:

$$\det A=\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & -3 \\ 5 & 0 & 4 \end{vmatrix}=$$

$$=3\cdot2\cdot4+1\cdot0\cdot2+(-1)\cdot(-3)\cdot5-5\cdot2\cdot2-0\cdot(-3)\cdot3-1\cdot(-1)\cdot4=23.$$

n- tartibli determinant deb quyidagi simvol bilan belgilangan sonni ataymiz:

$$D=\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \dots & \dots\dots\dots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}.$$

n-tartibli determinant n ta satr, n ta ustun va n^2 ta elementga ega, bunda $a_{11},a_{22},a_{33},...,a_{nn}$ elementlar birinchi (bosh) diagnolni, $a_{1n}a_{2,n-1},...,a_{n1}$ elementlar esa ikkinchi (yordamchi) diagonalni tashkil etadi.

1.6. Determinantlarning asosiy xossalari

1⁰. Determinantda hamma satrlar mos ustunlar qilib yozilsa, ya'ni transponirlanganda, determinatning qiymati o'zgarmaydi, ya'ni

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} . \qquad (1.6.1)$$

Isboti. Bu xossani isbot etish uchun (1.6.1) tenglikning to'g'riligini ko'rsatish yetarli. Ammo (1.6.1) dagi har ikki determenantni uchburchak qoidasini qo'llab hisoblasak, bir xil natijaga kelamiz. ■

2^0. Determinantning biror satridagi (yoki biror ustunidagi) barcha elementlar nolga teng bo'lsa, bunday determinant nolga teng bo'ladi, ya'ni

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0, \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = 0. \qquad (1.6.2)$$

Isboti. (1.6.2) ni har ikki determenantni uchburchak qoidasini qo'llab hisoblasak, har bir hadida nol qatnashadi. Bunday determinant qiymati nolga teng bo'ladi. ■

3^0. Determinantda istalgan ikki satrni (yoki ikki ustunni) o'zaro almashtirsak, determinantning faqat ishorasi o'zgaradi.

Isboti. Bu xossaning to'g'riligiga berilgan determenantga va undan ikki satr yoki ikki ustunning o'rnini almashtirishda hosil bo'lgan determinantga uchburchak qoidasini bevosita qo'llash bilan ishonch hosil qilish mumkin. Jumladan, 1-va 3- ustunlarning o'rnini almashtirsak, ushbu

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{vmatrix}$$

tenglikka ega bo'lamiz. ■

4^0. Determinantning ikkita satri (yoki ikkita ustuni) teng bo'lsa, bu determinant nolga teng bo'ladi.

Isboti. (1.5.5) determinantning 1- va 2- satri elementlari mos ravishda bir-biriga teng bo'lsin,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta .$$

Shu determinantdagi bu satrlar o'rinlarini amlashtiramiz. U vaqtda, bir tomondan, 3^0-xossaga asosan, determinantning qiymati o'z ishorasini o'zgartiradi. Lekin ikkinchi tomondan, o'zaro almashtirilayotgan satrlar bir xil bo'lgani uchun ularni o'zaro almashtirish determinant qiymatini o'zgartirmaydi. Demak, $\Delta = -\Delta$ tenglikka egamiz, bundan $2\Delta = 0$ yoki $\Delta = 0$ kelib chiqadi. Determinantning elementlari teng ikki ustunining o'rinlarinialmashtirishga tegishli mulohazalar ham shunga o'xshash yuritiladi. ■

5^0. Determinantning ixtiyoriy satri (ustuni) elementlarini biror o'zgarmas k soniga ko'paytirilsa, determinantning qiymati ham k ga ko'paytiriladi.

Isboti. (1.5.5) determinantning birinchi satri elementlarini k ga ko'paytirib determenantni uchburchak qoidasini qo'llanib hisoblaymiz:

$$\Delta_1 = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}a_{22}a_{33} + ka_{12}a_{23}a_{31} + ka_{13}a_{21}a_{32} - ka_{13}a_{22}a_{31} -$$

$$- ka_{11}a_{23}a_{32} - ka_{12}a_{21}a_{33} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\Delta .$$

Demak, determinantnin satrdagi (yoki ustundagi) barcha elementlarning umumiy ko'paytuvchisini determinant belgisi ostidan chiqarish mumkin. ■

6^0. Biror satridagi barcha elementlari boshqa bir satrining mos elementlariga proporsional bo'lgan determinant nolga tengdir. Xuddi shunday xossa ustunlar uchun ham o'rinli.

Isboti. (1.5.5) determinintning, masalan, birinchi satri elementlari uning uchinchi satri elementlari bilan proporsional, ya'ni $a_{11} = ka_{31}, a_{12} = ka_{32}, a_{13} = ka_{33}$ munosabatlar o'rinli bo'lsin deylik. Bu munosabatlardan foydalanib, quyidagiga ega bo'lamiz:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{31} & ka_{32} & ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \cdot 0 = 0 .$$

Oxirgi determinantning birinchi va uchinchi satrlari elementlari bir hil bo'lgani uchun 4^0-xossaga ko'ra uning qiymati nolga teng. Qolgan hollarda ham mulohazalar shu kabi yuritiladi. ■

7^0. Agar determinantning i-satridagi barcha elementlar m ta qo'shiluvchidan iborat bo'lsa, u holda bu determinantni m ta determinantlarning yig'indisi ko'rinishida ifodalash mumkin bo'lib, bunda ularning i-dan farqli barcha satrlari berilgan deteminantdagidek, i- satri esa birinchi determinantda birinchi qo'shiluvchilardan ikkinchisida - ikkinchilaridan va h.k. tuzilgandir. Xuddi shunday, ustunlar uchun ham o'rinlidir.

Xususiy holda bitta satrga boshqa bir satrni (ustunni) qo'shish (yoki undan ayirish) mumkin, ya'ni

$$\begin{vmatrix} a_{11} & a_{12}+b & a_{13} \\ a_{21} & a_{22}+c & a_{23} \\ a_{31} & a_{32}+d & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b & a_{13} \\ a_{21} & c & a_{23} \\ a_{31} & d & a_{33} \end{vmatrix}.$$

8^0. Agar determinantning hech bo'lmaganda bitta satri boshqa satrlari orqali chiziqli bog'langan bo'lsa, bu determinant nolga tengdir. Aksincha, agar n- tartibli $(n>2)$ determinant nolga teng bo'lsa, u holda uning hech bo'lmaganda bitta satri boshqa satrlari orqali chiziqli ifodalangan bo'ladi. Xuddi shunday ustunlar uchun ham o'rinlidir.

9^0. Determinantda biror ustun (satr) ning hamma elemntlarini bitta k songa ko'paytirib, bu ko'paytmalarni boshqa ustun (satr) ning mos elementlariga qo'shsak, determinantning qiymati o'zgarmaydi.

Isboti. (1.5.5) determinintning bininchi satri elementlarini k ga ko'paytirib, ikkinchi satri elementlariga mos ravishda qo'shaylik (boshqa hollar uchun ham isbot shunga o'xshash bo'ladi). Ko'rilayotgan holda

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ka_{11} & a_{22}+ka_{12} & a_{23}+ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Bu determinant qiymatini $\widetilde{\Delta}$ deb belgilaylik. Biz $\Delta=\widetilde{\Delta}$ ekanini ko'rsatishimiz lozim. Determinintning 6^0 - va 7^0-xossalarga ko'ra, quyidagiga ega bo'lamiz:

$$\widetilde{\Delta} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ma_{11} & ma_{12} & ma_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta + 0 = \Delta. \blacksquare$$

1.7. Minorlar va algebraik to'ldiruvchilar

1.7.1–ta'rif. n- tartibli D determinantning istalgan k ta satri va k ta ustunlarini ajrataylik. Bu satrlar va ustunlarning kesishgan joylaridagi elementlarni D determinantdagidek tartibda olib, ulardan k- tartibli M determinantni tuzsak, u D ning *k- tartibli minori* deb ataladi.

1.7.2–ta'rif. D determinantda ajratilgan k ta satr va k ta ustunni o'chiraylik. D ning qolgan elementlarini shu D dagidek tartibda olib, ulardan $(n-k)$ tartibli $\overline{M}$ determinantni tuzsak, u, M ga *qo'shma minor* deyiladi.

1.7.3-misol. Ushbu

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}$$

determinantda 1- va 5- satrlarni, 3 va 4- ustunlarni ajratib minor va qo'shma minorlarni tuzing.

Yechilishi. Berilgan determinantning 1- va 5- satrlarni, 3 va 4- ustunlarni kesishgan joylaridagi elementlarni ajratamiz. Budan 2- tartibli

$$M = \begin{vmatrix} a_{13} & a_{14} \\ a_{53} & a_{54} \end{vmatrix}$$

minor tuziladi. Bu ajratilgan satr va ustunlarni o'chirsak, qolgan elementlardan ushbu

$$\overline{M} = \begin{vmatrix} a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \end{vmatrix}$$

qo'shimcha minor hosil bo'ladi. ■

1.7.4- ta'rif. Ushbu

$$(-1)^{\alpha_1+\alpha_2+...+\alpha_k, \beta_1+\beta_2+...+\beta_k}$$

darajaning $\overline{M}$ qo'shimcha minorga ko'paytmasi *k- tartibli M minorning algebraik to'ldiruvchisi* (yoki M minorga mos algebraik to'ldiruvchi) deb ataladi, bunda $\alpha_1, \alpha_2, ..., \alpha_k$ ва $\beta_1, \beta_2, ..., \beta_k$ mos ravishda, D determinantning M minorga tegishli satr va ustunlarining raqamlarini bildiradi.

Algebraik to'ldiruvchi, odatda, A harf bilan belgilanadi. Ta'rifga muvofiq:

$$A = (-1)^{\alpha_1+\alpha_2+...+\alpha_k+\beta_1+\beta_2+...+\beta_k} \cdot \overline{M}$$

ko'rinishda yoziladi.

Agar M minor bitta elementdan iborat, ya'ni $M = a_{kl}$ bo'lsa, unga mos algebraik to'ldiruvchini A_{kl} bilan belgilaydilar. Bu holda

$$A_{kl} = (-1)^{k+l} \cdot M_{kl}.$$

1.7.3-misoldagi $M = \begin{vmatrix} a_{13} & a_{14} \\ a_{53} & a_{54} \end{vmatrix}$ minorning algebraik to'ldiruvchisi

$$A = (-1)^{1+5+3+4} \begin{vmatrix} a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \end{vmatrix}$$

bo'ladi.

1.8. Laplas teoremasi

1.8.1-lemma. M minorning istalgan hadini shu minorga mos A algebraik to'ldiruvchining istalgan hadiga ko'paytirsa, D determinantning hadi hosil bo'ladi.

1.8.2-teorema (Laplas). n- tartibli D determinantda istalgan k ta satr (yoki ustun) ni ajratamiz $(1 \le k < n)$. Bu ajratilgan satr (yoki ustun) larning elementlaridan tuzilgan hamma k- tartibli minorlarni o'z algebraik to'ldiruvchilari ko'paytirib natijalarni qo'shsak, yig'indi D determinantga teng bo'ladi.

1.8.3-misol. Ushbu

$$D = \begin{vmatrix} 2 & 1 & 3 & 4 \\ -1 & 3 & -2 & 5 \\ 0 & 5 & 4 & 2 \\ 1 & 2 & -3 & 1 \end{vmatrix}$$

5- tartibli determinantni 2-tartibli minorlar bo'yicha yoyib hisoblang.

Yechilishi. Determinantda, masalan, 1 va 2 satrlarni ajratsak, ularning elementlaridan, hammasi bo'lib $C_4^2 = \frac{4!}{2!2!} = 6$ ta 2- tartibli minor tuziladi. Laplas teoremasiga asosan, quyidagini hosil qilamiz:

$$D = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \cdot (-1)^{1+2+1+2} \begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} \cdot (-1)^{1+2+1+3} \cdot \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} \cdot (-1)^{1+2+1+4} \cdot$$

$$\cdot \begin{vmatrix} 5 & 4 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} \cdot (-1)^{1+2+2+3} \cdot \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} (-1)^{1+2+2+4} \cdot \begin{vmatrix} 0 & 4 \\ 1 & -3 \end{vmatrix} +$$

$$+ \begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} \cdot (-1)^{1+2+2+3} \cdot \begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} = 7 \cdot 10 - 1(-1) + 14 \cdot (-23) - 11 \cdot (-2) -$$

$$-7 \cdot 4 + 23 \cdot (-5) = 70 + 1 - 322 + 22 - 28 - 115 = -372. \blacksquare$$

1.9. Determinantni satr yoki ustun elementlari bo'yicha yoyish

Agar Laplas teoremasisida $k = 1$ bo'lsa, ya'ni D determinantda bitta satr ajratilsa, u vaqtda $M_1, M_2, ..., M_t$ minorlar, birinchi tartibli minorlar sifatida, satrning elementlaridan iborat bo'ladi.

Masalan, bu minorlar, i-satrning $a_{i1}, a_{i2}, ..., a_{in}$ elementlarini bildirsa, u vaqtda $A_1, A_2, ..., A_t$ algebraik to'ldiruvchilar bu elementlarning

$$A_{i1}, A_{i2}, ..., A_{in}$$

algebraik to'ldiruvchilariga aylanadi va Laplas teoremasiga ko'ra,

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} \tag{1.9.1}$$

ko'rinishni oladi. Bu (1.9.1) tenglik D determinantning *i satr elementlari bo'yicha* yoyilmasi deyiladi.

Agar minorlar j–ustunning $a_{1j}, a_{2j}, ..., a_{nj}$ elementlarini ifodalasa, u holda Laplas teorimasiga ko'ra, D determinantning qiymati

$$D = a_{1j}A_{1j} + a_{2j}A_{2j} + ... + a_{nj}A_{nj} \tag{1.9.2}$$

bo'ladi. Bu (1.9.2) tenglikka D determinantning *j-ustun elementlari bo'yicha* yoyilmasidir.

1.9.3-misol. Ushbu $\Delta=\begin{vmatrix}1 & 2 & -2\\ 1 & 0 & 5\\ 3 & 0 & 6\end{vmatrix}$ determinantni ikkinchi va uchinchi ustun bo'yicha hisoblang.

Yechilishi. Berilgan Δ determinantni (1.9.2)-formula bo'yicha hisoblaymiz:

$$\Delta=-2(-1)^{1+3}\begin{vmatrix}1 & 0\\ 3 & 0\end{vmatrix}+5(-1)^{2+3}\begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}+6(-1)^{3+3}\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}=18,$$

$$\Delta=2(-1)^{1+2}\begin{vmatrix}1 & 5\\ 3 & 6\end{vmatrix}+0(-1)^{2+2}\begin{vmatrix}1 & -2\\ 3 & 6\end{vmatrix}+0(-1)^{3+2}\begin{vmatrix}1 & -2\\ 1 & 5\end{vmatrix}=18.$$

Demak, berilgan Δ determinantnng qiymati bir xil chiqdi. ■

1.9.4-teorema. Determinantda i- satr (yoki j- ustun) ning a_{ij} dan boshqa, hamma elementlari nolga teng bo'lsa, bu determinant o'sha a_{ij} element bilan unga mos algebraik to'ldiruvchining ko'paytmasiga teng bo'ladi, ya'ni

$$D=a_{ij}A_{ij}.$$

1.9.4-teoremaga asosan, ushbu $\Delta=\begin{vmatrix}4 & 1 & 3\\ 0 & 0 & 5\\ -1 & 6 & 1\end{vmatrix}=5(-1)^{2+3}\begin{vmatrix}4 & 1\\ -1 & 6\end{vmatrix}=-125,$ determinantni hisoblash engil.

1.9.5–teorema. D determinantning bitta satri (yoki ustuni) dagi hamma elementlarini boshqa satr (ustuni) dagi mos elementlarining algebraik to'ldiruvchilariga ko'paytirib, natijalarni qo'shsak, yig'indi nolga teng bo'ladi, ya'ni

$$a_{i1}A_{j1}+a_{i2}A_{j2}+...+a_{in}A_{jn}=0 \quad (i\neq j),$$
$$a_{1s}A_{1t}+a_{2s}A_{2t}+...+a_{ns}A_{nt}=0 \quad (s\neq t).$$

1.9.6–teorema. D determinantning bitta satri (yoki ustuni) dagi hamma elementlarini shu satr (ustuni) dagi mos elementlarining algebraik to'ldiruvchilariga ko'paytirib, natijalarni qo'shsak, yig'indi D determinantga teng bo'ladi, ya'ni

$$a_{i1}A_{i1}+a_{i2}A_{i2}+...+a_{in}A_{in}=D, \quad a_{1s}A_{1s}+a_{2s}A_{2s}+...+a_{ns}A_{ns}=D.$$

1.10. Teskari matritsa haqida tushuncha

1.10.1–ta'rif. Agar n - tartibli A va B kvadrat matritsalar orasida $AB=BA=E$ (E– birlik matritsa) munosabat o'rinli bo'lsa, u holda B matritsani A matritsaga (va aksincha) *teskari matritsa* deyiladi.

A matritsa uchun teskari matritsasini A^{-1} orqali belgilanadi. U holda o'zaro teskari matritsalar uchun ushbu munosabat o'rinli:

$$AA^{-1}=A^{-1}A=E.$$

Berilgan kvadrat matritsaga teskari matritsa har doim ham mavjud bo'lavermaydi. Bunday matritsa mavjud bo'lganda, uni topish ko'p masalalarni hal etishda muhim ahamiyat kasb etadi.

1.10.2–ta'rif. Agar A kvadrat matritsaning determenanti nolga teng bo'lsa, u holda A matritsani maxsus, aks holda, maxsusmas matritsa deyiladi.

1.10.3-teorema. Ixtiyoriy maxsusmas matritsa uchun unga teskari matritsa mavjud.

Isboti. Faraz qilaylik, $A=[a_{ij}]$ matritsa n tartibli kvadrat matritsa bo'lib, $D=\det A\neq 0$ bo'lsin. A matritsaning $a_{ij}(i,j=1,2,...,n)$ elementlariga mos keluvchi algebraik to'ldiruvchilardan tuzilgan ushbu matritsani qaraymiz:

$$\widetilde{A}=\begin{bmatrix}A_{11} & A_{12}...A_{in}\\ A_{21} & A_{22}...A_{2n}\\ \multicolumn{2}{c}{..................}\\ A_{n1} & A_{n2}...A_{nn}\end{bmatrix}.$$

Agar bu matritsani transponirlasak,

$$\widetilde{A}^{T}=\begin{bmatrix}A_{11} & A_{21}...A_{n1}\\ A_{12} & A_{22}...A_{n2}\\ \multicolumn{2}{c}{....................}\\ A_{n1} & A_{2n}...A_{nn}\end{bmatrix}$$

matritsaga ega bo‘lamiz. $\widetilde{A}^T$ matritsani odatda $\widetilde{A}$ matritsaga qo‘shma matritsa deyiladi. Qo‘shma matritsaning barcha elementlarini A matritsaning determinantiga bo‘lib, quyidagi matritsani hosil qilamiz:

$$B=\begin{vmatrix} \frac{A_{11}}{\det A} & \frac{A_{21}}{\det A} \cdots \frac{A_{n1}}{\det A} \\ \frac{A_{12}}{\det A} & \frac{A_{22}}{\det A} \cdots \frac{A_{n2}}{\det A} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ \frac{A_{1n}}{\det A} & \frac{A_{2n}}{\det A} \cdots \frac{A_{nn}}{\det A} \end{vmatrix}.$$

Hosil bo‘lgan bu B matritsani A matritsaga teskari ekanini, ya’ni $B=A^{-1}$ ekanligini isbot qilamiz. Buning uchun determinantning xossalariga asoslanib, A va B matritsalarning ko‘paytmasini hisoblaymiz:

$$AB=\begin{bmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2}\ldots a_{nn} \end{bmatrix}\cdot\begin{vmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta}\cdots\frac{A_{n1}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta}\cdots\frac{A_{n2}}{\Delta} \\ \cdots\cdots & \cdots\cdots\cdots \\ \frac{A_{1n}}{\Delta} & \frac{A_{2n}}{\Delta}\cdots\frac{A_{nn}}{\Delta} \end{vmatrix}=\begin{bmatrix} 1 & 0\ldots 0 \\ 0 & 1\ldots 0 \\ \cdots & \cdots \\ 0 & 0\ldots 0 \end{bmatrix}.$$

Demak, $B=A^{-1}$. Bundan $\det A^{-1}=\frac{1}{D}A$ ekani kelib chiqadi. ■

Eslatmalar.

1. Berilgan maxsusmas A matritsa uchun uning teskari A^{-1} matritsasi yagonadir.
2. Maxsus kvadrat matritsa uchun teskari matritsa mavjud emas.

Endi ushbu

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

matritsa uchun teskari matritsani topaylik. Buning uchun avval $\det A$ determinantni tuzamiz va uni hisoblaymiz.

$$D=\det A=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 5 \end{vmatrix}=\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{vmatrix}=-1\neq 0.$$

Demak, A maxsusmas matritsa.

Endi qo‘shma matritsani tezamiz. Buning uchun A matritsaning satr elementlarining algebraik to‘ldiruvchilarini topamiz va ularni mos ravishda ustunlarga joylashtiramiz:

$$A_{11}=4\cdot 6-5\cdot 5=-1, \qquad A_{12}=-(2\cdot 6-3\cdot 5)=3,$$
$$A_{21}=-(2\cdot 6-5\cdot 3)=3, \qquad A_{22}=1\cdot 6-3\cdot 3)=-3,$$
$$A_{31}=2\cdot 5-4\cdot 3=-2, \qquad A_{32}=-(1\cdot 5-3\cdot 2)=1,$$

$$A_{13}=2\cdot 5-3\cdot 4=-2$$
$$A_{23}=-(5\cdot 1-3\cdot 2)=1$$
$$A_{33}=1\cdot 4-2\cdot 2=0$$

Shunday qilib,

$$\widetilde{A}^T=\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Nihoyat, A ning barcha elementlarini $\Delta=-1$ ga bo‘lamiz, u holda teskari matritsa ushbu ko‘rinishga ega bo‘ladi:

$$A^{-1}=\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Tekshirish ko‘rsatadiki, $A\cdot A^{-1}=E$. Haqiqatan,

$$AA^{-1}=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}\cdot\begin{bmatrix}1&-3&2\\-3&3&-1\\2&-1&9\end{bmatrix}=$$

$$=\begin{pmatrix}1\cdot1+\ \ 2(-3)+3\cdot2 & 1(-3)+2\cdot3+3(-1) & 1\cdot2+2(-1)+3\cdot0\\ 2\cdot1+4(-3)+5\cdot2 & 2(-3)+4\cdot3+5(-1) & 2\cdot2+4(-1)+5\cdot0\\ 3\cdot1+5(-3)+6\cdot2 & 3(-3)+5\cdot3+6(-1) & 3\cdot2+5(-1)+6\cdot0\end{pmatrix}=$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=E.$$

Shu yo‘l bilan $A^{-1}A=E$ ekanini isbotlash mumkin. ■

Teskari matritsa quyidagi xossalarga ega:

1^0. Teskari matritsaning determinanti berilgan matritsa determinantining teskari qiymatiga teng, ya’ni $\det(A^{-1})=\dfrac{1}{\det A}$.

2^0. Kvadrat matritsalar ko‘paytmasi AB uchun teskari ikkinchi B matritsaga teskari matritsaning birinchi A matritsaga teskari matritsaga ko‘paytmasiga teng, ya’ni

$$(AB)^{-1}=B^{-1}A^{-1}.$$

3^0. Transponirlangan teskari matritsa berilgan transponirlangan matritsaning teskarisiga teng, ya’ni

$$(A^{-1})^T=(A^T)^{-1}.$$

4^0. Teskari matritsaning teskarisi berilgan matritsaning o‘ziga teng, ya’ni

$$(A^{-1})^{-1}=A.$$

2×2 o‘lchamli matritsa uchun teskari matritsa quy’dagicha aniqlanadi. Agar $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ va $ad-bc\neq0$ bo‘lsa, u holda $A^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$ bo‘ladi.

1.11.Matrisa rangi. *Matrisa rangi* deb, uning noldan farqli minorlari tartibining eng kattasiga aytiladi va u $r(A)$ yoki $rang(A)$ kabi belgilanadi.

Agar A matrisaning rangi r ga teng bo‘lsa bu narsa A matrisada hech bo‘lmaganda bitta noldan fvrqli r tartibli minor birligini, biroq r dan katta tartibli har qanday minor nolga tengligini anglatadi.

Matrisa rangini aniqlashda odatda, ko‘p sondagi diterminantlarni hisoblashga to‘g‘ri keladi. Bu ishni osonlashtirish uchun maxsus usullarning bayonidan oldin matrisaning elementar almashtirishlarini keltiramiz:

1.Matrisani biror satri elementlarini noldan farqli songa ko‘paytirish;

2.Matrisaning biror satr elementlariga boshqa satrning mos elementlarini biror songa ko‘paytirib qo‘shish;

3.Matrisa satrlari o‘rnini almashtirish;

4.Matrisada, barcha elementlari nol bo‘lgan satrini tashlab yuborish;

Bir biridan almashtirishlar orqali hosil, qilinadigan matrisalar ekvivalent matrisalar deb ataladi. Ekvivalent matrisalar umuman aytganda bir- biriga teng emas lekin ularning ranglari teng bo‘lishidan foydalanish mumkin.

Matrisa rangi uchun quyidagi xossa o‘rinli:

1^0. $rang(A+B)\le rang(A)+rang(B)$;

2^0. $rang(A\cdot B)\le min\{rang(A),rang(B)\}$;

3^0. $rang(A\cdot B)+rang(BC)-r(B)\le rang(ABC)$.

1.3-§. Chiziqli algebraik tenglamalar sistemasi

Yuqoridagi paragraflarda chiziqli algebraning asosiy tushunchalaridan bo‘lgan matritsalar, determinantlar va ularning asosiy xossalarini o‘rgandik. Endi ular asosida chiziqli algebraik tenglamalar sistemalarini batafsil o‘rganamiz.

1.12. Chiziqli tenglamalar sistemasi. Kroneker-Kapelli teoremasi. n noma’lumli m ta chiziqli tenglamalar sistemasi berilgan bo‘lsin:

$$\begin{cases}a_{11}x_1+a_{12}x_2+...+a_{m}x_n=b_1,\\ a_{21}x_1+a_{22}x_2+...+a_{2n}x_n=b_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n=b_m,\end{cases}\qquad(1.12.1)$$

bunda tenglamalar soni noma’lumlar sonidan kichik, teng yoki undan katta $m<n, m=n, m>n$) bo‘lishi mumkin. (1.12.1) tenglamalar sistemasida

a_{ij} - koeffisiyentlar x_i-noma'lumlar, b_i-ozod hadlar $(i=1,2,...,m, \; j=1,2,...,n)$ deyiladi.

1.12.2-ta'rif. Agar $x_1,x_2,...,x_n$ noma'lumlarga berilgan $\alpha_1,\alpha_2,...,\alpha_n$ qiymatlarni qo'yganda (1.12.1) sistemasining hamma tenglamalarini qanoatlantirsa, $\alpha_1,\alpha_2,...,\alpha_n$ sonlar to'plami (1.12.1) *sistemaning yechimi* deyiladi.

(1.12.1) tenglamalar sistemasi koeffisiyentlaridan tuzilgan ushbu

$$A=\begin{pmatrix} a_{11} & a_{12}\dots a_{1n} \\ a_{21} & a_{22}\dots a_{2n} \\ \dots\dots & \dots\dots \\ a_{m1} & a_{m2}\dots a_{mn} \end{pmatrix}$$

matritsani va A matritsaga unga ozod hadlar ustunini qo'shish bilan hosil qilingan ushbu

$$B=\begin{pmatrix} a_{11} & a_{12}\dots a_{1n} & b_1 \\ a_{21} & a_{22}\dots a_{2n} & b_2 \\ \dots\dots & \dots\dots & \dots \\ a_{m1} & a_{m2}\dots a_{mn} & b_m \end{pmatrix}$$

matritsani qaraymiz. A matritsa (1.12.1) tenglamalar sistemasining *matritsasi yoki asosiy matritsasi*, B esa *kengaytirilgan matritsasi* deyiladi. Bu matritsalarning ranglari $rangA \le rangB$ tengsizlik bilan bog'langanligi ravshan.

Agar chizqli tenglamalar sistemasi yechimga ega bo'lsa, u *birgalikda*, agar yechimga ega bo'lmasa, *u birgalikda emas* deyiladi.

Birgalikda chiziqli tenglamalar sistemasi yagona yechimga ega bo'lsa, u *aniqlangan*, agar cheksiz ko'p yechimlarga ega bo'lsa, u *aniqmas sistema* deb ataladi.

Agar ikkita birgalikdagi tenglamalar sistemasidan birining har bir yechimi ikkinchisining yechimi va aksincha, ikkinchisining har bir yechimi birinchisining yechimi bo'lsa, bu sistemalar *teng kuchli* sistemalar deb ataladi.

Quyidagi almashtirishlar tenglamalar sistemasidan unga teng kuchli sistemaga o'tkazishni isbotlash mumkin:

1. Istalgan ikkita tenglamaning o'rinlarini almashtirish;
2. Tenglamalardan istalgan birining ikkala tomonini noldan farqli istalgan songa ko'paytirish;

3. Sistema tenglamalardan birining ikkala tomoniga boshqa bir tenglamaning istalgan haqiqiy songa ko'paytirilgan mos qismini qo'shish.

Endi (1.12.1) chiziqli tenglamalar sistemasining yechilishi alomatini qaraymiz.

1.12.2-teorema (Kroneker-Kapelli teoremasi). (1.12.1) chiziqli tenglamalar sistemasi birgalikda bo'lishi uchun, asosiy matritsa bilan kengaytirilgan matritsaning ranglari teng, ya'ni $rangA = rangB$ bo'lishi zarur va yetarlidir.

(1.12.1) sistemani tekshirish va yechish quyidagi tartibda amalga oshiriladi.

Tekshirish: (1.12.1) sistema asosiy va kengaytirilgan matritsalarining ranglari topiladi. Bunda:

1. Agar $rang(B) \ne rang(A))$ bo'lsa, (1.12.1) sistema birgalikda bo'lmaydi;
2. Agar $rang(B)=rang(A)=n$, ya'ni (1.12.1) sistemaning rangi uning noma'lumlari soniga teng bo'lsa, (1.12.1) sistema birgalikda va aniq bo'ladi;
3. Agar $rang(B)=rang(A)<n$ bo'lsa, sistema birgalikda va aniqmas bo'ladi.

1.13. Kramer qoidasi. n ta nomalumli n ta chiziqli tenglamalar sistemasi deb, ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \qquad (1.13.1.)$$

ko'rinishdagi chiziqli tenglamalarning chekli to'plamiga aytiladi. Bunda $a_{ij}\left(i,j=\overline{1,n}\right)$ sonlar berilgan sistemaning *koeffitsientlari*, $b_i\left(i=\overline{1,n}\right)$ esa, sistemaning *ozod hadlari* deyiladi. Bu (1.12.1) sestemadagi noma'lumlarning koeffisiyentlaridan n- tartibli ushbu detrminantni tuzamiz:

$$D=\begin{vmatrix} a_{11} & a_{12}\cdots a_{1s}\cdots a_{1n} \\ a_{21} & a_{22}\cdots a_{2s}\cdots a_{2n} \\ \dots\dots & \dots\dots\dots\dots \\ a_{n1} & a_{n2}\cdots a_{ns}\cdots a_{nn} \end{vmatrix}. \qquad (1.13.2)$$

Bu determinant (1.13.1) sistemaning determinanti deyiladi. Bunda ikki hol bo'lishi mumkin: $D \ne 0$ va $D=0$. Biz hozircha $D \ne 0$ bo'lgan holni ko'raylik.

(1.13.1) sistemaning birinchi tenglamasini $A_{1s}\ (s=\overline{1,n})$ algebraik to'ldiruvchiga, ikkinchisini A_{2s}ga,..., n–sini A_{ns} ga ko'paytirib, natijalarni hadlab qo'shamiz:

$$\begin{aligned}&(a_{11}A_{1s}+a_{21}A_{2s}+\cdots+a_{n1}A_{ns})x_1+\\&+(a_{12}A_{1s}+a_{22}A_{2s}+\cdots+a_{n2}A_{ns})x_2+\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\\&+(a_{1n}A_{1s}+a_{2n}A_{2s}+\cdots+a_{nn}A_{ns})x_n=\\&=b_1A_{1s}+b_2A_{2s}+\cdots+b_nA_n.\end{aligned} \qquad (1.13.3)$$

1.9-bandagi determinantni satr yoki ustun elementlari bo'yicha yoyish, **1.9.5**- va **1.9.6**-teoremalarga asosan,

$$a_{1s}A_{1s}+a_{2s}A_{2s}+\ldots+a_{ns}A_{ns}=D \qquad (1.13.4)$$

bo'lib,

$$a_{1k}A_{1s}+a_{2k}A_{2s}+\ldots+a_{nk}A_{ns}=0\ (k\neq s) \qquad (1.13.5)$$

ko'rinishidagi hamma yig'indilar nolga teng.

Demak, (1.13.4) va (1.13.5) ga asosan, (1.13.3) tenglikdan

$$D\cdot x_s=b_1A_{1s}+b_2A_{2s}+\ldots+b_nA_{ns} \qquad (1.13.6)$$

shaklini oladi. (1.13.6) tenglikning o'ng tomonidagi yig'indini (1.13.4) bilan solishtirib, D determinantning s- ustundagi $a_{1s}, a_{2s},..., a_{ns}$ elementlarini, mos ravishda, $b_1, b_2,..., b_n$ ozod hadlar bilan almashtirsa, determinant kelib chiqadi. Shunday qilib, bu determinant quydagi

$$D_s=\begin{vmatrix}a_{11} & a_{12}\cdots a_{1,s-1} & b_1 & a_{1,s+1}\cdots a_{1n}\\ a_{21} & a_{22}\cdots a_{2,s-1} & b_2 & a_{2,s+1}\cdots a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2}\cdots a_{n,s-1} & b_n & a_{n,s+1}\cdots a_{nn}\end{vmatrix}$$

ko'rinishga ega bo'ladi.

Demak, (1.13.6) ni $D\cdot x_s=D_s$shaklda yozib, bundan

$$x_s=\frac{D_s}{D}\qquad (s=1,2,...,n) \qquad (1.13.7)$$

tengliklarga ega bo'lamiz. Bu tengliklar *Kramer formulasi* deyiladi.

1.13.8-teorema. Agar (1.13.1) sistemaning (1.13.2) determinanti noldan farqli bo'lsa, ya'ni $D\neq 0$, u holda bu (1.13.1) sistema yechimga ega va bu yechim yagonadir. Bu yechim (1.13.7) formulalar bo'yicha, ya'ni Kramer qoidasi bo'yicha hosil qilinadi.

Agar (1.13.1) sistemadagi $b_1, b_2,..., b_n$ ozod hadlar nolga teng bo'lsa, ya'ni $b_1=b_2=...=b_n=0$, u holda bunday sistema *bir jinsli sistema* deyiladi.

1.13.9-teorema. n noma'lum n ta bir jinsli tenglamalar sistemasi noldan farqli yechimlarga ega bo'lishi uchun, sistema determinanti nolga teng bo'lishi zarur va yetarlidir.

Bir jinslimas va bir jinsli sistemalarning yechimlari orasidagi bog'lanish

1-tasdiq. Ixtiyoriy bir jinsli bo'lmagan sistemaning istalgan yechimi bilan unga mos bir jinsli sistemaning istagan yechimining yigindisi bir jinsli bo'lmagan sistemaning yechimi bo'ladi.

2-tasdiq. Ixtiyoriy bir jinsli bo'lmagan sistemaning istalgan ikkita yechimlarining ayirmasi unga mos bir jinsli sistemaning yechimi bo'ladi.

3-tasdiq. Ixtiyoriy bir jinsli bo'lmagan sistemaning umumiy yechimi X_u unga mos bir jinsli sistemaning umumiy yechimi X bilan o'zining istalgan bitta X_0 xususiy yechimining yig'indisiga teng bo'ladi, ya'ni $X_u=X+X_0$.

1.14. Gauss usuli. Bizga m nomu'lum n ta chiziqli tenglamalar sistemasi berilgan bo'lsin, ya'ni

$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1m}x_m=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2m}x_m=b_2\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nm}x_m=b_n\end{cases} \qquad (1.14.1)$$

(1.14.1) sistemaning yechimini *noma'lumlarini ketma-ket yo'qotish usuli* yoki *Gauss usuli* bilan topamiz. Noma'lumlarni ketma-ket yo'qotish bilan berilgan sistema uchburchak shaklga kelib qoladi.

Agar (1.14.1) sistemadagi biror tenglamani ikkinchisiga qo'shganda yoki har qanday haqiqiy songa ko'paytirganda, (1.14.1) sistemaga ekvivaliyent tenglamalar sistemasiga ega bo'lamiz.

Faraz qilaylik, (1.14.1) dagi $a_{11} \neq 0$ bo'lsin. (1.14.1) sistemadagi birinchi tengalamani a_{11} ga bo'lamiz. U holda

$$x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3 + \cdots + \frac{a_{1n}}{a_{11}} x_n = \frac{b_1}{a_{11}} \qquad (1.14.2)$$

hosil qilingan (1.14.2) tenglamaga $-a_{21}$, $-a_{31},\dots,-a_{n1}$ sonlarni ketma-ket ko'paytirib sistemaning tenglamalariga qo'shamiz, u holda ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1, \\ c_{12}x_2 + \cdots + c_{2m}x_m = d_2, \\ c_{32}x_2 + \cdots + c_{3m}x_m = d_3, \\ \dots\dots\dots\dots\dots\dots \\ c_n = x_2 + \cdots + c_{nm}x_m = d_n, \end{cases} \qquad (1.14.3)$$

sistemaga ega bo'lamiz. (1.14.3) tenglamalar sistemasi (1.14.1) sistemasiga ekvivaliyent ekanligi bizga ma'lum.

Bunda birinchi tenglamaga mutlaqo tegmaymiz va (1.14.3) sistemaning birinchi tenglamasidan tashqari barcha tenglamalaridan iborat qismini almashtirish kerak deb hisoblaymiz.

Bunda bu tenglamalar ichida chap tomonlarining barcha koeffisiyentlari nolga teng bo'lgan tenglamalar mavjud emas deb hisoblaymiz, albatta, bunday tenglamalarni, agar ularning ozod hadlari ham nolga teng bo'lsa, tashlab yuborgan bular edik, aks holda esa sistemaning birgalikda emasligini isbot qilgan bo'lar edik.

Shunday qilib, $c_{22}, c_{32,\dots,}c_{n2}$ koeffisentlar orasida noldan farqlilari bor; aniqlik uchun $c_{22} \neq 0$ deb qabul qilamiz. (1.14.3) sistemaning ikkinchi tenglamasining hamma hadlarini c_{22} ga bo'lamiz, so'ngra uni mos ravishda $-c_{22}, -c_{32}, \dots - c_{n2}$ sonlarga ko'paytirib uchinchi, turtinchi va boshqa tenglamalarga qo'shamiz, u holda

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m = b_1, \\ c_{22}x_2 + c_{23}x_3 + \cdots + c_{2m}x_m = d_2, \\ p_{33}x_3 + \cdots + p_{3m}x_m = q_3, \\ \dots\dots\dots\dots\dots\dots \\ p_{n3}x_3 + \cdots + p_{nm}x_m = q_n, \end{cases} \qquad (1.14.4)$$

sistemaga ega bo'lamiz.

Agar bu tenglamalardan biri noldan farqli ozod hadga ega bo'lib, chap tomonidagi barcha koeffisiyentlari esa nolga teng bo'lgan sistemaga ega bo'lib qolsa, u holda bu sistema yechimga ega bo'lmaydi.

Agar o'zgaruvchilar soni bilan tenglamalar soni $(m = n)$ teng bo'lib va (1.14.1) sistema birgalikda (yechimga ega) bo'lsin, u holda (1.14.4) sistema quyidagi ko'rinishga ega bo'ladi:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\ c_{22}x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2, \\ p_{33}x_3 + \cdots + p_{3n}x_n = q_3, \\ \dots\dots\dots\dots\dots\dots \\ r_{nn}x_n = \mu_n, \end{cases} \qquad (1.14.5)$$

bunda $a_{11}, c_{22}, p_{33}, \dots, r_{nn}$ koeffisiyentlar hammasi noldan farqli. (1.14.1) sistemaning oxirgi tenglamasidan x_n noma'lum uchun $x_n = \frac{\mu_n}{r_{nn}}$ tayin bir qiymat hosil qilamiz. Bu qiymatni oxiridan ikkinchi tenglamaga qo'yib, x_{n-1} noma'lum uchun bir qiymatli aniqlangan tayin qiymatni topamiz. Shunday davom ettirib, (1.14.1) sistemaning $x_1, x_2, \dots, x_n$ yagona yechimiga ega bo'lamiz.

Agar o'zgaruvchilar soni tenglamalar sonidan ko'p $(n < m)$ bo'lsa, u holda almashtirishlar yordamida (1.14.4) sistema quyidagi trapetsiya ko'rinishga keladi:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m = b_1, \\ c_{22}x_2 + c_{23}x_3 + \cdots + c_{2m}x_m = b_2, \\ \dots\dots\dots\dots\dots\dots \\ p_{nn}x_n + \cdots + p_{nm}x_m = q_n, \end{cases} \qquad (1.14.6)$$

(1.14.6) cistemadagi $x_{n+1}, x_{n+2}, \ldots x_m$ noma'lumlarni o'ng tomonga o'tkazib, quyidagi sistemani hosil qilamiz:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 - a_{1,n+1}x_{n+1} - \cdots - a_{1m}x_m \\ \qquad c_{22}x_2 + \cdots + a_{2n}x_n = d_2 - c_{2,n+1}x_{n+1} - \cdots - a_{2m}x_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad p_{nn}x_n = q_n - p_{n,n+1}x_{n+1} - \cdots - p_{nm}x_m \end{cases}$$

Bunda $x_{n+1}, x_{n+2}, \ldots x_n$ lardan iborat ozod noma'lumlarga ixtiyoriy qiymatlar berib, uchburchakli sistemani hosil qilamiz, so'ngra yuqoridagi uslub bilan ketma – ket $x_n, x_{n-1}, \cdots, x_1$ noma'lumlarni aniqlaymiz.

Agar $x_{n+1}, x_{n+2}, \cdots, x_m$ ga ixtiyoriy qiymatlar berish mumkinligini e'tiborga olsak, bu holda berilgan (1.14.1) sistema cheksiz ko'p yechimga ega bo'ladi.

1.15. Chiziqli tenglamalar sistemasini yechishning matritsa usuli

Ushbu n ta noma'lumli n ta chiziqli tenglamalar sistemasini ko'raylik:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \qquad (1.15.1)$$

Ushbu belgilashlarni kiritamiz:

$$A = \begin{pmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \qquad (1.15.2)$$

U holda (1.15.1) sistemani matritsalarni ko'paytirish qoidasidan foydalanib, ushbu ekvivalent shaklda yozish mumkin:

$$AX = B, \qquad (1.15.3)$$

bu yerda A – noma'lumlar oldidagi koeffisiyentlardan tuzilgan matritsa, B – ozod hadlardan tuzilgan ustun matritsa, X – noma'lumlardan tuzilgan ustun matritsa.

Agar A matritsa xosmas, ya'ni $\det A \neq 0$ bo'lsa, u holda uning uchun A^{-1} teskari matritsa mavjud. (1.15.3) matritsali tenglamaning ikkala qismini A^{-1} ga chapdan ko'paytirib, quyidagini hosil qilamiz:

$$A^{-1}(AX) = A^{-1}B$$

yoki

$$A^{-1}AX = A^{-1}B, \; A^{-1}A = E, EX = X$$

ekanligini hisobga olib,

$$X = A^{-1}B \qquad (1.15.4)$$

ni topamiz. (1.15.4) formula A matritsa xosmas bo'lganda n noma'lumli n ta chiziqli tenglamalar sistemasi yechishning matritsali yozuvini beradi:

$$x_1 = \frac{1}{\Delta}\{A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n\} = \frac{\Delta_1}{\Delta},$$

$$x_2 = \frac{1}{\Delta}\{A_{12}b_1 + A_{22}b_2 + \ldots + A_{n2}b_n\} = \frac{\Delta_2}{\Delta},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n = \frac{1}{\Delta}\{A_{1n}b_1 + A_{2n}b_2 + \ldots + b_n A_{nn}\} = \frac{\Delta_n}{\Delta},$$

bunda $\Delta = \det A$.

1-bob bo‘yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

1.1. Matritsa tushunchasi ([7], 1-q., 68-70 betlar; [14], 195-197 betlar; [4] [6]).

1. 2. Matritsalar ustida chiziqli amallar([1], [6], [10]; [14], 197-199 betlar).

1.3. Matritsalarni ko‘paytirish([1], [6], [10]; [3], 199-200 betlar).

1. 4. Transponirlangan matritsa([1], [6], [10]; [14], 202-204 betlar).

1.5. Elementar almashtirishlar ([1], [6], [10] [14], 195-197 betlar).

1.6. Ikkinchi, uchinchi va yuqori tartibli determinantlarning ta’riflari([7], 1-q., 28-36 betlar; [1], [6], [10]; [14], 15-20 va 28-30 betlar).

1.7. Determinantlarning asosiy xossalari([7], 1-q., 36-46 betlar; [14], 15-20 betlar).

1.8. Minorlar va algebraik to‘ldiruvchilar([7], 1-q., 46-8 betlar; [14], 20-22 betlar).

1.9. Laplas teoremasi([7], 1-q., 48-52 betlar; ([1], [6], [10]).

1.10. Determinantni satr yoki ustun elementlari bo‘yicha yoyish([7], 1-q., 52-57 betlar; [1], [6], [10]).

1.11. Teskari matritsa haqida tushuncha ([1], [6], [10] [14], 204-208 betlar).

1.12.Matrisa rangi([7], 1-q., 70-76 betlar; [14], 212-218 betlar).

1.13. Chiziqli tenglamalar sistemasi. Kroneker-Kapelli teoremasi ([1], [6], [10]; [14], 212-218 betlar).

1.14. Kramer qoidasi([7], 1-q., 83-87 betlar; [14], 209-212 betlar).

1.15.Gauss usuli([7], 1-q., 87-90 betlar; [14], 221-226 betlar).

1.16. Chiziqli tenglamalar sistemasini yechishning matritsa usuli([4], 1-q., 68-70 betlar; [14], 209-211 betlar).

1-amaliy mashg‘ulot
2-, 3-, $n-$ tartibli determinantlar va ularni hisoblash usullari

1-misol. Determinantni hisoblang

$$A=\begin{vmatrix}2 & 4\\ 6 & 8\end{vmatrix}.$$

Yechilishi. Ta’rifga ko‘ra,

$$A=\begin{vmatrix}2 & 4\\ 6 & 8\end{vmatrix}=2\cdot 8-4\cdot 6=16-24=-8\ \blacksquare$$

2-misol. Quyidagi $d=\begin{vmatrix}1 & 0 & 1\\ 2 & 1 & 3\\ 5 & 0 & -1\end{vmatrix}$ determinantni hisoblang.

Yechilishi. *1-usul.* Uchinch tartibli determinantni hisoblash uchun uchburchak qoudasidan foydalanamiz:

$$d=\begin{vmatrix}1 & 0 & 1\\ 2 & 1 & 3\\ 5 & 0 & -1\end{vmatrix}=1\cdot 1\cdot(-1)+0\cdot 3\cdot 5+1\cdot 0\cdot 2-1\cdot 1\cdot 5-3\cdot 0\cdot 1-(-1)\cdot 2\cdot 0=-6.$$

2-usul. Uchinch tartibli determinantning ikkinchi ustun elementlari bitta elementdan tashqari barchasi nol. Qulaylik uchun ikkinchi ustun bo‘yicha yoyishdan foydalanib hisoblaymiz

$$d=0\cdot(-1)^{1+2}\begin{vmatrix}2 & 3\\ 5 & -1\end{vmatrix}+1\cdot(-1)^{2+2}\begin{vmatrix}1 & 1\\ 5 & -1\end{vmatrix}+0\cdot(-1)^{3+2}\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}=\begin{vmatrix}1 & 1\\ 5 & -1\end{vmatrix}=-6.\ \blacksquare$$

3-misol. Determinant xossalaridan satr yoki ustun bo‘yicha yoyishdan foydalanib, quyidagi ayniyatlarni isbotlang:

$$\begin{vmatrix}\cos\frac{\alpha-\beta}{2} & \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2}\\ \cos\frac{\beta-\gamma}{2} & \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2}\\ \cos\frac{\gamma-\alpha}{2} & \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2}\end{vmatrix}=\frac{1}{2}[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\lambda)].$$

Yechilishi. Determinantni birinchi ustun bo‘yicha yoysak, quyidagilar hosil bo‘ladi:

$$\cos\frac{\alpha-\beta}{2}\begin{vmatrix}\sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2}\\ \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2}\end{vmatrix}-\cos\frac{\beta-\gamma}{2}\begin{vmatrix}\sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2}\\ \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2}\end{vmatrix}+$$

$$+\cos\frac{\gamma-\alpha}{2}\begin{vmatrix}\sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2}\\ \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2}\end{vmatrix}=$$

$$=\cos\frac{\alpha-\beta}{2}\left(\sin\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}-\sin\frac{\gamma+\alpha}{2}\cos\frac{\beta+\gamma}{2}\right)-$$

$$-\cos\frac{\beta-\gamma}{2}\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma+\alpha}{2}-\sin\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\right)+$$

$$=\cos\frac{\gamma-\alpha}{2}\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}-\sin\frac{\beta+\gamma}{2}\cos\frac{\alpha+\beta}{2}\right)=$$

$$=\cos\frac{\alpha-\beta}{2}\sin\frac{\beta-\alpha}{2}-\cos\frac{\beta-\gamma}{2}\sin\frac{\beta-\gamma}{2}+\cos\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\gamma}{2}=$$

$$=\frac{1}{2}\left[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\gamma)\right] \blacksquare$$

4-misol. Ushbu $d=\begin{vmatrix}1 & -1 & 2 & 1\\ 3 & 4 & -1 & 0\\ 2 & 1 & 3 & 1\\ -5 & 0 & 1 & -2\end{vmatrix}$ determinantni xossalaridan foydalanib hisoblang.

Yechilishi. Determinantda quyidagi almashtirishlarni bajaramiz :

a) 1- ustun elementlarini 2- ustunning mos elementlariga qo'shamiz;

b) (-2) ga ko'paytirilgan 1-ustun elementlarini 3-ustunning mos elementlariga qo'shamiz;

c) (-1) ga ko'paytirilgan 1-ustun elementlarini 4-ustunning mos elementlariga qo'shamiz.

Natijada determinantning qiymati o'zgarmaydi:

$$\begin{vmatrix}1 & -1 & 2 & 1\\ 3 & 4 & -1 & 0\\ 2 & 1 & 3 & 1\\ -5 & 0 & 1 & -2\end{vmatrix}=\begin{vmatrix}1 & 0 & 0 & 0\\ 3 & 7 & -7 & -3\\ 2 & 3 & -1 & -1\\ -5 & -5 & 11 & 3\end{vmatrix}.$$

Birinchi satrda uchta nol bo'lganligi sababli bu determinantni 1-satr elementlari bo'yicha yoyish qulaydir. Shunday qilib,

$$d=1\cdot(-1)^{1+1}\begin{vmatrix}7 & -7 & -3\\ 3 & -1 & -1\\ -5 & 11 & 3\end{vmatrix}=-21-99-35+15+77+63=0. \blacksquare$$

5-misol. $\Delta=\begin{vmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{vmatrix}$ determinantni hisoblang.

Yechilishi. Har bir keyingi ustundan 1-ustunni ayiramiz.

$$\Delta=\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & -2 & 0 & 0\\ 1 & 0 & -2 & 0\\ 1 & 0 & 0 & -2\end{vmatrix}=1\cdot(-1)^{1+1}\begin{vmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{vmatrix}=-8 \blacksquare.$$

6-misol. Laplas teoremasidan foydalanib determinantni hisoblang:

$$d=\begin{vmatrix}2 & 1 & 4 & 3 & 5\\ 3 & 4 & 0 & 5 & 0\\ 3 & 4 & 5 & 2 & 1\\ 1 & 5 & 2 & 4 & 3\\ 4 & 6 & 0 & 7 & 0\end{vmatrix}.$$

Yechilishi. Ikkinchi va beshinchi satrlardagi ikkinchi tartibli o'nta minorlardan faqat uchtasi noldan farqli. Shu satrlar bo'yicha yoyamiz:

$$d=(-1)^{10}\begin{vmatrix}3 & 4\\ 4 & 6\end{vmatrix}\begin{vmatrix}4 & 3 & 5\\ 5 & 2 & 1\\ 2 & 4 & 3\end{vmatrix}+(-1)^{12}\begin{vmatrix}3 & 5\\ 4 & 7\end{vmatrix}\cdot\begin{vmatrix}1 & 4 & 5\\ 4 & 5 & 1\\ 5 & 2 & 1\end{vmatrix}+(-1)^{18}\begin{vmatrix}4 & 5\\ 6 & 7\end{vmatrix}\begin{vmatrix}2 & 4 & 5\\ 3 & 5 & 1\\ 1 & 2 & 3\end{vmatrix}=$$

$$=2\cdot 49+1\cdot(-100)+2(-1)=-4. \blacksquare$$

Mustaqil yechish uchun misollar

1. Determinantlarni hisoblang:

a) $\begin{vmatrix}3 & 5\\ 5 & 8\end{vmatrix}$; b) $\begin{vmatrix}ab & ac\\ bd & cd\end{vmatrix}$; c) $\begin{vmatrix}\sin\alpha & \sin\beta\\ \cos\alpha & \cos\beta\end{vmatrix}$; d) $\begin{vmatrix}\log_b a & 1\\ 1 & \log_a b\end{vmatrix}$;

$$e)\begin{vmatrix}\cos\alpha+i\sin\alpha & 1\\ 1 & \cos\alpha-i\sin\alpha\end{vmatrix};\quad f)\begin{vmatrix}a+bi & c+di\\ -c+di & a-bi\end{vmatrix}.$$

2. Determinantlarni hisoblang:

$$a)\begin{vmatrix}-1 & 5 & 4\\ 3 & -2 & 0\\ -1 & 3 & 6\end{vmatrix};\quad b)\begin{vmatrix}0 & 2 & 2\\ 2 & 0 & 2\\ 2 & 2 & 0\end{vmatrix};\quad c)\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix};$$

$$d)\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix};\quad e)\begin{vmatrix}0 & a & 0\\ b & c & d\\ 0 & e & 0\end{vmatrix};\quad f)\begin{vmatrix}\sin\alpha & \cos\alpha & 1\\ \sin\beta & \cos\beta & 1\\ \sin\gamma & \cos\gamma & 1\end{vmatrix};\quad g)\begin{vmatrix}1 & 0 & 1+i\\ 0 & 1 & i\\ 1-i & -i & 1\end{vmatrix};$$

3. Determinantlarni yoymasdan turib, quyidagi ayniyatlarni isbotlang:

$$a)\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=(b-a)(c-a)(c-b);$$

$$b)\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix}=(ab+ac+bc)(b-a)(c-a)(c-b);$$

$$c)\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix};\quad d)\begin{vmatrix}1 & a & a^3\\ 1 & b & b^3\\ 1 & c & c^3\end{vmatrix}=(a+b+c)\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}.$$

4. a) $\begin{vmatrix}5x & 1 & 2 & 3\\ x & x & 1 & 2\\ 1 & 2 & x & 3\\ x & 1 & 2 & 2x\end{vmatrix}$ determinantning x^4 va x^3 ni saqlovchi hadlarni toping.

b) $\begin{vmatrix}x & 1 & 2 & 3\\ 1 & x & 3 & 2\\ 3 & 1 & x & 2\\ 5 & 3 & 1 & x\end{vmatrix}$, determinantning x^4, x^3 va x^2 ni saqlovchi hadlarni toping.

5. a) $\begin{vmatrix}1 & 1 & 1 & a\\ 2 & 2 & 1 & b\\ 3 & 2 & 1 & c\\ 1 & 2 & 3 & d\end{vmatrix}$ determinantni 4-ustun elementlari bo‘yicha yoying;

b) $\begin{vmatrix}a & 1 & 1 & 1\\ b & 0 & 1 & 1\\ c & 1 & 0 & 1\\ d & 1 & 1 & 0\end{vmatrix}$ determinantni 1-ustun elementlari bo‘yicha yoying;

c) $\begin{vmatrix}1 & 2 & -1 & -2\\ 2 & -1 & -2 & 1\\ a & b & c & d\\ -2 & -1 & 1 & 2\end{vmatrix}$ determinantni 3-satr elementlari bo‘yicha yoying.

6. Determinantni uni yoymasdan turib hisoblang:

$$\begin{vmatrix}a & b & c & 1\\ b & c & a & 1\\ c & a & b & 1\\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1\end{vmatrix}.$$

7. Determinantning xossalaridan foydalanib (satr va ustun bo‘yicha yoyishni ham hisobga olganda) ayniyatlarni isbotlang:

$$a)\begin{vmatrix}(a+b)^2 & c^2 & c^2\\ a^2 & (b+c)^2 & a^2\\ b^2 & b^2 & (c+a)^2\end{vmatrix}=2abc(a+b+c)^3;$$

$$b)\begin{vmatrix}\frac{1}{a+x} & \frac{1}{a+y} & 1\\ \frac{1}{b+x} & \frac{1}{b+y} & 1\\ \frac{1}{c+x} & \frac{1}{c+y} & 1\end{vmatrix}=\frac{(a-b)(a-c)(b-c)(x-y)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)};$$

8. Determinantlarni hisoblang:

$$a)\begin{vmatrix} -1 & 2 & 7 & 5 \\ 1 & 3 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -5 & 2 & -1 & 3 \end{vmatrix};\ b)\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix};\ c)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

9. Laplas teoremasidan foydalanib, quyidagi determinantlarni hisoblang:

$$\text{a)}\begin{vmatrix} 1 & 0 & 0 & -1 \\ 2 & 3 & 4 & 7 \\ -3 & 4 & 5 & 9 \\ -4 & -5 & 6 & 1 \end{vmatrix};\ \text{b)}\begin{vmatrix} 5 & 62 & -79 & 4 \\ 0 & 2 & 3 & 0 \\ 6 & 183 & 201 & 5 \\ 0 & 3 & 4 & 0 \end{vmatrix};\ \text{c)}\begin{vmatrix} 3 & -1 & 5 & 2 \\ 2 & 0 & 7 & 0 \\ -3 & 1 & 2 & 0 \\ 5 & -4 & 1 & 2 \end{vmatrix}.$$

10. Quyidagi determinantlarni avval almashtirishlar bajarib soddalashtiring, so'ng Laplas teoremasidan foydalanib hisoblang:

$$\text{a)}\begin{vmatrix} 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -8 & 5 & 9 & 5 \\ -11 & 7 & 7 & 4 \end{vmatrix};\ \text{b)}\begin{vmatrix} 9 & 7 & 9 & 7 \\ 8 & 6 & 8 & 6 \\ -9 & -7 & 9 & 7 \\ -8 & -6 & 8 & 6 \end{vmatrix};\ \text{c)}\begin{vmatrix} 6 & 8 & -9 & -12 \\ 4 & 6 & -6 & -9 \\ -3 & -4 & 6 & 8 \\ -2 & -3 & 4 & 6 \end{vmatrix}.$$

Mustaqil yechish uchun misollarning javoblari

1. a) -1; b) 0; s) $sin(\alpha - \beta)$; d) 0; e) 0; f) $a^2 + b^2 + c^2 + d^2$. **2.** a) -50; b) 16; c) 0; в) 3abc $-a^3 - b^3 - c^3$; e) 0; f) $sin(\beta - \delta) + sin(\delta - \alpha) + sin(\alpha - \beta)$; g) -2; $\sqrt{3}$. **4.** a) $10x^4, -2x^2, -3x^3$; b) $x^4, -x^3, -32x^2$;**5.** a) $4a - c - d$; b) $2a + b - c - d$; c) $-5a - 5b - 5c - 5d$. **6.** 0. **8.** a) -252; b) -3; c) -3. **9.** a) 216; b) 1; c) -106.**10.** a) -12; b)16; c) 1.

2-amaliy mashg'ulot.

Matritsalar va ular ustida amallar. Teskari matritsa, matritsa rangi.

1-misol.

1) $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ va $B=\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ berilgan bo'lsa, u holda

$$A+B=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}+\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}=\begin{pmatrix} 1+2 & 3-2 \\ 4+3 & 5+4 \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ 7 & 9 \end{pmatrix}$$ bo'ladi. ■

2) $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ va $\alpha = 2$ berilgan bo'lsa, u holda

$$A\alpha = \alpha A = 2\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}=\begin{pmatrix} 2\cdot 2 & 3\cdot 2 \\ 4\cdot 2 & 5\cdot 2 \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix}.■$$

2-misol. Quyidagi

$$A=\begin{bmatrix} 3 & 2 & 8 & 1 \\ 1 & -4 & 0 & 3 \end{bmatrix} \text{ va } B=\begin{bmatrix} 2 & -1 \\ 1 & -3 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$$

to'g'ri burchakli matritsalar ko'paytmasini toping.

Yechilishi. Ta'rifga ko'ra,

$$C = AB = \begin{bmatrix} 3\cdot 2+2\cdot 1+8\cdot 0+1\cdot 3 & 3\cdot(-1)+2\cdot(-3)+8\cdot 1+1\cdot 1 \\ 1\cdot 2+(-4)\cdot(-3)+0\cdot 0+3\cdot 3 & 1\cdot(-1)+(-4)\cdot(-3)+0\cdot 1+3\cdot 2 \end{bmatrix}=\begin{bmatrix} 11 & 0 \\ 7 & 14 \end{bmatrix}.■$$

3-misol. Ushbu $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ matritsa uchun teskari matritsani toping.

Yechilishi. Buning uchun avval $\det A$ determinantni tuzamiz va uni hisoblaymiz.

$$D = \det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{vmatrix} = -1 \neq 0.$$

Demak, A maxsusmas matritsa. Endi qo'shma matritsani tuzamiz. Buning uchun A matritsaning satr elementlarining algebraik to'ldiruvchilarini topamiz va ularni mos ravishda ustunlarga joylashtiramiz:

$$A_{11} = 4\cdot 6 - 5\cdot 5 = -1,\quad A_{12} = -(2\cdot 6 - 3\cdot 5) = 3,\quad A_{13} = 2\cdot 5 - 3\cdot 4 = -2,$$
$$A_{21} = -(2\cdot 6 - 5\cdot 3) = 3\quad A_{22} = 1\cdot 6 - 3\cdot 3) = -3,\quad A_{23} = -(5\cdot 1 - 3\cdot 2) = 1,$$
$$A_{31} = 2\cdot 5 - 4\cdot 3 = -2,\quad A_{32} = -(1\cdot 5 - 3\cdot 2) = 1,\quad A_{33} = 1\cdot 4 - 2\cdot 2 = 0\ .$$

Shunday qilib,

$$\widetilde{A}^T=\begin{bmatrix}-1 & 3 & -2\\ 3 & -3 & 1\\ -2 & 1 & 0\end{bmatrix}.$$

Nihoyat, $\widetilde{A}^T$ ning barcha elementlarini $\Delta = -1$ ga bo‘lamiz, u holda teskari matritsa ushbu ko‘rinishga ega bo‘ladi:

$$A^{-1}=\begin{bmatrix}1 & -3 & 2\\ -3 & 3 & -1\\ 2 & -1 & 0\end{bmatrix}.$$

Tekshirish ko‘rsatadiki, $A\cdot A^{-1}=E$. Haqiqatan,

$$AA^{-1}=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}\cdot\begin{bmatrix}1 & -3 & 2\\ -3 & 3 & -1\\ 2 & -1 & 9\end{bmatrix}=$$

$$=\begin{pmatrix}1\cdot 1+.\ 2(-3)+3\cdot 2 & 1(-3)+2\cdot 3+3(-1) & 1\cdot 2+2(-1)+3\cdot 0\\ 2\cdot 1+4(-3)+5\cdot 2 & 2(-3)+4\cdot 3+5(-1) & 2\cdot 2+4(-1)+5\cdot 0\\ 3\cdot 1+5(-3)+6\cdot 2 & 3(-3)+5\cdot 3+6(-1) & 3\cdot 2+5(-1)+6\cdot 0\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.\blacksquare$$

4-misol. Satrlarning elementar almashtirishlari yordamida teskari matritsa A^{-1} ni toping

$$A=\begin{pmatrix}3 & -4 & 5\\ 2 & -3 & 1\\ 3 & -5 & -1\end{pmatrix}.$$

Yechilishi. Quyidagilarni hosil qilamiz:

$$\left(\begin{array}{ccc|ccc}3 & -4 & 5 & 1 & 0 & 0\\ 2 & -3 & 1 & 0 & 1 & 0\\ 3 & -5 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow{R_1-R_2}\left(\begin{array}{ccc|ccc}1 & -1 & 4 & 1 & -1 & 0\\ 2 & -3 & 1 & 0 & 1 & 0\\ 3 & -5 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow{R_2-2R_1}$$

$$\xrightarrow{R_2-2R_1}\left(\begin{array}{ccc|ccc}1 & -1 & 4 & 1 & -1 & 0\\ 0 & -1 & -7 & -2 & 3 & 0\\ 3 & -5 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow{R_3-3R_1}\left(\begin{array}{ccc|ccc}1 & -1 & 4 & 1 & -1 & 0\\ 0 & -1 & -7 & -2 & 3 & 0\\ 0 & -2 & -13 & -3 & 3 & 1\end{array}\right)\xrightarrow{R_3-2R_2}$$

$$\xrightarrow{R_3-2R_2}\left(\begin{array}{ccc|ccc}1 & -1 & 4 & 1 & -1 & 0\\ 0 & -1 & -7 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & -3 & 1\end{array}\right)\xrightarrow{R_1-4R_3}\left(\begin{array}{ccc|ccc}1 & -1 & 0 & 3 & 11 & -4\\ 0 & -1 & -7 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & -3 & 1\end{array}\right)\xrightarrow{R_2+7R_3}$$

$$\xrightarrow{R_2+7R_3}\left(\begin{array}{ccc|ccc}1 & -1 & 0 & -3 & 11 & -4\\ 0 & -1 & 0 & 5 & -18 & 7\\ 0 & 0 & 1 & 1 & -3 & 1\end{array}\right)\xrightarrow{R_1-R_2}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -8 & 29 & -11\\ 0 & -1 & 0 & 5 & -18 & 7\\ 0 & 0 & 1 & 1 & -3 & 1\end{array}\right)\xrightarrow{(-1)R_2}$$

$$\xrightarrow{(-1)R_2}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -8 & 29 & -11\\ 0 & 1 & 0 & -5 & 18 & -7\\ 0 & 0 & 1 & 1 & -3 & 1\end{array}\right).$$

Bunda R_i matritsaning i-satri.
Shunday qilib, teskari matritsa quyidagi ko‘rinishga ega bo‘ladi:

$$A^{-1}=\begin{pmatrix}-8 & 29 & -11\\ -5 & 18 & -7\\ 1 & -3 & 1\end{pmatrix}.\blacksquare$$

Mustaqil yechish uchun misollar

1. Matritsalarning chiziqli kombinatsiyasi topilsin:

a) $3\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}-\begin{pmatrix}3 & 2\\ 3 & 2\end{pmatrix}-4\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$; b) $2\begin{pmatrix}2\\ 2\\ 1\end{pmatrix}-3\begin{pmatrix}0\\ 5\\ 6\end{pmatrix}$;

c) $2\begin{pmatrix}1 & 8 & 7 & -15\\ 1 & -5 & -6 & 11\end{pmatrix}-\begin{pmatrix}5 & 24 & -7 & -1\\ -1 & 2 & 7 & 3\end{pmatrix}$;

d) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$; e) $\begin{pmatrix}1 & 5\\ 5 & 1\end{pmatrix}-\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}$;

f) $2\begin{pmatrix}0 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}1 & 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 1\end{pmatrix}$.

2. Qanday shartlarda quyidagi ayniyatlar o‘rinli bo‘ladi:

a) $A+B=B+A$; b) $A+(B+C)=(A+B)+C$;

c) $\alpha(\beta A)=(\alpha\beta)A$; d) $\alpha(A+B)=\alpha A+\alpha B$;

e) $(\alpha+\beta)A=\alpha A+\beta A$.

3. Matritsalarning ko‘paytmasini hisoblang:

a) $(2\ \ -3\ \ 0)\begin{pmatrix}4\\3\\1\end{pmatrix}$; b) $\begin{pmatrix}4\\3\\1\end{pmatrix}(2\ \ -3\ \ 0)$; c) $\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}3&5\\5&9\end{pmatrix}$;

d) $(1\ \ 0)\begin{pmatrix}1&1\\1&1\end{pmatrix}$; g) $\begin{pmatrix}3&1\\2&1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}$; h) $(1\ \ 1\ \ 1)\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}$;

e) $(0\ \ 1\ \ 0\ \ 0)\begin{pmatrix}1&4&3\\0&3&2\\0&1&0\\0&2&1\end{pmatrix}$; f) $\begin{pmatrix}3&3&-4&-3\\0&6&1&1\\5&4&2&1\\2&3&3&2\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; i) $\begin{pmatrix}1&1&1&1\\1&0&1&1\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$.

4. Ko‘paytmaning mavjudligini tekshiring va mavjud bo‘lganda hisoblang:

a) $\begin{pmatrix}1&2\\3&4\end{pmatrix}(1\ \ 2)\begin{pmatrix}2\\1\end{pmatrix}$; b) $\begin{pmatrix}2\\4\end{pmatrix}(1\ \ 2)\begin{pmatrix}2\\1\end{pmatrix}$;

c) $(1\ \ 2)\begin{pmatrix}2\\1\end{pmatrix}(2\ \ 4)$; d) $(-12\ \ 13)\begin{pmatrix}13547&13647\\28423&28523\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}(-12\ \ 13)$.

5. Hisoblang:

a) $\begin{pmatrix}1&1\\1&1\end{pmatrix}^n$; b) $\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}^n$; c) $\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}^3$;

d) $\begin{pmatrix}1&-1\\1&-1\end{pmatrix}^n$; e) $\begin{pmatrix}1&1\\0&1\end{pmatrix}^n$; f) $\begin{pmatrix}1&0\\1&0\end{pmatrix}^n$.

6. Ayniyatlarning to‘g‘riligini tekshiring:

a) $(\alpha A)^T=\alpha A^T$; b) $(AB)^T=B^TA^T$;

c) $(ABC)^T=C^TB^TA^T$; d) $(A+B)^T=A^T+B^T$.

7. $f(A)$ ni hisoblang, agar

a) $f(x)=x^2-2x+1,\ A=\begin{pmatrix}1&0\\1&1\end{pmatrix}$; b) $f(x)=x^2-2x+1,\ A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$;

c) $f(x)=x^2-3x+2,\ A=\begin{pmatrix}0&-2\\1&3\end{pmatrix}$; d) $f(x)=(x-\varepsilon)^2,\ A=\begin{pmatrix}\varepsilon&1\\-1&\varepsilon\end{pmatrix}$.

8. A matritsa bilan o‘rin almashinuvchi bo‘lgan hamma matritsalar topilsin, agar:

a) $A=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&5\end{pmatrix}$; b) $A=\begin{pmatrix}2&-1\\3&-1\end{pmatrix}$; c) $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

9. $\begin{pmatrix}17&-6\\35&-12\end{pmatrix}=\begin{pmatrix}2&3\\5&7\end{pmatrix}\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}-7&3\\5&-2\end{pmatrix}$ tenglikdan foydalanib, $\begin{pmatrix}17&-6\\35&-12\end{pmatrix}^5$ ni hisoblang.

10. $\begin{pmatrix}4&3&-3\\2&3&-2\\4&4&-3\end{pmatrix}=\begin{pmatrix}1&3&1\\2&2&1\\3&4&2\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&2&-1\\1&1&-1\\-2&-5&4\end{pmatrix}$ tenglikdan foydalanib, $\begin{pmatrix}4&3&-3\\2&3&-2\\4&4&-3\end{pmatrix}^6$ ni hisoblang.

11. $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ matritsa $x^2-(a+d)x+ad-bc=0$ tenglamasini qanoatlantirishini isbotlang.

12. Teskari matritsani topish formulasidan foydalanib quyidagi matritsalar uchun teskari matritsani toping:

a) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$; b) $\begin{pmatrix}3 & 4\\ 5 & 7\end{pmatrix}$; c) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$; d) $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$;

e) $\begin{pmatrix}2 & 7 & 3\\ 3 & 9 & 4\\ 1 & 5 & 3\end{pmatrix}$; f) $\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$; g) $\begin{pmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{pmatrix}$.

13. Matritsaviy tenglamalar sistemasini yeching:

a) $\begin{cases} X+Y=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \\ 2X+3Y=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{cases}$; b) $\begin{cases} 2X-Y=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\ -4X-2Y=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}\end{cases}$.

14. Elementar almashtirishlar yordamida berilgan matritsa uchun teskari matritsani toping:

a) $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}$; b) $\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}$; c) $\begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$.

15. Quyidagi tenglamalardan X matritsani toping:

a) $\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}X=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$; b) $X\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$.

16. Hoshiyalovchi minor usulidan foydalanib matritsa rangini toping:

a) $\begin{pmatrix}2 & -4 & 3 & -3 & 5\\ 1 & -2 & 1 & 5 & 3\\ 1 & -2 & 4 & -34 & 0\end{pmatrix}$; b) $\begin{pmatrix}2 & -1 & 3 & -2 & 4\\ 4 & -2 & 5 & 1 & 7\\ 2 & -1 & 1 & 8 & 2\end{pmatrix}$;

c) $\begin{pmatrix}2 & 3 & 1 & -1\\ 3 & 1 & 4 & 2\\ 1 & 2 & 3 & -1\\ 1 & -4 & -7 & 5\end{pmatrix}$; d) $\begin{pmatrix}1 & 2 & 3 & -2\\ 2 & -3 & 1 & -4\\ 1 & 9 & 8 & -2\\ 1 & -12 & -7 & -2\end{pmatrix}$.

17. Elementar almashtirishlar yordami bilan quyidagi matritsalarning rangini toping:

a) $\begin{pmatrix}17 & 51 & 27 & 31\\ 93 & 25 & 14 & 121\\ 94 & 27 & 15 & 120\\ 18 & 53 & 28 & 30\end{pmatrix}$; b) $\begin{pmatrix}25 & 31 & 17 & 43\\ 75 & 94 & 53 & 132\\ 75 & 94 & 54 & 134\\ 25 & 32 & 20 & 48\end{pmatrix}$.

Mustaqil yechish uchun misollarning javoblari

1. a) 0; b) $\begin{pmatrix}4\\ -11\\ -16\end{pmatrix}$; c) $\begin{pmatrix}-3 & -8 & 21 & -29\\ 3 & -8 & -19 & 19\end{pmatrix}$;

d) $\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}$; e) $\begin{pmatrix}0 & 4\\ 4 & 0\end{pmatrix}$; f) $\begin{pmatrix}-1 & 0 & 1 & 1 & 1\\ 0 & 1 & -1 & -1 & 1\end{pmatrix}$.

2. a), b), d) lar o‘rinlidir, agar matritsalar bir xil o‘lchovli bo‘lsa; c), e) lar doimo o‘rinli.

3. a) (-1); b) $\begin{pmatrix}8 & -12 & 0\\ 6 & -9 & 0\\ 2 & 3 & 0\end{pmatrix}$; c) $\begin{pmatrix}8 & 14\\ 8 & 14\end{pmatrix}$;

d) (1 1); e) $\begin{pmatrix}4 & 4\\ 3 & 3\end{pmatrix}$; f) (6, 9, 12);

g) $\begin{pmatrix}0 & 3 & 2\end{pmatrix}$; h) $\begin{pmatrix}3\\ 6\\ 4\\ 3\end{pmatrix}$; i) $\begin{pmatrix}4\\ 3\\ 1\\ 2\end{pmatrix}$.

4. a) mavjud emas; b) $\begin{pmatrix} 8 \\ 16 \end{pmatrix}$; c) $\begin{pmatrix} 8 & 16 \end{pmatrix}$; d) $\begin{pmatrix} -1200 & 1300 \end{pmatrix}$.

5. a) $2^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; c) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

d) 0 agar $n > 1$ bo‘lsa; e) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

6. b), c), d) lar o‘rinli, agar ularda foydalanilgan amallar bajarilsa; a) doimo o‘rinlidir.

7. a) 0; b) 0; c) 0; d) $-E$.

8. a) $\left\{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in C\right\}$; b) $\left\{\begin{pmatrix} \alpha & \beta \\ -3\beta & \alpha + 3\beta \end{pmatrix} \mid \alpha, \beta \in C\right\}$;

c) $\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in C\right\}$;

9. $\begin{pmatrix} 3197 & -1266 \\ 7385 & -922 \end{pmatrix}$.

10. $\begin{pmatrix} 190 & 189 & -189 \\ 126 & 127 & -126 \\ 252 & 252 & -251 \end{pmatrix}$.

12. a) $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$; b) $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$;

c) $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; d) $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$;

e) $\begin{pmatrix} -\frac{7}{3} & 2 & -\frac{1}{3} \\ \frac{5}{3} & -1 & -\frac{1}{3} \\ -2 & 1 & 1 \end{pmatrix}$; f) $\frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$; g) $\frac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$;

13. a) $X = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, Y = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$; b) $Y = 2X + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, bunda X – ikkinchi tartibli ixtiyoriy matritsa.

14. a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; c) $\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$;

15. a) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$;.

16. a) $r = 2$; b) $r = 2$; c) r=4; d) r=2.

17. a) r=3; b) r=3;

3-amaliy mashg‘ulot.
Chiziqli algebraik tenglamalar sistemani yechishda Kramer, matritsa va Gauss usullari

1-misol. Ushbu

$$\begin{cases} 2x_1 + x_2 - 5x_3 + x_4 = 8, \\ x_1 - 3x_2 \qquad - 6x_4 = 9, \\ \qquad 2x_2 - x_3 + 2x_4 = -5, \\ x_1 + 4x_2 - 7x_3 + 6x_4 = 0 \end{cases}$$

sistemani yeching.

Yechilishi. Bu sistemaning D determinantini tuzamiz.

$$D=\begin{vmatrix}2 & 1 & -5 & 1\\ 1 & -3 & 0 & -6\\ 0 & 2 & -1 & 2\\ 1 & 1 & -7 & 6\end{vmatrix}=27\neq 0.$$

$D\neq 0$ bo'lgani uchun sistemaga Kramer qoidasini qo'llash mumkin. D_1, D_2, D_3, D_4 larni tuzamiz.

$$D_1=\begin{vmatrix}8 & 1 & 5 & 1\\ 9 & 3 & 0 & 6\\ 5 & 2 & 1 & 2\\ 0 & 1 & 7 & 6\end{vmatrix}=81,\qquad D_2=\begin{vmatrix}2 & 8 & 5 & 1\\ 1 & 9 & 0 & 6\\ 0 & 5 & 1 & 2\\ 1 & 0 & 7 & 6\end{vmatrix}=108,$$

$$D_3=\begin{vmatrix}2 & 1 & 8 & 1\\ 1 & -3 & 9 & -6\\ 0 & 2 & -5 & 2\\ 1 & 1 & 0 & 6\end{vmatrix}=-27,\qquad D_4=\begin{vmatrix}2 & 1 & -5 & 8\\ 1 & -3 & 0 & 9\\ 0 & 2 & -1 & -5\\ 1 & 0 & -7 & 6\end{vmatrix}=27.$$

(1.13.7) formulaga asosan, berilgan sistemaning yechimi $x_1=3$, $x_2=-4$, $x_3=-1$, $x_4=1$ bo'ladi.■

2-misol. Ushbu

$$\begin{cases}2x_1+7x_2+13x_3=0,\\ 3x_1+14x_2+12x_3=18,\\ 5x_1+25x_2+16x_3=39\end{cases}$$

sistemani Gauss usulidan foydalanib yeching.

Yechilishi. Bunda $a_{11}=2\neq 0$. Shuning uchun birinchi tenglamaning hamma hadlarini 2 ga bo'lamiz. Natijada berilgan tenglamaga ekvivalent bo'lgan ushbu tenglamalar sistemasi hosil bo'ladi:

$$\begin{cases}x_1+\frac{7}{2}x_2+\frac{13}{2}x_3=0,\\ 3x_1+14x_2+12x_3=18,\\ 5x_1+25x_2+16x_3=39\end{cases}$$

hosil bo'lgan sistemaning birinchi tenglamasini –3 ga ko'paytirib, ikkinchi tenglamaga, so'ngra –5 ga ko'paytirib uchinchi tenglamaga qo'shamiz. Natijada berilgan sistemaga ekvivalent bo'lgan quyidagi sistemani hosil qilamiz:

$$\begin{cases}x_1+\frac{7}{2}x_2+\frac{13}{2}x_3=0,\\ \frac{7}{2}x_2-\frac{15}{2}x_3=18,\\ \frac{15}{2}x_2-\frac{33}{2}x_3=39\end{cases}$$

shu bilan birinchi qadam tugadi.

Ikkinchi qadamda $a_{22}=\frac{7}{2}\neq 0$ ekanligidan foydalanib, ikkinchi tenglamaning hamma hadlarini $\frac{7}{2}$ ga bo'lib, hosil bo'lgan tenglamani $-\frac{15}{2}$ ga ko'paytirib, uchinchi tenglamaga qo'shamiz:

$$\begin{cases}x_1+\frac{7}{2}x_2+\frac{13}{2}x_3=0,\\ x_2-\frac{15}{7}x_3=\frac{36}{7},\\ -\frac{3}{7}x_3=\frac{3}{7}.\end{cases}$$

Bunda uchburchak sistema hosil bo'ldi. Uchinchi tenglamadan $x_3=-1$. ikkinchi tenglamadan $x_2=\frac{15}{7}(-1)+\frac{36}{7}=\frac{36}{7}-\frac{15}{7}=3$. Birinchi tenglamadan esa, $x_1=\frac{7}{2}\cdot 3-\frac{13}{2}\cdot(-1)=-4$.

Demak, berilgan sistemaning yagona yechimi $x_1=-4,\ \ x_2=3,\ \ x_3=-1$ bo'ladi. ■

3-misol. Ushbu

$$\begin{cases}x_1-2x_2+x_3=5\\ 2x_1-x_3=0\\ -2x_1+x_2+x_3=-1\end{cases}$$

sistemani yeching.

Yechilishi. Bu misolda (1.15.2) belgilashga asosan,

$$A=\begin{pmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{pmatrix},\ B=\begin{pmatrix}5\\0\\-1\end{pmatrix},\ X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

ko'rinishda bo'ladi. U holda berilgan tenglamalar sistemasini (1.15.3) ko'rinishda yozish mumkin. Endi $\det A$ ni topamiz:

$$\Delta=\det A=\begin{vmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{vmatrix}=3\neq 0.$$

A matritsa maxsusmas ekan. Bu A matritsa uchun teskari matritsa mavjud. Teskari matritsani topamiz

$$A^{-1}=\begin{pmatrix}\frac{1}{3}&1&\frac{2}{3}\\0&1&1\\\frac{2}{3}&1&\frac{4}{3}\end{pmatrix}.$$

(1.15.4) formulaga asosan, berilgan sistemani ham $X=A^{-1}B$ ko'rinishda yozish mumkin. Endi x_1,x_2,x_3 larni topamiz:

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{1}{3}&1&\frac{2}{3}\\0&1&1\\\frac{2}{3}&1&\frac{4}{3}\end{pmatrix}\begin{pmatrix}5\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}.$$

Bundan $x_1=1,\ \ x_2=-1,\ \ x_3=2$ bo'ladi. ■

4-misol. Tenglamalar sistemasini yeching:

$$\left.\begin{array}{r}2x_1+3x_2+5x_3+x_4=3,\\3x_1+4x_2+2x_3+3x_4=-2,\\x_1+2x_2+8x_3-x_4=8,\\7x_1+9x_2+x_3+8x_4=0.\end{array}\right\}$$

Yechilishi. Matritsaning shaklini almashtiramiz:

$$\left(\begin{array}{cccc|c}2&3&5&1&3\\3&4&2&3&-2\\1&2&8&-1&8\\7&9&1&8&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&2&8&-1&8\\0&-2&-22&6&-26\\0&-1&-11&3&-13\\0&-5&-55&15&-56\end{array}\right)\sim$$

$$\left(\begin{array}{cccc|c}1&2&8&-1&8\\0&1&11&-3&13\\0&0&0&0&0\\0&0&0&0&9\end{array}\right)\sim\left(\begin{array}{cccc|c}1&2&8&-1&8\\0&1&11&-3&13\\0&0&0&0&1\\0&0&0&0&0\end{array}\right).$$

Dastlab birinchi va uchinchi satrlarning o'rinlarini almashtiramiz, hosil bo'lgan birinchi satrni mos ravishda 3, 2, 7 larga ko'paytirib, qolgan satrlardan ayiramiz. Ikkinchi satrni (-2) ga bo'lib, mos ravishda 1 va 5 larga ko'paytirib qolgan satrlarga qo'shamiz, uchinchi qadamda esa, to'rtinchi satrni 9 ga bo'lib uchinchi satr bilan o'rinlarini almashtiramiz.

Sistemada 0=1 ko'rinishdagi tenglama hosil bo'ladi. Demak, berilgan sistema yechimga ega emas. ■

5- misol. Ushbu

$$\begin{cases}x_1+x_2-3x_3=-1,\\2x_1+x_2-2x_3=1,\\x_1+x_2+x_3=3,\\x_1+2x_2-3x_3=1\end{cases}$$

sistema birgalikdami?

Yechilishi. Asosiy va kengaytirilgan matritsalarni tuzamiz.

$$A=\begin{pmatrix}1&1&-3\\2&1&-2\\1&1&1\\1&2&-3\end{pmatrix}\qquad B=\begin{pmatrix}1&1&-3&1\\2&1&-2&1\\1&1&1&3\\1&2&-3&1\end{pmatrix}.$$

Asosiy matritsaning rangini topish uchun, uni elementar almashtirishlar yerdamida topamiz:

$$A\approx\begin{pmatrix}1&1&-3\\0&-1&4\\0&0&4\\0&1&0\end{pmatrix}\approx\begin{pmatrix}1&1&-3\\0&1&0\\0&-1&4\\0&0&4\end{pmatrix}\approx\begin{pmatrix}1&1&-3\\0&1&0\\0&0&4\\0&0&4\end{pmatrix}\approx\begin{pmatrix}1&1&-3\\0&1&0\\0&0&4\\0&0&0\end{pmatrix}\approx\begin{pmatrix}1&1&-3\\0&1&0\\0&0&4\end{pmatrix}=A_1$$

hosil bo'lgan ekvivalient A_1 matritsaning rangi $rangA_1=3$, chunki

$$\begin{vmatrix}1&1&-3\\0&1&0\\0&0&4\end{vmatrix}=4\neq 0.$$

Demak, A matritsaning rangi ham 3 ga teng: $rangA = 3$.

Endi kengaytirilgan matritsaning rangini topish uchun elementar almashtirishlar bajaramiz:

$$B\approx\begin{pmatrix}1&1&-3&1\\0&-1&4&3\\0&0&4&4\\0&1&0&2\end{pmatrix}\approx\begin{pmatrix}1&1&-3&1\\0&-1&4&3\\0&0&1&1\\0&0&4&4\end{pmatrix}\approx\begin{pmatrix}1&1&-3&-1\\0&-1&4&3\\0&0&1&1\\0&0&0&1\end{pmatrix}=B_1.$$

Ekvivalient B_1 matritsaning rangi $rangB_1 = 4$ teng, chunki

$$\begin{vmatrix}1&1&-3&-1\\0&-1&4&3\\0&0&0&1\end{vmatrix}=\begin{vmatrix}-1&4&3\\0&1&1\\0&0&1\end{vmatrix}=-1\neq 0.$$

B matritsaning rangi ham 4 ga teng, $rangB = 4$. Matritsaning ranglari har xil. Demak, berilgan sistema birgalikda emas. ■

6- misol. Ushbu

$$\begin{cases}x_1-2x_2+3x_3+x_4=1\\x_1-2x_2+3x_3-x_4=-1\\x_1-2x_2+x_3+5x_4=5\end{cases}$$

sistema birgalikdami?

Yechilishi. A matritsaning rangini hisoblaymiz.

$$A=\begin{pmatrix}1&-2&1&1\\1&-2&1&-1\\1&-2&1&5\end{pmatrix}\approx\begin{pmatrix}1&-2&1&1\\0&0&0&-2\\0&0&0&4\end{pmatrix}\approx\begin{pmatrix}1&-2&1&1\\0&0&0&-2\\0&0&0&0\end{pmatrix}\approx\begin{pmatrix}1&-2&1&1\\0&0&0&1\end{pmatrix}$$

$rangA = 2$ chunki $\Delta=\begin{vmatrix}1&1\\0&1\end{vmatrix}=1\neq 0$. Endi kengaytirilgan B matritsaning rangini hisoblaymiz

$$B=\begin{pmatrix}1&-2&1&1&1\\1&-2&1&-1&-1\\1&-2&1&5&5\end{pmatrix}\approx\begin{pmatrix}1&-2&1&1&1\\0&0&0&-2&-2\\0&0&0&4&4\end{pmatrix}\approx$$

$$\approx\begin{pmatrix}1&-2&1&1&1\\0&0&0&-2&-2\\0&0&0&0&0\end{pmatrix}\approx\begin{pmatrix}1&-2&1&1&1\\0&0&0&1&1\end{pmatrix}$$

$rangB = 2$, chunki $\Delta=\begin{vmatrix}1&1\\0&1\end{vmatrix}=1\neq 0$. Sistema birgalikda, chunki $rangA =$ $= rangB = 2$. Rang noma'lumlar sonidan kichik bo'lgani uchun sistema cheksiz ko'p yechimlarga ega. Ushbu

$$\begin{cases}x_3+x_4=1-x_1-2x_2,\\x_3-x_4=-1-x_1-2x_2\end{cases}$$

sistemani yechamiz. x_1 va x_2 ozod noma'lumlarga ixtiyoriy qiymatlarni, masalan, $x_1=1, x_2=0$ qiymatlarni beramiz. Sistema ushbu ko'rinishni oladi

$$\begin{cases}x_3+x_4=0,\\x_3-x_4=-2.\end{cases}$$

Uni yechib, $x_3=-1, x_4=1$ ni topamiz.

Demak, berilgan sistemaning cheksiz ko'p yechimlaridan biri $x_1=1, x_2=0,$ $x_3=-1, x_4=1$ aniqlanadi. ■

Mustaqil yechish uchun misollar

1. Tenglamalar sistemasini Kramer qoidasi va matritsa usullarida yeching:

a) $\begin{cases} x+4y=-10, \\ 3x-y=9; \end{cases}$ b) $\begin{cases} 2x-5y=1, \\ ax-5y=-2a-5; \end{cases}$

c) $\begin{cases} x+2y-z=12, \\ 3x-y+4z=-13, \\ -x+5y-z=27; \end{cases}$ d) $\begin{cases} 2x+4y+3z=14, \\ 3x-y+4z=-13, \\ -x+5y-z=27; \end{cases}$

e) $\begin{cases} 2x_1+2x_2-x_3+x_4=4, \\ 4x_1+3x_2-x_3+2x_4=6, \\ 8x_1+5x_2-3x_3+4x_4=12, \\ 3x_1+3x_2-2x_3+2x_4=6; \end{cases}$ f) $\begin{cases} 2x_1+3x_2+11x_3+5x_4=2, \\ x_1+x_2+5x_3+2x_4=1, \\ 2x_1+x_2+3x_3+2x_4=-3, \\ x_1+x_2+3x_3+4x_4=-3 \ . \end{cases}$

2. Tenglamalar sestemasini Gauss usulida yeching:

a) $\begin{cases} x-2y=3, \\ 2x-4y=6; \end{cases}$ b) $\begin{cases} x-2y=3, \\ 2x-4y=5; \end{cases}$

c) $\begin{cases} x-2y=1-i, \\ 2x-4y=2-2i; \end{cases}$ d) $\begin{cases} x-2y=1-i, \\ 2x-4y=2+i; \end{cases}$

e) $\begin{cases} x+2y-z=5, \\ 3x-y+z=-4; \end{cases}$ f) $\begin{cases} 2z_1-(1+i)z_2-3iz_3=b_1, \\ -2z_1+(1+i)z_2+(1+3i)z_3=b_2, \\ z_3=b_3; \end{cases}$

g) $\begin{cases} 2x+3y-z=0, \\ x-2y+4z=9, \\ y+z=2; \end{cases}$ h) $\begin{cases} 2x+3y-z=-1, \\ x+2y-4z=9, \\ -x-12y+14z=1; \end{cases}$

i) $\begin{cases} 3x_1-2x_2-5x_3+x_4=3, \\ 2x_1-3x_2+x_3+5x_4=-3, \\ x_1+2x_2-4x_4=-3, \\ x_1-x_2-4x_3+9x_4=22; \end{cases}$ j) $\begin{cases} 2x_1+7x_2+3x_3+x_4=5, \\ x_1+3x_2+5x_3-2x_4=3, \\ x_1+5x_2-9x_3+8x_4=1, \\ 5x_1+18x_2+4x_3+5x_4=12 \ . \end{cases}$

3. Agar $f(1)=-1,\ f(-1)=9,\ f(2)=-3$ bo'lsa, $f(x)$ kvadrat ko'phadni toping.

4. Agar $f(-1)=0,\ f(1)=4,\ f(2)=3,\ f(3)=16$ bo'lsa, uchinchi darajali ko'phadni toping.

5. k ning qanday haqiqiy qiymatlarida sistema nol yechimga ega bo'lmaydi:

a) $\begin{cases} kx_1+x_2+x_3+x_4=0, \\ x_1+(1+k)x_2+x_3+x_4=0, \\ x_1+x_2+(2+k)x_3+x_4=0, \\ 2x_1+2x_2+(3+k)x_3+2x_4=0 \end{cases}$ b) $\begin{cases} kx_1+x_2+x_3=0, \\ x_1+kx_2+x_3=0, \\ x_1+x_2+kx_3=0 \end{cases}$?

Mustaqil yechish uchun berilgan misolning javoblari

1. a) $\{(2,-3)\}$; b) $\{(-2,-1)\,|\,a\neq-2\}$; c) $\{(0,5,-2)\}$;
d) Крамер qoidasi bo'yicha sistemani yechish mumkin emas;
e) $\{(1,1,-1,-1)\}$; f) $\{(-2,0,1,-1)\}$.

2. a) $\{(3+2\alpha,\alpha)\,|\,\alpha\in\mathbf{R}\}$; b) Yechimi yo'q; c) $\{(1-i+2\alpha,\alpha)\,|\,\alpha\in\mathbf{C}\}$;
d) Yechimi yo'q; e) $\{(\alpha,1-4\alpha,-3-7\alpha)\,|\,\alpha\in\mathbf{R}\}$;

f) Agar $b_3\neq b_1+b_2$, u holda yechimi yo'q;
$\left\{\left(\frac{1}{2}(1+i)\alpha+b_1+3ib_3),\alpha,b_3\right)\Big|\,\alpha\in C, b_3=b_1+b_2\right\}$; g) $\{(1,0,2)\}$;
h) $\left(\frac{13}{3},-\frac{13}{3},-\frac{10}{3}\right)$; i) $\{(-1,3,-2,2)\}$;
j) $\{(6-28\alpha+17\beta,-1+7\alpha-5\beta,\alpha,\beta)\,|\,\alpha,\beta\in\mathbf{R}\}$.

3. $f(x)=x^4-5x+3$.

4. $f(x)=2x^3-5x^2+7$. **5.** a) $-1;\ 0;1$; b) $-2;1$.

1-bob bo‘yicha amaliy mashg‘ulotlarni mustahkamlash uchun nazorat topshiriqlari

1.1.-misol. Berilgan tenglamalar sistemasini

a) Kramer qoidasidan foydalanib;

b) Gauss usuli;

c) Maple tizimidan foydalanib yeching.

1.1.1. $\begin{cases} 2x+2y-z+t=4, \\ 4x+3y-z+2t=6, \\ 8x+5y-3z+4t=12, \\ 3x+3y-2z+2t=6. \end{cases}$

1.1.2. $\begin{cases} 2x+3y+11z+5t=2, \\ x+y+5z+2t=1, \\ 2x+y+3z+2t=-3, \\ x+y+3z+4t=-3. \end{cases}$

1.1.3. $\begin{cases} 2x+y+4z+8t=-1, \\ x+3y-6z+2t=3, \\ 3x-2y+2z-2t=8, \\ 2x-y+2z=4. \end{cases}$

1.1.4. $\begin{cases} 2x-y-6z+3t+1=0, \\ 7x-4y+2z-15t+32=0, \\ x-2y-4z+9t-5=0, \\ x-y+2z-6t+8=0. \end{cases}$

1.1.5. $\begin{cases} 2x+5y+4z+t=20, \\ x+3y+2z+t=11, \\ 2x+10y+9z+7t=40, \\ 3x+8y+9z+2t=37. \end{cases}$

1.1.6. $\begin{cases} 6x+5y-2z+4t+4=0, \\ 9x-y+4z-t-13=0, \\ 3x+4y+2z-2t-1=0, \\ 3x-9y+2t-11=0. \end{cases}$

1.1.7. $\begin{cases} 7x+9y+4z+2t-2=0, \\ 2x-2y+z+t-6=0, \\ 5x+6y+3z+2t-3=0, \\ 2x+3y+z+t=0. \end{cases}$

1.1.8. $\begin{cases} 3x+4y+z+2t+3=0, \\ 3x+5y+3z+5t+6=0, \\ 6x+8y+z+5t+8=0, \\ 3x+5y+3z+7t+8=0. \end{cases}$

1.1.9. $\begin{cases} 3x-2y-5z+t=3, \\ 2x-3y+z+5t=-3, \\ x+2y-4t=-3, \\ x-y-4z+9t=22. \end{cases}$

1.1.10. $\begin{cases} 4x-3y+z+5t-7=0, \\ x-2y-2z-3t-3=0, \\ 3x-y+2z+1=0, \\ 2x+3y+2z-8t+7=0. \end{cases}$

1.1.11. $\begin{cases} 2x-2y+t+3=0, \\ 2x+3y+z-3t+6=0, \\ 3x+4y-z+2t=0, \\ x+3y+z-t-2=0. \end{cases}$

1.1.12. $\begin{cases} x+y-6z-4t=6, \\ 3x-y-6z-4t=2, \\ 2x+3y+9z+2t=6, \\ 3x+2y+3z+8t=-7. \end{cases}$

1.1.13. $\begin{cases} 2x-3y+3z+2t-3=0, \\ 6x+9y-2z-t+4=0, \\ 10x+3y-3z-2t-3=0, \\ 8x+6y+z+3t+7=0. \end{cases}$

1.1.14. $\begin{cases} x+2y+5z+9t=79, \\ 3x+13y+18z+30t=263, \\ 2x+4y+11z+16t=146, \\ x+9y+9z+9t=92. \end{cases}$

1.1.15. $\begin{cases} x+2y+3z-2t=6, \\ 2x-y-2z-3t=8, \\ 3x+2y-z+2t=4, \\ 2x-3y+2z+t=-8 \end{cases}$

1.1.16. $\begin{cases} x+2y+3z+4t=5, \\ 2x+y+2z+3t=1, \\ 3x+2y+z+2t=1, \\ 4x+3y+2z+t=-5 \end{cases}$

1.1.17. $\begin{cases} y-3z+4t=-5, \\ x-2z+3t=-4, \\ 3x+2y-5t=12, \\ 4x+3y-5z=5 \end{cases}$

1.1.18. $\begin{cases} -3x+9y+3z+6t=2, \\ -5x+8y+2z+7t=3, \\ 4x-5y-3z-2t=0, \\ 7x-8y-4z-5t=4 \end{cases}$

1.1.19. $\begin{cases} 3x-3y-5z+8t=1, \\ -3x+2y+4z-6t=2, \\ 2x-5y-7z+5t=4, \\ -4x+3y+5z-6t=4 \end{cases}$

1.1.20. $\begin{cases} 2x-5y+4z+3t-2=0, \\ 3x-4y+7z+5t+4=0, \\ 4x-9y+8z+5t-4=0, \\ -3x+2y-5z+3t+6=0 \end{cases}$

1.1.21. $\begin{cases} 2x-5y+z+2t=4, \\ -3x+7y-z+4t=1, \\ 5x-9y+2z+7t=2, \\ 4x-6y+z+2t=4 \end{cases}$

1.1.22. $\begin{cases} 3x-5y-2z+2t=6, \\ -4x+7y+4z+4t=4, \\ 4x-9y-3z+7t=6, \\ 2x-6y-3z+2t=1 \end{cases}$

1.1.23. $\begin{cases} 3x+2y+2z+2t=1, \\ 9x-8y+5z+10t=2, \\ 5x-8y+5z+8t=4, \\ 6x-5y+4z+7t=1 \end{cases}$

1.1.24. $\begin{cases} 7x+6y+3z+7t=4, \\ 3x+5y+7z+2t=3, \\ 5x+4y+3z+5t=1, \\ 5x+6y+5z+4t=6 \end{cases}$

1.1.25. $\begin{cases} 6x-5y+8z+4t=1, \\ 9x+7y+5z+2t=4, \\ 7x+5y+3z+7t=1, \\ -4x+8y-8z-3t=2 \end{cases}$

1.1.26. $\begin{cases} 7x+3y+2z+6t=4, \\ 8x-9y+4z+9t=1, \\ 7x-2y+7z+3t=4, \\ 5x-3y+3z+4t=6 \end{cases}$

1.1.27-misol. Ushbu

$$\begin{cases} 2x+3y+4z+5t=3, \\ 5x+4y+3z+2t=-3, \\ 3x+2y+5z+4t=-3, \\ 5x+2y+3z+4t=-1 \end{cases} \qquad (1)$$

sistemani a) Kramer qoidasidan foydalanib; b) Gauss usuli; c) Maple tizimidan foydalanib yeching.

Yechilishi. *a)* **Kramer qoidasi.** Berilgan (1) sistemaning yechimini Kramer qoidasidan foydalanib topamiz. (1) sistemaning koeffisiyentlaridan tuzilgan determinantni Laplas teoremasiga asosan, quyidagi yoyilmaga ega bo‘lamiz:

$$\Delta=\det A=\begin{vmatrix}2&3&4&5\\5&4&3&2\\3&2&5&4\\5&2&3&4\end{vmatrix}=\begin{vmatrix}2&3\\5&2\end{vmatrix}(-1)^{1+2+1+2}\begin{vmatrix}5&4\\3&4\end{vmatrix}+\begin{vmatrix}2&4\\5&3\end{vmatrix}(-1)^{1+2+1+3}\begin{vmatrix}2&4\\2&4\end{vmatrix}+$$

$$+\begin{vmatrix}2&5\\5&2\end{vmatrix}(-1)^{1+2+1+4}\begin{vmatrix}2&5\\2&3\end{vmatrix}+$$

$$+\begin{vmatrix}3&4\\4&3\end{vmatrix}(-1)^{1+2+2+3}\begin{vmatrix}3&4\\5&4\end{vmatrix}+\begin{vmatrix}3&4\\4&2\end{vmatrix}(-1)^{1+2+2+4}\begin{vmatrix}3&5\\5&3\end{vmatrix}+\begin{vmatrix}4&5\\3&2\end{vmatrix}(-1)^{1+2+3+4}\begin{vmatrix}3&2\\5&2\end{vmatrix}=$$

$$=-56+84-56-224+28=-112\neq 0.$$

Endi $\Delta_x,\Delta_y,\Delta_z,\Delta_t$ determinantlarni hisoblash uchun Δ ning, navbat bilan, 1-,2-,3- va 4- ustunlari o'rniga ozod hadlarni qo'yamiz. Determinantni hisoblashda tartibni pasaytirish uslubidan foydadanamiz:

$$\Delta_x=\begin{vmatrix}3&3&4&5\\-3&4&3&2\\-3&2&5&4\\-1&2&3&4\end{vmatrix}=\begin{vmatrix}0&9&13&17\\0&-2&-6&-10\\0&-4&-4&-8\\-1&2&3&4\end{vmatrix}=(-1)(-1)^{4+1}\begin{vmatrix}9&13&17\\-2&-6&-10\\-4&-4&-8\end{vmatrix}=$$

$$=8\begin{vmatrix}9&13&17\\1&3&5\\1&1&2\end{vmatrix}=8\cdot 14=112,$$

bunda ikkinchi sirtdan -2, uchinchi sirtdan -4 determinant belgisidan tashqari chiqargan.

$$\Delta_y=\begin{vmatrix}2&3&4&5\\5&-3&3&2\\3&-3&5&4\\5&-1&3&4\end{vmatrix}=\begin{vmatrix}17&0&13&17\\-10&0&-6&-10\\-12&0&-4&-8\\5&-1&3&4\end{vmatrix}=(-1)(-1)^{4+2}\begin{vmatrix}17&13&17\\-10&-6&-10\\-12&-4&-8\end{vmatrix}=-112,$$

$$\Delta_z=\begin{vmatrix}2&3&3&5\\5&4&-3&2\\3&2&-3&4\\5&2&-1&4\end{vmatrix}=\begin{vmatrix}17&9&0&17\\-10&-2&0&-10\\-12&-4&0&-8\\5&2&-1&4\end{vmatrix}=(-1)(-1)^{4+3}\begin{vmatrix}17&9&17\\-10&-2&-10\\-12&-4&-8\end{vmatrix}=224,$$

$$\Delta_t=\begin{vmatrix}2&3&4&3\\5&4&3&-3\\3&2&5&-3\\5&2&3&-1\end{vmatrix}=\begin{vmatrix}17&9&13&0\\-10&-2&-6&0\\-12&-4&-4&0\\5&2&3&-1\end{vmatrix}=(-1)(-1)^{4+4}\begin{vmatrix}17&9&13\\-10&-6&-6\\-12&-4&-4\end{vmatrix}=-224.$$

Kramer formulalardan foydalanib x,y,z,t ni topamiz:

$$x=\frac{\Delta_x}{\Delta}=\frac{112}{-112}=-1,\ y=\frac{\Delta_y}{\Delta}=\frac{-112}{-112}=1,$$
$$z=\frac{\Delta_z}{\Delta}=\frac{224}{-112}=-2,\ t=\frac{\Delta_t}{\Delta}=\frac{-224}{-112}=2.$$

Shunday qilib, (1) sistemaning yechimi (-1,1,-2,2) bo'ladi. ■

b) **Gauss usuli.** Berilgan (1) sistema noma'lumlarni ketma-ket yo'qotish usuli ya'ni Gauss usuli bilan yechimini topamiz. Berilgan sistemaning determinanti $\Delta=\det A=-112\neq 0$ bo'lganligi uchun, bu sistema yagona yechimga ega.

Avval –5 ga ko'paytirilgan birinchi tenglamani ko'paytirilgan ikkinchi tenglamani qo'shib, so'ngra – 3 ga ko'paytirilgan birinchi tenglamani 2 ga ko'paytirilgan uchinchi tenglamani qo'shib, undan keyin –5 ga ko'paytirilgan ko'paytirilgan birinchi tenglamani 2 ga ko'paytirilgan to'rtinchi tenglamaga qo'shib, ushbuni hosil qilamiz:

$$\begin{cases}2x+3y+4z+5t=3,\\ \quad -7y-24z-21t=-21,\\ \quad -5y-2z-7t=-15,\\ \quad -11y-14z-17t=-17\end{cases}$$

Bu sistemaning ikkinchi tenglamasini –7 ga qisqartirib, uchinchi va to'rtinchi tenglamalarni esa –1 ga ko'paytiramiz, natijada, ushbu

$$\begin{cases}2x+3y+4z+5t=3,\\ \quad y+2z+3t=3,\\ \quad 5y+2z+7t=15,\\ \quad 11y+14z+17t=17\end{cases}$$

sistemani hosil qilamiz.

Ikkinchi tenglamani –5 va –11 ga ko'paytirib, mos ravishda uchinchi va to'rtinchi tenglamalarga qo'shamiz:

$$\begin{cases} 2x+3y+4z+5t=3, \\ y+2z+3t=3, \\ -8z-8t=0, \\ -8z-16t=-16 \end{cases}$$

Bu sistemaning uchinchi va to'rtinchi tenglamalarni –8 ga qisqartirib, uchinchi tenglamani to'rtinchi tenglamadan ayiramiz:

$$\begin{cases} 2x+3y+4z+5t=3, \\ y+2z+3t=3, \\ z+t=0, \\ t=2 \end{cases}$$

To'rtinchi tenglamadagi $t=2$ qiymatini uchinchi tenglamaga qo'yib, z ni aniqlaymiz: $z=-t,\ z=2$. z va t ning qiymatilarini ikkinchi tenglamaga qo'yib, y ni aniqlaymiz: $y=3-2z-3t=3+4-6=1$. Nihoyat, birinchi tenglamadan x ni topamiz: $2x-3y-4z-5t=3-3+8-10=-2;\ 2x=-2;\ x=-1$

Shunday qilib, (1) sistemaning yechimi (-1,1,-2,2) bo'ladi. ■

c) Maple tizimidan foydalanib misolni yechish:

```
> with(Student[LinearAlgebra]):
> A := <<2,5,3,5>|<3,4,2,2>|<4,3,5,3>|<5,2,4,4>>:
b := <3,-3,-3,-1>:
LinearSolve(A, b);
```

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 2 \end{pmatrix}.■$$

1.2-misol. Berilgan tenglamalar sistemani matritsa usulida yeching.

1.2.1. $\begin{cases} 2x+y+3z=7. \\ 2x+3y+z=1. \\ 3x+2y+z=6. \end{cases}$ **1.2.2.** $\begin{cases} 3x-y+z=12. \\ x+2y+4z=6. \\ 5x+y+2z=3. \end{cases}$

1.2.3. $\begin{cases} 2x-y+2z=3. \\ x+y+2z=-4. \\ 4x+y+4z=-3. \end{cases}$ **1.2.4.** $\begin{cases} 2x-y+3z=-4. \\ x+3y-z=11. \\ x-2y+2z=-7. \end{cases}$

1.2.5. $\begin{cases} 3x-2y+4z=12. \\ 3x+4y-2z=6. \\ 2x-y-z=-9. \end{cases}$ **1.2.6.** $\begin{cases} 8x+3y-6z=-4. \\ x+y-z=2. \\ 4x+y-3z=-5. \end{cases}$

1.2.7. $\begin{cases} 4x+y-3z=9. \\ x+y-z=-2. \\ 8x+3y-6z=12. \end{cases}$ **1.2.8.** $\begin{cases} 2x+3y+4z=33. \\ 7x-5y=24. \\ 4x+11z=39. \end{cases}$

1.2.9. $\begin{cases} 2x+3y+4z=12. \\ 7x-5y+z=-33. \\ 4x+z=-7. \end{cases}$ **1.2.10.** $\begin{cases} 3x-2y+4z=21. \\ 3x+4y-2z=9. \\ 2x-y-z=10. \end{cases}$

1.2.11. $\begin{cases} 4x+y+4z=19. \\ 2x-2+2z=11. \\ x+y+2z=8. \end{cases}$ **1.2.12.** $\begin{cases} 2x-y+2z=8. \\ x+y+2z=11. \\ 4x+y+4z=22. \end{cases}$

1.2.13. $\begin{cases} 2x-y-3z=0. \\ 3x+4y+2z=1. \\ x+5y+z=-3. \end{cases}$ **1.2.14.** $\begin{cases} 3x+y+z=-4. \\ -3x+5y+6z=36. \\ x-4y-2z=-19. \end{cases}$

1.2.15. $\begin{cases} 3x-y+z=9. \\ 5x+y+2z=11. \\ x+2y+4z=19. \end{cases}$ **1.2.16.** $\begin{cases} 2x+3y+z=12. \\ 2x+y+3z=16. \\ 3x+2y+z=8. \end{cases}$

1.2.17. $\begin{cases} x+4y-z=6. \\ 5y+4z=-20. \\ 3x-2y+5z=-22. \end{cases}$ **1.2.18.** $\begin{cases} 3x-2y-5z=5. \\ 2x+3y-4z=12. \\ x-2y+3z=-1. \end{cases}$

1.2.19. $\begin{cases} 2x-y+2z=0. \\ 4x+y+4z=6. \\ x+y+2z=4. \end{cases}$ **1.2.20.** $\begin{cases} 2x-y-3z=-9. \\ x+5y+z=20. \\ 3x+4y+2z=15. \end{cases}$

1.2.21. $\begin{cases} -3x+5y+6z=-8. \\ 3x+y+z=-4. \\ x-4y-2z=-9. \end{cases}$ **1.2.22.** $\begin{cases} 3x-y+z=-11. \\ 5x+y+2z=8. \\ x+2y+4z=16. \end{cases}$

1.2.23. $\begin{cases} 2x+3y+z=4. \\ 2x+y+3z=0. \\ 3x+2y+z=1. \end{cases}$ **1.2.24.** $\begin{cases} x-2y+3z=14. \\ 2x+3y-4z=-16. \\ 3x-2y-5z=-8. \end{cases}$

1.2.25. $\begin{cases} 3x+y-z=11. \\ 2x-y-z=4. \\ 3x-2y+4z=11. \end{cases}$ **1.2.26.** $\begin{cases} x+5y-6z=-15. \\ 3x+y+4z=13. \\ 2x-3y+3z=9. \end{cases}$

1.2.27-misol. Ushbu $\begin{cases} x_1-2x_2+x_3=5, \\ 2x_1-x_3=0, \\ -2x_1+x_2+x_3=-1 \end{cases}$

sistemani matritsa usulida yeching.

Yechilishi. Bu misolda (2.2) belgilashga asosan

$$A=\begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix},\ B=\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix},\ X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

ko'rinishda bo'ladi. U holda berilgan tenglamalar sistemasini

$$AX=B$$

ko'rinishda yozish mumkin. Endi $\det A$ ni topamiz:

$$\Delta=\det A=\begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix}=3\neq 0.$$

A matritsa maxsusmas ekan. Bu A matritsa uchun teskari matritsa mavjud.

Endi A matritsa elementlarining algebraik to'ldiruvchilarini hisoblab, so'ngra teskari matritsani topamiz:

$$\begin{matrix} A_{11}=1, & A_{21}=3, & A_{31}=2, \\ A_{12}=0, & A_{22}=3, & A_{32}=3, \\ A_{13}=2, & A_{23}=3, & A_{33}=4, \end{matrix}\quad A^{-1}=\begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix}.$$

(1.15.4) formulaga asosan, berilgan sistemani $X=A^{-1}B$ ko'rinishda yozish mumkin. Endi x_1, x_2, x_3 larni topamiz:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

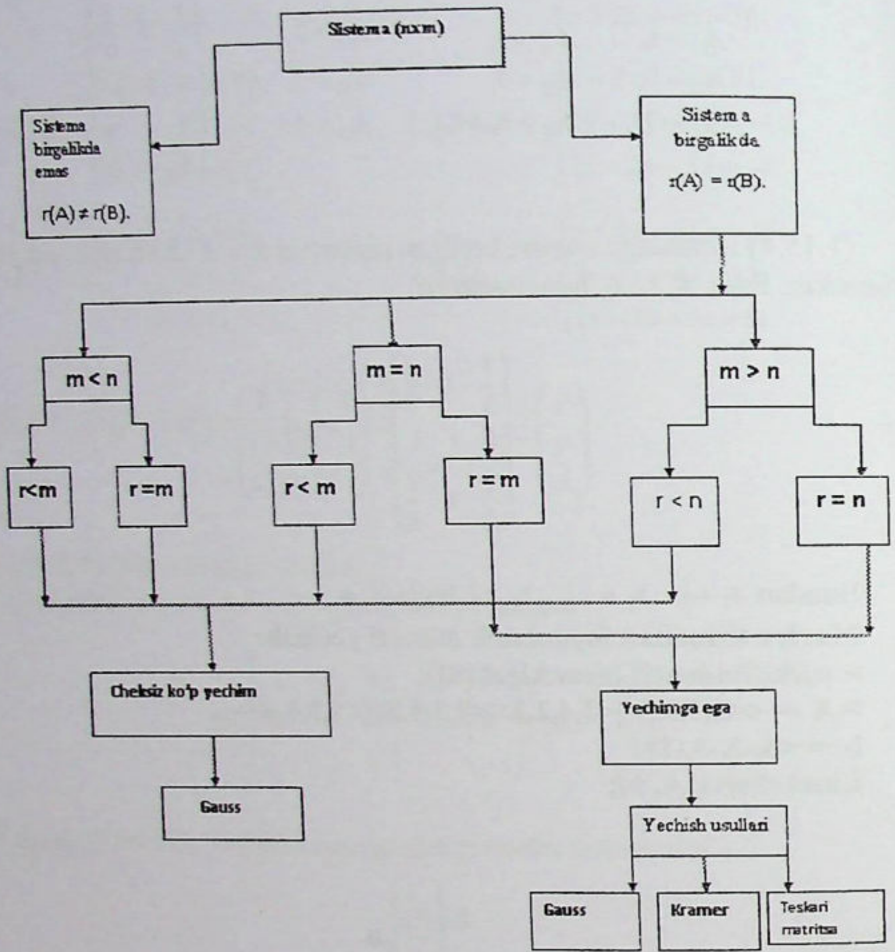
Bundan $x_1=1,\ x_2=-1,\ x_3=2$ bo'ladi. ■

Marlye tizimidan foydalanib misolni yechish:

```
> with(Student[LinearAlgebra]):
> A := <<2,5,3,5>|<3,4,2,2>|<4,3,5,3>|<5,2,4,4>>:
b := <3,-3,-3,-1>:
LinearSolve(A, b);
```

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$ ■

Chiziqli algebraik tenglamalar sistemasini yechishning umumiy sxemasi:



2-bob. VEKTOR ALGEBRASI ELEMENTLARI

2.1-§. Vektorlar ustida amallar. Ikki vektorning skalyar va vektor ko‘paytmalari

2.1. Vektorning ta’rifi, asosiy tushunchalar. Matematika, fizika, mexanika, elektrotexnika, radiotexnika va shunga o‘xshash soxalarda ikki xil miqdorlar uchrab turadi. Bu miqdorlarning bir turi uzining son qiymati bilan to‘la aniqlanadi. Masalan, shaklning yuzi, jismning xajmi, temperatura, elektr kattalik, zichlik kabi miqdorlar.Bunday miqdorlar *skalyar miqdorlar* deyiladi. Ikkinchi tur miqdorlar o‘zining son qiymati bilan to‘la aniqlanmaydi, ularni to‘la aniqlash uchun son qiymatlari bilan bir qatorda yo‘nalishlari ham berilgan bo‘lishi kerak. Masalan, kuch, tezlik, tezlanish kabi miqdorlar.

Vektorlar – uzunlik va yo‘nalishga ega bo‘lgan miqdorlardir. Vektorlarning uzunligi uning *moduli yoki absolyut miqdori* deyiladi. Nul vektorning o‘ziga xos xususiyatlari shundan iboratki, uning uzunligi nolga teng, u yo‘nalishga ega emas.

Nolga teng bo‘lmagan vektorlar yo‘naltirilgan kesmalar ko‘rinishida ifodalanadi va $\vec{AB}$ kabi yoziladi, bunda A - berilgan vektorning boshi, B – uning oxiridan iborat. $\vec{AB}$ vektorning uzunligi (moduli) $|\vec{AB}|$ kabi yoziladi. Bundan buyon vektorlar yoki ikkita harflar bilan (masalan $\vec{AB}$), yoki bitta harf bilan (masalan, $\vec{a},\ \vec{b},\ \vec{c}$) belgilanadi.

2.1.1-ta’rif. Agar ikkita nolga teng bo‘lmagan $\vec{a}$ va $\vec{b}$ vektorlarning uzunliklari teng, $\left|\vec{a}\right| = \left|\vec{b}\right|$ va ular bir xil yo‘nalishga ega bo‘lsalar, $\vec{a}\uparrow\uparrow\vec{b}$, bu vektorlar *o‘zaro teng* deyiladi va $\vec{a} = \vec{b}$ kabi yoziladi.

2.1.2-ta’rif. Agar ikkita nolga teng bo‘lmagan $\vec{a}$ va $\vec{b}$ vektorlarning uzunliklari teng, $\left|\vec{a}\right| = \left|\vec{b}\right|$, va ular qarama-qarshi yo‘nalishlarga ega bo‘lsalar, bu vektorlar *qarama-qarshi* deyiladi va $\vec{a} = -\vec{b}$ kabi yoziladi.

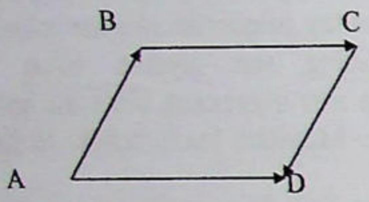
2.1.3-ta’rif. Agar $\vec{a}$ va $\vec{b}$ vektorlar bitta to‘g‘ri chiziqda yoki parallel to‘g‘ri chiziqlarda yotsalar, ular *kollinear* deyiladi.

2.1.4-ta’rif. Bitta tekislikda yoki bir necha parallel tekisliklarda yotgan uchta yoki undan ortiq vektorlar *komplanar vektorlar* deyiladi.

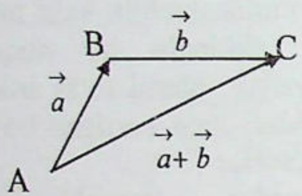
Fazoda ikki, uch, yoki ulardan ko‘p vektorlar berilgan bo‘lsin. Ularning uzunliklarini o‘zgartirmasdan, barcha vektorlarni o‘z-o‘ziga parallel ravishda bitta umumiy A nuqtaga ko‘chiramiz. Vektorlar ustida bunday amal – vektorlarni umumiy boshlang‘ich nuqtaga keltirish deyiladi.

2.1.5-masala. $ABCD$ parallelogramm berilgan. Parallelogrammning tomonida: a) teng; b) qarama-qarshi vektorlar ko‘rsatilsin.

Yechilishi. Parallelogrammning qarama-qarshi tomonlari teng va parallel bo‘lganligidan, javoblar quyidagicha bo‘ladi (2.1-chizma):



2.1-чизма.



2.2-чизма.

a) $\vec{AB}=\vec{DC}$, $\vec{BC}=\vec{AD}$, $\vec{BA}=\vec{CD}$, $\vec{CB}=\vec{DA}$;

b) $\vec{AB}=-\vec{CD}$, $\vec{AD}=-\vec{CB}$, $\vec{DA}=-\vec{BC}$, $\vec{BA}=-\vec{DC}$;

2.2. Vektorlar ustida amallar. Fazodagi vektorlarni qo‘shish, ayirish va vektorni skalyarga ko‘paytirish amallari tekislikda vektorlar ustidagi shunday amallarga o‘xshashdir.

1. Ikkita, $\vec{a}$ va $\vec{b}$ vektorlarni yoki uchburchak qoidasi bo‘yicha, yoki parallelogramm qoidasi bo‘yicha qo‘shish mumkin.

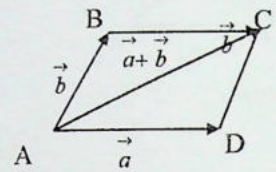
a) **Uchburchak qoidasi.** $\vec{a}=\vec{AB}$ vektorning boshini A nuqtaga joylashtirib, $\vec{b}$ vektorning boshini B nuqtaga joylashtiramiz (2.2-chizma) va $\vec{BC}=\vec{b}$ vektorni yasaymiz. Unda $\vec{a}$ vektorning boshini $\vec{b}$ vektorning oxiri bilan tutashtiruvchi $\vec{AC}$ vektor, $\vec{a}$ va $\vec{b}$ vektorlarning yiђindisidan iborat bo‘ladi:

$$\vec{AC}=\vec{AB}+\vec{BC}, \quad \vec{AC}=\vec{a}+\vec{b}.$$

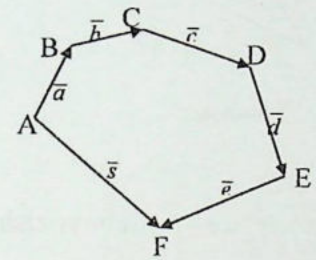
b) **Parallelogramm qoidasi.** Berilgan ikkita, $\vec{a}$ va $\vec{b}$ vektorlarning boshini bitta umumiy A nuqtaga keltiramiz va $\vec{AD}=\vec{a}$, $\vec{AB}=\vec{b}$ vektorlarni

yasaymiz (2.3-chizma). So‘ngra, $\vec{AB}$ va $\vec{AD}$ vektorlarni tomonlar sifatida qarab, $ABCD$ parallellogrammni yasaymiz. Unda AC diagonalda yotuvchi va umumiy A nuqtadan chiquvchi $\vec{AC}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning yiђindisidan iborat bo‘ladi:

$$\vec{AC}=\vec{a}+\vec{b}$$



2.3-чизма.



2.4-чизма

c) **Ko‘pburchak qoidasi.** Bir nechta, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$ vektorni qo‘shish uchun, har bir keyingi vektorning boshini undan oldingi vektorning oxiriga keltiramiz (2.4-chizma). Unda birinchi $\vec{a}$ vektorning A boshini oxirgi $\vec{e}$ vektorning oxiri bilan tutashtiruvchi $\vec{s}$ vektor berilgan $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$ vektorlarning yig‘indisi deyiladi va $\vec{s}=\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$ bo‘ladi.

Vektorlarni qo‘shish amali quyidagi xossalarga ega:

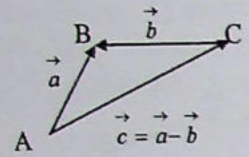
1^0. Qo‘shishning o‘rin almashtirish xossasi: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

2^0.Qo‘shishning guruhlash xossasi: $\vec{a}+\vec{b}+\vec{c}=\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{a}+\vec{c}\right)$.

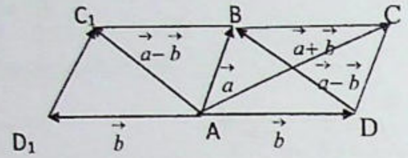
2. Vektorlarni ayirish. Ikkita $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasi deb, $\vec{b}+\vec{c}=\vec{a}$ shartni qanoatlantiruvchi $\vec{c}$ vektorga aytiladi va u $\vec{c}=\vec{a}-\vec{b}$ kabi yoziladi.

Shunday qilib, $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasini topish uchun, $\vec{a}$ vektorning oxiriga $\vec{b}$ vektorning oxirini ko‘chirish lozim. Unda, birinchi $\vec{a}$ vektorning boshini ikkinchi vektorning boshi bilan tutashtiruvchi $\vec{AC}=\vec{c}$ vektor, $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasidan iborat bo‘ladi: $\vec{AC}=\vec{a}-\vec{b}$. $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasini $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$ tenglikdan topish mumkin.

Buning uchun, $\vec{a}$ va $\vec{b}$ vektorlarni A nuqtaga keltirib, ularda $ABCD$ parallelogramm yasaymiz (2.6-chizma). So'ngra $-\vec{b}$ vektorni yasaymiz va $\vec{a}$ va $-\vec{b}$ vektorlarda ABC_1D_1 parallelogramm yasaymiz.



2.5-чизма.



2.6-чизма.

Unda $\vec{AC}=\vec{a}-\vec{b}$ deb yozish mumkin. Nixoyat, $\vec{AC}$ vektorni o'ziga parallel ravishda D nuqtaga ko'chiramiz. Unda $\vec{DB}=\vec{AC_1}=\vec{a}-\vec{b}$ bo'ladi.

Shunday qilib, agar $\vec{a}$ va $\vec{b}$ vektorlarda $ABCD$ parallelogramm yasalgan bo'lsa, uning bitta diagonalida ularning yig'indisi vektori, ikkinchisida esa, ularning ayirmasi vektori yotadi: $\vec{AC}=\vec{a}+\vec{b}$; $\vec{DB}=\vec{a}-\vec{b}$.

2.3. Vektorni songa ko'paytirish

2.3.1-ta'rif. $\vec{b}$ vektorning λ *songa ko'paytmasi* deb: 1) $|\vec{a}|=|\lambda||\vec{b}|$ va 2) $\lambda>0$ bo'lganda $\vec{a}\uparrow\uparrow\vec{b}$; $\lambda<0$ bo'lganda $\vec{a}\uparrow\downarrow\vec{b}$ shartlarni qanoatlantiruvchi $\vec{a}$ vektorga aytiladi, bunda $\uparrow\uparrow$-vektorlarning yo'nalishdoshligini, $\uparrow\downarrow$ - vektorlar qarama-qarshi yo'nalganligini anglatadi.

Vektorning songa ko'paytmasi quyidagi xossalarga ega:

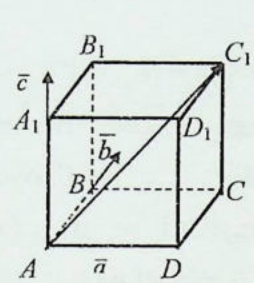
$$1^0.\ 1\cdot\vec{b}=\vec{b},\quad 2^0.\ \lambda\left(\mu\vec{b}\right)=(\lambda\mu)\vec{b};$$

$$3^0.\ (\lambda+\mu)\vec{b}=\lambda\vec{b}+\mu\vec{b};\quad 4^0.\ \lambda\left(\vec{b}+\vec{c}\right)=\lambda\vec{b}+\lambda\vec{c}.$$

Bu xossalarning isbotlari planimetriyadagi shu kabi xossalarning isbotiga o'xshashdir. Ushbu $\vec{a}=\lambda\vec{b}$ tenglikni $\vec{a}$ va $\vec{b}$ vektorlar kollinearligining zaruriy va yetarli sharti sifatida qarash mumkin.

2.4. Fazodagi bazis haqida

2.4.1.Vektorning koordinatalari. Biz tekislikdagi bazisni kollinear bo'lmagan vektorlar jufti shaklida kiritgan edik. Unda har qanday uchinchi vektorni bazisning ikkita vektori orqali ifodalash mumkin bo'lgan edi.



2.7-чизма.

Fazodagi vektorlarning yuqoridagiga o'xshash xossasini qarab chiqamiz. Fazoda uchta komplanar bo'lmagan $\vec{a},\vec{b},\vec{c}$ vektorlar berilgan bo'lsin. Bunda ixtiyoriy to'rtinchi $\vec{d}$ vektorni $\vec{a},\vec{b}$ va $\vec{c}$ vektorlar orqali ifodalash mumkinligini isbotlaymiz.

$\vec{a},\vec{b},\vec{c},\vec{d}$ vektorlarni umumiy A nuqtaga keltiramiz (2.7-chizma). $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar bo'lmaganligidan $\left(\vec{a}\,\vec{b}\right)$, $\left(\vec{a}\,\vec{c}\right)$, $\left(\vec{b}\,\vec{c}\right)$ vektorlar juftining har biri tekislikni aniqlaydi. C_1 uch orqali mos ravishda $(ABCD)$, $(ABBA_1)$ va (ADD_1A_1) tekisliklarga parallel tekisliklar o'tkazamiz. Natijada $ABCDA_1B_1C_1D_1$ prizmani hosil qilamiz. Vektorlarni qo'shish qoidasi bo'yicha

$$\vec{AC_1}=\vec{AD}+\vec{DC}+\vec{CC_1}$$

deb yozish mumkin. $\vec{AD}\parallel\vec{a}$ bo'lganligidan, ikki vektorning kollinearlik shartiga ko'ra, $\vec{AD}=x\,\vec{a}$ deb yozish mumkin. Shunga o'xshash, $\vec{AB}\parallel\vec{b}$ va $\vec{AA_1}\parallel\vec{c}$ munosabatlardan,

$$\vec{AB}\parallel y\,\vec{b}\ \text{va}\ \vec{AA_1}\parallel z\,\vec{c}$$

bo'lishi kelib chiqadi. Bu ifodalarni o'rniga keltirib qo'yib, $\vec{d}=\vec{AC_1}$ vektor uchun

$$\vec{d}=x\vec{a}+y\vec{b}+z\vec{c} \qquad (2.4.2)$$

tenglikni hosil qilamiz.

(2.4.2) tenglik $\vec{d}$ vektorning uchta komplanar bo'lmagan $\vec{a},\vec{b},\vec{c}$ vektorlar bo'yicha yoyilmasi deyiladi. Bu holda $\vec{a},\vec{b},\vec{c}$ vektorlar fazoda bazis hosil qiladi deyishadi, x,y,z koeffisiyentlar esa, $\vec{d}$ vektorning bu bazisdagi koordinatalari deyiladi va u $\vec{d}(x,y,z)$ kabi yoziladi. (2.4.2) yoyilmadagi $x\vec{a}$, $y\vec{b}$, $z\vec{c}$ qo'shiluvchilar $\vec{d}$ vektorning (2.4.2) yoyilmasini tashkil etuvchilar deyiladi.

Berilgan bazisda $\vec{d}$ vektor yoyilmasining yagonaligini isbotlaymiz. $\vec{d}$ vektorning (1) yoyilmasidan boshqa, yana koeffisiyentlari boshqa bo'lgan,

$$\vec{d}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c} \qquad (2.4.3)$$

yoyilmasi ham mumkin bo'lsin.

Modomiki, yoyilmalar har xil ekan, ularning koeffisiyentlari uchun

$$x\neq x_1,\quad y\neq y_1,\quad z\neq z_1$$

shartlardan hyech bo'lmaganda bittasi bajariladi. (2.4.2) tenglikdan (2.4.3) tenglikni ayirib,

$$(x-x_1)\vec{a}+(y-y_1)\vec{b}+(z-z_1)\vec{c}=0 \qquad (2.4.4)$$

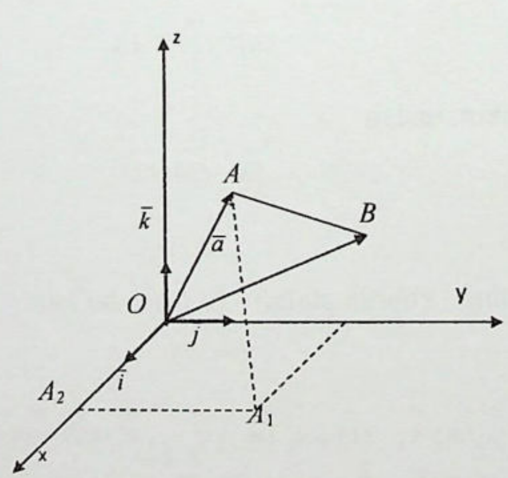
munosabatni olamiz. (2.4.4) dan $x-x_1\neq 0$ bo'lganda $\vec{a}$ vektorni $\vec{b}$ va $\vec{c}$ vektorlar orqali quyidagicha ifodalash mumkin:

$$\vec{a}=-\frac{y-y_1}{x-x_1}\vec{b}-\frac{z-z_1}{x-x_1}\vec{c} \qquad (2.4.5)$$

Bunday yoyilma esa, faqat $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar komplanar bo'lganda mumkin bo'ladi, bu esa, $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning komplanar emasligi shartiga ziddir. Demak, (2.4.4) tenglik, faqat $x-x_1=0$, $y-y_1=0$, $z-z_1=0$ va $x=x_1$, $y=y_1$, $z=z_1$ bo'lgandagina o'rinli bo'ladi. (2.4.2) yoyilmaning yagonaligi isbotlandi.

2.5. Vektorning dekart koordinatalari

Fazoda to'g'ri burchakli $Oxyz$ koordinatalar sistemasi berilgan bo'lsin. Unda bazis vektorlar sifatida Ox o'q bo'yicha $\vec{i}$, Oy o'q bo'yicha $\vec{j}$, Oz o'q bo'yicha $\vec{k}$ birlik vektorlarni kiritamiz (2.8-chizma). Fazoda $A(x_1,y_1z_1)$ nuqta berilgan bo'lsin. A va O nuqtalarni tutashtirib, $\vec{a}=\vec{AO}$ vektorni yasaymiz. A nuqtadan xOy tekislikka AA_1 perpendikulyarni va Ox o'qqa A_1A_2 perpendikulyar o'tkazamiz. Uch perpendikulyar haqidagi teoremaga ko'ra, $AA_2\perp Ox$, yani $OA_2-\bar{a}$ vektorning Ox o'qqa proyeksiyasidan iborat bo'ladi.



2.8-чизма.

Vektorlarni qo'shish qoidasidan foydalanib,

$$\vec{OA}=\vec{OA_2}+\vec{A_2A_1}+\vec{A_1A}$$

kabi yozishimiz mumkin. Modomiki, $\vec{OA_2} \parallel \vec{i}$ ekan, $\vec{OA_2}=OA_2\cdot\vec{i}=x_1\vec{i}$ bo'ladi, shunga o'xshash, $\vec{A_2A_1}\parallel\vec{j}$ va $\vec{A_1A}\parallel\vec{k}$ bo'lganligidan, $\vec{A_2A_1}=y_1\vec{j}$ va $\vec{A_1A}=z_1\vec{k}$ bo'lishi kelib chiqadi. Natijada,

$$\vec{a}=\vec{OA}=x\vec{i}+y_1\vec{j}+z\vec{k}$$

yoyilmaga ega bo'lamiz.

Shunday qilib, agar $\vec{a}=\vec{OA}$ vektor koordinatalar sistemasi boshidan chiqsa, $\vec{a}$ vektorning koordinatalari bu vektorning oxiri A nuqtaning koordinatalari bilan ustma-ust tushadi.

$\vec{AB}$ vektor o'zining boshi $A(x_1,y_1,z_1)$ va oxiri $B(x_2,y_2,z_2)$ koordinatalari bilan berilgan bo'lsin. $\vec{OA}$ va $\vec{OB}$ vektorlarni yasaymiz.

Vektorli ΔOAB dan

$$\vec{OB}=\vec{OA}+\vec{AB}$$

kabi yozish mumkin, undan

$$\vec{AB}=\vec{OB}-\vec{OA}$$

bo'ladi.

$\vec{AB}$ vektorning koordinatalari (x,y,z) bo'lsin. Unda vektorlarning yoyilmalari

$$\vec{AB}=x\vec{i}+y\vec{j}+z\vec{k},\quad \vec{OA}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k},\quad \vec{OB}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k},$$

ko'rinishda bo'ladi. Bu ifodalarni (2.4.4) ga keltirib qo'yib,

$$x\vec{i}+y\vec{j}+z\vec{k}=\vec{OA}=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k}$$

munosabatni olamiz.

Vektorlar o'zaro teng bo'lganligidan, ularning mos koordinatalari ham o'zaro teng bo'ladi:

$$x=x_2-x_1,\quad y=y_2-y_1,\quad z=z_2-z_1.$$

Shunday qilib, agar vektorning uchlari koordinatalari ma'lum bo'lsa, vektorning koordinatalari vektorning boshi va oxiri mos koordinatalari ayirmasiga teng ekan.

Endi o'z koordinatalari bilan berilgan vektorlar ustida amallarni qarab chiqamiz. Bizga $\vec{a}(x_1,y_1,z_1)$ va $\vec{b}(x_2,y_2,z_2)$ vektorlar berilgan bo'lsin. Ularning yoyilmalari

$$\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k},\qquad \vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$$

bo'ladi. Bu vektorlarning yig'indisi va ayirmasini topamiz:

$$\vec{a}+\vec{b}=(x_1+x_2)\vec{i}+(y_1+y_2)\vec{j}+(z_1+z_2)\vec{k},$$
$$\vec{a}-\vec{b}=(x_1-x_2)\vec{i}+(y_1-y_2)\vec{j}+(z_1-z_2)\vec{k},$$

ya'ni vektorlarni qo'shishda (ayirishda) ularning mos koordinatalari qo'shiladi (ayiriladi). Bu xulosa koordinataning geometrik ma'nosidan ham kelib chiqadi. Vektorlar (kesmalar) yig'indisi (ayirmasi) proyeksiyasi vektorlar (kesmalar) proyeksiyalari yig'indisiga (ayirmasiga) tengligidan, vektorlarni (kesmalarni) qo'shishda (ayirishda) ularning mos koordinatalari qo'shiladi (ayiriladi).

Agar $\vec{b}$ vektor λ songa ko'paytirilsa, uning har bir koordinatasi ana shu songa ko'paytiriladi: $\lambda\vec{b}=(\lambda x_2,\lambda y_2,\lambda z_2)$, bu proyeksiyaning xossasidan kelib chiqadi.

Agar $\vec{a}=\lambda\vec{b}$ bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'ladi, buning koordinatalar vositasidagi ifodasi

$$x_1 = \lambda x_2, \quad y_1 = \lambda y_2, \quad z_1 = \lambda z_2$$

ko'rinishni oladi, ya'ni agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsalar, ularning mos koordinatalari proporsionaldirlar:

$$\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

2.6. Chiziqli bog'liq va chiziqli erkli vektorlar. Agar $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorlar berilgan bo'lsin.

2.6.1-ta'rif. Ushbu $\lambda \vec{a}_1 + \lambda \vec{a}_2 + ... + \lambda \vec{a}_n$ ifoda $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorlarning $\lambda_1, \lambda_2, ..., \lambda_n$ koeffisentli chiziqli *kombinasiyasi* deyiladi, bunda $\lambda_1, \lambda_2, ..., \lambda_n$ biror haqiqiy sonlar.

2.6.2-ta'rif. Agar berilgan $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorlar uchun kamida bittasi noldan farqli shunday $\lambda_1, \lambda_2, ..., \lambda_n$ sonlar mavjud bo'lsinki, ushbu $\lambda \vec{a}_1 + \lambda \vec{a}_2 + ... + \lambda \vec{a}_n = \vec{0}$ tenglik o'rinli bo'lsa, $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorlar *chiziqli bog'loq* deyiladi. Agar bunday $\lambda_1, \lambda_2, ..., \lambda_n$ sonlar mavjud bo'lmasa, $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorlar *chiziqli erkli* deyiladi.

2.6.3-misol. Kollinear vektorlar chiziqli bog'liqdir.

Xaqiqatan, $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsin. U holda $\vec{a} // \vec{b} \Rightarrow \vec{a} = \lambda \vec{b} \Rightarrow \vec{a} - \lambda \vec{b} = \vec{0}$ bo'ladi. Agar $\lambda_1 = 1, \lambda_2 = -\lambda$ desak, $\lambda_1 \vec{a} + \lambda_2 \vec{b} = \vec{0}$ kelib chiqadi.

2.7. Ikki vektorning skalyar ko'paytmasi

2.7.1-teorema. Nul bo'lmagan ikkita $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko'paytmasi deb, bu vektorlar uzunliklarining ular orasidagi burchak kosinusiga ko'paytmasiga aytiladi:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi, \qquad (2.7.2)$$

bunda φ – $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak.

Agar $\vec{a}$ va $\vec{b}$ vektorlardan hyech bo'lmaganda bittasi nul vektor bo'lsa, ularning skalyar ko'paytmasi nulga tengdir: $\vec{a} \cdot \vec{b} = 0$.

Ikki vektor skalyar ko'paytmasining ba'zi xususiy hollari haqida to'xtalamiz:

1. Agar $\vec{b} = \vec{a}$ bo'lsa, $\varphi = 0$, $|\vec{b}| = |\vec{a}|$ bo'ladi. Unda ta'rifdan, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ bo'lishi kelib chiqadi.

$\vec{a} \cdot \vec{a} = \vec{a}^2$ ko'paytma - $\vec{a}$ vektorning skalyar kvadrati deb ataladi. Bundan vektorning uzunligini aniqlash uchun

$$|\vec{a}| = \sqrt{\vec{a}^2}$$

formulani olamiz, ya'ni vektorning uzunligi bu vektorning skalyar kvadratidan olingan kvadrat ildizga tengdir.

2.Agar nul bo'lmagan $\vec{a}$ va $\vec{b}$ vektorlar uchun $\vec{a} \cdot \vec{b} = 0$ bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlar bir-biriga perpendikulyardirlar.

Nul bo'lmagan $\vec{a}$ va $\vec{b}$ vektorlar uchun $\vec{a} \cdot \vec{b} = 0$ bo'lishi faqat $\cos\varphi = 0$ va unda $\varphi = 90^0$ bo'lishi mumkinligini bildiradi.

Skalyar ko'paytmaning asosiy xossalari quyidagilardan iborat:

1^0. O'rin almashtirish qonuni: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

2^0. Guruhlash qonuni: $\lambda \vec{a} \cdot \vec{b} = \left(\lambda \vec{a}\right) \cdot \vec{b} = \vec{a} \cdot \left(\lambda \vec{b}\right)$.

3^0. Taqsimot qonuni: $\left(\vec{a} + \vec{b}\right)\vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$.

4^0. $\vec{a} \uparrow\uparrow \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}|$, $\vec{a} \uparrow\downarrow \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = -|\vec{a}| \cdot |\vec{b}|$.

Bu xossalarning isboti planimetriyadagi shunday xossalarning isbotiga o'xshashdir.

Skalyar ko'paytmaning fizikaga tadbiqi quyidagicha: $\vec{S}$ siljishda o'zgarmas $\vec{F}$ kuch bajaradigan A ish bu vektorlarning skalyar ko'paytmasiga teng:

$$A = \vec{F} \cdot \vec{S} = |\vec{F}| \cdot |\vec{S}| \cdot \cos\varphi,$$

bunda φ – $\vec{F}$ va $\vec{S}$ vektorlar yo'nalishlari orasidagi burchakdir.

2.8. Skalyar ko'paytmaning vektorlar koordinatalari orqali ifodasi

Agar $\vec{a}(x_1,y_1,z_1)$ va $\vec{b}(x_2,y_2,z_2)$ vektorlarning koordinatalari ma'lum bo'lsa, ularning yoyilmalari

$$\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k},\quad \vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$$

ko'rinishni oladi.

$\vec{a}\cdot\vec{b}$ skalyar ko'paytmani hisoblashdan oldin, birlik vektorlarning skalyar ko'paytmalarini topamiz:

$$(\vec{i}\cdot\vec{i})=(\vec{j}\cdot\vec{j})=(\vec{k}\cdot\vec{k})=1,\quad (\vec{i}\cdot\vec{j})=(\vec{j}\cdot\vec{k})=(\vec{i}\cdot\vec{k})=0.$$

$\vec{a}\cdot\vec{b}$ skalyar ko'paytma vektorlar yoyilmalari orqali

$$\vec{a}\cdot\vec{b}=(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})$$

ko'rinishda yoziladi.

Skalyar ko'paytmaning taqsimot xossasidan foydalanib, oxirgi ifodaning o'ng tomonini ko'phadlar kabi ko'paytiramiz:

$$\vec{a}\cdot\vec{b}=x_1x_2\vec{i}^2+y_1y_2\vec{j}^2+z_1z_2\vec{k}^2+x_1x_2(\vec{i}\,\vec{j})+x_1z_2(\vec{i}\,\vec{k})+y_1x_2(\vec{j}\,\vec{i})+$$
$$+y_2z_1(\vec{j}\,\vec{k})+x_2z_1(\vec{i}\,\vec{k})+z_2y_1(\vec{j}\,\vec{k})$$

yoki

$$(\vec{a}\,\vec{b})=x_1x_2+y_1y_2+z_1z_2.$$

Shunday qilib, ikki vektorning skalyar ko'paytmasi ko'paytiriluvchi vektorlar mos koordinatalari ko'paytmalari yig'indisiga teng ekan.

$\vec{a}$ vektorning skalyar kvadrati

$$\vec{a}=x_1^2+y_1^2+z_1^2$$

bo'ladi. $|\vec{a}|^2=\vec{a}^2$ bo'lganligidan, $\vec{a}$ vektorning uzunligini topish formulasi

$$|\vec{a}|=\sqrt{x_1^2+y_1^2+z_1^2}$$

ko'rinishda yoziladi, ya'ni vektorning uzunligi vektor koordinatalari kvadratlari yig'indisidan olingan kvadrat ildizga teng.

Ikki vektorning skalyar ko'paytmasi formulasidan foydalanib, $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchakni hisoblashimiz mumkin:

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

yoki

$$\cos\varphi=\frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}}.$$

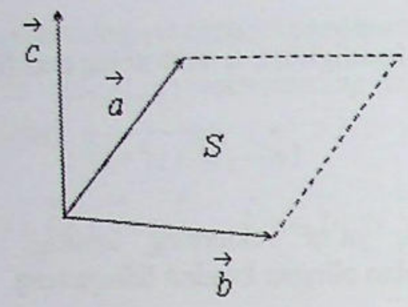
Ikkita $\vec{a}$ va $\vec{b}$ vektorlarning perpendikulyarlik sharti

$$\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2=0$$

ko'rinishda yoziladi.

2.9. Ikki vektorning vektor ko'paytmasi. Vektor ko'paytma ta'rifini kiritishdan avval, biz uchta o'zaro nokomplonar vektor uchligining fazoda joylashishi bilan bog'liq bo'lgan zarur bir tushunchani kiritamiz. Shuni aytib o'tamizki, keyingi bandlarda yuritiladigan mulohazalar faqat uch o'lchovli fazoga doir bo'ladi.

2.9.1-ta'rif. Agar komplanar $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar boshi umumiy nuqtaga keltirilgandan so'ng $\vec{c}$ vektorning oxiridan (uchidan) qaralganda $\vec{a}$ vektordan $\vec{b}$ vektorga qarab π dan kichik burchakka burish soat strelkasiga teskari bo'lsa, bu $\vec{a}$, $\vec{b}$ va $\vec{c}$ uchlik *o'ng uchlik*, aks holda *chap uchlik* deyiladi.Chap yoki o'ng uchlikni tashkil etadigan uchlik *tartiblangan uchlik* deb yuritiladi (2.9-chizma).



2.9-chizma.

2.9.2-ta'rif. $\vec{a}$ va $\vec{b}$ *vektorlarning vektor ko'paytmasi* deb quyidagi shartlarni qanoatlantiradigan $\vec{c}$ vektorga aytiladi:

1. $\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarga perpendikulyar(ortogonal)

2. $\vec{c} = |\vec{a}||\vec{b}|\sin(\widehat{\vec{a},\vec{b}})$;

3. $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning tartiblangan uchligi o'ng uchlikni tashkil etadi.

Bu ta'rifda $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ deb faraz qilinadi $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko'paytmasi $\vec{a} * \vec{b}$ yoki $[\vec{a}, \vec{b}]$ ko'rinishda yoziladi.

Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lmasa, u holda $|\vec{c}| = |[\vec{c}, \vec{b}]|$ son $\vec{a}$ va $\vec{b}$ vektorlarga yasalgan parallelogrammning yuzasiga teng bo'ladi.

Haqiqatan, maktab geometriya kursidan ma'lumki. $\vec{a}$ va $\vec{b}$ vetorlarga yasalgan parallelogrammning S yuzi uning tomonlari uzunliklarini shu tomonlar orasidagi burchak sinusi bilan ko'paytmasiga teng.

Demak,

$$S = |\vec{a}||\vec{b}|\sin(\widehat{\vec{a},\vec{b}}) = \left\|[\vec{a}, \vec{b}]\right\|.$$

Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa,u holda $|\vec{a} * \vec{b}| = 0$, chunki

$$(\widehat{\vec{a}, \vec{b}}) = 0 \text{ yoki } \pi \text{ va } \sin(\widehat{\vec{a}, \vec{b}}) = 0.$$

Vektor ko'paytma quyidagi qonunlarga bo'ysunadi.

1^0. Vektor ko'paytmada ko'paytuvchilar o'rnini almashtirilsa, uning ishorasi o'zgaradi, ya'ni $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$.

Haqiqatdan ham,agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa, bu ravshan. $\vec{a}$ va $\vec{b}$ vektorlar kollinear emas deb faraz qilaylik. Bu holda ikki vektorning vector ko'paytmasi ta'rifiga ko'ra $\vec{c_1} = [\vec{a}, \vec{b}]$ hamda $\vec{c_2} = [\vec{b}, \vec{a}]$ vektorlarning uzunligi $\vec{a}$ va $\vec{b}$ vektorlarga yasalgan parallelogrammning yuziga teng bo'lgani uchun bir xil; ammo $\vec{c_1}$ va $\vec{c_2}$ vektorlar bir biriga qarama-qarshi yo'nalgan.

Demak $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$.

2^0. Har qanday $\lambda \in R$, $\vec{a}$ va $\vec{b}$ vektorlar uchun vektor ko'paytma skalyar ko'paytuvchiga nisbatan guruhlash qonuniga bo'ysunadi, ya'ni

$$[\vec{a}, \lambda\vec{b}] = [\lambda\vec{a}, \vec{b}] = \lambda[\vec{a}, \vec{b}].$$

3^0. $\vec{a}$ va $\vec{b}$ vektorlar yig'indisi bilan $\vec{c}$ vektorning vektor ko'paytmasi taqsimot qonuniga bo'ysunadi, ya'ni

$$[\vec{a} + \vec{b}, \vec{c}] = [\vec{a}, \vec{c}] + [\vec{b}, \vec{c}].$$

2.10.Vektorlarial ko'paytmani ko'paytuvchi vektorlarning oordinatalari orqali ifodalash. Quyidagi teorema, ikki vektorning koordinatalarini, ya'ni to'g'ri burchakli Dekart koordinatalari sistemasi o'qlariga proeksiyalarini bilgan holda, ularning vektorial ko'paytmasini hisoblash mumkin.

2.10.1-teorema. Agar $\vec{a}$ va $\vec{b}$ vektorlar o'zlarining, $\vec{a} = \{x_1, y_1, z_1\}$, $\vec{b} = \{x_2, y_2, z_2\}$ koordinatalari bilan berilgan bo'lsa, $\vec{a}$ vektorning $\vec{b}$ vektorga vektorial ko'paytmasi

$$[\vec{a},\vec{b}]=\left\{\begin{vmatrix} y_1 z_1 \\ y_2 z_2\end{vmatrix}; -\begin{vmatrix} x_1 z_1 \\ x_2 z_2\end{vmatrix}; \begin{vmatrix} x_1 y_1 \\ x_2 y_2\end{vmatrix}\right\}=\begin{vmatrix} y_1 z_1 \\ y_2 z_2\end{vmatrix}\vec{i}-\begin{vmatrix} x_1 z_1 \\ x_2 z_2\end{vmatrix}\vec{j}+\begin{vmatrix} x_1 y_1 \\ x_2 y_2\end{vmatrix}\vec{k}$$

formula bilan aniqlanadi.

Isboti. Avvalo koordinata o'qlarining $\vec{i},\vec{j},\vec{k}$ koordinatalar uchun quyidagi munosabatlar o'rinli bo'lishini eslatib o'tamiz :

$$\begin{array}{lll} [\vec{i},\vec{i}]=0, & [\vec{i},\vec{j}]=\vec{k}, & [\vec{i},\vec{k}]=-\vec{j}, \\ [\vec{j},\vec{i}]=-\vec{k}, & [\vec{i},\vec{i}]=0, & [\vec{j},\vec{k}]=\vec{i}, \\ [\vec{k},\vec{i}]=\vec{j}, & [\vec{k},\vec{j}]=-\vec{l}, & [\vec{i},\vec{i}]=0. \end{array} \qquad (2.10.2)$$

$\vec{a}$ va $\vec{b}$ vektorlar Dekart koordinatalar sistemasida mos ravishda $\{x_1,y_1,z_1\}$ va $\{x_2,y_2,z_2\}$ koordinatalarga ega bo'lsin, ya'ni

$$\vec{a}=\{x_1,y_1,z_1\}=x_1\vec{l}+y_1\vec{j}+z_1\vec{k},\ \vec{b}=\{x_2,y_2,z_2\}=x_2\vec{l}+y_2\vec{j}+z_2\vec{k}.$$

$[\vec{a},\vec{b}]$ ko'paytmani (2.10.2) ni hamda vektor ko'paytmaning xossalarini e'tiborga olib topamiz:

$$[\vec{a},\vec{b}]=(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})=x_1x_2[\vec{i},\vec{i}]+y_1x_2[\vec{j},\vec{i}]+z_1x_2[\vec{k},\vec{i}]+$$
$$x_1y_2[\vec{i},\vec{j}]+y_1y_2[\vec{j},\vec{j}]+z_1y_2[\vec{k},\vec{j}]+x_1z_2[\vec{i},\vec{k}]+y_1z_2[\vec{j},\vec{k}]+z_1z_2[\vec{k},\vec{k}]$$

yoki

$$[\vec{a},\vec{b}]=-y_1x_2\vec{k}+z_1x_2\vec{j}+x_1y_2\vec{k}-z_1y_2\vec{i}-x_1z_2\vec{j}+y_1z_2\vec{i}.$$

Bir xil ortoginalga ega bo'lgan qo'shiluvchilarni gruppalab yozamiz:

$$[\vec{a},\vec{b}]=(y_1z_2-z_1y_2)\vec{i}-(x_1z_2-z_1x_2)\vec{j}+(x_1y_2-y_1x_2)\vec{k}=$$
$$=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2\end{vmatrix}\vec{i}-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2\end{vmatrix}\vec{j}+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}\vec{k}.$$

Bu ifodani yana uchunchi tartibli determinant ko'rinishda yozish mumkin

$$[\vec{a},\vec{b}]=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2\end{vmatrix}.$$

2.10.3-teorema. $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lishi uchun $[\vec{a},\vec{b}]=0$ bo'lishi zarur va yetarli.

2.10.4-natija. Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa,u holda ularni koordinatalari proporsional bo'ladi, ya'ni $\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}$

2.10.5- teorema (Uchburchak yuzining formulasi). $\vec{a}$ va $\vec{b}$ vektorlarga uchburchak yasalgan bo'lsin,u holda bu uchburchakning yuzi:

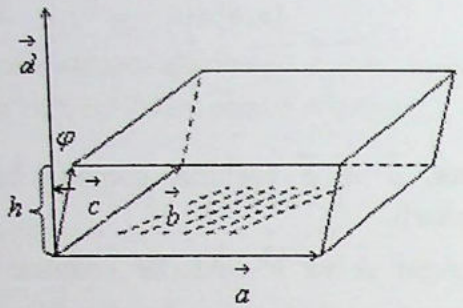
$$S=\frac{1}{2}\left|[\vec{a},\vec{b}]\right|=\frac{1}{2}\mathrm{mod}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2\end{vmatrix}$$

formula bilan topiladi.

2.11. Vektorlarning aralash ko'paytmasi

2.11.1-ta'rif. $\vec{a},\vec{b}$ va $\vec{c}$ vektorlar tartiblangan uchligining *aralsh ko'paytmasi* deb, $[\vec{a},\vec{b}]$ vektor bilan $\vec{c}$ vektorning skalyar ko'paytmasiga teng bo'lgan songa aytiladi va $(\vec{a}*\vec{b})\cdot\vec{c}$ yoki $[\vec{a},\vec{b}]\cdot\vec{c}$ kabi belgilanadi.

Aralash ko'paytmaning miqdor nuqtai nazaridan ma'nosini tekshiramiz. $\vec{a},\vec{b}$ va $\vec{c}$ vektorlar komplanar bo'lmagan vektorlar bo'lsin. $[\vec{a},\vec{b}]=\vec{d}$ deb belgilasak, $\vec{d}$ vektor miqdori $\vec{a}$ va $\vec{b}$ vektorlardan yasalgan parallelogrammning yuziga teng $[\vec{a},\vec{b}]\cdot\vec{c}=\vec{d}\cdot\vec{c}$ bo'lgani uchun skalyar ko'paytma ta'rifiga ko'ra $\vec{d}\cdot\vec{c}=\left|\vec{d}\right|np_{\vec{d}}\vec{c}$. Ammo $np_{\vec{d}}\vec{c}=h$ miqdorning moduli, ya'ni $|h|$ son $\vec{a},\vec{b}$ va $\vec{c}$ vektorlarga yasalgan parallelepipedning balandligini anglatadi. $\vec{c}$ bilan $\vec{d}$ orasidagi burchak o'tkir bo'lsa, h musbat ishora bilan, o'tmas burchak esa manfiy ishorasi bilan olinadi (2.10-chizma).



2.10-chizma.

Shunday qilib $\vec{d}\cdot\vec{c} = |\vec{d}|np_{\vec{d}}\vec{c} = \pm|\vec{d}|h$.

Aralash ko'paytmaning absolyut qiymati shu $\vec{a},\vec{b}$ va $\vec{c}$ vektorlarga yasalgan parallelepiped hajmiga teng, ya'ni $V = |[\vec{a},\vec{b}]\cdot\vec{c}|$.

Endi aralash ko'paytmaning ba'zi xossalarini keltiramiz.

1^0.Ko'paytmada ikki qo'shni vektorning o'rinlari almashtirilsa, aralsh ko'paytmaning ishorasi teskariga almashadi,ya'ni quyidagi tengliklar o'rinli.

$$[\vec{a},\vec{b}]\cdot\vec{c} = -[\vec{b},\vec{a}]\cdot\vec{c} = -[\vec{a},\vec{c}]\cdot\vec{b} = -[\vec{c},\vec{b}]\cdot\vec{a}.$$

2^0. $\vec{a},\vec{b}$ va $\vec{c}$ vektorlarning o'rinlari doiraviy siklda almashtirilsa, aralash ko'paytma o'z ishorasini o'zgartirmaydi, ya'ni ushbu tenglilar o'rinli

$$[\vec{a},\vec{b}]\cdot\vec{c} = [\vec{b},\vec{c}]\cdot\vec{a} = [\vec{c},\vec{a}]\cdot\vec{b}.$$

3^0.Agar $\vec{a},\vec{b}$ va $\vec{c}$ vektorlardan istalgan ikkitasi bir-biriga teng yoki parallel (kollinear) bo'lsa,ularning aralash ko'paytmasi nolga teng bo'ladi, ya'ni $[\vec{a},\vec{b}]\cdot\vec{c} = 0$.

4^0.Agar $\vec{a},\vec{b}$ va $\vec{c}$ vektorlar o'zaro komplanar vektorlar bo'lsa,ularning aralash ko'paytmasi nolga teng bo'ladi, ya'ni $[\vec{a},\vec{b}]\cdot\vec{c} = 0$.

2.12. Proeksiyalari bilan berilgan vektorlarning aralsh ko'paytmasi.

$\vec{a}\{x_1,y_1,z_1\}$, $\vec{b}\{x_2,y_2,z_2\}$, $\vec{c}\{x_3,y_3,z_3\}$ vektorlar o'zlarining koordinatalari bilan berilgan bo'lsin. Bu vektorlarning aralsh ko'paytmasini topaylik. $[\vec{a},\vec{b}]$ vektor bilan $\vec{c}$ vektorning skalyar ko'paytmasi bu vektorlarga mos proeksiyalari ko'paytmalarining yig'indisiga teng ekanini bilamiz. Shuning uchun

$$[\vec{a},\vec{b}]\cdot\vec{c} = x_3\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} + y_3\begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} + z_3\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

yoki

$$[\vec{a},\vec{b}]\cdot\vec{c} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Demak, proeksiyalari bilan berilgan uchta vektorlarning aralsh ko'paytmasi bu vektorlar proeksiyalaridan tuzilgan uchinchi tartibli determinantga teng.

$\vec{a},\vec{b}$ va $\vec{c}$ vektorlarning komplanar bo'lishi uchun

$$[\vec{a},\vec{b}]\cdot\vec{c} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

tenglik bajarilishi zaruriy va yetarli.

2-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

2.1. Vektorning ta'rifi, asosiy tushunchalar([14], 37-39 betlar, [4],[6], [9]).

2.2. Vektorlar ustida amallar([14], 39-40 betlar, [5],[4], [9]).

2.3.Vektorni songa ko'paytirish([14], 44-45 betlar, [4],[6], [9]).

2.4. Fazodagi bazis haqida([14], 51-52 betlar, [5],[4], [9])

2.5. Vektorning dekart koordinatalari([14], 51-53 betlar, [4],[6], [9]).

2.6. Chiziqli bog'liq va chiziqli erkli vektorlar([14], 48-50 betlar, [4],[6], [9]).

2.7. Ikki vektorning skalyar ko'paytmasi([14], 65-66 betlar, [4],[6], [9])

2.8. Skalyar ko'paytmaning vektorlar koordinatalari orqali ifodasi([14], 68-69 betlar, [5],[6], [9])

2.9. Ikki vektorning vektor ko'paytmasi([14], 71-75 betlar, [],[6], [9])

2.10.Vektorlarial ko'paytmani ko'paytuvchi vektorlarning oordinatalari orqali ifodalash ([14], 72-75 betlar, [5],[6], [9]).

2.11.Vektorlarning aralash ko'paytmasi([14], 78-79 betlar, [],[6], [9])

2.12. Proeksiyalari bilan berilgan vektorlarning aralsh ko'paytmasi([14], 79-80 betlar, [4],[6], [9]).

2.1-amaliy mashg'ulot.

VEKTORLAR

1-misol Agar $A(3;4;1)$ va $B(6;8;6)$ bo'lsa, $\vec{AB}$ vektorning koordinata o'qlari bilan tashkil etgan burchaklarning kosinuslarini toping.

Yechilishi. $\vec{AB}$ vektorning Ox, Oy, Oz o'qlardagi proyeksiyalarini topamiz:

$$np_{Ox}\vec{AB}=6-3=3;\ np_{Oy}\vec{AB}=8-4=4;\ np_{Oz}\vec{AB}=6-1=5.$$

$\vec{AB}$ vektorning modulini topamiz: $\left|\vec{AB}\right|=\sqrt{3^2+4^2+5^2}=\sqrt{50}=5\sqrt{2}$.

Ushbu

$$\cos\alpha=\frac{a_x}{\sqrt{a_x^{\ 2}+a_y^{\ 2}+a_z^{\ 2}}},\quad \cos\beta=\frac{a_y}{\sqrt{a_x^{\ 2}+a_y^{\ 2}+a_z^{\ 2}}},\quad \cos\gamma=\frac{a_z}{\sqrt{a_x^{\ 2}+a_y^{\ 2}+a_z^{\ 2}}}$$

formulalarga asosan, vektorning yo'naltiruvchi kosinuslarini topamiz:

$$\cos\alpha=\frac{3}{5\sqrt{2}},\ \cos\beta=\frac{4}{5\sqrt{2}},\cos\gamma=\frac{1}{\sqrt{2}}.\blacksquare$$

2-misol. Agar $\left|\vec{a}\right|=3,\ \left|\vec{b}\right|=4$ va $\varphi=\frac{\pi}{3}$ bo'lsa, $2\vec{a}+\lambda\vec{b}$ va $\vec{a}-4\vec{b}$ vektorlar λ ning qanday qiymatlarida o'zaro perpendikulyar bo'ladi?

Yechilishi. Berilgan vektorlarning skalyar ko'paytmasini topamiz:

$$\left(2\vec{a}+\lambda\vec{b}\right)\left(\vec{a}-4\vec{b}\right)=2\vec{a}\vec{a}-8\vec{a}\vec{b}+\lambda\vec{a}\vec{b}-4\lambda b^2$$

$$=2\cdot|a|^2-8\left|\vec{a}\right|\left|\vec{b}\right|\cos\frac{\pi}{3}+\lambda\left|\vec{a}\right|\left|\vec{b}\right|\cos\frac{\pi}{3}-4\lambda|b|^2=$$

$$=18-8\cdot3\cdot4\cdot\frac{1}{2}+\lambda\cdot3\cdot3\cdot\frac{1}{2}-4\cdot\lambda16=18-48+62-64\lambda=$$

$$=-30-58\lambda$$

Ikki vektor perpendikulyar bo'lishi uchun ularning skalyar ko'paytmasi nolga teng bo'lishi shartidan topamiz: $30-58\lambda=0\Rightarrow\lambda=\frac{15}{29}.\blacksquare$

3-misol. $\vec{F}=\{3;2;-4\}$ kuch $A(2;-1;1)$ nuqtaga qo'yilgan. Bu kuchning koordinatalar boshiga nisbatan momentini aniqlang.

Yechilishi. Agar $\vec{F}$ vektor A nuqtaga qo'yilgan bo'lsa, $\vec{a}$ vektor O nuqtadan A nuqtaga yo'nalgan, ya'ni $\vec{OA}=\vec{a}$ bo'ladi. $[\vec{a},\vec{F}]$vektor ko'paytma esa, $\vec{F}$kuchning Onuqtaga nisbatan kuch momentini ifodalaydi.

Demak,

$$\vec{a}=\vec{OA}=\{2;-1;1\},\quad [\vec{OA},\vec{F}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{\kappa}\\ 2 & -1 & 1\\ 3 & 2 & -4\end{vmatrix}=2\vec{i}+11\vec{j}+7\vec{\kappa}.\blacksquare$$

Mustaqil yechish uchun berilgan misol va masalalar

1. Quyida berilgan vektorlarning uzunligi va vektor yo'nalishidagi birlik vektorni toping.

1) $\vec{a} = \{2;-6;3\}$;

2) $\vec{b} = \{4;-5;2\}$;

3) $\vec{c} = \{6;10;0\}$.

2. Ushbu $\vec{a} = \{-2;11;z\}$ vektorning uzunligi 15 ga teng bo'lsa, z ni toping.

3. $A(-2; 5; -4)$, $B(3; -7; 8)$, $C(2; 4; 0)$ nuqtalar berilgan. $\overline{AB}$, $\overline{BA}$, $\overline{AC}$, $\overline{BC}$ va $\overline{CA}$ vektorlarni toping.

4. Agar $\vec{a} = \{-1; 3; 7\}$ vektor va $M(4; -3; 0)$ nuqta berilgan bo'lib, $a = MN$ bo'lsa, N nuqtaning koordinatalarini toping.

5. Agar $|\vec{a}| = 4$ bo'lib, $\vec{a}$ vektorning Ox, Oy, Oz o'qlari bilan mos ravishda $\alpha = 60^0, \beta = 45^0, \gamma = 60^0$ tashkil etsa, $\vec{a}$ vektorining koordinata o'qlaridagi proeksiyalarini toping.

6. Quyida berilgan vektorlarning yo'naltiruvchi kosinuslarini toping.

1) $\vec{a} = \{-3;12;-4\}$;

2) $\vec{b} = \{3;-4;5\}$.

7. Vektor koordinata o'qlari bilan quyidagi burchaklarni tashkil etishi mumkinmi?

1) $\alpha = 225^0, \beta = 240^0, \gamma = 120^0$; 2) $\alpha = 60^0, \beta = 45^0, \gamma = 60^0$;

3) $\alpha = 60^0, \beta = 90^0, \gamma = 120^0$.

8. $\vec{a}$ vektor Ox va Oy o'qlari bilan $\alpha = 120^0$ va $\beta = 60^0$ tashkil etadi. Agar $|\vec{a}| = 5$ bo'lsa, uning koordinatalarini toping.

9. Agar $M(x,y)$ nuqtani aniqlovchi radius vektori koordinata o'qlari bilan bir xil burchakni tashkil qilib, uning uzunligi $5\sqrt{3}$ ga teng bo'lsa, M nuqtaning koordinatalarini toping.

10. $\vec{a}$ va $\vec{b}$ vektorlar berilgan. Quyidagi berilgan:

1) $2\vec{a}$; 2) $-0{,}5\vec{b}$; 3) $-\vec{a} + \vec{b}$;

4) $0{,}5\vec{a} - 3\vec{b}$ vektorlarni yasang.

11. $|\vec{a}| = 15$, $|\vec{b}| = 25$ va $|\vec{a} + \vec{b}| = 32$ bo'lsa, $|\vec{a} - \vec{b}|$ ni toping.

12. $\vec{a}$ va $\vec{b}$ ***b*** vektorlar 60^0 burchakni tashkil etadi. $|\vec{a}| = 6$, $|\vec{b}| = 3$ ekanligini bilgan holda $(2\vec{a} + \vec{b})(2\vec{a} - 3\vec{b})$ vektorni hisoblang.

13. ABC uchburchakda $\overline{AB}$ vektor $\vec{m}$ ga va $\overline{AC}$ vektor $\vec{n}$ ga teng bo'lsa, quyidagi vektorlarni yasang:

1) $\vec{m} + \frac{\vec{n}}{2}$; 2) $\vec{n} - \frac{\vec{m}}{2}$; 3) $\frac{\vec{m} - \vec{n}}{2}$.

14. $\vec{a} = \{5; -3; 7\}$ va $\vec{b} = \{3; -1; -2\}$ vektorlar berilgan. Quyida berilgan vektorlarning koordinata o'qlaridagi proeksiyalarini toping:

1) $\vec{a} + \vec{b}$; 2) $\vec{a} - \vec{b}$; 3) $-3\vec{a}$;

4) $\frac{1}{3}\vec{b}$; 5) $2\vec{a} - 3\vec{b}$; 6) $\frac{1}{3}\vec{a} + \vec{b}$.

15. α va β ning qanday qiymatlarida $\vec{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + \alpha\boldsymbol{k}$ va $\vec{b} = \beta\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$ vektorlar kolleniar bo'ladi?

16. $\vec{c} = 11\vec{i} + 1\vec{j} - 10\vec{k}$ vektor berilgan. $\vec{c}$ vektorga parallel, unga qarama-qarshi yo'nalgan va uzunligi 45 ga teng bo'lgan $\vec{d}$ vektorni toping.

17. $\vec{AB} = \{4; 6; 2\}$ va $\vec{AC} = \{-8; 10; -12\}$ vektorlar ABC uchburchakning tomonlari bilan ustma- ust tushadi. Boshlari uchburchakning uchlarida va medianalar bilan ustma- ust tushgan vektorlarning koordinatalarini toping.

18. $\vec{a} = \{1; -2\}$, $\vec{b} = \{-2; 3\}$, $\vec{c} = \{-4; 7\}$ vektorlar berilgan. Har bir vektorni, qolgan ikki vektorlarni bazis deb olganda, yoyilmalarini aniqlang.

19. Tekislikda $A(2; -1)$, $B(-1; -2)$, $C(-2; -3)$, $D(-3; 2)$ nuqtalar berilgan. $\vec{AB}$ va $\vec{AC}$ vektorlarni bazis vektorlari deb, quyidagi vektorlarning yoyilmalarini toping:

1) $\vec{AD}$; 2) $\vec{BD}$; 3) $\vec{CD}$;

4) $\vec{BD} + \vec{CD}$; 5) $\vec{AB} + \vec{BD} + \vec{CD}$.

20. $\vec{a} = \{1; -3; 2\}$, $\vec{b} = \{-2; 1; 3\}$, $\vec{c} = \{1; -2; -1\}$ vektorlar berilgan. $\vec{d} = \{-6; 5; 11\}$ vektorning $\vec{a}$, $\vec{b}$, $\vec{c}$ bazis vektorlari bo'yicha yoyilmasini toping.

21. $\vec{a} = \{2; -1; 2\}$, $\vec{b} = \{1; 0; 2\}$, $\vec{c} = \{7; -7; 3\}$, $\vec{d} = \{-1; 2; 1\}$ vektorlar berilgan. Bu vektorlarning har birining yo'yilmasini qolgan uchta vektorlarni bazis vektori deb qabul qilgan holda toping.

22. $\vec{a}$ va $\vec{b}$ vektorlar $\frac{\pi}{3}$ burchakni tashkil qiladi. $|\vec{a}| = 5$, $|\vec{b}| = 6$ ekanligini bilgan holda quyidagilarni hisoblang.

1) $(\vec{a}, \vec{b})$; 2) $\vec{a}^2$; 3) $\vec{b}^2$;

4) $(\vec{a} + \vec{b})^2$; 5) $(\vec{a} - \vec{b})^2$;

6) $(2\vec{a} - 3\vec{b}, 2\vec{a} + 3\vec{b})$; 7) $(2\vec{a} - 3\vec{b})^2$.

23. $\vec{a}$ va $\vec{b}$ vektorlar o'zaro ortogonal bo'lib, $\vec{c}$ vektor bilan esa $\frac{2\pi}{3}$ burchakni tashkil etadi. $|\vec{a}| = 2, |\vec{b}| = 4, |\vec{c}| = 8$ ekanini bilgan holda, quyidagilarni hisoblang.

1) $(2\vec{a} + 3\vec{b}, 2\vec{b} - 3\vec{c})$;

2) $(\vec{a} + \vec{b} - \vec{c})^2 \vec{a}$;

3) $(2\vec{a} - 3\vec{b} + 4\vec{c})^2$;

4) $(\vec{a} - \vec{b} - \vec{c})^2$

24. Ushbu $(\bar{a} + \bar{b})^2 + (\bar{a} - \bar{b})^2 = 2(|\bar{a}|^2 + |\bar{b}|^2)$ ayniyatni isbotlang va uning geometrik ma'nosini aniqlang.

25. Ushbu $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ shartni qanoatlantiruvchi birlik $\vec{a}, \vec{b}$ va $\vec{c}$ vektorlar berilgan. $(\bar{a}, \bar{b}) + (\bar{b}, \bar{c}) + (\bar{c}, \bar{a})$ ni hisoblang.

26. $|\bar{a}| = 7$, $|\bar{b}| = 7$ ekanini bilgan holda, α ning qanday qiymatlarida $\bar{a} + \alpha \cdot \bar{b}$ va $\bar{a} - \alpha \cdot \bar{b}$ vektorlar perpendikulyar bo'ladi.

27. *ABC* uchburchakning tomonlari bilan ustma-ust tushgan $\bar{b} = \vec{AB}$, $\bar{c} = \vec{AC}$ vektorlar berilgan. Boshi *C* uchida va *CD* balandlik bilan ustma-ust tushgan vektorning $\bar{b}, \bar{c}$ bazis vektorlari bo'yicha yoyilmasini toping.

28. $\bar{a}$ va $\bar{b}$ vektorlar orasidagi burchak $\frac{2\pi}{3}$ burchakni tashkil etadi. $|\bar{a}| = 3$, $|\bar{b}| = 4$ ekanini bilgan holda, $\bar{a} - \bar{b}$ va $\bar{a} + \bar{b}$ vektorlar orasidagi burchakni toping.

29. $\bar{a} = \{2; -4; -2\}, \bar{b} = \{4; -3; 2\}$ vektorlar berilgan. Quyidagilarni hisoblang:

1) $(\bar{a}, \bar{b})$; 2) $\sqrt{\bar{a}^2}$; 3) $\sqrt{\bar{b}^2}$;

4) $(3\bar{a} - \bar{b}, 2\bar{a} + 3\bar{b})$; 5) $(\bar{a} + \bar{b})^2$; 6) $(\bar{a} - \bar{b})^2$

30. To'rtburchakning *A*(3; -1; 2), *B*(2; 1; -3), *C*(5; -4; 7) va *D*(6; 7; -1) uchlari berilgan. Uning *AC* va *BD* diagonallari perpendikulyar ekanligini isbotlang.

31. *ABC* uchburchakning *A*(2; -3; 1), *B*(5; -3; 5), *C*(9; -3; 2) uchlari be rilgan. *A* uchidagi ichki burchakni toping.

32. *ABC* uchburchakning *A*(4; -2; 7), *B*(2; 1; 1) va *C*(-1; 7; 3) uchlari berilgan. *ABC* uchburchakning ichki burchaklarini hisoblab, uning teng yonli uchburchak ekanligini isbotlang.

33. $\vec{a} = \{2; -6; 3\}$ va $\bar{x}$ vektorlar o'zaro kollenear bo'lib, *Oz* o'qi bilan o'tkir burchakni tashkil qiladi. $|\vec{x}| = 42$ ekanini bilgan holda $\bar{x}$ vektorning koordinatalarini toping.

34. $\bar{a} = \{2; -3; 4\}, \bar{b} = \{-1; 2; 3\}$ vektorlar berilgan. $(\bar{x}, \bar{a}) = -7, (\bar{x}, \bar{b}) = 2$ ekanligini bilgan holda *Oz* o'qqa perpendikulyar bo'lgan $\bar{x}$ vektorni toping.

35. $\vec{c} = \{3; 6; 3\}$ vektorning koordinata o'qlari bilan teng o'tkir burchaklarni tashkil etuvchi o'qdagi proeksiyalarini toping.

36. $\vec{a} = \{4; -2; -5\}$, $\vec{b} = \{6; 1; -3\}$, $\vec{c} = \{4; -12; 3\}$ vektorlar berilgan. $\vec{a} + \vec{b}$ vektorning $\vec{c}$ vektordagi proeksiyalarini toping.

37. $\vec{a} = \{4; -2; -4\}$, $\vec{b} = \{-1; -4; 1\}$ va $\vec{c} = \{5; -8; -4\}$ vektorlar berilgan. $\vec{a}$ vektorning $\vec{b} + \vec{c}$ vektorlardagi proeksiyalni toping.

39. $\vec{a}$ va $\vec{b}$ vektorlar $\frac{5\pi}{6}$ burchakni tashkil etadi. $|\bar{a}| = 3$ va $|\bar{b}| = 4$ ekanini bilgan holda $|[\bar{a}, \bar{b}]|$ ni hisoblang.

40. $|\vec{a}| = 6, |\vec{b}| = 12$ va ular 36^0 burchak tashkil etishini bilgan holda $|[\bar{a}, \bar{b}]|$ ni hisoblang.

41. O'zaro ortogonal bo'lgan $\vec{a}$ va $\vec{b}$ vektorlarning uzunliklari $|\vec{a}| = 2, |\vec{b}| = 3$ ekanini bilgan holda quyidagilarni hisoblang:

1) $|[2\bar{a} - \bar{b}, \bar{a} - 2\bar{b}]|$; 2) $[\bar{a} - 3\bar{b}, 3\bar{a} + \bar{b}]^2$.

42. Ushbu $[\bar{a},\bar{b}]^2+(\bar{a},\bar{b})^2=a^2b^2$ ayniyatni isbotlang.

43. $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ shartni qanoatlantiradi. Ushbu $[\bar{a},\bar{b}]=[\bar{b},\bar{c}]=[\bar{c},\bar{a}]$ o'rinli ekanligini isbotlang.

44. $\vec{a}=\{2;1;-3\}$ va $\vec{b}=\{1;-2;-1\}$ vektorlar berilgan. Quyidagi vektor ko'paytmalarining koordinatalarini toping.

1) $[\bar{a}-\bar{b}]$; 2) $[\bar{a},\bar{a}-2\bar{b}]$; 3) $[\bar{a}-2\bar{b},\bar{a}+2\bar{b}]$; 4) $[3\bar{a}-2\bar{b},2\bar{a}+3\bar{b}]$.

45. ABC uchburchakning $A(1;2;-1)$, $B(3;-1;2)$ va $C(-2;3;5)$ uchlari berilgan. ABC uchburchakning yuzini hisoblang.

46. ABC uchburchakning $A(1;-2;-1)$, $B(6;-2;-5)$, $C(-3;1;-1)$ uchlari berilgan. B uchudan AC tomoniga tushirilgan balandlikning uzunligini toping.

47. $\vec{a}$ vektor $\vec{b}$ va $\vec{c}$ vektorlar bilan o'zaro perpendikulyar bo'lib, $\vec{b}$ va $\vec{c}$ vektorlar $\frac{5\pi}{6}$ burchakni tashkil qiladi. $|\vec{a}|=5, |\vec{b}|=8, |\vec{c}|=3$ ekanligini bilgan holda $(\vec{a},[\vec{b},\vec{c}])$ aralash ko'paytmani hisoblang.

48. $\vec{a}=\{2;-3;1\}$, $=\{-1;2;4\ \vec{b}\}$, $\vec{c}=\{3;-5;2\}$ vektorlar berilgan. $([\vec{a},\vec{b}],\vec{c})$aralash ko'paytmani hisoblang.

49. Ushbu $A(1;2;3)$, $B(1;0;5)$, $C(2;1;-1)$ va $D(2;-1;1)$ nuqtalarning bitta tekislikda yotishini isbotlang.

50. Uchlari $A(2;-1;3)$, $B(1;3;4)$, $C(-1;1;2)$, $D(5;4;5)$ nuqtalarda joylashgan tetraedrning hajmini toping.

51. Piramidaning $A(3;7;6)$, $B(3;1;2)$, $C(-4;8;-5)$, $D(1;-2;4)$ uchlari berilgan. C uchudan tushirilgan piramidaning balandligini toping.

52. $ABCD$ tetraedrning uchta $A(1;-3;-2)$, $B(3;-1;4)$, $C(2;-3;4)$ uchlari va uning hajmi 3 ga teng. Tetraedrning D uchi Ox o'qiga tegishli ekanini bilgan holda, uning koordinatalarini toping.

Mustaqil yechish uchun misol va masalalarning javoblari

1. 1)7;$\{\frac{2}{7},-\frac{6}{7},\frac{3}{7}\}$; 2)13;$\{\frac{4}{13},-\frac{3}{13},-\frac{12}{13}\}$; 3)15; $\{\frac{2}{15},\frac{2}{3},\frac{11}{15}\}$. **2.** ±10.

3. a) $\{5,-12,12\}$; b) $\{-5,12,-12\}$; c) $\{4,-1,4\}$; d) $(-1,11,-8)$; e) $\{-4,1,-4\}$.

4. $N(3,0,7)$. **5.** $\{2;2\sqrt{2};2\}$. **6.** 1) $-\frac{3}{13};\frac{12}{13};-\frac{4}{13}$; 2) $\frac{3}{5\sqrt{2}};-\frac{4}{5\sqrt{2}};\frac{1}{\sqrt{2}}$;

7. 1) Ha; 2) Ha; 3) Yo'q. **8.** $(-\frac{5}{2};\frac{5}{2};\pm\frac{5}{\sqrt{2}})$. **9.** $\pm\{5;5;5\}$. **11.** 26.

12 81. **14.** 1) $\{8;-4;5\}$;2) $\{2;-2;9\}$; 3) $\{-15;9;-21\}$; 4) $\{1;-\frac{1}{3};-\frac{2}{3}\}$;

5) $\{1;-3;20\}$;6) $\{\frac{14}{3};-2;\frac{1}{3}\}$. **15.** 4; -4,5. **16.** $\{-33;-6;30\}$. **17.** $\{-2;2;-5\}$; $\{-8;11;-8\}$; $\{10;-13;13\}$. **18.** $\vec{a}=0{,}5\vec{b}-0{,}5\vec{c}$; $\vec{b}=2\vec{a}+\vec{c}$; $\vec{c}=-2\vec{a}+\vec{b}$. **19.** 1) $\{11;-7\}$;2) $\{10;-7\}$; 3) $\{11;-8\}$; 4) $\{21;-15\}$; 5) $\{32;-22\}$. **20.** $\vec{c}=\vec{p}+2\vec{q}-3\vec{r}$. **21.** $\vec{a}=-2\vec{b}+\vec{c}+3\vec{d}$, $\vec{b}=-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{c}+\frac{3}{2}\vec{d}$, $\vec{c}=\vec{a}+2\vec{b}-3\vec{d}$, $\vec{b}=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}-\frac{1}{3}\vec{d}$. **22.** 1) 15; 2) 25; 3) 36; 4) 91; 5) 31;6) -118; 7) 244 **23.** 1) 240; 2) 132;3) 1440;4) 68 . **25.** -1,5.

26. $\alpha=\pm1$. **27.** $\frac{(\vec{b},\vec{c})}{|b|^2}\vec{b}-\vec{c}$. **28.** $\arccos(-\frac{7}{\sqrt{481}})$. **29.** 1) 16; 2) $\sqrt{24}$; 3) $\sqrt{29}$; 4) 169; 5) 85; 6) 21. **31.** $\frac{\pi}{4}$. **33.** $\{12;-36;18\}$. **34.** $\{-8;-3;0\}$.

35. $\{3\sqrt{2};3\sqrt{2};3\sqrt{2}$. **37.** 4. **39.** 6. **40.** $36\sqrt{3}$. **41.** 1) 18; 2) 5184.

44. 1) $\{7;1;5\}$; 2) $\{14;2;10\}$; 3) $\{-28;-4;-20\}$; 4) $\{-98;-13;-69\}$.

46. $h=5$. **47.** 60. **48.** 5. **50.** 3. **51.** $\frac{22}{3}$. **52.** $\frac{13}{3}$.

2-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

$\bar{a},\bar{b}$ va $\bar{c}$ vektorlar berilgan. Quyidagilarni bajaring: a) ikki vektorning skalyar ko'paytmasini hisoblang; b) berilgan vektorlarning kollinear yoki ortogonal bo'lish yoki bo'lmasligini tekshiring; c) vektor ko'paytmasining uzunligini toping; g) uch vektorning aralash ko'paytmasini hisoblang; d) berilgan vektorlar komplanar bo'lish yoki bo'lmasligini tekshiring.

1. $\vec a = 2\vec i - 3\vec j + \vec k$, $b=\vec j + 4\vec k$, $\vec c = 5\vec i + 2\vec j - 3\vec k$:

a) $\vec b, -4\vec c$; b) $\vec a, \vec c$; c) $3\vec a, 2\vec c$; g) $\vec a, 3\vec b, \vec c$; d) $\vec a, 2\vec b, \vec c$.

2. $\vec a = 3\vec i + 4\vec j + \vec k$, $\vec b = \vec i - 2\vec j + 7\vec k$, $\vec c = 3\vec i - 6\vec j + 21\vec k$:

a) $\vec a, \vec c$, b) $\vec b, \vec c$; c) $4\vec b, 2\vec c$; g) $5\vec a, 2\vec b, \vec c$; d) $2\vec a, -3\vec b, \vec c$.

3. $\vec a = 2\vec i - 4\vec j - 2\vec k$, $\vec b = 7\vec i + 3\vec j$, $\vec c = 3\vec i + 5\vec j - 7\vec k$:

a) $-2\vec a\vec c$, b) $\vec a, \vec c$; c) $3\vec a, -7\vec b$; g) $\vec a, 2\vec b, 3\vec c$; d) $3\vec a, 2\vec b, 3\vec c$.

4. $\vec a = -7\vec i + 2\vec k$, $\vec b = 2\vec i - 6\vec j + 4\vec k$, $\vec c = \vec i - 3\vec j + 2\vec k$:

a) $2\vec a, -7\vec c$, b) $\vec b, \vec c$; c) $4\vec b, 3\vec c$; g) $\vec a, -2\vec b, -7\vec c$; d) $2\vec a, 4\vec b, 3\vec c$.

5. $\vec a = -4\vec i + 2\vec j - \vec k$; $\vec b = 3\vec i + 5\vec j - 2\vec k$, $\vec c = \vec j + 5\vec k$:

a) $\vec a, -4\vec c$, b) $\vec a, \vec b$; c) $2\vec b, 2\vec a$; g) $\vec a, 6\vec b, 3\vec c$; d) $\vec a, 6\vec b, 3\vec c$.

6. $\vec a = 3\vec i - 2\vec j + \vec k$, $\vec b = 2\vec j - 3\vec k$, $\vec c = 3\vec i + 2\vec j \quad \vec k$:

a) $2\vec a, 4\vec b$, b) $\vec a, \vec c$; c) $5\vec a, 3\vec c$; g) $\vec a, 3\vec b, 2\vec c$; d) $5\vec a, 4\vec b, 3\vec c$.

7. $\vec a = 4\vec i - \vec j + 3\vec k$, $\vec b = 2\vec i + 3\vec j - 5\vec k$, $\vec c = 7\vec i + 2\vec j + 4\vec k$:

a) $2\vec b, 4\vec c$; b) $\vec b, \vec c$; c) $3\vec b, 5\vec c$; g) $7\vec a, 4\vec b, 2\vec c$; d) $7\vec a, 2\vec b, 5\vec c$.

8. $\vec a = 4\vec i + 2\vec j - 3\vec k$, $\vec b = 2\vec i + \vec k$, $\vec c = 12\vec i - 6\vec j + 9\vec k$:

a) $\vec b, 4\vec c$; b) $\vec a, \vec c$; c) $4\vec a, 3\vec b$; g) $2\vec a, 3\vec b, \vec c$; d) $2\vec a, 3\vec b, -4\vec c$.

9. $\vec a = -\vec i + 5\vec k$, $\vec b = -3\vec i + 2\vec j + 2\vec k$, $\vec c = -2\vec i - 4\vec j + \vec k$:

a) $3\vec a, 2\vec b$; b) $\vec b, \vec c$; c) $7\vec a, -3\vec c$; g) $3\vec a, -4\vec b, 2\vec c$; d) $7\vec a, 2\vec b, -3\vec c$.

10. $\vec a = 6\vec i - 4\vec j + 6\vec k$, $\vec b = 9\vec i - 6\vec j + 9\vec k$, $\vec c = \vec i - 8\vec k$:

a) $3\vec a, -5\vec c$, b) $\vec a, \vec b$; c) $3\vec b, -9\vec c$; g) $2\vec a, -4\vec b, 3\vec c$; d) $3\vec a, -4\vec b, -9\vec c$.

11. $\vec a = 5\vec i - 3\vec j + 4\vec k$, $\vec b = 2\vec i - 4\vec j - 2\vec k$, $\vec c = 3\vec i + 5\vec j - 7\vec k$:

a) $-3\vec a, 6\vec c$, b) $\vec b, \vec c$; c) $-2\vec b, 4\vec c$; g) $\vec a, -4\vec b, 2\vec c$; d) $\vec a, -2\vec b, 6\vec c$.

12. $\vec a = -4\vec i + 3\vec j - 7\vec k$, $\vec b = 4\vec i + 6\vec j - 2\vec k$, $\vec c = 6\vec i + 9\vec j - 3\vec k$:

a) $5\vec a, -3\vec b$, b) $\vec b, \vec c$; c) $4\vec b, 7\vec c$; g) $-2\vec a, \vec b, -2\vec c$; d) $-2\vec a, 4\vec b, 7\vec c$.

13. $\vec a = -5\vec i + 2\vec j - 2\vec k$, $\vec b = 7\vec i - 5\vec k$, $\vec c = 2\vec i + 3\vec j - 2\vec k$:

a) $8\vec a, -6\vec c$, b) $\vec a, \vec c$; c) $3\vec b, 11\vec c$; g) $2\vec a, 4\vec b, -5\vec c$; d) $8\vec a, -3\vec b, 11\vec c$.

14. $\vec a = -4\vec i - 6\vec j + 2\vec k$, $\vec b = 2\vec i + 3\vec j - \vec k$, $\vec c = -\vec i + 5\vec j - 3\vec k$:

a) $3\vec a, -7\vec c$, b) $\vec a, \vec b$; c) $-4\vec i, 11\vec a$; g) $5\vec a, -\vec b, 3\vec c$; d) $3\vec a, -9\vec b, 4\vec c$.

15. $\vec a = -4\vec i + 2\vec j - 3\vec k$, $b = -3\vec j + 5\vec k$, $\vec c = 6\vec i + 6\vec j - 4\vec k$:

a) $3\vec a, 9\vec b$, b) $\vec a, \vec c$; c) $-7\vec a, 4\vec c$; g) $\vec a, 3\vec b, \vec c$; d) $\vec a, 2\vec b, \vec c$.

16. $\vec a = -3\vec i + 8\vec j$, $\vec b = 2\vec i + 3\vec j - 2\vec k$, $\vec c = 8\vec i + 12\vec j - 8\vec k$:

a) $3\vec b, -8\vec c$ b) $\vec b, \vec c$; c) $-7\vec a, 9\vec c$; g) $4\vec a, -6\vec b, 6\vec c$; d) $4\vec a, -6\vec b, 9\vec c$.

17. $\vec a = 2\vec i - 4\vec j - 2\vec k$, $\vec b = -9\vec i + 4\vec k$, $\vec c = 3\vec i + 5\vec j - 7\vec k$:

a) $3\vec b, -8\vec b$ b) $\vec a, \vec c$; c) $-5\vec a, 4\vec b$; g) $7\vec a, 5\vec b, -\vec c$; d) $\vec a, 2\vec b, \vec c$.

18. $\vec a = 9\vec i - 3\vec j + \vec k$, $\vec b = -9\vec i + 4\vec k$, $\vec c = \vec i - 5\vec j + 7\vec k$:

a) $7\vec a, 5\vec b$, b) $\vec b, \vec c$; c) $-6\vec a, 4\vec c$; g) $2\vec a, -7\vec b, 3\vec c$; d) $2\vec a, -7\vec b, 4\vec c$.

19. $\vec a = -2\vec i + 4\vec j - \vec k$, $\vec b = 5\vec i + \vec j - 2\vec k$, $\vec c = 7\vec i + 4\vec j - \vec k$:

a) $-9\vec a, 7\vec c$, b) $\vec a, \vec b$; c) $-8\vec b, 5\vec c$; g) $\vec a, -6\vec b, 2\vec c$; d) $\vec a, -6\vec b, 5\vec c$.

20. $\vec a = -9\vec i + 4\vec j - 5\vec k$, $\vec b = \vec i - 2\vec j + 4\vec k$, $\vec c = -5\vec i + 10\vec j - 20\vec k$:

a) $9\vec a, 4\vec c$, b) $\vec b, \vec c$; c) $-6\vec b, 7\vec c$; g) $-2\vec a, 7\vec b, 5\vec c$; d) $-2\vec a, 7\vec b, 4\vec c$.

21. $\vec a = 2\vec i - 7\vec j + 5\vec k$, $\vec b = -\vec i + 2\vec j - 6\vec k$, $\vec c = 3\vec i + 2\vec j - 4\vec k$:

a) $7\vec a, -4\vec b$, b) $\vec b, \vec c$, c) $5\vec b, 3\vec c$; g) $-3\vec a, 6\vec b, -\vec c$; d) $7\vec a, -4\vec b, 3\vec c$.

22. $\vec a = 7\vec i - 4\vec j - 5\vec k$, $\vec b = \vec i - 11\vec j + 3\vec k$, $\vec c = 5\vec i + 5\vec j + 3\vec k$:

a) $-4\vec a, -5\vec c$, b) $\vec a, \vec c$; v) $3\vec b, 6\vec c$; g) $3\vec a, -7\vec b, 2\vec c$; d) $-4\vec a, 2\vec b, 6\vec c$.

23. $\vec a = 3\vec i - \vec j + 2\vec k$, $\vec b = -\vec i + 5\vec j - 4\vec k$, $\vec c = 6\vec i - 2\vec j + 4\vec k$:

a) $-5\vec a, 4\vec c$, b) $\vec a, \vec b$; v) $6\vec a, -7\vec b$; g) $6\vec a, 3\vec b, 8\vec c$; d) $-5\vec a, 3\vec b, 4\vec c$.

24. $\vec a = 3\vec i - \vec j + 2\vec k$, $\vec b = -\vec i + 5\vec j - 4\vec k$, $\vec c = 6\vec i - 2\vec j + 4\vec k$:

a) $-2\vec a, 5\vec b$, b) $\vec a, \vec c$; c) $6\vec a, -4\vec c$; g) $4\vec a, -7\vec b, -2\vec c$; d) $6\vec a, -7\vec b, -2\vec c$.

25. $\vec a = -3\vec i - \vec j - 5\vec k$, $\vec b = 2\vec i - 4\vec j + 6\vec k$, $\vec c = \vec i - 2\vec j + 3\vec k$:

a) $\vec a, -4\vec c$, b) $\vec a, \vec c$; c) $6\vec b, 3\vec c$; g) $-3\vec a, 4\vec b, -5\vec c$; d) $-2\vec a, 5\vec b, -6\vec c$.

26. $\vec a = -3\vec i + 2\vec j + 7\vec k$, $\vec b = \vec i - 5\vec k$, $\vec c = 6\vec i + 4\vec j - \vec k$:

a) $3\vec b, \vec c$ b) $\vec a, \vec c$; c) $5\vec a, -2\vec c$; g) $-2\vec a, \vec b, 7\vec c$; d) $-2\vec a, 3\vec b, 7\vec c$.

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II BO'LIM. ANALITIK GEOMETRIYA

3-bob. TEKISLIKDA ANALITIK GEOMETRIYA

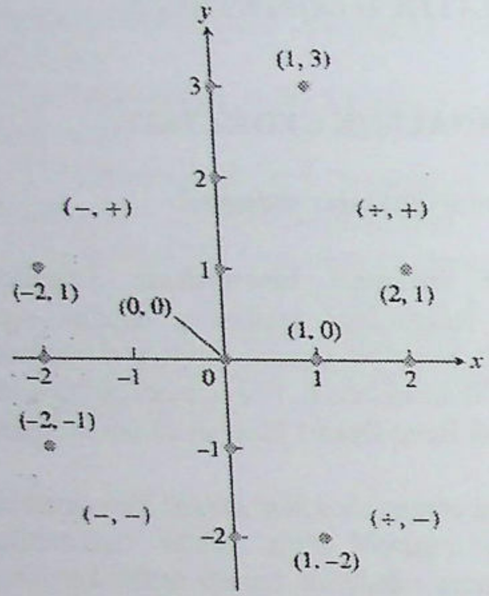
3.1-§.Tekislikda koordinatalar sistemasi

3.1.Tekislikda koordinatalar sistemasi tushunchasi. Tekislikda koordinatalari sistemasi deganda tekislikdagi nuqtaning holatini sonli ifodalash tushuniladi. Bu usullar turlicha bo'lib, ulardan biri to'g'ri burchakli (dekart) koordinatalari sistemasi bo'lib hisoblanadi. Uni birinchi bo'lib XVII asrda fransuz matematigi va faylasufi Rene Dekart kiritgan va uni ko'pgina masalalarga qo'llagan.

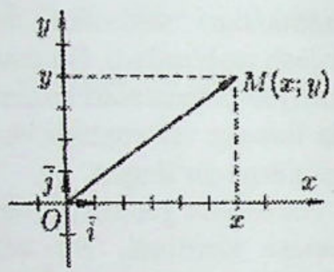
Tekislikda geometrik obyektlarni o'rganish uchun Dekart koordinatalar sistemasi kiritiladi. Bu tushuncha o'quvchi o'rta maktab matematika kurslarida, oliy matematikaning hamma sohalarida hamda ushbu kursning yuqoridagi bo'limlarda berilgan (ishlatilishga qaramasdan uning matematika sohasida muhim urin tutishi e'tiborga olgan holda uni yana bu bandda ham qisqacha keltirib o'tishni lozim topdik. Tekislikda Dekart koordinatalari sistemasi quydagicha kiritiladi: tekislikda o'zaro perpindikulyar bo'lgan ikkita to'g'ri chiziq olinadi, ulardan biri gorizontal, ikkinchisi vertikal joylashgan bo'lib, ularning kesishish nuqtasini O harfi bilan belgilab, uni *koordinatalar boshi* deb ataymiz. O nuqta gorizontal to'g'ri chiziqni ikki qismga ajratib, ularning o'ng tomondagisini musbat yo'nalish, chap tamondagisi esa, manfiy yo'nalish deb hisoblaymiz. Xuddi shunday O nuqta vertikal to'g'ri chiziqni ikki qismga ajratadi.Ularning yuqoridagi qismini musbat yo'nalish, pastdagi qismini manfiy yo'nalish deb qaraymiz (3.1-chizma).

Odatda gorizontal chiziq Ox (yoki *abssissa) o'qi*, vertikal chiziq esa Oy (yoki *ordinata) o'qi* deyiladi. Koordinatalar o'qlari tekislikni to'rta choraklarga ajratadi. Ular 3.2-chizmada ko'rsatilganidek masshtab birligini tayinlab tekislikda ixtiyoriy M nuqtani olamiz.Uning holati bitta son bilan emas, balki ikkita son bilan aniqlanadi. M nuqtani Ox va Oy o'qlariga perpendikulyar tushiramiz va ularning koordinata o'qlari bilan kesishish nuqtalari mos ravishda M_x va M_y bilan belgilaymiz.

Ox o‘qdagi M_x nuqtani ifodalovchi soni x bilan (x son M_x nuqta O nuqtadan o‘ngda bo‘lsa, musbat, chapda bo‘lsa, manfiy bo‘ladi)



3.1-chizma



3.2-chizma

Xuddi shunday Oy o‘qidagi M_y nuqtani ifodalovchi soni y deymiz (y son M_y nuqta Oy o‘qida yuqorida bo‘lsa, musbat, pastda bo‘lsa manfiy bo‘ladi).

M_x va M_y nuqtalar mos ravishda Ox va Oy o‘qlarida x va y sonlari aniqlaydi. Bu x va y sonlaridan tashkil topgan (x,y) juftlik M nuqaning koordinatalari, ya’ni x ga M nuqtaning *abssissasi* , y ga esa *ordinatasi* deyiladi va u $M = M(x,y)$ kabi yoziladi.

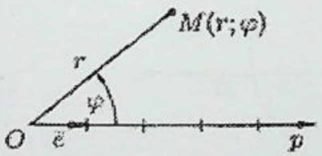
3.1.1-eslatma. Ox o‘qidagi nuqtalarning koordinatalari $(x,0)$, Oy ordinata o‘qidagi nuqtalarning koordinatalari $(0,y)$, koordinatalar boshining koordinatasi $(0,0)$ bo‘ladi .

Shunday kilib, tekislikdan olingan har bir nuqta x va y xaqiqiy sonlardan tuzilgan (x,y) -juftlikni aniqlayda va aksincha har bir (x,y)-juftlik tekislikdagi bita nuqtani ifoda qiladi. Bunday tuzilgan sistema odatda *Dekart koordinatalari sistemasi* deb yuritiladi. Koordinata o‘qlarining birlik vektorlarini $\vec{i}$ va $\vec{j}$ bilan belgilaymiz.

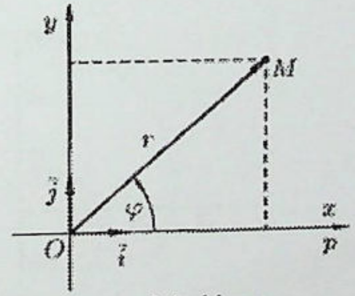
Dekart koordinatalari sistemasidan tashqari amalyotda ko‘p qo‘llaniladigan qutb koordinatalari deb ataluvchi sistemani qisqacha keltiramiz.

3.2. Qutb koordinatalar sistemasi haqida tushuncha. Tekislikdagi nuqtalarning o‘rnini qutb koordinatalar sistemasida aniqlash uchun biror O nuqta olib, bu nuqtadan Ox to‘g‘ri chiziqni o‘tkazamiz va bu to‘g‘ri chiziqda musbat yo‘nalishni belgilaymiz (3.3-chizma). O nuqtani *qutb* deb, Ox o‘qini esa, *qutb o‘qi* deb ataymiz.

Endi ma’lum masshtab birligini olib, tekislikdagi ixtiyoriy M nuqtaning o‘rnini O qutbga va Ox qutb o‘qiga nisbatan aniqlaymiz. Buning uchun M nuqta bilan O qutbni tutashtiramiz. Natijada qutbdan M nuqtagacha bo‘lgan $|OM|$ masofa va qutb o‘qi bilan $\vec{OM}$ yo‘nalgan kesma orasida $\angle xOM = \varphi$ burchak hosil bo‘ladi. Bunda $\rho = \left|\vec{OM}\right|$, M nuqtaning *qutb radiusi*, φ burchak esa, M nuqtaning *qutb burchagi* deyiladi. φ burchakni trigonometriyadagi burchak deb tushinishga kelishib olamiz, ya’ni bu burchakni ishorasi bilan $\pm 2\pi k$ qo‘shiluvchi aniqligida qaraymiz. ρ va φ ni M nuqtaning qutb koordinatalari deb ataymiz va $M(\rho,\varphi)$ shaklda yozamiz. M nuqta qutb burchagining $-\pi < \varphi < \pi$ tengsizliklarni qanoatlantiradigan qiymati, uning qutb burchagining *bosh qiymati* deb ataladi.



3.3-chizma.



3.4-chizma.

Yuqorida aytilganlarga binoan, qutb koordinatalari uchun $\rho \geq 0$ va $-\pi < \varphi < \pi$ tengsizlik o‘rinli bo‘ladi. Qutb koordinatalariga bunday shartni qo‘ymaslik ham mumkin. Bu holda ρ bilan φ ni umuman $-\infty$ dan $+\infty$ ga o‘zgaradi deb qaraladi. φ $-\infty$ dan $+\infty$ gacha o‘zgarganda qutb koordinatalar sistemasi umumlashgan *qutb koordinatalari* sistemasi deyiladi.

Agar M nuqtaning qutb koordinatalari ma'lum bo'lsa (3.4-chizma), u holda uning Dekart koordinatalari $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ formulalar orqali ifodalanadi va

$$\cos\varphi=\frac{x}{\sqrt{x^2+y^2}},\ \sin\varphi=\frac{y}{\sqrt{x^2+y^2}},\ \ \rho=\sqrt{x^2+y^2},\ \ \varphi=arctg\frac{y}{x} \qquad (3.2.1)$$

formulalar orqali topiladi.

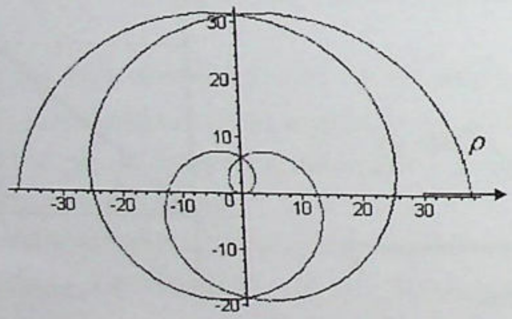
1-misol. $A\left(3;\frac{\pi}{3}\right)$ nuqta yasalsin.

Yechilishi. Ox qutb o'qini o'tkazib, uni $\frac{\pi}{3}$ burchakka buramiz va shu bilan $\overline{OM}$ musbat yo'nalishni aniqlaymiz. Endi $\rho=3$ bo'lgani uchun OM ning 3 birlik masshtabni olamiz, bu kesmaning uchi, izlanayotgan $A\left(3;\frac{\pi}{3}\right)$ nuqtani beradi.■

2-misol. Ushbu $\rho=a\varphi$ $(a>0)$ funksiya grafigini chizing.

Yechilishi. Qutb koordinatalar sistemasida ba'zan funksiyalarning grafiklari nuqtalar bo'yicha chiziladi.

$\varphi>0$ qiymat uchun jadval tuzamiz:



3.5- chizma.

φ	O	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
ρ	O	0,8a	1,6a	2,5a	3,1a	3.9a	4,7a	5,5a	6,3a	7,1a	7,9a	8,7a	9,5a

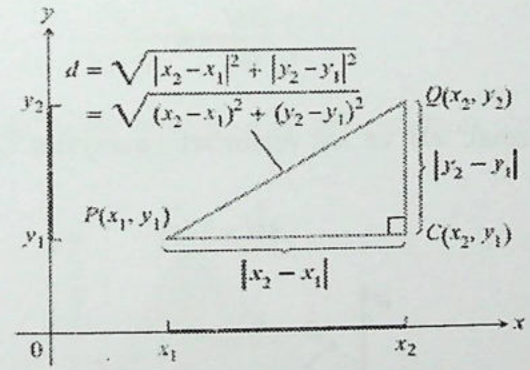
Koordinatalar tekisligida yuqorida topilgan nuqtalarning o'rinlarini topib, ularni chiziqlar bilan birlashtirish, natijada $\varphi>0$ bo'lganda funksiyaning grafigi hosil bo'ladi (3.5-chizma).■

3.3. Tekislikda koordinatalar usulining qo'llanilishi

1⁰.Teislikdagi ikki nuqta orasidagi masofa. Tekislikda ixtiyoriy ikki $P(x_1,y_1)$ va $Q(x_2,y_2)$ nuqtalar berilgan bo'lsin. Berilgan $P(x_1,y_1)$ va $Q(x_2,y_2)$ nuqtalarni to'g'ri chiziq bilan birlashtirib natijada AB kesmani hosil qilamiz. PQ kesmaning uzunligi $P(x_1,y_1)$ va $Q(x_2,y_2)$ nuqtalar orasidagi masofani ifodalaydi va uni d bilan belgilaymiz (3.6-chizma). Berilgan nuqtalarning koordinatalariga ko'ra, ular orasidagi masofa d ni topamiz. 3.6-chizmadagi PQC-to'g'ri burchakli uchburchakdan $PC=x_2-x_1$, $QC=y_2-y_1$. U holda Pifagor teoremasiga asosan, topish talab qilingan masofa

$$d^2=(x_2-x_1)^2+(y_2-y_1)^2, d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad (3.3.1)$$

topiladi.



3.6-chizma

1-misol. A(2,3) va V(3,4) nuqtalar orasidagi masofa topilsin.

Yechilishi. Berilgan nuqtalar orasidagi masofani (3.3.1) formula orqali topamiz:

$$d=\sqrt{(3-2)^2+(4-3)^2}=\sqrt{2}.■$$

2-misol. Qutb koordinatalari sistemasida berilgan $M_1(\rho_1,\varphi_1)$ $M_2(\rho_2,\varphi_2)$ nuqtalar orasidagi masofa topilsin.

Yechilishi. Misol shartiga ko'ra,

$$M_1M_2 = d,\ OM_1 = \rho_1,\ OM_2 = \rho_2,$$
$$\angle POM_1 = \varphi_1,\ \angle M_1OM_2 = \varphi_2 - \varphi_1$$

M_1OM uchburchakga kosinuslar teoremasini qo'llab topamiz.

$$d^2 = \rho_1^{\ 2} + \rho_2^{\ 2} - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1).$$

Bundan

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_2)}\,.\blacksquare$$

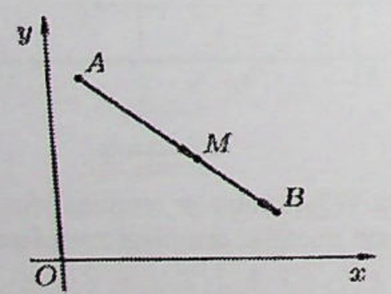
2^0. ***Kesmani berilgan nisbatda bo'lish.*** Tekislikda $A(x_1,y_1)$ va $B(x_2,y_2)$ nuqtalar berilgan bo'lib, ularni to'g'ri chiziq bilan birlashtirish natijasida AB kesma hosil qilingan hamda biror $\lambda\ (\lambda > 0)$ soni berilgan bo'lsin.

AB kesmada shunday $M(x,y)$ nuqtani topish kerakki

$$\frac{AM}{MB} = \lambda \tag{3.3.2}$$

bo'lsin (3.7-chizma). $\vec{AM}$ va $\vec{MB}$ vektorlarini qaraymiz. (3.3.2) dan

$$\vec{AM} = \lambda\vec{MB}.$$



3.7-chizma

Ravshanki, $\vec{AM} = (x - x_1)\vec{i} + (y - y_1)\vec{j}$, $\vec{MB} = (x_2 - x)\vec{i} + (y_2 - y)\vec{j}$,

$$(x - x_1)\vec{i} + (y - y_1)\vec{j} = \lambda(x_2 - x)\vec{i} + \lambda(y_2 - y)\vec{j}.$$

Ikki vektorning tenglik alomatiga asosan, $x - x_1 = \lambda(x_2 - x_1)$, bundan

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \tag{3.3.3}$$

$y - y_1 = \lambda(y_2 - y)$, bunda

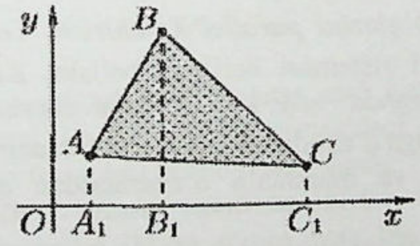
$$y = \frac{y_1 + \lambda y_2}{1+\lambda}. \tag{3.3.4}$$

(3.3.3) va (3.3.4) formulalarga *kesmasi berilgan nisbatta bo'lish formulalari* deyiladi,

Xususiy holda, $\lambda = 1$ bo'lganda, ya'ni $AM = MB$ bo'lganda, ular

$$y = \frac{y_1 + y_2}{2},\ x = \frac{x_1 + x_2}{2} \tag{3.3.5}$$

ko'rinishda bo'ladi. Bu holda $M(x,y)$ nuqta AB kesmaning o'rtasi bo'ladi.



3.8-chizma

3.3.6-eslatma. Agar $\lambda = 0$bo'lsa, u holda A va M nuqtalar ustma-ust tushadi. $\lambda < 0$bo'lsa, u holda M nuqta AB kesmadan tashqarida yotadi.

3⁰. Uchburchakning yuzi. Tekislikda $A(x_1, y_1), B(x_2, y_2)$ va $C(x_3, y_3)$ nuqtalar berilgan bo'lib, uchlari shu nuqtalarida bo'lgan, uchburchakning yuzini topish talab qilingan bo'lsin. Uchburchakning $A(x_1, y_1), B(x_2, y_2)$ va $C(x_3, y_3)$ uchlaridagi Ox o'qiga perpendikulyar tushiramiz: AA_1, BB_1, CC_1 (3.8-chizma). Bunda $OA_1 = x$, $OB_1 = x, OC_1 = x$ va $AA_1 = y$.

Ravshanki,

$$S_{ABC} = \frac{y_1+y_2}{2}(x_2-x_1) + \frac{y_2+y_3}{2}(x_3-x_2) - \frac{y_1+y_3}{2}(x_3-x_1) =$$
$$= \frac{1}{2}(x_2y_1 - x_1y_1 + x_2y_2 - x_1y_2 + x_1y_2 - x_2y_2 + x_3y_3 - x_2y_3 -$$
$$x_3y_1 + x_1y_1 - x_3y_3 + x_1y_3) =$$
$$= \frac{1}{2}[(y_2-y_1)(x_3-x_1) - (y_3-y_1)(x_2-x_1)] = \frac{1}{2}\mathrm{mod}\begin{vmatrix} x_3-x_1 & x_2-x_1 \\ y_3-y_1 & y_2-y_1 \end{vmatrix}$$

tenglik o'rinli bo'ladi .

Demak,

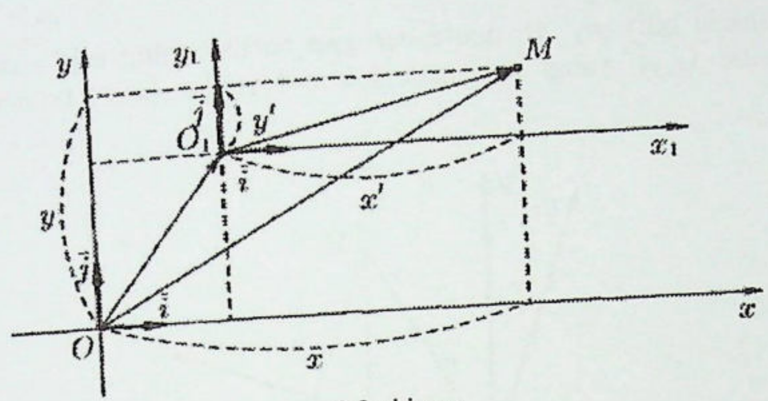
$$S_\Delta = \frac{1}{2}\mathrm{mod}\begin{vmatrix} x_3-x_1 & x_2-x_1 \\ y_3-y_1 & y_2-y_1 \end{vmatrix}. \qquad (3.3.7)$$

(3.3.7)- formulaga *uchburchakning yuzini topish* haqidagi formula deyiladi.

3.4. Koordinatalar sistemalarini almashtirish. Bir koordinatalar sistemasidan ikkinchi koordinatalar sistemasiga o'tishini *koordinata sistemasining almashtirish* deyiladi.

1⁰. Koordinata o'qlarini parallel ko'chirish. Tekislikda xOy to'g'ri burchak koordinatalari sistemasi berilgan bo'lsin. Koodinatalar o'qlarini parallel ko'chirish deganda: xOy koordinatalar sistemasidan yangi $x_1O_1y_1$ koordinatalar sistemasiga o'tishda koordinatoa boshlang'ich holati o'zgaradi, o'qlarning yo'nalishi va masshtabi o'zgarmasdan qoladigan ko'chirish tushuniladi.

Yangi $x_1O_1y_1$ koordinatalar sistemasi boshi O_1 ning koordinatalar (x_0, y_0) bo'lsin, ya'ni $O_1(x_0, y_0)$ tekislikdagi ixtiyoriy M nuqtaning xOy koordinatalar sistemalardagi koordinatalari (x, y), yangi $x_1O_1y_1$ koordinatalar sistemalardagi koordinatalari (x_1, y_1) bo'lsin (3.9-chizma) .



3.9-chizma

Ushbu

$$\vec{OM} = x\vec{i} + y\vec{j}, \quad \vec{OO_1} = x_0\vec{i} + y_0\vec{j}, \quad \vec{O_1M_2} = x'\vec{i} + y'\vec{j}$$

vektorlarni qaraymiz.

Ravshanki , $\vec{OM} = \vec{OO_1} + \vec{O_1M}$, ya'ni

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i} + y'\vec{j}.$$

Bundan $x\vec{i} + y\vec{j} = (x_0 + x')\vec{i} + (y_0 + y')\vec{j}$,

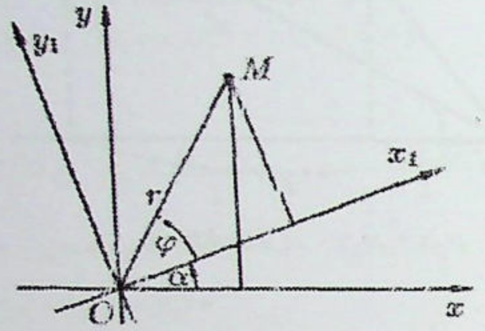
$$\begin{cases} x = x_0 + x', \\ y = y_0 + y'. \end{cases} \qquad (3.4.1)$$

(3.4.1)-formula tekislikdagi nuqtaning eski kordanatlar sistemasidagi koordinatalarini yangi koordinatlar sistemasidagi koordinatalar orqali ifodalash va aksincha formulasi bo'lib hisoblanadi.

2⁰. Koordinatalar o'qlarini burib ko'chirish. Koordinatalar o'qlarini bo'rib ko'chirish deganda: ikkala o'qini ham bir xil burchakga burish, koordinatalar boshi va masshtab birligi o'zgarmasdan qiladigan ko'chirish tushuniladi.

Faraz qilaylik yangi $x_1O_1y_1$ koordinatalar sistemasi eski xOy koordinatalar sistemani α burchakka burish natijasida hosil bo'lgan bo'lsin.

Tekislikda ixtiyoriy M nuqta berilgan bo‘lib, uning eski sistemadagi koordinatalar (x,y), yangi sistemadagi koordinatalar (x',y') bo‘lsin (3.10-chizma).



3.10-chizma

Ikkita O umumiy qutbga (masshtabi bir xil) qutb o‘qlari Ox va Ox_1 bo‘lgan qutb koordinatalari sistemasini qaraymiz. Ikkala sistema uchun ham qutb radiusi r, qutb burchaklari mos ravishda $\alpha+\varphi$ va φ bo‘lsin, bunda φ yangi qutb koordinatali sistemasidagi qutb burchagi.

Qutb koordinatalar sistemasidan to‘g‘ri burchakli Dekart koordinatalari sistemasiga o‘tish formulasiga asosan,

$$\begin{cases} x = r\cos(\alpha+\varphi), \\ y = r\sin(\alpha+\varphi), \end{cases} \quad \text{ya'ni} \quad \begin{cases} x = r\cos\varphi\cos\alpha - r\sin\varphi\sin\alpha, \\ y = r\sin\alpha\cos\varphi + r\sin\varphi\cos\alpha. \end{cases}$$

Bunda $r\cos\varphi = x'$, $r\sin\varphi = y'$ bo‘lgani uchun

$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha, \\ y = x'\sin\alpha + y'\cos\alpha. \end{cases} \qquad (3.4.2)$$

(3.4.2) formula o‘qlari *burish formulasi* deb ataladi. Bu formula tekislikdagi ixtiyoriy $M(x,y)$ nuqtaning eski koordinatalar sistemasidagi koordinatalari (x,y) larni yangi sistemadagi (x',y') koordinatalari orqali ifodalash formulasi bo‘lib hisoblanadi va aksincha.

3.2-§. Tekislikdagi to‘g‘ri chiziq tenglamalari

3.5. To‘g‘ri chiziqning umumiy tenglamasi. Tekislikdagi ixtiyoriy $M_1(x_1,y_1)$ va $M_2(x_2,y_2)$ nuqtalar berilgan bo‘lsin. Bu nuqtalardan teng masofalarda joylashgan nuqtalar to‘plamini qaraylik. Ikki nuqta orasidagi masofoni topish formulasiga ko‘ra,

$$M_1M = \sqrt{(x-x_1)^2+(y-y_1)^2}, \; M_2M = \sqrt{(x-x_1)^2+(y-y_1)^2}$$

bo‘ladi. Shartga ko‘ra, $M_1M = M_2M$ bo‘lgani uchun

$$\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}.$$

Bundan $(x-x_1)^2+(y-y_1)^2 = (x-x_2)^2+(y-y_2)^2$,

$$x^2-2xx_1+x_1^2+y^2-2yy_1+y_1^2 = x^2-2xx_2+x_2^2+y^2-2yy_2+y_2^2,$$
$$2(x_2-x_1)x+2(y_2-y_1)y+(x_1^2+y_1^2-x_2^2-y_2^2)=0$$

bo‘ladi.

Agar $A=2(x_2-x_1)$, $B=2(y_2-y_1)$, $C=x_1^2+y_1^2-x_2^2-y_2^2$ deb belgilasak, u holda keyingi tenglikdan

$$Ax+By+C=0 \qquad (3.5.1)$$

bo‘ladi. (3.5.1) tenglamaga to‘g‘ri chiziqning *umumiy tenglamasi* deyiladi. A,B,C sonlar (3.5.1) tenglamani *koeffisiyentlari* deyiladi. Ular turli qiymatlarga teng bo‘lganda, ularga mos turli to‘g‘ri chiziqlar hosil bo‘ladi.

Demak, to‘g‘ri chiziqning tekislikdagi holati A,B,C sonlar bilan to‘liq aniqlanadi.

Endi, ushbu $Ax+By+C=0$ to‘g‘ri chiziq tenglamasining xususiy hollarini qaraylik.

1^0. (3.5.1) tenglamada $A\neq 0$, $B\neq 0, C=0$ bo‘lsin. U holda (3.5.1) ning ko‘rinishi

$$Ax+By=0 \qquad (3.5.2)$$

bo'ladi. $O(0,0)$ nuqta, ya'ni koordinatalar boshining koordinatalari (3.5.2) tenglamaning qanoatlantiradi. Bu holda, to'g'ri chiziq koordinatalar boshidan o'tadi, ya'ni (3.5.2) tenglama biln ifodalanuvchi to'g'ri chiziqlar koordinatalar boshidan o'tuvchi to'g'ri chiziq lar oilasini tashkil qilar ekan.

2^0.(3.5.1) tenglamada $A=0$, $B\neq 0$, $C\neq 0$ bo'lsin. U holda (3.5.1) tenglama

$$By+C=0 \qquad (3.5.3)$$

ko'rinishga keladi. Bundan $y=-\frac{C}{B}=a$ desak, u holda, (3.5.3) ning ko'rinishi $y=a$ ko'rinishda bo'ladi. Bu holda (3.5.3) to'g'ri chiziq Ox o'qiga parallel bo'lgan to'g'ri chiziq lar oilasini ifoda kilar ekan.

3^0. (3.5.1) tenglamada $B=0, A\neq 0, C\neq 0$ bo'lsin. Bu holda (3.5.1) tenglama

$$Ax+C=0 \qquad (3.5.4)$$

ko'rinishda bo'ladi . Bundan, $x=-\frac{C}{A}=b$ deb belgilasak, (3.5.4) tenglama $x=b$ ko'rinishga keladi. Bu holda (3.5.4) to'g'ri chiziq Oy o'qiga parallel bo'lgan to'g'ri chiziq lar oilasini ifoda qilar ekan.

4^0. (3.5.1) tenglamada $B=0, C=0, A\neq 0$ bo'lsin. Bu holda (3.5.1) tenglamaning ko'rinishi

$$Ax=0, x=0 \qquad (3.5.5)$$

ko'rinishga keladi.

Demak, (3.5.5) to'g'ri chiziq ordinata o'qini (Oy) ifoda qilar ekan.

5^0. (3.5.1) tenglamada $A=0, C=0, B\neq 0$ bo'lsin. Bu holda (3.5.1) tenglamaning ko'rinishi.

$$By=0, y=0 \qquad (3.5.6)$$

ko'rinishga keladi.

Demak, (3.5.6) to'g'ri chiziq abssissa o'qini (Ox) ifoda qilar ekan.

3.6.To'g'ri chziqning burchak koeffitsentli tenglamasi. Tekislikda Dekart koordinatalar sistemasi berilgan bo'lib, bu sistemada Ox o'qini $A(0,0)$ nuqtada kesib o'tuvchi ixtiyoriy l to'g'ri chiziq berilgan bo'lsin (3.11-chizma). Ox o'qini A nuqta atrofida soat strelkasi harakatiga teskari yo'nalishda l to'g'ri chiziq bilan ustma-ust tushguncha aylantirishdan hosil bo'lgan $\varphi\,(0\leq\varphi\leq\pi)$, l to'g'ri chiziq bilan Ox o'qi orasidagi burchak deyiladi.

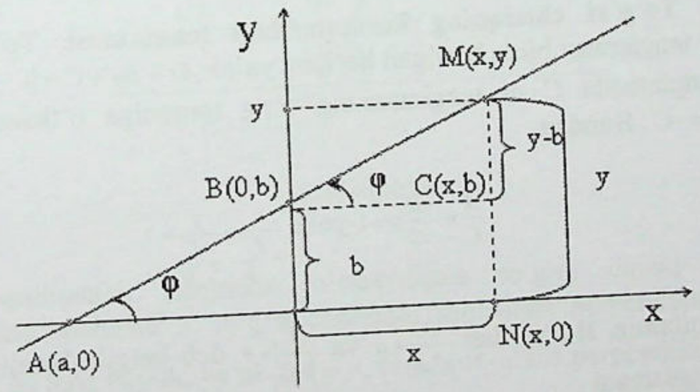
Agar l to'g'ri chiziq Ox o'qiga parallel bo'lsa, u holda bu to'gri chiziq bilan Ox o'qi orasidagi burchak nolga teng deb hisoblanadi. Keyingi mulohazalarimizda avval $\varphi\neq\frac{\pi}{2}$ holni qaraymiz.

Agar l bilan Ox o'qi orasidagi φ burchak va l to'g'ri chiziqning Oy o'qi bilan kesishish $B(0,b)$ nuqtasining ordinatasi b ma'lum bo'lsa, u holda l to'g'ri chiziq tekislikda bir qiymatli aniqlangan bo'ladi.

$M(x,y)\in l$ to'g'ri chiziqning ixtiyoriy nuqtasi bo'lsin. $\varphi\neq\frac{\pi}{2}$ bo'lgani uchun tangensning ta'rifiga ko'ra, $tg\varphi=\frac{MC}{BC}$ yoki $tg\varphi=\frac{y-b}{x}$. Bundan $y=xtg\varphi+b$ bo'lib, $k=tg\varphi$ desak,

$$y=kx+b \qquad (3.6.1)$$

kelib chiqadi.



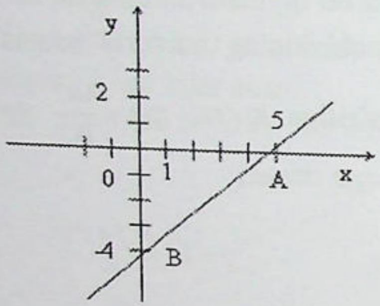
3.11-chizma.

Demak, $y=kx+b$ tenglama l to'g'ri chiziqning tenglamasidir. $k{=}tg\varphi$

miqdorni l to‘gri chiziqning burchak koeffitsenti, (3.6.1) tenglama esa to‘g‘ri chiziqning *burchak koeffitsentliu tenglamasi* deyiladi. (3.6.1) tenglamadagi b son l to‘g‘ri chiziqning Oy o‘qidan ajratgan kesmaning miqdorini anglatadi.

Agar B nuqta Ox o‘qidan yuqorida joylashsa, $b>0$; agar B nuqta Ox o‘qidan pastda joylashsa, $b<0$ bo‘ladi. Agar $b=0$ bo‘lsa, to‘g‘ri chiziq koordinata boshidan o‘tadi va uning tenglamasi $y=kx$ bo‘ladi. Agar $k=0$ bo‘lsa, Ox o‘qiga parallel to‘g‘ri chiziq tenglamasiga ega bo‘lamiz: $y=b$.

Agar $\varphi=\frac{\pi}{2}$ bo‘lsa, bu holda $k=tg\varphi$ son aniqlanmagan bo‘lib, AB to‘g‘ri chiziq Ox o‘qiga perpendikulyar bo‘ladi, uning tenglamasi $x=a$ $y=0\,(x=0)$ tenglamasi Ox (Oy) o‘qining tenglamasi bo‘ladi.



3.12-chizma

3.7. To‘g‘ri chiziqning kesmalardagi tenglamasi. To‘g‘ri chiziq umumiy tenglamasi bilan berilgan bo‘lsin, ya’ni $Ax+By+C=0$ $(A^2+B^2\neq 0)$. Bu tenglamada C ni tenglamaning o‘ng tomoniga o‘tkazaylik, ya’ni $Ax+By=-C$. Bundan

$$-\frac{A}{C}x-\frac{B}{C}y=1 \text{ yoki } \frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1 \qquad (3.7.1)$$

ni hosil qilamiz. Bu yerdan $-\frac{C}{A}=a$ va $-\frac{C}{B}=b$ deb belgilashlarni kiritamiz, (3.7.1) tenglamani

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (3.7.2)$$

ko‘rinishga keltiramiz. (3.7.2) tenglamaga to‘g‘ri chiziqning *kesmalarga nisbatan tenglamasi* deyiladi.

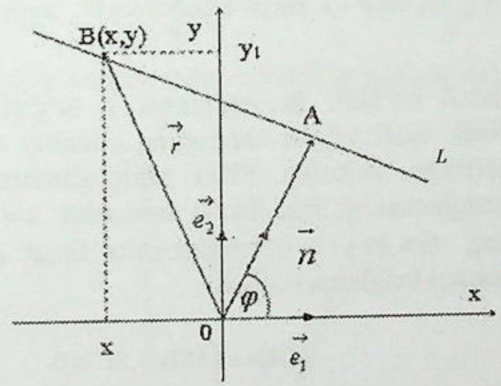
3.7.3. Eslatma. Koordinatalar boshidan va koordinatalar o‘qlariga parallel bo‘lgan to‘g‘ri chiziq tenglamalarini to‘g‘ri chiziqning kesmalardagi tenglamasi ko‘rinishda tasvirlash mumkin emas.

Misol. Ox o‘qidan $OA=5$, Oy o‘qidan $OB=-4$ birlikda kesib o‘tgan AB to‘g‘ri chiziq tenglamasini yozing.

Yechilishi. Faraz qilaylik, (3.7.2) tenglamada $a=5$, $b=-4$ bo‘lsin (3.12-chizma), u holda

$$\frac{x}{5}+\frac{y}{-4}=1 \text{ yoki } \frac{x}{5}-\frac{y}{4}=1. ■$$

3.8. To‘g‘ri chiziqning normal tenglamasi. Koordinatalar boshidan o‘tmaydigan to‘g‘ri chiziqlar uchun ko‘pincha $Ax+Bx+C=0$ tenglamaning maxsus formasidan foydalaniladi.



3.13-chizma

L koordinatalar boshidan o‘tmaydigan to‘g‘ri chiziq bo‘lsin. Koordinatalar boshidan L to‘g‘ri chiziqqa tushirilgan perpendikulyarning uzunligi p ga teng deylik, ya’ni $|\vec{OA}|=p$, bunda A-o‘sha perpendikulyarning asosi. $\vec{n}$ va $\vec{OA}$ vektorlar kolliniar va bir xil yo‘nalgan bo‘lib, $\vec{n}$ vektor

$\left(\left|\vec{n}\right|=1\right)$ $\vec{OA}$ ning birlik vektori bo'lsin. Bu holda $\vec{OA}=p\cdot\vec{n}$ deb yozish mumkin. $\vec{OA}$ vektor bilan $\vec{e_1}$ vektor o'zaro φ burchak hosil qilsin (3.13-chizma). Odatda, $\vec{OA}$ vektorni l to'g'ri chiziqning normali deyiladi. Agar ixtiyoriy $B(x,y)\in l$ nuqtani olsak, u holda $\vec{OB}=\vec{r}$ radius – vektorning koordinatalari (x_1,y_1) bo'ladi, ya'ni $\vec{OB}=\vec{r}=x\vec{e_1}+y\vec{e_2}$. 3.13-chizmada $\vec{AB}=\vec{r}-\vec{OA}$. Bu vektor $\vec{OA}$ vektorga perpendikulyar bo'lgani uchun ushbu tenglikni yoza olamiz:

$$(\vec{r}-\vec{OA})\cdot\vec{n}=0 \text{ yoki } \vec{r}\cdot\vec{n}-\vec{OA}\cdot\vec{n}=0.$$

Agar $\vec{n}$ birlik vektor uchun $\vec{n}=\vec{e_1}\cos\varphi+\vec{e_2}\sin\varphi$ ekanini e'tiborga olsak, yuqoridagi tenglikdan

$$\left(x\vec{e_1}+y\vec{e_2}\right)(\vec{e_1}\cos\varphi+\vec{e_2}\sin\varphi)=\left|\vec{OA}\right|\left|\vec{n}\right|\cos 0^0,\quad x\cos\varphi+y\sin\varphi=p\cdot 1$$

Tenglama hosil bo'ladi. Bu tenglama l to'g'ri chiziqning normal tenglamasi deyiladi. Endi to'g'ri chiziqning umumiy tenglamasini normal tenglama ko'rinishiga keltirish bilan shug'ullanamiz. Buning uchun $Ax+Bx+C=0$ tenglamaning har ikkala tomonini $\lambda\neq 0$ ga ko'paytiramiz (to'g'ri chiziqning $Ax+Bx+C=0$ tenglamasini faqat $A^2+B^2\neq 0$ va $C<0$ bo'lgandagina normal tenglama bo'ladi).

$$(\lambda A)x+(\lambda B)y+\lambda C=0 \qquad (3.8.1)$$

Endi λ ni quyidagicha tanlab olamiz: $\lambda A=\cos\varphi$, $\lambda B=\sin\varphi$, $\lambda C=-P$. Bundan

$$\lambda=\pm\frac{1}{\sqrt{A^2+B^2}} \qquad (3.8.2)$$

O'ng tomondagi $\pm$ ishoralaridan qaysi birini olish $\lambda C=-P$, $P>0$ tenglikka bog'liq bo'ladi. Boshqacha aytganda, λ va C ishorasi qarama – qarshi qilib tanlanadi. λ ning topilgan qiymatini (3.8.1) ga qo'yamiz:

$$\frac{Ax+By+C}{\pm\sqrt{A^2+B^2}}=0$$

(3.8.2) tenglik bilan aniqlangan λ to'g'ri chiziqning *normal ko'paytuvchi* deyiladi.

Misol. Ushbu $3x+4y-5=0$ to'g'ri chiziqning umumiy tenglamasidan normal tenglamasini tuzing.

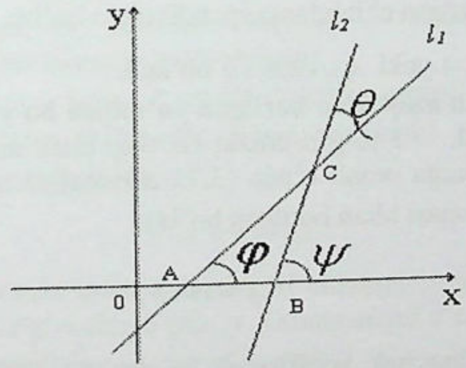
Yechilishi. Avvalo $C=-5<0$ bo'lgani uchun $\lambda>0$ bo'lishi lozim. Shuning uchun (3.8.2) formuladan "+" ishorasini olib quyidagini topamiz:

$$\lambda=\frac{1}{+\sqrt{3^2+42}}=\frac{1}{5}.$$

Berilgan tenglamaning ikkala tomonini bu ko'paytuvchiga ko'paytiramiz

$$\frac{3}{5}x+\frac{4}{5}y-1=0.$$

Bu to'g'ri chiziqning izlanayotgan normal tenglamasidir.■



3.14-chizma

3.9. Ikki to'g'ri chiziq orasidagi burchak. Oy o'qiga parallel bo'lmagan ikkita l_1 va l_2 to'g'ri chiziqlarni qaraymiz. Ular burchak koeffitsentli tenglamalari bilan berilgan bo'lsin:

$l_1: \ y_1 = k_1x + b_1$, bunda $k_1 = tg\varphi$,
$l_2: \ y = k_2x + b_2$, bunda $k_2 = tg\psi$.

l_1 va l_2 to‘g‘ri chiziqlar orasida burchak θ ni topish talab etiladi (3.14-chizma).

l_1 va l_2 to‘g‘ri chiziqlarning kesishish nuqtasini C, Ox o‘q bilan kesishish nuqtalarini esa A va B deb belgilaymiz. Elementar geometriya fanidan ma’lum, ABC uchburchakning tashqi burchagi ψ, o‘ziga qo‘shni bo‘lmagan ikkita ichki burchaklar yig‘indisiga teng bo‘ladi: $\psi = \theta + \varphi$. Bu yerdan $\theta = \psi - \varphi$. Trigonometrik formuladan foydalanib θ burchakni topamiz:

$$tg\theta = tg(\psi - \varphi) = \frac{tg\psi - tg\varphi}{1 + tg\varphi tg\psi} = \frac{k_2 - k_1}{1 + k_1k_2}. \qquad (3.9.1)$$

(3.9.1) formulaga to‘g‘ri chiziqli burchak koeffitsenti bilan berilganda ikki to‘g‘ri chiziq orasidagi burchakni topish formulasi deyiladi.

Agar l_1 va l_2 to‘g‘ri chiziqlar parallel bo‘lsa, u holda $\psi = \varphi$ bo‘ladi.

Demak $k_1 = k_2$, yoki $\frac{A_1}{A_2} = \frac{B_1}{B_2}$.

Agar l_1 va l_2 to‘g‘ri chiziqlar perpendikulyar bo‘lsa, u holda $\varphi = \frac{\pi}{2}$ bo‘ladi.

Demak, $k_2k_1 = -1$ yoki $A_1A_2 + B_1B_2 = 0$ bo‘ladi.

3.10. Berilgan nuqtadan berilgan yo‘nalish bo‘yicha o‘tuvchi to‘g‘ri chiziq tenglamasi. l to‘g‘ri chiziq Ox o‘qi bilan musbat burchak tashkil etib va $P(x_1, y_1)$ nuqta orqali o‘tsin (3.15-chizma). l to‘g‘ri chiziq burchak koeffitsenti tenglamasi bilan berilgan bo‘lsin:

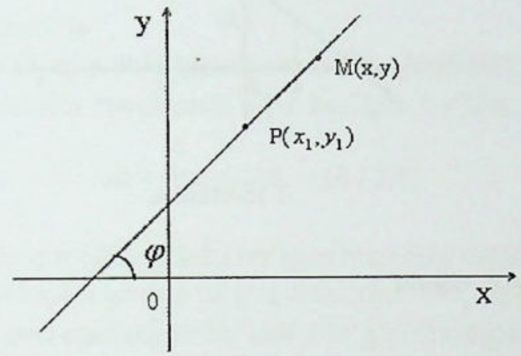
$$y = kx + b, \qquad (3.10.1)$$

bunda $k = tg\varphi$ - burchak koeffitsenti, b esa Oy o‘qini kesib o‘tgandagi kesmaning uzunligi. $P(x_1, y_1)$ nuqta l to‘g‘ri chiziq yotganligi uchun, u (3.10.1) tenglamani qanoatlantiradi:

$$y_1 = kx_1 + b. \qquad (3.10.2)$$

(3.10.1) tenglamadan (3.10.2) tenglamani hadma-had ayiramiz.

$$y - y_1 = k(x - x_1) \qquad (3.10.3)$$



3.15-chizma

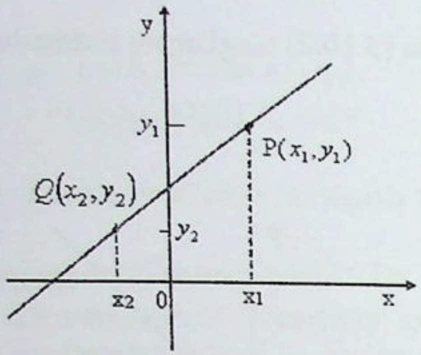
Bu (3.10.3) tenglama esa, izlanayotgan to‘g‘ri chiziq tenglamasidir. Agar l to‘g‘ri chiziq Oy o‘qiga parallel bo‘lib, $P(x_1, y_1)$ nuqta orqali o‘tuvchi to‘g‘ri chiziq bo‘lsa, tenglamasi $x = x_1$ ko‘rinshda bo‘ladi.

3.11. Berilgan ikki nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasi. Berilgan ikki $P(x_1, y_1)$ va $Q(x_2, y_2)$ nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini topish talab qilamiz (3.15-chizma).

Faraz qilaylik $x_1 \neq x_2$, ya’ni P va Q nuqtalar orqaqli o‘tuvchi to‘g‘ri chiziq Oy o‘qiga parallel bo‘lmasin. Bizga ma’lumki $P(x_1, y_1)$ nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasi

$$y - y_1 = k(x - x_1) \qquad (3.11.1)$$

ko‘rinishda bo‘ladi, bunda k-berilgan to‘g‘ri chiziqning nomalum burchak koeffitsenti. Berilgan to‘g‘ri chiziq $Q(x_2, y_2)$ nuqta orqali o‘tadi, u holda bu Q nuqta (3.11.1) tenglamani qanoatlantiradi: ya’ni $y_2 - y_1 = k(x_2 - x_1)$.



3.15-chizma.

Demak, $x_1 \neq x_2$ uchun

$$k = \frac{y_2 - y_1}{x_2 - x_1} \qquad (3.11.2)$$

ega bo'lamiz. Topilgan (3.11.2) koeffitsentni (3.11.1) ga olib borib qo'yamiz, u holda P va Q nuqtalar o'tuvchi to'g'ri chiziq tenglamasiga ega bo'lamiz:

$$y_2 - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Bu tenglamada $y_1 \neq y_2$ uchun quyidagi ko'rinishda yozish mumkin:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \qquad (3.11.3)$$

(3.11.3) tenglamaga ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi deyiladi.

Agar $x_1 = x_2$ teng bo'lsa, P va Q nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi Oy o'qiga parallel bo'ladi va uning tenglamasi: $x = x_1$ ko'rinishda bo'ladi.

Misol. $P(4;-2)$ va $Q(3;-1)$ nuqtalar orqali o'tuvchi to'g'ri chiziq tenglamasini yozing.

Yechilishi. (3.11.3) tenglamaga asosan, ushbu

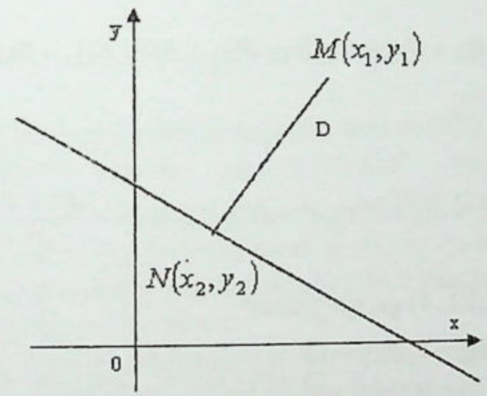
$$\frac{x-4}{3-4} = \frac{y+2}{-1+2} \Rightarrow y = -x + 2$$

tenglamaga ega bo'lamiz.■

3.12. Nuqtadan to'g'ri chiziqqacha masofa. Qandaydir $M(x_1, y_1)$ nuqta va l to'g'ri chiziq umumiy tenglamasi bilan berilgan bo'lsin, ya'ni

$$Ax + By + C = 0 \qquad (3.12.1)$$

M nuqtadan l to'g'ri chiziqgacha bo'lgan masofani topish talab qilingan bo'lsin. Bu masofani topish ucun l to'g'ri chiziqdan $N(x_2, y_2)$ nuqtani olamiz (3.16-chizma.). MN perpendikulyar bo'ladi l to'g'ri chiziqqa, bunda $d = MN$ izlanayotgan masofa.



3.16-chizma

MN nuqtaiardan o'tuvchi l to'g'ri chiziq perpendikulyar to'g'ri chiziq tenglamasi.

$$B(x - x_1) - A(y - y_1) = 0 \qquad (3.12.2)$$

bo'ladi. Bu yerda perpendikulyar tenglamasi $N(x_2, y_2)$ o'tadi:

$$B(x_2 - x_1) - A(y_2 - y_1) = 0$$

Demak,

$$\frac{x_2 - x_1}{A} = \frac{y_2 - y_1}{B} = t \qquad (3.12.\ 3)$$

deb belgilaymiz. Bunda t-proportsionallik koeffitsenti. Shuning uchun

$$d = NM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(At)^2 + (Bt)^2} = \sqrt{A^2 + B^2}|t| \qquad (3.12.4)$$

$N(x_2,\ y_2)$ l to‘g‘ri chiziqqa tegishli bo‘lgani uchun, (3.12.3) tenglikdan

$$x_2 = x_1 + At, \qquad y_2 = y_1 + Bt$$

ega bo‘lamiz. $N(x_2,\ y_2)$ (3.12. 1) qanoatlantiradi.

$$Ax_2 + By_2 + C = 0 \Rightarrow A(x_1 + At) + B(y_1 + Bt) + C = 0$$

yoki

$$Ax_1 + By_1 + C + (A^2 + B^2)t = 0 \Rightarrow t = -\frac{Ax_1 + By_1 + C}{A^2 + B^2} \qquad (3.12.\ 5)$$

(3.12. 5) ni (3.12. 4) ga qo‘yamiz

$$d = \sqrt{A^2 + B^2}\ \frac{|Ax_1 + By_1 + C|}{A^2 + B^2} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \qquad (3.12.\ 6)$$

(3.12. 6) ga $M(x_1, y_1)$ nuqtadan l to‘g‘ri chiziq bo‘lgan masofani topish formulasi deyiladi.

Xususiy holda, $x_1 = 0,\ \ y_1 = 0$ bo‘lsa, u holda koordinata boshidan to‘g‘ri chiziqqacha bo‘lgan masofa

$$d_0 = \frac{|C|}{\sqrt{A^2 + B^2}}$$

formula bilan aniqlanadi.

3.12.6-misol. $M(-2, 7)$ nuqtadan $24x + 7y - 2 = 0$to‘g‘ri chiziqqacha bo‘lgan masofani toping.

Yechilishi. (3.12.6) tenglamaga asosan, masofani topamiz:

$$d = \frac{|24x_1 + 7y_1 - 2|}{\sqrt{24^2 + 7^2}} = \frac{|24(-2) + 7 \cdot 7 - 2|}{\sqrt{625}} = \frac{1}{25} = 0{,}04.\ \blacksquare$$

3.3-§. Tekislikda ikkinchi tartibli chiziqlar

3.13. Ikkinchi tartibli chiziq ning umumiy tenglamasi. Ushbu

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ky + F = 0 \qquad (3.13.1)$$

yoki

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \qquad (2)$$

ko‘rinishdagi tenglamalarga tekislikdagi *ikkinchi tartibli chiziq ning umumiy tenglamasi* deyiladi. Bunda A, B, C, D, K, F va $a_{11}, a_{12}, a_{22}, a_{13}, a_{23}, a_{33}$ koeffisentlar bo‘lib, A, B, C yoki a_{11}, a_{12}, a_{22} koeffisentlarning kamida bittasi noldan farqli (aks holda biz birinchi tartibli chiziq qa ega bo‘lib qolamiz). Endi tenglamalarning xususiy xoldagi ko‘rinishi bilan tanishamiz. Masalan, aylana, ellips, giperbola va parabola.

3.14. Aylana. 3.14.1. Aylananing kanonik tenglamasi. Markaz deb ataluvchi (x_0, y_0) nuqtada bir xil R masofada joylashgan tekslikning nuqtalar to‘plamiga, markazi (x_0, y_0) nuqtada bo‘lgan R radiusli *aylana* deyiladi. Biz qisqacha uni $N(x_0, y_0, R)$ deb belgilaymiz, ya’ni

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \qquad (3.14.2)$$

Aylana tekislikda o‘z markazining koordinatalari va radiusi bilan bir qiymatli aniqlanadi. (3.14.2) dan quyidagini hosil qilamiz:

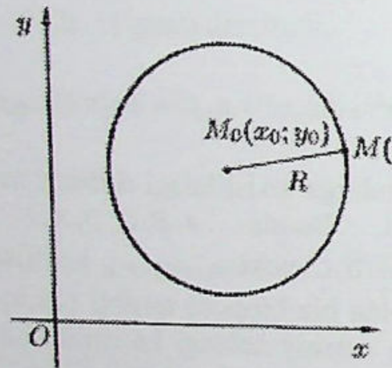
$$x^2 + y^2 - 2x_0x - 2y_0y + (x_0^{\ 2} + y_0^{\ 2} - R^2) = 0.$$

Agar ushbu $-2x_0 = 2D, -2y_0 = 2K$, $x_0^2 + y_0^2 - R^2 = F$ belgilashlar kiritsak, u holda aylananing tenglamasi

$$x^2 + y^2 + 2Dx + 2Ey + F = 0 \qquad (3.14.3)$$

ko'rinishda bo'ladi. Bundan aylana ikkinchi tartibli chiziq ekani kelib chiqadi, chunki oxirgi (3.14.3) tenglama (3.13.1) bilan $A=1, B=0, C=1$ bo'lganda ustma-ust tushadi. Endi (3.13.1) tenglama qachon aylanani tasvirlaydi, degan savol qo'yamiz. (3.13.1) tenglama aylanani ifoda etishi uchun $A=C$, $B=0$ $M=\frac{D^2}{4A^2}+\frac{K^2}{4A^2}-\frac{F}{A}>0$ bo'lishi yetarli.

Haqiqatdan ham, $A=C$, $B=0$ bo'lgani uchun egri chiziq tenglamasi $Ax^2 + Cy^2 + Dx + Ky + F = 0$ ko'rinishga keladi.



3.17-chizma

A soni nolga teng emasligi turgan gap, chunki $B=0$. Shuning uchun oxirgi tenglamaning har ikki tomonini A ga bo'lsak,

$$x^2+y^2+\frac{D}{A}x+\frac{K}{A}y+\frac{F}{A}=0.$$

Oxirgi tenglamani quyidagi ko'rishda yozamiz:

$$(x+\frac{D}{2A})^2+(y+\frac{K}{2A})^2=M.$$

Ammo $M>0$ bo'lgani uchun $M=R^2$, $a=-\frac{D}{2A}, b=-\frac{K}{2A}$ desak, markazi (a,b) nuqtada va radiusi R ga teng bo'lgan aylana tenglamasi hosil bo'ladi.

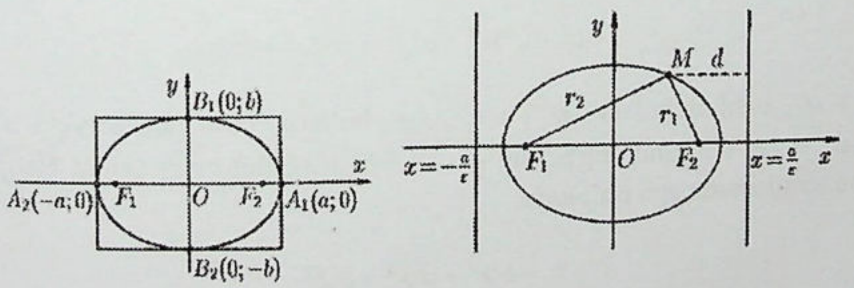
Agar $A=C, B=0, M=0$ bo'lsa, tenglamaning ko'rinishi $(x-x_0)^2+(y-y_0)^2=0$ bo'lib, bu tenglama tekislikda faqat bitta nuqtani tasvirlaydi (boshqacha qilib aytsak, nol radiusli aylanani tasvirlaydi). Agar $A=C, B=0, M<0$ bo'lsa, u holda tenglamaning ko'rinishi $(x-x_0)^2+(y-y_0)^2=-R^2$ bo'lib, bu tenglama mavhum aylanani tasvirlaydi. Boshqacha aytganda, bu tenglamani qanoatlantiruvchi haqiqiy (x,y) nuqtalar to'plami bo'sh to'plamdir.

Agar aylana markazi koordinatalar boshida bo'lsa, u holda $x_0=y_0=0$ bo'ladi, tenglama $x^2+y^2=R^2$ ko'rinishni oladi.

3.15. Ellips. 3.15.1. Ellipsning kanonik tenglamasi.

3.15.2-ta'rif. Tekislikning shunday nuqtalaridan fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalar yig'indisi o'zgarmas bo'lib, u fokuslar orasidagi masofadan katta bo'lgan, tekislikning shunday nuqtalar to'plamiga *ellips* deb ataladi.

Quyida Dekart koordinatalari sistemasida ellips tenglamasini keltirib chiqarish bilan shug'ullanamiz. Ellips fokuslarini F_1 va F_2 bilan belgilaymiz. Dekart koordinatalar sistemasini Ox o'q fokuslaridan o'tadigan, Oy o'qi esa $[F_1F_2]$ kesmani teng ikkiga bo'ladigan qilib kiritamiz. Fokuslar orasidagi masofani 2c orqali belgilaymiz, ya'ni $d(F_1F_2)=2c$. U holda F_1 va F_2 nuqtalarning koordinatalari mos ravishda $F_1(-c,0)$ va $F_2(c,0)$ bo'ladi (3.18-chizma).



3.18-chizma.

$M(x,y)$-ellipsning ixtiyoriy nuqtasi bo'lsin. Ellipsning ta'rifiga ko'ra,

$$d(F_1M)+d(F_2M)=2a$$

yoki

$$d(F_1M)=\sqrt{(x+c)^2+y^2}, d(F_2M)=\sqrt{(x-c)^2+y^2}$$

ifodaning o'rniga qo'yib

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a \qquad (3.15.3)$$

tenglamaga ega bo'lamiz. Bu ellipsning Dekart koordinatalar sistemasidagi tenglamasidir. Tenglamani soddalashtirish uchun uni radikallardan qutqarish kerak. Bitta radikalni tenglamaning o'ng tomoniga o'tkazamiz va hisoblashlar bajaramiz:

$$(\sqrt{(x+c)^2+y^2})^2=(2a-\sqrt{(x-c)^2+y^2})^2$$

$$\Rightarrow (x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$\Rightarrow 4a\sqrt{(x-c)^2+y^2}=4a^2-4xc \Rightarrow (a\sqrt{(x-c)^2+y^2})^2=(a^2-xc)^2 \Rightarrow$$

$$\Rightarrow a^2[(x-c)^2+y^2]=a^4-2xca^2+x^2c^2$$

$$\Rightarrow a^2x^2+a^2c^2-2a^2xc+a^2y^2=a^4-2xca^2+x^2c^2 \Rightarrow x^2(a^2-c^2)+a^2y^2=a^4-a^2c^2$$

yoki

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

F_1MF_2 uchburchakda $MF_1+MF_2>F_1F_2$ bo'ladi, ya'ni $2a>2c$ yoki $a>c$, demak $a^2-c^2>0$. Shuning uchun $a^2-c^2=b^2, b\neq 0$ deb belgilaymiz. Natijada ushbu tenglamada ega bo'lamiz:

$$b^2x^2+a^2y^2=a^2b^2$$

yoki

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (3.15.4)$$

Bu (3.15.4) tenglamada ellipsning *kanonik* tenglamasi deyiladi. Demak (3.15.4) tenglama (3.15.3) tenglamadan ikki marta kvadratga ko'tarib, radikaldan qutqarish yo'li bilan hosil qilindi. Ko'rsatish mumkinki, bu amallarni bajarishda tegishli nuqtalar to'plamiga yangi nuqtalar kirib qolmaydi. Hamda hech bir nuqta yo'qotilmaydi ham.

3.15.4. Kanonik tenglamasi bo'yicha ellips shaklini tekshirish. Ellipsning shaklini (grafigini) quyidagi sxema bo'yicha tekshiramiz:

1) Agar ellipsning fokuslari deb ataluvchi F_1 va F_2 nuqtalar ustma-ust tushib qolsa, u holda $c=0$ bo'ladi. Bunda $a^2=b^2$ bo'lib, ellipsning kanonik tenglamasi $x^2+y^2=a^2$ ko'rinishga keladi. Demak, aylana ellipsning xususiy holidir.

2) Endi ellipsning koordinata o'qlari bilan kesishish nuqtalarini aniqlaymiz. Ox o'qi bilan kesishish nuqtalarini topamiz:

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=\pm a \\ y=0 \end{cases}.$$

Demak, ellips Ox o'q bilan ikki $A_1(a,0)$ va $A_2(-a,0)$ nuqtada kesishadi. Shuningdek, ravshanki ellips Oy o'qi bilan ham ikki $B_1(0,b)$ va $B_2(0,-b)$ nuqtada kesishadi.

Bunda A_1A_2, B_1B_2 nuqtalarni ellipsning uchlari deb ataladi; $[A_1A_2]$, $[B_1B_2]$ kesmalarni ellipsning o'qlari deyiladi. Bizga $|A_2A_1|=2a, |B_2B_1|=2b$, $a>b$ ekanligi ma'lum bo'lganligi uchun $[A_1A_2]$ kesmani ellipsning *katta o'qi*, $[B_1B_2]$ kesmani esa ellipsning *kichik o'qi* deyiladi. Demak, a va b sonlar ellips yarim o'qlarining uzunligi bo'ladi.

3) Ellipsning koordinata o'qlariga nisbatan simmetrikligini tekshiramiz. Ellipsning kanonik tenglamasini ushbu shaklda yozamiz:

$$F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

Ravshanki, x va y ning istalgan qiymatlari uchun ushbuga egamiz:

$$F(-x,-y=F(x,y); F(-x,y)=F(x,y); F(x,-y)=F(x,y).$$

Bu munosabatlarning o'rinli bo'lishiga sabab tenglamada o'zgaruvchi koordinatalarning faqat kvadratlari ishtirok etadi va $F(x,y)$ funksiya x va y ga nisbatan juft funksiyadir.

Demak, ellips Ox va Oy o'qlariga nisbatan simmetrik ravishda joylashgan. Bundan tashqari $F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ tenglamada ishtirok etayotgan x va y o'zgaruvchilarning o'zgarish sohalari $-a\le x\le a, -b\le y\le b$ tengsizliklar bilan aniqlanadi.

3.15.5. Ellipsning ekssentrisiteti. Ellipsning fokuslari orasidagi masofaning uning katta o'qi uzunligiga nisbati ellipsning ekssentrisiteti deb ataladi va u ε harfi bilan belgilanadi. Ta'rifga ko'ra $\varepsilon=\frac{2c}{2a}=\frac{c}{a}$ yoki $\varepsilon=\frac{c}{a}$, bunda $0\le c<a$ ga asosan ekssentrisitet uchun $0\le\varepsilon<1$ tengsizlik o'rinli.

Ekssentrisitet ellipsning cho'ziqlik darajasini xarakterlaydi. Ekssentrisitet qancha katta bo'lsa, ellips shuncha cho'ziq bo'ladi. $\varepsilon=0$ da fokuslar ustma-ust tushadi va yarim o'qlar teng bo'lib qoladi, bu holda ellips aylanaga o'tadi. Endi b/a ni ε orqali ifodalaylik. Buning uchun $a^2-c^2=b^2$ tenglikdan foydalanamiz. Sodda almashtirishlar ko'rsatadiki,

$$c=\varepsilon a, a^2-c^2=a^2-\varepsilon^2a^2=a^2(1-\varepsilon^2), b^2=a^2(1-\varepsilon^2)\Rightarrow\frac{b}{a}=\sqrt{1-\varepsilon^2}.$$

Agar b ning qiymati a dan nolgacha kamaysa, ε ning qiymati 0 dan 1 gacha o'sib boradi. Shunday qilib ellipsning ekssentrisiteti nolga qancha yaqin bo'lsa, ellipsning shakli aylanaga shuncha yaqin va ekssentrisiteti qancha katta bo'lsa, uning shakli shuncha cho'ziq bo'ladi.

3.15.6. Ellipsning fokal radiuslari. Ellipsning ixtiyoriy nuqtasidan uning fokuslarigacha bo'lgan masofalar bu nuqtaning *fokal radiuslari* deyiladi.

Bu ta'rifga qaraganda $\vec{F_1M}$ bilan $\vec{F_2M}$ vektorlar ellipsdagi M nuqtaning fokal radiuslaridir, bularni mos ravishda $\vec{r_1}$ va $\vec{r_2}$ bilan belgilaymiz, bu holda $|F_1M|+|F_2M|=2a$ formulaga asosan:

$$|\vec{r_1}|=|\vec{F_1M}|=\sqrt{(x-c)^2+y^2},\quad |\vec{r_2}|=|\vec{F_2M}|=\sqrt{(x+c)^2+y^2}.$$

Fokal radiuslar uchun soddaroq formula topish maqsadida bu tenglamalarning ikkala tomonini kvadratga ko'tarib, chiqqan natijaning 2- sidan 1-sini hadlab ayirsak, $r_2^2-r_1^2=4cx$ tenglik hosil bo'ladi. Buni quyidagi ko'rinishda yozish mumkin:

$$(r_2-r_1)(r_2+r_1)=4cx \text{ yoki } 2a(r_2-r_1)=4cx\Rightarrow r_2-r_1=2\frac{c}{a}x.$$

Bu tenglik bilan $r_2+r_1=2a$ ni birgalikda yechsak, hamda $\varepsilon=\frac{c}{a}$ ni e'tiborga olsak, $r_1=a-\varepsilon x, r_2=a+\varepsilon x$ formulalarga ega bo'lamiz. Bu formulalar fokal radiuslarni x orqali chiziqli ifodalaydi.

3.15.7. Eellipsning direktrisasi. Ellipsning *direktrisasi* deb, uning katta o'qiga perpendikulyar bo'lgan va markazdan masofasi $\frac{a}{\varepsilon}$ ga teng bo'lgan 2 ta to'g'ri chiziqqa aytiladi.

Bu ta'rifga muvofiq, ellips direktrisalarining tenglamasi $x=\frac{a}{\varepsilon}, x=-\frac{a}{\varepsilon}$ bo'ladi. Ellipsda $\varepsilon<1$ bo'lgani sababli $\frac{a}{\varepsilon}>a$.

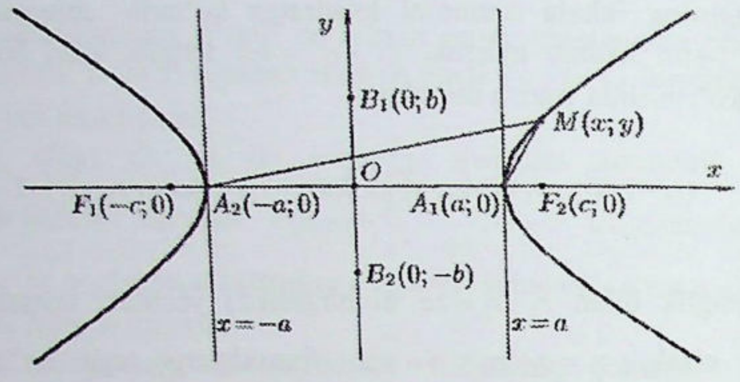
Demak, direktrisalar ellipsning A_1 va A_2 uchlaridan tashqarida joylashgan. $x=\pm\frac{a}{\varepsilon}$ to'g'ri chiziqlar ushbu xossaga ega. Ellipsning har qanday nuqtasdidan o'ng fokusigacha va unga nuqtadan ungacha mos direktrisagacha bo'lgan masofalar nisbati o'zgarmas miqdor bo'lib, ε ga teng.

3.16.Giperbola. 3.16.1. Giperbola1ning kanonik tenglamasi

3.16.2-ta'rif. Tekislikning shunday nuqtalaridan fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalar ayirmasining absolyut qiymati o'zgarmas bo'lib, u $2a$ teng bo'lgan, tekislikning shunday nuqtalar to'plamiga *giperbola* deb ataladi.

Ta'rifda aytilgan ayirma fokuslar orasidagi masofadan kichik bo'lishi hamda noldan farqli bo'lishi shart.

Giperbolaning sodda tenglamasini keltirib chiqarish uchun dekart koordinatalar sistemasini ellips tenglamasini keltirib chiqargandek tanlaymiz. Aniqroq aytganda, absissalar o'qini giperbolaning F_1 va F_2 fokuslar orqali o'tkazamiz, koordinatalar boshi deb $[F_1F_2]$ kesmaning o'rtasini olamiz (3.19-chizma).



3.19-chizma.

$M(x,y)$ giperbolaning ixtiyoriy nuqtasi bo‘lsin. $[F_1M]$ va $[F_2M]$ kesmalarning uzunliklarini mos ravishda r_1 va r_2 bilan belgilaymiz. Bu holda

$$r_1=\sqrt{(x+c)^2+y^2},\ r_2=\sqrt{(x-c)^2+y^2}.$$

r_1 va r_2 ning qiymatlariga ko‘ra, ta’rif bo‘yicha ushbu tenglamani yozish mumkin:

$$\left|\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}\right|=2a$$

yoki

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$$

(agar $MF_1>MF_2$ bo‘lsa, o‘ng tomonda “+” ishora, $MF_1<MF_2$ bo‘lsa, “–“ ishora olinadi). Bu holda ham ellips tenglamasini keltirib chiqarishda qilingan almashtirishlarni bajarib, quyidagi tenglamani hosil qilasiz

$$\pm a\sqrt{(x-c)^2+y^2}=cx-a^2 \Rightarrow (c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2).$$

$c>0$ bo‘lgani sababli $b=\sqrt{c^2-a^2}>0$ belgilashni kiritib, giperbola tenglamasini ushbu ko‘rinishda yozamiz

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ . \qquad (3.16.3)$$

Bu (3.16.3) tenglamaga giperbolaning *kanonik tenglamasi* deyiladi.

3.16.3. Kanonik tenglamasi bo‘yicha giperbola shaklini (grafigini) tekshirish. Giperbola shaklini uning (3.16.2) tenglamasiga ko‘ra tekshiramiz. Bu tenglamaga x,y ning kvadrati kiradi, shuning uchun agar (x,y) nuqta giperbola nuqtasi bo‘lsa, $(\pm x,\pm y)$ nuqtalar ham giperbolaning nuqtalari bo‘ladi, bu esa giperbolaning nuqtalarining koordinata o‘qlariga nisbatan simmetrik joylashganini bildiradi. Simmetriya o‘qlarining kesishgan nuqtasi giperbolaning markazi deyiladi. Ravshanki, markaz giperbolaga tegishli emas.

Koordinata o‘qlari bilan kesishish nuqtalari tekshiramiz. Giperbolaning Ox o‘qi bilan kesishish nuqtasining koordinatalarini topish uchun

$$\begin{cases}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\\ y=0\end{cases}$$

sistemasini yechamiz. Bunda $x=\pm a$. Demak, giperbola Ox o‘q bilan $A_1(a,0)$, $A_2(-a,0)$ nuqtalarda kesishadi. Bu nuqtalarni, odatda giperbolaning uchlari deyiladi.

$[A_1,A_2]$ kesmani esa giperbolaning haqiqiy o‘qi deyiladi.

Shunga o‘xshash, giperbolaning Oy o‘qi bilan kesishish nuqtasini topish uchun

$$\begin{cases}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\\ x=0\end{cases}$$

sistemasini yechamiz. Bunda $y^2=-b^2, b\neq 0$. Bu tenglama haqiqiy yechimlarga ega emas.

Demak, giperbola ordinatalar o‘qi bilan kesishmaydi. Chiqarilgan natijalarga muvofiq $[B_1,B_2]$ kesmani giperbolaning mavhum o‘qi deb ataladi. a va b sonlar mos ravishda haqiqiy va mavhum yarim o‘qlar deyiladi.

Giperbolaning ikki tarmog'i (shohi) mavjud bo'lib, ulardan biri $x \le -a$ yarim tekislikda, ikkiinchisi esa, $x \ge a$ yarim tekislikda yotadi.

3.16.4. Giperbolaning asimptotalari. Giperbolaning grafigini yana ham ochiqroq tasavvur qilish uchun y bilan o'zaro bog'liq bo'lgan asimptotalar deb ataluvchi ikki to'g'ri chiziqni ko'zdan kechiramiz.

x yoki y ni musbat deb faraz qilib. Giperbolaning (3.16.3) tenglamasini y ga nisbatan yechamiz:

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1 \text{ yoki } y=\frac{b}{a}\sqrt{x^2-a^2}.$$

Bu tenglamani $y=\frac{b}{a}x$ to'g'ri chiziqning tenglamasi bilan solishtirib ko'ramiz. Shu to'g'ri chiziq va giperbolaning biror vertikal to'g'ri chiziq bilan kesishish nuqtalarini ya'ni $N(x,y)$ va $M(x,y)$ nuqtalarni mos nuqtalar deb ataymiz. Ravshanki, ko'rilayotgan holda $Y > y$ va chizmadan $|MN| = Y - y$ ga egamiz. Endi biz x cheksiz o'sganda bu ayirmaning nolga intilishini ko'rsatamiz. Buning uchun $Y-y=\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}$ fuksiyaning $x\to\infty$ dagi nisbatini hisoblaymiz:

$$\lim_{x\to\infty}(Y-y)=\frac{b}{a}\lim_{x\to\infty}(x-\sqrt{x^2-a^2})=$$

$$=\frac{b}{a}\lim_{x\to\infty}\frac{(x-\sqrt{x^2-a})(x+\sqrt{x^2-a^2})}{(x+\sqrt{x^2-a^2})}=\frac{b}{a}\lim_{x\to\infty}\frac{a^2}{x+\sqrt{x^2-a^2}}=0.$$

Demak, $\lim_{x\to\infty}(Y-y)=0$. Bu oxirgi natijadan ko'ramizki, x absissa cheksiz ortib borganda $|MN|$ masofa kamaya boradi va nolga intiladi. Buning ma'nosi quyidagidan iborat: birinchi chorakda giperbolaning tarmog'i $x\to\infty$ da $y=\frac{b}{a}x$ to'g'ri chiziqqa intiladi.

Ushbu $y=\frac{b}{a}x, y=-\frac{b}{a}x$ to'g'ri chiziqlar *giperbolaning asimptotalari* deyiladi.

Giperbolaning uning tenglamasi bo'yicha chizish uchun oldin uning asimptotalarini yasash qulaylik tug'diradi.

3.16.5. Teng tomonli giperbola. $a=b$ bo'lgan holda giperbola teng tomonli deb ataladi uning tenglamasi (3.16.3) ko'ra :

$$x^2-y^2=a^2$$

ko'rinishda bo'ladi. Ochiq ko'rinadiki, asimptotalarining burchak koeffisientlari $\left(k=\pm\frac{b}{a}\right)$ teng tomonli giperbola uchun ± 1 ga teng. Demak, teng tomonli giperbolaning asimptotalari o'zaro perpendikulyar.

3.16.6. Giperbolaning ekssentrisiteti. Giperbola fokuslari orasidagi masofaning giperbolaning haqiqiy o'qi uzunligiga nisbati giperbolaning ekssentrisiteti deyiladi va ε orqali belgilanadi. Ta'rifga ko'ra $\varepsilon=\frac{2c}{2a}=\frac{c}{a}$.

Giperbolada $c>a$ bo'lgani sababli $\varepsilon>1$. Demak, giperbolaning ekssentrisiteti hamma vaqt birdan katta bo'ladi. Ekssentrisitet formulasini $c=\sqrt{a^2+b^2}$ ekanidan foydalanib, $E=\frac{\sqrt{a^2+b^2}}{a}=\sqrt{1+\left(\frac{b}{a}\right)^2}$ formula ko'rinishda yozish mumkin. Ellipsga o'xshash, ε birga qancha yaqin bo'lsa, giperbolaning tarmoqlari shuncha siqiq va ε birdan qancha katta bo'lsa, giperbola tarmoqlari shuncha yoyiq joylashgan bo'ladi.

3.16.7. Giperbolaning fokal radiuslari. Giperbolaning istalgan $M(x,y)$ nuqtasidan uning $F_1(c,0)$ va $F_2(-c,0)$ fokuslarigacha bo'lgan $r_1=d(F_1M)$, $r_2=d(F_2M)$ masofalar shu M nuqtaning fokal radiuslari deyiladi.

Ellipsning fokal radiuslarini hisoblashga o'xshash, giperbolaning fokal radiuslari uchun quyidagi formulalar o'rinli:

$x<0$ da $\begin{cases} r_1=a-\varepsilon x, \\ r_2=-a-\varepsilon x; \end{cases}$ (chap tarmoq uchun)

$x>0$ da $\begin{cases} r_1=-a+\varepsilon x, \\ r_2=a+\varepsilon x. \end{cases}$ (o'ng tarmoq uchun).

3.16.8. Giperbolaning direktrisalari. Giperbolaning *direktrisalari* deb, uning markazidan $\pm\frac{a}{\varepsilon}$ masofada fokal o'qiga perpendikulyar bo'lib o'tadigan ikki to'g'ri chiziqqa aytiladi.

Bu ta'rifga ko'ra, giperbola direktrisalarining tenglamalari quyidagi ko'rinishda bo'ladi:

$$x=\pm\frac{a}{\varepsilon} \text{ va } x=\frac{-a}{\varepsilon}.$$

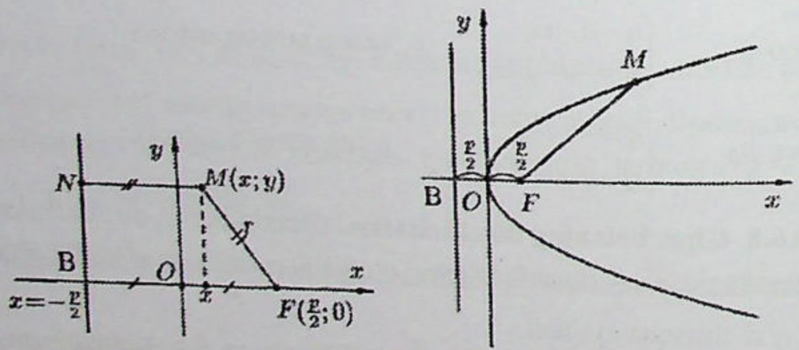
Giperbolada $\varepsilon>1$ bo'lgani sababli $\frac{a}{\varepsilon}<a$ bo'ladi. Demak, giperbolaning direktrisalari uning markazi 0 bilan A va A_1 uchlari orasida joylashgan.

Giperbolaning ixtiyoriy nuqtalaridan fokusgacha bo'lgan masofaning mos direktrisagacha bo'lgan masofa nisbati o'zgarmas va ε ga teng.

3.17. Parabola. 3.17.1. Parabolaning kanonik tenglamasi. *Parabola* deb, tekislikning shunday nuqtalar to'plamiga aytiladiki, bu nuqtalar fokus deb ataluvchi berilgan nuqtadan va direktrisa deb ataluvchi berilgan to'g'ri chiziqdan teng uzoqlashgan bo'ladi.

Parabolaning tenglamasini keltirib chiqarish uchun, ellips va giperbola tenglamalarini chiqarishda qilingandek. Dekart koordinatalar sistemasi maxsus tanlanadi. Boshqacha aytganda, fokus deb ataluvchi F nuqtadan o'tuvchi va berilgan L to'g'ri chiziqqa (direkrisaga) perpendikulyar bo'lgan to'g'ri chiziqni OX o'qi deb qabul qilamiz, berilgan F nuqtadan ε to'g'ri chiziqqacha bo'lgan masofani $|R|$ deb belgilaymiz. Buni $d(F,B)=|p|$ kabi yozamiz (3.20-chizma).

$[F,B]$ kesmaning o'rtasini koordinatalar boshi O deb qabul qilamiz. U holda F nuqtaning koordinatasi $F\left(\frac{p}{2}:0\right)$. Direktrisaning tenglamasi esa $x=-\frac{p}{2}$ va $x=\frac{p}{2}$ ko'rinishda yoziladi. Parabolaning ixtiyoriy $M(x,y)$ nuqtasi uchun uning ta'rifiga binoan $|MN|=|MF|$.



3.20-chizma.

Agar

$$|MF|=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2} \text{ va } |MN|=\left|x+\frac{p}{2}\right|$$

ekanini hisobga olsak, yuqoridagi tenglikni quyidagicha yozish mumkin:

$$\left|x+\frac{p}{2}\right|=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}.$$

Bu tenglikning ikkala tomonini mos ravishda kvadratga ko'tarib va sodda almashtirish bajarib

$$x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2$$

yoki

$$y^2=2px \tag{3.17.2}$$

tenglamaga ega bo'lamiz. Bu (3.17.2) tenglama parabolaning *kanonik tenglamasi* deyiladi.

Ushbu $y^2=-2px$, $x^2=2py$, $x^2=-2py\ (p>0)$ ko'rinishdagi tenglamalar ham parabolaning *kanonik tenglamalari* deyiladi.

(3.17.2) tenglama bilan berilgan parabolaning ba'zi sodda xossalarini keltiramiz:

1^0. Parabola koordinatalar boshidan o'tadi, ya'ni $O(0,0)$ nuqta parabola tenglamasini qanoatlantiradi;

2^0. Parabola koordinata o'qlari bilan faqat va faqat koordinatalar boshida kesishadi, shuning uchun $O(0,0)$ nuqtani parabolaning uchi deyiladi;

3^0.(3.17.2) parabola Ox o'qqa nisbatan simmetrik;

4^0. Parabola $x\geq 0$ yarim tekislikda joylashgan;

5^0. Parabolaning shakli (grafigi) u tenglamadan $y=\pm\sqrt{2px}$ ekani ko'rinadi. Bundan agar $p>0$ bo'lsa, $x\geq 0$ ekani, agar $p<0$ bo'lsa, $x\leq 0$ ekani kelib chiqadi.

3.17.3. Parabolaning ekssentrisiteti va direktrisasi. Parabolaning ixtiyoriy nuqtasidan uning fokusigacha bo'lgan masofani r bilan, direktrisagacha bo'lgan masofani k bilan belgilasak, parabola ta'rifidan $r = k$ deb yozish mumkin.

Parabolaning *ekssentrisiteti* deb, $\varepsilon = \frac{r}{k}$ songa aytiladi. Bu holda, ravshanki,

$$\varepsilon = \frac{r}{k} = 1.$$

3-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

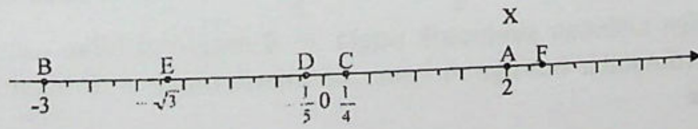
3.1.Tekislikda koordinatalar sistemasi tushunchasi([3], [5], [6],[9]).
3.2. Qutb koordinatalar sistemasi haqida tushuncha([3], [5], [6],[9]).
3. 3. Tekislikda koordinatalar usulining qo'llanilishi([3], [5], [6],[9]).
3.4. Koordinatalar sistemalarini aniqlash([3], [5], [6],[9]).
3.5. Тўғри чизиқнинг умумий тенгламаси([3], [5], [6],[9]).
3.6.To'g'ri chziqning burchak koeffitsentli tenglamasi([3], [5], [6],[9]).
3.7.To'g'ri chiziqning kesmalardagi tenglamasi([3], [5], [6],[9]).
3.8.To'g'ri chiziqning normal tenglamasi([3], [5], [6],[9]).
3.9.Ikki to'g'ri chiziq orasidagi burchak([3], [5], [6],[9]).
3.10. Berilgan nuqtadan berilgan yo'nalish bo'yicha o'tuvchi to'g'ri chiziq tenglamasi([3], [5], [6],[9]).
3.11. Berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi([3], [5], [6],[9]).
3.12. Nuqtadan to'g'ri chiziqqacha masofa([3], [5], [6],[9]).
3.13. Иккинчи тартибли чизиқ нинг умумий тенгламаси([3], [5], [6],[9]).
3.14. Aylana([3], [5], [6],[9]).
3.15. Ellips([3], [5], [6],[9]).
3.16.Giperbola([3], [5], [6],[9]).
3.17.Parabola([3], [5], [6],[9]).

3.1-amaliy mashg'ulot.
Tekislikda koordinatalar sistemasi

1. Son o'qi.

1– misol. Masshtab kesmani 1 sm ga teng qilib olib, quyidagi $A(2)$, $B(-3)$, $C(\frac{1}{4})$, $D(-\frac{1}{5})$, $E(-\sqrt{3})$, $F(2\sqrt{3})$ nuqtalarni yasang.

Yechilishi. Dastlab masshtab kesmani 1 sm ga teng qilib olib son o'qi yasaymiz, so'ngra nuqtalarni son o'qida joylashtiramiz



■

2– misol. $A(-2)$ va $B(5)$ nuqtalar berilgan. AB kesmaning uzunligini toping.

Yechilishi. AB kesmaning uzunligini topamiz:

$$AB = |x_A - x_B| = |-2-5| = |-7| = 7. \blacksquare$$

3–misol. $C(4)$ va D (9) nuqtalar bilan chegaralangan kesmani $\lambda = \frac{2}{3}$ nisbatda bo'luvchi N nuqtaning koordinatasini toping.

Yechilishi. (3.3.3) formulaga ko'ra quyidagiga ega bo'lamiz:

$$x_N = \frac{x_A + \lambda x_B}{1+\lambda} = \frac{4 + \frac{2}{3}\cdot 9}{1 + \frac{2}{3}} = \frac{10}{\frac{5}{3}} = 6.$$

Demak, $N(6)$ bo'ladi.■

4–misol. $A(4)$ nuqta uchun $B(-2)$ nuqtaga nisbatan simmetrik bo'lgan A' nuqtani toping.

Yechilishi. B nuqta AA' kesmaning o'rtasidir (3.3.5) formulaga ko'ra,

$$x_B = \frac{x_A + x_{A'}}{2}$$

ga ega bo'lamiz. Bu tenglikka $x_A = 4$ va $x_B = -2$ ni qo'yib, A' nuqtaning koordinatasi $x_{A'}$ ni topamiz: $-2 = \frac{4 + x_{A'}}{2}$ yoki $x_{A'} = -8$. ■

2. Tekislikda ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo'lish.

5-misol. $A(-3;-1)$ va $B(0;-4)$ nuqtalarning abssissalar o'qiga, ordinatalar o'qiga va koordinatalar boshiga nisbatan simmetrik bo'lgan nuqtalarni toping.

Yechilishi. O nuqta BB' kesma o'rtasi $x_0 = \frac{x_B + x_{B'}}{2} = 0$, chunki $O(0;0)$ shunigdek $x_B = x_{B'} = 0$, $y_0 = \frac{y_B + y_{B'}}{2} = 0$; $\frac{-4 + y_{B'}}{2} = 0$, $y_{B'} = 4$. Demak, $B'(0;4)$ ordinata o'qiga nisbatan simmetrik nuqta B' B ning o'zi bilan ustma – ust tushadi. Koordinatalar boshiga nisbatan simmetrik nuqta B' B bilan ustma – ust tushadi. ■

6– misol. $A(-2;5)$ va $B(8;2)$ nuqtalar orasidagi masofani toping.

Yechilishi. AB kesmaning uzinligini (3.3.1) formula bo'yicha hisoblaymiz:

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = \sqrt{(-2-8)^2 + (5-(-2))^2} = \sqrt{100+49} = \sqrt{149}. \blacksquare$$

7–misol. Ordinatalar o'qida $A(-6;1)$ va $B(3;4)$nuqtalardan baravar uzoqlashgan nuqtani toping.

Yechilishi. Izlanayotgan M nuqta ordinatalar o'qida yotadi, shuning uchun uning abssissasi $x_M = 0$. nuqtaning ordinatasi y_M ni topish uchun $MA = MB$ ekanligidan foydalanamiz. (3.3.1) formulaga ko'ra ushbuga egamiz:

$$MA = \sqrt{(x_M - x_A)^2 + (y_m - y_A)^2} = \sqrt{(0+6)^2 + (y_m - 1)^2} = \sqrt{36 + (y_M - 1)^2},$$

$$MB = \sqrt{(x_M - x_B)^2 + (y_m - y_B)^2} = \sqrt{(0-3)^2 + (y_m - 4)^2} = \sqrt{9 + (y_M - 14)^2}.$$

bu yerda y_M ga nisbatan tenglamani hosil qilamiz:

$$\sqrt{36 + (y_M - 1)^2} = \sqrt{9 + (y_M - 4)^2}$$

Endi tenglamani yechib, M nuqtaning y_M ordinatasini topamiz:

$$36 + y^2 - 2y + 1 = 9 + y^2 - 8y + 16, \ 6y = -12,$$

bu yerda $y_M = -2$. Shunday qilib izlanayotgan nuqta M (0; -2) bo'ladi. ■

8– misol. $A(2, 4)$ va $B(8, -20)$ nuqtalar bilan chegaralangan kesma uchta teng bo'lakka bo'lingan. C va D bo'linish nuqtalarning koordinatalarini toping.

Yechilishi. C nuqta AB kesmani $\lambda = \frac{AC}{CB} = \frac{1}{2}$ nisbatda bo'ladi. C nuchtaning koordinatasini (3.3.3) formulaga ko'ra,

$$x_C = \frac{x_A + \lambda x_B}{1+\lambda} = \frac{2+\frac{1}{2}\cdot 8}{1+\frac{1}{2}} = \frac{6\cdot 2}{3} = 4, \qquad y_C = \frac{y_A + \lambda y_B}{1+\lambda} = \frac{4+\frac{1}{2}\cdot(-20)}{1+\frac{1}{2}} = -4.$$

Shunday qilib, (4, -4) D nuqta AB kesmani $\lambda = \frac{AD}{DB} = \frac{2}{1} = 2$ nisbatda bo'ladi. Bu yerdan

$$x_D = \frac{x_A + \lambda x_B}{1+\lambda} = \frac{2+8\cdot 2}{1+2} = \frac{18}{3} = 6, \qquad y_D = \frac{y_A + \lambda y_B}{1+\lambda} = \frac{4+2\cdot(-20)}{1+2} = -12.$$

Demak, D(6;-12). ■

9–misol. ABC uchburchakning $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_c, y_C)$ uchlari berilgan. Bu uchburchakning medianalar kesishish nuqtasining koordinatalarini toping.

Yechilishi. AB kesmaning o'rtasini D deb belgilaymiz. D nuqtaning koordinatalarini topamiz:

$$x_D = \frac{x_A + x_B}{2}, \quad y_D = \frac{y_A + y_B}{2}.$$

Ma'lumki, uchburchakning medianalari bir nuqtada kesishib, shu nuqtada medianalarning har biri 2:1 nisbatda bo'linadi (uchburchakning tegishli uchidan hisoblaganda.).

Medianalar kesishadigan M nuqtani topamiz, buning uchun CD medianani nisbatda bo'lamiz (C dan D ga qarab)

$$x_M = \frac{x_C + \lambda x_D}{1+\lambda} = \frac{x_C + \frac{x_A + x_B}{2}}{1+2} = \frac{x_A + x_B + x_C}{3},$$

$$y_M = \frac{y_C + \lambda y_D}{1+\lambda} = \frac{y_C + \frac{y_A + y_B}{2}}{1+2} = \frac{y_A + y_B + y_C}{3}.$$

Shunday qilib, uchburchak medianalarining kesishgan nuqtalarning koordinatalari uning uchlari bir qismli koordinatalarining o'rta arifmetik qiymatiga teng ekan. ■

10–misol. A (-5, 6) va B(1, -3) nuqtalar orqali o'tuvchi to'g'ri chiziqda abssissasi $x_C = -3$ bo'lgan C nuqtani toping.

Yechilishi. $x_A < x_C < x_B$ bo'lganligi sababli C nuqta AB kesmaning ichida yotadi. C nuqta AB kesmani bo'ladigan $\lambda = \frac{AC}{CB}$ nisbatini topamiz. Buning uchun (3.3.3) formuladan foydalanamiz:

$$x_C = \frac{x_A + \lambda x_B}{1+\lambda},$$

Bu munosabatga A, B va C nuqtalarning abssissalarini qo'yib, λ ga nisbatan ushbu tenglamani hosil qilamiz:

$$-3 = \frac{-5+\lambda\cdot 1}{1+\lambda},$$

Bu yerdan $\lambda = \frac{1}{2}$. Endi C nuqtaning ordinatasini y_C topish mumkin.

$$y_C = \frac{y_A + \lambda y_B}{1+\lambda} = \frac{6+\frac{1}{2}(-3)}{1+\frac{1}{2}} = 3.$$

Shunday qilib, C(-3,3). ■

3. Uchburchakning yuzi

11–misol. Burchaklarining uchlari A(-4, -2), B (3, 4) va C(-2, 4) nuqtalardan bo'lgan uchburchakning yuzini toping.

Yechilishi. Berilgan misolda:

$x_A = -4, x_B = 3, x_C = -2, y_A = -2, y_B = 4, y_C = 4$

(3.3.7) formulaga asosan, ABC uchburchakning yuzini topamiz.

$$S_\Delta = \frac{1}{2}\left|(x_B - x_A)(y_C - y_A) - (x_C - x_A)(y_B - y_A)\right| =$$

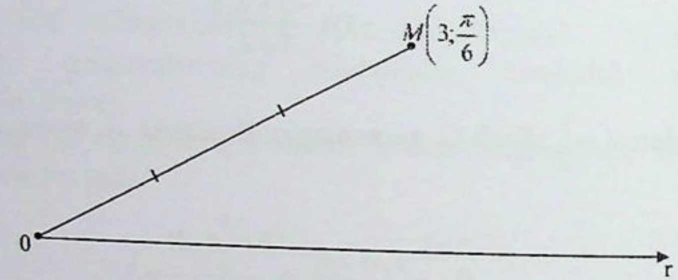
$$= \frac{1}{2}\left|(3-(-4))(4-(-2)) - (-2-(-4))(4-(-2))\right| =$$

$$=\frac{1}{2}|7\cdot 6-2\cdot 6|=3\cdot 5=15\ (kv.birlik).$$ ■

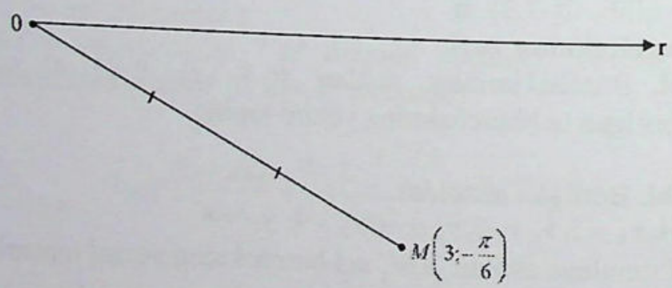
4. Qutb koordinatalar sistemasi

12–misol. Qutb koordinatalar sistemasida $M(3;\frac{\pi}{6})$ nuqtani yasang.

Yechilishi. *O* qutb orqali *Or* qutb o'qini o'tkazamiz. *Or* o'qidan *O* qutb atrofida soat strelkasiga qarama-qarshi yo'nalishda $\frac{\pi}{6}$ burchak yasaymiz. Burchakning ikkinchi tomonida *O nuqtadan boshlab 3* masshtab birligiga teng qilib *OM* kesmani qo'yamiz. *M* nuqta izlanayotgan nuqtadir (1-chizma). ■



1-chizma



2-chizma.

14-misol. Qutb koordinatalar sistemasida $N(3;-\frac{\pi}{3})$ nuqtani yasang.

Yechilishi. *Or* o'qidan *O* qutb atrofida soat strelkasi yo'nalishda $-\frac{\pi}{3}$ burchak yasaymiz. Burchakning ikkinchi tomonida *O nuqtadan boshlab 3* masshtab birligiga teng qilib *OM* kesmani qo'yamiz. *M* nuqta izlanayotgan nuqtadir (2 – chizma). ■

14-misol. Nuqtaning to'g'ri burchakli dekart koordinatalari (-2; 2). Bu nuqtaning qutb koordinatalari toping.

Yechilishi. Berilgan misolda $x = -2$, $y = 2$. Endi r va φ larni (2) formulalarga asosan topamiz:

$$r=\sqrt{4+4}=2\sqrt{2},\ \varphi=\text{arctg}\left(-\frac{2}{2}\right)=arctg(-1)=\frac{3\pi}{4}$$

Demak, $\left(2\sqrt{2};\ \frac{3\pi}{4}\right)$. ■

15– misol. Nuqtaning qutb koordinatalari $\left(2;\ \frac{\pi}{4}\right)$. Bu nuqtaning to'g'ri burchakli koordinatalari topilsin.

Yechilishi. Misol shartiga ko'ra $r=2$, $\varphi=\frac{\pi}{4}$. Endi (1) formulalarga ko'ra x va y larni topamiz: $x=2\cos\frac{\pi}{4}=2\frac{\sqrt{2}}{2}=\sqrt{2}$, $y=2\sin\frac{\pi}{4}=2\frac{\sqrt{2}}{2}=\sqrt{2}.$

Demak $(\sqrt{2},\sqrt{2})$. ■

17–misol. Nuqtaning qutb koordinatalari $\left(4;\ -\frac{\pi}{3}\right)$. buning to'g'ri burchakli dekart koordinatalari topilsin.

Yechilishi. Berilgan misolda $r = 4$, $\varphi=-\frac{\pi}{3}$. Bularni $x=r\cos\varphi$, $y=r\sin\varphi$, formulalarga qo'yilsa: $x=4\cos(-\frac{\pi}{3})=2$,

$$y=4\sin(-\frac{\pi}{3})=-2\sqrt{3}.$$ Demak $(2;-2\sqrt{3})$ ■

5. Dekart koordinatalar sistemasini almashtirish.

17–misol. $A(3;2)$ nuqta berilgan. Agar koordinata o'qlarini yo'nalishlarini saqlagan holda koordinatalar boshini $O_1(-5;3)$ nuqtaga ko'chirilsa, bu nuqtaning koordinatalarini toping.

Yechlishi. (3.4.1) formulalarda $x=3, y'=-2,\ x'=-5, y'=3$ deb topamiz.

$$x_0=3-(-5)=8,\quad y_0=-2-3=-5.$$

Shunday qilib, yangi sistemada A nuqta (8;-5) koordinatalarga ega ekan.

18-misol. Agar koordinata o'qlari 75^0 ga burilgan bo'lsa, u holda koordinata almashtirish formulalari yozilsin.

Yechilishi. Koordinata o'qlari 75^0 burilganligi uchun (2) formulaga asosan, koordinata almashtirish formulasi quyidagicha bo'ladi:

$$x=x_1\cos 75^0-y_1\sin 75^0,\quad y=x_1\sin 75^0+y_1\cos 75^0$$

$sin75^0$, $cos75^0$ ning qiymatlarini hisoblaymiz:

$$sin75^0=sin(30^0+45^0)=sin30^0cos45^0+cos30^0sin45^0=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}}{4}(1+\sqrt{3}),$$

$$cos75^0=cos(30^0+45^0)=cos30^0cos45^0+sin30^0sin45^0=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\frac{\sqrt{2}}{4}(\sqrt{3}-1).$$

Topilgan qiymatlarni yuqoridagi formulaga qo'ysak, u holda koordinatalar almashtirish formulalari quyidagicha bo'ladi:

$$x=\frac{\sqrt{2}}{4}(\sqrt{3}-1)x_1-\frac{\sqrt{2}}{4}(1+\sqrt{3})\,y_1,\quad y=\frac{\sqrt{2}}{4}(\sqrt{3}+1)\,x_1-\frac{\sqrt{2}}{4}(1-\sqrt{3})\,y_1.\ ■$$

19-misol. Umumiy O koordinatalar boshiga ega bo'lgan ikkita Ox, Oy va Ox', Oy' koordinatalar sistemasi biri ikkinchisiga nisbatan biror α burchakka burilgan. $A(\sqrt{3};1)$ nuqtaning koordinatalari birinchi sistemaga nisbatan olingan

Ox' o'qining musbat yo'nalishi OA kesmaning yo'nalishi bo'yicha aniqlanadi. Shu shartlarda koordinatalar almashtirish formulasini keltirib chiqaring.

Yechilishi. $\overline{OA}$ kesmaning Ox o'qi musbat yo'nalishlari bilan tashkil qilingan burchagi α ni topamiz: $\text{tg}\,\alpha=\frac{1}{\sqrt{3}}$, bundan $\alpha=30^0$. Ox' o'qining musbat yo'nalishi OA kesmaning yo'nalishiga mos tushgani uchun, yangi sistema, eski sistemani $\alpha=30^0$ burchakga burish natijasida hosil qilinadi.

Shuning uchun (3.4.2) formulaga binoan koordinatalar almashtirish formulasi quyidagicha bo'ladi:

$$x=x_1cos30^0-y_1sin30^0=\frac{\sqrt{3}}{2}x_1-\frac{1}{2}y_1,\ y=x_1sin30^0+y_1cos30^0=\frac{1}{2}x_1+\frac{\sqrt{3}}{2}y_1$$

.■

Mustaqil yechish uchun misollar

1. Masshtab kesmani 1 sm ga teng qilib olib quyidagi nuqtalarni yasang: $A(4)$, $B(5)$, $C(\frac{1}{2})$, $D(-\frac{1}{6})$, $E(-\sqrt{5})$, $F(3\sqrt{5})$.

2. $A(-6)$ va B (2) nuqtalar berilgan. AB kesmaning uzunligini toping.

3. $A(4)$ va $B(8)$ nuqtalar bilan chegaralangan kesmani $\lambda=\frac{1}{4}$ nisbatda bo'luvchi N nuqtaning koordinatalarini toping.

4. $M(-3)$ va $N(4)$ nuqtalar berilgan. MN kesmani 1) $\lambda=2$; 2) $\lambda=\frac{3}{4}$; 3) $\lambda=1$; 4) $\lambda=0$ nisbatda bo'luvchi K nuqtani toping.

5. Quyidagi hollarning har birida AB kesmaning o'rtasini toping. 1) $A(-7)$, $B(-3)$; 2) $A(6)$, $B(-3)$; 3) $A(-2)$, $B(5)$.

6. $A(-5)$ nuqtadan $B(-2)$ nuqtaga nisbatan uch marta katta masofada $\underline{AB}$ kesmadan tashqarida yotuvchi N nuqtani toping.

7. A (-3) va $B(7)$ nuqtalarda og'irliklari 6 H va 18 H bo'lgan yuklar qo'yilgan. Teng ta'sir etuvchi qo'yilgan nuqtani toping

8. To'g'ri burchakli Dekart koordinatalar sistemasida quyidagi $A(5, 3)$, $B(-3; 4)$, $C(6,-4)$, $D(-3,-2)$, $E(-4,0)$, $F(0,-2)$ nuqtaning yasang.

9. Agar $ABCDE$ beshburchakning uchlari $A(6, 0)$, $B(5; 2)$, $C(0,3)$, $D(-7,1)$, $E(-4,-6)$ bo'lsa, shu beshburchakni yasang.

10. Agar: 1) nuqtaning ordinatasi manfiy; 2) nuqtaning abssissasi musbat bo'lsa, shu nuqta qaysi choraklarda yotishi mumkin.

11. Kvadratning tomoni 5 birlikka teng. Uning ikki qo'shni tomoni koordinatalar o'qlarida yotadi. Kvadrat uchlarini koordinatalarini toping (q hol qarab chiqilsin).

12. Tomoni 2 birlikka teng bo'lgan kvadratning diagonallari koordinata o'qlarida yotadi. Uning uchlarining koordinatalarini toping.

13. Agar: 1) $A(3, -4)$, $B(-6; 8)$; 2) $A(6, 0)$, $B(3; -6)$; 3) $A(-2, -4)$, $B(-1; 8)$; 4) $A(6, -5)$, $B(-2;1)$ bo'lsa, A va B nuqtalar orasidagi masofani toping.

14. Uchlari $A(3;-2)$, $B(-1;3)$ va $C(2;5)$ nuqtalarda joylashgan uchburchakning perimetrini toping.

15. Quyidagi hollarning har birida ABC uchburchak o'tkir burchakli, o'tmas burchakli yoki to'g'ri burchakli bo'lishini aniqlang:

1) $A(1;4)$, $B(5;8)$, $C(3;2)$; 2) $A(2;1)$, $B(-2;5)$, $C(-1;3)$;
3) $A(1;-1)$, $B(-2;1)$, $C(1;2)$.

16. $A(2;4)$, $B(3;9)$ va $C(2;-4)$ nuqtalardan baravar uzoqlashgan D nuqtani toping.

17. Koordinata o'qlaridan va $C(-2;4)$ nuqtadan baravar uzoqlashgan nuqtani topnig.

18. AB kesmani (bu yerda $A(4;6)$ va $B(2;-5)$ 1) $\lambda=2$; 2) $\lambda=1$; 3) $\lambda=\frac{1}{3}$;

4) $\lambda=\frac{2}{3}$ nisbatda bo'luvchi nuqtaning koordinatalarini toping.

19. ABC uchburchak berilgan: $A(-1;4)$, $B(2;3)$, $C(7;-4)$. A burchak bissektrisasining uzunligini toping.

20. ABC uchburchakning ikkita uchi: $A(3;0)$, $B(-5;7)$ va og'irlik markazining koordinatalari $M(1;14)$) ma'lum. Uchburchakning uchinchi uchi C ning koordinatalarini toping.

21. ABC parallelogramda $A(1;2)$, $B(-2;3)$, $C(5;6)$ uchlari ma'lum. D uchuni toping.

22. $A(5;-3)$ va $B(7;6)$ nuqtalarni tutashtiruvchi kesmada ordinatasi $y_C = 2$ bo'lgan C nuqtani toping.

23. $A(2;-3)$ va $B(0;-9)$ nuqtalarda o'tuvchi to'g'ri chiziqda teng koordinatalarga ega bo'lgan C nuqtani toping.

24. Uchlari $A(-3;-1)$, $B(2;3)$ va $C(-1;5)$ nuqtalardan bo'lgan ABC uchburchakning yuzini toping. (J: 11kv. birlik)

25. Burchaklari uchlaridan ikkitasining koordinadalari (-2;2) va (4;5) bo'lib, uchinchisi koordinatalar boshida bo'lgan uchburchakning yuzini toping.

26. Burchaklari uchlarining koordinatalari $A(4;6)$, $B(2;-4)$, $C(-3;2)$ va D(-3;2) bo'lgan to'rtburchakning yuzini toping.

27. Burchaklarining uchlari: $A(-2,3)$, $B(4,1)$, $C(2,5)$, $D(-2,1)$ va $E(-1,-2)$ nuqtalarda bo'lgan beshburchakning yuzini toping.

28. Uchlari $A(1;5)$, $B(2;7)$, $C(4;11)$ nuqtalar bo'lgan ABC uchburchakning yuzini toping.

Qutb koordinatalar sistemasida quyidagi nuqtalarni yasang:

29. $A\left(3;\ \frac{\pi}{3}\right)$, $B\left(4;\ \frac{3\pi}{4}\right)$, $C\left(2;-\frac{\pi}{3}\right)$, $D\left(5;\ -\frac{3\pi}{4}\right)$.

30. Ushbu $\left(2;\ \frac{\pi}{4}\right)$, $\left(4;\ \frac{2\pi}{3}\right)$ va $\left(1;-\frac{\pi}{6}\right)$nuqtalarga a) qutbga nisbatan;

b) qutb o'qiga nisbatan simmetrik bo'lgan nuqtaning qutb koordinatalarini toping.

31. $A\left(4;\ \frac{\pi}{3}\right)$ nuqtaning to'g'ri burchakli koordinatalarini toping.

32. Quyidagi nuqtalarning to'g'ri burchakli koordinatalarini toping.

1) $A\left(2;\ \frac{\pi}{2}\right)$; 2) $B\left(2\sqrt{3};\ -\frac{\pi}{3}\right)$, 3) $C\left(\sqrt{2};-\frac{\pi}{4}\right)$; 4) $D\left(2;\ -\frac{3\pi}{4}\right)$.

33. To'g'ri burchakli koordinatalari $(-3;\ 3\sqrt{3})$ bo'lgan nuqtaning qutb koordinatalarini toping.

34. Quyidagi nuqtalarning qutb koordinatalarini toping.

1) $A(0;5)$; 2) $B(0;-\sqrt{3})$, 3) $C(-\sqrt{3};-1)$; 4) $D(1;\ -1)$

35. Qutb koordinata sistemasining boshi O'(4;7) nuqtada bo'lib, qutb o'qi burchakli dekart koordinata sistemasi bilan $\alpha=\frac{\pi}{3}$ burchak tashkil qiladi. A(6;9) nuqtaning koordinatalari Dekart koordinatalar sistemasida berilgan. A nuqtaning qutb koordinatalar sistemasidagi koordinatalari topilsin.

36. Quyidagi funksiyalarning grafiklarini chizing.

1. $\rho=a\sin 3\varphi$ 2. $\rho=2+6\cos\varphi$. 3. $\rho=3+4\cos\varphi$. 4. $\rho=\frac{2}{\sin\varphi}+4$

5. $\rho=2\frac{\sin\varphi}{\varphi}$. 6. $\rho=6\sin\frac{\varphi}{3}$. 7. $\rho=\sqrt{\frac{\pi}{\varphi}}$. 8. $\rho=\frac{3}{\varphi}$.

37. Ikkita $M_1(3;1)$ va $M_2(-9;-4)$ nuqtalar berilgan. Koordinatalar boshi M_1 nuqtaga ko'chirilgan. Koordinatalar o'qlari shunday berilganki, natijada yangi abssissa o'qining musbat yo'nalishi, $\overline{M_1M_2}$ yo'nalgan kesmaning yo'nalishi bilan mos tushadi. Koordinatalar almashtirish formulalarini keltirib chiqaring.

38. A nuqtaning Oxy koordinata sistemasidagi koordinatalari 2 va 4. Koordinata o'qlarining y o'nalish o'zgartirilmasdan koordinata boshi $O'(3;4)$

nuqtaga ko'chirilgan. A nuqtaning yangi koordinata sistemasidagi koordinatalari topilsin.

39. A nuqtaning Oxy koordinata sistemasidagi koordinatalari 3 va –2, $O'x'y'$ sistemadagi koordinatalari –3 va –5. Eski sistemaga nisbatan yangi sistema koordinata boshining koordinatalari topilsin.

40. $A(5;3)$, $B(-2;4)$, $C(3;-7)$ nuqtalar berilgan. Agar o'qlarning yo'nalishini o'zgartirmasdan koordinatalar boshi A nuqtaga keltirilgan bo'lsa, u vaqtda berilgan nuqtalarning yangi sistemadagi koordinatalari topilsin.

41. M (2;4) nuqta Oxy koordinata sistemasida berilgan. Agar $O'x'y'$ koordinata sistemasining boshi $O'(3;5)$ nuqta bo'lib, $O'x'$ o'qining yo'nalishi Ox o'qining yo'nalishi bilan bir xil, $O'y'$ o'qining yo'nalishi esa Oy o'qining yo'nalishiga qarama-qarshi yo'nalgan bo'lsa, M nuqtaning $O'x'y'$ koordinata sistemasidagi koordinatalari topilsin.

42. $A(4;6)$ nuqta berilgan. Agar koordinata o'qlari 60^0 ga burilgan bo'lsa, u holda A nuqtaning yangi sistemadagi koordinatalari topilsin.

43. $M(4:-2)$ nuqta Oxy koordinata sistemasida berilgan. Koordinata o'qlari O nuqta atrofida $\alpha=45^{\circ}$ ga burilganda M nuqtaning $Ox'y'$ sistemadagi koordinatalari topilsin.

44. $M(2;3)$ nuqta Oxy koordinata sistemasida berilgan. Koordinata o'qlarini $\alpha=\frac{\pi}{6}$ ga burib, koordinata boshini $O'(4;-1)$ nuqtaga ko'chirganda, M nuqtaning $O'x'y'$ sistemasidagi koordinatalari topilsin.

45. Koordinatalar boshi O^0 (-2;4) nuqtaga ko'chirilgan bo'lib, koordinatalar o'qlari esa $\alpha=60^0$ ga burilgan. $A(3;5)$ nuqtaning koordinatalari yangi sistemaga nisbatan aniqlangan. A nuqtaning eski sistemaga nisbatan koordinatalari aniqlansin.

46. $A(\sqrt{2};-3), B(\sqrt{3};3)$ va $C(0;-2\sqrt{2})$. nuqtalar yangi sistemada berilgan. Koordinata o'qlari $\alpha=30^{\circ}$ ga burilganda A,B va C nuqtalarning eski sistemasidagi koordinatalari topilsin.

47. Koordinata o'qlari $\alpha=30^0$ ga burilgan. $M(3;5)$ nuqtaning koordinatalari yangi sistemada aniqlangan. Bu nuqtaning koordinatalarining eski sistemada hisoblansin.

3.2-amaliy mashg'ulot.
Tekislikda to'g'ri chiziq

1- misol. Ox o'qining musbat yo'nalishi bilan 45^0 burchak tashkil qilgan va koordinata boshidan o'tgan to'g'ri chiziq tenglamasini toping.

Yechilishi. berilgan masalani yechish uchun to'g'ri chiziqning $y=kx+b$ ko'rinishdagi tenglamasidan foydalanish qulay. Shartga ko'ra to'g'ri chiziq koordinata boshidan o'tadi, u holda b=0. To'g'ri chiziqning burchak koeffitsienti $k=\mathrm{tg}135^0=-1$ bo'ladi.

Demak, izlanayotgan to'g'ri chiziq tenglamasi $y=-x$ bo'ladi.

2-misol. M(2,1) nuqtadan o'tgan: 1) $y=3x-4$ to'g'ri chiziqqa parallel; 2) $4x-5y+6=0$ to'g'ri chiziqqa perpendikulyar bo'lgan to'g'ri chiziqni toping.

Yechilishi. Berilgan nuqtadan berilgan yo'nalishda o'tgan to'gri chiziqning

$y-y0=k(x-x0)$ tenglamasiga ko'ra M(2,1) nuqtadan o'tgan to'g'ri chiziq $y-1=k(x-2)$ bo'ladi. k koeffitsientni topamiz.

1) $y=3x-4$ to'g'ri chiziq uchun burchak koeffitsienti $k=3$. berilgan to'g'ri chiziqqa izlanayotgan to'g'ri chiziq parallel. Parallellik shartiga ko'ra, burchak koeffitsientlari teng bo'ladi: $k1=k=3$. Demak, M(2,1) nuqtadan o'tgan $y=3x-4$ to'g'ri chiziqqa parallel to'g'ri chiziq tenglamasi $y-1=3(x-2)$ yoki $3x-y-5=0$ ko'rinishda bo'ladi.

2) $4x-5y+6=0$ to'g'ri chiziq umumiy tenglamasi bilan berilgan. Bu to'g'ri chiziqning burchak koeffitsientini topish uchun y ga nisbatan yechamiz: $y=\frac{4}{5}x+\frac{6}{5}$. Berilgan to'g'ri chiziqning burchak koeffitsienti $k=0{,}8$. izlanayotgan to'g'ri chiziq berilgan o'g'ri chiziqqa perpendikulyar. To'g'ri chiziqlarning perpendikulyarlik sharti $k1k=-1$ ko'ra $k_1=-\frac{1}{k}=-\frac{5}{4}$.

Demak, M(2;1) nuqtadan o'tgan $4x-5y+6=0$ to'g'ri chiziqqa perpendikulyar bo'lgan to'g'ri chiziq tenglamasi: $y-1=-\frac{5}{4}(x-2)$ yoki $5x+4y-14=0$ ko'rinishda bo'ladi.

3-misol. Ushbu $6x-2y+7=0$ va $4x+2y-9=0$ to'g'ri chiziqlar orasidagi o'tkir burchakni toping.

Yechilishi. Berilgan to'g'ri chiziqlarning burchak koeffitsientlarini topamiz:

$$6x-2y+7=0; y=3x+\frac{7}{2}, k_1=3;$$

$$4x+2y-2=0; y=-2x+\frac{9}{2}, k=-2.$$

k_1 va k_2 ning qiymatlarini $tg\varphi=\left|\frac{k_2-k_1}{1+k_1k_2}\right|$ formulaga qo'yamiz:

$tg\varphi=\left|\frac{-2-3}{1-3\cdot 2}\right|=1, \varphi=\frac{\pi}{4}$. Demak, berilgan to'g'ri chiziqlar orasidagi o'tkir burchak $k=\frac{\pi}{4}$ ga teng ekan.

Mustaqil yechish uchun misollar

1. Ushbu 1) P(4,0) va Q(3,1), 2) C(-1,1) va D(2,7), 3) A(2,-4) va B(-3,11) nuqtalardan o'tgan to'g'ri chiziqning burchak koeffitsienti va ordinatalar o'qidan ajratgan kesmasini toping.

2. To'g'ri burchakli dekart koordinatalar sistemasining boshidan o'tuvchi va x o'qiga: 1) 45^0, 2) 60^0, 3) 135^0, 4) 180^0 og'ma bo'lgan to'g'ri chiziq tenglamasini yozing.

3. To'g'ri burchakli koordinatalar sistemasiga nisbatan, koordinatalar boshidan o'tuvchi va

1) $y=\frac{1}{4}x+1$ $y=3x+5$ to'g'ri chiziqqa parallel bo'lgan;

2) to'g'ri chiziqqa perpendikulyar bo'lgan;

3) $y=2x+5$ to'g'ri chiziq bilan 45^0 burchak tashkil qilgan;

4) $y=x-1$ to'g'ri chiziqqa 60^0 li burchak ostida o'gma bo'lgan to'g'ri chiziqning tenglamasini yozing.

4. Uchburchakning uchlari berilgan: A(2,3), B(-2,-1) va C(4,-2).

1) Uning uchala tomonining;

2) *C* uchidan o'tkazilgan medianasining;

3) *A* uchidan BC tomoniga tushirilgan balandligining tenglamasini tuzing.

5. Berilgan uchta nuqtaning bir to'g'ri chiziqda yotishi yoki yotmasligini tekshiring:

1) (1,3), (5,7) va (10,12) 2) (2,4), (4,-1) va (0,3).

6. 1) A(-2,-3) nuqtadan o'tuvchi va burchak koeffitsienti $k=1$ bo'lgan to'g'ri chiziq tenglamasini tuzing; 2) (-2,0) nuqtadan o'tuvchi va burchak koeffitsienti $k=-2$ ga teng bo'lgan to'g'ri chiziq tenglamasini tuzing.

7. (-3,-2) nuqtadan o'tuvchi va *Ox* o'qi bilan arctg2 burchak tashkil etuvchi to'g'ri chiziq tenglamasini tuzing.

8. 1) C(3,1) va D(4,-2), 2) A(2,3) va B(-3,1) nuqtalardan o'tuvchi to'g'ri chiziqning Ox o'qqa o'g'ish burchagini toping.

9. A(6,2) va (-3,8) nuqtalardan o'tuvchi to'g'ri chiziqning koordinata o'qlarida ajratuvchi kesmalarini toping.

10. Quyidagi to'g'ri chiziqlarning kesishish nuqtalarini toping:

1) $y=5x$ va $x+y-12=0$, 2) $x-4y-7=0$ va $x+2y-4=0$.

11. Ushbu to'g'ri chiziqlar orasidagi o'tkir burchakni toping:

1) $y=3x$ va $y=-x$ 2) $2x-3y+6=0$ va $3x-y-3=0$

3) $\frac{x}{5}+\frac{y}{2}=1$ va $\frac{x}{3}+\frac{y}{4}=1$.

12. $5x-12y-16=0$ va $3x+4y-12=0$ to'g'ri chiziqlar orasidagi o'tkir burchakni toping:

13. Uchlari A(-6,-1), B(4,6) va C(2,1) bo'lgan uchburchak berilgan. Bu uchburchakning ichki burchaklarini toping.

14. Uchburchakning A(2,-1), B(-7,3) va C(-1,-5) uchlari berilgan. C burchak bissektrisasining tenglamasini tuzing:

15. 1) A(-7,3) nuqtadan $5x-7y+21=0$ to'g'ri chiziqqa parallel holda o'tuvchi to'g'ri chiziq tenglamasini tuzing; 2) A(-1,-4) nuqtadan $\frac{x}{4}+\frac{y}{3}=1$ to'g'ri chiziqqa parallel holda o'tuvchi to'g'ri chiziq tenglamasini tuzing.

16. 1) $B(5;-2)$ nuqtadan $6x-12y+5=0$ to'g'ri chiziqqa perpendikulyar holda o'tuvchi to'g'ri chiziq tenglamasini tuzing; 2) $M(-4;1)$ nuqtadan $\frac{x}{5}-\frac{y}{6}=1$ to'g'ri chiziqqa perpendikulyar holda o'tuvchi to'g'ri chiziq tenglamasini tuzing.

17. 1) $M(6;8)$ nuqtadan $4x+3y+2=0$ to'g'ri chiziqqacha bo'lgan masofani toping;

2) $N(4;6)$ nuqtadan $3x+4y+14=0$ to'g'ri chiziqqacha bo'lgan masofani toping;

3) Ikkita parallel $4x+3y-8=0$ va $4x+3y-33=0$ to'g'ri chiziqlar orasidagi masofani toping.

18. To'g'ri burchakli dekart koordinatalar sistemasida berilgan to'g'ri chiziqlarning tenglamalari normal shaklga keltiring:

1) $4x-3y+10=0$, 2) $6x+8y-15=0$

3) $y-x\sqrt{3}=4$ 4) $x\cos 10^0+y\sin 10^0+4=0$

19. $7x-y+3=0$ va $3x+5y-4=09$ to'g'ri chiziqlarning kesishish nuqtasidan va $A(2,-1)$ nuqtadan o'tuvchi to'g'ri chiziqning tenglamasini yozing.

20. m va n ning qanday qiymatlarida $mx+8y+n=0$ va $2x+my-1=0$ to'g'ri chiziqlar: 1) parallel; 2) ustma-ust; 3) perpendikulyar bo'ladi?

21. Ushbu $(x+2y-7)+\lambda(3x-y+5)=0$ dastaga tegishli va dastaning asosiy to'g'ri chiziqlaridan har biriga perpendikulyar bo'lgan to'g'ri chiziqlarning tenglamasini toping.

22. Teng tomonli to'g'ri burchakli uchburchak gipotenuzasi tenglamasi $y=7x-4$ va uning to'g'ri burchak uchi $C(3,4)$ nuqtada bo'lganda uchburchak katetlarining tenglamasini tuzing.

23. Quyidagi to'g'ri chiziqlarning parametrik tenglamasini yozing:

1) $y=2x-3$, 2) $y=0{,}5x+1$, 3) $6x+11y+9=0$,

4) $\frac{x}{3}-\frac{y}{4}=1$; 5) $\frac{x-1}{2}=\frac{y}{3}$; 6) $4y+5=0$.

24. μ va λ koeffitsientlar qanday shartni qanoatlantirganda $\lambda x+\mu y+2=0, 3x-2y+3=0, y-1=0$ to'g'ri chiziqlar bir nuqtada kesishadi?

25. Agar $A_1x+B_1y+C_1=0, A_2x+B_2y+C_2=0, A_3x+B_3y+C_3=0$, to'g'ri chiziqlar bir nuqtada kesishsa, $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=0$ bo'lishini isbotlang.

26. M nuqtaning $5x-12y-13=0$ va $3x-4y-19=0$ to'g'ri chiziqlardan chetlanishi mos ravishda -3 va -5 ga teng, M nuqtaning koordinatalarini toping.

Mustaqil yechish uchun misollarning javoblari

1.1) $k=-1, b=4$; **2)** $k=2, b=3$; **3)** $k=-3$, $b=2$. **2. 1)** $y=x$; **2)** $y=\sqrt{3}x$; **3)** $y=-x$; **4)** $y=0$. **3. 1)** $y=3x$; 2). $y=-4x$; 3). $y=-3x$ *yoki* $y=\frac{1}{3}x$. **4)** $y=-(2+\sqrt{3})x$ yoki $y=-(2-\sqrt{3})x$. **4.** 1). $AB: y=x+1$; AC: $y=-\frac{5}{2}x+8$. $BC: y=-\frac{1}{6}x-\frac{4}{3}$. **2)** $y=-\frac{3}{4}x+1$, **3)** $y=6x-9$.

5. 1). $y=x-1$; 2). $y=-2x-4$. **6.** $2x-y+4=0$.

7. 1) $\alpha=arctg0{,}4=21^0 48'$; 2) $180^0-arctg3\approx 108^0 26'$. **8.** $x=9, y=6$

9. 1) $(2;10)$ 2) $(5;-0{,}5)$.

10. 1) $\alpha=arctg2\approx 63^0 26'$; 2) $\alpha=arctg\frac{7}{9}\approx 37^0 52'$; 3) $\alpha=arctg\frac{14}{23}\approx 31^0 20'$.

11. $\varphi=\arccos\frac{33}{65}\approx 59^0 29'$. **12.** $tgA=0{,}383; A=20^0 57'$; $tgB=0{,}6545; B=33^0 12'$; $tgC=1{,}3846, C=125^0 50'$.

13. $x+1=0$. **14.1)** $5x-7y+56=0$, **2)** $3x+4y+19=0$.

15. 1) $2x+y-89=0$, **2)** $5x+6y+14=0$. **16. 1)** 10, **2)** 10, **3)** 5.

17. 1) $\frac{4x-3y+10}{-5}=0$; 2) $0{,}6x+0{,}8y-1{,}5=0$ 3) $\frac{y}{2}-\frac{\sqrt{3}}{2}x-2=0$;

4) $x\cos 100-y\sin 100=0$. **18.** $25x+29y-21=0$.

19. 1) $m=-4$, $n\neq 2$ *yoki* $m=4$, $n\neq -2$;
2) $m=-4$, $n=2$, $m=4$, $n=-2$; 3) $m=0$, $n-ixtiyoriy$

20. $14x-7y+32=0$, $7x+21y-75=0$.

21. $y=\frac{3}{4}x+\frac{7}{4}; y=-\frac{4}{3}x+8$. **22. 1)** $x=2+t, y=1+2t$,

2) $x=2+2t, y=2+t$; **3)** $x=-7+11t, y=3-6t$, **4)** $x=3t; y=-4+4t$

5) $x=2-2t, y=3t$; **6)** $x=t, y=-1{,}25$

23. $-\lambda+3\mu+6=0$. **25.** M(2;3).

3.3-amaliy mashg'ulot.
Ikkinchi tartibli chiziqlar
1. Ellips

1-misol. Ellipsning fokuslari koordinatalar boshiga nisbatan simmetrik bo'lib, *OX* o'qida joylashgan. Quyidagi berilganlarga ko'ra ellipsning tenglamasini tuzing: 1) uning yarim o'qlari 7 va 4 ga teng; 2) uning katta o'qi 20, fokuslari orasidagi masofa 2c = 6 ga teng; 3) uning kichik o'qi 8, fokuslari orasidagi masofa 2c=6 ga teng; 4) fokuslari orasidagi masofa 2c = 8 ga teng, ekssentrisiteti esa $\varepsilon = \frac{4}{5}$ ga teng; 5) uning katta o'qi 20, ekssentrisiteti esa $\varepsilon = \frac{4}{5}$ ga teng ; 6) uning kichik o'qi 40, ekssentrisiteti esa $\varepsilon = \frac{3}{5}$ ga teng; 7) uning direktrisalari orasidagi masofa $\frac{50}{3}$ va fokuslari orasidgi masofa esa 2c = 6 ga teng; 8) uning katta o'qi 10, direktrisalari orasidagi masofa 20 ga teng; 9) uning kichik o'qi 8, direktrisalari orasidagi masofa esa $\frac{50}{3}$ ga teng; 10) uning direktrisalari orasidagi masofa $\frac{49\sqrt{10}}{10}$ va $\varepsilon = \frac{2\sqrt{10}}{7}$ ga teng.

Yechilishi. Shartga ko'ra ellipsning fokuslari koordinata boshiga nisbatan simmetrik bo'lib, abssissa o'qida joylashgani uchun uning kanonik tenglamasi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ko'rinishda bo'ladi, bu yerda a uning katta yarim o'qi, b esa uning kichik yarim o'qi.

1). a=7, b=4 bo'lgan uchun ellips tenglamasi $\frac{x^2}{49}+\frac{y^2}{16}=1$ ko'rinishda bo'ladi.

2). Shartga ko'ra 2à=20; 2ñ=12. Bulardan $a = 10$; $c = 6$. b yarim o'qni topish uchun $b=\sqrt{a^2-c^2}$ formuladan foydalanamiz. $b=\sqrt{100-36}=\sqrt{64}=8$. Shunday qilib, ellips tenglamasi $\frac{x^2}{100}+\frac{y^2}{64}=1$ ko'rinishda bo'ladi.

3). Shartga ko'ra $2b = 8$, $2c = 6$. Bulardan $b = 4$, $c = 3$. a katta yarim o'qni topish uchun $a=\sqrt{b^2+c^2}$ formuladan foydalanamiz: $a=\sqrt{b^2+c^2}=\sqrt{16+9}=5$. Shunday qilib, ellips tenglamasi $\frac{x^2}{25}+\frac{y^2}{16}=1$ ko'rinishda bo'ladi.

4). Shartga ko'ra $2a = 8$, $\varepsilon=\frac{4}{5}$. Birinchi tenglikdan $c = 4$, $\varepsilon=\frac{4}{5}$ bo'lgani uchun $a = 5$. b yarim o'qni topish uchun $b=\sqrt{a^2+c^2}$ formuladan foydalanamiz $b=\sqrt{25-16}=3$. Demak, ellips tenglamasi $\frac{x^2}{25}+\frac{y^2}{9}=1$ ko'rinishda bo'ladi.

5). Shartga ko'ra $2a = 16$, $\varepsilon=\frac{3}{4}$. Birinchi tenglikdan $a=8, \varepsilon=\frac{c}{a}$ bo'lgani uchun $c=a\varepsilon=8\cdot\frac{3}{4}=6$. b yarim o'qni $b=\sqrt{a^2+c^2}$ formuladan topamiz: $b=\sqrt{64-36}=\sqrt{28}=2\sqrt{7}$. Demak, ellips tenglamasi $\frac{x^2}{64}+\frac{y^2}{28}=1$ ko'rinishda bo'ladi.

6). Shartga ko'ra 2b = 40, $\varepsilon=\frac{3}{5}$. Bulardan $b = 20$, $\varepsilon=\frac{c}{a}$, $c=\sqrt{a^2-b^2}$ bo'lgani uchun $\varepsilon^2=\frac{a^2-b^2}{a^2}=1-\left(\frac{b}{a}\right)^2$ keyingi tenglikdan: $\frac{b}{a}=\sqrt{1-\varepsilon^2}$, bundan $a=\frac{b}{\sqrt{1-\varepsilon^2}}=\frac{20}{\sqrt{1-\frac{9}{25}}}=\frac{20\cdot5}{4}=25$. Demak, ellips tenglamasi $\frac{x^2}{625}+\frac{y^2}{400}=1$ ko'rinishda bo'ladi.

7). Shartga ko'ra $\frac{2a}{\varepsilon}=\frac{50}{3}$, $2c = 6$. Ikkinchi tenglikdan $c = 3$, $\varepsilon=\frac{c}{a}$, va $\frac{2a}{\varepsilon}=\frac{50}{3}$ tengliklardan $a^2=\frac{25}{3}$, $c=\frac{25}{3}3=25$, $b=\sqrt{a^2-c^2}$ formulaga asosan $b=\sqrt{25-9}=4$. Demak, ellipis tenglamasi $\frac{x^2}{25}+\frac{y^2}{16}=1$ ko'rinishda bo'ladi.

8). Shartga ko'ra $2a = 10$, $\frac{2a}{\varepsilon}=20$. Bulardan $a=5$, $\varepsilon=\frac{a}{10}=\frac{5}{10}=\frac{1}{2}$. $\varepsilon=\frac{c}{a}$ bo'lgani uchun, bundan $c=a\varepsilon=\frac{5}{2}$. b ni topish uchun $b=\sqrt{a^2-c^2}$ formuladan foydalanamiz. $B=\sqrt{25-\frac{25}{4}}=\frac{5\sqrt{3}}{2}$. Demak, ellips tenglamasi

$\frac{x^2}{25}+\frac{y^2}{\frac{75}{4}}=1$ ko'rinishda bo'ladi.

9). Shartga ko'ra $2b=8$, $\frac{2a}{\varepsilon}=\frac{50}{3}$. Bulardan $b=4$, $\frac{a}{\varepsilon}=\frac{25}{3}$, $\varepsilon=\frac{c}{a}$, bundan esa $\frac{a^2}{c}=\frac{a^2}{\sqrt{a^2-16}}=\frac{25}{3}$, bundan $9a^4-625a^2+10000=0$. Bu tenglamani yechish natijasida $a^2=25$ va $a^2=\frac{400}{9}$, $a_1=5$, $a_2=\frac{20}{3}$. Demak, ellips tenglamasi $\frac{9x^2}{400}+\frac{y^2}{16}=1$ ko'rinishda bo'ladi.

10). Shartga ko'ra $\frac{2a}{\varepsilon}=\frac{49\sqrt{10}}{10}$, $\varepsilon=\frac{2\sqrt{10}}{7}$. Bulardan $a=\frac{49\sqrt{10}}{2\ \ 10}$

$\varepsilon=\frac{49\sqrt{10}}{20}\frac{2\sqrt{10}}{7}=7$, $\varepsilon=\frac{c}{a}$ bundan $c=a\varepsilon=\frac{2\sqrt{10}}{7}7=2\sqrt{10}$,

$b=\sqrt{49\ \ 40}=3$.

Demak, ellips tenglamasi $\frac{x^2}{49}+\frac{y^2}{9}=1$ ko'rinishda bo'ladi.■

2. Giperbola.

2-misol. $25x^2-144y^2=3600$ giperbola berilgan. Uning 1) a va b yarim o'qlari; 2) fokuslari; 3) ekssentrisiteti; 4) asimptotalarining tenglamasi; 5) direktrisalarining tenglamasini toping.

Yechilishi. 1). Berilgan giperbolaning tenglamasini sodda ko'rinishga keltiramiz: $\frac{x^2}{144}-\frac{y^2}{25}=1$ bundan $a=12$, $b=5$.

2). 1)-shartga ko'ra $a=12$; $b=5$. Ushbu $b=\sqrt{c^2\ \ a^2}$ yoki $c=\pm\sqrt{b^2+a^2}$ formulalarga asosan giperbolaning fokuslarini topamiz: $c=\pm 13$.

3). Ushbu $\varepsilon=\frac{2c}{2a}=\frac{c}{a}$ formulalarga asosan giperbolaning ekssentrisiteti topamiz: $\varepsilon=\frac{13}{12}$.

4). 1-shartga ko'ra $a=12$; $b=5$. Ushbu $y=\frac{b}{a}x$, $y=-\frac{b}{a}x$ to'g'ri chiziqlar giperbolaning asimptotalari. U holda giperbolaning asimptotalari tenglamasi $y=\frac{5}{12}x$, $y=-\frac{5}{12}x$ ko'rinishda bo'ladi.

5). 1)-3)-shartlarga ko'ra $a=12$; $\varepsilon=\frac{13}{12}$. Giperbolaning direktrisalari tenglamasi $x=\frac{a}{\varepsilon}=\frac{144}{13}$ va $x=\frac{-a}{\varepsilon}=\frac{-144}{13}$ ko'rinishda bo'ladi.■

3. Parabola.

3-misol. Parabola uchi koordinatalar boshida yotsa va quyidagilarni bilgan holda parabola tenglamasi tuzilsin:

1) parabola OX o'qiga nisbatan simmetrik joylashgan va u A(2;4) nuqtadan o'tadi;

2) parabola OX o'qiga nisbatan simmetrik joylashgan va u $B(-4;12)$ nuqtadan o'tadi;

3) parabola OY o'qiga nisbatan simmetrik joylashgan va u $C(6;9)$ nuqtadan o'tadi;

4) parabola OY o'qiga nisbatan simmetrik joylashgan va u $D(10;-2)$ nuqtadan o'tadi.

Yechilishi. 1) parabolaning uchi koordinata boshida va OX o'qiga nisbatan simmetrik joylashgani va $A(2;4)$ nuqtadan o'tgani uchun o'ng yarim tekislikda yotgan bo'ladi hamda uning kanonik tenglamasini ko'rinishi

$$y^2=2rx$$

shaklida bo'ladi. Bu tenglamadagi x va y larning o'rniga $A(2;4)$ nuqtaning koordinatalarini qo'ysak, natijada $4r=16$ tenglamani hosil qilamiz. Bundan esa

$r=4$ ni topamiz. Demak, parabolaning kanonik tenglamasi $y^2=8x$ ko'rinishda bo'ladi.

2) parabolaning uchi koordinatalar boshida, o'zi OX o'qiga nisbatan simmetrik joylashgani va $B(-4;12)$ nuqtadan o'tgani uchun u chap yarim tekislikda joylashgan bo'ladi va uning kanonik tenglamasining ko'rinishi

$$y^2=-2rx$$

shaklida bo‘ladi. Bu tenglamadagi x va y larning o‘rniga $B(-4;12)$ nuqtaning koordinatalarini qo‘yib, natijada $8r = 144$ tenglamani hosil qilamiz. Bundan esa $r = 18$ ni topamiz. Demak, parabolaning kanonik tenglamasi: $y^2 = -36x$.

3) parabolaning uchi koordinata boshida, o‘zi OY o‘qiga nisbatan simmetrik joylashgani va $C(6; 9)$ nuqtadan o‘tgani uchun u yuqori yarim tekislikda joylashgan bo‘ladi va uning kanonik tenglamasining ko‘rinishi

$$x^2 = 2ry$$

ko‘rinishda bo‘ladi. Bu tenglamadagi x, y larning o‘rniga $C(6; 9)$ nuqtaning koordinatalarini qo‘ysak, natijada $18r = 36$ tenglamani hosil qilamiz. Bundan esa

$r = 2$ ni topamiz. Demak, parabolaning kanonik tenglamasi $x^2 = 4y$ ko‘rinishda bo‘ladi.

4) parabolaning uchi koordinata boshida, o‘zi esa OY o‘qiga nisbatan simmetrik joylashgani va $D(10; -2)$ nuqtadan o‘tgani uchun u pastki yarim tekislikda joylashgan bo‘ladi. Uning kanonik tenglamasining ko‘rinishi

$$x^2 = -2ry$$

shaklida bo‘ladi. Bu tenglamadagi x, y larning o‘rniga D(-10;-2) nuqtaning koordinatalarini qo‘ysak, natijada $4r = 100$ tenglamani hosil qilamiz. Bundan esa

$r = 25$ ni topamiz. Demak, parabolaning kanonik tenglamasi: $x^2 = -50y$.

Mustaqil yechish uchun misol va masalalar
Ellips

1. Quyidagi ma’lumotlarga ko‘ra ellips fokuslari abssissa o‘qida, koordinata boshiga nisbatan simmetrik bo‘lgan ellipsning eng sodda tenglamasini tuzing: 1) Yarim o‘qlari $a = 16$ va $b = 8$ ga teng; 2) Fokuslari orasidagi masofa $2c = 12$ va katta o‘qi $2a = 20$ ga teng; 3) Katta yarim o‘qi $a = 10$ ga, ekssentrisiteti esa $\varepsilon = 0{,}4$ ga teng; 4) kichik o‘qi $2b = 36$, fokuslari orasidagi masofa esa $2c = 20$ ga teng; 5) Uning katta o‘qi $2a = 24$, direktrisalar orasidagi masofa esa, $D = 32$ ga teng; 6) uning kichik yarim o‘qi $b = 6$, direktrisalar orasidagi masofa esa 26 ga teng; 7) direktrisalar orasidagi masofa $D = 36$, ekssentrisiteti esa $\varepsilon = 1/3$ ga teng:

2. Ellips tenglamasi berilgan: $16x^2 + 25y^2 = 400$. 1) O‘qlarining uzunlik-lari; 2) fokuslarining koordinatalari; 3) ekssentrisitetini hisoblang.

3. $\frac{x^2}{30} + \frac{y^2}{24} = 1$ ellipsda uning kichik o‘qidan 5 birlik masofadagi nuqtani toping.

4. Ellips $A(\sqrt{5};-3)$ va $B(-2\sqrt{5};2)$ nuqtalardan o‘tadi. Ellipsning tenglamasini tuzing.

5. $\frac{x^2}{12} + \frac{y^2}{6} = 1$ ellipsning $x - y - 3 = 0$ to‘g‘ri chiziq bilan kesishish nuqtalarini toping.

6. $\frac{x^2}{49} + \frac{y^2}{24} = 1$ ellipsga ichki to‘g‘ri to‘rtburchak chizilgan, uning ikkita qarama-qarshi tomoni fokuslaridan o‘tadi. Shu to‘g‘ri to‘rtburchakning yuzini toping.

7. Quyida tenglamasi bilan berilgan chiziqlarni aniqlang va chizing.

1) $y = \frac{3}{4}\sqrt{16 - x^2}$; 2) $y = -\frac{5}{4}\sqrt{16 - x^2}$

3) $y = \frac{9}{7}\sqrt{49 - x^2}$; 4) $y = -\frac{4}{3}\sqrt{9 - x^2}$

8. $\frac{x^2}{100} + \frac{y^2}{25} = 1$ ellipsning $x + 2y - 14 = 0$ to‘g‘ri chiziq bilan kesishish nuqtalarining koordinatalarini toping.

9. Agar fokuslari Ox o‘qida yotuvchi ellips $A(\sqrt{3};\sqrt{6})$ va $B(3;\sqrt{2})$ nuqtadan o‘tsa, shu ellipsning tenglamasini tuzing.

10. $\frac{x^2}{16} + \frac{y^2}{12} = 1$ ellipsga (2;-3) nuqtada urinuvchi to‘g‘ri chiziqning tenglamasini tuzing.

11. $x=\pm 8$ to'g'ri chiziqlar kichik o'qi 8 ga teng bo'lgan ellipsning direktrisalaridir. Shu ellipsning tenglamasini va ekssentrisitetini toping.

12. Ekssentrisiteti $\varepsilon=\frac{4}{5}$ bo'lgan ellips koordinata o'qlariga simmetrik bo'lib, $M(4;-2,8)$ nuqtadan o'tadi. M nuqtaning fokal radiuslarini aniqlang.

13. $\frac{x^2}{30}+\frac{y^2}{24}=1$ ellipsning $2x-y+17=0$ to'g'ri chiziqqa parallel bo'lgan urinmalarini toping.

Giperbola

14. Quyidagilarni bilgan holda fokuslari abssissa o'qida koordinata boshiga nisbatan simmetrik joylashgan giperbolaning eng sodda tenglamasini tuzing:

1) haqiqiy o'qi $2a = 20$ va mavhum o'qi esa $2b=16$ ga teng; 2) fokuslar orasidagi masofa $2c = 20$, mavhum o'qi esa $2b = 12$ ga teng; 3) fokuslar orasidagi masofa $2c = 10$, ekssentrisiteti esa $\varepsilon=\frac{5}{4}$ teng; 4) haqiqiy o'qi 2a = 8, ekssentrisiteti esa $\varepsilon=\frac{3}{2}$ ga teng; 5) asimptotalari $y=\pm\frac{4}{3}x$ tenglamalar bilan berilgan fokuslari orasidagi masofa esa $2c = 10$ teng; 6) direktrisalar orasidagi masofa $\frac{225}{16}$, fokuslar orasidagi masofa esa $2c = 32$ teng; 7) direktrisalar orasidagi masofa $\frac{32}{5}$, mavhum o'qi esa $2b = 16$ ga teng; 8) direktrisalar orasidagi masofa $\frac{24}{5}$, ekssentrisiteti esa $\varepsilon=\frac{5}{2}$ ga teng; 9) asimptota tenglamalari $y=\pm\frac{3}{4}x$, direktrisalari orasidagi masofa $\frac{64}{5}$ ga teng.

15. $\frac{x^2}{81}-\frac{y^2}{144}=1$ giperbolaning uchlari, fokuslari va asimptotalarini toping.

16. $16x^2-25y^2=400$ giperbola berilgan. 1) a va b; 2) fokuslari; 3) ekssentrisiteti; 4) asimptota tenglamalari; 5) direktrisalarini toping.

17. Fokuslarining koordinatalari $F_1(-20;0)$ va $F_2(20;0)$, $\varepsilon=\frac{5}{3}$ ekssentrisiteti bo'yicha giperbola tenglamasini tuzing.

18. Haqiqiy va mavhum o'qlarining yig'indisi 14 ga, fokuslari orasidagi masofa esa 20 ga teng bo'lib, fokuslari Ox o'qida yotgan giperbolaning tenglamasini tuzing:

19. $\frac{x^2}{16}-\frac{y^2}{9}=1$ giperbolaga $M_1(5;\frac{9}{4})$ nuqta tegishli. M_1 nuqtaning fokal radiuslarini toping.

20. Quyidagi shartda giperbolaning ekssentrisitetini hisoblang:
1) asimptotalar orasidagi burchak 60^0 ga teng;
2) asimptotalar orasidagi burchak 90^0 ga teng;

21. Quyidagi tenglamasi bilan berilgan chiziqlarni aniqlang va chizing:

1) $y=\frac{4}{5}\sqrt{x^2-25}$; 3) $y=\frac{4}{15}\sqrt{x^2+225}$

2) $y=\frac{4}{3}\sqrt{x^2-9}$; 4) $y=4\sqrt{x^2+1}$.

22. Agar giperbolaning asimptotalari $y=\pm\frac{\sqrt{6}}{3}x$ tenglamalar bilan berilgan bo'lsa, y (6;-4) nuqtadan o'tsa, shu giperbolaning tenglamasini tuzing.

23. $9x+2y-24=0$ to'g'ri chiziq va $\frac{x^2}{4}-\frac{y^2}{9}=1$ giperbolaning asimptotalari bilan chegaralangan uchburchakning yuzini hisoblang:

24. $\frac{x^2}{5}-\frac{y^2}{4}=1$ giperbolaga (5;4) nuqtada urinuvchi to'g'ri chiziq tenglamasini tuzing.

25. Quyida berilganlarga ko'ra koordinata boshiga nisbatan simmetrik, fokuslari abssissa o'qida yotgan giperbola tenglamasini tuzing:

1) $M_1(5;\frac{9}{4}), M_2(\ 8;3\sqrt{3})$ giperbola nuqtalari;

2) $M_1(\ 5;3)$ giperbola nuqtasi, $\varepsilon = \sqrt{2}$ esa uning ekssentrisiteti;

3) $M(4,5;-1)$ giperbola nuqtasi, $y = \pm\frac{2}{3}x$ to‘g‘ri chiziqlar esa uning asimptotalari;

4) $M(-3;2,5)$ giperbola nuqtasi, $x = \pm\frac{4}{3}$ esa uning direktrisa tenglamalari.

26. $\frac{x^2}{15} - \frac{y^2}{6} = 1$ giperbolaga 1) $x + y - 7 = 0$ to‘g‘ri chiziqqa parallel;

2) $x - 2y = 0$ to‘g‘ri chiziqqa perpendikulyar bo‘lgan urinmalarni o‘tkazing.

27. $\frac{x^2}{49} + \frac{y^2}{24} = 1$ ellips bilan umumiy fokuslarga ega va ekssentrisiteti $\varepsilon = 1{,}25$ bo‘lgan giperbolaning tenglamasini tuzing.

Parabola

28. Quyida berilganlarga ko‘ra parabolaning eng sodda tenglamasini tuzing: 1) fokusi $F(6;0)$ nuqtada, uchi koordinatalar boshida; 2) direktrisasi $x = -5$ to‘g‘ri chiziqdan iborat va uchi koordinatalar boshida;3) direktrisasi $y=-4$ to‘g‘ri chiziqdan iborat va uchi koordinatalar boshida; 4) parabola y o‘qiga nisbatan simmetrik bo‘lib, fokusi (0;6) nuqtada va uchi koordinatalar boshida;

29. $y^2 = 16x$ parabolada fokal radius vektori 29 ga teng bo‘lgan nuqta topilsin.

30. Uchi koordinatalar boshida bo‘lib, Ox o‘qiga nisbatan simmetrik bo‘lgan va quyidagi nuqtalardan o‘tuvchi parabolaning tenglamasini tuzing: 1) (10;-3); 2) (-8;6); 3) (-4;4).

31. Parabolaning tenglamasi berilgan: $y^2 = 6x$. Uning direktrisasi tenglamasini tuzing.

32. Parabolaning berilgan tenglamasiga ko‘ra uning fokusi koordinatalarini hisoblang:

1) $y^2 = 6x$; 2) $y^2 = -4x$; 3) $x^2 = 14y$; 4) $x^2 = -5y$.

33. $y^2 = 16x$ parabolaning $4x - 3y + 8 = 0$ to‘g‘ri chiziq bilan kesishish nuqtalarini toping.

34. Uchi $A(2;3)$ nuqtada, fokusi $F(6;3)$ nuqtada bo‘lgan parabola tenglamasini tuzing.

35. $y^2 + 4y - 24x + 76 = 0$ parabola fokusining koordinatalarini toping:

36. $y^2 = 12x$ parabolaning $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ellips bilan kesishish nuqtalarini toping.

37. $y^2 = 18x$ parabola bilan $(x+6)^2 + y^2 = 100$ aylana umumiy vatarining tenglamasini tuzing.

38. $y^2 = 3x$ parabolaning $\frac{x^2}{20} - \frac{y^2}{5} = 1$ giperbola bilan kesishish nuqtalarini toping.

39. $y^2 = 2px$ parabolaga muntazam uchburchak ichki chizilgan. Uchburchak uchlarining koordinatalarini aniqlang.

Mustaqil yechish uchun berilgan misollarning javoblari

Ellips

1. 1) $\frac{x^2}{256} + \frac{y^2}{64} = 1$; **2)** $\frac{x^2}{100} + \frac{y^2}{64} = 1$; **3)** $\frac{x^2}{100} + \frac{y^2}{84} = 1$;

4) $\frac{x^2}{424} + \frac{y^2}{324} = 1$; **5)** $\frac{x^2}{144} + \frac{y^2}{63} = 1$; **6)** $\frac{x^2}{52} + \frac{y^2}{36} = 1, \frac{x^2}{234} + \frac{y^2}{36} = 1$;

7) $\frac{x^2}{36} + \frac{y^2}{32} = 1$. **2.** 1) $2a = 10,\ 2b = 8$; 2) $F(-3,0)$; 3) $\varepsilon = \frac{3}{5}$. **3.** $(\pm5;\pm2)$.

4. $\frac{x^2}{32}+\frac{3y^2}{32}=1$. **5.** $(2\pm\sqrt{3};-1\pm\sqrt{3})$. **6.** $S=64\frac{4}{7}$ *кв.бир*. **8.** (8;3), (6;4).

9. $\frac{x^2}{12}+\frac{y^2}{8}=1$. **7.10.** x-2y-8=0. **11.** $\frac{x^2}{32}+\frac{y^2}{16}=1, \varepsilon=\frac{\sqrt{2}}{2}$. **12.** $r_1=\frac{9}{5}, r_2=8\frac{1}{5}$.

13. 2x-y+12=0 va 2x-y-12=0.

Giperbola

14. 1) $\frac{x^2}{100}-\frac{y^2}{64}=1$; 2) $\frac{x^2}{64}-\frac{y^2}{36}=1$; 3) $\frac{x^2}{16}-\frac{y^2}{9}=1$;

4) $\frac{x^2}{16}-\frac{y^2}{20}=1$; 5) $\frac{x^2}{9}-\frac{y^2}{16}=1$; 6) $\frac{x^2}{225}-\frac{y^2}{31}=1$; 7) $\frac{x^2}{16}-\frac{y^2}{9}=1$;

8) $\frac{x^2}{36}-\frac{y^2}{185}=1$; 9) $\frac{x^2}{64}-\frac{y^2}{36}=1$.

15. (-9;0),(9;0), $F_1(-15;0), F_2(15;0), y=\pm\frac{4}{3}x$

16. 1) $a=5, b=4$; 2) $F_1(-\sqrt{41};0), F_2(\sqrt{41};0)$; 3) $\varepsilon=\frac{\sqrt{41}}{5}$; 4) $y=\pm\frac{4}{5}x$;

5) $x=\pm\frac{a}{\varepsilon}=\pm\frac{25}{\sqrt{41}}$ **17.** $\frac{x^2}{144}-\frac{y^2}{256}=1$. **18.** $\frac{x^2}{36}-\frac{y^2}{64}=1, \frac{x^2}{64}-\frac{y^2}{36}=1$. **20.**

1) $\varepsilon=\frac{2}{3}\sqrt{3}$; 2) $\varepsilon=\sqrt{2}$. **22.** $\frac{x^2}{12}\ \frac{y^2}{8}=1$. **23.** $S_\Delta = 12 kv.bir.$

24. $x+y=1$. **25.** 1) $\frac{x^2}{16}-\frac{y^2}{9}=1$, 2) $x^2-y^2=16$; 3) $\frac{x^2}{18}-\frac{y^2}{8}=1$;

4) $\frac{x^2}{4}-\frac{y^2}{5}=1$ yoki $\frac{9x^2}{61}-\frac{16y^2}{305}=1$. **26.** 1) $x+y+3=0, x+y-3=0$;

2) $2x+y+\sqrt{54}=0$, $2x+y-\sqrt{54}=0$. **27.** $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Parabola

28. 1) $y^2=24x$; 2) $y^2=10x$; 3) $x^2=16y$; 4) $x^2 = 24y$. **29.** $A(25;-20)$; $B(25;20)$. **30.** 1) $y^2=0,9x$; 2) $y^2=-4,5x$; **31.** $x=-1,5$. **32.** 1) (1,5;0); 2) (-1;0);3) (0;3,5); 4) (0;-1,25). **33.** (4;8) yoki (1;4). **34.** $(y-3)^2 = 16(x-2)$. **35.** F(9;-2). **36.** $(\frac{5}{4};\sqrt{15}), (\frac{5}{4};-\sqrt{15})$. **37.** x-2=0. **38.** $(\frac{5}{4};\sqrt{15}), (\frac{5}{4};-\sqrt{15})$.

39. $O(0;0), A(6;2\sqrt{3}), B(6;-2\sqrt{3})$.

3-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

3.1-masala. *ABC* uchburchakning uchlari berilgan. Quyidagilarni toping: 1) *AB* tomon tenglamasini; 2) *C* uchidan *AB* tomonga tushurilgan mediana tenglamasini; 3) *B* burchak bissektrisasining tenglamasini; 4) *A* uchidan *BC* tomonga tushurilgan balandligining tenglamasini; 5) *C* nuqtadan o'tuvchi *AB* tomoniga parallel to'g'ri chiziq tenglamasini; 6) *C* nuqtadan *AB* to'g'ri chiziqqacha bo'lgan masofani; 7) *AB* va *AC* tomonlari orasidagi burchakni;

3.1.1. *A*(4,-5), *B*(6,9), *C*(-4,-1).
3.1.2. *A*(1,-3), *B*(-5,4), *C*(-2,10).
3.1.3. *A*(1,8), *B*(-5,-4), *C*(-1,-3).
3.1.4. *A*(0,4), *B*(5,-3), *C*(-6,-2).
3.1.5. *A*(6,-4), *B*(-8,3), *C*(-2,-7).
3.1.6. *A*(2,3), *B*(-4,-7), *C*(2,0).
3.1.7. *A*(-4,-8), *B*(4,1), *C*(0,7).
3.1.8. *A*(4,-2), *B*(7,0), *C*(-3,1).
3.1.9. *A*(4,1), *B*(-2,8), *C*(1,-5).
3.1.10. *A*(4,0), *B*(1,-3), *C*(5,2).
3.1.11. *A*(7,10), *B*(1,3), *C*(4,-2).
3.1.12. *A*(8,6), *B*(1,3), *C*(-2,-3).
3.1.13. *A*(11,-3), *B*(-1,-3), *C*(7,1).
3.1.14. *A*(5,9), *B*(4,-1), *C*(0,1).
3.1.15. *A*(7,3), *B*(1,7), *C*(-2,1).
3.1.16. *A*(1,6), *B*(6,1), *C*(-3,-2).
3.1.17. *A*(2,6), *B*(6,-6), *C*(2,-4).
3.1.18. *A*(10,1), *B*(3,7), *C*(-3,4).
3.1.19. *A*(8,3), *B*(2,8), *C*(-4,4).

3.1.20. $A(7,7)$, $B(-7,5)$, $C(-3,-3)$.
3.1.21. $A(3,-3)$, $B(4,3)$, $C(-6,1)$.
3.1.22. $A(6,2)$, $B(-6,8)$, $C(2,-4)$.
3.1.23. $A(7,5)$, $B(-4,0)$, $C(2,-5)$.
3.1.24. $A(8,-1)$, $B(2,6)$, $C(-4,4)$.
3.1.25. A(-5,0), $B(2,-6)$, $C(8,-3)$.
3.1.26. $A(1,-4)$, $B(-1,10)$, $C(-9,6)$.

Quyidagi masalani echishdagi har bir bajarilgan amalni **Maple13** dasturida bajarilishini ko'rsatib boramiz.

2.1-masala. Tekislikda A(10;3), B(-4;7), C(4;8) nuqtalar koordinatalari bilan berilgan. Quyidagilarni toping.

1) Δ ABC tomonlarining tenglamalarini; 2) AB va AC tomonlar orasidagi burchakni; 3) AD balandlik va uning uzunligini; 4) AE mediana va AN bissektrisani; 5) Δ ABC ni yuzasini hisoblang; 6)Tekislikda ΔABC ni quring.

Yechilishi. 1) Δ ABC tomonlari tenglamalarini berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$ formulasiga asosan topamiz.

AB: $\frac{y-3}{7-3}=\frac{x-10}{-4-10}$, $\frac{y-3}{4}=\frac{x-10}{-14}$ bundan 2x + 7y - 41, $k_1=-\frac{2}{7}$.

> with(geometry):
point(A,10,3), point(B,-4,7):
l1:=line(AB,[A,B]); *l1:= AB*
> Equation(AB,[x,y]); 82 K 4 x K 14 y = 0

AC: $\frac{x-10}{4-10}=\frac{y-3}{8-3}$, $\frac{y-3}{5}=\frac{x-10}{-6}$ bundan 5x + 6y – 68 = 0 $y=-\frac{5}{6}x+\frac{34}{3}$, $k_2=-\frac{5}{6}$

> with(geometry):
point(A,10,3),point(S,4,8):
l2:=line(AC,[A,S]); *l2:= AC*
> Equation(AC,[x,y]); 68 K 5 x K 6 y = 0

BC: $\frac{x+4}{4+4}=\frac{y-7}{8-7}$, $\frac{x+4}{8}=\frac{y-7}{1}$ bundan x – 8y + 60 = 0 , $y=\frac{1}{8}x+\frac{15}{2}$, $k_3=\frac{1}{8}$

> with(geometry):
point(B,-4,7),point(S,4,8):
l3:=line(BC,[B,S]); *l3:= BC*

> Equation(BC,[x,y]); K 60 K x C 8 y = 0

Δ ABC tomonlarining uzunliklari:

> with(geometry):
> triangle(ABC, [point(A,10,3), point(B,-4,7), point(C,4,8)]):
distance(A,B);distance(A,C);distance(B,C); $\sqrt{212}$ $\sqrt{61}$ $\sqrt{65}$

> sides(ABC); $[\sqrt{212}, \sqrt{65}, \sqrt{61}]$

2)AB va AC tomonlar orasidagi burchak

$$tg\varphi=\frac{k_2-k_1}{1+k_1k_1}$$

formulaga asosan topiladi:

$$k_1=-\frac{2}{7},\ k_2=-\frac{5}{6};\quad tg=\frac{-\frac{2}{7}+\frac{5}{6}}{1-\frac{2}{7}\cdot(-\frac{5}{6})}=\frac{23}{52},\ \varphi=arctg(\frac{23}{52})\approx 42^0 15'.$$

> with(geometry):
EnvHorizontalName := 'x': _EnvVerticalName := 'y':
line(AB, 82-4*x-14*y=0), line(AC, 68-5*x-6*y = 0); *AB, AC*

> hi:=FindAngle(AB, AC); f: = $\arctan\frac{23}{52}$

> hi:=evalf(hi); f: = 0,4164386172

3) AD balandlikni topish uchun A(10;3) nuqtadan o'tuvchi to'g'ir chiziqlar dastasi tenglamasini yozamiz, undan BC tomonga perpendikulyar bo'lgan to'g'ri chiziqni ajratamiz va tenglamasini yozamiz.

$y - y_1 = k (x - x_1)$ da $x_1 = 10$, $y_1 = 3$ bo'lsa, $y - 3 = k (x - 10)$

bu to'g'ri chiziqlar dastasidan BC($k_3=\frac{1}{8}$) ga perpendikulyarini ajratish uchun k ni quyidagi perpendikulyarlik shartidan foydalanib topamiz:

$$k=-\frac{1}{k_3}=\frac{1}{-\frac{1}{8}},\ \text{dan } k=-8$$

Bu holda AD balandlik tenglamasi:

$$y-3=8(x-10),\ 8x+y-83=0\ (AD)$$

> with(geometry):
triangle(ABC,[point(A,10,3), point(B,-4,7), point(C,4,8)]):
altitude(hA1,A,ABC); hA1
> form(hA1); *line2d*
> detail(hA1);
name of the object: hA1
form of the object: line2d
*equation of the line: -83+8*x+y = 0*

Bu AD balandlik uzunligini topishda BC tomondan A(10;3) nuqtagacha bo'lgan masofani, ya'ni AD balandlik uzunligini hisoblaymiz.

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}=\frac{|1\cdot 10+8\cdot 3+60|}{\sqrt{1^2+(-8)^2}}=\frac{46}{\sqrt{65}}=5.7$$

> with(geometry):
> assume(m<>0):
line(BC, x-8*y=-60,[x,y]): distance(A, BC); $\frac{46}{65}\sqrt{65}$

4) AE mediana tenglamasini yozish uchun BC tomon o'rtasi E nuqtaning koordinaialarini

$$x_0=\frac{x_1+x_2}{2},\ y_0=\frac{y_1+y_2}{2}$$

formulaga asosan topamiz:

$$x_E=\frac{x_B+x_C}{2}=\frac{-4+4}{2}=0,\ y_E=\frac{y_B+y_C}{2}=\frac{7+8}{2}=\frac{15}{2},\quad E(0;\ \frac{15}{2})$$

Bu holda AE ning tenglamasi:

$$\frac{x-10}{0-10}=\frac{y-3}{\frac{15}{2}-3},\quad \frac{x-10}{-10}=\frac{y-3}{\frac{9}{2}},\quad \frac{x-10}{-20}=\frac{y-3}{9},\quad 9x+20u-150=0$$

> with(geometry):
triangle(ABC, [point(A,10,3), point(B,-4,7), point(C,4,8)]):
median(mA, A, ABC); *mA*

> form(mA); *line2d*
> detail(mA);
name of the object: mA
form of the object: line2d
*equation of the line: 75-9/2*x-10*y = 0*
> median(mA, A, ABC, E);
> form(mA); *segmebt2d*

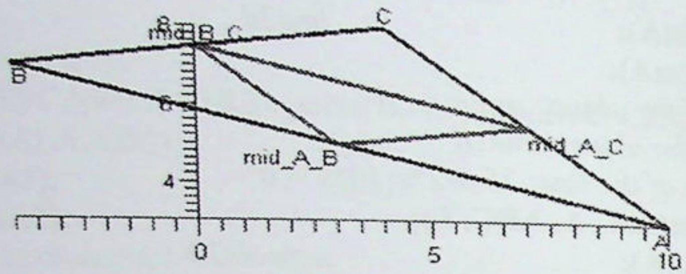
> coordinates(E); $\left[0, \frac{15}{2}\right]$

> detail(mA);
name of the object: mA
form of the object: segment2d
the two ends of the segment: [[10, 3], [0, 15/2]]
<u>*ABC uchburchk og'irlik markazi:*</u>
> with(geometry):
s:=[point(A,10,3), point(B,-4,7), point(C,4,8)]; *ps := [A, B, C]*

> centroid(G,s); *G*
> form(G); *point2d*

> coordinates(G); $\left[\frac{10}{3}, 6\right]$

> detail(G); *name of the object: G*
form of the object: point2d
coordinates of the point: [10/3, 6]
<u>*Uchlari uchburchak tomonlarining o'rtalarida bo'lgan uchburchak.*</u>
> with(geometry):
triangle(T, [point(A,10,3), point(B,-4,7), point(C,4,8)]):
medial(mT,T); *mT*
> detail(mT); *name of the object: mT*
form of the object: triangle2d
method to define the triangle: points
the three vertices: [[3, 5], [7, 11/2], [0, 15/2]]
> draw({T(color=blue),mT(color=red)},style=line,axes=NONE, rinttext=true);



Endi AN bissektrisa tenglamasini ikki xil usulda topish mumkin:

a) $$\frac{A_1x+B_1y+C}{\sqrt{A_1^2+B_2^1}}=\pm\frac{A_2x+B_2y+C}{\sqrt{A_2^2+B_2^2}}$$

tenglamaga asosan aniqlaymiz.

Bu bissektrisa AB va AC tomonlar orasida bo'lganligi uchun bu tomonlar tenglamalari

$$AB: 2x + 7y - 41 = 0 \text{ va } AC: 5x + 6y - 68 = 0.$$

ga asosan:

$$\frac{2x+7y-41}{\sqrt{2^2+7^2}}=\pm\frac{5x+6y-68}{\sqrt{5^2+6^2}},$$

bundan:

$$\frac{2x+7y-41}{\sqrt{53}}=+\frac{5x+6y-68}{\sqrt{61}}, \quad (1)$$

$$\frac{2x+7y-41}{\sqrt{53}}=-\frac{5x+6y-68}{\sqrt{61}} \quad (2)$$

tenglamalardan qaysi biri A burchakning ichki burchagining bissektirsasi AD ekanligini aniqlaymiz. (1) tenglamaga B va C nuqtalarning koordinatalarini qo'yganda chap va o'ng kasrlar ishoralari har xil bo'ladi. Bundan (1) tenglama ΔABC ning A ichki burchagining bissektrisasi bo'ladi. (2) tenglamaga B va C nuqtalarning koordinatalari qo'yilganda bir xil ishorali bo'lgani uchun (2) tenglama qo'shni burchak bissektrisasi bo'ladi.

<u>*Ichki burchak bissektrisasi:*</u>

> with(geometry):
triangle(ABC, [point(A,10,3), point(B,-4,7), point(C,4,8)]):
define the ``line" bisector bA
> bisector(bA, A, ABC); *bA*
> Equation(bA,[x,y]); 1630.535418 K 104.0420976 x K 196.7048141 y = 0

> detail(bA); *name of the object: bA*
form of the object: line2d
*equation of the line: 1630.535418-104.0420976*x-196.7048141*y = 0*

<u>*Qo'shni burchakning (qo'shma) bissektrisasi;*</u>

> with(geometry):
triangle(ABC,[point(A,10.,3),point(B,-4,7),point(C,4,8)]):
define the external bisector bA
> ExternalBisector(bA, A, ABC); *bA*
> Equation(bA,[x,y]); 1654.921848 K 196.7048141 x C 104.0420976 y = 0

> detail(bA); *name of the object: bA*
form of the object: line2d
*equation of the line: 1654.921848-196.7048141*x+104.0420976*y = 0*
> bisector(ibA,A,ABC):
ArePerendicular(bA,ibA); *true*

b) A burchak bissektrisasini BC tomon bilan kesishish nuqtasi N ning koordinatalarini topamiz. Geometriya kursidan ma'lumki, uchburchak burchagining bissektrisasi burchak qarshisidagi tomonni burchakka yopishgan tomonlarga proportsional bo'laklarga bo'ladi.

Demak: $\lambda=\frac{|CN|}{|NB|}=\frac{|AC|}{|AB|}$ dan λ dani son qiymatini topib($\lambda=\frac{|AC|}{|AB|}=\frac{\sqrt{212}}{\sqrt{61}}$),

$$x_N=\frac{x_C+\lambda x_B}{1+\lambda},\ y_N=\frac{y_C+\lambda y_B}{1+\lambda}$$

formulalarga asosan N(x, y) nuqtani topamiz va ikki nuqtadan o'tuvchi to'g'ri chiziq formulasiga asosan: AN bissektrisa tenglamasini tuzamiz:

$$\frac{x-x_A}{x_N-x_A}=\frac{y-y_A}{y_N-y_A}.$$

bA bissektrisani BC tomon bilan kesishish nuqtasi N koordinatalari:
> restart;
> with(geometry):
triangle(ABC, [point(A,10.,3.), point(B,-4,7), point(C,4,8)]):
define the ``segment'' bisector bA
> bisector(bA, A, ABC,N); *bA*
> form(bA); *segmebt2d*
> OnSegment(N, B, C, sqrt(212/61.)); *N*
> coordinates(N); [1.206942951, 7.650867870]
> detail(bA); *name of the object: bA*
form of the object: segment2d
the two ends of the segment: [[10., 3.], [1.206942951, 7.650867870]]

5) Δ ABC ning yuzini quyidagicha hisoblaymiz.

a) $S=\frac{1}{2}ah$, (bunda *a*- asos, *h*-asosga tushirilgan balandlik) formulasiga asosan hisoblaymiz. Asos uchun BC tomon uzunligi:

$$|BC|=\sqrt{(4+4)^2+(8-7)^2}=\sqrt{65}=8.062 \text{ ni}$$

va balandlik uchun 3) punktdagi AD uzunligi $|AD|=\frac{46}{\sqrt{65}}=5.7$ ekanidan,

Δ ABC ning yuzasi: $S=\frac{1}{2}|BC||AD|=\frac{1}{2}\sqrt{65}\frac{46}{\sqrt{65}}=23$,

b) $$S=\pm\frac{1}{2}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}$$

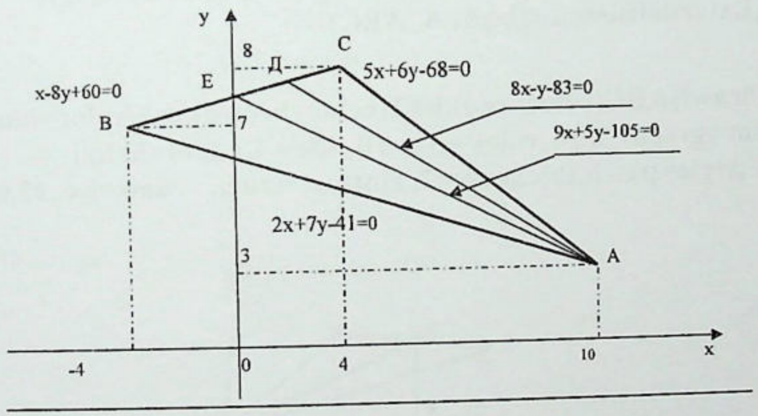
formulaga ABC uchburchakning uchlarining koordinatalarini qo'yib uni yuzasini topamiz.

$$S=\pm\frac{1}{2}\begin{vmatrix} -4-10 & 7-3 \\ 4-10 & 8-3 \end{vmatrix}=\pm\frac{1}{2}\begin{vmatrix} -14 & 4 \\ -6 & 5 \end{vmatrix}=23$$

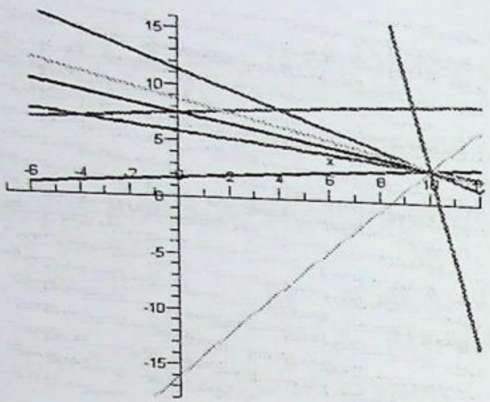
> with(geometry):

triangle(ABC, [point(A,10,3), point(B,-4,7), point(C,4,8)]):
area(ABC); *23*

<u>*A nuqtadan BC tamonga parallel:*</u>
> with(geometry):
triangle(ABC, [point(A,10,3), point(B,-4,7), point(C,4,8)]): line(BC, [B,C]); *BC*
> arallelLine(lA,A,BC); *lA*
> Equation(lA,[x,y]); K 14 K x C 8 y = 0
> detail(lA); *name of the object: lA*
form of the object: line2d
*equation of the line: -14-x+8*y = 0*
ABC uchburchakni qurish:



ABC uchburchakni tamonlari tenglamalariga asosan qurish;
> with(plots):
> plot([(41-2*x)/7,(68-5*x)/6,(60+x)/8,(75-9/2*x)/10, (14+x)/8,(197*x-1655.)/104.1, (-104.1*x+1631.)/187, (83-8*x)], x=-6..12, color=[red,red,red,blue,blue,green,green, brown], style=[line], thickness=2, view=[-7..12,-18..16]);

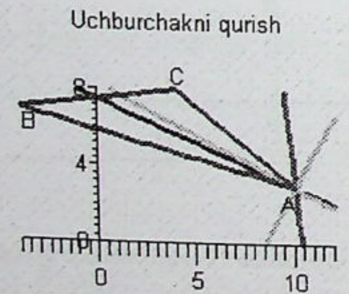


ABC uchburchakni uchining koordinatalariga asosan kesmalari bo'yicha qurish:

> triangle(ABC,[point(A,10,3),point(B,-4,7),point(C,4,8)]): altitude(hA1,A,ABC), median(mA, A, ABC), bisector(bA, A, ABC),ExternalBisector(bqA, A, ABC);

hA1, mA, bA, bqA

> draw([ABC(color=red),hA1(color=brown),mA(color=blue), bA(color=green), bqA(color=green)],title=`Uchburchakni qurish`,style=patch,thickness=2, rinttext=true, view=[-4..12,0..8]);



3.2-masala. Quyidagi ma'lum: *A,B* egri chiziqda yotuvchi nuqtalar; *F* fokus; *a*-katta yarim o'q(yoki yarim haqiqiy o'q); *b*-kichik (yoki mavhum) yarim o'q; *e*-ekssentristet; *D* egri chiziq direktrisasi : 2c-fokuslar orasidagi masofasi ; $u = \pm kx$ – giperbola asimptotalari tenglamalari .a) ellipsning ; b) giperbolaning; c) parabolaning kanonik tenglamasini tuzing.

3.2.1. a) $a = 9, e = \frac{\sqrt{17}}{9}$; b) $b = 7$; F($-\sqrt{130}$:0);

c) simmetriya o'qi *Oy*, *A*(-4;3).

3.2.2. a) $b = 3$, F($-\sqrt{55}$:0); b) $a = 8$; $e = \frac{5}{4}$; c) *D*: $x = 3$.

3.2.3. *A*(3,0), $B(2, \frac{\sqrt{5}}{3})$; b) $k = \frac{3}{4}$; $e = \frac{5}{4}$; c) *D*: $y = -2$.

3.2.4. a) $e = \frac{\sqrt{21}}{5}$, A(-5, 0); b) $A(\sqrt{80}, 3)$; $B(4\sqrt{6}; 3\sqrt{2})$; c) *D*: $y = 1$.

3.2.5. a) $2a = 22$, e=$\frac{\sqrt{57}}{11}$; b) $k = \frac{2}{3}$,; $2c = 10\sqrt{13}$;

c) simmetriya o'qi *Ox*, *A*(27;9).

3.2.6. a) $b = \sqrt{15}$, $e = \frac{\sqrt{10}}{25}$; b) $k = \frac{3}{4}$; 2a=16;

c) simmetriya o'qi *Ox*, *A*(4;-8).

3.2.7. a) $a = 4$, *F*(3: 0); b) $b = 2\sqrt{10}$; *F*(-11: 0); c) *D*: $x = -2$.

3.2.8.a) $b = 4$, *F*(3: 0); b) $a = 5$; $e = \frac{7}{5}$; c) *D*: $x = 6$.

3.2.9. a) $A(0:\sqrt{3})$; $B(\sqrt{\frac{14}{3}}:1)$; b) $k = \frac{\sqrt{21}}{10}$, $e = \frac{11}{10}$; c) *D*: $y = -3.2$.

3.2.10. a) *A*(8: 0); $e = \frac{7}{8}$ b) $A(3: -\sqrt{\frac{3}{5}})$; B($\sqrt{\frac{13}{5}}$: 6); c) *D*: $y = 3.2$.

3.2.11. a) $2a = 24$, *ye*=$\frac{\sqrt{22}}{6}$; b) $b = 7$; F($-\sqrt{130}$: 0);

c) simmetriya o'qi *Oy*, *A*(-4; 3).

3.2.12. a) $a = 9$, $ye = \frac{\sqrt{17}}{9}$; b) $k = \sqrt{\frac{2}{3}}$: 2*c*=10;

c) simmetriya o'qi *Ox*, *A*(-7;-4).

3.2.13. a) $a = 6$, *F*(-4: 0); b) $b = 3$; *F*(7: 0); c) *D*: $x = -7$.

3.2.14.a) $b = 7$, $F(5:0)$; b) $a = 11$; $e = \frac{12}{11}$; c) $D: x = 0$.

3.2.15. a) $A(-\sqrt{\frac{17}{3}}; \frac{1}{3})$, $B = (\sqrt{\frac{21}{2}}; \frac{1}{2})$; b) $k = \frac{1}{2}$; $e = \frac{\sqrt{5}}{2}$; c) $D: y = -1$.

3.2.16. a) A(0;8), $e=\frac{3}{5}$ b) $A(\sqrt{6};0)$; $B(-2\sqrt{2};1)$; c) D: y=9.

3.2.17. a) 2a=22, $ye=\frac{10}{11}$; b) $k=\frac{\sqrt{11}}{5}$; 2c=12;

c) simmetriya o‘qi Ox, A(-7;5).

3.2.18. a) b=5, $ye=\frac{12}{13}$; b) $k=\frac{1}{3}$; 2a=6;

c) simmetriya o‘qi Oy, A(-9;6).

3.2.19.a) a=9, F(7;0); b) b=6; F(12;0); c) D: $x=-\frac{1}{4}$.

3.2.20. a) b=5, F(-10;0); b) a=9; $e=\frac{4}{3}$; c) D: x=12.

3.2.21. a) A(0;-2), $B(\frac{\sqrt{15}}{2};1)$; b) $k=\frac{2\sqrt{10}}{9}$; $e=\frac{11}{9}$; c) D: y=5.

3.2.22. a) A(-6;0), $e=\frac{2}{3}$; b) $A(\sqrt{8}:0)$; $B(-\frac{\sqrt{20}}{3}:2)$; c) D: y=1.

3.2.23. a) 2a=50, $e=\frac{2}{3}$; b) $k=\frac{\sqrt{29}}{14}$; 2c=30; c) simmetriya o‘qi Oy, A(4;1).

3.2.24. a) $b=2\sqrt{15}$, $ye=\frac{7}{8}$; b) $k=\frac{5}{6}$; 2a=12;

c) simmetriya o‘qi Oy, $A(-2;3\sqrt{2})$.

3.2.25. a) a=13, F(-5;0); b) b=44; F(-7;0); c) D: $x=-\frac{3}{8}$.

3.2.26. a) b=7, F(13;0) b) b=4; F(-11:0); c) D: x=13.

4-bob. FAZODA ANALITEK GEOMETRIYA

4.1-§. Fazoda tekislik

4.1. Tekislikning umumiy tenglamasi.

4.1.1-teorema. Agar $Oxyz$ to‘g‘ri burchakli dekart koordinata sistemasida, ixtiyoriy π tekislik berilgan bo‘lsa, u holda bu sistemada berilgan π tekislik x, y, va z o‘zgaruvchilarga nisbatan birinchi darajali algebraik tenglama bilan tasvirlanadi.

Isboti. $Oxyz$ to‘g‘ri burchakli dekart koordinata sistemasida π tekislikga perpindikulyar bo‘lgan $\vec{n} \neq \vec{0}$ vektor berilgan bo‘lsin. $\vec{n}$ vektorni π tekislikning *normali* deb ataymiz. π tekislikda yotuvchi $M_0(x_0, y_0, z_0)$ ixtiyoriy nuqtani olamiz. Agar $\vec{n}$ normalning shu koordinatalar sistemasidagi koordinatalari

$$\vec{n} = \{A, B, C\} = A\vec{i} + B\vec{j} + C\vec{k}$$

bo‘lib, $M(x, y, z)$ fazoning ixtiyoriy nuqtasi bo‘lsa, u holda $M \in \pi$ bo‘lishi uchun $\overrightarrow{M_0M}$ vektor $\vec{n}$ vektorga perpindikulyar, ya’ni ularning skalyar ko‘paytmasi $\overrightarrow{M_0M} \cdot \vec{n} = 0$ bo‘lishi zarur va yetarli. $\overrightarrow{M_0M}$ vektorning koordinatalari

$$\overrightarrow{M_0M} = \{x - x_0, y - y_0, z - z_0\} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$$

bo‘lgani uchun

$$\overrightarrow{M_0M} \cdot \vec{n} = A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

bundan

$$Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$$

yoki

$$Ax + By + Cz + D = 0,$$

bu yerda $D = -Ax_0 - By_0 - Cz_0$.■

4.1.2-teorema. Agar $Oxyz$ to'g'ri burchakli dekart koordinata sistemasida x, y, z o'zgaruvchilarga nisbatan birinchi darajali algebraik tenglama berilgan bo'lsa, u holda bu tenglama fazoda tekislikni tasvirlaydi.

x, y, z o'zgaruvchilarga nisbatan birinchi darajali algebraik tenglamaning umumiy ko'rinishi

$$(\pi): \quad Ax + By + Cz + D = 0 \tag{4.1.3}$$

bo'lib, unda A, B, C, D – o'zgarmas sonlar. (4.1.3) tenglamaga tekislikning umumiy tenglamasi deyiladi.

Isboti. Faraz qilaylik (x_0, y_0, z_0) uchlik (4.1.3) tenglamaning yechimi bo'lsin, u holda x, y, z o'zgaruvchilar o'rniga (x_0, y_0, z_0) yechimini qo'yib, undan

$$Ax_0 + By_0 + Cz_0 + D = 0 \qquad yoki \qquad D = -Ax_0 - By_0 - Cz_0$$

ega bo'lamiz. D ning topilgan bu qiymatini (4.1.3) tenglamaga qo'ysak,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{4.1.4}$$

yoki vektor ko'rinishda $\overrightarrow{M_0M} \cdot \vec{n} = 0$ tenglamaga ega bo'lamiz. Bu (4.1.4) tenglama (4.1.3) tenglamaga teng kuchli. Bu munosabatdan $M_0(x_0, y_0, z_0) \in \pi$ ekani kelib chiqadi. Bu mulohazalar teoremani to'la isbotlaydi.■

4. 2. Tenglikning koordinata o'qlariga nisbatan joylashuvi. Tekislikning umumiy tenglamasidagi ba'zi koeffisentlar nolga teng bo'lgan holda tekislikning fazoda qanday joylashishini tekshiramiz.

1) $D = 0$ bo'lsa, (4.1.3) tenglama $Ax + By + Cz = 0$ ko'rinishni oladi. Bu tenglama koordinatalar boshidan o'tgan tekislikni tasvirlaydi, chunki $O(0,0,0)$ ning koordinatalari bu tenglamani qanoatlantiradi.

2) a) $C = 0$ bo'lsa, (4.1.3) tenglama $Ax + By + D = 0$ ko'rinishni oladi va uning normal vektori $\vec{n} = \{A, B, O\}$ Oz o'qiga bo'ladi. Demak, tekislik Oz o'qiga parallel bo'ladi.

b) $B = 0$ bo'lsa, (4.1.3) tenglama $Ax + Cz + D = 0$ ko'rinishni oladi va uning normal vektori $\vec{n} = \{A, O, C\}$ Oy o'qiga bo'ladi (4.2- chizma). Demak, tekislik Oy o'qiga parallel bo'ladi.

c) $A = 0$ bo'lsa, (4.1.3) tenglama $By + Cz + D = 0$ ko'rinishni oladi va uning normal vektori $\vec{n} = \{O, B, C\}$ Ox o'qiga bo'ladi. Demak, tekislik Ox o'qiga parallel bo'ladi (4.3-chizma).

3. a) Agar $C = 0$, va $D = 0$ bo'lsa , $Ax + By = 0$ tenglama Oz o'qiga o'tgan tekislikni tasvirlaydi (4.4-chizma).

b) Agar $A = 0$ *va* $D = 0$ bo'lsa, $By + Cz = 0$ tenglama Ox o'qidan o'tgan tekislikni tasvirlaydi (4.5-chizma).

c) Agar $B = 0$ *va* $D = 0$ bo'lsa, $Ax + Cz = 0$ tenglama Oy o'qidan o'tgan tekislikni tasvirlaydi (4.6-chizma).

4. a) Agar $A \neq 0$, $B = 0$, $C = 0$, $D \neq 0$ bo'lsa, $Ax + D = 0$ yoki $x = -\frac{D}{A}$ tenglama (Oyz) koordinatalar tekisligiga parallel yoki undan $R = \left|-\frac{D}{A}\right|$ ga teng masofada yotgan tekislikni aniqlaydi (4.7-chizma).

b) Agar $A = 0, C = 0$ bo'lsa, $By + D = 0$ tenglama yoki $y = -\frac{D}{B}$ tenglama (Oxz) tekislikka parallel yoki undan $R = \left|-\frac{D}{B}\right|$ ga teng masofada yotgan tekislikni aniqlaydi (4.8-chizma).

c) Agar $A = 0, B = 0$ bo'lsa, $Cz + D = 0$ *yoki* $z = -\frac{D}{C}$ tenglama Oxy koordinatalar tekisligiga parallel yoki undan $R = \left|-\frac{D}{C}\right|$ ga teng masofada yotgan tekislikni aniqlaydi (4.9-chizma).

5. Quyidagi

a) $A \neq 0$, $B = 0$, $C = 0$, $D = 0$;

b) $A = 0$, $B \neq 0$, $C = 0$, $D = 0$;

c) $A = 0$, $B = 0$, $C \neq 0$, $D = 0$;

hollarda mos ravishda $Ax = 0$, $By = 0$, $Cz = 0$ yoki $x = 0$, $y = 0$, $z = 0$ tenglamalarga egamiz. Ular mos ravishda $(Oyz), (Oxz), (Oxy)$, koortdinata tekisliklarini tasvirlaydi (4.10-chizma).

Ravshanki tekislikning umumiy tenglamasida barcha koeffisentlari nolga teng bo'lmaganda tekislik barcha koordinata o'qlarini kesib o'tadi. Tekislikni yasash uchun bu nuqtalarni topish lozim. Endi tekislikning umumiy tenglamasini quyidagicha yozaylik:

$$Ax + By + Cz = -D \qquad yoki \qquad \frac{Ax}{-D} + \frac{By}{-D} + \frac{Cz}{-D} = 1$$

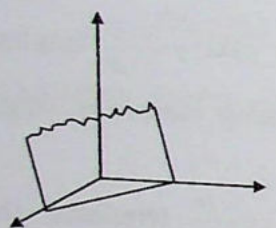
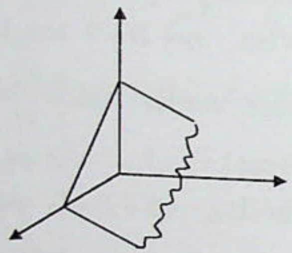
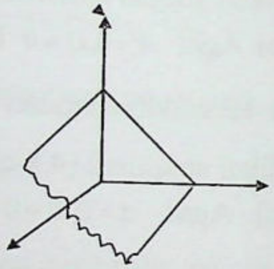
Bundan

$$\frac{x}{\frac{-D}{A}}+\frac{y}{\frac{-D}{A}}+\frac{z}{\frac{-D}{C}}=1$$

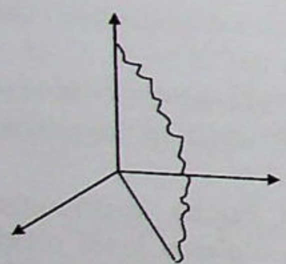
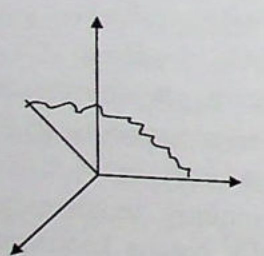
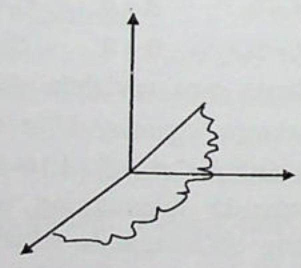
Agar $-\frac{D}{A}=a,\quad -\frac{D}{B}=b,\quad -\frac{D}{C}=c$ belgilashlarni kiritsak

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (4.2.1)$$

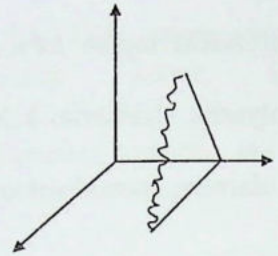
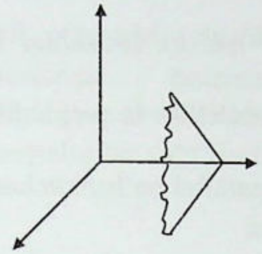
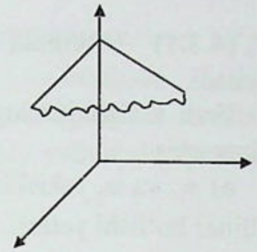
Ko'rinishdagi tenglamaga ega bo'lamiz. Bu (4.2.1) tenglamani tekislikning *kesmalarga nisbatan tenglamasi* deb ataladi (4.11-chizma).

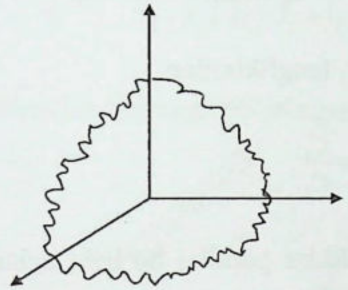
4.1-chizma. $\pi \,//\, Oz$, $c=0$. 4.2-chizma. $\pi \,//\, Oy$, $B=0$. 4.3-chizma. $\pi \,//\, Ox$, $A=0$.

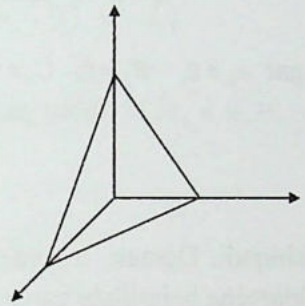
4.4-chizma. $C=0$, $D=0$. 4.5-chizma. $B=0$, $D=0$. 4.6-chizma. $A=0$, $D=0$.

4.7-chizma. $B\neq 0, C=0, A\neq 0, D\neq 0$. 4.8-chizma. $A=0, C=0$. 4.9-chizma. $A=0, B=0$.



4.10-chizma.



4.11-chizma. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

4.3. Ikki tekislik orasidagi burchak. Tekisliklarning parallellik va perpindikulyarlik sharti. Umumiy tenglamalari

$$A_1x+B_1y+C_1z+D_1=0 \text{ va } A_2x+B_2y+C_2z+D_2=0$$

bilan berilgan π_1 va π_2 tekisliklarni qaraymiz (4.12-chizma).

Ikki tekislik orasida φ burchak deyilganda bu tekisliklar bilan hosil qilingan ikkita ikki yoqli burchakdan biri tushiniladi. π_1 va π_2 tekisliklar fazoda har qanday joylashganda ham ular orasidagi φ burchaklardan biri $\vec{n_1}=\{A_1,B_1,C_1\}$ va $\vec{n_2}=\{A_2,B_2,C_2\}$ normal vektorlar orasidagi burchakka teng. Shu sababli bu burchak ikki vektor orasidagi burchak formulasiga ko'ra hisoblanadi:

$$\cos y=\frac{\vec{n_1}\cdot\vec{n_2}}{\left|\vec{n_1}\right|\cdot\left|\vec{n_2}\right|}=\frac{A_1\cdot A_2+B_1\cdot B_2+C_1\cdot C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\,\sqrt{A_2^2+B_2^2+C_2^2}}. \qquad (4.3.1)$$

(4.3.1) – formula *ikki tekislik orasidagi burchakni topish formulasi* deyiladi.

Endi tekisliklarning parallellik va perpindikulyarlik shartlarini keltirib chiqaramiz.

a) π_1 va π_2 tekisliklar parallel bo'lishi uchun ularning normal vektorlari kollinar bo'lishi yetarli, ya'ni

$$\vec{n_1} = \lambda \cdot \vec{n_2} \text{ yoki } A_1 = \lambda A, \quad B_1 = \lambda B_2, \quad C_1 = \lambda C_2.$$

Agar $A_2 \neq 0, \quad B_2 \neq 0, \quad C_2 \neq 0$ bo'lsa , tengliklardan

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \tag{4.3.2}$$

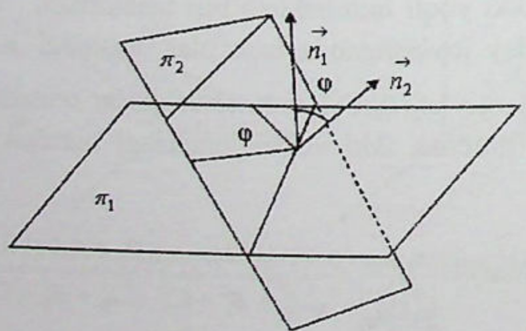
kelib chiqadi. Demak, π_1 va π_2 tekisliklar parallel bo'lishi uchun (4.3.2) tekisliklarning bajarilishi zarur va yetarlidir.

b) π_1 va π_2 tekisliklar perpindikulyar bo'lishi uchun ularning normal vektorlari o'zaro perpindikulyar bo'lishi yetarli, ya'ni $\vec{n_1} \perp \vec{n_2} \Rightarrow \vec{n_1} \cdot \vec{n_2} = 0$

Bundan quyidagi tenglik kelib chiqadi:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0 \tag{4.3.3}$$

Demak, π_1 va π_2 tekisliklar perpindikulyar bo'lishi uchun (4.3.3) tenglikning bajarilishi zarur va yetarlidir.



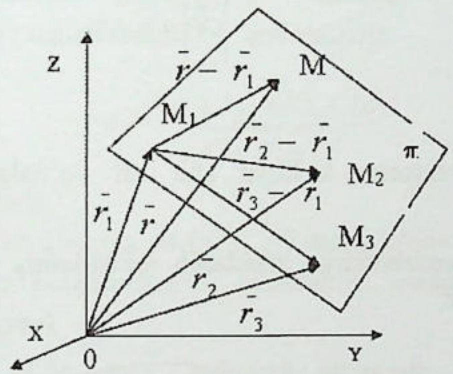
4.12-chizma

4.4. Uch nuqta orqali o'tuvchi tekislik tenglamasi. $Oxyz$ to'g'ri burchakli dekart koordinata sistemasida berilgan uchta $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$ nuqtadan o'tuvchi yagona π tekislik tenglamasini topaylik. Bu nuqtalar bir to'g'ri chiziqda yotmaydi. Quyidagi vektorlarni qaraymiz.

$$\vec{r_2} - \vec{r_1} = \overrightarrow{M_1M_2} = \{x_2 - x_1, \quad y_2 - y_1, \ z_1 - z_2\}$$

$$\vec{r_3} - \vec{r_1} = \overrightarrow{M_1M_3} = \{x_3 - x_1, \quad y_3 - y_1, \ z_2 - z_1\}$$

Bu vektorlar kolliniar bo'lmagan vektorlar, ya'ni $\overrightarrow{M_1M_2} \neq \overrightarrow{M_1M_3}$ (4.13-chizma).



4.13-chizma.

$M(x, y, z)$ nuqta M_1, M_2, M_3 nuqtalar bilan birgalikda π tekislikda yotishi uchun $\overrightarrow{M_1M}, \overrightarrow{M_1M_2}, \overrightarrow{M_1M_3}$ vektorlarning komplanar bo'lishi zarur va yetarlidir, ya'ni ularning aralash ko'paytmasi nolga teng bo'lishi zarur va shartlidir. Uch vektorning komplanarlik shartidan foydalanib, quyidagiga ega bo'lamiz:

$$\left[\overrightarrow{M_1M}, \overrightarrow{M_1M_2}\right]\overrightarrow{M_1M_3} = 0,$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \tag{4.4.1}$$

(4.4.1)– tenglama uch nuqta orqali o'tuvchi *tekislik tenglamasi* deyiladi.

4.5. Tekislikning normal tenglamasi. Koordinatalar boshidan o'tmaydigan tekisliklar uchun ko'pincha tekislikning normal tenglamasidan foydalaniladi. $\vec{n_b}$ birlik normal vektor berilgan bo'lsin. $\vec{n_b}$ vektor Ox, Oy, Oz koordinatalar o'qlari bilan mos ravishda α,β,γ burchaklar tashkil qilsa, u holda

$$\vec{n_b}=\{\cos\alpha,\cos\beta,\cos\gamma\}=\vec{i}\cos\alpha+\vec{j}\cos\beta+\vec{k}\cos\gamma$$

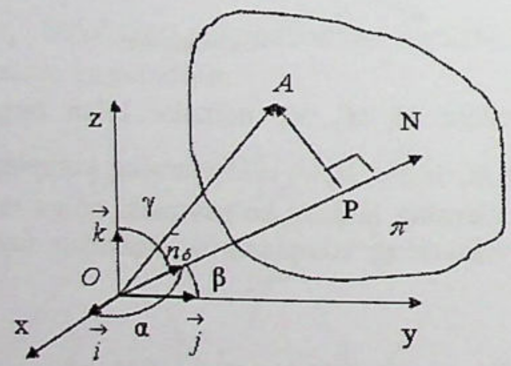
Bo'lishi ma'lum. $\vec{OP}$ koordinatalar boshidan π tekislikka tushirilgan perpindikulyar bo'lib, uning uzunligi

$$\left|\vec{OP}\right|=p$$

ga teng bo'lsin.

$\vec{n}$ va $\vec{OP}$ vektorlar kollinar (bir xil yo'nalgan) bo'lgani uchun $\vec{OP}=p\cdot\vec{n_b}$.

Quyidagi munosabatlarga asoslanib tekislikning normal tenglamasini keltirib chiqaraylik:



4.14-chizma.

$$A(x,y,z)\in\pi,\ \vec{n}\perp\pi;\ \vec{n_b}\perp\pi;\ \vec{PA}\perp\vec{n},\ \left|\vec{n_b}\right|=1,\quad ON\cap\pi=P$$

Ravshanki, 4.14-chizmadan $\vec{PA}=\vec{OA}-\vec{OP}$ va $\vec{PA}\perp\vec{n_b}$, u holda $\left(\vec{OA}-p\vec{n_b}\right)\vec{n_b}=0$ yoki $\vec{OA}\vec{n_b}=p\vec{n_b}^2$.

Agar $\vec{OA}=\{x,y,z\}$ ekanligini e'tiborga olsak, quyidagiga ega bo'lamiz:

$$x\cos\gamma+y\cos\beta+z\cos\gamma=p$$

yoki

$$x\cos\gamma+y\cos\beta+z\cos\gamma-p=0. \qquad (4.5.1)$$

(4.5.1)- tenglamaga tekislikning *normal tenglamasi* deyiladi.

4.6. Nuqtadan tekislikkacha bo'lgan masofa. Fazoda Dekart koordinatalar sistemasida $M_1(x_1,y_1,z_1)$ nuqtadan $Ax+By+Cz+D=0$ tenglama bilan berilgan tekislikkacha bo'lgan masofa

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}} \qquad (4.6.1)$$

formula bo'yicha topiladi.

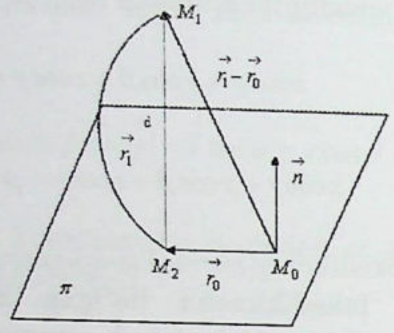
Isboti. Fazodan taiynlangan $M_1(x_1,y_1,z_1)$ nuqta olamiz. Bu $M_1(x_1,y_1,z_1)$ nuqtadan π tekislikka perpendikulyar tushiramiz, uning asosini M_2 deb belgilaymiz (4.15-chizma).

$\vec{M_1M_2}\perp\pi, M_2\in\pi, \vec{M_0M_1}=\vec{r_1}-\vec{r_0}$ vektorni qaraymiz, bunda $\vec{r_1}-\vec{r_0}$ lar $M_0\in\pi$ va M_1 nuqtalarning radius vektorlari. $d=\left|\vec{M_1M_2}\right|$ biz izlayotgan masofa bo'lsin. π tekslikning $\vec{n}=\{A,B,C\}$ normal vektorini o'tkazamiz. $M_0M_1M_2$ uchburchakdan, $\vec{M_1M_2}//\vec{n}$ hisobga olsak, u holda

$$d=\left|np_{\vec{n_0}}\left(\vec{r_1}-\vec{r_0}\right)\right|=\left|\vec{n_0}\cdot\left(\vec{r_1}-\vec{r_0}\right)\right|=\left|\frac{\vec{n}}{\left|\vec{n}\right|}\cdot\left(\vec{r_1}-\vec{r_0}\right)\right|=$$

$$=\frac{\left|\vec{n}\left(\vec{r_1}-\vec{r_0}\right)\right|}{\sqrt{A^2+B^2+C^2}}=\frac{|A_1x_1+By_1+Cz_1+D|}{\sqrt{A_2+B^2+C^2}}$$

bo'ladi, bunda $\vec{n}_0 = \frac{\vec{n}}{|\vec{n}|} = \frac{\{A,B,C\}}{\sqrt{A^2+B^2+C^2}}, D = -Ax_0 - By_0 - Cz_0.$

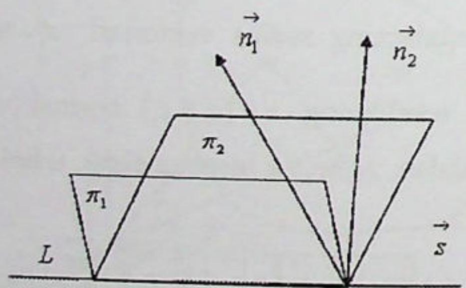


4.15-chizma

4.2-§. Fazoda to'g'ri chiziq

4.7. Fazoda to'g'ri chiziqning umumiy tenglamasi. Quyidagi birinchi darajali tenglamalar sistemasini qaraylik:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, & (\pi_1). \\ A_2x + B_2y + C_2z + D_2 = 0, & (\pi_2). \end{cases} \quad (4.7.1)$$



4.16- chizma.

Bu tenglamalarning har biri tekislikni ifodalaydi. Agar bu $\vec{n}_1$ va $\vec{n}_2$ tekisliklar parallel bo'lmasa (ya'ni ularning $\vec{n}_1$ va $\vec{n}_2$ normal vektorlari

kolliniar bo'lmasa), u holda (4.7.1) sistema ikki tekislikni kesishish chizig'i sifatida ya'ni fazoning koordinatalari (4.7.1) sistemaning har bir tenglamasini qanoatlantiradigan nuqtalari geometrik o'rni sifatida biror L to'g'ri chiziqni aniqlaydi (4.16- chizma). (4.7.1) tenglamalar fazoda *to'g'ri chiziqning umumiy tenglamalari* deb ataladi.

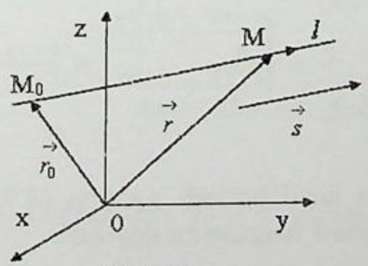
(4.7.1) to'g'ri chiziqqa parallel yoki unda yotadigan istalgan vektor bu to'g'ri chiziqning *yo'naltiruvchi vektori* deb ataladi.

To'gri chiziq tekisliklarning $\vec{n}_1$ va $\vec{n}_2$ normal vektorlariga perpindikulyar bo'lgani uchun (bu yerda $\vec{n}_1 = \{A_1, B_1, C_1\}$ va $\vec{n}_2 = \{A_2, B_2, C_2\}$) L to'g'ri chiziqning $\vec{s}$ yo'naltiruvchi vektori (u holda $\vec{n}_1$ va $\vec{n}_2$ vektorlarga perpindikulyar) sifatida $\left[\vec{n}_1, \vec{n}_2\right]$ vektor ko'paytmani olish mumkin

$$\vec{s} = \left[\vec{n}_1, \vec{n}_2\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}.$$

4.8. To'g'ri chiziqning vektor tenglamasi. ℓ ixtiyoriy to'g'ri chiziq bo'lsin. Bu to'g'ri chiziqning fazodagi holati biror tayin $M_0 \in \ell$ nuqta va ℓ da yotuvchi yoki unga parallel bo'lgan $\vec{s}$ vektor bilan to'la aniqlanadi. Berilgan M_0 va ixtiyoriy M nuqtalarning radius-vektorlarini mos ravishda $\vec{r}_0$ va $\vec{r}$ bilan belgilaymiz. Endi quyidagicha mulohaza yuritamiz (4.17-chizma).

$$M_0\left(\vec{r}_0\right) \in \ell, \quad M\left(\vec{r}\right) \in \ell \Rightarrow \overrightarrow{M_0M} \in \ell$$



4.17-chizma.

yoki shartga ko'ra $\overrightarrow{M_0M} // \vec{s}$ bo'lgani uchun, $\overrightarrow{M_0M} = \lambda \vec{s}$ yoki $\vec{r} - \vec{r}_0 = \lambda \vec{s}, \lambda \neq 0$ bo'ladi. Bundan $\vec{r} = \vec{r}_0 + \lambda \vec{s}$ tenglama kelib chiqadi. Uni to'g'ri chiziqning vektor tenglamasi deyiladi.

4.9. To'g'ri chiziqning parametrik va kanonik tenglamalari. ℓ to'g'ri chiziq o'zining vektor tenglamasi $\vec{r} = \vec{r}_0 + \lambda \vec{s}$ bilan berilgan bo'lsin. $M_0(x_0, y_0, z_0)$ nuqta $\vec{r}_0$ radius vektorning oxiri, $M(x, y, z)$ esa ℓ to'g'ri chiziqning o'zgaruvchi $\vec{r}$ radius vektorining oxiri bo'lsin, $\vec{s}$ vektorning Ox, Oy, Oz, koordinata o'qlariga proyeksiyalarini mos ravishda m, n, p bilan belgilaymiz. U holda

$$\vec{r} - \vec{r}_0 = \lambda \vec{s} \Leftrightarrow \begin{cases} x - x_0 = \lambda m, \\ y - y_0 = \lambda n, \\ z - z_0 = \lambda p. \end{cases}$$

Bundan ushbu tengliklarga ega bo'lamiz:

$$\begin{cases} x(\lambda) = x_0 + \lambda m, \\ y(\lambda) = y_0 + \lambda n, \\ z(\lambda) = z_0 + \lambda p. \end{cases} \qquad (4.9.1)$$

(4.9.1) tenglamalar to'plami to'g'ri chiziqning parametrik tenglamasi deyiladi. Parametr $\lambda \in (-\infty, \infty)$ turli qiymatlar qabul qilganda ℓ to'g'ri chiziqning turli nuqtalari va faqqat shu to'g'ri nuqtalari hosil bo'ladi. Ravshanki, (4.9.1) munosabatlardan parametr λ ni chiqarish mumkin. Natijada

$$\begin{cases} \frac{x-x_0}{m} = \lambda, \\ \frac{y-y_0}{n} = \lambda, \\ \frac{z-z_0}{p} = \lambda, \end{cases} \Rightarrow \begin{cases} \frac{x-x_0}{m} = \frac{y-y_0}{n}, \\ \frac{y-y_0}{n} = \frac{z-z_0}{p}, \end{cases} \Rightarrow \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p} \qquad (4.9.2)$$

tenglamalar sistemasi hosil bo'ladi. Odatda (4.9.2) tenglamalar sistemasi to'g'ri chiziqning *kanonik tenglamasi* deyiladi.

4.10. Berilgan ikki nuqta orqali o'tuvchi to'g'ri chiziq tenglamasi. Fazoda $M_0(x_1, y_1, z_1)$ va $M_2(x_2, y_2, z_2)$ nuqtalar berilgan bo'lib, bu nuqtalar orqali o'tuvchi to'g'ri chiziq tenglamasi ushbu

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z} \qquad (4.10.1)$$

ko'rinishda bo'ladi.

Agar boshlang'ich nuqtasi M_1 nuqtada bo'lsa, $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$ vektorni to'g'ri chiziqning yo'naltiruvchi vektori sifatida qarash mumkin.

4.11. Ikki to'g'ri chiziq orasidagi burchak. Fazodagi ikki to'g'ri chiziq orasidagi burchak deb, fazoning ixtiyoriy nuqtasidan berilgan to'g'ri chiziqlarga parallel qilib o'tkazilgan ikki to'g'ri chiziq orasidagi burchakka aytiladi.

Boshqacha aytganda, to'g'ri chiziqlarning yo'naltiruvchi vektorlari orasidagi burchak berilgan to'g'ri chiziqlar *orasidagi burchak* deb ataladi.

To'g'ri chiziqlar kanonik tenglamalari bilan berilgan bo'lsin:

$$\ell_1: \quad \frac{x - x_1}{m_1} = \frac{y - y_1}{n_1} = \frac{z - z_1}{p_1}, \qquad M_1(x_1, y_1, z_1) \in \ell_1,$$

$$\ell_2: \quad \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2} = \frac{z - z_2}{p_2}, \qquad M_2(x_2, y_2, z_2) \in \ell_2$$

bu to'g'ri chiziqlarning yo'naltiruvchi vektorlari $\vec{s}_1 = \{m_1, n_1, p_1\}, \vec{s}_2 = \{m_2, n_2, p_2\}$ bo'lsin. Bu holda $\vec{s}_1, \vec{s}_2$ vektorlar orasidagi burchak (4.18-chizmada) bunday deb topiladi.

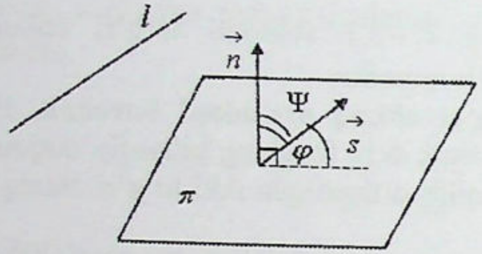
$$\cos\left(\vec{s}_1, \vec{s}_2\right) = \frac{\vec{s}_1 \cdot \vec{s}_2}{\left|\vec{s}_1\right| \left|\vec{s}_2\right|} = \frac{m_1 m_2 + n_1 n_2 + p_1 p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2} \sqrt{m_2^2 + n_2^2 + p_2^2}} \qquad (4.11.1)$$

(4.11.1) – formulaga fazoda *ikki to'g'ri chiziq orasidagi burchak formulasi* deyiladi.

Agar l_1 va l_2 to‘g‘ri chiziqlar o‘zaro parallel bo‘lsa, $\vec{s_1}$ va $\vec{s_2}$ vektorlar o‘zaro kolliniar bo‘ladi. Bundan ikki to‘g‘ri chiziqning parallellik sharti ushbu

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2} \qquad (4.11.2)$$

ko‘rinishda bo‘ladi.



4.18-chizma.

Demak, l_1 va l_2 to‘g‘ri chiziqlar parallel bo‘lishi uchun (4.11.2) tenglikning bajarilishi zarur va yetarlidir.

Agar l_1 va l_2 to‘g‘ri chiziqlar o‘zaro fogonal bo‘ladi. Bundan ikki to‘g‘ri chiziqning perpindikulyarlik sharti ushbu

$$m_1m_2+n_1n_2+p_1p_2=0 \qquad (4.11.3)$$

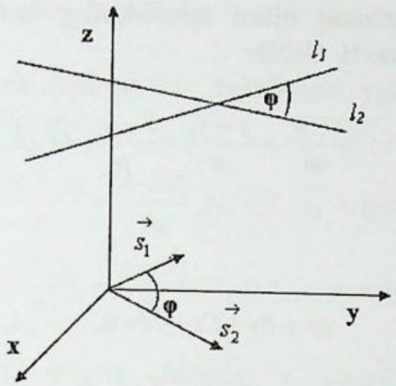
ko‘rinishda bo‘ladi.

Demak, l_1 va l_2 to‘g‘ri chiziqlar perpindikulyar bo‘lishi uchun (4.11.3) tenglikning bajarilishi zarur va yetarlidir.

4.12. To‘g‘ri chiziq bilan tekislik orasidagi burchak. To‘g‘ri chiziq bilan uning tekislikdagi proyeksiyasi orasidagi burchak *to‘g‘ri chiziq va tekislik orasidagi burchak* deyiladi (4.19-chizma).

Tekislik va to‘g‘ri chiziq quyidagi tenglamalari bilan berilgan bo‘lsin:

$$\pi: Ax+By+Cz+D=0, \vec{n}=\{A,B,C\},$$
$$\ell: \frac{x-x_1}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p}, \vec{s}=\{m,n,p,\}, M_1(x_1,y_1,z_1)\in\ell.$$



4.19-chizma.

Ikki vektorning skalyar ko‘paytmasi formulasidan $\vec{n}\cdot\vec{s}=|\vec{n}|\cdot|\vec{s}|\cos\Psi$ quyidagiga ega bo‘lamiz (bunda $\Psi=\frac{\pi}{2}-\varphi$):

$$\cos\psi=\sin\varphi=\frac{Am+Bn+Cp}{\sqrt{A^2+B^2+C^2}\,\sqrt{m^2+n^2+p^2}}. \qquad (4.12.1)$$

(4.12.1) formulaga ℓ to‘g‘ri chiziq va π tekislik *orasidagi burchak formulasi* deyiladi.

Agar ℓ to‘g‘ri chiziq va π tekislik parallel bo‘lsa, u holda (4.12.1) dan ushbu

$$Am+Bn+Cp=0 \qquad (4.12.2)$$

tenglikka ega bo‘lamiz.

Demak, ℓ to‘g‘ri chiziq bilan π tekislik parallel bo‘lishi uchun (4.12.2) tenglikning bajarilishi zarur va yetarlidir.

Agar ℓ to‘g‘ri chiziq va π tekislik perpindikulyar bo‘lishi uchun

$$\frac{A}{m}=\frac{B}{n}=\frac{C}{p}$$

tenglikning bajarilishi zarur va yetarlidir.

4.13. To'g'ri chiziq bilan tekislikning kesishishi. Bir chiziqning tekislikda yotish sharti. Ushbu

$$\ell: \quad \frac{x-x}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p}, \ M_1(x_1,y_1,z_1)\in \ell \qquad (4.13.1)$$

to'g'ri chiziq bilan

$$\pi: \quad Ax+By+Cz+D=0, \ \vec{n}=\{A,B,C\} \qquad (4.13.2)$$

tekislikning kesishish nuqtasini (bunday nuqta mavjud bo'lsa) topish bilan shug'ullanamiz. Buning uchun ℓ to'g'ri chiziqning (4.13.1) tenglamasini parametrik ko'rinishda yozamiz:

$$\left.\begin{array}{l} x=x_1+\lambda m \\ y=y_1+\lambda n \\ z=z_1+\lambda p \end{array}\right\} \qquad (4.13.3)$$

x,y,z ning bu qiymatlarini (4.13.2) tenglamaga qo'yamiz:

$$A(x_1+\lambda m)+B(y_1+\lambda n)+C(z_1+\lambda p)+D=0$$

yoki

$$Ax_1+By_1+Cz_1+D+A\lambda m+B\lambda n+C\lambda p=0.$$

Bu tenglamadan λ ga nisbatan yechamiz:

$$\lambda=\frac{Ax_1+By_1+Cz_1+D}{Am+Bn+Cp}. \qquad (4.13.4)$$

λ ning bu qiymatini (4.13.3) tenglamaga qo'yib, berilgan to'g'ri chiziq bilan tekislikning kesishish nuqtasining koordinatalarini topamiz.

Agar (4.13.4) formulada $Am+Bn+Cp\neq 0$ bo'lsa, ℓ to'g'ri chiziq bilan π tekislik *bitta nuqtada* kesishadi.

Agar $Am+Bn+Cp=0$ bo'lib, $Ax_1+By_1+Cz_1+D\neq 0$ bo'lsa, ℓ to'g'ri chiziq bilan π tekislik bilan *parallel* bo'ladi.

Agar $Am+Bn+Cp=0$ va $Ax_1+By_1+Cz_1+D=0$ bo'lsa, ℓ to'g'ri chiziq π tekislikda yotadi.

4.14. Ikki to'g'ri chiziqning bir tekislikda yotish sharti. Ushbu to'g'ri chiziqlar berilgan bo'lsin:

$$(\ell_1): \frac{x-x_1}{m_1}=\frac{y-y_1}{n_1}=\frac{z-z_1}{p_1}, M_1\in \ell_1, \vec{s_1}=\{m_1,n_1,p_1\}\in \ell,$$

$$(\ell_2): \frac{x-x_2}{m_2}=\frac{y-y_2}{n_2}=\frac{z-z_2}{p_2}, M_2\in \ell_2, \vec{s_2}=\{m_2,n_2,p_2\}\in \ell.$$

Ravshanki, l_1 va l_2 to'g'ri chiziqlar π tekislikda yotish uchun $\overrightarrow{M_1M_2}, \vec{s_1}, \vec{s_2}$ uchta vektor kolliniar bo'lishi zarur va yetarlidir.

Uch vektorning aralash ko'paytmasi nolga teng bo'lishidan foydalanib, ushbuga ega bo'lamiz:

$$\left[\overrightarrow{M_1M_2}, \vec{s_1}\right]\cdot \vec{s_2}=0$$

yoki

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{vmatrix}=0 . \qquad (4.14.1)$$

(4.14.1)-tenglikning bajarilishi l_1 va l_2 to'g'ri chiziqlar bir tekislikda yotishi uchun zarur va yetarli bo'lishi shartdir.

4.3-§. Ikkinchi tartibli sirtlar

4.15. Ikkinchi tartibli sirtning umumiy tenglamasi. Fazoning biror Dekart koordinatalar sistemasida a_{11}, a_{22}, a_{33}, a_{12}, a_{23}, a_{13} koeffisientlardan kamida biri noldan farqli bo'lgan

$$a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{23}yz+2a_{13}xz+$$

$$+2a_{14}x+2a_{24}y+2a_{34}z+a_{44}=0 \qquad (4.15.1)$$

tenglama bilan beriladigan nuqtalari to'plami *ikkinchi tartibli sirt* deyiladi.

Bizning asosiy maqsadimiz quyidagilardan: agar nuqtalarning geometrik o'rni sifatida sirt berilgan bo'lsa, uning tenglamasini tuzish yoki aksincha, agar $Oxyz$ koordinatalar sistemasida tenglama berilgan bo'lsa, shu tenglama bilan tasvirlanadigan sirtning shaklini tekshirish kerak.

(4.15.1) tenglamaga ikkinchi tartibli sirtning *umumiy tenglamasi* deyiladi.

4.16. Aylanma sirtlar. Biror tekis L chiziqning l o'qi atrofida aylanishidan hosil bo'lgan nuqtalar to'plami *aylanma sirt* deyiladi. L chiziq aylanma sirtning meridiani, l o'qi esa, uning aylanish o'qi deyiladi. Shuni ta'kidlab o'tamizki, meridianning aylanish o'qi atrofida aylanishida uning har bir nuqtasi aylana chizadi.

Aylanish o'qi Oz o'qidan iborat bo'lgan L meridian esa Oyz tekislikda yotib,

$$\begin{cases} F(y,z) = 0, \\ x = 0 \end{cases} \qquad (4.16.1)$$

tenglama bilan berilgan holni qarashdan boshlaymiz, S berilgan L chiziqning Oz o'qi atrofida aylanishidan hosil bo'lgan sirti va $M(x,y,z)\in S$ bo'lsin. M nuqta orqali Oz o'qiga perpendikulyar qilib o'tgazilgan Q tekislik S sirtning markazi Oz o'qida yotuvchi K nuqtada bo'lgan aylana buyicha kesadi.

Aylana bilan L chiziqning kesishish nuqtasini N bilan belgilaymiz. N nuqta $N(0,y_1,z)$ koordiniatarlarida K nuqta esa $(0,0,z)$ koordinatariga ega. $|KM|=|KN|=|y_1|$. Ikki nuqta orasidagi masofani topish formulasiga ko'ra

$$d=|KM|=\sqrt{(x-0)^2+(y-0)^2+(z-z)^2}=\sqrt{x^2+y^2}$$

bo'ladi. Bundan $|y_1|=\sqrt{x^2+y^2}$ yoki $y_1=\pm\sqrt{x^2+y^2}$. Buni e'tiborga olsak, (4.16.1) tenglamasi quyidagi

$$F(\pm\sqrt{x^2+y^2},z)=0 \qquad (4.16.2)$$

ko'rinishga keladi. Shunday qilib, S aylanma sirtiga tegishli ixtiyoriy M nuqtaning koordinatalari (4.16.2) tenglamani qanoatlantiradi. Ox va Oy o'qi atrofida aylantirishdan hosil bo'lgan aylanma sirtning tenglamasi mos ravishda

$$F(x,\pm\sqrt{y^2+z^2})=0 \text{ yoki } F(\pm\sqrt{x^2+z^2},y)=0$$

ko'rinishida bo'ladi.

Ellipsoid. Ellipsni simmetriya o'qi atrofida aylantirishdan hosil bo'lgan aylanma sirti *ellipsoid* deyiladi.Ushbu

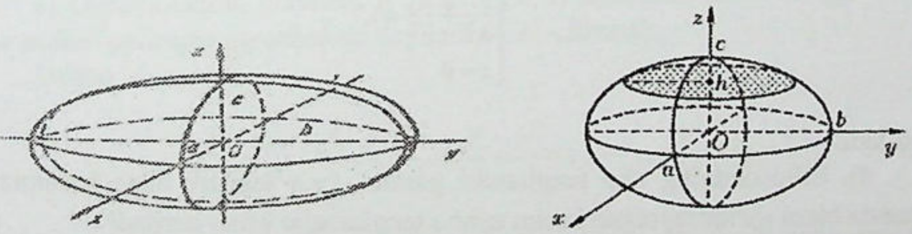
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

tenglamaga ellipsoidning kanonik tenglamasi deyiladi, bunda a,b,c musbat sonlar. Odatda a,b,c musbat sonlar ellipsoidning yarim o'qlari deyiladi (4.20-chizma).

Agar $a\neq b, b\neq c, a\neq c$ bo'lsa, u holda elipsoid uch o'qli elipsoid deyiladi.

Masalan. Oyz koordinatalar sistemasida $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ tenglama bilan berilgan ellipsning o'z o'q atrofida aylantirishdan hosil bo'lgan sirtning tenglamasini tuzing.

Yechilishi. Aylanma sirtning tenglamasini tuzish qoidasiga asosan, quyidagiga ega bo'lamiz:



4.20-chizma.

$$\frac{(\pm\sqrt{x^2+y^2})^2}{b^2}+\frac{z^2}{c^2}=1 \text{ yoki } \frac{x^2}{b^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad (4.16.3)$$

(4.16.3) tenglama bilan berilgan sirt *aylanma ellipsoid* deyiladi.

Agar $b>c$ bo'lsa, (4.16.3) tenglama qisilangan aylanma ellipsoydani (2-chizma), $b<c$ bo'lsa, chuzilgan aylanma ellipsoydaning (4.20-chizma), $b=c$ bo'lganda esa, tenglama $x^2+y^2+z^2=b^2$ ko'rinishini oladi va bu tenglama sferani ifodalaydi. Umumiy holda ellipsondning formasini tekshirish qulay.

1). Ellipsondning $z=0$ tekislik bilan kesishish chizig'i ushbu tenglamalar sistemasi bilan beriladi

$$\begin{cases} \dfrac{x^2}{y^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1, \\ z=0. \end{cases}$$

Bu chiziq yarim o'qlari a va b dan iborat bo'lib Oxy va Oxz tekisliklari simmetrik bo'lgan $A_1B_1A_2B_2$ ellipsdir (4.20-chizma)

2). Ellipssoidning $y=0$ tekislik bilan kesimning tenglamasi

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=1, \\ y=0 \end{cases}$$

ko'rinishda bo'ladi.

3).Ellipsondning $x=0$ tekislik bilan kisimning tenglamasi

$$\begin{cases} \dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1, \\ x=0 \end{cases}$$

bo'ladi.

4). Ellipsondning Oxy tenglamasi paralel $z=h$ tekislik bilan kesamiz. Bunda bizni qiziqtirayotgan kesim ushbu tenglamalar bilan beriladi

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1+\dfrac{h^2}{c^2}, \\ z=h, \end{cases}$$

bu yerda uchta hol bo'ishi mumkin.

$a)$agar $|h|<c$, bo'lsa $1-\dfrac{h^2}{c^2}>0$ bo'lib, kesimda yarim o'qlari $a_h=a\sqrt{\frac{h^2}{c^2}}$, $b_h=b\sqrt{1-\frac{h^2}{c^2}}$ va markazi Oz o'qidagi (0, 0, h) nuqtada bo'lgan

$$\begin{cases} \dfrac{x^2}{a_h^2}+\dfrac{y^2}{b_h^{\ 2}}=1, \\ z=h \end{cases}$$

ellips hosil bo'ladi.

$b)$agar $h=c$ yoki $h=-c$ bo'lsa, u holda kesimda ellipsoidning c_1 va c_2 o'qlari hosil bo'ladi.

$c)$agar $h<-c$ va $h>c$ bo'lsa, u holda kesimlar mavhum ellipslar hosil bo'ladi, ya'ni $z=h$ tekslik bilan ellipsoid umumiy nuqtaga ega bo'lmaydi.

7) Ellipsoidning tenglamasida x, y, z o'zgaruvchilari faqat juft darajada qatnashadi, bunda elipsoid kordinatalar boshiga nisbatan simmetrik bo'ladi. Kordiratalar boshi ellipsoidning markazi deyiladi.

8) Ellipsoidning o'zgarish sohasi $-a\le x\le a,\quad -b\le y\le b,\quad -c\le z\le c$ b¢ladi.

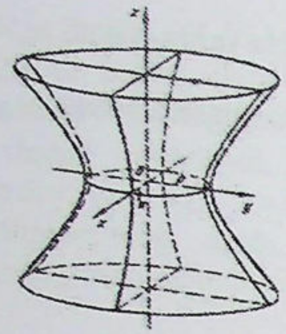
Gipgerboloidlar. Giperboloidlar ikki xil, ya'ni bir pallali gepirboloid va ikki palalli gepirboloid bo'ladi.

1) Giperbolaning mavhum o'qi atrofida aylanishidan hosil bo'lgan sirt *bir pallali aylanma giperboloid* deyiladi (4.21-chizma).

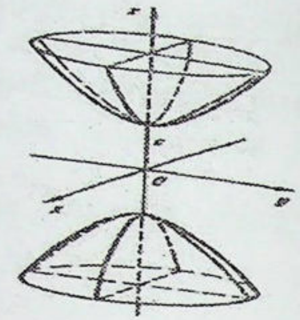
Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (4.16.4)$$

ko'rinishdagi tenglamaga bir pallali gepirboloidning kononik tenglamasi deyiladi. Koordinatalar boshi bir pallali gepirboloidning markazi, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ tenglama uchun a va b sonlar bir pallali gepirboloidning haqiqiy yarim o'qi , c esa mavhum yarim o'qi deyiladi.



4.21-chizma.



4.22-chizma.

Bir pallali giperboloudning tekisliklar bilan kesimlarini qaraymiz:
1) Giperboloudni Oxy, Oxz, Oyz tekisliklar bilan kesak, kesimda

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \\ z=0 \end{cases} \text{-ellips; } \begin{cases} \frac{x^2}{a^2}-\frac{z^2}{c^2}=1, \\ y=0 \end{cases} \text{- giperbola; } \begin{cases} \frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \\ x=0 \end{cases} \text{- giperbola.}$$

2) Agar giperboloudni Oxy tekislikka paralel bo'lgan $z=h$ tekislik bilan kessak, rasmda

$$\begin{cases} \frac{x^2}{a_h^2}+\frac{y^2}{b_h^2}=1, \\ z=h. \end{cases}$$

Ellips hosil bo'ladi, bunda $a_h=\frac{a}{c}\sqrt{c^2+h^2}$, $b_h=\frac{b}{c}\sqrt{c^2+h^2}$. $h=0$ bo'lganda eng kichik yarim o'qli ellips hosil bo'ladi.

2). Giperbolani o'zining haqiqiy o'qi atrofida aylanishidan hosil bo'lgan sirt *ikki pallali aylanma giperboloid* deyiladi (4.22-chizma).

Ushbu $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ko'rinishdagi tenglama *ikki pallali giperboloidning* kanonik tenglamasi deyiladi.

Bu sirtning turli tekisliklari bilan kesilganligini qaraymiz.

1) Oyz tekislik bilan kesim

$$\begin{cases} -\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \\ x=0 \end{cases}$$

tenglamalar sestimasi bilan beriladi. Bu tenglamalar sestimasi yechimga ega emas, shuning uchun Oyz tekislik bilan kesishmaydi.

2) Oyz tekislik bilan kesim

$$\begin{cases} \frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{h^2}{a^2}-1, \\ x=h \end{cases}$$

tenglamalar sestimasi bilan berildi.

Agar $|h|>a$ bo'lsa, gepirbolaning $x=h$ tekislik bilan kesimida $\frac{y^2}{b_h^2}+\frac{z^2}{c_h^2}=1$ ellipis xosil bo'ladi, bunda

$$b_h=b\sqrt{\frac{h^2}{a^2}-1},\ c_h=c\sqrt{\frac{h^2}{a^2}-1}\ .$$

Agar $|h|<a$ bo'lsa, $x=h$ tekislik va gepirboloid kesishmaydi. Boshqacha aytganda $x=-h$ va $x=h$ paralel tekisliklarning orasidagi polosada gepiboloidning nuqtalari yo'q. Shuni alohida takidlaymizki, ushbu

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

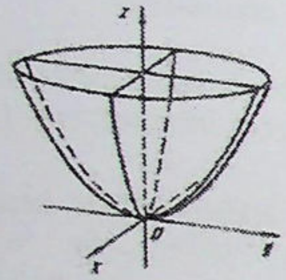
tenglamalar ham ikki pallali gepirboloidlarni ifodalaydi.

Paraboloidlar. Paraboloidlarning o'z o'qlari atirofida aylanishidan hosil bo'lgan sirt *aylanma paraboloid* deyiladi. To'g'riburchakli dekart koordinatalari sistemasida

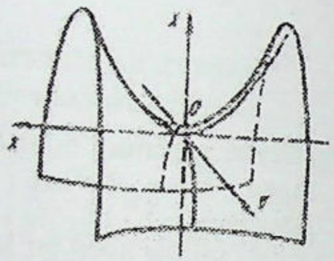
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$$

tenglama bilan tasvirlangan sirt elliptik paraboloid deb aytiladi (4.23-chizma). Elliptik paraboloidni uning o'qiga perpendikulyar bo'lgan $z=h$ tekislik bilan kessak, kesim quyidagi tenglama bilan aniqlanadi:

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2h, \\ z=h. \end{cases}$$



4.23-chizma.



4.24-chizma.

Agar $h>0$ bo'lsa, kesim ellips, agar $h<0$ bo'lsa, sirt bilan tekislik kesishmaydi, agar $h=0$ bo'lsa, kesim $O(0,0,0)$ nuqtadan iborat.

Elliptik paraboloidani Ox va Oy o'qlariga perpendikulyar bo'lgan $x=h$ va $y=h$ tekisliklar bilan kesak, kesimda parabola hosil bo'ladi.

$Oxyz$ tekistlikda

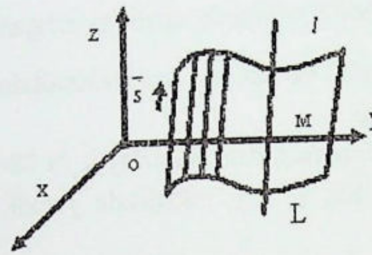
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$$

ko'rinishdagi tenglama bilan tasvirlangan sirt *gepirbolik paraboloid* deyiladi (4.24-chizma).

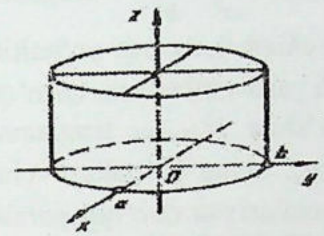
Gepirbolik paraboloidni Oxy tekistlikka parallel bo'lgan $z=h$ $(h>0)$ tekistlik bilan kesak, kesim quyidagi tenglama bilan aniqlanadi:

$$\begin{cases} \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2h, \\ z=h. \end{cases}$$

4.17. Silindrik sirtlar. Berilgan $\vec{s}$ vektor yo'nalishiga parallelligini saqlagan holda, berilgan L chiziqni kesadigan to'g'ri chiziqlar to'plami *silindrik sirt* deyiladi (4.25-chizma). Bunda L chiziq silindrlik sirtning yo'naltiruvchisi, $\vec{s}$ vektorga parallel l chiziqlar silindrik sirtining yasovchlari deyiladi.



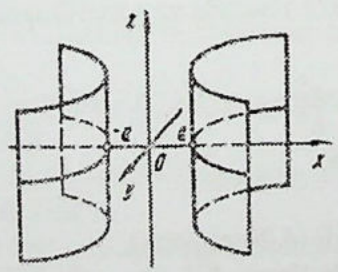
4.25-chizma.



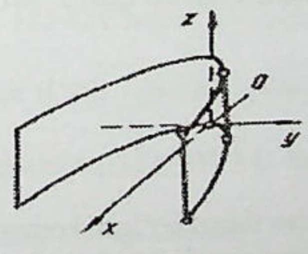
2.26-chizma.

Yasovchilari Oz o'qlarida paralil bo'lgan silindrlik sirtni qaraylik. Bu sirtni sirtning yasovchisi l to'g'ri chmziqing tenglamasi $y=a(a>0)$ bo'lib, u Oyz tekistlikda yotadi. Aylanma sirt tenglamasini etiborga olsak bu sirtning tenglamasi.

$$\pm\sqrt{x^2+y^2}=a \text{ yoki } x^2+y^2=a^2.$$



4.27-chizma.



4.28-chizma.

Bu sirtni to'g'ri doiraviy silindr deyiladi.
Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

tenglama fazoda silindrlik sirtni tasvirlaydi, uning yo'naltiruvchisi ellips bo'lib, u Oxy tekistlikda yotadi, yasovchisi esa, Oz o'qqa paralel bo'ladi. Bunday sirt *elliptik silindr* deyladi(4.26-chizma).

Ushbu $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tenglama fazoda *gepirbolik silindrik* sirtni tasvirlaydi (4.27-chizma), uning yo'naltiruvchisi gepirbola bo'lib, u Oxy tekistlikda yotadi yasovchilari esa, Oz o'qqa paralil bo'ladi.

Ushbu $y^2=2px$ tenglama *paraboloik silindrik* sirtni tasvirlaydi (4.28-chizma), uning yo'naltiro'vchisi parabola bo'lib, u Oxy tekislikda yotadi, yasovchilari esa Oz o'qqa parallel bo'ladi.

4.18. Konus sirtlar. Fazoda biror qo'zg'almas $M_0(x_0,y_0,z_0)$ nuqtadan utib, berilgan L chiziqni kesuvchi l chiziqning harakatidan hosil bo'lgan sirt *konus sirt* deyiladi.

$M_0(x_0,y_0,z_0)$ nuqta konus sirtining *uchi*, L chiziq uning *yunaltiruvchisi*, l to'g'ri chiziq esa, konus sirtining *yasovchisi* deyiladi.

Dekart koordinatalar sitemasida

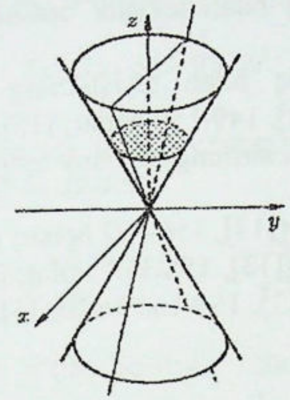
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \qquad (4.18.1)$$

tenglamaga ega bo'lgan K sirt uchi $O(0,0,0)$ nuqtada va yunaltiruvchisi

$$L: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \\ z=h \end{cases}$$

ellipsdan iborat bo'lgan konusni ifodalaydi (4.29-chizma).

Tenglamasi (4.18.1) ko'rinishda bo'lgan *konusni* tekisliklar bilan kesilsa, kesimda ikkinchi tartibli chiziqlar hosil bo'ladi, ya'ni konusni $y=h,\ z=h,\ \frac{x}{a}-\frac{z}{c}=h\ (h>0)$ tekisliklar bilan kessak, kesimda mos ravishda giperbola, ellips, parabola hosil bo'ladi.



4.29-chizma.

4-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

4.1. Tekislikning umumiy tenglamasi ([13], 129-130 betlar; [5], [6], [9], [10]).

4. 2. Tenglikning koordinata o'qlariga nisbatan joylashuvi([13], 130-132 betlar; [5], [6], [9], [10]).

4.3. Ikki tekislik orasidagi burchak. Tekisliklarning parallellik va perpindikulyarlik sharti([13], 138-140 betlar; [5], [6], [9], [10]).

4.4. Uch nuqta orqali o'tuvchi tekislik tenglamasi([13], 137-139 betlar; [5], [6], [9], [10]).

4. 5. Tekislikning normal tenglamasi([13], 135-137 betlar; [5], [6], [9], [10]).

4.6. Nuqtadan tekislikkacha bo'lgan masofa ([13], 136-137 betlar; [5], [6], [9], [10]).

4.7. Fazoda to'g'ri chiziqning umumiy tenglamasi([13], 140-141 betlar; [5], [6], [9], [10]).

4.8. To'g'ri chiziqning vektor tenglamasi([13], 140-141 betlar; [15], [6], [9], [10]).

4.9. To'g'ri chiziqning parametrik va kanonik tenglamalari([13], 141-142 betlar; [5], [6], [9], [10]).

4.10. Berilgan ikki nuqta orqali o'tuvchi to'g'ri chiziq tenglamasi([13], 145 bet; [5], [6], [9], [10]).

4.11. Ikki to'g'ri chiziq orasidagi burchak([13], 145-146 betlar; [5], [6], [9], [10]).

4.12. To'g'ri chiziq bilan tekislik orasidagi burchak ([13], 147-148 betlar; [5], [6], [9], [10]).

4.13. To'g'ri chiziq bilan tekislikning kesishishi. Bir chiziqning tekislikda yotish sharti([3], 149-150 betlar; [15], [6], [9], [10]).

4.17. Ikkinchi tartibli sirtning umumiy tenglamasi([3], 156 bet; [5], [6], [9], [10]).

4.16. Aylanma sirtlar([13], 156-157 betlar; [5], [6], [9], [10]).

4.17. Silindrik sirtlar([13], 158-159 betlar; [5], [6], [9], [10]).

4.18.Konus sirtlar ([13], 159-160 betlar; [5], [6], [9], [10]).

4.1-amaliy mashg'ulot.
Fazoda tekislik va to'gri chiziq

1-misol. $M_0(3;-4,2)$ nuqtadan o'tuvchi va $\vec{n}=\{1,-2,3\}$ vektorga perpendikulyar bo'lgan tekislik tenglamasini tuzing.

Yechilishi. Bu yerda A=1, B=-2, C=3. (4.1.4) tenglamaga asosan, izlanayotgan tenglamani hosil qilamiz :

$$1(x-3)-2(y+4)+3(z-2)=0 \qquad yoki \qquad x-2y+3z-17=0. ■$$

2-misol. $ABCD$ piramidaning uchlari berilgan: $A(4;7;8)$ $B(-1;13;0)$, $C(2;4;9)$, $D(1;8;9)$.Quyidagilarni toping: 1) ABC tekislik tenglamasini; 2) AB qirra tenglamasini; 3) D uchidan o'tuvchi ABC yoqqa perpendikulyar to'g'ri chiziq tenglamasini;4) C uchidan o'tuvchi AB qirraga parallel to'g'ri chiziq tenglamasini; 5) D uchidan o'tuvchi AB qirraga perpendikulyar tekislik tenglamasini; 6) AB qirra bilan ABS yo'q orasidagi burchak sinusini; 7) ABC va Oxy tekisliklar orasidagi burchak kosinusini; 8) D uchdan ABC yoqqacha bo'lgan masofani.

Yechilishi.1) ABC tekislik tenglamasini tuzish uchun (4.4.1) formuladan foydalanamiz

$$\begin{vmatrix} x-4 & y-7 & z-8 \\ -5 & 6 & -8 \\ -2 & -3 & 1 \end{vmatrix}=0.$$

Bu yerdan izlanayotgan ABC tekislik tenglamasiga ega bo'lamiz:

$$6x-7y-9z+97=0.$$

2) Ikki nuqta orqali o'tuvchi to'g'ri chiziqning (4.10.1) formulasiga ko'ra, AB qirra teglamasini hosil qilamiz:

$$\frac{x-4}{-1-4}=\frac{y-7}{13-7}=\frac{z-8}{0-8} \qquad yoki \qquad \frac{x-4}{5}=\frac{y-7}{-6}=\frac{z-8}{8}.$$

3) D (1,8,9) nuqtadan o'tib, berilgan ABC yoqqa perpendikulyar bo'lgan to'g'ri chiziq teglamasini tuzish uchun $\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C}$ formuladan foydalanamiz. ABC tenglikning normal vektor $\vec{n}=\{6,-7,-9\}$,

$$\frac{x-1}{6}=\frac{y-8}{-7}=\frac{z-9}{-9}.$$

4) $C(2,4,9)$ nuqtadan o'tib, AB qirra $\frac{x-4}{5}=\frac{y-7}{-6}=\frac{z-8}{8}$ teglamasiga paralel to'g'ri chiziq teglamasi $\frac{x-2}{5}=\frac{y-4}{-6}=\frac{z-9}{8}$ ko'rinishda bo'ladi.

5) Berilgan $D(1,8,9)$ nuqtadan o'tib, AB qirraga $\frac{x-4}{5}=\frac{y-7}{-6}=\frac{z-8}{8}$ perpendikulyar tekislik tenglamasi $5(x-1)-6(y-8)+8(z-9)=0$ yoki $5x-6y+8z-29=0$ bo'ladi.

6) AB qirra bilan ABC yoq orasidagi burchak sinusini topish uchun (4.12.1) formuladan foydalanamiz:

$$\sin\varphi=\frac{|5\cdot 6-7\cdot(-6)-9\cdot 8|}{\sqrt{5^2+6^2+8^2}\sqrt{6^2+(-7)^2+(-9)^2}}=0, \quad \varphi=0^0.$$

7) ABC va $Oxy: 6x-7y-9z+17=0,\ z=0$ tekisliklar orasidagi burchakni topish uchun (4.3.1) formulasidan foydalanamiz:

$$\cos\varphi = \frac{\vec{n}_1\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{0\cdot 6 + 0\cdot(-7) + 1\cdot(-9)}{\sqrt{1}\sqrt{6^2+(-7)^2+(-9)^2}} = -\frac{9}{\sqrt{166}} = -0{,}7.$$

8) $D(1,8,9)$ nuqtadan $6x-7y-9z+97=0$ tenglikkacha bo‘lgan masofani (4.6.1) formuladan foydalanamiz:

$$d = \frac{6\cdot 1 - 7\cdot 8 - 9\cdot 9 + 97}{\sqrt{6^2+(-7)^2+(-9)^2}} = \frac{50}{\sqrt{166}}.$$

Mustaqil yechish uchun misol va masalalar
Fazoda tekislik

1. Ushbu $A(3;2;-2)$, $B(-2;0;0)$, $C(-3;1;0)$, $D(-4;-2;2,5)$ nuqtalar berilgan. Bu nuqtalardan qaysilari $2x - 3y + 2z + 4 = 0$ tekislikka tegishli bo‘lishini ko‘rsating.

2. 1) $M(-3,0,2)$ nuqtadan o‘tuvchi va $\boldsymbol{n}=(1,3,4)$ vektorga perpendikulyar tekislikning tenglamasini tuzing.

2) $M(6,4,5)$ nuqtadan o‘tuvchi va $\boldsymbol{n}=(-1,-3,2)$ vektorga perpendikulyar tekislikning tenglamasini tuzing.

3) $A(4;-2;3)$ va $B(1;4;2)$ nuqtalar berilgan. A nuqtadan o‘tuvchi va AB vektorga perpendikulyar bo‘lgan tekislikning tenglamasini tuzing.

3. 1) Ox o‘qdan va $M(3,2,4)$ nuqtadan o‘tuvchi;

2) Oy o‘qdan va $M(-2,-3,-4)$ nuqtadan o‘tuvchi;

3) Oz o‘qdan va $M(1,1,1)$ nuqtadan o‘tuvchi tekislik tenglamasini tuzing.

4. $M(2,-1,3)$ nuqtadan o‘tuvchi va $\boldsymbol{a} = (3,0,-1)$ hamda $\boldsymbol{b} = (-3,2,2)$ vektorlarga parallel ravishda o‘tuvchi tekislikning tenglamasini tuzing.

5. 1) $M(-2,3,4)$ nuqtadan o‘tuvchi va $x + 2y - 3z + 4=0$ tekislikka parallel bo‘lgan tekislikning tenglamasini tuzing.

2) $M_1(-2, -3, 1)$ va $M_2(1, 4, -2)$ nuqtalardan o‘tuvchi va $2x - 3y - z + 4 = 0$ tekislikka perpendikulyar bo‘lgan tekislikning tenglamasini tuzing.

6. Quyidagi tekisliklarning koordinata o‘qlaridan ajratgan kesmalarini hisoblang:

1) $4x - 3y - z + 12=0$; 2) $5x + y - 4z - 20=0$;

3) $x - 8z - 16 = 0$; 4) $y - 7 = 0$.

7. Quyidagi berilgan tekislik tenglamalarini normal shaklga keltiring.

1) $2x - 9y + 6z - 22=0$; 2) $5x + \sqrt{8}y - 4z + 5 = 0$;

3) $4x + 3y + 12z + 6 = 0$.

8. 1) $A(2,3,4)$ nuqtadan $4x + 3y + 12z - 5 = 0$ tekislikgacha

2) $B(3, 1, -1)$ nuqtadan $3x - y + 2z + 1 = 0$ tekislikgacha

3) $C(2, 0, -1/2)$ nuqtadan $4x - 4y + 2z + 17 = 0$ tekislikgacha bo‘lgan masofani toping.

9. Quyida berilgan tekisliklar orasidagi o‘tkir burchaklarni toping.

1) $2x - 3y + 4z - 1 = 0$ va $3x - 4y - z + 3 = 0$;

2) $x - y + z + 1 = 0$ va $2x + 3y + z - 3 = 0$;

3) $4x - 5y + 3z - 1 = 0$ va $x - 4y - z + 9 = 0$.

10. Quyidagi 1) $11x - 2y - 10z + 75 = 0$ va $11x - 2y - 10z - 45 = 0$;

2) $2x - 3y + 6z + 28 = 0$ va $2x - 3y + 6z - 14 = 0$ parallel tekisliklar orasidagi masofani toping.

11. Quyida berilgan uchta tekislikning kesishish nuqtasini toping.

1) $3x - 5y + 3z - 1 = 0$, $x + 2y + z - 4 = 0$, $2x + 7y - z - 8 = 0$;

2) $2x - 4y + 9z - 28 =0$, $7x + 9y - 9z - 5 = 0$, $7x + 3y - 6z + 1 = 0$;

3) $2x + y - 5 = 0$, $x + 3z - 16 = 0$, $5y - z - 10 = 0$.

12. Kubning ikkita yog‘i $2x - 2y + z - 1 = 0$ va $2x - 2y + z + 5 = 0$ tekisliklarda yotadi. Bu kubning hajmini hisoblang.

13. $M_1(3, 4, -5)$ nuqtadan o‘tgan, $\boldsymbol{a}_1 = \{3, 1, -1\}$ va $\boldsymbol{a}_2 = \{1, -2, 1\}$ vektorlarga parallel bo‘lgan tekislik tenglamasini tuzing.

14. $M_1(3, -1, 2)$, $M_2(4, -1, -1)$ va $M_3(2, 0, 2)$ nuqtalar orqali o‘tgan tekislik tenglamasini tuzing.

15. $M_1(2, -1, 3)$ va $M_2(3, 1, 2)$ nuqtalar orqali o‘tgan $\boldsymbol{a} = \{3, -1, 4\}$ vektorga parallel bo‘lgan tekislik tenglamasini tuzing.

Fazoda to‘g‘ri chiziq

16. Ozod had D ning qanday qiymatlarida quyidagi $\begin{cases} 3x - y + 2z - 6 = 0, \\ 2x + 3y - z + D = 0 \end{cases}$ to‘g‘ri chiziq:

1) Ox

2) Oy

3) Oz o‘qini kesadi.

17. $6x - 17y + 12z - 13 = 0$ tekislik bilan koordinatalar tekisligining kesishishidan hosil bo‘lgan to‘g‘ri chiziq tenglamalarini tuzing.

18. $M_1(3,5,0)$ nuqtadan o‘tgan: 1) $a = (8, -3, 2)$ vektorga;

2) $\frac{x+1}{6}=\frac{y+2}{5}=\frac{z-6}{-3}$ to‘g‘ri chiziqqa; 3) Ox o‘qiga; 4) Oy o‘qiga; 5) Oz o‘qiga parallel bo‘lgan to‘g‘ri chiziq tenglamasini tuzing.

19. Quyida berilgan ikki nuqta orqali o‘tgan to‘g‘ri chiziq tenglamasini tuzing:

1) (2; -2; 3), (3; 4; -1); 2) (-8;-1;6), (4;0;-5);

3) (6; -2;5), (8;-3;4); 4) (7;-2;-4), (5;2;6).

20. Quyida berilgan to‘g‘ri chiziqlarning kanonik tenglamasini tuzing.

1) $2x + 2y + 3z - 4 = 0,\ x + 2y - 3z + 4 = 0;$

2) $5x + y + z = 0,\ 2x + 3y - 2z + 5 = 0;$

3) $2x - y + 3z - 2 = 0,\ 3x + y - 4z - 8 = 0.$

21. Quyida berilgan to‘g‘ri chiziqlarning parametrik tenglamasini tuzing.

1) $2x + 3y - z - 4 = 0,\ 3x - 5y + 2z - 1 = 0;$

2) $x + 2y - z - 6 = 0,\ 2x - y + z + 1 = 0.$

22. Quyida berilgan to‘g‘ri chiziqlarning parallelligini isbotlang.

1) $\frac{x+2}{3}=\frac{y-1}{-2}=\frac{z}{1}$ va $\begin{cases} x+y-z=0, \\ x-y-3z-2=0. \end{cases}$

2) $\begin{cases} x=2t+5 \\ y=-t+2 \\ z=t-7 \end{cases}$ va $\begin{cases} x+3y+z+2=0, \\ x-y-3z-2=0. \end{cases}$

3) $\begin{cases} x+y-3z+1=0, \\ x-y+z+3=0, \end{cases}$ va $\begin{cases} x+2y-5z-1=0, \\ x-2y+3z-9=0. \end{cases}$

23. Quyidagi berilgan to‘g‘ri chiziqlarning perpendikulyarligini isbotlang.

1) $\frac{x}{1}=\frac{y-1}{-2}=\frac{z}{3}$ va $\begin{cases} 3x+y-5z+1=0, \\ 2x+3y-8z+3=0. \end{cases}$

2) $\begin{cases} x=2t+1 \\ y=3t-2 \\ z=-6t+1 \end{cases}$ va $\begin{cases} 2x+y-4z+2=0, \\ 4x-y-5z+4=0. \end{cases}$

3) $\begin{cases} x+y-3z-1=0, \\ 2x-y-9z-2=0 \end{cases}$ va $\begin{cases} 2x+y+2z+5=0, \\ 2x-2y-z+2=0. \end{cases}$

24. Quyida berilgan ikki to‘g‘ri chiziq orasidagi o‘tkir burchakni toping.

1) $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z+4}{2}$ va $\frac{x+1}{12}=\frac{y+3}{3}=\frac{z-2}{4}.$

2) $\frac{x-3}{1}=\frac{y+2}{-1}=\frac{z}{\sqrt{2}}$ va $\frac{x+2}{1}=\frac{y-3}{1}=\frac{z+5}{\sqrt{2}}.$

3) $\frac{x-1}{3}=\frac{y+4}{-2}=\frac{z-2}{4}$ va $\frac{x+3}{2}=\frac{y-1}{3}=\frac{z+1}{-2}$

25. Uchburchakning A(3,6,-7), B(-5,2,3) va C(4,-7,-2) uchlari berilgan. Uning C uchidan tushirilgan medianasining parametrik tenglamasini tuzing.

Tekislik va to‘g‘ri chiziq

26. $\frac{x+2}{4}=\frac{y-1}{4}=\frac{z+3}{2}$ to‘g‘ri chiziq bilan $12x + 3y - 4z + 4 = 0$ tekislik orasidagi burchakni hisoblang.

27. $\frac{x+4}{3}=\frac{y-1}{2}=\frac{z-3}{4}$ to‘g‘ri chiziq bilan $2x - 3y - 2z + 5 = 0$ tekislik orasidagi burchakni hisoblang.

28. M(2,-3,4) nuqtadan $\frac{x-3}{3}=\frac{y+1}{2}=\frac{z+4}{4}$ to‘g‘ri chiziqqa perpendikulyar holda o‘tuvchi tekislik tenglamasini tuzing.

29. N(-1,2,-3) nuqtadan $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+3}{2}$ to‘g‘ri chiziqqa perpendikulyar holda o‘tuvchi tekislik tenglamasini tuzing.

8.30. Quyida berilgan to‘g‘ri chiziq bilan tekislik kesishish nuqtasini toping.

1) $\frac{x-2}{4}=\frac{y-3}{2}=\frac{z+1}{5}$, $x+2y-3z-4=0$.

2) $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z}{6}$, $2x+3y+z-1=0$.

3) $\frac{x+2}{-2}=\frac{y-1}{3}=\frac{z-3}{2}$, $x+2y-2z+6=0$.

31. $M(1,3,2)$ nuqtadan o‘tib, $x-2y+2z-3=0$ tekislikka perpendikulyar bo‘lgan to‘g‘ri chiziq tenglamasini tuzing.

32. $M(-1,1,-2)$ nuqtadan o‘tib, $4x-5y-z-3=0$ tekislikka perpendikulyar ravishda o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

33. Quyida berilgan to‘g‘ri chiziqning tekislikda yotishini tekshiring.

1) $\frac{x+3}{4}=\frac{y-1}{2}=\frac{z+2}{3}$, $2x-y-2z-9=0$.

2) $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-4}{5}$, $3x-4y-2z-7=0$.

3) $\frac{x+3}{3}=\frac{y-2}{-1}=\frac{z+1}{-5}$, $x-2y+z-15=0$.

34. m ning qanday qiymatida $\frac{x+1}{3}=\frac{y-2}{m}=\frac{z+3}{-2}$ to‘g‘ri chiziq

$x-3y+6z+7=0$ tekislikka parallel bo‘ladi?

35. C ning qanday qiymatida $\begin{cases}3x-2y+z+3=0,\\4x-3y+4z+1=0\end{cases}$ to‘g‘ri chiziq

$2x-y+Cz-2=0$ tekislikka parallel bo‘ladi?

36. Ushbu $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}$ to‘g‘ri chiziqqa nisbatan $P(4;3;10)$ nuqtaga simmetrik bo‘lgan nuqtani toping.

37. $P(7;9;7)$ nuqtadan $\frac{x-2}{4}=\frac{y-1}{3}=\frac{z}{2}$ to‘g‘ri chiziqqacha bo‘lgan masofani toping.

38. $P(1,-1,-2)$ nuqtadan $\frac{x-2}{4}=\frac{y-1}{3}=\frac{z}{2}$ to‘g‘ri chiziqqacha bo‘lgan masofani toping.

39. Quyidagi ikki parallel to‘g‘ri chiziq orasidagi masofani toping.

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z}{2} \text{ va } \frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$$

40. $x=3t-2$, $y=-4t+1$, $z=4t-5$ to‘g‘ri chiziq bilan $4x-3y-6z-5=0$ tekislikning parallel ekanligini isbotlang.

Mustaqil yechish uchun misol va masalalarning javoblari

Fazoda tekislik

1. A va B nuqtalar. **2**. 1) $x+3y+4z-5=0$; 2) $x+3y-2z-8=0$; 3) $3x-6y+z-27=0$. **3**. 1) $2y+z=0$; 2) $2x-z=0$; 3) $x-y=0$. **4**. $2x-3y+6z-25=0$. **5**. 1) $x+2y-3z+8=0$; 2) $2x-3y-5z=0$. **6**. 1) -3; 4; 12; 2) 4;20;-5; 3) 16; 0; -2; 4) 0; 7; 0. **8.7**. 1) $\frac{2}{11}x-\frac{9}{11}y+\frac{6}{11}z-2=0$; 2) $\frac{5}{7}x+\frac{\sqrt{8}}{7}y-\frac{4}{7}z+\frac{5}{7}=0$; 3) $\frac{4}{13}x+\frac{3}{13}y+\frac{12}{13}z+\frac{6}{13}=0$. **8**. 1) $d=\frac{60}{13}$; 2) $d=\frac{4}{\sqrt{35}}$; 3) $d=4$. **9**. 1) $\varphi=\arccos\frac{14}{\sqrt{29\ \ 26}}=\arccos 0{,}5098$; 2) $\varphi=90^\circ$; 3) $\varphi=\arccos 0{,}7$.

10. 1) $d=8$; 2) $d=6$. **11**. 1) (1,1,1); 2) (2,3,4); 3) (1,3,5). **12**. V = 8. **13**. $x+4y+7z+16=0$. **14**. $3x+3y+z-8=0$. **15**. $x-y-z=0$.

Fazoda to‘g‘ri chiziq

16. 1) $D = -4$; 2) $D = 9$; 3) $D = 3$.

17. $6x - 17y + -13 = 0,\ z = 0$;

$6x + 12z - 13 = 0,\ y = 0;\ -17y + 12z - 13 = 0,\ x = 0$.

18. 1) $\frac{x-3}{8} = \frac{y}{-3} = \frac{z-4}{2}$; 2) $\frac{x-3}{6} = \frac{y}{5} = \frac{z-4}{-3}$; 3) $\frac{x-3}{1} = \frac{y}{0} = \frac{z-4}{0}$; 4) $\frac{x-3}{0} = \frac{y}{0} = \frac{z-4}{1}$.

19. 1) $\frac{x-1}{1} = \frac{y+2}{6} = \frac{z-3}{-4}$; 2) $\frac{x+4}{12} = \frac{y+1}{1} = \frac{z-6}{-11}$;

3) $\frac{x-6}{2} = \frac{y+2}{-1} = \frac{z-5}{-1}$; 4) $\frac{x-7}{-2} = \frac{y+2}{4} = \frac{z+4}{10}$.

20. 1) $\frac{x}{-12} = \frac{y}{9} = \frac{z-\frac{4}{3}}{2}$; 2) $\frac{x}{-5} = \frac{y+1}{12} = \frac{z-1}{13}$; 3) $\frac{x-2}{1} = \frac{y-2}{17} = \frac{z}{5}$;

21. 1) $x = t + 1,\ y = -7t,\ z = -19t - 3$; 2) $x = -t + 1,\ y = 3t + 2,\ z = 5t - 1$.

24. 1) $= \arccos 0{,}8974 = 26^0 71'$; 2) $\varphi = 60^0$; 3) $\varphi = 68^0 53'$.

25. $x = 5t + 4,\quad y = -11t - 7,\ z = -2$.

Tekislik va to‘g‘ri chiziq

26. $\varphi = \arcsin\frac{2}{3}$. **27.** $\varphi = \arcsin 0{,}3604$. **28.** $3x+2y+4z-16=0$.

29. $4x + 3y + 2z + 4 = 0$.

30. 1) (6,5,4); **2)** (2,-3,6); **3)** To‘g‘ri chiziq tekislikda yotadi.

31. $\frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-2}{2}$. **32.** $\frac{x+1}{4} = \frac{y-1}{-5} = \frac{z+2}{-1}$. **33.** 1) Yotadi;

2) Yotadi; 3) Yotmaydi. **34.** $m = -3$. **35.** $c = -2$. **36.** (2,9,6). **37.** $d = \sqrt{22}$.

38. $d = 7$. **39.** $d = 3$.

4.2-amaliy mashg‘ulot.
Ikkinchi tartibli sirtlar

1-misol. Oyz koordinatalar sistemasida $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ tenglama bilan berilgan ellipisning o‘z o‘qi atrofida aylantirishdan hosil bo‘lgan sirtning tenglamasi tuzing.

Yechilishi. Aylanma sirtning tenglamasini tuzish qoidasidan foydalanib quyidagiga ega bo‘lamiz:

$$\frac{\left(\pm\sqrt{x^2 + y^2}\right)^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ yoki } \frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

(1) tenglama bilan berilgan sirt aylanma ellipsoid ifodalaydi. ■

2-misol. Ushbu $9x^2 + 12y^2 + 4z^2 - 36 = 0$ ko‘rinishdagi tenglama qanday sirtni aniqlaydi.

Yechilishi. Berilgan tenglamaning ozod hadini o‘ng tomonga o‘tkazib, tenglamaning har bir hadini shu ozod hadga bo‘lamiz. Sungra tenglamani sodda ko‘rinishga keltirib, ushbu natijaga ega bo‘lamiz:

$$\frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \text{ yoki } \frac{x^2}{2^2} + \frac{y^2}{\left(\sqrt{3}\right)^2} + \frac{z^2}{3^2} = 1.$$

Bu tenglamaning yarim o‘qlari $a = 2,\ b = \sqrt{3},\ c = 3$ bo‘lgan ellipsoidni aniqlaydi. Oxy, Oxz, Oyz koordinatalar tekisliklarida uning kesimlari mos ravishda ellipsni beradi:

$$\left.\begin{matrix}\frac{x^2}{4} + \frac{y^2}{3} = 1\\ z = 0\end{matrix}\right\},\quad \left.\begin{matrix}\frac{x^2}{4} + \frac{z^2}{9} = 1\\ y = 0\end{matrix}\right\} \text{ va } \left.\begin{matrix}\frac{y^2}{3} + \frac{z^2}{9} = 1\\ x = 0\end{matrix}\right\}. ■$$

3-misol. $\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$ sirtning $\frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}$ to‘g‘ri chiziq bilan kesishish nuqtalarini toping.

Yechilishi. Sirt bilan to‘g‘ri chiziqning kesishish nuqtalarini topish uchun to‘g‘ri chiziqning parametrik tenglamasini tuzamiz: $x = 4 + 2t,\ y = -3t - 6,\ z = -2t - 2$. To‘g‘ri chiziqning parametrik tenglamasini $\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$ sirtning tenglamasiga qo‘yib $t^2 + 3t + 2 = 0$ tenglamaga ega bo‘lamiz, bunda $t_1 = -1,\ t_2 = -2$. Demak, $M(2;-3;0)$, $N(0;0;2)$ nuqtalar sirt bilan to‘g‘ri chiziqning kesishish nuqtalari bo‘ladilar. ■

Mustaqil yechish uchun misollar

Quyida berilgan tenglamalar qanday sirtlarni aniqlaydi. Kesimlar usulida bu sirtlarni tekshiring va ularni chizing.

1. $x^2+2y^2-6z^2=0$.

2. $3x^2+2y^2-4z^2=12$.

3. $3x^2-4y^2+24z=0$.

4. $x^2+14y=0$.

5. $4y^2+z^2+8x=0$.

6. $x^2-9y^2-4z^2+1=0$.

7. $z=4-x^2-y^2$.

8. $x^2+4y^2+8z^2-16=0$.

9. $x^2+3y^2-9z=0$.

10. $4x^2+y^2=9$.

11. $6x^2-y^2+4z^2=0$.

12. $10x^2+5y^2-2z^2-50=0$.

13. $4z^2-5y^2+40=0$.

14. $z^2-6y^2=12x$.

15. $4x^2-12y^2-6z^2=12$.

16. $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{1}=1$ sirtning $\frac{x-4}{4}=\frac{y+3}{0}=\frac{z-1}{1}$ to'g'ri chiziq bilan kesishish nuqtalarini toping.

17. $\frac{x^2}{9}-\frac{y^2}{6}+\frac{z^2}{4}=1$ sirtning $\frac{x}{3}=\frac{y-2}{4}=\frac{z-3}{2}$ to'g'ri chiziq bilan kesishish nuqtalarini toping.

Mustaqil yechish uchun misollarning javoblari

1. Konus. **2.** Bir pallali giperboloid. **3.** Konis. **4.** Parabolik silindr.

5. Elliptik paraboloida. **6.** Ikki pallali giperboloid. **7.** Elliptik paraboloida. **8.** Ellipsoida. **9.** Elliptik paraboloida. **10.** Elliptik silindr. **11.** Konus. **12.** Bir pallali giperboloid. **13.** Giperbolik silindr. **14.** Giperbolik paraboloid. **15.** Ikki pallali giperboloid. **16.** Kesishmaydi.

4-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

ABCD piramidaning uchlari berilgan. Quyidagilarni toping:

1) *ABC* tekislik tenglamasini; 2) *AB* qirra tenglamasini; 3) *D* uchidan o'tuvchi *ABC* yoqqa perpendikulyar to'g'ri chiziq tenglamasini;

4) *C* uchidan o'tuvchi *AB* qirraga parallel to'g'ri chiziq tenglamasini; 5) *D* uchidan o'tuvchi *AB* qirraga perpendikulyar tekislik tenglamasini; 6) *AB* qirra bilan *ABC* yo'q orasidagi burchak sinusini ; 7) *ABC* va *Oxy* tekisliklar orasidagi burchak kosinusini; 8) *D* uchdan *ABC* yoqqacha bo'lgan masofani;

1. *A*(7,3,5),	*B*(5,3,2),	*C*(10, 2, 4),	*D*(7,-2,1).
2. *A*(-8,-6,-3),	*B*(4,2,1),	*C*(0, 5,2),	*D*(0,2,5).
3. *A*(7,-3,14),	*B*(-6,0,5),	*C*(1,2,1),	*D*(-2,-1,2).
4. *A*(5,5,-6),	*B*(-4,-8,4),	*C*(1,2,1),	*D*(-2,-1,2).
5. *A*(7,-8,1),	*B*(-3,-6,-2),	*C*(2,-3,-5),	*D*(5,4,14).
6. *A*(16,-8,1-13),	*B*(6,2,5),	*C*(-3,0,3),	*D*(0,2,1).
7. *A*(7,3,-5),	*B*(1,2,3),	*C*(-1,2,1),	*D*(2,-1,2).
8. *A*(8,3,2),	*B*(4,-2,2),	*C*(3,1,-1),	*D*(2,1,1).
9. *A*(8,-4,-5),	*B*(7,3,6),	*C*(-2,1,4),	*D*(1,3,2).
10. *A*(6,-7,-3),	*B*(1,2,3),	*C*(1,3,2),	*D*(2,1,1).
11. *A*(-12,7,-1),	*B*(0,-2,-5),	*C*(-4,5,1),	*D*(-7,4,-3).
12. *A*(-5,-6,1),	*B*(-2,1,2),	*C*(0,-1,4),	*D*(-3,2,-1).
13. *A*(-1,0,-7),	*B*(4,-5,3),	*C*(-2,1,-9),	*D*(1,-1,-3).
14. *A*(2,4,-2),	*B*(-1,1,2),	*C*(3,0,-2),	*D*(1,-1,1).
15. *A*(4,-1,2),	*B*(-1,1,0),	*C*(2,-1,1),	*D*(0,2,1).
16. *A*(16,-9,-5),	*B*(1,-2,2),	*C*(-1,2,1),	*D*(2,0,1).
17. *A*(-9,-2,3),	*B*(6,-1,-2),	*C*(1,0,1),	*D*(-3,2,1).
18. *A*(-10,7,-6),	*B*(-3,0,-6),	*C*(-5,3,-2),	*D*(-1,10,3).
19. *A*(5,3,-2),	*B*(-1,0,3),	*C*(-4,-2,-1),	*D*(4,2,-1).
20. *A*(-12,7,-1),	*B*(0,-2,-5),	*C*(-4,5,1),	*D*(-7,4,-3).
21. *A*(0,3,4),	*B*(1,0,3),	*C*(2,-1,4),	*D*(0,3,1).
22. *A*(-16,20,-21),	*B*(-4,1,3),	*C*(2,3,0),	*D*(-1,-1,-2).
23. *A*(2,-1,1),	*B*(3,7,-2),	*C*(3,6,-3),	*D*(-7,5,1).
25. *A*(8,-10,2),	*B*(-3,3,-1),	*C*(0,-6,5),	*D*(-3,-4,2).
24. *A*(7,2,-3),	*B*(4,1,1),	*C*(2,1,2),	*D*(2,-1,1).
26. *A*(5,-4,5),	*B*(1,0,-1),	*C*(1,2,2),	*D*(6,3,1).

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III BO'LIM. MATEMATIK ANALIZ

5-bob. HAQIQIY VA KOMLEKS SONLAR

5.1-§. To'plam. Haqiqiy sonlar

5.1. To'plam tushunchasi. To'plamlar ustida amallar. "To'plam" tushunchasi matematikaning ta'rifsiz qabul qilingan asosiy tushunchalaridan biri bo'lib, ba'zi belgilariga asoslanib birgalikda qaraladigan obyektlar yoki narsalar (predmetlar) majmuasidir. To'plamni tashkil qiluvchi har bir obyekt yoki narsa uning "elementi" deyiladi. To'plam tushunchasi misollar yordamida tushuntiriladi. Masalan, Samarqand shahridagi oliy ta'lim muassasalarida o'qiydigan talabalar, barcha butun sonlar, kutubxonadagi kitoblar va hakozolar to'plamni tashkil etadi.

To'plamlar lotin yoki grek alfavitining bosh harflari bilan, uning elementlari esa, kichik harflar bilan belgilanadi. Masalan, $A,B,C,D,...,X,Y,Z$ lar bilan to'plamni, $a,b,c,d,...,x,y,z$ lar balan esa to'plamning elementlari belgilanadi.

Agar A to'plamning elementi a bo'lsa, $a \in A$ kabi yoziladi va "a element A to'plamga tegishli" deb o'qiladi. Aks holda, ya'ni a element A to'plamga tegishli bo'lmasa, unda $a \notin A$ (yoki $a \overline{\in} A$) kabi yoziladi va "a element A to'plamga tegishli emas" deb o'qiladi. Masalan, $A=\{2,4,6,8\}$ bo'lsa, u holda $4 \in A, 3 \notin A$.

Bitta ham elementga ega bo'lmagan to'plam *bo'sh to'plam* deyiladi va $\varnothing$ kabi belgilanadi. Bo'sh to'plamlarga quyidagilar misol bo'la oladi: $a)$ $x^2+4=0$ tenglamaning haqiqiy ildizlari to'plami; $b)$ o'zaro parallel ikkita turli to'g'ri chiziqning umumiy nuqtalari to'plami; $c)$ $|x-4|<-2$ tengsizlikning yechimlari to'plami va h.k.

Ko'pincha to'plamlar, ularning elementlari chekli yoki cheksiz bo'lishidan qat'iy nazar, simvolik ravishda doirachalar bilan tasvirlanadi. Bu tasvirlash to'plamlar ustida bajariladigan amallarni tasavvur qilishda va ular orasidagi munosabatlarni o'rganishda ancha qulayliklar tug'diradi.

5.1.1-ta'rif. Agar A to'plamning har bir elementi B to'plamning ham elementi bo'lsa, A to'plam B to'plamning *qism yoki qism to'plami* (to'plam osti) deb ataladi va $A \subset B$ kabi belgilanadi . Bu quyidagicha o‘qiladi: "B to‘plam A to‘plamni o‘z ichiga oladi".

5.1.2-eslatma. Bo‘sh to‘plam har qanday A to‘plamning qism to‘plami hisoblanadi: $\varnothing \subset A$. Har qanday A to‘plam o‘z-o‘zining qism to‘plami hisoblanadi: $A \subset A$.

5.1.3-eslatma. n ta elementdan iborat bo‘lgan to‘plamning qism to‘plamlari soni 2^n ga teng.

5.1.4-eslatma. Agar $A, B, C, ...$ to‘plamlarning har biri J to‘plamning qism to‘plamlari bo‘lsa, J to‘plamga *universal to‘plam* deyiladi.

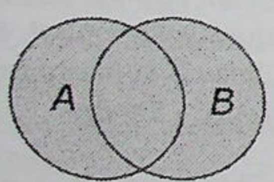
5.1.5-ta’rif. Agar A to‘plam B to‘plamning qismi, B to‘plam A to‘plamning qismi bo‘lsa, ya’ni $A \subset B, B \subset A$ bo‘lsa, u holda A va B *to‘plamlar bir-biriga teng* deyiladi va $A = B$ kabi yoziladi.

5.1.6-ta’rif. B ixtiyoriy to‘plam bo‘lib, A to‘plam uning biror qismi bo‘lsin. B to‘plamning A ga kirmagan barcha elementlaridan tashkil topgan to‘plam A ning B ga qadar *to‘ldiruvchisi* deyiladi va u $C_B(A)$ kabi belgilanadi.

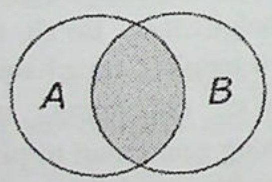
5.1.7-ta’rif. A va B to‘plamlarning barcha elementlaridan tashkil topgan C to‘plam, A va B to‘plamlarning *yig‘indisi (birlashmasi)* deyiladi va u $A \cup B = C$ kabi belgilanadi (5.1-chizma) .

5.1.8-eslatma. Shuni qayd qilib o‘tish kerakki, agar biror element ham A to‘plamga, ham B to‘plamga qarashli bo‘lsa, bu element yig‘indi C to‘plamda bir marta hisoblanadi.

5.1.9-ta’rif. A va B to‘plamlarning umumiy elementlaridan tashkil topgan C to‘plam, A va B to‘plamlarning *umumiy qismi yoki ko‘paytmasi (kesishmasi)* deyiladi va $C = A \cap B$ kabi belgilandi (5.2-chizma) .



5.1-chizma.



5.2-chizma.

5.1.10-eslatma. Biz to‘plamlarning yig‘indisi hamda ko‘paytmasi ta’riflarini ikkita to‘plam uchun keltirdik. Agar $A_1, A_2, ..., A_n$ to‘plamlar berilgan bo‘lsa, ularning $A_1 \cup A_2 \cup ... \cup A_n$ yig‘indisi hamda $A_1 \cap A_2 \cap ... \cap A_n$ ko‘paytmasi ham yuqoridagiga o‘xshash ta’riflanadi.

5.1.11-ta’rif. A to‘plamning B to‘plamga tegishli bo‘lmagan barcha elementlaridan tuzilgan C to‘plam A va B to‘plamlarning *ayirmasi* deyiladi va $C = A \setminus B$ kabi belgilanadi (5.3-chizma).

Yuqoridagi **5.1.7-**, **5.1.9-**, **5.1.11**-ta’riflardan to‘plamlarning quyidagi xossalari kelib chiqadi:

1^0. $A \cup A = A$. 2^0. $A \cup B = B \cup A$. 3^0. $A \cup \varnothing = A$.

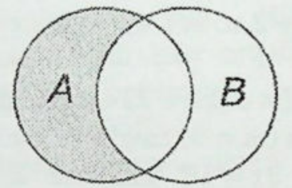
4^0. Agar $A \subset B$ bo‘lsa, $A \cup B = B$ bo‘ladi.

5^0. $A \cap A = A$. 6^0. $A \cap B = B \cap A$. 7^0. $A \cap \varnothing = \varnothing$.

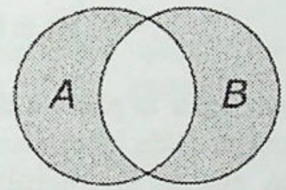
8^0. Agar $A \subset B$ bo‘lsa, u holda $A \cap B = A$ bo‘ladi.

9^0. $A \setminus \varnothing = A$. 10^0. $\varnothing \setminus A = \varnothing$. 11^0. $A \setminus A = \varnothing$.

5.1.12-ta’rif. A to‘plamning B to‘plamga tegishli bo‘lmagan elementlaridan va B to‘plamning A to‘plamga tegishli bo‘lmagan elementlaridan tuzilgan C to‘plam A va B to‘plamlarning *simmetrik ayirmasi* deb ataladi va $C = A \Delta B$ kabi belgilanadi, ya’ni $A \Delta B = (A \setminus B) \cup (B \setminus A)$(5.4-chizma).



5.3-chizma.



5.4-chizma.

5.1.13-ta’rif. Birinchi elementi X to‘plamga va ikkinchi elementi Y to‘plamga tegishli barcha (x, y) juftlardan iborat bo‘lgan nuqtalar to‘plami X va Y to‘plamlarning *Dekart (to‘g‘ri) ko‘paytmasi* deyiladi va u $[X, Y]$ (yoki $X \times Y$) kabi belgilanadi, ya’ni $C = X \times Y = \{(x, y) : x \in X, y \in Y\}$.

5.1.14-eslatma. A to‘plamni o‘z-o‘ziga Dekart ko‘paytmasi quyidagicha belgilanadi: $A^2 = A \times A = \{(x, y) : x \in A, y \in A\}$.

5.2. Natural, butun, ratsional sonlar to‘plami. Haqiqiy sonlar va ularning xossalari kitobxonlarga o‘rta maktab, litsey, kollej matematika kurslaridan ma’lum ekanligini qayd qilib, “Oliy matematika” kursi davomida haqiqiy sonlardan keng foydalanilishini e’tiborga olgan holda, ularning asosiy hossalarini isbotsiz qisqacha bayon etish bilan kifoyalanamiz.

Son tushunchasi matematikaning asosiy va dastlabki tushunchalaridan biri bo'lib, u uzoq o'tmishdan ma'lum. O'tmishda insonlar sanash taqozasiga ko'ra, *natural sonlar* deb ataluvchi $1,2,3,\dots,n,\dots$ sonlardan foydalanib kelishgan. Odatda natural sonlar to'plami $\mathbb{N}=\{1,2,3,\dots,n,\dots\}$ kabi belgilanadi.

Natural sonlarning oldiga minus "-" ishorasini qo'yishdan hosil bo'lgan ifodalar *manfiy natural sonlar* deyiladi.

Barcha manfiy natural sonlar, nol soni va barcha natural sonlardan iborat bo'lgan to'plam *butun sonlar to'plami* deyiladi va u

$$\mathbb{Z}=\{\dots,-n,\dots,-3,-2,-1,0,1,2,3,\dots,n,\dots\}$$

kabi belgilanadi. Ravshanki, $\mathbb{N}\subset\mathbb{Z}$.

Butun sonlar to'plami ham tartiblangan to'plamga misol bo'la oladi. $\mathbb{Z}$ to'plamda ham qo'shish, ayirish va ko'paytirish amallari aniqlangan, ya'ni

$$\forall n,m\in\mathbb{Z}\Rightarrow n+m\in\mathbb{Z}, n\cdot m\in\mathbb{Z}.$$

Butun sonlar to'plami $\mathbb{Z}$ da $m+x=n\ (m\in\mathbb{Z},n\in\mathbb{Z})$ tenglama har doim yechimga ega, lekin bu to'plamda $m\cdot x=n\ (m\neq 0)$ tenglama har doim ham yechimga ega bo'lavermaydi. Masalan, $3x=18$ tenglama $\mathbb{Z}$ da $x=6$ yechimga ega, lekin $3x=5$ tenglama $\mathbb{Z}$ da yechimga ega emas. Bundan butun natural sonlar to'plamini ham kengaytirish zaruriyati kelib chiqadi.

Ma'lumki, $r=\frac{m}{n}\ (m\in\mathbb{Z},n\in\mathbb{N})$ ko'rinishda ifodalanadigan son *ratsional son* deyiladi (m va n lar 1 dan boshqa umumiy bo'luvchilarga ega emasligi qisqacha $(m,n)=1$ kabi belgilanadi). Ratsional sonlar to'plami $\mathbb{Q}=\left\{r: r=\frac{m}{n},(m,n)=1,m\in\mathbb{Z},n\in\mathbb{N}\right\}$ kabi belgilanadi.

Har qanday ratsional $\frac{m}{n}$ son chekli o'nli kasr yoki cheksiz davriy o'nli kasr orqali ifodalanishini va aksincha, har bir chekli o'nli kasr yoki cheksiz davriy o'nli kasr $\frac{m}{n}$ ko'rinishini qayd qilib o'tamiz.

Ravshanki, $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}$ munosabat o'rinli.

$\mathbb{Q}$ to'plam quyidagi xossalarga ega:

a) $\mathbb{Q}$ to'plam ham, $\mathbb{N}$ va $\mathbb{Z}$ to'plamlar singari tartiblangan to'plam. Unda qo'shish, ayirish, ko'paytirish amallari bilan birga bo'lish amali ham aniqlangan, ya'ni

$$r,t\in\mathbb{Q}\Rightarrow r\pm t\in\mathbb{Q},\quad r\cdot t\in\mathbb{Q},\quad \frac{r}{t}\in\mathbb{Q}\ (t\neq 0).$$

b) $\mathbb{Q}$–zich to'plam. Faraz qilaylik, $r\in\mathbb{Q}$, $t\in\mathbb{Q}$ bo'lib, $r<t$ bo'lsin. U holda, $\frac{r+t}{2}\in\mathbb{Q}$, $r<\frac{r+t}{2}<t$ munosabat o'rinli bo'ladi. Bundan esa ixtiyoriy r va t ratsional sonlar orasida $\frac{r+t}{2}$ ratsional son borligini aniqlaymiz. Agar $\frac{r+t}{2}=s$ deb olsak, r va s sonlar orasida joylashgan $\frac{r+s}{2}$, hamda s va t ratsional sonlari orasida joylashgan $\frac{s+t}{2}$ ratsional son borligini ko'ramiz, ya'ni quyidagi munosabat o'rinli

$$r<\frac{r+s}{2}<\frac{r+t}{2}<\frac{s+t}{2}<t.$$

Bu jarayonni cheksiz davom ettirish yo'li bilan ixtiyoriy r va t ratsional sonlar orasida cheksiz ko'p ratsional sonlar borligiga ishonch hosil qilamiz. Mana shu xossaga ratsional sonlar to'plami $\mathbb{Q}$ ning *zichlik xossasi* deyiladi. Biz yuqorida ixtiyoriy ratsional son cheksiz davriy o'nli kasr orqali ifodalanishini qayd qilib o'tdik, biroq cheksiz davriy bo'lmagan o'nli kasrlar ham mavjud. Masalan, $1{,}4142135\dots$, $3{,}141583,\dots$.

5.2.1-ta'rif. Cheksiz davriy bo'lmagan o'nli kasrga *irratsional son* deyiladi. Masalan, $\sqrt{2}=1{,}4142\dots$, $\pi=3{,}141583\dots$ – irratsional sonlardir. Barcha irratsional sonlari to'plamini $\mathbb{I}$ bilan belgilaymiz.

5.3. Haqiqiy sonlar. Rasional hamda irrasional son umumiy nom bilan *haqiqiy son* deyiladi. Barcha haqiqiy sonlar to'plamini $\mathbb{R}$ harfi bilan belgilanadi. Demak, $\mathbb{R}=\mathbb{Q}\cup\mathbb{I}$.

Shunday qilib, $\mathbb{Q}$–rasional sonlar to'plami haqiqiy sonlar to'plami $\mathbb{R}$ gacha kengaytirildi, ya'ni ushbu $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$ munosabat o'rinli.

Haqiqiy sonlar to'plami quyidagi muhim xossalarga ega:

1^0. Haqiqiy sonlar to'plami tartiblangan to'plam;

2^0. Haqiqiy sonlar to'plami zich to'plam;

3^0. Haqiqiy sonlar to'plami to'liq to'plam.

Biror l to'g'ri chiziqda ixtiyoriy O nuqtani belgilab (O nuqta sanoq boshi), so'ngra $[0;1]$ birlik kesmani tanlaymiz va yo'nalishni belgilaymiz. Bunday holda koordinata to'g'ri chizig'i, ya'ni *son o'qi berilgan* deyiladi . Har bir natural yoki kasr songa l to'g'ri chiziqda bitta nuqta mos keladi.

5.3.1-eslatma. Koordinata boshi O nuqtaga mos kelgan "0" (nol) soni musbat ham, manfiy ham hisoblanmaydi, u koordinata to‘g‘ri chizig‘idagi musbat koordinatali nuqtalarni manfiy koordinatali nuqtalardan ajratib turadi.

Koordinata to‘g‘ri chizig‘idagi berilgan yo‘nalishni (odatda u o‘ng tomonga yo‘nalgandir) musbat, berilgan yo‘nalishga qarama-qarshi yo‘nalishni esa manfiy yo‘nalish deyiladi.

5.4. Haqiqiy sonning absolyut qiymati va uning xossalari. $x\in \mathrm{R}(x\neq 0)$ son berilgan bo‘lsin. Bunda $x,-x$ sonlardan biri albatta musbat bo‘ladi. Bu musbat son, x sonning absolyut qiymati deb ataladi va u $|x|$ kabi belgilanadi. Nol sonning absolyut qiymati $|0|=0$ deb olinadi.

Haqiqiy sonning absolyut qiymati quyidagi xossalarga ega:

1^0. $x\in \mathrm{R}$ son uchun $|x|\geq 0$, $|x|$ munosabatlar o‘rinli.

2^0. Agar $0<a\in \mathrm{R}$ berilganda $x\in \mathrm{R}$ son uchun $|x|\leq a \Leftrightarrow a\leq x\leq a$ munosabat o‘rinli.

3^0. $x,y\in \mathbb{R}$, $|x+y|<|x|+|y|$.

4^0. $x,y\in \mathrm{R}$, $\big||x|-|y|\big|\leq |x-y|$.

5^0. $x,y\in \mathrm{R}$ uchun $|x\cdot y|=|x|\cdot|y|$.

6^0. $x,y\in \mathrm{R}, y\neq 0$ uchun $\left|\frac{x}{y}\right|=\frac{|x|}{|y|}$ tenglik o‘rinli.

5.5. To‘plamning aniq yuqori va aniq quyi chegaralari. $a<b$ shartni qanoatlantiradigan a va b sonlarni olamiz va ularni koorrdinata to‘g‘ri chizig‘ida nuqtalar bilan belgilaymiz.

Amalda "interval", "kesma", "yarim interval", "nur" terminlaridan ko‘pincha foydalanmasdan, ular bir nom bilan, "sonli oraliq" deb ishlatiladi.

Oraliqlar turi	Geometrik tasviri	Belgilanishi	Tengsizliklar yordamida yozilishi
Interval	a b	(a,b)	$a<x<b$
Kesma	a b	$[a,b]$	$a\leq x\leq b$
Yarim interval	a b	$(a,b]$	$a<x\leq b$
Yarim interval	a b	$[a,b)$	$a\leq x<b$
Nur	a	$[a,+\infty)$	$x\geq a$
Nur	b	$(-\infty;b]$	$x\leq b$
Ochiq nur	a	$(a,+\infty)$	$x>a$
Ochiq nur	b	$(-\infty;b)$	$x<b$
Son o‘qi		$(-\infty;\infty)$	$-\infty<x<\infty$

Biror X to‘plam ($X\subset R$) berilgan bo‘lsin.

5.5.1-ta’rif. Agar shunday M son (m son) mavjud bo‘lib, $\forall x\in X$ uchun $x\leq M$ $(x\geq m)$ tengsizlik bajarilsa, u holda X to‘plam *yuqoridan (quyidan) chegaralangan va M soni(m soni) uning yuqori(quyi) chegarasi* deyiladi. Masalan, $X=\left\{\frac{1}{2n}: n=1,2,3,....\right\}$ to‘plam yuqoridan $\frac{1}{2}$ (quyidan 0) bilan chegaralangan.

5.5.2-ta’rif. Agar $\forall M$ son ($\forall m$ son) olinganda ham shunday $x_0\in X$ topilsaki, $x_0>M$ $(x_0<m)$ tengsizlik bajarilsa, X to‘plam *yuqoridan (quyidan) chegaralanmagan* deyiladi. .

Masalan, $Z=\{....,-n,...,-3,-2,-1,0,1,2,3,...,n,....\}$ to‘plam ham yuqoridan ham quyidan chegaralanmagan.

5.5.3-ta’rif. Agar X to‘plam ham quyidan, ham yuqoridan chegaralangan bo‘lsa *X to‘plam chegralangan* deyiladi.

5.5.4-teorema. Har qanday yuqoridan chegaralangan to‘plam uchun uni yuqoridan chegaralovchi sonlar ichida eng kichigi mavjud.

5.5.5-ta’rif. Yuqoridan chegaralangan X to‘plam uchun yuqoridan chegaralovchi sonlarning ichida eng kichigi, X to‘plamning *aniq yuqori chegarasi* deyiladi va $\sup X$ kabi belgilanadi.

5.5.6.-teorema. Har qanday quyidan chegaralangan to‘plam uchun uni quyidan chegaralovchi sonlar ichida eng kattasi mavjud.

5.5.7-ta’rif. Quyidan chegaralangan X to‘plam uchun quyidan chegaralovchi sonlarning ichida eng kattasi, X to‘plamning *aniq quyi chegarasi* deyiladi va $\inf X$ kabi belgilanadi.

To‘plamning aniq yuqori hamda aniq quyi chegaralarini mos ravishda quyidagicha ham ta’riflash mumkin:

5.5.8-ta’rif. Agar $\forall x\in X$ uchun $x\leq a$ tengsizlik o‘rinli bo‘lib, $\forall\varepsilon>0$ olinganda ham, X to‘plamda $\exists x'$ element topilsaki, $x'>a-\varepsilon$ o‘rinli bo‘lsa, unda a son X ning *yuqori chegarasi* deyiladi va $a=\sup X$ deb yoziladi.

5.5.9-ta'rif. Agar $\forall x \in X$ uchun $b \le x$ tengsiz o'rinli bo'lib, $\forall \varepsilon > 0$ olinganda ham X da $\exists x'$ element topilsaki, $x' < b + \varepsilon$ tenglik o'rinli bo'lsa, unda a son X ning *aniq quyi chegarasi* deyiladi va $b = \inf X$ deb yoziladi.

To'plamning aniq quyi va aniq yuqori chegaralari quyidagi xossalarga ega:

1^0. Agar $X(X \subset R)$ to'plam yuqoridan chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\sup X_1 \le \sup X$ bo'ladi.

2^0. Agar $X(X \subset R)$ to'plam quyidan chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\inf X_1 \ge \inf X$ bo'ladi.

3^0. Agar $X(X \subset R)$ to'plam chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\inf X \le \inf X_1 \le \sup X_1 \le \sup X$ bo'ladi.

4^0. Agar $\forall x \in X$ uchun $x \le a$ $(x \ge b)$ tengsizlik bajarilsa, $\sup X \le a$ $(\inf X \ge b)$ tengsizlik o'rinli bo'ldi.

5.6. Matematik induksiya usuli. Har qanday matematik izlanishning asosida deduktiv va induktiv uslublar yotadi.

Umumiy xulosadan xususiy xulosalarni keltirib chiqarish usuli, *deduktiv usul* deyiladi.

Xususiy tasdiqdan umumiy tasdikni keltirib chiqarish usuli *induktiv yoki induksiya usuli* deyiladi.

Induksiya usuli to'la va to'la bo'lmasligi mumkin.

Agar tasdiq kuzatishga ulgura olinmagan hollarga ham tegishli bo'lsa, bunga to'la bo'lmagan *matematik induksiya* usuli deyiladi.

Agar mulohazalar ro'y berishi mumkin bo'lgan barcha hollarni o'z ichiga olsa va shu asosda xulosa qilinsa, bunday induksiya to'la *matematik induksiya usuli* deyiladi.

To'la matematik induksiya usuliga quyidagi tamoyil (prinsip) asos qilib olinadi: biror $p(n)$ tasdiq berilganda:

1) $n = 1$ uchun $p(n)$ tasdiqning to'g'riligi tekshiriladi;

2) $n = k(k \in N)$ uchun $p(k)$ tasdiq to'g'ri deb faraz qilinganda, undan $n = k+1$ uchun $p(k+1)$ tasdiqning to'g'riligi kelib chiqsa, bu $p(n)$ tasdiq har qanday natural n uchun o'rinli bo'ladi, deb xulosa chiqariladi. Bu prinsipning 1) bandi induksiya bazisi, 2) bandi esa, *induksiya qadami* deyiladi.

Ba'zi hollarda, $p(n)$ tasdiqni n ning faqat natural qiymatlari uchungina emas, balki uning Z to'plamga qarashli barcha qiymatlari uchun ham

to'g'riligini isbotlash talab qilinadi. Bunday hollarda yuqoridagi to'la matematik induksiya usulidan foydalanish maqsadga muvofiq bo'ladi.

5.6.1-misol. Barcha n $(n \in N)$ lar uchun ushbu

$$S_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad (5.6.2)$$

tenglikning to'g'riligini isbotlang.

Yechilishi. I. $n=1$ bo'lganda , $S_1 = 1 = \frac{1(1+1)(2+1)}{6} = 1$ bo'ladi. Demak, $p(1)$ tasdiq to'g'ri.

II.Endi ixtiyoriy k natural son uchun $p(k)$ tasdiq , ya'ni

$$S_k = 1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad (5.6.3)$$

tenglik o'rinli bo'lsin, deb faraz qilib, ushbu

$$S_{k+1} = 1^2 + 2^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \qquad (5.6.4)$$

tenglikning to'g'riligini isbotlaymiz. (5.6.3) tenglikning ikkala tomoniga $(k+1)^2$ ni qo'shib,

$$S_k + (k+1)^2 = 1^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad (5.6.5)$$

ni hosil qilamiz. Bu tenglikning o'ng tomonini quyidagicha shakl almashtiramiz:

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \qquad (5.6.6)$$

Shunday qilib, (5.6.5) va (5.6.6) dan (5.6.4) tenglikka ega bo'lamiz, ya'ni $p(k+1)$ tasdiqning o'rinli ekanligi isbot bo'ladi.

Demak, matematik induksiya usuliga ko'ra (5.6.2) tenglik $\forall n$ $(n \in N)$ lar uchun to'g'ri ekan.■

5.2-§. Funksiya

5.7. Funksiya tushunchasi. Matematika fanida eng asosiy tushunchalardan biri, *funksiyadir*. Ikkita to'plam elementlari orasidagi bog'lanishni o'rganish jarayoni funksiya tushunchasi orqali amalga oshiriladi.

Agar to'plamlarning elementlari haqiqiy sonlardan iborat bo'lsa, ularning elementlari orasidagi bog'lanishni o'rganadigan *sonli funksiya* deb ataluvchi funksiya (qisqacha funksiya) orqali amalga oshiriladi. Kelgusida, biz sonli funksiya(funksiya)ni o'rganamiz.

5.8. Funksiyaning ta'rifi. X va Y to'plamlar berilgan bo'lib, ular R ning bo'sh bo'lmagan qism to'plamlari $(X \subset R, Y \subset R)$, x va y lar esa, mos ravishda ularning elementlari $(x \in X,\ y \in Y)$ bo'lsin.

5.8.1-ta'rif. Agar X to'plamdagi har bir x songa biror f qoida yoki qonunga ko'ra Y to'plamdan olingan biror y son mos qo'yilsa, X to'plamda f *funksiya berilgan (aniqlangan)* deb ataladi va u simvolik ravishda $f: X \to Y$ yoki $y = f(x)$ kabi belgilanadi.

Bunda x- argument yoki erkli o'zgaruvchi, y- funksiya yoki erksiz o'zgaruvchi, f - xarkteristika (qonun yoki qoida); X - to'plam funksiyaning *aniqlanish sohasi*, $Y = \{y : y = f(x),\ x \in X\}$ to'plam esa, uning qiymatlari *to'plami (o'zgarish sohasi)* deyiladi. Bundan keyin biz funksiyaning *aniqlanish sohasini* $D(f)$, *qiymatlar to'plamini esa,* $E(f)$ bilan belgilaymiz.

5.9. Funksiyaning berilish usullari. Funksiya umumiy holda *analitik, jadval, grafik va so'z usullari* bilan berilishi mumkin.

Analitik usul. Ko'pincha x va y o'zgaruvchilar orasidagi bog'lanish formulalar yordamida ifodalanadi. Bunda argument x ning har bir qiymatiga mos keladigan y funksiyaning qiymati, x ustida analitik amallar - qo'shish, ayirish, ko'paytirish, bo'lish, darajaga ko'tarish, ildizdan chiqarish, logarifmlash va h.k. amallarni bajarish natijasida topiladi. Odatda bunday usul - funksiyaning *analitik usulda* berilishi deyiladi.

Masalan, $y = 6x - 2$, $y = x^2 + \ln x$ funksiyalar analitik usulda berilgan.

Jadval usul. Ba'zi hollarda $x \in X$ ва $y \in Y$ o'zgaruvchilar orasidagi bog'lanish formulalar yordamida berilmasdan, jadval orqali berilgan bo'lishi ham mumkin. Masalan, t- yanvar oyining birinchi dekadasi (10 kunligi) kunlari nomeri bo'lsa, T- shu nomerli kuni soat 16^{00} da Samarqand shahrida kuzatilgan havo haroratini bildirsin, natijada quyidagi jadvalga kelamiz:

t	1	2	3	4	5	6	7	8	9	10
T	-3^0	-5^0	$+2^0$	$+5^0$	$+1^0$	0^0	-2^0	-5^0	-3^0	-1^0

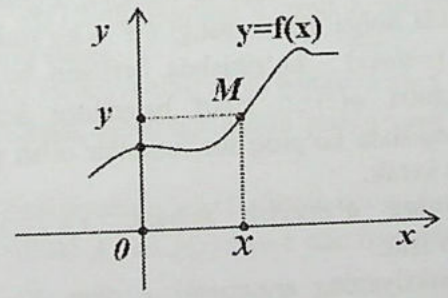
bunda, t- argument, T- funksiya bo'ladi. Bog'lanishning bunday berilishi, funksiyaning jadval usulda berilishi deb ataladi. Bu usuldan, ko'pincha, miqdorlar orasida tajribalar o'tkazish jarayonida foydalaniladi.

Jadval usulining qulayligi shundan iboratki, argumentning u yoki bu aniq qiymatlarida funksiyani hisoblamasdan, uning qiymatlarini aniqlash mumkin. Jadval usulining qulay bo'lmagan tomoni shundan iboratki, argumentning o'zgarishi bilan funksiyaning o'zgarish xarakterini to'liq aniqlab bo'lmaydi.

Grafik usul. xOy koordinatalar tekisligida x ning X to'plam $(X = D(f))$ dan olingan har bir qiymati uchun $M(x, y)$ nuqta yasaladi, bunda x nuqtaning abssissasi, ordinatasi esa, y bo'lib u, f funksiyaning x ga mos kelgan qiymatiga teng. Yasalgan nuqtalarni birlashtirsak, natijada biror chiziq hosil bo'ladi, hosil bo'lgan bu chiziq berilgan funksiyaning *grafigi* deb qaraladi (5.5-chizma).

5.9.1-ta'rif. Tekislikning $(x, f(x))$ kabi aniqlangan nuqtalaridan iborat ushbu $\{(x, f(x))\} = \{(x, f(x)) : x \in X,\ y = f(x) \in Y\}$ to'plam, $y = f(x)$ funksiyaning *grafigi* deb ataladi.

Funksiyaning grafik usulda berilishining kamchiligi shundan iboratki, argumentning sonli qiymatida berilgan funksiyaning aniq qonuniyatini (formulasini) har doim topib bo'lmaydi, lekin bu usulning boshqa usullardan afzalligi uning ta'sviri yaqqol ko'zga ko'rinib turishidadir.



5.5-chizma.

So'zlar orqali ifodalanadigan usul. Bu usulda ($x \in X$, $y \in Y$) o'zgaruvchilar o'rtasidagi funksional bog'lanish faqat so'zlar orqali ifodalanadi. Masalan: har bir rasional songa 1 ni, har bir irrasional songa 0 ni mos qo'yish natijasida ham funksiya hosil bo'ladi. Bu funksiya, odatda, *Dirixle funksiyasi* deyiladi va $D(x)$ kabi belgilanadi:

$$D(x)=\begin{cases}1, & x \in Q, \\ 0, & x \in I.\end{cases}$$

5.10. Funksiyaning aniqlanish sohasi

5.10.1-ta'rif. Argumentning funksiya ma'nosini yo'qotmaydigan (ya'ni cheksiz yoki mavhumlikka aylantirmaydigan) hamma qiymatlari to'plami, shu *funksiyaning aniqlanish sohasi* deyiladi.

Agar funksiya jadval usuli bilan berilganda, uning aniqlanish sohasi x ning jadvalda ko'rsatilgan qiymatlaridan iborat bo'ladi.

Agar funksiya grafik shaklida berilsa, uning aniqlanish sohasi funksiya grafigining Ox o'qdagi proyeksiyasidan iborat bo'ladi. Funksiya analitik usulida berilganda esa, x ning funksiyani aniqlaydigan formula ma'noga ega bo'ladigan qiymatlari to'plami, shu funksiyaning aniqlashish sohasi bo'ladi. Masalan, agar funksiya kasr shaklida berilgan bo'lsa, uning aniqlanish sohasi argumentning kasr maxrajini noldan farqli qiladigan qiymatlari to'plamidan iborat bo'ladi.

Agar funksiya ushbu $f(x)=\log_{\varphi(x)} g(x)$ ko'rinishda berilganda $g(x)>0$, $\varphi(x)>0$ va $\varphi(x)\neq 1$ bo'lishi talab qilinadi, agar funksiya $f(x)=\varphi(x)\pm\psi(x)$ ko'rinishda berilganda, uning aniqlanish sohasi $D(f)=D(\varphi)\cap D(\psi)$ dan iborat bo'ladi, funksiya $f(x)=u(x)^{v(x)}$ $(u(x)\geq 0)$ ko'rinishda berilganda asos va daraja ko'rsatgich argument x ning bir xil qiymatlarida bir vaqtda nolga aylanmasligi (0^0 – ko'rinishdagi aniqmaslik) kerak, funksiya $f(x)=\sqrt[m]{\varphi(x)}$ - ko'rinishda berilgan bo'lib, m–juft, ya'ni $m=2n$ $(n \in N)$ bo'lganda $\varphi(x)\geq 0$ shart bajarilishi kerak. Funksiyaning aniqlanish sohasini topishda ko'proq ma'lumotlar olish uchun ([33], 17-21 betlarga q.) ga qarash kerak.

5.11. Funksiyaning o'zgarish sohasi. $y=f(x)$ funksiya $x \in X$ to'plamda berilgan bo'lsin.

5.11.1-ta'rif. Funksiyaning argumenti x ning $X=D(f)$ dagi barcha qiymatlarni qabul qilganda funksiyaning unga mos kelgan qiymatlar

to'plami $E(f)$, shu funksiyaning *o'zgarish (qiymatlar to'plami) sohasi* deyiladi.

Funksiyaning o'zgarish sohasi diskret nuqtalardan, nuqtadan, oraliq, segment, bir necha oraliqlardan va h.k. iborat bo'lishi mumkin. Jadval yoki grafik usulda berilgan funksiyalarning o'zgarish sohalari o'z-o'zidan ma'lum. Analitik usulda, ya'ni $y=f(x)$ shaklda berilganda funksiyaning o'zgarish sohasini topish uchun y ning $f(x)=y$ tenglama haqiqiy yechimga ega bo'ladigan barcha qiymatlarini topish talab qilinadi.

Funksiyaning o'zgarish sohasini topishda quyidagi tasdiqlarni e'tiborga olish lozim:

1^0. Agar berilgan funksiya (bu yerda uzluksiz funksiya nazarda tutiladi) qaralayotgan sohada, eng kichik va eng katta qiymatga erishsa, $f(x)$ funksiyaning o'zgarish sohasi, uning eng kichik va eng katta qiymati hamda ular orasidagi barcha sonlar to'plamidan iborat bo'ladi.

2^0. Agar funksiya eng katta (eng kichik) qiymatiga ega bo'lmasa (ya'ni u cheksiz ortsa (kamaya) borsa), u holda funksiyaning o'zgarish sohasi funksiyaning eng kichik (eng katta) qiymati va shu qiymatdan katta (kichik) barcha sonlar to'plamidan iborat bo'ladi.

3^0. Agar $y=f(x)$ funksiya berilgan bo'lib, bu tenglamani x ga nisbatan yechish mumkin bo'lsa, ya'ni $x=\varphi(y)$ ko'rinishda yozish mumkin bo'lsa, u holda $y=f(x)$ funksiyaning o'zgarish sohasini topish uchun $x=\varphi(y)$ funksiyaning aniqlanish sohasini topish yetarli. Demak, bu holda $D(\varphi)=E(f)$ bo'ladi.

4^0. Umumiy holda, $y_1=f(x)$ funksiyaning $E(f)$ sohasi $a\leq y_1\leq b$ $(b>0)$, $y_2=\varphi(x)$ funksiyaning $E(f)$ sohasi esa $c\leq y_2\leq d$ $(d>0)$ bo'lganda $f(x)+\varphi(x)$ funksiyaning o'zgarish sohasini $a+c\leq y_1+y_2\leq b+d$ kabi aniqlash, $f(x)\cdot\varphi(x)$ funksiyaning o'zgarish sohasini esa, $ac\leq y_1y_2\leq bd$ kabi aniqlash mumkin emas.

5^0. Agar $y=f(x)$ funksiyaning $E(f)$ sohasi $a\leq y\leq b$ bo'lsa, $g(x)=mf(x)$ funksiyaning $E(g)$ sohasi, $m>0$ bo'lganda $ma\leq g\leq mb$; $m<0$ bo'lganda esa $ma\geq g\geq mb$ bo'ladi.

Agar $y=f(x)$ funksiyaning $E(f)$ sohasi $a\leq y\leq b$ bo'lsa, $g(x)=n+f(x)$ funksiyaning $E(g)$ sohasi $n+a\leq g(x)\leq n+b$ dan iborat bo'ladi.

6^0. Agar $y=f(x)$ funksiyaning $E(f)$ sohasi $-a\le y\le a\ (-\infty\le y\le +\infty)$ bo'lsa, $y_1=|f(x)|$ yoki $y_2=f^2(x)$ funksiyaning o'zgarish sohasi $0\le y_1\le a$ yoki $0\le y\le +\infty$ bo'ladi. ([33], 21-25 betlarga qaralsin).

5.12. Funksiyaning sinflari. Odatda funksiyalar quyidagi sinflarga ajratiladi: juft va toq, davriy, bir qiymatli va ko'p qiymatli, chegaralangan va chegaralanmagan, monoton, teskari, murakkab va elementar funksiyalar.

a) Juft va toq funksiyalar

5.12.1-ta'rif. Agar istalgan $x\in X\ (X\subset R)$ uchun $-x\in X$ bo'lsa, u holda X to'plam $O(0,0)$ nuqtaga (koordinatalar boshiga) nisbatan *simmetrik to'plam* deyiladi.

Butun sonlar to'plami Z, $[-a,a]$, $(-a,a)$, $(-\infty,\infty)$ kabi to'plamlar koordinatalar boshiga nisbatan simmetrik to'plamlardir.

$y=f(x)$ funksiya $O(0,0)$ nuqtaga nisbatan simmetrik bo'lgan X to'plamda aniqlangan bo'lsin.

5.12.2-ta'rif. Agar istalgan $x\in X$ uchun $f(-x)=f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *juft funksiya* deyiladi.

$y=x^2$, $y=\cos x$, $y=|x|$, $y=f(|x|)$ funksiyalar koordinatalar boshiga nisbatan simmetrik bo'lgan to'plamlarda qaralgan bo'lsa, ular juft funksiyalar bo'ladi. Ta'rifda X to'plamning koordinatalar boshiga nisbatan simmetrikligi muhimdir. Masalan, $y=x^2$, $x\in[-1,2]$ funksiya berilgan bo'lsa, u juft funksiya bo'lmaydi, chunki $[-1,2]$ to'plam koordinatalar boshiga nisbatan simmetrik emas.

Juft funksiyalarning grafigi ordinata o'qiga nisbatan simmetrik bo'ladi (5.6-chizma).

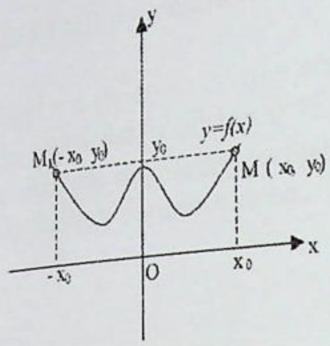
5.12.3-ta'rif. Agar istalgan $x\in X$ uchun $f(-x)=-f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *toq funksiya* deyiladi.

$y=x^3$, $y=tg\,x$, $y=\dfrac{|x|}{2x}$ funksiyalar o'zlarining aniqlanish sohalarida toq funksiyalar bo'ladi.

Toq funksiyaning grafigi koordinatalar boshiga nisbatan simmetrik bo'ladi (5.7-chizma).

Agar istalgan $x\in X$, $-x\in X$ lar uchun $f(-x)\ne\pm f(x)$ shartlar o'rinli bo'lsa, u holda $y=f(x)$ funksiya X to'plamda juft ham emas, toq ham emas deyiladi.

Ushbu $f(x)=x^2-x$, $\varphi(x)=\sin x-\cos x$ funksiyalar o'zlarining aniqlanish sohasida juft ham, toq ham emas.



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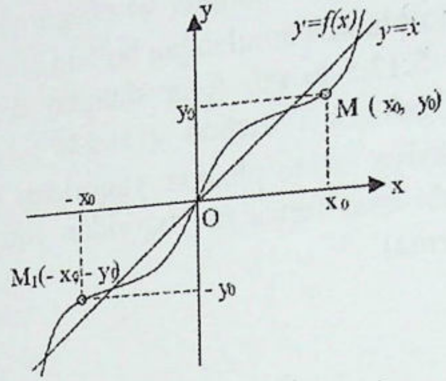

5.7-chizma.

Juft funksiyaning grafigini chizishda argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni chap tomonga y o'qiga nisbatan simmetrik ravishda ko'chirish yetarli.

Toq funksiyaning grafigini chizishda esa, argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni koordinata boshiga nisbatan simmetrik ko'chirish yetarli.

Juft va toq funksiyalar quyidagi xossalarga ega:

1^0. Ikkita juft funksiyalarning yig'indisi, ayirmasi, ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda), yana juft funksiya bo'ladi.

2^0. Ikkita toq funksiyalarning yig'indisi va ayirmasi, yana toq funksiya bo'ladi.

3^0. Ikkita toq funksiyalarning ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda) juft funksiya bo'ladi.

Simmetrik bo'lmagan to'plamda aniqlangan funksiyalarning juft va toqligi to'g'risida so'z yuritish ma'noga ega emas.

Aniqlanish sohasining koordinata boshiga nisbati simmetrikligi funksiyaning juft va toqligi uchun zaruriy shart bo'lib, yetarli shart bo'la olmaydi. Masalan, $y=x+3$ va $y=3^x$ funksiyalar $D(f)=(-\infty,\infty)$ simmetrik to'plamda aniqlangan, lekin ular juft ham emas, toq ham emas.

5.12.4-teorema. Koordinata boshiga nisbatan simmetrik bo'lgan X to'plamda aniqlangan har qanday $f(x)$ funksiya juft va toq funksiyalar yig'indisi ko'rinishda ifodalanadi:

$$F(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2},$$

Bunda birinchi had- juft funksiya, ikkinchi had esa – toq funksiyadir.

6^0. Agar $y=f(x)$ funksiyaning $E(f)$ sohasi $-a\le y\le a\ (-\infty\le y\le+\infty)$ bo'lsa, $y_1=|f(x)|$ yoki $y_2=f^2(x)$ funksiyaning o'zgarish sohasi $0\le y_1\le a$ yoki $0\le y\le+\infty$ bo'ladi. ([33], 21-25 betlarga qaralsin).

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5.12.2-ta'rif. Agar istalgan $x\in X$ uchun $f(-x)=f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *juft funksiya* deyiladi.

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Juft funksiyalarning grafigi ordinata o'qiga nisbatan simmetrik bo'ladi (5.6-chizma).

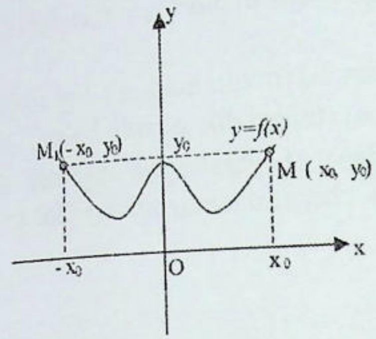
5.12.3-ta'rif. Agar istalgan $x\in X$ uchun $f(-x)=-f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *toq funksiya* deyiladi.

$y=x^3$, $y=tg\,x$, $y=\dfrac{|x|}{2x}$ funksiyalar o'zlarining aniqlanish sohalarida toq funksiyalar bo'ladi.

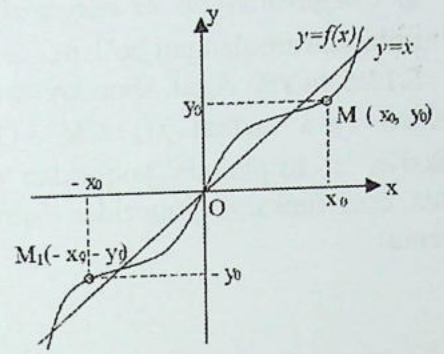
Toq funksiyaning grafigi koordinatalar boshiga nisbatan simmetrik bo'ladi (5.7-chizma).

Agar istalgan $x\in X$, $-x\in X$ lar uchun $f(-x)\ne\pm f(x)$ shartlar o'rinli bo'lsa, u holda $y=f(x)$ funksiya X to'plamda juft ham emas, toq ham emas deyiladi.

Ushbu $f(x)=x^2-x$, $\varphi(x)=\sin x-\cos x$ funksiyalar o'zlarining aniqlanish sohasida juft ham, toq ham emas.



5.6-chizma.



5.7-chizma.

Juft funksiyaning grafigini chizishda argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni chap tomonga y o'qiga nisbatan simmetrik ravishda ko'chirish yetarli.

Toq funksiyaning grafigini chizishda esa, argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni koordinata boshiga nisbatan simmetrik ko'chirish yetarli.

Juft va toq funksiyalar quyidagi xossalarga ega:

1^0. Ikkita juft funksiyalarning yig'indisi, ayirmasi, ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda), yana juft funksiya bo'ladi.

2^0. Ikkita toq funksiyalarning yig'indisi va ayirmasi, yana toq funksiya bo'ladi.

3^0. Ikkita toq funksiyalarning ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda) juft funksiya bo'ladi.

Simmetrik bo'lmagan to'plamda aniqlangan funksiyalarning juft va toqligi to'g'risida so'z yuritish ma'noga ega emas.

Aniqlanish sohasining koordinata boshiga nisbati simmetrikligi funksiyaning juft va toqligi uchun zaruriy shart bo'lib, yetarli shart bo'la olmaydi. Masalan, $y=x+3$ va $y=3^x$ funksiyalar $D(f)=(-\infty,\infty)$ simmetrik to'plamda aniqlangan, lekin ular juft ham emas, toq ham emas.

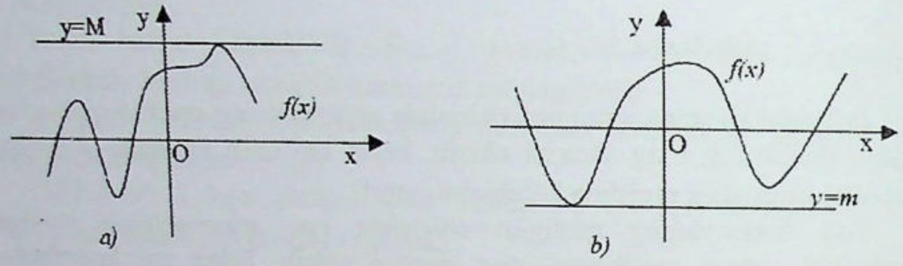
5.12.4-teorema. Koordinata boshiga nisbatan simmetrik bo'lgan X to'plamda aniqlangan har qanday $f(x)$ funksiya juft va toq funksiyalar yig'indisi ko'rinishda ifodalanadi:

$$F(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2},$$

Bunda birinchi had- juft funksiya, ikkinchi had esa – toq funksiyadir.

b) Chegaralangan va chegaralanmagan funksiyalar. $y = f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

5.12.5-ta'rif. Agar shunday o'zgarmas M (o'zgarmas m) son topilib, istalgan $x \in X$ uchun $f(x) \le M$ $(f(x) \ge m)$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda yuqoridan (quyidan) chegaralangan deyiladi, aks holda esa, funksiya yuqoridan (quyidan) chegaralanmagan deyiladi (5.8-chizma).



a) yuqoridan chegaralangan funksiya b) quyidan chegaralangan funksiya

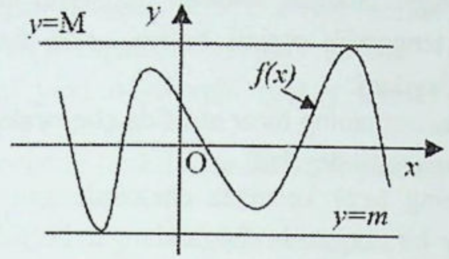
5.8-chizma.

5.12.6-ta'rif. Agar $f(x)$ funksiya X to'plamda ham yuqoridan, ham quyidan chegaralangan bo'lsa, ya'ni shunday o'zgarmas M va m sonlar mavjud bo'lib, istalgan $x \in X$ uchun

$$m \le f(x) \le M \qquad (5.12.7)$$

tengsizlik o'rinli bo'lsa, u holda $f(x)$ funksiya X to'plamda *chegaralangan* deyiladi (5.9-chizma).

$m = \inf\limits_{x\in X}\{f(x)\}$ son $f(x)$ funksiyaning X to'plamdagi aniq quyi chegarasi, $M = \sup\limits_{x\in X}\{f(x)\}$ son esa, $f(x)$ funksiyaning X to'plamdagi aniq yuqori chegarasi deyiladi. $M - m$ ayirma $f(x)$ funksiyaning X *to'plamdagi tebranishi* deb ataladi.



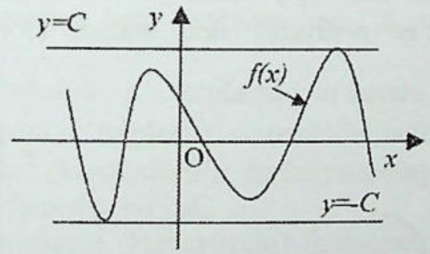
5.9-chizma.

Agar $f(x)$ funksiya chegaralangan bo'lib, m va M sonlar uning aniq quyi va aniq yuqori chegaralari bo'lsa, u holda

$$|f(x)| \le C \qquad (5.12.8)$$

tengsizlik o'rinli bo'ladi, bunda $C = \max\{|m|, |M|\}$. (5.12.7) bilan (5.12.8) tengsizliklar o'zaro teng kuchlidir (5.9-chizma).

Demak, (5.12.8) tengsizlik funksiyaning chegaralanganlik shartini ifodalaydi.



5.10-chizma.

Chegaralangan funksiyalarning grafigi Ox o'qqa parallel bo'lgan $y = C$ va $y = -C$ to'g'ri chiziqlar orasida bo'ladi (5.10-chizma).

Quyidan chegaralangan $(f(x) \ge m)$ funksiyaning grafigi Ox o'qqa parallel bo'lgan $y = m$ to'g'ri chiziqdan yuqorida joylashgan bo'ladi (5.8-b) chizma).

Yuqoridan chegaralangan funksiyaning grafigi $(f(x) \le M)$ Ox o'qqa parallel bo'lgan $y = M$ to'g'ri chiziqdan pastda joylashadi (5.8-a) chizma).

5.12.9-ta'rif. Agar istalgan musbat $C>0$ son uchun shunday $x_c \in X$ topilib, $|f(x_c)|>C$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda *chegaralanmagan* deyiladi.

$f(x)$ funksiya x_0 nuqtaning biror atrofida chegaralanmagan bo'lsa, u x_0 *nuqtada chegaralanmagan* deyiladi.

$f(x)$ funksiyaning $[a;b]$ kesmada chegaralangan bo'lishi uchun uning $[a;b]$ kesmaning har bir nuqtasida chegaralangan bo'lishi zarur va yetarlidir.

Chegaralangan funksiya quyidagi xossalarga ega:

$f(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan bo'lib, ular shu to'plamda chegaralangan bo'lsa, u holda

a) $f(x)\pm g(x)$; *b)* $f(x)\cdot g(x)$; *c)* $\dfrac{f(x)}{g(x)}, (g(x)\neq 0, x\in X)$; *d)* $|f(x)|$, $|g(x)|$

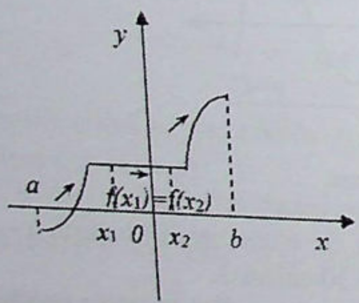
funksiyalar ham X to'plamda chegaralangan bo'ladi.

c) Monoton funksiyalar. $y=f(x)$ funksiya $X=[a,b]$ $(X\subset R)$ to'plamda berilgan bo'lsin.

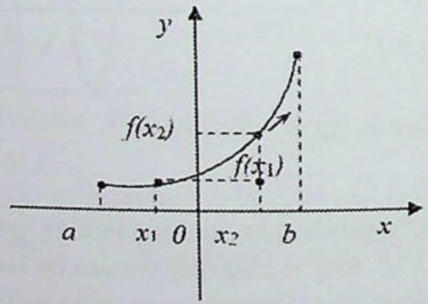
5.12.10-ta'rif. Agar istalgan $x_1,x_2\in X$ lar uchun $, x_1<x_2$ bo'lganda

$$f(x_1)\le f(x_2)\ (f(x_1)<f(x_2))$$

tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda o'suvchi yoki kamaymovchi (qat'iy o'suvchi) deb ataladi (5.11-*a)* chizma, 5.11-*b*) chizma).



a)

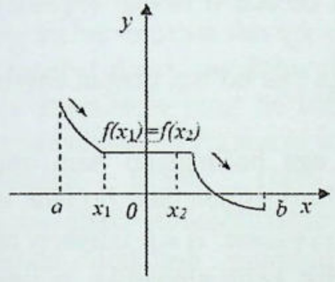


b)

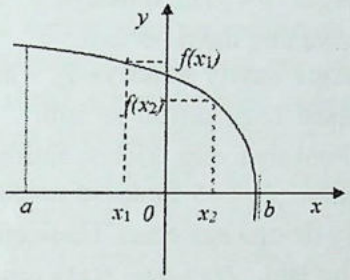
5.11-chizma.

5.12.11-ta'rif. Agar istalgan $x_1,x_2\in X$ lar uchun $x_1<x_2$ bo'lganda $f(x_1)\ge f(x_2)$ $(f(x_1)>f(x_2))$tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda kamayuvchi yoki o'smovchi (qat'iy kamayuvchi) deb ataladi (5.12-*a*) chizma, 5.12-*b)* chizma).

O'suvchi va kamayuvchi funksiyalar *monoton funksiyalar* deb ataladi.



a)



b)

5.12-chizma.

Funsiyani monotonlikka tekshirishda quyidagi umumiy tasdiqlar muhim ahamiyatga ega:

1. Agar $f(x)$ funksiya X to'plamda o'suvchi bo'lsa, u holda $f(x)+C$ (C- ixtiyoriy o'zgarmas son) funksiya ham X to'plamda o'suvchi bo'ladi.

2. Agar $f(x)$ funksiya X to'plamda o'suvchi bo'lsa, u holda $Cf(x)$ $(C>0)$ funksiya ham X to'plamda o'suvchi bo'ladi.

3. Ikkita o'suvchi (kamayuvchi) funksiyalarning yig'indisi yana o'suvchi (kamayuvchi) funksiya bo'ladi.

4. Ikkita musbat o'suvchi (kamayuvchi) funksiyalarning ko'paytmasi yana o'suvchi (kamayuvchi) bo'ladi.

5. Agar $f(x)$ funksiya X to'plamda o'suvchi, musbat va $n\in N$bo'lsa, $f^n(x)$ funksiya ham X to'plamda o'suvchi bo'ladi.

6. Agar $f(x)$ funksiya o'suvchi bo'lsa, $-f(x)$ funksiya kamayuvchi bo'ladi va aksincha.

d) Davriy funksiyalar. $f(x)$ funksiya $X\,(X\subset R)$ to'plamda aniqlangan bo'lsin.

5.12.12-ta'rif. Agar shunday o'zgarmas $T(T\neq 0)$ son mavjud bo'lsaki, istalgan $x, x+T\in X$ lar uchun

$$f(x+T)=f(x) \qquad (5.12.13)$$

tenglik o'rinli bo'lsa, $f(x)$ *davriy funksiya* deyiladi, bunda T son, *funksiyaning davri* deb ataladi.

(5.12.13) shartni qanoatlantiruvchi musbat T larning eng kichigi (agar u mavjud bo'lsa) funksiyaning *asosiy davri* deb ataladi.

Agar $y=f(x)$ funksiya T davrga ega bo'lsa, u holda $nT(n\in Z)$ ham funksiyaning davri bo'ladi.

Agar davriy funksiya T_0 – asosiy davrga ega bo'lsa, qolgan davrlarning hammasi T_0 ga karrali bo'ladi.

Funksiya eng kichik musbat davrga ega bo'lmasligi ham mumkin. Masalan, $f(x)=5$ funksiya uchun ixtiyoriy haqiqiy son davr bo'ladi, lekin u asosiy davrga ega emas. Haqiqatan ham, $f(x)=const$, $\alpha\neq 0$ ixtiyoriy haqiqiy son bo'lsin. $f(x+\alpha)=f(x)=const$. Bu yerdan kelib chiqadiki, α davr eng kichik musbat davr emas.

5.12.14-ta'rif. Agar

$$f(x+\omega)=-f(x),\ (\omega\neq 0)$$

bajarilsa, u holda $f(x)$ *anti davriy funksiya* deyiladi.

Davriy funksiyalar quyidagi xossalarga ega:

1^0. Ikkita T davrga ega bo'lgan funksiyaning yig'indisi, ko'paytmasi yana davriy funksiya bo'ladi va uning davri T ga teng bo'ladi.

2^0. Agar $T(T\neq 0)$ $f(x)$ va $g(x)$ funksiyalarning eng kichik musbat davri bo'lsa, bu son $f(x)\pm g(x)$, $f(x)\cdot g(x)$ uchun eng kichik musbat davr bo'lmasligi ham mumkin. Masalan, 1) $f(x)=3\sin x+2$, $g(x)=2-3\sin x$ funksiyalar eng kichik musbat $T=2\pi$ davrga ega, lekin ularning yig'indisi $f(x)+g(x)=4$ esa, eng kichik asosiy davrga ega emas. 2) $f(x)=\sin x+1$, $g(x)=1-\sin x$ funksiyalarning eng kichik musbat davri $T=2\pi$, lekin $f(x)\cdot g(x)=\cos^2 x=\frac{1}{2}(1+\cos 2x)$ ko'paytmaning eng kichik musbat davri $T=\pi$ bo'ladi.

3^0. Agar $f(x)$ funksiya T davrga ega bo'lsa, u holda $f(ax)$, $f(ax)+b$ funksiyalar $\tau=\frac{T}{a}$ davrga ega (bunda $a\neq 0$ ixtiyoriy haqiqiy son, x, $ax\in X$) bo'ladi.

4^0. Agar $f(x)$ funksiya T davrga ega bo'lsa, u holda $Af(ax+b)$ $(A=const,\ a>0)$ ham davriy funksiya bo'ladi va uning davri $\tau=\frac{T}{a}$ ga teng bo'ladi.

Agar istalgan $x\in X$ va ba'zi bir T lar uchun $f(x+T)=\frac{1}{f(x)}$ $(T\neq 0)$ bo'lsa, u holda $f(x)$ funksiya $2T$ davrga ega bo'ladi.

Eng kichik musbat davrga ega bo'lgan funksiyalar yig'indisining eng kichik musbat davri, qo'shiluvchi funksiyalarning davrlarining eng kichik umumiy karralisiga teng bo'ladi. Bunda, shakl almashtirishlar natijasida nolga aylanadigan qo'shiluvchi funksiyalarning davri hisobga olinmaydi.

Masalan, ushbu $f(x)=2\sin 4x+ctg3x+3\sin x+\sin(x-\pi)+2\sin(x+\pi)$ funksiyani shakl almashtirishlar natijasida quyidagi $f(x)=2\sin 4x+ctg3x$, ko'rinishda ifodalash mumkin, bunda $3\sin x+\sin(x-\pi)+2\sin(x+\pi)=0$. *a)* $g(x)=2\sin 4x$ funksiyaning eng kichik musbat davri, 3^0- xossaga asosan, $T_1=\frac{2\pi}{4}=\frac{\pi}{2}$. *b)* $h(x)=ctg3x$ funksiyaning eng kichik musbat davri $T_2=\frac{\pi}{3}$. $\frac{\pi}{2}$ va $\frac{\pi}{3}$ sonlarning eng kichik karralisi π.

e) Teskari funksiyalar. $f(x)$ funksiya X to'plamda aniqlangan bo'lib, funksiyaning o'zgarish (qiymatlari) sohasi Y bo'lsin.

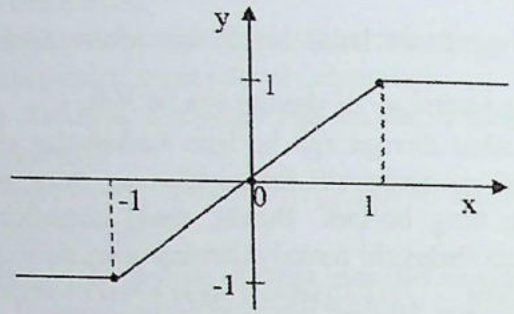
5.12.15-ta'rif. $y=f(x)$ funksiyaning har bir $y\in Y$ qiymatiga biror, g munosabatga ko'ra, X dan faqat bitta x qiymat mos kelsa, Y to'plamda funksiya aniqlangan bo'ladi, va u $y=f(x)$ ga nisbatan *teskari funksiya* deyiladi va $x=f^{-1}(y)=g(y)$ ko'rinishda belgilanadi.

Odatdagidek, funksiyani y bilan, argumentni esa, x bilan belgilashlarga muvofiq, $x=f^{-1}(y)$ ko'rinishda yozishadi. $f^{-1}(x)=g(x)$ desak, $y=g(x)$ bo'ladi.

5.12.16-teorema. $f(x)$ funksiya $D(f)$ to'plamda teskari $g(x)$ funksiyaga ega bo'lishi uchun o'z aniqlanish sohasidagi argumentning har xil qiymatiga funksiyaning ham har xil qiymati mos kelishi zarur va yetarli, ya'ni $\forall x_1,x_2\in D(f)$ lar uchun $x_1\neq x_2 \Leftrightarrow f(x_1)\neq f(x_2)$.

5.12.17-teorema. Agar $y=f(x)$ funksiya X da aniqlangan qat'iy monoton o'suvchi (kamayuvchi) bo'lsa, Y da $y=f(x)$ ga teskari funksiya mavjud, bu funksiya ham qat'iy monoton o'suvchi (kamayuvchi) bo'ladi.

5.12.18-eslatma. Agar funksiya monoton bo'lib, lekin qat'iy monoton bo'lmasa, bu funksiyaning teskarisi mavjud bo'lmaydi. Buni, masalan, 5.13-chizmada ko'rsatilgan,



5.13-chizma.

$$f(x)=\begin{cases} -1, \ x<-1 \ bo'lganda, \\ x, -1\le x\le 1 \ bo'lganda, \\ 1, x>1 \ \ bo'lganda \end{cases}$$

funksiya misolida ko'rish mumkin.

5.12.19-eslatma. Juft funksiyaning teskarisi mavjud emas. Xususiy holda, aniqlanish sohasining funksiya qat'iy monoton bo'lgan qismlarida teskari funksiya mavjud bo'ladi. Masalan, $y=x^2$ funksiya uchun $[0,+\infty)$ da $y=\sqrt{x}$ teskari funksiya bo'ladi.

5.12.20-eslatma. Davriy funksiyaning teskarisi mavjud emas. Xususiy holda, aniqlanish sohasining funksiya qat'iy o'suvchi (kamayuvchi) bo'lgan qismlarida teskari funksiyalar mavjud bo'ladi.

Masalan, $f_1(x)=\sin x \quad (x\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right])$; $f_2(x)=\cos x \quad (x\in[0,\pi])$; $f_3(x)=tgx$ $(x\in(-\frac{\pi}{2},\frac{\pi}{2}))$; $f_4(x)=ctgx(x\in(0,\pi))$ funksiyalar uchun ko'rsatilgan oraliqlarda $g_1(y)=\arcsin y \ (y\in[-1,1])$, $g_2(y)=\arccos y \ (y\in[-1,1])$; $g_3(y)=arctgy \ (y\in(-\infty,+\infty))$; $g_4(y)=arcctgy \ (y\in(-\infty,+\infty))$ teskari funksiyalar mavjud, chunki bu oraliqlarda ular qat'iy monotondir:

5.12.21-eslatma. $y=f(x)$ funksiya va bu funksiyaga teskari bo'lgan $x=f^{-1}(y)$ funksiyaning aniqlanish sohasi va o'zgarish sohasi o'z rollarini

almashtiradi, ya'ni $y=f(x)$ funksiyaning aniqlanish sohasi teskari funksiyaning o'zgarish sohasi bo'ladi, $y=f(x)$ funksiyaning o'zgarish sohasi esa, teskari funksiyaning aniqlanish sohasi bo'ladi.

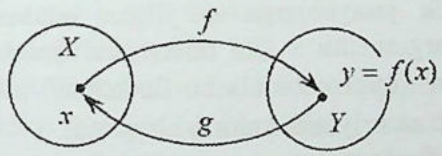
$y=f(x)$ funksiya biror X to'plamda aniqlangan bo'lib, uning qiymatlari to'plami Y bo'lsin. $g(y)$ funksiya Y to'plamda aniqlangan bo'lib, X to'plam esa, uning qiymatlari to'plami bo'lsin.

5.12.22-teorema. $g(y)$ funksiya $y=f(x)$ ga teskari funksiya bo'lishi uchun

$$g(f(x))=x \quad (x\in X) \quad (f(g(y))=y \quad (y\in Y)) \qquad (5.12.23)$$

shartning bajarilishi zarur va yetarlidir (5.14-chizma).

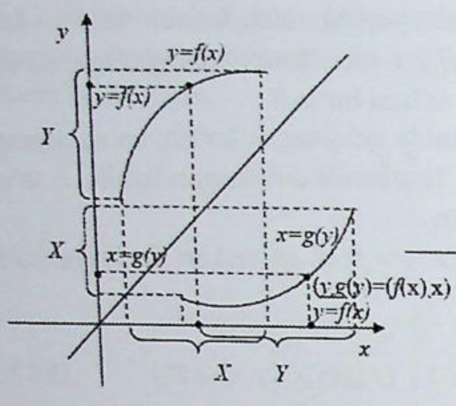
Misollar: **1)** $f(x)=\frac{1}{x}$, $g(x)=\frac{1}{x}$ $(x\ne 0)$ funksiyalar (5.12.23) shartni qanoatlantiradi. Haqiqatan, ham $f(g(x))=\frac{1}{\frac{1}{x}}=x$. Demak, ular bir-biriga teskari funksiyalar bo'ladi.



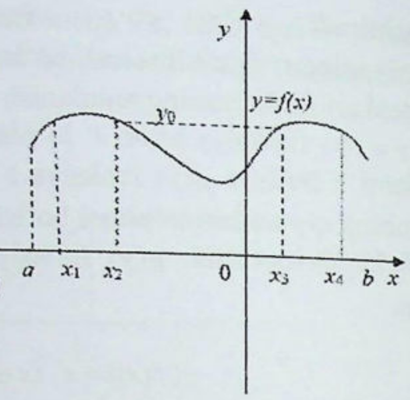
5.14-chizma.

$y=f(x)$ to'g'ri funksiyadan $x=f^{-1}(y)$ teskari funksiyaga o'tish va uning grafigini chizish uchun quyidagi amallarni bajarish maqsadga muvofiq:

1. $y=f(x)$ tenglama x o'zgaruvchiga nisbatan (agar tenglamani x ga nisbatan yechish mumkin bo'lsa) yechiladi: $x=f^{-1}(y)=g(y)$.
2. x ni y bilan y ni x bilan almashtiriladi: $y=f^{-1}(x)=g(x)$.
3. $y=f(x)$ to'g'ri funksiyaning grafigi chiziladi.
4. Hosil qilingan $y=f(x)$ funksiyaning grafigini I va III chorak koordinatalar burchaklaridan o'tuvchi bissektrisaga nisbatan simmetrik almashtirish natijasida teskari funksiya grafigi hosil qilinadi.



5.15-chizma.



5.16-chizma.

$f(x)$ funksiyaning grafigi $\{(x,y): x \in X, y \in Y\}$ nuqtalar to'plamidan, $g(y)$ funksiyaning grafigi esa, $\{(y,g(y)) = (f(x),x)\}$ nuqtalar to'plamidan tuziladi (5.15-chizma).

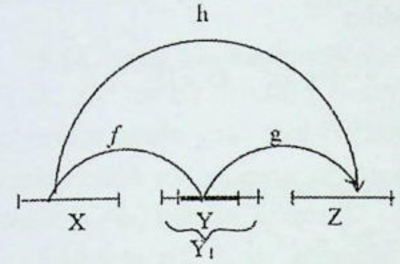
5.16-chizmadagi $y = f(x)$ funksiya uchun teskari funksiya mavjud emas, chunki $x_1 < x_2 < x_3 < x_4$ da $y_0 = f(x_1) = f(x_2) = f(x_3) = f(x_4)$ bo'ladi, bu esa teskari funksiya mavjud bo'lishi shartiga ziddir.

Ba'zi bir fizik jarayonlarga qo'yilgan talablar argumenti x, o'z navbatida yangi t-argumentli $x = \varphi(t)$ funksiyani ifodalaydigan $y = f(x)$, ya'ni $y = f[\varphi(x)]$ funksiyani (matematikada bu funksiyani murakkab funksiya deb yuritiladi) o'rganish zaruriyatini keltirib chiqaradi.

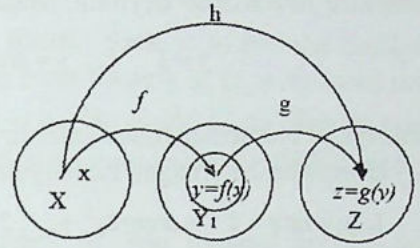
h) **Murakkab funksiyalar.** f va g funksiyalar, mos ravishda, X va Y_1 to'plamlarda berilgan bo'lib, f funksiyaning qiymatlari to'plami $E(f) = Y$, g funksiyaning qiymatlari to'plami $E(g) = Z$ va $Y \subseteq Y_1$ (f funksiyaning qiymatlari to'plami, g fūnksiyaning aniqlanish sohasida saqlansin) shart bajarilganda X to'plamda $F = g(f(x)) = h(x)$ $(F = g(y), y = f(x))$ murakkab funksiya yoki g va f *funksiyalarning kompozisiyasi aniqlangan* deyiladi va u $z = g \circ f$ kabi belgilanadi (5.17, 5.18-chizmalar).

Demak, $\forall x \in X$ uchun f funksiya yordamida bitta $y \in Y$ mos qo'yiladi, so'ngra $\forall y \in Y$ uchun g funksiya yordamida bitta $z \in Z$ mos qo'yiladi. Shunday qilib, $z = g(f(x))$ funksiyaning aniqlanish sohasi $y = f(x)$ funksiyaning aniqlanish sohasiga ustma-ust tushadi yoki uning qismi bo'ladi. Bunda f funksiyaning qiymatlari sohasi, g funksiyaning aniqlanish

sohasida yotishi muhim, aks holda g va f funksiyalarning kompozisiyasi aniqlanmaydi.



5.17-chizma.



5.18-chizma.

1^0. $u = \varphi(x)$ davriy funksiya bo'lsin. Agar $y = f(u)$ funksiya qat'iy monoton bo'lsa, u holda $y = f(\varphi(x))$ murakkab funksiya ham davriy va ularning davrlari bir-biriga teng bo'ladi.

2^0.Agar $y = f(u)$ funksiya qat'iy monoton bo'lsa, u holda $y = f(\varphi(x))$ murakkab funksiyaning davri $u = \varphi(x)$ funksiyaning davridan kichik bo'lishi ham mumkin.

3^0. Agar $y = f(x), x = \varphi(t)$ toq funksiya bo'lsa, u holda $y = f(\varphi(t))$ myrakkab funksiya ham toq funksiya bo'ladi.

4^0. Agar $y = f(x)$ juft funksiya, $x = \varphi(t)$ esa, toq (juft) funksiya bo'lsa, u holda $y = f(\varphi(t))$ murakkab funksiya ham juft funksiya bo'ladi.

k) Elementar funksiyalar. Matematikaning ko'p masalalarida qo'llaniladigan quyidagi funksiyalarga *asosiy elementar* funksiyalar deyiladi. Ular

1. $y = b$ - o'zgarmas funksiya $(b = const)$, $b \in R$.
2. $y = x^\alpha$ - darajali funksiya, α - haqiqiy son.
3. $y = a^x$ - ko'rsatkichli funksiya, bunda $a > 0$, $a \neq 1$.
4. $y = \log_a x$ - logarifmik funksiya $a > 0$, $a \neq 1$, $x > 0$.
5. $y = \sin x$, $y = \cos x$, $y = tgx$, $y = ctgx$, $y = \dfrac{1}{\cos x} = \sec x$, $y = \dfrac{1}{\sin x} = \operatorname{cosec} x$ - trigonometrik funksiyalar.
6. $y = \arcsin$, $y = \arccos x$, $y = arctgx$, $y = arcctgx$ - teskari trigonometrik funksiyalar.

Asosiy elementar funksiyalarning ustida chekli sondagi qo'shish, ayirish, ko'paytirish, bo'lish amallarini bajarish va murakkab funksiya hosil qilish natijasida yuzaga keladigan (analitik usulda berilgan) funksiyalar *elementar funksiyalar* deyiladi. Masalan, ushbu

$$y = 3^{x+\mathrm{tg}\frac{1}{x}}, \quad y = \arcsin x^2, \quad y = \log_3\left(x^2 - 4x + 3\right)$$

funksiyalar elementar funksiyalardir.

Elementar bo'lmagan funksiyalarga misollar keltiramiz:

1. $y = \sin x + 3^{4x} + arctg 2x^2 + \ldots$; 2. $y = x + 2x^2 + \frac{1}{4}x^3 + \ldots$;

3. Dirixle funksiyasi: $D(x) = \begin{cases} 0, & x \in Q, \\ 1, & x \in I. \end{cases}$

1, 2 - misollarda amallar chekli marta bajarilmagan, 3 - misolda esa, elementar funksiyalar qatnashgani yo'q.

Elementar funksiyalar va ularning xossalari o'rta maktab matematika kursida batafsil o'rganilganligini hisobga olgan holda, ularning ta'riflarini, o'rganilgan asosiy xossalarini, grafiklarini mustaqil o'rganishni o'quvchilarning o'zlariga havola qilamiz.

5.3-§. Sonlar ketma-ketligi

5.13. Nuqtaning atrofi.

5.13.1-ta'rif. Quyidagi $U_\varepsilon(a) = \{x : x \in R,\ a - \varepsilon < x < a + \varepsilon\}$ to'plam, a nuqtaning ε *atrofi* deyiladi, ε *- son esa, atrofning radiusi* deyiladi.

Ushbu $U_\varepsilon^+(a) = \{x : x \in R,\ a < x < a + \varepsilon\}$ to'plam, a nuqtaning *o'ng atrofi*, $U_\varepsilon^-(a) = \{x : x \in R,\ a - \varepsilon < x < a\}$ to'plam esa, a nuqtaning *chap atrofi* deyiladi.

Ushbu $0 < |x - a| < \varepsilon$ tengsizlik, $a - \varepsilon < x < a + \varepsilon,\ x \neq a$ tengcizliklarga teng-kuchli bo'lib, ularning har ikkalasini a nuqtaning $\mathring{U}_\varepsilon(a)$ atrofi shaklida ifodalash mumkin: $\mathring{U}_\varepsilon(a) = \{x : x \in R,\ a - \varepsilon < x < a + \varepsilon,\ x \neq a\}$. Ba'zi hollarda, $\mathring{U}_\varepsilon(a)$ atrof, a nuqtaning *teshik (o'yilgan) atrofi* deb ham yuritiladi.

Haqiqiy sonlar to'plami R tarkibiga $-\infty$ va $+\infty$ simvollarni $\forall x \in R$ uchun $-\infty < x$ va $x < +\infty$ xususiyat orqali qo'shib, $\overline{R}$ to'plamni hosil qilamiz: $\overline{R} = R = \{-\infty\} \cup \{+\infty\}$.

$\overline{R}$ da $-\infty$ va $+\infty$ «nuqta» larning atrofi tushunchalari quyidagicha kiritiladi: $U_c(+\infty) = \{x : x, c \in R,\ c < x < +\infty\}$, $U_c(-\infty) = \{x : x, c \in R,\ -\infty < x < c\}$,

$$U_c(\infty) = \{x : x \in R,\ c \in R,\ |x| > c\}.$$

5.14. Sonli ketma-ketlik va uning limiti. N va R to'plamlar berilgan bo'lib, f – har bir natural $n\ (n \in N)$ songa biror haqiqiy $x_n\ (x_n \in R)$ sonni mos qo'yuvchi qoida yoki usul bo'lsin: $f : n \to x_n$. Bu holda N *to'plamda sonli ketma - ketlik yoki natural argumentli funksiya aniqlangan* deyiladi va u $\{x_n\}$ yoki $x_n = f(n)$ kabi belgilanadi.

5.14.1-ta'rif. Agar shunday o'zgarmas M son mavjud bo'lib, $\{x_n\}$ ketma – ketlikning har bir hadi shu M sondan katta bo'lmasa, ya'ni $\forall n\ (n \in N)$ uchun $x_n \le M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ *ketma - ketlik yuqoridan chegaralangan* deb ataladi.

5.14.2-ta'rif. Agar shunday o'zgarmas m son mavjud bo'lib, $\{x_n\}$ ketma-ketlikning har bir hadi shu m sondan kichik bo'lmasa, ya'ni $\forall n\ (n \in N)$ uchun $m \le x_n$ tengsizlik bajarilsa, $\{x_n\}$ *ketma-ketlik quyidan chegaralangan* deyiladi.

5.14.3-ta'rif. Agar $\{x_n\}$ ketma-ketlik ham quyidan, ham yuqoridan chegaralangan bo'lsa, ya'ni shunday M va m o'zgarmas sonlar mavjud bo'lib, $\forall n\ (n \in N)$ uchun $m \le x_n \le M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ *ketma-ketlik chegaralangan* deyiladi.

5.14.4-teorema. $\{x_n\}$ ketma-ketlik chegaralangan bo'lishi uchun, shunday $A > 0$ son mavjud bo'lib, $\forall n\ (n \in N)$ uchun $|x_n| \le A$ tengsizlikning bajarilishi zarur va yetarlidir.

5.14.5-ta'rif. Agar ixtiyoriy $A > 0$ (istalgancha katta) son olinganda ham, $\{x_n\}$ ketma-ketlikning hech bo'lmaganda bitta x_{n_0} elementi topilib, $|x_{n_0}| \ge A$ tengsizlik bajarilsa, $\{x_n\}$ *ketma-ketlik chegaralanmagan ketma-ketlik* deyiladi.

5.14.6-ta'rif. Agar $\forall A > 0$ son uchun $\exists n_0(A)$ nomer mavjud bo'lib, $n \ge n_0(A)$ dan boshlab, $\{x_n\}$ ketma-ketlikning barcha elementlari $|x_n| > A$ tengsizlikni qanoatlantirsa, $\{x_n\}$ *cheksiz katta ketma-ketlik* deyiladi.

5.14.7-ta'rif. Agar $\forall \varepsilon > 0$ $\exists n_0(\varepsilon)$ nomer mavjud bo'lib, $n \ge n_0(\varepsilon)$ dan $|x_n| < \varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ *cheksiz kichik ketma-ketlik* deyiladi.

5.15. Cheksiz kichik ketma-ketliklarning asosiy xossalari

1⁰. Ikkita $\{x_n\}$ va $\{y_n\}$ cheksiz kichik ketma-ketliklarning yig'indisi va ayirmasi $\{x_n \pm y_n\}$, yana cheksiz kichik ketma-ketlik bo'ladi. Bu xossadan ushbu natija kelib chiqadi:

2⁰. Chekli sondagi cheksiz kichik ketma-ketliklarning algebraik yig'indisi, yana cheksiz kichik ketma-ketlik bo'ladi.

3⁰. Cheksiz kichik ketma-ketlik bilan chegaralangan ketma-ketlikning ko'paytmasi, cheksiz kichik ketma-ketlik bo'ladi.

4⁰. Har qanday cheksiz kichik ketma-ketlik chegaralangan ketma – ketlik bo'ladi.

5⁰. Chekli sondagi cheksiz kichik ketma-ketliklarning ko'paytmasi, cheksiz kichik ketma-ketlik bo'ladi.

6⁰. Agar $\{x_n\}$ ketma-ketlik cheksiz katta ketma-ketlik bo'lsa, u holda biror n_0 nomerdan boshlab $\left\{\frac{1}{x_n}\right\}$ ketma-ketlik aniqlangan bo'ladi va u cheksiz kichik ketma-ketlik bo'ladi. Agar $\{y_n\}$ cheksiz kichik ketma-ketlikning hamma hadlari noldan farqli bo'lsa, u holda $\left\{\frac{1}{y_n}\right\}$ ketma – ketlik cheksiz katta ketma-ketlik bo'ladi.

5.16. Yaqinlashuvchi ketma-ketliklar va ularning xossalari

5.16.1– ta'rif. Agar shunday haqiqiy a son mavjud bo'lib, $\forall\varepsilon>0$ uchun shunday $n_0(\varepsilon)$ nomer topilib, $\forall n\geq n_0(\varepsilon)$ lar uchun $|x_n-a|<\varepsilon$ tengsizlik bajarilsa, a songa $\{x_n\}$ ketma-ketlikning *limiti* deyiladi va $\lim\limits_{n\to\infty}x_n=a$ kabi yoziladi.

Bu ta'rifni qisqacha,

$$\left(\lim_{n\to\infty}x_n=a\right)\Leftrightarrow\left(\forall\varepsilon>0,\quad \exists n_0=n_0(\varepsilon)\in N:\forall n\geq n_0\to|x_n-a|<\varepsilon\right)$$

kabi ifodalash ham mumkin

5.16.2- ta'rif. Agar a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olinganda ham, $\{x_n\}$ ketma-ketlikning biror hadidan keyingi barcha hadlari shu atrofga joylashsa, *a son, $\{x_n\}$ ketma-ketlikning limiti* deyiladi.

5.16.3-ta'rif. Agar $\{x_n\}$ ketma-ketlikning limiti mavjud va chekli bo'lsa, $\{x_n\}$ ketma – ketlik *yaqinlashuvchi ketma-ketlik* deyiladi.

Yuqoridagi ta'riflar o'zaro ekvivalent ta'riflardir.

5.16.4- ta'rif. Agar ixtiyoriy a son va ixtiyoriy n_0 son olinganda ham shunday ε_0 son va shunday $n>n_0$ natural son topilib, $|x_n-a|\geq\varepsilon_0$ bo'lsa, $\{x_n\}$ ketma-ketlik *limitga ega emas* deyiladi.

Bu ta'rifni qisqacha quyidagicha ta'riflash mumkin:

$$\forall n_0\in N,\ \exists\varepsilon_0,\ \exists n\in N: n>n_0\Rightarrow|x_n-a|\geq\varepsilon_0.$$

5.16.5-ta'rif. Agar $\{x_n\}$ ketma-ketlik limitga ega bo'lmasa, u *uzoqlashuvchi ketma-ketlik* deyiladi.

5.16.6-ta'rif. Limiti nolga teng bo'lgan $\{x_n\}$ ketma-ketlik $\left(\lim\limits_{n\to\infty}x_n=0\right)$ *cheksiz kichik ketma-ketlik* deyiladi.

5.16.7-ta'rif. Cheksiz katta bo'lgan $\{x_n\}$ ketma-ketlik *cheksiz limitga ega deyiladi va* $\lim\limits_{n\to\infty}x_n=\infty$ kabi belgilanadi.

Yaqinlashuvchi ketma-ketliklar quyidagi xossalarga ega:

1⁰. Yaqinlashuvchi ketma-ketlik yagona limitga ega bo'ladi.

2⁰. Har qanday yaqinlashuvchi ketma-ketlik chegaralangan ketma – ketlik bo'ladi, aks holda ketma-ketlik chegaralanmagan bo'ladi.

3⁰. $\{x_n\}$va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, ular, mos ravishda, a va b limitlarga ega bo'lsa, u holda $\{x_n\pm y_n\}$, $\{x_n\cdot y_n\}$, $\left\{\frac{x_n}{y_n}\right\}$ ketma – ketliklar ham yaqinlashuvchi bo'ladi va

$$\lim_{n\to\infty}(x_n\pm y_n)=a\pm b;\qquad \lim_{n\to\infty}(x_n\cdot y_n)=a\cdot b;\qquad \lim_{n\to\infty}\frac{x_n}{y_n}=\frac{a}{b}\ (b\neq0)$$

munosabatlar o'rinli bo'ladi.

4⁰. $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, $\lim\limits_{n\to\infty}x_n=a$, $\lim\limits_{n\to\infty}y_n=b$ bo'lsin. Agar $\forall n(n\in N)$ uchun $x_n\leq y_n$ $(x_n\geq y_n)$ bo'lsa, u holda $a\leq b$ $(a\geq b)$ bo'ladi.

5⁰. $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, $\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}y_n=a$ bo'lsin. Agar $\forall n(n\in N)$ uchun $x_n\leq z_n\leq y_n$ tengsizlik o'rinli bo'lsa, u holda $\{z_n\}$ ketma-ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty}z_n=a$ bo'ladi.

5.16.8–eslatma. Agar yaqinlashuvchi ketma-ketlikning hamma elementlari $x_n > b$ qat'iy tengsizlikni qanoatlantirsa, u holda, bu ketma-ketlikning limiti x har doim $x > b$ bo'lmaydi.

6^0. Agar yaqinlashuvchi $\{x_n\}$ ketma-ketlikning hamma hadlari $[a, b]$ kesmaning ichida joylashsa. U holda uning limiti x ham $[a, b]$ kesmaning ichida joylashadi.

5.16.9–eslatma. Ikkita $\{x_n\}$ va $\{y_n\}$ ketma – ketliklarning yig'indisi, ayirmasi, ko'paytmasi va nisbatidan iborat bo'lgan ketma – ketlikning yaqinlashuvchi bo'lishidan, bu $\{x_n\}$ va $\{y_n\}$ ketma – ketliklarning har birining yaqinlashuvchi bo'lishi har doim ham kelib chiqavermaydi.

5.17. Monoton ketma-ketlikning ta'riflari

5.17.1-ta'rif. Agar $\{x_n\}$ ketma-ketlikning elementlari $\forall n \in N$ uchun $x_n \le x_{n+1}$ $(x_n < x_{n+1})$ tengsizlikni qanoatlantirsa, $\{x_n\}$ *o'suvchi (qat'iy o'suvchi) ketma-ketlik* deyiladi.

5.17.2-ta'rif. Agar $\{x_n\}$ ketma-ketlikning elementlari $\forall n \in N$ uchun $x_n \ge x_{n+1}$ $(x_n > x_{n-1})$ tengsizlikni qanoatlantirsa, $\{x_n\}$ *kamayuvchi (qat'iy kamayuvchi) ketma-ketlik* deyiladi.

5.17.3-ta'rif. O'suvchi va kamayuvchi ketma-ketliklar umumiy nom bilan *monoton ketma-ketliklar* deb ataladi.

5.18. Monoton ketma-ketlikning yaqinlashishi haqidagi teoremalar

5.18.1- teorema. Agar $\{x_n\}$ ketma-ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, u yaqinlashuvchi (chekli limitga ega) bo'ladi; agar $\{x_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo'lsa, u holda u uzoqlashuvchi (limiti $+\infty$) bo'ladi.

5.18.2- teorema. Agar $\{x_n\}$ ketma-ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, u yaqinlashuvchi (chekli limitga ega) bo'ladi; agar $\{x_n\}$ ketma-ketlik quyidan chegaralanmagan bo'lsa, u holda u uzoqlashuvchi (limiti $-\infty$) bo'ladi.

Bu teoremalardan quyidagi natijalar kelib chiqadi.

5.18.3-natija. O'suvchi ketma-ketlik yaqinlashuvchi bo'lishi uchun, uning yuqoridan chegaralangan bo'lishi zarur va yetarli.

5.18.4-natija. Kamayuvchi ketma-ketlik yaqinlashuvchi bo'lishi uchun, uning quyidan chegaralangan bo'lishi zarur va yetarli.

5.18.5-teorema. Monoton $\{x_n\}$ ketma-ketlik yaqinlashuvchi (chekli limitga ega) bo'lishi uchun, uning chegaralangan bo'lishi zarur va yetarli.

5.18.6-eslatma. Har qanday yaqinlashuvchi ketma-ketlik monoton ketma-ketlik bo'lavermaydi.

5.18.7-eslatma. Yuqoridagi teoremalardan quyidagi xulosani chiqarish mumkin: yuqoridan chegaralangan o'suvchi $\{x_n\}$ ketma-ketlikning hamma hadlari uning limiti $\bar{x}$ dan katta $(x_n \le \bar{x})$ bo'la olmaydi. Xuddi shunday, quyidan chegaralangan kamayuvchi $\{x_n\}$ ketma-ketlikning hamma hadlari uning limiti $\underline{x}$ dan kichik $(\underline{x} \le x_n)$ bo'la olmaydi.

5.18.8-teorema. Ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar berilgan bo'lsin. Agar: 1) $\{x_n\}$ o'suvchi, $\{y_n\}$ kamayuvchi ketma-ketlik; 2) $\forall n \in N$ uchun $x_n < y_n$; 3) $\lim\limits_{n\to\infty}(y_n - x_n) = 0$ bo'lsa, u holda $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$ tenglik o'rinli bo'ladi.

5.18.9-teorema. Agar $[a_1;b_1] \supset [a_2;b_2] \supset [a_3;b_3] \supset \ldots \supset [a_n;b_n] \supset \ldots$ munosabatda bo'lgan, $[a_1;b_1], [a_2;b_2], [a_3;b_3], \ldots, [a_n;b_n] \ldots$ kesmalar ketma-ketligi uchun $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ shart o'rinli bo'lsa, u holda $\{a_n\}$ va $\{b_n\}$ ketma-ketliklar bitta limitga ega bo'ladi, hamda bu limit barcha kesmalarga tegishli bo'lgan yagona nuqta bo'ladi.

5.19. Ixtiyoriy ketma-ketliklarning quyi va yuqori limitlari. Aniqmas ifodalar. Ushbu ixtiyoriy $x_1, x_2, \ldots x_n, \ldots$ ketma – ketlik va butun musbat o'suvchi $k_1, k_2, \ldots, k_n, \ldots$ ixtiyoriy sonlar ketma – ketliklarini qaraylik. $\{x_n\}$ ketma – ketlik elementlari ichidan $k_1, k_2, \ldots, k_n, \ldots$ nomerdagilarini olib, ularni indeksning o'sish tartibida joylashtiramiz. Natijada, biz $x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots$ ketma – ketlikga ega bo'lamiz va uni $\{x_n\}$ ketma – ketlikning qismiy ketma – ketligi deb ataymiz. Xususiy holda $\{x_n\}$ ketma – ketlikning o'zi $(k_n = n)$ ham qismiy.

5.19.1- teorema. Agar $\{x_n\}$ ketma-ketlik a limitga ega bo'lsa, u holda uning har qanday qismiy ketma-ketligi ham shu a limitga ega bo'ladi.

5.19.2-eslatma. $\{x_n\}$ ketma-ketlik qismiy ketma-ketliklarining limitga ega bo'lishidan, berilgan ketma-ketlikning limitga ega bo'lishi har doim ham kelib chiqavermaydi.

5.19.3-ta'rif. $\{x_n\}$ ketma-ketlik qismiy ketma-ketligining limiti, $\{x_n\}$ ketma-ketlikning *qismiy limiti* deyiladi.

5.19.4-teorema. Agar $\{x_n\}$ ketma-ketlikning hamma qismiy ketma-ketliklari yaqinlashuvchi bo'lib, ularning har biri bitta a limitga ega bo'lsa, u holda $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, u ham shu a limitga ega bo'ladi.

5.19.5-ta'rif. Agar $x\,(x\in(-\infty;+\infty))$ nuqtaning ixtiyoriy ε atrofida $\{x_n\}$ ketma-ketlikning cheksiz ko'p elementlari joylashsa, x nuqta $\{x_n\}$ ketma-ketlikning *limit nuqtasi* deyiladi.

5.19.6-teorema. Har qanday haqiqiy sonlar (chegaralangan va chegaralanmagan) ketma-ketligidan chekli songa, $+\infty$ yoki $-\infty$ larga intiluvchi qismiy ketma-ketlik ajratish mumkin.

5.19.7-teorema (Bolsano-Veyershtrass). Har qanday chegaralangan $\{x_n\}$ ketma-ketlikdan yaqinlashuvchi $\{x_{n_k}\}$ qismiy ketma-ketlik ajratish mumkin.

5.19.8-ta'rif. Agar $\{x_n\}$ ketma-ketlikdan $x\,(x\in(-\infty;+\infty))$ nuqtaga yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, x nuqta $\{x_n\}$ ketma-ketlikning *limit nuqtasi* deyiladi.

5.19.9-ta'rif. $\{x_n\}$ ketma-ketlik limit nuqtalarining eng kattasiga, bu ketma-ketlikning *yuqori limiti* deyiladi va $\overline{x}=\overline{\lim\limits_{n\to\infty}}x_n=\lim\limits_{n\to\infty}\sup\limits_{k>n}x_n$ kabi belgilanadi.

5.19.10-ta'rif. $\{x_n\}$ ketma-ketlik limit nuqtalarining eng kichigiga, bu ketma-ketlikning *quyi limiti* deyiladi va $\underline{x}=\underline{\lim\limits_{n\to\infty}}x_n=\lim\limits_{n\to\infty}\inf\limits_{k>n}x_k$ kabi belgilanadi.

Agar $\{x_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo'lsa, $\overline{\lim\limits_{n\to\infty}}x=+\infty$ bo'ladi.

Agar $\{x_n\}$ ketma-ketlik quyidan chegaralanmagan bo'lsa, $\underline{\lim\limits_{n\to\infty}}x_n=-\infty$ bo'ladi.

5.19.11- teorema. Har qanday $\{x_n\}$ ketma-ketlik yuqori (quyi) (chekli, $+\infty$ yoki $-\infty$) limitga ega.

5.19.12-natija. Agar $\{x_n\}$ ketma-ketlik chegaralangan bo'lsa, uning quyi va yuqori limitlari chekli bo'ladi.

Ketma-ketlikning quyi va yuqori limitlari quyidagi xossalarga ega.

Ixtiyoriy $\{x_n\}$ ketma-ketlik uchun $\overline{\lim\limits_{n\to\infty}}x_n=\overline{x}$ bo'lsin. U holda $\forall\varepsilon<0$ son olinganda ham:

1^0. $\exists n_0\in N$ son topiladiki, $\forall n>n_0$ uchun $x_n<\overline{x}+\varepsilon$ bo'ladi.

2^0. $\forall n_1\in N$ son uchun ε va n_1 larga bog'liq, shunday natural son $n'>n_1$ topiladiki, $x_{n'}>\overline{x}-\varepsilon$ bo'ladi.

Agar $\overline{x}$ son 1^0 va 2^0 shartlarni qanoatlantirsa, u holda $\overline{x}=\lim\limits_{n\to\infty}\sup\limits_{n>k}\{x_n\}=\overline{\lim\limits_{n\to\infty}}x_n$ bo'ladi.

Endi $\{x_n\}$ ketma-ketlik uchun $\underline{x}=\underline{\lim\limits_{n\to\infty}}x_n$ bo'lsin. U holda, $\forall\varepsilon>0$ son olinganda ham:

1^0. $\exists n_0\in N$ son topiladiki, $\forall n>n_0$ uchun $x_n>\underline{x}-\varepsilon$ bo'ladi.

2^0. $\forall n_1\in N$ son uchun ε va n_1 larga bog'liq natural son $n'>n_1$ topiladiki, $x_{n'}<\underline{x}+\varepsilon$ bo'ladi.

Agar $\underline{x}$ son 1^0 va 2^0 shartlarni qanoatlantirsa, u holda $\underline{x}=\lim\limits_{k\to\infty}\inf\limits_{k>n}\{x_k\}=\underline{\lim\limits_{n\to\infty}}x_n$ bo'ladi.

5.19.13-teorema. $\forall\{x_n\}$ ketma-ketlik a limitga ega bo'lishi uchun $\underline{\lim\limits_{n\to\infty}}x_n=\overline{\lim\limits_{n\to\infty}}x_n=a$ tenglikning o'rinli bo'lishi zarur va yetarli.

Agar:1) $x_n\to 0,\ y_n\to 0$ bo'lganda, $\dfrac{y_n}{x_n}$ va $x_n^{\,y_n}$ ketma-ketliklarning; 2) $x_n\to\infty,\ y_n\to\infty$ bo'lganda, $\dfrac{y_n}{x_n}$ ketma-ketlikning; 3) $x_n\to 0,\ y_n\to\infty$ bo'lganda x_ny_n va $y_n^{\,x_n}$ ketma-ketliklarning; 4) $x_n\to\infty,\ y_n\to\infty$ bo'lganda, x_n+y_n ketma-ketlikning va 5) $x_n\to 1,\ y_n\to\infty$ bo'lganda $x_n^{\,y_n}$ ketma-ketlikning limitini topish talab etilsa, ular to'g'risida aniq javob aytib bo'lmaydi, chunki natijalar x_n va y_n ketma-ketliklarning intilish tezligiga bog'liq bo'ladi. Masalan: 1) $x_n=\frac{1}{n}\to 0,\ y_n=\frac{1}{n^2}\to 0$ bo'lishidan $\frac{x_n}{y_n}=n\to\infty$, 2) $x_n=\frac{1}{n^2}\to 0,\ y_n=\frac{1}{n}\to 0$ bo'lishidan $\frac{x_n}{y_n}=\frac{1}{n}\to 0$; 3) $x_n=\frac{1}{n}\to 0,\ y_n=\dfrac{(-1)^n}{n}\to 0$ bo'lishidan, $\frac{x_n}{y_n}=(-1)^n\ \to?$ mavjud emasligi kelib chiqadi. Bu misollardan ko'rinadiki, $x_n\to 0,\ y_n\to 0$ bo'lishini bilgan holda, $\dfrac{y_n}{x_n}$ ketma-ketlikning limiti to'g'risida aniq xulosa chiqarib bo'lmaydi. Xuddi shunday, yuqorida keltirilgan boshqa hollarga ham misollar keltirish mumkin. Shuning uchun $\dfrac{0}{0},\dfrac{\infty}{\infty},\infty-\infty,\ 0\cdot\infty,0^0,\infty^0,1^\infty$ ko'rinishdagi ular *aniqmas ifodalar* yoki *aniqmasliklar* deyiladi. Bu yozuvlarni mos ravishda: "cheksiz kattalarning ayirmasi", "cheksiz kichik bilan chelsiz kattaning ko'paytmasi", "cheksiz kichik taqsim cheksiz kichik", "cheksiz katta taqsim cheksiz katta", "cheksiz kichik darajasi cheksiz kichik", "cheksiz katta darajasi cheksiz kichik", "bir darajasi cheksiz katta" deb o'qish maqsadga muvofiq bo'ladi. Aniqmasliklarning o'ziga xos xususiyatlarini inobatga olib va turli usullardan foydalanib ularning limitlarini topishga *aniqmasliklarni ochish* deyiladi.

5.19.14-teorema (Koshi kriteriysi). $\forall\{x_n\}$ ketma-ketlik yaqinlashuvchi (chekli limitga ega) bo'lishi uchun, $\forall\varepsilon>0$ son olinganda ham, shunday $n_0\in N$ son mavjud bo'lib, $\forall n>n_0$ va $\forall m>n_0$ lar uchun

$$|x_m - x_n| < \varepsilon \tag{5.19.20}$$

tengsizlikning bajarilishi zarur va yetarli.

Agarda (5.19.20) shart bajarilmasa, ya'ni

$$\exists\varepsilon_0>0:\ \forall k\in N,\ \exists n\ge k,\ \forall m\ge k:\ |x_m-x_n|\ge\varepsilon_0$$

bo'lsa, $\{x_n\}$ ketma-ketlik uzoqlashuvchi bo'ladi.

5.4-§. Kompleks sonlar

5.20. Kompleks sonlar va ular ustida arifmetik amalllar. a va b haqiqiy sonlar berilgan bo'lsin.

5.20.1-ta'rif. a va b haqiqiy sonlarning (a,b) tartiblangan juftligiga *kompleks son* deb aytiladi.

(a,b) va (c,d) juftliklar ularning mos koordintalari teng bo'lgandagina *teng* deyiladi, ya'ni

$$(a,b)=(c,d) <=> \begin{cases} a=c, \\ b=d. \end{cases}$$

Kompleks sonlarni qo'shish va ko'paytirish amallari quyidagi tengliklar yordamida kiritiladi:

$$(a,b)+(c,d)=(a+c,b+d), \tag{5.20.2}$$

$$(a,b)\cdot(c,d)=(ac-bd,ad+bc). \tag{5.20.3}$$

$(0,1)$ kompleks soni i harfi orqali belgilash va uni *mavhum bir* deb atash qabul qilingan, ya'ni $i=(0,1)$. (5.20.3) formulaga asosan, $i\cdot i=i^2$ ko'paytmani hisoblaymiz: $i^2=i\cdot i=(0,1)\cdot(0,1)=(-1,0)=-1$. (5.20.2), (5.20.3) formulalardan

$$(a,0)+(c,0)=(a+c,\,0),\ (a,0)\cdot(c,0)=(ac,\,0),$$
$$(0,b)=(0,1)(b,0)=ib,\ (a,b)=(a,0)+(0,b)=a+ib$$

tengliklar kelib chiqadi.

Shunday qilib, har bir (a,b) kompleks sonni $a+ib$ ko'rinishda yozish mumkin.

Kompleks son tushunchasi yuqori darajali algebraik tenglamalarni, jumladan, kvadrat tenglamalarni yechish jarayonida yuzaga kelgan. Ushbu $ax^2+bx+c=0$ kvadrat tenglamani qaraylik, bunda $a\neq 0,b,c$ lar haqiqiy sonlar. Agar $b^2-4ac<0$ bo'lsa, kvadrat tenglama ikkita kompleks ildiz ga ega bo'ladi.

5.20.4-ta'rif. Ushbu $a+ib$ ko'rinishdagi ifoda *kompleks son yoki kompleks sonning algebraik shakli* deyiladi, bu yerda a,b lar haqiqiy sonlardir.

Kompleks sonlarni bitta harf bilan ham belgilash mumkin, masalan: $\alpha=a+ib$, bu yerda a son α ning haqiqiy, b esa mavhum qismi deb ataladi va mos ravishda $\text{Re}\,\alpha=a$ va $\text{Im}\,\alpha=b$ ko'rinishda yoziladi (lotincha reabis – haqiqiy, imaginarins–mavhum demakdir). Masalan:

$$\text{Re}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})=-\frac{1}{2},\text{Im}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})=\frac{\sqrt{3}}{2}.$$

Ushbu $\alpha=a+ib$ va $\beta=c+id$ kompleks sonlar berilgan bo'lsin.

5.20.5-ta'rif. Agar α va β kompleks sonlar uchun $a=c$ va $b=d$ yoki $\text{Re}\,\alpha=\text{Re}\,\beta$ va $\text{Im}\,\alpha=\text{Im}\,\beta$ tengliklar o'rinli bo'lsa, bu *kompleks sonlar teng* deyiladi va $\alpha=\beta$ kabi yoziladi.

Har qanday ikkita $\alpha=a+ib$ va $\beta=c+id$ kompleks son uchun qo'shish va ayirish amallari quyidagi qoidalar asosida aniqlanadi:

$$\alpha\pm\beta=(a+ib)\pm(c+id)=(a\pm c)+i(b\pm d).$$

Masalan, 1) $(2-4i)+(5+8i)=(2+5)+i(-4+8)=7+4i$;

2) $(3-8i)-(2-5i)=(3-2)+i(-8-(-5))=1-3i$.

Har qanday ikkita kompleks sonni ko'paytirish ko'phadlarni kupaytirish qoidasiga binoan bajariladi, ya'ni

$$\alpha\cdot\beta=(a+ib)(c+id)=ac+iad+ibc+i^2bd.$$

Ma'lumki, $i^2=-1$, shuning uchun $\alpha\cdot\beta=(as-bd)+i(bc+ad)$ bo'ladi.

5.20.6-ta'rif. Agar $a=c, b=-d$ bo'lsa, u holda β kompleks son α kompleks songa *qo'shma kompleks son* deyiladi va $\overline{\alpha}$ kabi belgilanadi.

Ta'rifga ko'ra, $\alpha=a+ib$ bo'lsa, u holda $\overline{\alpha}=\overline{a+ib}=a-ib$ kompleks son o'zaro qo'shma kompleks son deyiladi. Qo'shma kompleks sonlarning yig'indisi ham, ko'paytmasi ham haqiqiy son bo'ladi: $\alpha+\overline{\alpha}=2\alpha$ va $\alpha\overline{\alpha}=a^2+b^2$. Masalan:

1) $(4+i3)+(4-i3)=8$; 2) $(4+i3)\cdot(4-i3)=4^2-(i3)^2=16-i^2 9=25$.

Ikkita $\alpha=a+ib$ va $\beta=c+id$ kompleks sonning birini ikkinchisiga bo'lish uchun kasrning surat va maxrajining $\overline{\beta}=c-id$ qo'shma kompleks songa ko'paytiramiz. U holda

$$\frac{\alpha}{\beta}=\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}.$$

Masalan, $\frac{1-i2}{1+i2}=\frac{(1-i2)^2}{(1+i2)(1-i2)}=\frac{1-i4+i^2 4}{1+4}=-\frac{3}{5}-i\frac{4}{5}$.

5.20.7-eslatma. $k=0,1,2,...$ sonlar uchun

$$i^{4k}=1,\quad i^{4k+1}=i,\quad i^{4k+2}=-1,\quad i^{4k+3}=-i$$

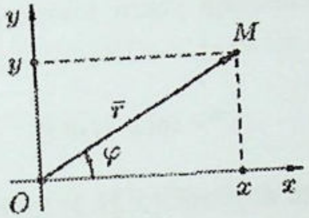
tengliklar o'rinli bo'ladi.

5.21. Kompleks sonning trigonometrik shakli. Tekislikda Dekart koordinatalari sistemasi va ixtiyoriy $z=x+iy$ kompleks son berilgan bo'lsin. Koordinatalari x va y bo'lgan (x,y)juftlik tekislikda bitta nuqtani ifodalaydi, biz uni M deylik: $M(x,y)$. $z=x+iy$ kompleks sonning ikki xil geometrik ma'nosi bor:

a) $z=x+iy$ kompleks son tekislikda $M(x,y)$ nuqtani tasvirlaydi;

b) $O(0,0)$ nuqta bilan $M(x,y)$ nuqtani tutashtiruvchi vektorni tasvirlaydi (5.19-chizma).

Demak, har bir kompleks songa tekislikda bitta M nuqta va aksincha tekislikdagi har bir M nuqtaga bitta kompleks son mos quyiladi. Natijada kompleks sonlar to'plami bilan tekislik nuqtalri orasida o'zaro bir qiymatli moslik o'rnatilgan bo'ladi. Buni e'tiborga olgan holda xOy tekislikni *kompleks sonlar tekisligi* deb ham yuritiladi.



5.19-chizma.

Koordinatalar boshi O nuqta bilan M nuqtani birlashtiruvchi OM kesma uzunligi r ga z kompleks soning *moduli* deyiladi va u $|z|$ kabi belgilanadi. Pifagor teoremasiga ko'ra, $r=|z|=\sqrt{x^2+y^2}$ bo'ladi.

Kompleks sonlar tasvirlanadigan tekislik $z=x+iy$ kompleks o'zgaruvchining tekisligi deyiladi. Ox o'qda yotuvchi nuqtalarga haqiqiy sonlar mos keladi (bunda $y=0$), Oy o'qda yotuvchi nuqtalar sof mavhum sonlarni tasvirlaydi (bu holda $x=0$). Shu sababli, Ox haqiqiy o'q, Oy esa, mavhum o'q deyiladi.

$\vec{OM}$ vektor bilan Ox o'qining musbat yo'nalishi orasidagi φ burchakka $z=x+iy$ kompleks sonning *argumenti* deyiladi va $\varphi=\arg z$ kabi belgilanadi. $z\neq 0$ bo'lgan $z=0$ kompleks sonning argumenti aniqlanmagan bo'ladi. $2\pi k$, $k=0,\pm1,\pm2,...$ kompleks sonning argumenti ko'p qiymatli bo'lib, u $2\pi k$, $k=0,\pm1,\pm2,...$, bunda $\arg z$— aniqlikda bo'ladi: $Arg z=\arg z+2\pi k=\varphi+2\pi k$, $k=0,\pm1,\pm2,...$ munosabatda argumentning bosh qiymati deyiladi va uni $0\le\arg z\le 2\pi$ qaraymiz.

5.19-chizmadan $x=r\cos\varphi, y=r\sin\varphi$ tengliklarni hisobga olib, z kompleks sonni bunday ifodalash mumkin:

$$z=x+iy=r(\cos\varphi+i\sin\varphi) \tag{5.21.1}$$

bunda $r=|z|=\sqrt{x^2+y^2}$ va

$$\varphi = \arg z = \begin{cases} arctg\dfrac{y}{x}, & x \geq 0, y \geq 0 \ \ bo'lganda, \\ arctg\dfrac{y}{x} + \pi, & x < 0 \ \ bo'lganda, \\ arctg\dfrac{y}{x} + 2\pi, & x \geq 0, \ y < 0 \ \ bo'lganda. \end{cases} \qquad (5.21.2)$$

(5.21.1)–formulaga $z = x + iy$ kompleks sonning *trigonometrik shakli* deyiladi. $z = x + iy$ ko'rinishdagi yozuv kompleks sonning *algebraik shakli* deyiladi. Ushbu

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \qquad (5.21.3)$$

formulaga *Eyler formulasi* deyiladi. (5.21.3) formuladan foydalanib, (5.21.1) ni

$$z = re^{i\varphi} \qquad (5.21.4)$$

ko'rinishda yozish mumkin. (5.21.4) ga kompleks sonning *ko'rsatkichli shakli* deyiladi.

5.21.5-misol. Ushbu $z = -1 + i$ sonni trigonometrik shaklga keltiring.

Yechilishi. $x = -1, \ \ y = 1, \qquad r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. Berilgan son ikkinchi chorakda yotadi. (5.21.2) formuladan foydalanib φ burchakni topamiz:

$$\varphi = \pi + arctg\frac{1}{-1} = \pi - arctg 1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

bo'ladi. Demak, (5.21.1) formulaga asosan,

$$z = -1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}). \blacksquare$$

5.22. Kompleks sonlarni ko'paytirish va darajaga ko'tarish

Ushbu

$$z_1 = r(\cos\varphi + i\sin\varphi) \text{ va } z_2 = \rho(\cos\psi + i\sin\psi)$$

trigonometrik shaklda berilgan kompleks sonlarni o'zaro ko'paytirish va darajaga ko'tarish qo'yidagicha bajariladi:

$$z_1 z_2 = r(\cos\varphi + i\sin\varphi)\rho(\cos\psi + i\sin\psi\varphi) = r\rho[\cos(\varphi+\psi)i\sin(\varphi+\psi)], \ (5.22.1)$$

ya'ni ikki kompleks sonning ko'paytmasi yana kompleks son bo'lib, uning moduli ko'paytuvchilar modullarining ko'paytmasiga, argumenti ko'paytuvchilar argumentlarining yig'indisiga teng.

Agar $z_1 = z_2$ bo'lsa, (5.22.1) dan quydagi

$$z_1^n = [r(\cos\phi + i\sin\phi)]^n = r^n(\cos n\phi + i\sin n\phi)$$

bo'ladi. Umumiy holda qo'yidagi

$$z_1^n = [r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad (5.22.2)$$

formula o'rinli bo'ladi. Agar $r = 1$ bo'lsa, (5.22.2) dan

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi \qquad (5.22.3)$$

bo'ladi. (5.22.3) ga *Muavr formulasi* deyiladi.

5.22.4-misol. Ushbu $(1+i)^{25}$ ni hisoblang.

Yechilishi. Dastlab $1 + i$ ni trigonometrik shaklga keltirib olamiz:

$$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}).$$

Endi (5.22.2) formulaga asosan,

$$\begin{aligned}(1+i)^{25} &= \left[\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\right]^{25} = (\sqrt{2})^{25}(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}) = \\ &= (\sqrt{2})^{25}\left[\cos(6\pi + \frac{\pi}{4}) + i\sin(6\pi + \frac{\pi}{4})\right] = (\sqrt{2})^{25}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \\ &= (\sqrt{2})^{25}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2^{12}(1+i).\end{aligned}$$

Demak, $(1+i)^{25} = 2^{12}(1+i). \blacksquare$

5.23. Kompleks sondan ildiz chiqarish. Ildizdan chiqarish amali darajaga ko'tarish amaliga teskari amaldir. Kompleks sonning n – darajali ildizi $\sqrt[n]{z}$ deb shunday W songa aytiladiki, bu sonning n- darajasi ildiz ostidagi songa tengdir, ya'ni agar $W=\sqrt[n]{z}$ bo'lsa, $W^n=z$ bo'ladi.

Agar $z=r(\cos\varphi+i\sin\varphi)$ va $\rho(\cos\varphi+i\sin\varphi)$ bo'lsa, u holda

$$\sqrt[n]{z}=\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\rho(\cos\varphi+i\sin\gamma)=W\ .$$

Muavr formulasiga ko'ra:

$$z=r(\cos\varphi+i\sin\varphi)=\rho^n(\cos n\psi+i\sin n\psi),$$

bundan $\rho^n=r,\ \ n\psi=\varphi+2\pi k\ \ \rho^n=r,\ k$ butun son. ρ va ψ ni topamiz:

$$\rho=\sqrt[n]{r},\ \ \psi=\frac{\varphi+2\pi k}{n},$$

bunda k – istalgan butun son.

Demak,

$$\sqrt[n]{z}=\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}(\cos\frac{\varphi+2\pi k}{n}+i\sin\frac{\varphi+2\pi k}{n}),\quad (5.23.1)$$

bunda $k=0,1,2,...,n-1$. $k=n,n+1,n+2,...$ deb faraz qilinsa, oldingi ildizlar kelib chiqadi.

Shunday qilib, $W_0,W_1,W_2,...,W_{n-1}$ ildizlar hosil bo'ladi, n ta ildizning hammasi markazi koordinatalar boshida bo'lib, radiusi $\sqrt[n]{r}$ ga teng aylana ichiga chizilgan muntazam n tomonli ko'pburchak uchlarida yotadi.

5.23.2-misol. Ushbu $\sqrt[4]{-i}$ ni hisoblang.

Yechilishi. $-i=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$, shu sababli, (5.23.1) formulaga ko'ra, ushbu

$$W_k=\sqrt[4]{-i}=\cos\frac{\frac{3\pi}{2}+2\pi k}{4}=i\sin\frac{\frac{3\pi}{2}+2\pi k}{4}=$$

$$=\cos\frac{(4k+3)\pi}{8}+i\sin\frac{(4k+3)\pi}{8},k=0,1,2,3$$

tenglik o'rinli bo'ladi.■

5-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

5.1. To'plam tushunchasi. Qism to'plam ta'rifi ([1],1-t., 3-bo'lim; [3],1-q., 5-6 betlar; [12], 1-q., 5-9 betlar; [9], 1-t., 1-bo'lim).

5.2. To'plamlar ustida amallar: to'plamlarning tengligi, to'plamlarning yig'indisi (birlashmasi), ko'paytmasi (kesishmasi), ayirmasi, simmetrik ayirmasi, Dekart ko'paytmasi ([1],1-t., 3-bo'lim; [3],1-q., 6-8 betlar; [12], 1-q.,8-11 betlar; [9], 1-t., 1-bo'lim).

5.3. To'plamlarni taqqoslash: to'plamlar orasida o'zaro bir qiymatli moslik (o'.b.m), ekvivalent to'plamlar, sanoqli va sanoqsiz to'plamlar ([1],1-t., 3-bo'lim; [3],1-q., 13-15 betlar; [12], 1-q.,16-18 betlar; [9], 1-t., 1-bo'lim).

5.4. Sonli to'plamlar. Son o'qi ([1],1-t., 3-bo'lim; [3],1-q., 5 bet; [12], 1-q.,20-21 betlar; [9], 1-t., 1-bo'lim).

5.5. Haqiqiy sonlarning (Dedekind bo'yicha) ta'rifi ([5], 1-t.,11-26 betlar; [12], 1-q.,34-38 betlar).

5.6. Haqiqiy sonlar to'plamining xossalari:

1^0. Haqiqiy sonlar to'plami – tartiblangan to'plam. ([3],1-q., 16-21 betlar; [5], 1-t.,27-42 betlar; [12], 1-q.,34-35 betlar).

2^0. Haqiqiy sonlar to'plami – zich to'plam ([5], 1-t.,27-42 betlar; [12], 1-q.,35 bet).

3^0. Haqiqiy sonlar to'plami –to'liq (uzluksiz) ([5], 1-t.,27-42 betlar; [12], 1-q.,35-38 betlar).

5.7. Matematik induksiya usuli. Uni qo'llash bosqichlari ([14], 152-210 betlar, [20], 31-42 betlar).

5.8. Sonlar ketma – ketligining ta'rifi ([3],1-q., 37 bet; [5], 1-t.,41-45 betlar; [12], 1-q.,64-65 betlar, [15], 26-27 betlar). Chegaralangan va chegaralanmagan ketma – ketliklar va ularga misollar ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 37-38 betlar; [5], 1-t.,83-89 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.9. Cheksiz katta va cheksiz kichik ketma – ketliklarning ta'riflari va ularga misollar ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 49-51 betlar; [5], 1-t.,47-56 betlar; [12], 1-q.,70-72, 81-82 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.10. Cheksiz kichik ketma – ketliklarning xossalari ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 49-51 betlar; [5], 1-t.,47-52 betlar; [12], 1-q.,72-75 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.11. Yaqinlashuvchi ketma – ketliklar va ularning xossalari ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 43-46 betlar; [5], 1-t.,46-54 betlar; [12], 1-q.,72-74 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.12. Ketma – ketlik limitining ta'riflari ([3],1-q., 38-42 betlar; [5], 1-t.,46-47 betlar; [12], 1-q.,68-69 betlar).

5.13. Tengsizliklarda limitga o'tish ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 44-46 betlar; [5], 1-t.,56-57 betlar; [12], 1-q.,79-81 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.14. Monoton ketma – ketliklar va ularning chekli limitga ega bo'lishi haqidagi teoremalar ([1],1-t., 5-, 6-bo'limlar; [3],1-q., 51-53 betlar; [5], 1-t.,70-82 betlar; [12], 1-q.,86-89 betlar; [9], 1-t., 3-bo'lim, [30], 11-bo'lim).

5.15. Ixtiyoriy ketma – ketlikning limitga ega bo'lishi haqidagi Koshi teoremasi ([3],1-q., 56-57 betlar; [5], 1-t.,83-85 betlar; [12], 1-q.,101-104 betlar, [9], 1-t., 3-bo'lim).

5.16. Qismiy ketma – ketliklar. Bolsano – Veyershtrass teoremasi ([3],1-q., 57-60 betlar; [5], 1-t.,83-89 betlar; [12], 1-q.,98-101 betlar, [9], 1-t., 3-bo'lim).

5.17. Ketma – ketlikning yuqori va quyi limitlari ta'rifi va ularga misollar ([3],1-q., 60-62 betlar; [5], 1-t.,83-89 betlar; [12], 1-q.,104-108 betlar, [9], 1-t., 3-bo'lim).

5.18. Kompleks sonlar va ular ustida arifmetik amalllar

5.19. Kompleks sonning trigonometrik shakli.

5.20. Kompleks sonni ko'paytirish va darajaga ko'tarish.

5.21. Kompleks sondan ildiz chiqarish

5.1-amaliy mashg'ulot.
To'plamlar

1-misol. $A=\{0,2,4,6,8,10\}$, $B=\{1,2,3,4,5,6\}$ bo'lsa, $C=A\cup B=\{0,1,2,3,4,5,6,8,10\}$ bo'ladi.

2-misol. $A=\{\pm 2,\pm 4,\pm 6,\pm 8,\pm 10,...\}$, $B=\{\pm 3,\pm 6,\pm 9,\pm 12,....\}$ bo'lsa, $C=A\cap B=\{\pm 6,\pm 12,....\}$ bo'ladi.

3-misol. $A=\{1,2,3,4,5,6,7\}$, $B=\{0,2,4,6,8\}$ bo'lsa, $C=A\setminus B=\{1,3,5,7\}$ bo'ladi.

4-misol. $A=\{1,2,3,4,5,6,7\}$, $B=\{6,7,8,9,10\}$ bo'lsa, $A\Delta B=\{1,2,3,4,5\}\cup\{8,9,10\}=\{1,2,3,4,5,8,9,10\}$ bo'ladi.

Mustaqil yechish uchun misollar

Quyidagi berilgan A va B to'plamlarga ko'ra, $A\cup B$, $A\cap B$, $A\setminus B$, $B\setminus A$ to'plamlarni toping:

1. $A=\{2,4,6,8,10\}$, $B=\{4,8,12,16\}$.

2. $A=\{1,3,5,7,9,...,2n-1,...\}$, $B=\{3,6,9,...,3n,...\}$.

3. $A=\{x:(x-2)(x-3)=0\}$, $B=\{x:(x-2)(x+4)=0\}$.

4. $A=\{x:x^2-4=0\}$, $B=\{x:x-2=0\}$.

Quyidagi berilgan A va B to'plamlarga ko'ra $A\cup B$, $A\cap B$, $A\setminus B$, $B\setminus A$ $A\Delta B$ to'plamlarni toping.

5. $A=\{x:0<x<2\}=(0;2)$, $B=\{x:1\le x\le 3\}=[1;3]$.

6. $A=\{x:x(x-3)<0\}$, $B=\{x:(x-3)(x-1)\ge 0\}$.

7. $A=\{x:x\in R\ x^2-8x+15\le 0\}$, $B=\{x:x\in N,\ x^2-6x<0\}$.

8. $A=\{x:x\in Z,-2<x<4\}$, $B=\{x:x\in N,\ 1\le x\le 7\}$.

Quyidagi berilgan A,B ва C to'plamlarga ko'ra , $A\cup B$, $A\cap C$, $A\cup(B\cap C)$, $(A\cap B)\cap C$ to'plamlarni toping:

9. $A=\{x:-3\le x\le 2\}=[-3;2]$, $B=[0;4]$, $C=[3;5]$.

10. $A=Q$, $B=Z$, $C=N$.

Mustaqil yechish uchun berilgan masalalarning javoblari

1. $A\cup B=\{2,4,6,8,10,12,16\}$. $A\setminus B=\{2,6,10\}$, $B\setminus A=\{12,16\}$, $A\cap B=\{4;8\}$. **2.** $A\cup B=\{1,3,5,6,7,9,...\}$, $A\cap B=\{3,9,27,...\}$. $A\setminus B=\{1,5,7,...\}$, $B\setminus A=\{6,12,...\}$. **3.** $A\cup B=\{-4,2,3\}$, $A\cap B=\{2\}$, $A\setminus B=\{-4\}$. **4.** $A\cup B=\{-2,2\}$, $A\cap B=\{2\}$, $A\setminus B=\{2\}$, $B\setminus A=\varnothing$.

5. $A\cup B=\{x:0<x\le 3\}$, $A\cap B=\{x:1\le x<2\}$, $A\setminus B=\{x:0<x<1\}$, $B\setminus A=\{x:2\le x\le 1\}$. $A\Delta B=\{x:0<x<1,\ 2\le x\le 3\}$. **6.** $A\cup B=\{x:-\infty<x\le\infty\}$, $A\cap B=\{x:0<x<1\}$, $A\setminus B=\{x:1<x<3\}$, $B\setminus A=\{x:-\infty<x<0,\ 3\le x<\infty\}$, $A\Delta B=\{x:-\infty<x\le 0,\ 1\le x\le\infty\}$. **7.** $A\cup B=\{1,2,3,4,5\}$, $A\cap B=\{3,4,5\}$, $A\setminus B=(3,4)\cup(4,5)$, $B\setminus A=\{1,2\}$, $A\Delta B=\{1,2\}\cup(3,4)\cup(4,5)$. **8.** $A\cup B=\{-1,0,1,2,3,4,5,6,7\}$, $A\cap B=\{1,2,3\}$, $A\setminus B=\{-1;0\}$, $B\setminus A=\{4,5,6,7\}$, $A\Delta B=\{-1,0,4,5,6,7\}$. **9.** $A\cup B=[-3;4]$, $A\cap C=\varnothing$, $A\cup(B\cap C)=[-3;2]\cup[3;4]$, $(A\cup B)\cap C=[0;2]\cap[3;5]=\varnothing$. **10.** $A\cap B=Q$, $A\cap C=N$, $(A\cup B)\cap C=Z\cap N=N$.

5.2-amaliy mashg'ulot.
Funksiya

Funksiyaning aniqlanish sohasini topish vaqtida formulani boshqa ko'rinishga keltirish tavsiya etilmaydi.

1-misol. $f(x)=\frac{x}{x^2-4}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. $\frac{x}{x^2-4}$ funksiyaning maxraji nolga aylanadigan nuqtalarda funksiya ma'noga ega emas. Demak, bu funksiyaning aniqlanish sohasini topishda quyidagi $x^2-4\neq 0$ ёки $x\neq\pm 2$ shartlar bajarilishini talab qilish kerak.

Shunday qilib, funksiyaning aniqlanish sohasi uchta oraliqlarning birlashmasidan iborat, ya'ni

$$D(f)=(-\infty;-2)\cup(-2;2)\cup(2;+\infty).$$

2-misol. $f(x)=\frac{\sqrt{9-x}}{\lg(x-3)}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Birinchidan, $9-x\ge 0$ ёки $x\le 9$; ikkinchidan $x-3>0$ ёки $x>3$; uchinchidan $\lg(x-3)\neq 0$ ёки $x\neq 4$. $x\le 9$, $x>3$

ва $x\neq 4$ shartlarni hisobga olsak, berilgan funksiyaning aniqlanish sohasi $D(f)=(3;4)\cup(4;9]$ dan iborat bo'ladi.

3-misol. $f(x)=\sin x+\cos x$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Berilgan funksiyani quyidagi ko'rinishda yozish mumkin:

$$f(x)=\sin x+\cos x=\sin x+\sin\left(\frac{\pi}{2}-x\right)=2\sin\frac{\pi}{4}\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right).$$

Ma'lumki, $-1\le\cos\left(x-\frac{\pi}{4}\right)\le 1$ bu yerdan,

$-\sqrt{2}\le\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\le\sqrt{2}$ yoki $-\sqrt{2}\le f\le\sqrt{2}$.

Bo'lgani uchun, funksiyaning o'zgarish sohasi $E(f)=\left[-\sqrt{2};\sqrt{2}\right]$ bo'ladi.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning aniqlanish sohalarini toping:

1. $f(x)=\frac{4x}{x^2-9}$.

2. $f(x)=\frac{1}{\sqrt{x^2-4x}}$.

3. $f(x)=\lg(1-\lg(x^2-5x+16))$.

4. $f(x)=\arcsin(5x-6)$.

5. $|f|=\lg x$.

6. $f(x)=\sqrt{|x|-8}$.

7. $f(x)=\log_2\log_3\sqrt{4x-4x^2}$.

8. $f(x)=\frac{\arccos x}{\ln\left(x+\frac{1}{2}\right)}$.

9. $f(x)=\log_{x-1}\left(x-\frac{1}{4}\right)$.

10. $f(x)=\sqrt{|x|-3}+\frac{1}{\sqrt{10-x}}$.

Quyidagi funksiyalarning o'zgarish sohalari topilsin.

11. $f(x)=\frac{3x}{x^2+1}$.

12. $f(x)=\frac{x+2}{2x-3}$.

13. $f(x)=ctgx\cdot tgx$.

14. $f(x)=3\sin x+4\cos x$.

15. $f(x)=x^2+|x|-2$.

16. $f(x)=10^{\frac{1}{x+2}}$.

Mustaqil yechish uchun berilgan misollarning javoblari

1. $(-\infty;-3)\cup(-3;3)\cup(3;\infty)$. **2.** $(-\infty;0)\cup(4;\infty)$. **3.** $(2;3)$. **4.** $[1;\frac{7}{5}]$. **5.** $[1;\infty)$. **6.** $(-\infty;-8]\cup[8;\infty)$. **7.** $\varnothing$. **8.** $\left(-\frac{1}{2};\frac{1}{2}\right)\cup\left(\frac{1}{2};1\right]$. **9.** $(1;2)\cup(2;\infty)$. **10.** $(-\infty-3]\cup[3;10)$. **11.** $[-1,5;1,5]$. **12.** $(-\infty;0,5)\cup(0,5;+\infty)$. **13.** $\{1\}$. **14.** $[-5;5]$. **15.** $[-2;+\infty)$. **16.** $(0;1)\cup(1;\infty)$.

5.3-amaliy mashg'ulot.
Sonlar ketma-ketligi va uning limiti

1– misol. Ushbu

$$a)\ x_n=\frac{8n^3-5}{3n^3+11};\quad b)\ x_n=(-1)^n\frac{6n^3-1}{2n^3+3}\cos n;\quad c)\ x_n=\frac{2n^2}{n-1}\cos\pi n, n\ge 2$$

ketma-ketliklarning qaysi biri chegaralangan, qaysi biri chegaralanmagan?

Yechilishi. *a)* Ravshanki, $0<\dfrac{8n^3-5}{3n^3+11}<\dfrac{8n^3}{3n^3}<\dfrac{8}{3}=2\dfrac{2}{3},\quad \forall n\in N.$

Demak, $x_n=\dfrac{8n^3-5}{3n^3+11}$ ketma-ketlik chegaralangan.

b) $x_n=(-1)^n\dfrac{6n^3-1}{2n^3+3}\cos n$ ketma-ketlik ham chegaralangan, chunki

$$|x_n|=\left|(-1)^n\frac{6n^3-1}{2n^3+3}\cos n\right|=\left|(-1)^n\right|\cdot\frac{6n^3-1}{2n^3+3}\cdot|\cos n|<\frac{6n^3}{2n^3+3}<3.$$

Demak, berilgan ketma-ketlik chegaralangan.

c) $|x_n|=\left|\dfrac{2n^2}{n-1}\cdot\cos\pi n\right|=\dfrac{2n^2}{n-1}\cdot|\cos\pi n|=\dfrac{2n^2}{n-1}>2n.$

Ixtiyoriy $A>0$ son uchun $n>\dfrac{A}{2}$ deb olsak, $\left(n=\left[\dfrac{A}{2}\right]+1\right)$ $|x_n|>2n>A$ tengsizlik bajariladi. Demak, berilgan ketma-ketlik chegaralanmagan ekan.■

2-misol. Ushbu 1) $x_n=\dfrac{(-1)^n4}{5\sqrt{n}+1}$; 2) $x_n=\dfrac{1}{n}\sin\left[(2n-1)\dfrac{\pi}{2}\right]$; 3) $x_n=2^{\sqrt{n}}$

ketma-ketliklarning qaysi biri $n\to\infty$ da cheksiz kichik, qaysi biri cheksiz katta ketma-ketlik bo'ladi?

Yechilishi. 1) $\forall\varepsilon>0$ sonni olamiz.

$$|x_n|=\left|\frac{(-1)^n\cdot 4}{5\sqrt{n}+3}\right|<\frac{4}{5\sqrt{n}+3}<\frac{4}{5\sqrt{n}}<\frac{4}{4\sqrt{n}}=\frac{1}{\sqrt{n}}$$

$n>\dfrac{1}{\varepsilon^2}$ bo'lganda $|x_n|<\varepsilon$ tengsizlik bajariladi $\left(n_0(\varepsilon)=\left[\dfrac{1}{\varepsilon^2}\right]+1\right)$. $\varepsilon=\dfrac{1}{10}$ deb olinganda, $|x_n|<\dfrac{1}{10}$ bo'lishi uchun, $\dfrac{1}{\sqrt{n}}<\dfrac{1}{10}$ yoki $n>100$ bo'lishi kerak. Shuning uchun $n_0(\varepsilon)=100$ deb olish mumkin. Shunday qilib, $x_n=\dfrac{(-1)^n\cdot 4}{5\sqrt{n}+3}$ ketma-ketlik cheksiz kichik ketma-ketlik ekan.

2) $\forall\varepsilon>0$ olamiz. $|x_n|=\left|\dfrac{1}{n}\sin\left[(2n-1)\dfrac{\pi}{2}\right]\right|<\dfrac{1}{n}$. $n>\dfrac{1}{\varepsilon}$ bo'lganda $|x_n|<\varepsilon$ tengsizlik bajariladi. $n_0(\varepsilon)=\left[\dfrac{1}{\varepsilon}\right]+1$, deb olish mumkin.

Demak, $n\ge n_0(\varepsilon)=\left[\dfrac{1}{\varepsilon}\right]+1$ dan boshlab $|x_n|<\varepsilon$ bajariladi. Shunday qilib, ta'rifga ko'ra, berilgan ketma-ketlik cheksiz kichik ketma-ketlik ekan.

3) $\forall A>0$ (istalgancha katta) son olamiz va $2^{\sqrt{n}}>M$ tengsizlikni yechamiz:

$$\sqrt{n}>\log_2 M,\qquad n>(\log_2 M)^2.$$

Agar $n_0(A)=\left[(\log_2 M)^2\right]$ deb olinsa, $n>n_0(A)$ dan boshlab $|x_n|>M$ bo'ladi. Demak, berilgan ketma-ketlik cheksiz katta ketma-ketlik ekan.■

3-misol. Ushbu $x_n=\dfrac{2n-1}{3n+1}$ ketma-ketlikning limiti $\dfrac{2}{3}$ ga teng ekanligini ta'rif bo'yicha isbot qiling va quyidagi jadvalni to'ldiring:

ε	0,1	0,01	0,001	0,0001	....
$n_0(\varepsilon)$					......

Yechilishi. Ixtiyoriy musbat ε sonni olamiz. Bu songa ko'ra, shunday $n_0(\varepsilon)$ nomerning mavjudligini ko'rsatish kerakki, $\forall n > n_0(\varepsilon)$ lar uchun $\left|x_n - \frac{2}{3}\right| < \varepsilon$ tengsizlik o'rinli bo'lsin. Buning uchun

$$\left|x_n - \frac{2}{3}\right| = \left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| = \left|-\frac{5}{3(3n+1)}\right| = \frac{5}{3(3n+1)} < \varepsilon$$

tengsizlikni n ga nisbatan yechish kerak:

$$\frac{5}{3\varepsilon} < 3n+1, \quad \frac{5}{3\varepsilon} - 1 < 3n, \qquad \frac{5-3\varepsilon}{9\varepsilon} < n$$

$n_0(\varepsilon)$ natural son (izlanayotgan nomer) sifatida $\left[\frac{5-3\varepsilon}{9\varepsilon}\right] = n_0(\varepsilon)$ son olinsa, u holda $\forall n > n_0$ uchun $\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| < \varepsilon$ tengsizlik bajariladi. Endi berilgan ε ga ko'ra, $n_0(\varepsilon)$ ni topib, jadvalni to'ldiramiz:

ε	0,1	0,01	0,001	0,0001
n_0	5	55	555	5555

4-misol. Quyidagi $\{x_n\}$ ketma-ketliklarning limitlarini hisoblang:

1) $x_n = \frac{2n^3}{2n^2+3} + \frac{1-5n^2}{5n+1}$; 2) $x_n = \sqrt{3n+5} - \sqrt{n-1}$;

3) $x_n = n^2(n - \sqrt{n^2+1})$; 4) $x_n = \sqrt[3]{n^2 - n^3} + n$.

Yechilishi. 1) Kasrlarni qo'shib,

$$x_n = \frac{2n^3 - 13n^2 + 3}{10n^3 + 2n^2 + 15n + 3}$$

ni hosil qilamiz. Bundan, yaqinlashuvchi ketma-ketliklarning xossalarini e'tiborga olib,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^3\left(2 - \frac{13}{n} + \frac{3}{n^3}\right)}{n^3\left(10 + \frac{2}{n} + \frac{5}{n^2} + \frac{3}{n^3}\right)} = \frac{2}{10} = \frac{1}{5}$$

ekanligini topamiz. Agar $y_n = \frac{2n^3}{2n^2+3}$, $z_n = \frac{1-5n^2}{5n+1}$ deb belgilasak, u holda bu ketma-ketliklar yig'indisining limiti, ya'ni $\lim_{n\to\infty}(y_n + z_n) = \frac{1}{5}$, lekin qo'shiluvchilarning harbiri uzoqlashuvchi (cheksiz katta) ketma- ketliklar (5.6.9-eslatmaga qarang).

Marlye tizimidan foydalanib misolni yechish:

> Limit((2*n^3- 13*n^2+3)/(10*n^3+2*n^2+15*n+3),n=infinity) =limit((2*n^3- 13*n^2+3)/(10*n^3+2*n^2+15*n+3),n=infinity);

$$\lim_{n\to\infty}\left(\frac{2n^3 - 13n^2 + 3}{10n^3 + 2n^2 + 15n + 3}\right) = \frac{1}{5}.$$

2) $x_n = \sqrt{3n+5} - \sqrt{n-1} = \sqrt{n}\left(\sqrt{3 + \frac{5}{n}} - \sqrt{1 - \frac{1}{n}}\right)$.

Bunda $\lim_{n\to\infty}\left(\sqrt{3+\frac{5}{n}} - \sqrt{1-\frac{1}{n}}\right) = \lim_{n\to\infty}\sqrt{3+\frac{5}{n}} - \lim_{n\to\infty}\sqrt{1-\frac{1}{n}} = \sqrt{3} - 1$

ekanligini e'xtiborga olsak, u holda, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{3+\frac{5}{n}} - \sqrt{1-\frac{1}{n}}\right) = +\infty$

ekanligini topamiz.

Marlye tizimidan foydalanib misolni yechish:

>Limit(sqrt(3*n+5)-sqrt(n-1),n=infinity)= limit(sqrt(3*n+5)-sqrt(n-1),n=infinity);

$$\lim_{n\to\infty}(\sqrt{3n+5}-\sqrt{n-1})=\infty.$$

3) $x_n=n^2(n-\sqrt{n^2+1})$, bundan ikkinchi ko'paytuvchini $n+\sqrt{n^2+1}$ ga kupaytirib, bo'lsak, u holda

$$x_n=-\frac{n^2}{n+\sqrt{n^2+1}}=-\frac{n}{1+\sqrt{1+\frac{1}{n^2}}}=-n\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}},$$

$$\lim_{n\to\infty}x_n=-\lim_{n\to\infty}n\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}}=-\infty$$ ekanligini topamiz.

Marlye tizimidan foydalanib misolni yechish:

>Limit(n^2*(n-sqrt(n^2+1)),n=infinity)= limit(n^2*(n-sqrt(n^2+1)),n=infinity);

$$\lim_{n\to\infty}(n^2(n-\sqrt{n^2+1})=-\infty.$$

4) $x_n=\sqrt[3]{n^2-n^3}+n=\dfrac{n^2}{(n^2-n^3)^{2/3}-n\sqrt[3]{n^2-n^3}+n^2}$ ekanligini ko'rish qiyin emas, bundan

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}-\left(\frac{1}{n}-1\right)^{1/3}+1}=\frac{1}{3}.$$

Marlye tizimidan foydalanib misolni yechish:

> Limit(surd(n^2- n^3,3)+n,n=infinity)=limit(surd(n^2-n^3,3)+n,n=infinity);

$$\lim_{n\to\infty}(\sqrt[3]{n^2-n^3}+n)=\frac{1}{3}.\ ■$$

Mustaqil yechish uchun misol va masalalar

Quyidagi ketma-ketliklarning dastlabki beshta hadini yozing:

1. $x_n=2+(-1)^n\dfrac{2}{n+1}$. **2.** $x_n=n(3-3(-1)^n)$. **3.** $x_n=\dfrac{4n+3}{3n-2}$.

Quyidagi ketma-ketliklarning umumiy hadini yozing.

4. $-\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ldots$. **5.** $0,\ 4,\ 0,\ 4,\ \ldots$. **6** $1,0,-3,0,5,0,-7,0,\ldots$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarni chegaralanganlikka tekshiring.

7. $\left\{\dfrac{3n+40}{n}\right\}$. **8.** $\left\{\dfrac{5n}{6n-7}\right\}$.

9. $\{(-1)^n(n^2+1)\}$ **10.** $\{(-1)^{n+1}[1+(-1)^n]\cdot n\}$. **11.** $\{5n^2\}$

Quyidagi ketma-ketliklarning chegaralanganligini isbotlang.

12. $\left\{\dfrac{4n^2-3}{3+n^2}\right\}$. **13.** $\left\{\dfrac{2n+(-1)^n}{3n+2}\right\}$. **14.** $\left\{\dfrac{n^2+5n-6}{(n+2)^2}\right\}$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarning cheksiz kichik ketma-ketlik ekanligini ta'rif bo'yicha ko'rsating.

15. $x_n=\dfrac{3}{n}$. **16.** $x_n=\dfrac{(-1)^{n+1}}{n}$. **17.** $x_n=\dfrac{1+(-1)^n}{3^n}$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarning cheksiz katta ketma-ketlik ekanligini ta'rif bo'yicha ko'rsating:

18. $x_n=n$. **19.** $x_n=4-3\cdot n$. **20.** $x_n=2^n$.

Ketma-ketlik limiti ta'rifidan foydalanib, quyidagi tengliklarni isbotlang.

21. $\lim\limits_{n\to\infty}\dfrac{4n-3}{4n+5}=1$. **22.** $\lim\limits_{n\to\infty}\dfrac{5n-2}{3n+4}=\dfrac{5}{3}$.

23. $\lim\limits_{n\to\infty}\dfrac{n^2+3}{n^2+2n+1}=1.$ **24.** $\lim\limits_{n\to\infty}\dfrac{5n^2+4}{8n^2+7}=\dfrac{5}{8}.$

25. Quyida berilgan $\{x_n\}$ ketma-ketliklarning yaqinlashuvchi ekanligini isbotlang:

1) $x_n=\dfrac{3+(-1)^n}{n};$ 2) $x_n=\dfrac{n+1}{n};$

3) $x_n=\sqrt[n]{2}\quad (a>0);$ 4) $x_n=\dfrac{1}{\sqrt[n]{n!}};$

26. Quyida berilgan $\{x_n\}$ ketma-ketliklarning uzoqlashuvchi ekanligini isbotlang.

1) $x_n=(-1)^n\cdot 2^n.$ 2) $x_n=n^{(-1)^n}.$ 3) $x_n=\dfrac{n^2-10}{n}.$

27. a soni $\{x_n\}$ ketma-ketliklarning limiti emasligini ta'rif yordamida ko'rsating.

1) $x_n=(-1)^n\cdot 2+2,\quad a=0.$ 2) $x_n=\dfrac{n^2-1}{n^2}\quad a=1.$

3) $x_n=\dfrac{(-1)^n}{n},\quad a=-1.$ 4) $x_n=\cos\dfrac{\pi n}{3},\quad a=\dfrac{1}{2}.$

Limitlarni toping

28. $\lim\limits_{n\to\infty}\left(\dfrac{1}{6}+\dfrac{1}{12}+\ldots+\dfrac{1}{n^2+3n+2}\right).$

29. $\lim\limits_{n\to\infty}\left(\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\ldots+\dfrac{1}{(3n-2)(3n+1)}\right).$

30. $\lim\limits_{n\to\infty}\left(\dfrac{1}{1\cdot 5}+\dfrac{1}{5\cdot 9}+\ldots+\dfrac{1}{(4n-3)(4n+1)}\right).$

31. $\lim\limits_{n\to\infty}\left(\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{1}{3\cdot 5\cdot 7}+\ldots+\dfrac{1}{(2n-1)(2n+1)(2n+3)}\right).$

32. $\lim\limits_{n\to\infty}\dfrac{\sqrt{4n^2+2}-2n}{\sqrt{n+2}-\sqrt{n}}.$ **33.** $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+7}-\sqrt{n^2-7}}{\sqrt{n^2+1}-n}.$

34. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n}-\sqrt[3]{n+1}}{\sqrt[4]{n+1}-\sqrt[4]{n}}.$ **35.** $\lim\limits_{n\to\infty}\dfrac{\sqrt[4]{n^3+n}-\sqrt{n}}{n+2+\sqrt{n+1}}.$

Quyidagi limitlarni toping:

36. $\lim\limits_{n\to\infty}\left(1+\dfrac{p}{m}\right)^{n\cdot q},\ p,q\in N.$ **37.** $\lim\limits_{n\to\infty}\left(\dfrac{3^n+1}{3^n}\right)^{3^n}.$ **38.** $\lim\limits_{n\to\infty}\left(\dfrac{n+2}{n+5}\right)^{n}.$

39. Quyida berilgan $\{x_n\}$ ketma-ketliklarni Koshi kriteriysidan foydalanib, yaqinlashuvchi ekanligini isbotlang.

1) $x_n=\dfrac{\cos a}{3}+\dfrac{\cos 2a}{3^2}+\dfrac{\cos 3a}{3^3}+\ldots+\dfrac{\cos na}{2^n},\ a\in R.$

2) $x_n=1+\dfrac{1}{2!}+\dfrac{1}{3!}+\ldots+\dfrac{1}{n!}.$

3) $x_n=a_1q+a_2q^2+\ldots+a_nq^n$, bunda $|q|<1,\ \forall n\in N$ uchun $|a_n|\le C,\ C=const.$

Mustaqil yechish uchun berilgan misollarning javoblari

1. $1,\ \dfrac{8}{3},\ \dfrac{3}{2},\ \dfrac{12}{5},\ \dfrac{5}{3},\ldots$ **2.** $6,\ 0,\ 18,\ 0,\ 30,\ldots$ **3.** $7,\ \dfrac{11}{4},\ \dfrac{15}{7},\ \dfrac{19}{10},\ldots$

4. $x_n=\dfrac{(-1)^n}{n+1}.$ **5.** $x_n=2(1+(-1)^n).$ **6.** $x_n=n\cos\dfrac{\pi(n-1)}{2}.$

7. Chegaralangan. **8.** Chegaralangan. **9.** Chegaralanmagan. **10.** Yuqoridan chegaralangan, quyidan chegaralanmagan. **11.** Quyidan chegaralangan, yuqoridan chegaralanmagan. **48.** $\frac{1}{2}$. **29.** $\frac{1}{3}$. **30.** $\frac{1}{4}$. **31.** $\frac{1}{12}$.

32. 0. **33.** 14. **34.** $-\infty$. **35.** 0. **36.** e^{pq}. **37.** e. **38.** e^{-3}.

5.4-amaliy mashg'ulot.
Kompleks sonlar

1-misol. Quyidagi tenglamadan x va y haqiqiy sonlarni toping:

$$(5x-3y)+(x-2y)i=6+(8-x+y)i.$$

Yechilishi. Kompleks sonlarning tenglik shartidan foydalanib,

$$\begin{cases} 5x-3y=6 \\ x-2y=8-x+y \end{cases}$$

sistemani hosil qilamiz. Bu sistemadan x va y noma'lumlarni topamiz:

$$x=-\frac{2}{3},\ \ y=-\frac{28}{9}. \blacksquare$$

2-misol. i ning darajalarini toping.

Yechilishi. Ta'rifga ko'ra $i^0=1$, $i^1=i$ va $i^2=-1$. Shuning uchun $i^3=i^2i=-i$, $i^4=i^3i=1$, $i^5=i^4\cdot i=i$.

Umuman olganda: $i^{4n}=1$, $i^{4n+1}=i$, $i^{4n+2}=-1$, $i^{4n+3}=-i$, $n\in N$. ■

3-misol. Darajaga ko'taring: $(1+i)^{20}$, $(1-i)^{210}$.

Yechilishi. Bu masalani Nyuton binomi formulasidan foydalanib hal qilsa bo'ladi, lekin uni quyidagicha yozish qulayroq:

$$(1+i)^2=2i,\ (1-i)^2=-2i. \text{ U holda}$$

$$(1+i)^{20}=\left[(1+i)^2\right]^{10}=(2i)^{10}=-2^{10},$$

$$(1-i)^{21}=\left[(1-i)^2\right]^{10}(1-i)=(-2i)^{10}(1-i)=-2^{10}(1-i). \blacksquare$$

Kompleks koeffitsientli istagan kvadrat tenglamani yechish uchun, avvalo kompleks sonning kvadrat ildizini topib olish kerak. Ta'rifga ko'ra, $x+yi$ son

$a+bi$ sonning kvadrat ildizi bo'lishi:

$$(x+yi)^2=a+bi \qquad (1)$$

tenglikning bajarilishiga teng kuchli.

(1) tenglik quyidagi formulalar yordamida topiladigan ikkita har xil yechimlarga ega bo'ladi:

$$x=\pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}};\ \ y=\pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}},$$

bu yerda radikal arifmetik ildizni bildiradi, agar $b>0$ bo'lsa, x va y larning ishoralari bir xil qilib, $b<0$ bo'lganda esa har xil qilib tanlanadi.

4-misol. Ildizdan chiqaring: $\sqrt{5+12i}$

Yechilishi. $\sqrt{5+12i}=x+yi$ bo'lsin. Ildizning ta'rifiga ko'ra

$$(x+yi)^2=5+12i \text{ yoki } x^2-y^2+2xyi=5+12i,$$

bundan

$$\begin{cases} x^2-y^2=5, \\ 2xy=12 \end{cases}$$

sistemani hosil qilamiz.

Bu sistemadagi ikkala tenglikni kvadratga ko'tarib va ularni qo'shib, $(x^2+y^2)^2=25+144$ va $x^2+y^2=13$ larni hosil qilamiz.

U holda $\begin{cases} x^2+y^2=13, \\ x^2-y^2=5 \end{cases}$ sistemadan x va y noma'lumlarni topamiz:

$$x=\pm3,\ y=\pm2.$$

Oldingi sistemaning ikkinchi tenglamasidan x va y larning bir xil ishorali bo'lishi kelib chiqadi. Shuning uchun $x_1=3$, $y_1=2$; $x_2=-3$, $y_2=-2$. Shunday qilib, $\sqrt{5+12i}$ ildiz ikkita $3+2i$ va $-3-2i$ qiymatlarga ega.■

Kompleks sonning kvadrat ildizini topishni bilgan holda aynan maktab matematika kursidekdagi kompleks koeffitsientli $ax^2 + bx + c = 0$ tenglamaning ildizlari

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

formula yordamida topilishini ko'rsatish mumkin.

5-misol. Sistemani yeching:

$$\begin{cases} (1+i)z_1 + (1-i)z_2 = 1+i \\ (1-i)z_1 + (1+i)z_2 = 1+3i \end{cases}.$$

Yechilishi. Sistemadagi birinchi tenglamaning ikkala tomonini $(1-i)$ ga, ikkinchi tenglamaning ikkala tomonini esa $(1 + i)$ ga ko'paytirib

$$\begin{cases} 2z_1 - 2iz_2 = 2 \\ 2z_1 + 2iz_2 = -2+4i \end{cases}$$

ni hosil qilamiz.

Bu tenglamalarni qo'shib, $4z_1 = 4i$ ga kelamiz. Bundan $z_1 = i$.Birinchi tenglamadan ikkinchisini ayirib $-4z_2 i = 4 - 4i$ ni hosil qilamiz. Bundan $z_2 = \frac{-1+i}{i} = 1+i$.■

6-misol. a ning qanday haqiqiy qiymatlarida

$$4i^4 - 3ai^3 + (2-a)i - 5 + a$$

son haqiqiy bo'ladi?

Yechilishi. $i^4 = 1$, $i^3 = -i$ bo'lganligi sababli

$$4i^4 - 3a^2 + (2-a)i - 5 + a = (2a+2)i + a - 1.$$

Shuning uchun $2a + 2 = 0$ bo'lganda bu son haqiqiy bo'ladi, ya'ni $a = -1$. ■

7-misol. $z\bar{z} + 2\bar{z} = 3 + 2i$ tenglamani yeching.

Yechilishi. $z = x + yi$ bo'lsin. U holda $x^2 + y^2 + 2x - 2yi = 3 + 2i$. Haqiqiy va mavhum qismlarini tenglashtirib

$$\begin{cases} x^2 + y^2 + 2x = 3 \\ -2y = 2 \end{cases}$$

sistemani hosil qilamiz. Bundan $y = -1,\ x = -1 \pm \sqrt{3}$. Natijada,

$$z_1 = (-1+\sqrt{3}) - i,\quad z_2 = (-1-\sqrt{3}) - i. ■$$

8-misol. Tekislikda $|z-i| + |z+i| < 4$ tengsizlikni qanoatlantiruvchi kompleks sonlarni tasvirlaydigan nuqtalar to'plamini aniqlang.

Yechilishi. $z = x + iy,\ x, y \in \boldsymbol{R}$ bo'lsin. U holda $|x+yi-i| + |x+yi+i| < 4$. Bundan $\sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + (y+1)^2} < 4$.

Bu tengsizlikni soddalashtirib, unga teng kuchli bo'lgan

$$4x^2 + 3y^2 < 12 \quad \text{yoki} \quad \frac{x^2}{3} + \frac{y^2}{4} < 1$$

tengsizlikni hosil qilamiz. Shunday qilib, izlanayotgan to'plam tekislikning $\frac{x^2}{3} + \frac{y^2}{4} < 1$ ellips bilan chegaralangan qismidan iborat. ■

9-misol. Hisoblang: $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}$.

Yechilishi. Ushbu

$$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}),\ 1-i = \sqrt{2}(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi),$$

tengliklar o‘rinli bo‘lganligi sababli quyidagiga ega bo‘lamiz:

$$\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}=\left(\frac{2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})}{\sqrt{2}(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4}))}\right)^{20}=\sqrt{2}^{20}\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)^{20}=$$

$$=\left(\sqrt{2}\left(\cos\frac{7}{12}\pi+i\sin\frac{7}{12}\pi\right)\right)^{20}=2^{10}\left(\cos\frac{140}{12}\pi+i\sin\frac{140}{12}\pi\right)=$$

$$=2^{10}\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)=2^{10}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=2^{9}\sqrt{2}(1-i).\ \blacksquare$$

10-misol. $\sqrt[4]{-16}$ ildizning barcha qiymatlarini toping.

Yechilishi. $z=-16$ ni trigonometrik shaklga keltiramiz:

$$z=-16=16(\cos\pi+i\sin\pi).$$

U holda kompleks sondan ildiz chiqarish formulasiga ko‘ra,

$$w_{\kappa}=2\left(\cos\frac{\pi+2\pi\kappa}{4}+i\sin\frac{\pi+2\pi\kappa}{4}\right),\ k=0,1,2,3.$$

Natijada,

$$w_0=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}+i\sqrt{2},\ w_1=2\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)=-\sqrt{2}+i\sqrt{2},$$

$$w_2=2\left(\cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\right)=-\sqrt{2}-i\sqrt{2},\quad w_3=2\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)=\sqrt{2}-i\sqrt{2}.$$

■

11-misol. $\sqrt[4]{\dfrac{-1+i}{1-i\sqrt{3}}}$ to‘plam elementlarining trigonometrik shaklini yozing.

Yechilishi. $-1+i=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$ va

$1-i\sqrt{3}=2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$ bo‘lganligi uchun

$$\frac{-1+i}{1-i\sqrt{3}}=\frac{\sqrt{2}}{2}\left(\cos\left(\frac{3}{4}\pi+\frac{\pi}{3}\right)+i\sin\left(\frac{3}{4}\pi+\frac{\pi}{3}\right)\right)=\frac{\sqrt{2}}{2}\left(\cos\frac{13}{12}\pi+i\sin\frac{13}{12}\pi\right).$$

Natijada,

$$\sqrt[4]{\frac{-1+i}{1-i\sqrt{3}}}=\frac{1}{\sqrt[8]{2}}\left(\cos\frac{13\pi+24\kappa\pi}{48}+i\sin\frac{13\pi+24\kappa\pi}{48}\right),\ \kappa=0,1,2,3.\ \blacksquare$$

Mustaqil yechish uchun misollar

1. Berilgan z_1 va z_2 kompleks sonlarning yig‘indisi va ko‘paytmasini toping:

a) $z_1=5+4i$, $z_2=-2+3i$; b) $z_1=-8-7i$, $z_2=-3i$;

c) $z_1=5+\sqrt{3}i$, $z_2=5-\sqrt{3}i$.

2. z_2-z_1 ayirmani va $\frac{z_2}{z_1}$ bo‘linmani toping:

$z_1=1+2i$, $z_2=5$; b) $z_1=-1+\sqrt{3}i$, $z_2=-\sqrt{2}+\sqrt{6}i$;

c) $z_1=a-\sqrt{b}i$, $\quad z_2=a+\sqrt{b}i$.

3. Hisoblang:

a) $(4+i)(5+3i)-(3+i)(3-i)$; b) $\dfrac{(5+i)(7-6i)}{3+i}$; *c)* $\dfrac{(5+i)(3+5i)}{2i}$;

d) $\dfrac{(1+3i)(8-i)}{(2+i)^2}$; *e)* $\dfrac{(2+i)(4+i)}{1+i}$; *f)* $\dfrac{(3-i)(1-4i)}{z-i}$; *g)* $(2+i)^3+(2-i)^3$;

4. Kompleks sonning haqiqiy qismini toping:

$z=\dfrac{(1+2i)^3}{i}+i^{19}$; b) $z=\dfrac{5+2i}{2-5i}-\dfrac{3-4i}{4+3i}+\dfrac{1}{i}$.

5. Kompleks sonning mavhum qismini toping:

$z=(2-i)^3(2+11i)$ b) $z=\dfrac{2-3i}{1+4i}+i^6$.

6. Tenglikni isbotlang:

a) $(1+i)^{8n}=2^{4n}\quad (n\in \mathbf{Z})$; *b)* $(1+i)^{4n}=(-1)^n 2^{2n}\quad (n\in \mathbf{Z})$.

7. Tenglamalar sistemasini yeching:

a) $\begin{cases} iz_1+(1+i)z_2=2+2i \\ 2iz_1+(3+2i)z_2=5+3i \end{cases}$; b) $\begin{cases} (1-i)z_1-3z_2=-i \\ 2z_2-(3+3i)z_2=3-i \end{cases}$;

c) $\begin{cases} 2z_1-(2+i)z_2=-i \\ (4-2i)z_1-5z_2=-1-2i \end{cases}$

8. Hisoblang:

a) $i^4+i^{14}+i^{24}+i^{34}+i^{44}$;

b) $i+i^2+i^3+\dots i^n,\ n>4$;

c) $i\cdot i^2\cdot i^3\cdot i^4\dots i^{50}$.

9. $\omega=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ bo'lganda quyidagilarni hisoblang:

a) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$; b) $(a+b)(a+b\omega)(a+b\omega^2)$;

c) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$.

10. Tenglamani yeching:

$(i-z)(1+2i)+(1-iz)(3-4i)=1+7i$; b) $z^2+\bar{z}=0$;

c) $(1-i)\bar{z}-3iz=2-i$; d) $z\bar{z}+3(z-\bar{z})=4+3i$;

e) $z\bar{z}+3(z+\bar{z})=7$; f) $z\bar{z}+3(z+\bar{z})=3i$.

11. Hisoblang:

$\sqrt{2i}$; b) $\sqrt{-8i}$; c) $\sqrt{3-4i}$; d) $\sqrt{-15+8i}$; e) $\sqrt{-11+60i}$;

f) $\sqrt{-8-6i}$; q) $\sqrt{2-3i}$; h) $\sqrt{1-i\sqrt{3}}$; i) $\sqrt[4]{2-i\sqrt{12}}$; j) $\sqrt[4]{-1}$.

12. Tenglamani yeching:

$x^2-(2+i)x+(-1+7i)=0$; b) $x^2-(3-2i)x+(5-5i)=0$;

c) $(2+i)x^2-(5-i)x+(2-2i)=0$; d) $x^4-6x^2+25=0$;

13. Quyidagi kompleks sonlarni ifodalovchi nuqtalarni yasang:
$1; -1; i; -i; -1+i; 2-3i; -6+3i; \cos 30^\circ - i\sin 30^\circ$;
$\cos 150^\circ + i\sin 150^\circ$.

14. Kompleks tekislikda berilgan z_1, z_2, z_3 nuqtalar parallelogramning ketma-ket uchlaridan iborat. Bu parallelogramning to'rtinchi uchini toping.

15. Kompleks tekislikda $z_1=6+8i$, $z_2=4-3i$ nuqtalar berilgan.
z_1 va z_2 vektorlar hosil qilgan burchak bissektrisasining nuqtalariga mos keluvchi kompleks sonlarni toping.

16. Tenglamani yeching:

a) $|z|-iz=1-2i$; b) $z^2+3|z|=0$; c) $z^2+|z|^2=0$.

17. Tenglamalar sistemasini yeching: $|z+1-i|=|3+2i-z|=|z+i|$.

18. Tenglamalar sistemasini yeching: $\begin{cases} |z+1|=|z+2| \\ |3z+9|=|5z+10i|. \end{cases}$

19. Tekislikda quyidagi shartlarni qanoatlanturvchi z kompleks sonlarga mos keladigan nuqtalar to'plamini tasvirlang:

$|z|=1$; b) $\arg z=\frac{\pi}{3}$; c) $|z|\le 2$; d) $|z-1-i|<1$; e) $|z+3+4i|\le 5$.

20. $|z+1-i|\le 1$ shartni qanoatlantiruvchi z kompleks sonlar ichidan eng kichik musbat argumentga ega bo'lgan sonni toping.

21. $|z-5i|\le 3$ shartni qanoatlantiruvchi z kompleks sonlar ichidan eng kichik musbat argumentga ega bo'lgan sonni toping.

22. *Oxy* tekislikdagi qanday *M(x,y)* nuqtalar uchun quyidagi tengliklar o'rinli:

a) $\left|\sqrt{2x+y}+i\sqrt{x+2y}\right|=\sqrt{3}$. b) $\left|\sqrt{x^2+4}+i\sqrt{y-4}\right|=\sqrt{10}$?

23. Kompleks sonlarni trigonometrik shaklga keltiring:
a) 7; b) i; c) -3; d) $-5i$; e) $1+i\sqrt{3}$; f) $-1+i\sqrt{3}$ g) $1-i\sqrt{3}$; h) $\sqrt{3}+i$;
i) $-\sqrt{3}+i$; j) $-\sqrt{3}-i$; k) $\sqrt{3}-i$; l) $1+i\dfrac{\sqrt{3}}{3}$;

24. Kompleks sonlarni algebraik va trigonometrik shaklga keltiring:
a) $\dfrac{i(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi)}{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}$; b) $\dfrac{1}{\cos\frac{4}{3}\pi-i\sin\frac{4}{3}\pi}$; c) $\dfrac{i}{(1+i)^2}$;

25. Kompleks sonlarni trigonometrik shaklga keltiring:

a) $\dfrac{5(\cos100^0+i\sin100^0)i}{3(\cos40^0-i\sin40^0)}$; b) $\dfrac{\sin\frac{2}{5}\pi+i(1-\cos\frac{2}{5}\pi)}{i-1}$.

26. Hisoblang: a) $\left(\dfrac{1+i\sqrt{3}}{1-i}\right)^{20}$; b) $\left(1-\dfrac{\sqrt{3}-i}{2}\right)^{24}$;
c) $\dfrac{\left(-1+i\sqrt{3}\right)^{15}}{(1-i)^{20}}+\dfrac{\left(-1-i\sqrt{3}\right)^{15}}{(1+i)^{20}}$; d) $\dfrac{(1+i)^{2n+1}}{(1-i)^{2n-1}}$, $n\in N$; e) $z=(tg1-i)^4$;

27. Isbotlang: $\left(\dfrac{1+itg\alpha}{1-itg\alpha}\right)^n=\dfrac{1+itg\alpha n}{1-itg\alpha n}$.

28. Ildizning qiymatlarini trigonometrik shaklda yozing:
a) $\sqrt[6]{i}$; b) $\sqrt[10]{512\left(1-i\sqrt{3}\right)}$; c) $\sqrt[8]{8\sqrt{2}(1-i)}$.

29. Ildizning qiymatlarini algebraik shaklda yozing:
a) $\sqrt[3]{1}$; b) $\sqrt[4]{1}$; c) $\sqrt[6]{1}$; d) $\sqrt[3]{i}$; e) $\sqrt[4]{-4}$; f) $\sqrt[6]{64}$; g) $\sqrt[8]{16}$; h) $\sqrt[6]{-27}$;

30. Tenglamani yeching: a) $z^5-1-i\sqrt{3}=0$; b) $z^6+64=0$.

Mustaqil yechish uchun berilgan misollarning javoblari

1. a) $3+7i$; $-22+7i$; b) $-8-10i$; $21-24i$; c) 10; 28.

2. a) $4-2i$; $1-2i$; b) $\left(1-\sqrt{2}\right)+\left(\sqrt{6}-\sqrt{3}\right)i$; $\sqrt{2}$; c) $2\sqrt{b}i$, $\dfrac{a^2-b}{a^2+b}+\dfrac{2a\sqrt{b}}{a^2+b}i$;

3. a) $7+17i$; b) $10-11i$; c) $14-5i$; d) $5+i$; e) $\dfrac{13}{2}-\dfrac{1}{2}i$; f) $\dfrac{11}{5}-\dfrac{27}{5}i$; d) 4; 1.

4. a) -2; b) 0.

5. a) 0; b) $-\dfrac{11}{17}$.

7. a) $z_1=2,\quad z_2=1-i$; b) $\varnothing$; c) $z_1=\dfrac{(2+i)z_2-i}{2}$.

8. a) 1; b) 0, agar $n=4\kappa$; i, agar $n=4\kappa+1$; i-1, agar $n=4\kappa+2$; -1, agar $n=4\kappa+3$; c) $-i$.

9. a) $a^2+b^2+c^2-(ab+bc+ac)$; b) a^3+b^3;
c) $2\left(a^3+b^3+c^3\right)-3\left(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\right)+12abc$.

10. a) $-1-i$; b) $0,-1$, $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$; c) i; d) $z_1=\dfrac{\sqrt{15}}{2}+\dfrac{1}{2}i$, $z_2=\dfrac{\sqrt{15}}{2}+\dfrac{1}{2}i$;
e) $\left\{x+yi\,|\,-7\le x\le 1,\ y=\pm\sqrt{7-6x-x^2}\right\}$; f) $\varnothing$,

11. a) $\pm(1+i)$; b) $\pm(2-2i)$; c) $\pm(2-i)$; d) $\pm(1+4i)$; e) $\pm(5+6i)$; f) $\pm(1-3i)$; g) $\pm\left(\sqrt{\dfrac{\sqrt{13}+2}{2}}-i\sqrt{\dfrac{\sqrt{13}-2}{2}}\right)$; h) $\pm\left(\sqrt{\dfrac{3}{2}}-i\sqrt{\dfrac{1}{2}}\right)$; i) $i^{\alpha}\left(\dfrac{1+\sqrt{3}}{2}+\dfrac{1-\sqrt{3}}{2}i\right)$, $\alpha=0,1,2,3$; j) $\dfrac{\sqrt{2}(\pm1\pm i)}{2}$.

12. a) $x_1=3-i$; $x_2=-1+2i$; b) $x_1=2+i$, $x_2=1-3i$; c) $x_1=1-i$; ;
d) $x_1=2-i$, $x_2=-2+i$, $x_3=2+i$, $x_4=-2-i$;

14. $z_4=z_1+z_3-2z_2$.

15. $t(7+i)$, t –ixtiyoriy musbat son.

16. a) $2-\frac{3}{2}i$; 0, $3i$, $-3i$; c) bi, $b\in \mathbf{R}$.

17. $\frac{7}{6}+\frac{5}{6}i$.

18. $-\frac{3}{2}-\frac{17}{4}i$; $-\frac{3}{2}-2i$.

19. a) $\pm\frac{1}{2}\pm\frac{1}{2}i$; b) -1, $\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; c) $4+i\sqrt{3}$, $2+2i\sqrt{3}$, $1+2i\sqrt{3}$, $i\sqrt{3}$, 1, 3.

20. i. **21.** $\frac{12}{5}+\frac{16}{5}i$.

23. a) $7(\cos 0+i\sin 0)$; b) $\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$; c) $3(\cos\pi+i\sin\pi)$;

d) $5\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$; e) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$; f) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$;

g) $2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$; h) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$; i) $2\left(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi\right)$;

j) $2\left(\cos\left(-\frac{5}{6}\pi\right)+i\sin\left(-\frac{5}{6}\pi\right)\right)$; k) $2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$;

l) $\frac{2}{\sqrt{3}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$;

24. a) $1=\cos 0+i\sin 0$; b) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4}{3}\pi+\sin\frac{4}{3}\pi$; c) $\frac{1}{2}=\frac{1}{2}(\cos 0+\sin 0)$;

25. a) $\frac{5}{3}(\cos 230^0+i\sin 230^0)$; b) $\sqrt{2}\sin\frac{\pi}{5}\left(\cos\frac{29}{20}\pi+i\sin\frac{29}{20}\pi\right)$.

26. a) $2^9(1-i\sqrt{3})$; b) $(2-\sqrt{3})^{12}$; c) – 64; d) 2, agar n – juft, –2, agar n – toq; e) $\frac{1}{\cos^4 1}(\cos 4+i\sin 4)$; f) $\frac{1}{\cos^4 2}(\cos 2+i\sin 2)$; g) $-32i\cos^5\frac{3\pi}{5}$.

28. a) $\cos\frac{(4k+1)\pi}{12}+i\sin\frac{(4k+1)\pi}{12}$ $(0\le k\le 5)$;

b) $(\cos\frac{(6k-1)\pi}{30}+i\sin\frac{(6k-1)\pi}{30}$ $(0\le k\le 9)$;

s) $\sqrt{2}\left(\cos\frac{(8k-1)\pi}{32}+i\sin\frac{(8k-1)\pi}{32}\right)(0\le k\le 7)$.

29. a) $\left\{1,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right\}$; b) $\{\pm 1,\pm i\}$; c) $\left\{\pm 1,\pm\frac{1+i\sqrt{3}}{2};\pm\frac{1-i\sqrt{3}}{2}\right\}$;

d) $\left\{\frac{\sqrt{3}}{2}+\frac{1}{2}i,-\frac{\sqrt{3}}{2}+\frac{1}{2}i,-i\right\}$; e) $\{1\pm i;-1\pm i\}$; f) $2\sqrt[6]{1}$(см(c));

g) $\{\pm\sqrt{2},\pm\sqrt{2}i,\pm\sqrt{2}(1+i),\pm\sqrt{2}(1-i)\}$; h) $\left\{\pm i\sqrt{3},\pm\frac{\sqrt{3}}{2}(\sqrt{3}+i),\pm\frac{\sqrt{3}}{2}(\sqrt{3}-i)\right\}$.

30. a) $\sqrt[5]{2}\left(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15}\right)$; $\sqrt[5]{2}\left(\cos\frac{7\pi}{15}+i\sin\frac{7\pi}{15}\right)$; $\sqrt[5]{2}\left(\cos\frac{13\pi}{15}+i\sin\frac{13\pi}{15}\right)$;

$\sqrt[5]{2}\left(\cos\frac{19\pi}{5}+i\sin\frac{19\pi}{5}\right)$; $\sqrt[5]{2}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

b) $\sqrt{3}+i$, 2i, $-\sqrt{3}+i$, $-\sqrt{3}-i$, $-2i$, $\sqrt{3}-i$.

5-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

5.1- masala. A va B to'plamlar berilganda, $A\cup B$, $A\backslash B$, $A\cap B$ larni toping.

5.1.1. $A=\{-10,-9,-8,-7,-6,-5,-4,-3,-2,-1, 0, 1, 3, 5, 7, 9\}$, $B=\{-8,-6,-4,-2, 0, 2, 4, 6, 8, 10\}$.

5.1.2. $A=\{2, 4, 6, 8, 10, 12,.......\}$, $B=\{3, 6, 9, 12, 15, 18,......\}$.

5.1.3 $A=\{x\in N: 2<x\le 6\}$, $B=\{x\in N: 1<x<4\}$, $C=\{x\in N: x^2-4=0\}$ to'plamlar berilganda: 1) $B\cup C$; 2) $A\cap B\cap C$; 3) $A\cup B\cup C$ larni toping.

5.1.4. A va B to'plamlar, mos ravishda, $a=4n+2$, $b=3n$, $n\in N$ elementlardan tashkil topgan. $A\cap B$ ni toping.

5.1.5. A, B va C to'plamlar, mos ravishda, $A=\{x: x\in N,\ \ 3\le x<7\}$,

$B=\{x: x\in N,\ 1<x<7\}$, $C=\{x: x\in N,\ x^2-5=0\}$ ko'rinishda berilganda: 1) $A\cup B\cup C$, 2) $A\cap C$; 3) $A\cap B\cap C$ to'plamlarni toping.

5.1.6. A va B mos ravishda, $A=\{x: x\in N;\ x^2-5x+6\le 0\}$, $B=\{x: x\in N: 2x^2-5x<0\}$ ko'rinishda berilganda: 1) $A\cup B$; 2) $A\cap B$; 3) A/B; 4) B/A to'plamlarni toping.

5.1.7. $B=\{(x,y)\in R\times R:\ x+y\ge 1\}$, $A=\{(x,y)\in R\times R:\ |x|+|y|\le 1\}$ to'plamlar berilganda: 1) $A\cup B$; 2) $A\cap B$; 3) $A\setminus B$, 4) $B|A$ to'plamlarni toping.

5.1.8. A, B va C to'plamlar, mos ravishda,

$A=\{x: x\in N,\ 2<x\le 6\}$, $B=\{x: x\in N, 1<x<4\}$, $C=\{x: x\in N,\ x^2-4=0\}$ ko'rinishda berilganda: 1) $B\cup C$; 2) $(A\cap B)\cup(B\cup C)$; 3) BxC; 4) CxB to'plamlarni toping.

5. 1.9. 1) $A\cup B=B$; 2) $A\cap B=A$ tengliklarning o'rinli bo'lishi uchun $A\subset B$ bo'lishi zarur va yetarli ekanligini isbotlang.

5.1.10. $A\setminus(B\setminus C)=(A\setminus B)\cup C$ tenglik o'rinli bo'lishi uchun $C\subset A$ bo'lishi zarur va yetarli ekanligini isbotlang.

5.1.11. $A=\{x: x\in R,\ x>1\}$, $B=\{x: x\in R,\ x<2\}$ bo'lganda $A\cup B$, $A\cap B$ to'plamlarni toping.

5.1.12. $A=\{1;2\}$, $B=\{2;3\}$ bo'lganda, $A\times B$ to'plamni toping.

5.1.13. $A=\{1;2\}$, $B=\{2;3\}$ bo'lganda, $B\times A$ to'plamni toping.

5.1.14. $A=\{1,2,3\}$, $B=\{2,3,4,5\}$ bo'lganda, $A\setminus B$ va $B\setminus A$ to'plamlarni toping.

5.1.15. $A=\{x: x\in R;\ x>1\}$, $B=\{x: x\in R;\ x<2\}$ bo'lganda, $A\setminus B$ va $B\setminus A$ to'plamlarni toping.

5.1.16. $A=\{\pm 2;\pm 4;\pm 6;\pm 8;\pm 10;\pm 12;\pm 14,...\}$
$B=\{\pm 3;\pm 6;\pm 9;\pm 12,......\}$ to'plamlar berilganda, $A\setminus B$ to'plamni toping.

5.1.17. $A=\{\pm 2;\pm 4;\pm 6;.......\}$, $B=\{\pm 3;\pm 6;\pm 9;\pm 12,....\}$ to'plamlar berilganda, $A\cup B$ to'plamni toping.

5.1.18. $\{a,b,c\}$ to'plamning kism to'plamlarini toping.

5.1.19. $A=\{-1;1\}$, B esa, $(x-1)^2(x+1)^2=0$ tenglamaning ildizlaridan iborat bo'lsa, u holda A va B to'plamlarni solishtiring.

5.1.20. A to'plam n elementdan, B to'plam m elementdan, $A\cap B$ to'plam esa k elementdan tashkil topgan bo'lsa, u xolda: 1) $A\cup B$; 2) $A\times B$ to'plamlarning elementlari sonini aniklang.

5.1.21. $A=\{x: x\in R,\ x>2\}$, $B=\{x: x\in R,\ x<3\}$ bo'lganda $A\cup B$, $A\cap B$ to'plamlarni toping.

5.1.22. $A=\{3;2\}$, $B=\{3;4\}$ bo'lganda, $A\times B$ to'plamni toping.

5.1.23. $A=\{5;6\}$, $B=\{6;7\}$ bo'lganda, $B\times A$ to'plamni toping.

5.1.24. $A=\{2,3,4\}$, $B=\{1,2,4,5\}$ bo'lganda, $A\setminus B$ va $B\setminus A$ to'plamlarni toping.

5.1.25. $A=\{x: x\in R;\ x>6\}$, $B=\{x: x\in R;\ x<7\}$ bo'lganda, $A\setminus B$ va $B\setminus A$ to'plamlarni toping.

5.1.26-misol. Agar $A=\{x: x\in N,\ x^2-4x\le 0\}$, $B=\{x: x\in Z,\ x^2-x-6\le 0\}$ bo'lsa, 1) $A\cup B$; 2) $A\cap B$; 3) $A\setminus B$, 4) $B\setminus A$ to'plamlarni toping.

Yechilishi ([1],1-t., 3-bo'lim; [2], 1-bo'lim; [3], 26-27 betlar). Ravshanki, $x^2-4x\le 0$ tengsizlikning natural yechimlari 1, 2, 3, 4 lardan iborat bo'lib, $A=\{1,2,3,4\}$ to'plamni tashkil qiladi. $x^2-x-6\le 0$ tengsizlikning butun yechimlari $-2,-1$, 0, 1, 2, 3 lardan iborat bo'lib, ular $B=\{-2,-1,0,1,2,3,\}$ to'plamni tashkil yetadi. 1.4-,1.5-,1.6-ta'riflarga ko'ra, 1) $A\cup B=\{-2;-1;0;1;2;3;4\}$; 2) $A\cap B=\{1;2;3\}$; 3) $A\setminus B=\{4\}$; 4) $B\setminus A=\{-2;-1;0\}$ dan iborat bo'ladi.

5.2- masala. Quyidagi sonlar ketma – ketligining limitini hisoblang.

5.2.1. $\lim\limits_{n\to\infty}\dfrac{(2-n)^2+(2+n)^2}{(2-n)^2-(2+n)^2}$.

5.2.2. $\lim\limits_{n\to\infty}\dfrac{(3-n)^4-(2-n)^4}{(2-n)^3-(2+n)^3}$.

5.2.3. $\lim\limits_{n\to\infty}\dfrac{(5-n)^2-(5+n)^2}{(5+n)^2-(2-n)^2}$.

5.2.4. $\lim\limits_{n\to\infty}\dfrac{(1+3n)^3-27n^3}{(1+2n)^2+6n^2}$.

5.2.5. $\lim\limits_{n\to\infty}\dfrac{(4-n)^2}{(n+1)^2-(n+1)^3}$.

5.2.6. $\lim\limits_{n\to\infty}\dfrac{(n+2)^3+(n+3)^3}{(n+1)^4-(n+2)^4}$.

5.2.7. $\lim\limits_{n\to\infty}\dfrac{6n^2-3n}{(n+1)^3-(n-1)^3}$.

5.2.8. $\lim\limits_{n\to\infty}\dfrac{(2n-3)^3-(n+5)}{(3n-1)^3+(2n+3)}$.

5.2.9. $\lim\limits_{n\to\infty}\dfrac{(2n+1)^3-(4n+3)^3}{(2n+1)^3-(n-2)^3}$.

5.2.10. $\lim\limits_{n\to\infty}\dfrac{(n+2)^3-(n-2)^3}{(n+1)^2-(n-1)^2}$.

5.2.11. $\lim\limits_{n\to\infty}\dfrac{(n+1)^3+(n-1)^3}{n^4+3n^2-1}$.

5.2.12. $\lim\limits_{n\to\infty}\dfrac{(3-n)^4-(2-n)^4}{(1-n)^4-(1+n)^4}$.

5.2.13. $\lim\limits_{n\to\infty}\dfrac{(1-n)^4-(1+n)^4}{(1+n)^3-(1-n)^3}$.

5.2.14. $\lim\limits_{n\to\infty}\dfrac{(n+2)^3-(n+2)^2}{(n-1)^3-(n+1)^3}$.

5.2.15. $\lim\limits_{n\to\infty}\dfrac{(4-3n)^2}{(n-3)^3-(n+3)}$.

5.2.16. $\lim\limits_{n\to\infty}\dfrac{(n+1)^2+(n-1)^2-(n+2)^3}{(3-n)^3}$.

5.2.17. $\lim\limits_{n\to\infty}\dfrac{(n+1)^4-(n-1)^4}{(n+1)^3+(n-1)^3}$.

5.2.18. $\lim\limits_{n\to\infty}\dfrac{(n+5)^2+(2n+1)^2}{(n+3)^3-(n+2)^3}$.

5.2.19. $\lim\limits_{n\to\infty}\dfrac{n^3-(n-1)^3}{(n+1)^4-n^4}$.

5.2.20. $\lim\limits_{n\to\infty}\dfrac{(n+2)^4-(n-2)^4}{(n+2)^3+(n-2)^3}$.

5.2.21. $\lim\limits_{n\to\infty}\dfrac{(3-n)^2+(3+n)^2}{(3-n)^2-(3+n)^2}$.

5.2.22. $\lim\limits_{n\to\infty}\dfrac{(7-n)^4-(4-n)^4}{(8-n)^3-(5+n)^3}$.

5.2.23. $\lim\limits_{n\to\infty}\dfrac{(9-n)^2-(7+n)^2}{(2+n)^2-(2-n)^2}$.

5.2.24. $\lim\limits_{n\to\infty}\dfrac{(1+4n)^3-7n^3}{(1+4n)^2+6n^2}$.

5.2.25. $\lim\limits_{n\to\infty}\dfrac{(4-2n)^2}{(3n+1)^2-(3n+1)^3}$.

5.2.26-misol. $\lim\limits_{x\to\infty}\dfrac{(n-3)^4+(n+2)^4}{(n+5)^4+3}$ limitni hisoblang.

Yechilishi ([2], 2-bo'lim; [3], 1-q., 37-44 betlar) Bu limitni hisoblashda $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}=\dfrac{\lim\limits_{n\to\infty}x_n}{\lim\limits_{n\to\infty}y_n}=\dfrac{b}{c}$ formulani bevosita qo'llab bo'lmaydi, chunki berilgan kasr ifodaning surati va maxraji chekli limitga ega emas, ya'ni $\lim\limits_{n\to\infty}(n-3)^4+(n+2)^4=\infty$, $\lim\limits_{n\to\infty}((n+5)^4+3)=\infty$. Shunday qilib, $\dfrac{(n-3)^4+(n+2)^4}{(n+5)^4+3}$ ifoda $n\to\infty$ da $\dfrac{\infty}{\infty}$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu limitni

hisoblashda, ya'ni aniqmaslikni ochish uchun kasr ifodaning surat va maxrajini n^4 ga bo'lamiz.

$$\lim_{n\to\infty}\frac{\left(1-\frac{3}{n}\right)^4+\left(1+\frac{2}{n}\right)^4}{\left(1+\frac{5}{n}\right)^4+\frac{3}{n^4}}.$$

Endi shakl o'zgartirish natijasida xosil bo'lgan kasr ifodaning limitini hisoblashda yuqorida keltirilgan formulani qo'llash mumkin. Shunday qilib,

$$\lim_{n\to\infty}\frac{(n-3)^4+(n+2)^4}{(n+5)^4+3}=\lim_{n\to\infty}\frac{\left(1-\frac{3}{n}\right)^4+\left(1+\frac{2}{n}\right)^4}{\left(1+\frac{5}{n}\right)^4+\frac{3}{n^4}}=\frac{(1-0)^4+(1+0)^4}{(1+0)^4+0}=2.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit(((n-3)^4+(n+4)^4)/((n+5)^4+3),n=infinity)=limit(((n-3)^4+(n+4)^4)/((n+5)^4+3),n=infinity);

$$\lim_{n\to\infty}\frac{(n-3)^4+(n+4)^4}{(n+5)^4+3}=2$$

5.3 - masala. Quyidagi sonlar ketma – ketligining limitini hisoblang.

5.3.1. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^3+1}-\sqrt{n-1}}{\sqrt[3]{n^3+1}-\sqrt{n-1}}$.

5.3.2. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+2}-\sqrt{n^2+2}}{\sqrt[4]{4n^4+1}-\sqrt{n^4-1}}$.

5.3.3. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^5+3}-\sqrt{n-3}}{\sqrt[5]{n^5+3}+\sqrt{n-3}}$.

5.3.3. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{2n^3-7}+\sqrt[3]{2n^2+3}}{\sqrt[4]{n^5+5}+\sqrt{n}}$.

5.3.5. $\lim\limits_{n\to\infty}\dfrac{4n^2-\sqrt[4]{n^3}}{\sqrt[3]{n^6+n^3+1}-5n}$.

5.3.6. $\lim\limits_{n\to\infty}\dfrac{n\sqrt[4]{11n}+\sqrt{25n^4-81}}{(n-7\sqrt{n})\sqrt{n^2-n+1}}$.

5.3.7. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^7+3}-\sqrt{n-3}}{\sqrt[7]{n^7+3}+\sqrt{n-3}}$.

5.3.8. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+2}-\sqrt[3]{n^3+2}}{\sqrt[7]{n+2}-\sqrt[5]{n^5+2}}$.

5.3.9. $\lim\limits_{n\to\infty}\dfrac{n^2-\sqrt{n^3+1}}{\sqrt[3]{n^6+2}-n}$.

5.3.10. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n-1}-\sqrt{n^2+1}}{\sqrt[3]{3n^3+3}+\sqrt[4]{n^5+1}}$.

5.3.11. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2-1}+7n^3}{\sqrt[4]{n^{12}+n+1}-n}$.

5.3.12. $\lim\limits_{n\to\infty}\dfrac{n\sqrt[5]{n}-\sqrt[3]{27n^6+n^2}}{\left(n+\sqrt[4]{n}\right)\sqrt{9+n^2}}$.

5.3.13. $\lim\limits_{n\to\infty}\dfrac{\sqrt{5n+2}-\sqrt[3]{8n^3+5}}{\sqrt[4]{n+7}-n}$.

5.3.14. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt{n^2-3}}{\sqrt[3]{n^5-4}-\sqrt[4]{n^4+1}}$.

5.3.15. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n}-9n^2}{3n-\sqrt[4]{9n^8+1}}$.

5.3.16. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^6+4}+\sqrt{n-4}}{\sqrt[6]{n^6+6}-\sqrt{n-6}}$.

5.3.17. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt[3]{8n^3+3}}{\sqrt[4]{n+4}-\sqrt[5]{n^5+5}}$.

5.3.18. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2}-\sqrt{n^2+5}}{\sqrt[5]{n^7}-\sqrt{n+1}}$.

5.3.19. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2+2}-5n^2}{n-\sqrt{n^4-n+1}}$.

5.3.20. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^8+6}-\sqrt{n-6}}{\sqrt[8]{n^8+6}+\sqrt{n-6}}$.

5.3.21. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+1}-\sqrt{n^2-1}}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}}$.

5.3.22. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+2}-\sqrt{n^2+2}}{\sqrt[4]{n^4+1}-\sqrt{n^4-1}}$.

5.3.23. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^3+3}-\sqrt{n-3}}{\sqrt[3]{n^3+3}+\sqrt{n-3}}$.

5.3.24. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{2n^3-9}+\sqrt[3]{2n^2+4}}{\sqrt[4]{n^3+5}+\sqrt{n}}$.

5.3.25. $\lim\limits_{n\to\infty}\dfrac{4n^2-\sqrt[4]{n^5}}{\sqrt[3]{n^3+n^2+1}-5n^2}$.

5.3.26-misol. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^4+3}-\sqrt{n^3-3}}{\sqrt[10]{n^{20}+3}+\sqrt{n-3}}$ limitni hisoblang.

Yechilishi ([12], 1-q., 3-bo'lim; [3], 1-q., 37-44 betlar). Bu limitni hisoblashda ham $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}=\dfrac{\lim\limits_{n\to\infty}x_n}{\lim\limits_{n\to\infty}y_n}=\dfrac{b}{c}$ formulani bevosita qo'llab bo'lmaydi, chunki berilgan kasr ifodaning surati va maxraji chekli limitga ega emas. Bu limitni hisoblashda kasr ifodaning surati va maxrajidagi irrasional ifodadan n ni chiqarib, uning eng katta darajasiga bo'lamiz:

$$\lim_{n\to\infty}\frac{n^2\sqrt{1+\dfrac{3}{n^4}}-n^{\frac{3}{2}}\sqrt{1-\dfrac{3}{n^3}}}{n^2\sqrt[10]{1+\dfrac{3}{n^{20}}}+n^{\frac{1}{2}}\sqrt{1-\dfrac{3}{n}}}=\lim_{n\to\infty}\frac{\sqrt{1+\dfrac{3}{n^4}}-n^{-\frac{1}{2}}\sqrt{1-\dfrac{3}{n^3}}}{\sqrt[10]{1+\dfrac{3}{n^{20}}}+n^{-\frac{3}{2}}\sqrt{1-\dfrac{3}{n}}}=1.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

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> Limit((sqrt(n^4+3)-sqrt(n^3-3))/((n^20+3)^(1/10)+sqrt(n-3)),n=infinity)=limit(((sqrt(n^4+3)-sqrt(n^3-3))/((n^20+3)^(1/10)+sqrt(n-3))),n=infinity);
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$$\lim_{n\to\infty}\frac{\sqrt{n^4+3}-\sqrt{n^3-3}}{(n^{20}+3)^{(1/10)}+\sqrt{n-3}}=1$$

6-bob. FUNKSIYANING LIMITI VA UZLUKSIZLIGI

6.1-§. Funksiyaning limiti

6.1. Ixtiyoriy argumentli funksiyaning limiti

$f(x)$ funksiya biror X ($X \subset R$)– haqiqiy sonlar to'plamida aniqlangan bo'lsin. a nuqta-biror son yoki $-\infty, +\infty, \infty$ simvollarning biri bo'lsin.

6.1.1-ta'rif. Agar $U_\varepsilon(a)$ $(\varepsilon > 0)$ atrofda hech bo'lmaganda X to'plamning a dan farqli bitta nuqtasi joylashsa, a nuqta X to'plamning *limit nuqtasi (quyuqlanish nuqtasi)* deb ataladi.

Misollar: 1) Ushbu $X = [0;3] = \{x : x \in R,\ 0 \le x \le 3\}$ to'plamning har bir nuqtasi uning limit nuqtasi bo'ladi.

2) $X = N = \{1, 2, 3, \ldots, n, \ldots\}$ to'plam limit nuqtaga ega emas.

3) $X=(0,1)=\{x :\ x \in R,\ 0<x<1\}$ to'plamning har bir nuqtasi shu to'plamning limit nuqtasi bo'ladi va yana 0 va 1 nuqtalar ham $(0;1)$ to'plamning limit nuqtalari bo'ladi, lekin to'plamga tegishli emas.

Limit nuqta quyidagi xossalarga ega:

1^0. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, a nuqtaning ixtiyoriy atrofida X to'plamning cheksiz ko'p nuqtalari (elementlari) joylashgan bo'ladi.

2^0. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, X to'plam nuqtalaridan (elementlaridan) har doim a ga intiluvchi $\{x_n\}$ $(x_n \in X, x_n \ne a; n = 1,2,\ldots)$ ketma-ketlik tuzish mumkin.

6.1.2-ta'rif (Geyne ta'rifi). Agar X to'plamning nuqtalaridan tuzilgan va a ga intiluvchi har qanday $\{x_n\}$ $(x_n \in X, x_n \ne a; n = 1,2,\ldots)$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona (chekli yoki cheksiz) b limitga intilsa, shu b limit $f(x)$ funksiyaning *a nuqtadagi (yoki $x \to a$ dagi) limiti* deb ataladi va u $\lim\limits_{x \to \infty} f(x) = b$ yoki $x \to a$ da $f(x) \to b$ kabi belgilanadi.

6.1.3-ta'rif (Koshi ta'rifi). Agar istalgan $\forall \varepsilon > 0$ con uchun shunday $\delta(\varepsilon) > 0$ son topilsaki, argument x ning $0 < |x - a| < \delta, x \in X$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning *a nuqtadagi ($x \to a$ dagi) limiti* deb ataladi.

6.1.4-eslatma. **6.1.2**-va **6.1.3**-ta'riflar o'zaro ekvivalent ta'riflardir.

Agar

$$\forall \varepsilon > 0\ \exists C = C(\varepsilon) > 0 : \forall x,\ x > C,\ x \in X,\ |f(x) - b| < \varepsilon;\ \forall x,\ x < -C,\ x \in X,\ |f(x) - b| < \varepsilon;$$
$$\forall x,\ |x| > C,\ x \in X,\ |f(x) - b| < \varepsilon$$

u holda ular, mos ravishda, $\lim\limits_{x \to +\infty} f(x) = b;\ \lim\limits_{x \to -\infty} f(x) = b;\ \lim\limits_{x \to \infty} f(x) = b$ kabi yoziladi.

6.1.5-ta'rif (funksiya limitining $x \to \infty$ da Geyne ta'rifi). Agar X to'plamning nuqtalaridan (elementlaridan) tuzilgan har qanday cheksiz katta $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b ga intilsa, shu b ga $f(x)$ funksiyaning *$x \to \infty$ dagi limiti* deyiladi va u $\lim\limits_{x \to \infty} f(x) = b$ kabi belgilanadi.

6.1.6-ta'rif (funksiya limitining $x \to +\infty$ $(x \to -\infty)$ da Geyne ta'rifi). Agar X to'plamning musbat (manfiy) elementlaridan tuzilgan har qanday cheksiz katta $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b limitga intilsa, shu b limit $f(x)$ funksiyaning *$x \to +\infty$ $(x \to -\infty)$ dagi limiti* deyiladi va u $\lim\limits_{x \to +\infty} f(x) = b$ $(\lim\limits_{x \to -\infty} f(x) = b)$ kabi belgilanadi.

6.1.7-ta'rif (funksiyaning chekli nuqtadagi cheksiz limitlari). Agar istalgan $\forall E > 0$ son uchun shunday $\exists \delta = \delta(E) > 0$ mavjud bo'lib, $\forall x \in \dot{U}_E(a) \cap X$ uchun $|f(x)| > E$ tengsizlik o'rinli bo'lsa, ya'ni $f(x) \in U_E(\infty) = \{x : x \in R, |x| > E\}$, u holda $f(x)$ funksiya *$x \to a$ $(x = a)$ da ∞ limitga ega* deyiladi va $\lim\limits_{x \to a} f(x) = \infty$ kabi yoziladi.

6.1.8-ta'rif. Agar $\forall E > 0$ son uchun shunday $\exists \delta = \delta(E) > 0 : \forall x \in \dot{U}_\delta(a) \cap X$ uchun $f(x) > E$ $(f(x) < -E)$ bo'lsa, u holda $f(x)$ funksiya $x \to a$ $(x = a)$ da $+\infty$ $(-\infty)$ *limitga ega* deyiladi va $\lim\limits_{x \to a} f(x) = +\infty$ $(\lim\limits_{x \to a} f(x) = -\infty)$ kabi yoziladi.

6.1.9-ta'rif (Geyne ta'rifi). Agar X to'plamning nuqtalaridan (elementlaridan) tuzilgan va har biri hadi a dan katta (kichik) bo'lib, a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b songa intilsa, shu b son $f(x)$ funksiyaning *a nuqtadagi o'ng (chap)* limiti deb ataladi.

6.1.10-ta'rif (Koshi ta'rifi). Agar istalgan $\forall \varepsilon > 0$ son uchun shunday $\delta = \delta(\varepsilon) > 0$ son topilsaki, argument x ning $U_\delta^+(a)$ $(U_\delta^-(a))$ atrofdagi barcha qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa. b son $f(x)$ funksiyaning a

nuqtadagi *o'ng (chap) limiti* deb ataladi va u, mos ravishda, quyidagicha yoziladi:

$$\lim_{x\to a+0} f(x)=b \text{ yoki } f(a+0)=b \quad (\lim_{x\to a-0} f(x)=b \text{ yoki } f(a-0)=b).$$

Ravshanki, **6.1.9**-va **6.1.10**-ta'riflar o'zaro ekvivalent ta'riflardir.

6.1.11-eslatma. Funksiyaning biror nuqtada bir tomonli limitlari mavjud bo'lishidan, uning shu nuqtada limitga ega bo'lishi har doim ham kelib chiqavermaydi.

6.1.12-teorema. $f(x)$ funksiyaning a nuqtada b limitga ega bo'lishi uchun, uning shu nuqtada o'ng va chap limitlari mavjud bo'lib,

$$f(a+0)=f(a-0)=b$$

tengliklarning o'rinli bo'lishi zarur va yetarli.

6.2. Funksiya limitga ega bo'lishining zaruriy va yetarli sharti (Koshi kriteriysi). $f(x)$ funksiya $X(X\subset R)$ to'plamda aniqlangan. a (chekli yoki cheksiz) nuqta X to'plamning limit nuqtasi bo'lsin.

6.2.1-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilib, argument x ning $0<|x'-a|<\delta,\ 0<|x''-a|<\delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy x' va x'' $x'\in X, x''\in X$ qiymatlarida

$$|f(x')-f(x'')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya uchun a nuqtada *Koshi sharti* bajariladi, deyiladi.

$f(x)$ funksiya uchun a nuqtada Koshi shartining bajarilmasligi quyidagicha ifodalanadi: $\forall\delta>0$ son olingada ham, shunday $\varepsilon>0$ va $0<|x'-a|<\delta,\ 0<|x''-a|<\delta$ tengsizliklarni qanoatlantiruvchi x', x'' $(x'\in X, x''\in X)$ lar uchun, $|f(x')-f(x'')|\geq\varepsilon$ tengsizlik o'rinli bo'ladi.

6.2.2-teorema (Koshi). $f(x)$ funksiyaning a nuqtada chekli limitga ega bo'lishi uchun, a nuqtada Koshi shartining bajarilishi zarur va yetarli.

6.2.3-teorema. Agar $f(x)$ funksiya X to'plamda o'suvchi (kamayuvchi) bo'lib, yuqoridan (quyidan) chegaralangan bo'lsa, $f(x)$ funksiya a nuqtada

chekli limitga ega bo'ladi va agar $f(x)$ funksiya yuqoridan (quyidan) chegaralanmagan bo'lsa, uning limiti $+\infty$ $(-\infty)$ bo'ladi.

6.3. Chekli limitga ega bo'lgan funksiyalar ustida arifmetik amallar. X $(X\subset R)$ to'plam berilgan bo'lib, a uning limit nuqtasi bo'lsin. $f(x)$, $h(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan bo'lsin.

6.3.1-teorema. Agar $f(x)$ va $g(x)$ funksiyalar a nuqtada limitga ega va ularning limitlari, mos ravishda, b va c bo'lsa, u holda $f(x)\pm g(x)$, $f(x)\cdot g(x)$, $\frac{f(x)}{g(x)}$, $(c\neq 0)$ funksiyalar ham shu a nuqtada chekli limitga ega bo'ladi va ushbu

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)=b\pm c, \quad (6.3.2)$$

$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)=b\cdot c, \quad (6.3.3)$$

$$\lim_{x\to a}(kf(x))=k\lim_{x\to a} f(x)=kb,$$

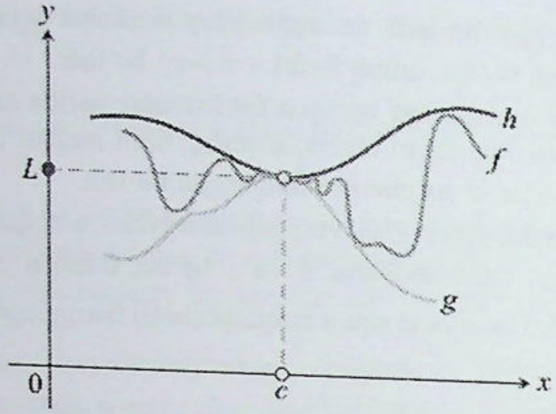
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{b}{c} \quad (6.3.4)$$

munosabatlar o'rinli.

6.3.5-eslatma. Yuqoridagi (6.3.2) va (6.3.3) qoidalar, qo'shiluvchilar va ko'paytuvchilar soni ixtiyoriy chekli bo'lganda ham o'rinli.

6.3.6-eslatma. (6.3.2), (6.3.3) va (6.3.4) qoidalarda $f(x)$ va $g(x)$ funksiyalarning yig'indisi, ko'paytmasi va nisbatidan iborat bo'lgan funksiyalarning limitga ega bo'lishidan, bu funksiyalar har birining limitga ega bo'lishi, har doim ham kelib chiqavermaydi.

6.3.7-teorema. Agar c nuqtaning biror $U_\delta(c)$ atrofidan olingan x ning barcha qiymatlarida $g(x)\leq f(x)\leq h(x)$ tengsizlik o'rinli bo'lib, $x\to c$ da $g(x)$ va $h(x)$ funksiyalar limitga ega bo'lib, $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} h(x)=l$ bo'lsa, u holda $f(x)$ funksiya ham c nuqtada limitga ega va $\lim\limits_{x\to c} f(x)=l$ bo'ladi (6.1-chizma).



6.1chizma.

6.4. Aniqmas ifodalar. Yuqoridagi **6.3.1**-teoremada $f(x)$ va $g(x)$ funksiyalardan quyidagi ikki shart bajarilishi talab qilingan edi: 1) $f(x)$ va $g(x)$ funksiyalar a nuqtada chekli limitga ega; 2) $\frac{f(x)}{g(x)}$ ning limitiga doir mulohazalarda esa, $\lim\limits_{x\to a} g(x)=c\neq 0$ bo'lsin, deb faraz qilingan edi. Agar $x\to a$ da bu shartlarning birortasi bajarilmasa, ya'ni jumladan: $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=0$ bo'lsa, ularning $\frac{f(x)}{g(x)}$ nisbati $\frac{0}{0}$ ko'rinishdagi aniqmaslikni ifodalaydi; 2) $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} g(x)=\infty$ bo'lsa, ularning $\frac{f(x)}{g(x)}$ nisbati $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslikni ifodalaydi; 3) $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=\infty$ bo'lsa, ularning $f(x)\cdot g(x)$ ko'paytmasi $0\cdot\infty$ ko'rinishdagi aniqmaslikni ifodalaydi. 4) $\lim\limits_{x\to a} f(x)=+\infty\ (-\infty)$, $\lim\limits_{x\to a} g(x)=-\infty\ (+\infty)$ bo'lsa, u holda $f(x)+g(x)$ ifoda $\infty-\infty$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu hollarda $x\to a$ va $f(x)$ va $g(x)$ funksiyalarning o'z limitlariga qanday intilish xususiyatlariga qarab, $\frac{f(x)}{g(x)}$, $f(x)\cdot g(x)$, $f(x)+g(x)$ ifodalarning xarakterini aniqlash, *aniqmaslikni ochish* deb yuritiladi.

6.5. Murakkab funksiyaning limiti. $t=\varphi(x)$ funksiya X to'plamda aniqlangan va bu funksiyaning qiymatlar to'plamida T da $y=f(t)$ funksiya

aniqlangan bo'lib, ular yordamida $y=f(\varphi(x))$ murakkab funksiya aniqlangan bo'lsin. a nuqta X to'plamning limit nuqtasi bo'lsin. Murakkab funksiya X to'plamida aniqlangan.

6.5.1-teorema. Agar: 1) $\lim\limits_{x\to a}\varphi(x)=c$ limit o'rinli bo'lib, a nuqtaning shunday $\dot{U}_\delta(a)$ atrofi mavjud bo'lsinki, barcha $x\in U_\delta(a)$ lar uchun $\varphi(x)\neq c$ bo'lsa; 2) c nuqta T to'plamning limit nuqtasi bo'lib, $\lim\limits_{t\to c} f(t)=b$ limit mavjud bo'lsa, u holda $x\to a$ da $y=f(\varphi(x))$ murakkab funksiya ham limitga ega va

$$\lim_{x\to a} f(\varphi(x))=\lim_{t\to c} f(t)=b \qquad (6.5.2)$$

bo'ladi.

$f(t)$ funksiya c nuqtada uzluksiz bo'lgan holda, (**6.5.2**) tenglikni

$$\lim_{x\to a} f(\varphi(x))=f\left(\lim_{x\to a}\varphi(x)\right)$$

ko'rinishda yozish mumkin.

6.5.3-eslatma. Teoremadagi a nuqtaning $\dot{U}_\delta(a)$ atrofida $\varphi(x)\neq c$ bo'lsin degan shartni $f(t)$ funksiya $t=c$ nuqtada aniqlangan va $\lim\limits_{t\to c} f(t)=f(c)=b$ tengliklar o'rinli bo'lsin degan shart bilan almashtirish mumkin.

6.5.4-eslatma. Yuqoridagi a, c va b larning biri chekli, ikkinchisi ∞ yoki barchasi cheksiz bo'lganda ham **6.6.1**-teorema o'rinli bo'ladi.

Funksiyaning limitini hisoblashda quyidagi ajoyib limitlar muhim rol o'ynaydi:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{\sin x}=1; \qquad (6.5.5)$$

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=\lim_{y\to 0}(1+y)^{\frac{1}{y}}=e \quad (e\approx 2{,}71828...) \qquad (6.5.6)$$

6.5.7-eslatma. Agar biror $\dot{U}_\delta(x_0)$ atrofda $\alpha(x)\neq 0$ va $\lim\limits_{x\to x_0}\alpha(x)=0$ bo'lsa,

u holda $\lim\limits_{x\to x_0}(1+\alpha(x))^{\frac{1}{\alpha(x)}}=e$ bo'ladi.

6.5.8-eslatma. Agar biror $\dot{U}_\delta(x_0)$ atrofda $\alpha(x)\neq 0$, $\beta(x)\neq 0$,

$\lim\limits_{x \to x_0} \alpha(x) = \lim\limits_{x \to x_0} \beta(x) = 0$ va $\exists \lim\limits_{x \to x_0} \frac{\alpha(x)}{\beta(x)} = \lambda$ bo'lsa, u holda $\lim\limits_{x \to x_0} (1 + \alpha(x))^{\frac{1}{\beta(x)}} = e^{\lambda}$ bo'ladi. Xususiy holda, $\lim\limits_{x \to x_0} (1 + \mu\alpha(x))^{\frac{1}{\alpha(x)}} = e^{\mu}$, $\mu = const$.

(6.5.6) formuladan natija sifatida kelib chiqadigan ushbu

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, \quad \lim_{x \to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln a}, \ a > 0, \ a \neq 1, \quad \lim_{x \to 0} \frac{(1+x)^{\mu} - 1}{x} = \mu,$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a, \quad a > 0,$$

(xususiy holda $a = e$ bo'lganda)

$$\lim_{x \to a} \frac{\ln(1+x)}{x} = 1, \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

formulalar funksiya limitini hisoblashda ko'p qo'llaniladi.

Ikkita $f(x)$ va $g(x)$ funksiyalar X $(X \subset R)$ to'plamda berilgan bo'lib, $f(x) > 0$ hamda a nuqta X to'plamning limit nuqtasi bo'lsin. Bu holda daraja ko'rsatkichli funksiyaning limiti

$$\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} e^{g(x) \ln f(x)} = e^{\lim\limits_{x \to a} [g(x) \ln f(x)]}$$

formula orqali topiladi, ya'ni daraja ko'rsatkichli funksiyaning limitini topish masalasi, $\lim\limits_{x \to a} [g(x) \ln f(x)]$ limitni topishga olib kelinar ekan. Bu limitni hisoblashda quyidagi hollar bo'lishi mumkin.

1). Agar $\lim\limits_{x \to a} g(x) = A$, $\lim\limits_{x \to a} \ln f(x) = B$ bo'lsa, u holda $\lim\limits_{x \to a} f(x) = e^B$ va $e^{\lim\limits_{x \to a} [g(x) \ln f(x)]} = e^{AB} = (e^B)^A = [\lim\limits_{x \to a} f(x)]^{\lim\limits_{x \to a} g(x)}$ formula o'rinli bo'ladi.

2). Agar $\lim\limits_{x \to a} [g(x) \ln f(x)] = +\infty$ bo'lsa, u holda $e^{\lim\limits_{x \to a} [g(x) \ln f(x)]} = +\infty$, $\lim\limits_{x \to a} [g(x) \ln f(x)] = -\infty$ bo'lganda esa, $e^{\lim\limits_{x \to a} [g(x) \ln f(x)]} = 0$ bo'ladi.

Agar $\lim\limits_{x \to a} [g(x) \ln f(x)] = \infty$ va $g(x) \ln f(x)$ ko'paytma funksiyaning biror $\dot{U}_{\delta}(a)$ atrofda ishorasi saqlanmasa, u holda $[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$ funksiya $x \to a$ da limitga ega bo'lmaydi.

3). $g(x) \ln f(x)$ ko'paytma funksiyada $x \to a$ da birining limiti nol, ikkinchisining limiti esa cheksiz bo'lsa, bu holda, quyidagi uch hol bo'lishi mumkin:

a) $\lim\limits_{x \to a} g(x) = 0$, $\lim\limits_{x \to a} f(x) = +\infty$ (∞^0),

b) $\lim\limits_{x \to a} g(x) = 0$, $\lim\limits_{x \to a} f(x) = 0$ (0^0),

c) $\lim\limits_{x \to a} g(x) = \infty$, $\lim\limits_{x \to a} f(x) = 1$ (1^{∞}).

6.5.8-eslatma. Agar $\lim\limits_{x \to a} f(x) = 1$, $\lim\limits_{x \to a} g(x) = \infty$ bo'lsa, u holda 1^{∞} ko'rinishdagi aniqmaslik, 1) holni e'tiborga olgan holda,

$$\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} \left\{ [1 + (f(x) - 1)]^{\frac{1}{f(x)-1}} \right\}^{(f(x)-1)g(x)} = e^{\lim\limits_{x \to a} (f(x)-1)g(x)}$$

formula yordamida ochiladi.

6.6. Cheksiz katta va cheksiz kichik funksiyalar. X to'plam berilgan bo'lib, a uning limit nuqtasi bo'lsin. X to'plamda $\alpha(x)$ va $\beta(x)$ funksiyalar berilgan bo'lsin.

6.6.1-ta'rif. Agar $\lim\limits_{x \to a} \alpha(x) = 0$ bo'lsa, $\alpha(x)$ funksiya $x \to a$ da *cheksiz kichik funksiya* deyiladi.

Agar X to'plamda aniqlangan $f(x)$ funksiya $x \to a$ da b limitga ega bo'lsa, u holda $\alpha(x) = f(x) - b$ funksiya $x \to a$ da cheksiz kichik funksiya bo'ladi. Chunki $\lim\limits_{x \to a} \alpha(x) = \lim\limits_{x \to a} (f(x) - b) = \lim\limits_{x \to a} f(x) - b = 0$.

Demak, b limitga ega bo'lgan har qanday $f(x)$ funksiyani

$$f(x) = b + \alpha(x)$$

ko'rinishda tasvirlash mumkin, bu yerda $\alpha(x)$ -cheksiz kichik funksiya.

6.6.2-ta'rif. Agar $\lim\limits_{x \to a} \beta(x) = \infty$ bo'lsa, $\beta(x)$ funksiya $x \to a$ da *cheksiz katta funksiya* deyiladi.

Cheksiz kichik va cheksiz katta funksiyalar quyidagi xossalarga ega:

1^0. Cheksiz kichik funksiyalarning yig'indisi (ayirmasi) cheksiz kichik funksiya bo'ladi.

2^0. Cheksiz kichik funksiya bilan chegaralangan funksiyalarning ko‘paytmasi cheksiz kichik bo‘ladi.

3^0. Agar $\alpha(x)$ $(\alpha(x)\neq 0)$ cheksiz kichik funksiya bo‘lsa, $\frac{1}{\alpha(x)}$ cheksiz katta funksiya bo‘ladi.

4^0. Agar $\beta(x)$ cheksiz katta funksiya bo‘lsa, $\frac{1}{\beta(x)}$ cheksiz kichik funksiya bo‘ladi.

5^0. Agar $x\to a$ da $f(x)$ cheksiz katta funksiya bo‘lsa, $g(x)$ esa biror $U_\delta(a)$ da $|g(x)|>c$ ($x\neq a$, c -biror musbat son) bo‘lsa, u holda $f(x)g(x)$ cheksiz katta funksiya bo‘ladi.

6.6.3-eslatma. Cheksiz katta funksiyalarning yig‘indisi (ayirmasi) va nisbati cheksiz katta funksiya bo‘lmasligi ham mumkin.

6.7. Funksiyalarni solishtirish. $O(f)$ va $o(f)$ belgilar. X to‘plamda $f(x)$ va $g(x)$ funksiyalar aniqlangan bo‘lsin. a nuqtaning biror $U_\delta(a)$ atrofida $f(x)$ va $g(x)$ funksiyalarni solishtiramiz.

6.7.1-ta’rif. Agar shunday $\delta>0$ va $C>0$ o‘zgarmas sonlar mavjud bo‘lib, $\forall x\in U_\delta(a)$ uchun $|f(x)|\leq C|g(x)|$ tengsizlik o‘rinli bo‘lsa, u holda $x\to a$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan chegaralangan deyiladi va $f(x)=O(g(x))$ kabi yoziladi.

Xuddi shunday $x\to a+0$, $x\to a-0$, $x\to\infty$, $x\to-\infty$ da ham, $f(x)=O(g(x))$ yozish saqlanadi.

Xususiy holda, $f(x)$ funksiya $\dot{U}_\delta(a)$ atrofda chegaralangan bo‘lsa, u $x\to a$ da $f(x)=O(1)$ kabi yoziladi.

6.7.2-ta’rif. Agar $x\to a$ da $f(x)$ va $g(x)$ funksiyalar uchun $f(x)=O(g(x))$ $g(x)=O(f(x))$ va munosabatlar o‘rinli bo‘lsa, u holda $x\to a$ da $f(x)$ va $g(x)$ funksiyalar bir xil tartibli funksiyalar deb ataladi va $f(x)\succ\prec g(x)$, $x\to a$ kabi belgilanadi.

6.7.3-teorema. Agar $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=k$ mavjud bo‘lib, $k\neq 0$ bo‘lsa, u holda $f(x)$ va $g(x)$ funksiyalar $x\to a$ da bir xil tartibli funksiyalar bo‘ladi.

6.7.4-ta’rif. Agar biror $\dot{U}_\delta(a)$ atrofda aniqlangan $f(x)$ va $g(x)$ cheksiz kichik funksiyalar uchun $f(x)=\varphi(x)g(x)$ tenglik o‘rinli bo‘lib, bunda $\lim\limits_{x\to a}\varphi(x)=0$ bo‘lsa, u holda $x\to a$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan

yuqori tartibli cheksiz kichik funksiya deyiladi va u $f(x)=o(g(x))$ kabi belgilanadi.

Xususiy holda, agar $g(x)=1$ bo‘lsa, $x\to a$ da $f(x)=o(1)$ ifoda $f(x)$ funksiyaning cheksiz kichik funksiya ekanligini anglatadi, ($x\to a$ da $f(x)\to 0$).

Xuddi yuqoridagidek, $f(x)=o(g(x))$ simvolik ifodaning mazmuni $x\to a-0$, $x\to a+0$, $x\to\infty$, $x\to+\infty$, $x\to-\infty$ da ham saqlanadi.

6.7.5-teorema. $x\to a$ da $f(x)$ funksiya $g(x)$ cheksiz kichik funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya bo‘lishi uchun, $\forall x\in\dot{U}_\delta(a)$ uchun $g(x)\neq 0$ bo‘lib, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=0$ tenglik o‘rinli bo‘lishi zarur va yetarli.

$o(f)$, $O(f)$ belgilar (simvol) bilan ishlash vaqtida quyidagi qoidalarga amal qilish kerak:

1) $o(cf)=o(f)$, c -o‘zgarmas son.
2) $o(f)+o(f)=o(f)$.
3) $o(f)\cdot o(g)=o(f\cdot g)$.
4) $o(o(f))=o(f)$.
5) $O(cf)=O(f)$.
6) $O(o(f)=O(f)$.
7) $O(o(f)=o(O(f))=o(f)$.
8) $g\cdot O(f)=O(g\cdot f)$, $g\cdot o(f)=o(g\cdot f)$.
9) $O(f)\cdot O(f)=O(f^2)$, $O(f)\cdot o(f)=o(f^2)$, $o(f)\cdot o(f)=o(f^2)$.

6.7.6-ta’rif. Agar biror $\dot{U}_\delta(a)$ atrofda aniqlangan $f(x)$ va $g(x)$ funksiyalar uchun $f(x)=\varphi(x)g(x)$ tenglik o‘rinli bo‘lib, $\lim\limits_{x\to a}\varphi(x)=1$ bo‘lsa, $x\to a$ da $f(x)$ va $g(x)$ funksiyalar *o‘zaro ekvivalent funksiyalar* deb ataladi va $x\to a$ da $f(x)\sim g(x)$ kabi belgilanadi.

6.7.7-teorema. $x\to a$ da $f(x)$ va $g(x)$ funksiyalar $\dot{U}_\delta(a)$ da $(g(x)\neq 0,\ f(x)\neq 0)$ o‘zaro ekvivalent bo‘lishi uchun,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{g(x)}{f(x)}=1$$

bo‘lishi zarur va yetarli.

Odatda, ekvivalentlik tushunchasi $f(x)$ va $g(x)$ funksiyalar cheksiz kichik va cheksiz katta bo‘lgan hollarda ishlatiladi.

Funksiyalarning ekvivalentlik tushunchasi quyidagi sodda xossalarga ega:

1) $x\to a$ da $f(x)\sim f(x)$,
2) agar $f(x)\sim g(x)$ bo‘lsa, $g(x)\sim f(x)$ ham bo‘ladi;

3) $x \to a$ da $f(x) \sim g(x)$, va $g(x) \sim h(x)$ bo'lsa, u holda $f(x) \sim h(x)$ bo'ladi.

4) agar $f(x) \sim g(x)$ bo'lsa, u holda $f(x) = O(g(x))$ bo'ladi.

5) agar $f(x) \sim g(x)$ va $h(x) \sim S(x)$ bo'lsa, $f(x)h(x) \sim g(x)S(x)$ bo'ladi.

6) agar $\lim_{x \to a} f(x) = k \neq 0$ bo'lsa, $f(x) \sim k$ bo'ladi.

Bu xossalardan quyidagi munosabat kelib chiqadi: agar $f(x) \sim g(x)$ bo'lsa, u holda

$$f(x) - g(x) = o(g(x)) \text{ yoki } \qquad f(x) = g(x) + o(g(x)) \qquad (1)$$

munosabat o'rinli. Agar $f(x)$ funksiya (1) ko'rinishda tasvirlangan bo'lsa, u holda $g(x)$, $x \to a$ da $f(x)$ ning *bosh qismi* deb ataladi.

Funksiyalarning limitini hisoblashda $x \to 0$ da ekvivalent quyidagi funksiyalar ko'proq ishlatiladi:

$$x \sim \sin x \sim tgx \sim \arcsin x \sim arctgx \sim \ln(1+x) \sim e^x - 1.$$

6.8. Funksiyaning yuqori va quyi limiti. $f(x)$ funksiya X $(X \subset R)$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin.

6.8.1-ta'rif. Agar X to'plamning elementlaridan tuzilgan va $x_n \to a$ shunday $\{x_n\}$ $(x_n \neq a,\ n = 1,2,...)$ ketma-ketlik mavjud bo'lib, $\lim_{n \to \infty} f(x_n) = b$ bo'lsa, b son $f(x)$ funksiyaning $x \to a$ dagi qismiy limiti deb ataladi. Cheksiz va bir tomonli qismiy limitlar ham, xuddi shunday ta'riflanadi.

Funksiyaning qismiy limitlari ichida har doim eng kattasi va eng kichigi topiladi. Ular, mos ravishda, funksiyaning yuqori va quyi limiti deb ataladi va $\overline{\lim_{x \to a}} f(x)$, $\underline{\lim_{x \to a}} f(x)$ kabi belgilanadi. $\overline{\lim_{x \to a}} f(x) = \underline{\lim_{x \to a}} f(x)$ tenglikning bajarilishi funksiyaning limitga ega bo'lishi uchun zaruriy va yetarli shartidan iborat.

6.2-§. Uzluksiz funksiya

$f(x)$ funksiya X $(X \subset R)$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi va $a \in X$. bo'lsin. $f(x)$ funksiyaning limiti to'g'risida quyidagi hollar bo'lishi mumkin:

1^0. $x \to a$ da $f(x)$ ning limiti mavjud, chekli va $\lim_{x \to a} f(x) = f(a)$;

2^0. $x \to a$ da $f(x)$ funksiyaning limiti mavjud, chekli va $\lim_{x \to a} f(x) = b \neq f(a)$;

3^0. $x \to a$ da $f(x)$ funksiyaning limiti mavjud va $\lim_{x \to a} f(x) = \infty$;

4^0. $x \to a$ da $f(x)$ funksiya limiti mavjud emas.

Bu hollarni batafsil ko'rib o'tamiz.

6.9. Uzluksiz funksiyaning ta'riflari. $f(x)$ funksiya $X \subset R$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi va $a \in X$. bo'lsin.

6.9.1-ta'rif. Agar $x \to a$ da $f(x)$ funksiyaning limiti mavjud va u $f(a)$ ga teng, ya'ni

$$\lim_{x \to a} f(x) = f(a) \qquad (6.9.2)$$

bo'lsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

$a = \lim_{x \to a} x$ ekanligini e'tiborga olgan holda, (6.9.2) tenglikni quyidagicha ham yozish mumkin: $\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$.

Demak, funksiya a nuqtada uzluksiz bo'lsa, "lim" belgisi bilan funksiyaning xarakteristikasi "f" ning o'rnini almashtirish mumkin.

6.9.3-ta'rif (Geyne). Agar X to'plamning elementlaridan tuzilgan va a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham, funksiyaning unga mos qiymatlaridan tuzilgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt $f(a)$ ga intilsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

6.9.4-ta'rif (Koshi). Agar istalgan $\varepsilon > 0$ son uchun shunday $\delta = \delta(\varepsilon, a) > 0$ son topilsaki, funksiya argumenti x ning $|x - a| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$|f(x) - f(a)| < \varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

6.9.5-ta'rif. Agar istalgan $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, funksiya argumenti x ning barcha $x \in U_\delta(a)$ qiymatlarida $f(x)$ funksiyaning mos qiymatlari $f(U_\delta(a)) \subset U_\varepsilon(f(a))$ bo'lsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

Matematik belgilardan foydalanib, 6.9.3-, 6.9.4-, 6.9.5-ta'riflarni, mos ravishda, quyidagi ko'rinishda yozish mumkin:

$$\forall \{x_n\} : \lim_{n \to \infty} x_n = a \to \lim_{n \to \infty} f(x_n) = f(a),$$

$$\forall \varepsilon > 0 \exists\ \delta > 0 : \ \forall x : |x - a| < \delta \to |f(x) - f(a)| < \varepsilon,$$

$$\forall \varepsilon > 0 \exists\ \delta > 0 : \ \forall x : U_\delta(a) \to f(x) \in U_\varepsilon(f(a)) .$$

Bunda $x-a$ ayirmaga argument orttirmasi, $f(x)-f(a)$ ayirmaga esa, funksiyaning a nuqtadagi orttirmasi deyiladi. Ular, mos ravishda, Δx va Δy yoki $\Delta f(a)$ kabi belgilanadi:

$$\Delta x = x - a, \quad \Delta y = \Delta f(a) = f(x) - f(a).$$

Argument va funksiya orttirmasini quyidagi ko'rinishda ham yozish mumkin:

$$x = a + \Delta x, \quad \Delta f(a) = f(a + \Delta x) - f(a) \qquad (6.9.6)$$

Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, (6.9.2) va (6.9.6) munosabatlardan $\lim_{\Delta x \to 0} \Delta f = 0$ bo'lishi kelib chiqadi. Bu esa, funksiya uzluksizligini quyidagicha ta'riflash ham mumkinligini ko'rsatadi.

6.9.7-ta'rif. Agar x argumentning a nuqtadagi Δx orttirmasi nolga intilganda, $f(x)$ funksiyaning unga mos Δf orttirmasi ham nolga intilsa, ya'ni $\lim_{\Delta x \to 0} \Delta f = 0$ bo'lsa, $f(x)$ funksiya a *nuqtada uzluksiz* deyiladi

6.9.8-eslatma. X to'plamda $f(x)$ funksiya aniqlangan bo'lib, a X to'plamning limit nuqtasi bo'lmasa, ya'ni $\lim_{x \to a} f(x)$ ma'noga ega bo'lmasa, $f(x)$ funksiyaning a nuqtada uzluksizligi haqida gapirishning ma'nosi yo'q.

$X \subset R$ to'plamda $f(x)$ funksiya aniqlangan bo'lib, $a \in X$ esa, X to'plamning o'ng (chap) limit nuqtasi bo'lsin.

6.9.9-ta'rif. Agar $x \to a+0$ $(x \to a-0)$ da $f(x)$ funksiyaning o'ng (chap) limiti mavjud va u $f(a)$ ga teng, ya'ni

$$\lim_{x \to a+0} f(x) = f(a+0), \; f(a+0) = f(a),$$

$$\left(\lim_{x \to a-0} f(x) = f(a-0), \; f(a-0) = f(a)\right)$$

bo'lsa, $f(x)$ funksiya a *nuqtada o'ngdan (chapdan) uzluksiz* deyiladi.

6.9.10-ta'rif (Geyne). Agar X to'plamning elementlari (nuqtalari) dan tuzilgan va har bir hadi $x_n > a$ $(x_n < a)$ $(n = 1,2,...)$ bo'lib, a ga intiluvchi har qanday $\{x_n\}$ ketma – ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ketma-ketlik hamma vaqt $f(a)$ qiymatga intilsa, $f(x)$ funksiya a *nuqtada o'ng (chap) dan uzluksiz* deyiladi.

6.9.11-ta'rif(Koshi). Agar $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topilib, argument x ning $a < x < a + \delta$ $(a - \delta < x < a)$ tengsizliklarni qanoatlantiruvchi qiymatlarida $|f(x) - f(a)| < \varepsilon$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya a *nuqtada o'ng (chap) dan uzluksiz* deyiladi.

Ma'lumki, funksiya limitining Geyne va Koshi ta'riflari o'zaro ekvivalent bo'lgani singari, funksiyaning nuqtadagi uzluksizligining Geyne va Koshi ta'riflari ham o'zaro ekvivalent bo'ladi.

6.9.12-ta'rif. Agar $f(x)$ funksiya $X \subset R$ to'plamning har bir nuqtasida uzluksiz bo'lsa, $f(x)$ funksiya X *to'plamda uzluksiz* deyiladi.

6.9.13-ta'rif. Agar $f(x)$ funksiya (a, b) oraliqda uzluksiz bo'lib, $x=a$ nuqtada o'ngdan, $x=b$ nuqtada esa chapdan uzluksiz bo'lsa, $f(x)$ funksiya $[a, b]$ *kesmada uzluksiz* deyiladi.

6.9.14-teorema. $f(x)$ funksiyaning a nuqtada uzluksiz bo'lishi uchun $f(a+0) = f(a-0) = f(a)$ tenglikning bajarilishi zarur va yetarli.

6.10. Funksiya uzilish nuqtalarining turlari. $f(x)$ funksiya X $(X \subset R)$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin, $a \in X$.

6.10.1-ta'rif. Agar $x \to a$ da $f(x)$ funksiyaning: 1) limiti mavjud va chekli bo'lib, $\lim_{x \to a} f(x) = b \neq f(x)$; 2) $\lim_{x \to a} f(x) = \infty$ $(\lim_{x \to a} f(x) = \pm\infty)$; 3) limiti mavjud bo'lmasa, $f(x)$ funksiya a *nuqtada uzilishga* ega deyiladi.

Funksiyaning berilgan nuqtada uzilishga ega bo'lish hollarini alohida qarab o'tamiz:

1-hol. Agar $f(x)$ funksiyaning a nuqtadagi o'ng $f(a + 8) = \lim_{x \to a+0} f(x)$ va chap $f(a - 8) = \lim_{x \to a+0} f(x)$ limitlari mavjud bo'lib, $f(a + 0) = f(a - 0) \neq f(a)$ munosabat o'rinli bo'lsa, $f(x)$ funksiya a *nuqtada tuzatiladigan (yo'qotilishi mumkin bo'lgan) uzilishga* ega deyiladi.

2-hol. Agar $x \to a$ da $f(x)$ funksiyaning o'ng va chap limitlari mavjud va chekli bo'lib, ular bir-biriga teng bo'lmasa $(f(a-0) \neq f(a+0))$, $f(x)$ funksiya a *nuqtada birinchi tur uzilishga* ega deyiladi.

Ushbu $f(a+0) - f(a-0)$ ayirmaga $f(x)$ funksiyaning a nuqtadagi *sakrashi* deyiladi.

3-hol. Agar $x \to a$ da $f(x)$ funksiyaning:

1) o'ng va chap limitlaridan hech bulmaganda bittasi mavjud bo'lmasa;

2) o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlari turli ishorali cheksizdan iborat bo'lsa;

3) (funksiyaning o'ng va chap limitlari cheksiz) limiti cheksiz bo'lsa, $f(x)$ funksiya a *nuqtada ikkinchi tur uzilishga* ega deyiladi.

Agar chap $f(a-0)$ yoki o'ng $f(a+0)$ limitlardan hech bo'lmaganda biri ∞ ga teng bo'lsa, $x=a$ nuqta $f(x)$ funksiyaning cheksiz uzilish nuqtasi deyiladi.

6.11. Nuqtada uzluksiz bo'lgan funksiyaning lokal xossalari. $f(x)$ funksiya X to'plamda aniqlangan bo'lsin. X to'plamdan biror $a\in X$ nuqta olib, bu nuqtaning shu X to'plamga tegishli bo'lgan yetarli kichik $U_\delta(a)$ atrofini qaraylik.

1^0. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, u holda a nuqtaning yetarli kichik atrofida funksiya chegaralangan bo'ladi, ya'ni

$\exists\delta>0\ \exists C>0:\forall x\in U_\delta(a)\to|f(x)|\le C.$

2^0. Agar $f(x)$ funksiya a nuqtada uzluksiz va $f(a)\neq 0$ bo'lsa, $f(a)$ son bilan a nuqtaning yetarli kichik atrofida $f(x)$ funksiyaning ishorasi bir xil bo'ladi, ya'ni $\exists\delta>0:\forall x\in U_\delta(a)\to sign f(x)=sign f(a)$.

6.11.1-natija. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lib, bu nuqtaning yetarli kichik atrofidan olingan x nuqtalarda ham musbat, ham manfiy ishorali qiymatlarni qabul qilaversa, funksiyaning a nuqtadagi qiymati nolga teng bo'ladi.

3^0. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, a nuqtaning yetarli kichik atrofidan olingan x' va x'' nuqtalar uchun $|f(x')-f(x'')|<\varepsilon$ tengsizlik o'rinli bo'ladi, bunda $\forall\varepsilon>0$ son.

Funksiyaning nuqta atrofidagi xususiyatlariga uning lokal xususiyatlari deyiladi.

6.12. Uzluksiz funksiyalar ustida amallar. Uzluksiz funksiyalar uchun quyidagi tasdiqlar o'rinli:

6.12.1-teorema. Agar $f(x)$ va $g(x)$ funksiyalar $X\subset R$ to'plamda aniqlangan bo'lib, ularning har biri $a\in X$ nuqtada uzluksiz, ya'ni $\lim\limits_{x\to a} f(x)=f(a)$, $\lim\limits_{x\to a} g(x)=g(a)$ bo'lsa, $f(x)\pm g(x)$, $f(x)\cdot g(x)$, $\frac{f(x)}{g(x)}$ $(g(a)\neq 0)$ funksiyalar ham shu nuqtada uzluksiz va

$$\lim_{x\to a}[f(x)\pm g(x)]=f(a)\pm g(a),$$

$$\lim_{x\to a} f(x)\cdot g(x)=f(a)\cdot g(a),$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}\ (g(a)\neq 0)$$

tengliklar o'rinli bo'ladi.

6.12.2-eslatma. Ikkita funksiyaning yig'indisi, ayirmasi, ko'paytmasi va nisbati uzluksiz bo'lishidan, bu funksiyalardan har birining uzluksiz bo'lishi har doim ham kelib chiqavermaydi.

6.13. Murakkab funksiyaning uzluksizligi. $y=f(x)$ funksiya X to'plamda, $z=\varphi(y)$ funksiya esa, Y to'plamda aniqlangan va $E(\varphi)\subseteq X$ bo'lsin. U holda, ular yordamida, $z=\varphi(f(x))$ murakkab funksiyani tuzish mumkin bo'ladi.

6.13.1-teorema. $y=f(x)$ funksiya $a\in X$ nuqtada, $z=\varphi(y)$ funksiya esa a nuqtaga mos kelgan $y_a=f(a)$ nuqtada uzluksiz bo'lsa, $z=\varphi(f(x))$ murakkab funksiya a nuqtada uzluksiz bo'ladi.

6.14. Monoton funksiyalarning uzluksizligi.

6.14.1-teorema. Agar $f(x)$ funksiya X oraliqda o'suvchi (kamayuvchi) bo'lib, uzilishga ega bo'lsa, uning uzilishi faqat birinchi tur uzilish bo'ladi.

Agar $f(x)$ funksiya X oraliqda o'suvchi (kamayuvchi) bo'lib, uning qiymatlari Y oraliqni tutash to'ldirsa (ya'ni funksiya har bir $y\in Y$ qiymatni hech bo'lmaganda bir marta qabul qilsa) bu funksiya X oraliqda uzluksiz bo'ladi.

6.15. Teskari funksiyaning uzluksizligi

6.15.1-teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan, uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'lsa, u holda $[f(a),f(b)]$ da teskari $f^{-1}(y)$ funksiya mavjud bo'lib, u uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'ladi.

6.16. Kesmada uzluksiz bo'lgan funksiyalarning xossalari (global xossalar). $[a;b]$ kesmada aniqlangan va uzluksiz bo'lgan funksiyalarni qaraymiz, bunda a va b nuqtalardagi uzluksizliklar mos ravishda, o'ngdan va chapdan uzluksizlik, deb qaraladi.

6.16.1-teorema (Veyershtrassning birinchi teoremasi). Agar $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsa, $f(x)$ funksiya shu kesmada chegaralangan bo'ladi, ya'ni $\exists C>0:\forall x\in[a;b]\to|f(x)|\le C$.

6.16.2-teorema (Veyershtrassning ikkinchi teoremasi). Agar $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsa, $f(x)$ funksiya shu kesmada o'zining aniq yuqori hamda aniq quyi chegaralariga erishadi, ya'ni $[a;b]$ kesmada shunday x_1 va x_2 nuqtalar topiladiki,

$$f(x_1)=\sup_{x\in[a,b]}\{f(x)\},\ f(x_2)=\inf_{x\in[a,b]}\{f(x)\}$$

tenglik o'rinli bo'ladi.

6.16.3-teorema (Bolsano-Koshining birinchi teoremasi). Agar $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lib, kesmaning chetki nuqtalarida har xil ishorali qiymatlarga ega bo'lsa, shunday x_0 $(a<x_0<b)$ nuqta topiladiki, unda $f(x)$ funksiya nolga aylanadi: $f(x_0)=0$.

6.16.4-teorema (Bolsano-Koshining ikkinchi teoremasi). Agar $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lib, kesmaning chetki nuqtalarida $f(a)=A$, $f(b)=B$ qiymatlar qabul qilsa, hamda $A\neq B$ bo'lsa, A va B sonlar orasidan ixtiyoriy C son olinganda ham, a bilan b orasida shunday c nuqta topiladiki, $f(c)=C$ bo'ladi.

6.16.5-eslatma. Veyershtrassning birinchi teoremasi $(a,\ b)$ oraliq uchun har doim ham o'rinli emas.

6.16.6-eslatma. Uzilishga ega bo'lgan funksiya $[a;b]$ kesmaning har bir nuqtasida aniqlangan, lekin bu kesmada chegaralanmagan bo'lishi mukin.

6.16.7-eslatma. $[a;b]$ kesmada uzluksiz bo'lmagan funksiya uchun Veyershtrassning ikkinchi teoremasi o'rinli bo'lmasligi mumkin.

6.16.8-eslatma. (a,b) oraliqda uzluksiz bo'lgan $f(x)$ funksiya uchun Veyershtrassning ikkinchi teoremasi o'rinli bo'lmasligi ham mumkin.

6.16.9-eslatma. Xususiy hollarda, $f(x)$ funksiya (a,b) oraliqda uzluksiz (uzilishga ega) bo'lsa, $f(x)$ funksiya shu oraliqda eng kichik va eng katta qiymatiga erishishi mumkin.

6-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

6.1. Funksiyaning ta'rifi. Funksiyaning aniqlanish sohasi va o'zgarish sohasi. Funksiyaning berilish usullari ([1], 1-t., 3-bo'lim, [3], 1-q., 63-65 betlar; [5], 1-t., 93-114 betlar; [12], 1-q.,109-112 betlar; [9], 1-t., 2-bo'lim).

6.2. Funksiyaning chegaralanganligi. Juft va toq funksiyalar, misollar ([1], 1-t., 3-bo'lim, [3], 1-q., 65-68 betlar; [5], 1-t., 93-114 betlar; [12], 1-q.,113-114 betlar; [9], 1-t., 2-bo'lim).

6.3. Davriy funksiyalar va ularning xossalari, misollar ([3], 1-q., 66-68 betlar; [5], 1-t., 93-114 betlar; [12], 1-q.,114-119 betlar).

6.4. Monoton funksiyalar. Elementar funksiyalar ([1], 1-t., 3-bo'lim, [3], 68-74 betlar; [5], 1-t., 93-114 betlar; [12], 1-q.,119-120 betlar; [9], 1-t., 2-bo'lim,

[30], 7-bo'lim).

6.5. Murakkab va teskari funksiyalar hamda misollar ([3],1-q., 69-70 betlar; [5], 1-t., 93-114 betlar; [12], 1-q.,120-121 betlar).

6.6. Funksiya limiti. To'plamning limit nuqtasi va uning xossalari. Funksiya limitining ta'riflari. Funksiyaning bir tomonli limitlari ([1], 1-t., 9-bo'lim, [3], 1-q., 75-84 betlar; [5], 1-t., 115-135 betlar; [12], 1-q.,127-133 betlar [9], 1-t., 3-bo'lim, [30], 2-bo'lim).

6.7. Chekli limitga ega bo'lgan funksiyaning xossalari ([1], 1-t., 9-bo'lim, [3], 1-q., 85-68 betlar; [5], 1-t., 115-135 betlar; [12], 1-q.,136-138 betlar; [30], 2-bo'lim).

6.8. Chekli limitga ega bo'lgan funksiyalar ustida arifmetik amallar ([1], 1-t., 9-bo'lim, [3], 1-q., 85-88 betlar; [5], 1-t., 115-135 betlar; [12], 1-q.,138-139 betlar; [9], 4-t., 3-bo'lim, [30], 2-bo'lim).

6.9. Murakkab funksiyaning limiti ([5], 1-t., 115-135 betlar; [12], 1-q.,139-141 betlar).

6.10. Monoton funksiyaning limiti ([1], 1-t., 9-bo'lim, [3],1-q., 88-89 betlar; [5], 1-t., 115-135 betlar; [12], 1-q.,141-142 betlar; [30], 2-bo'lim).

6.11. Cheksiz katta va cheksiz kichik funksiyalar ([1], 1-t., 9-bo'lim, [3],1-q., 91 bet; [5], 1-t., 115-135 betlar; [12], 1-q.,145 bet; [30], 2-bo'lim).

6.12. Funksiyalarni taqqoslash. «*O*» va «*o*» belgilar ([1], 1-t., 9-bo'lim, [3], 1-q., 92-96 betlar; [5], 1-t., 136-145 betlar; [12], 1-q.,146-150 betlar; [9], 1-t., 5-bo'lim).

6.13. Funksiya uzluksizligi ta'riflari ([3],1-q., 97-98 betlar; [5], 1-t., 146-147 betlar; [12], 1-q.,151-153 betlar; [30], 2-bo'lim).

6.14. Funksiyaning bir tomonli uzluksizliklari ta'riflari ([3],1-q., 98-99 betlar; [5], 1-t., 150-151 betlar; [12], 1-q.,153-155 betlar; [30], 2-bo'lim).

6.15. Funksiyaning uzilishlari va ularning turlari ([3],1-q., 101-102 betlar; [5], 1-t., 150-151 betlar; [12], 1-q.,155-158 betlar; [30], 2-bo'lim).

6.16. Monoton funksiyaning uzluksizligi va uzilishi haqidagi teorema ([3],1-q., 102-103 betlar; [5], 1-t., 154-155 betlar; [12], 1-q.,158-159 betlar; [30], 2-bo'lim).

6.17. Uzluksiz funksiyalar ustida arifmetik amallar haqidagi teoremalar ([3],1-q., 99-101 betlar; [5], 1-t., 148-149 betlar; [12], 1-q.,159-161 betlar; [30], 2-bo'lim).

6.18. Murakkab funksiyaning uzluksizligi haqidagi teorema ([3],1-q.., 104-105 betlar; [5], 1-t., 156-157 betlar; [12], 1-q.,151 bet; [30], 2-bo'lim).

6.19. Limitlarni hisoblashda funksiyaning uzluksizlikdan foydalanish ([12], 1-q.,162-164 betlar;).

6.20. Uzluksiz funksiyaning lokal xossalari ([3],1-q., 103-105 betlar; [12], 1-q.,164-165 betlar, [10], 1-q., 167-169 betlar, [28],101bet).

6.21. Uzluksiz funksiyaning global xossalari (Veyershtrassning birinchi va ikkinchi teoremalari) ([3],1-q., 105-108 betlar; [5], 1-t., 175-178 betlar; [12], 1-q.,168-169 betlar).

6.22. Uzluksiz funksiyaning global xossalari (Bolsano-Koshining birinchi va ikkinchi teoremalari) ([3],1-q., 108-109 betlar; [5], 1-t., 168-172 betlar; [12], 1-q.,165-168 betlar).

6.1-amaliy mashg'ulot
Funksiyaning limiti

1-misol. $\lim\limits_{x\to-2} x^2=4$ bo'lishini Koshi ta'rifi bo'yicha ko'rsating.

Yechilishi. *a) δ ni topish*. Faraz qilaylik ixtiyoriy $\varepsilon>0$ berilgan bo'lsin. Biz shunday $\delta=\delta(\varepsilon)>0$ sonni izlaymizki, x ning $0<|x-(-2)|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|x^2-4|<\varepsilon$ tengsizlik o'rinli bo'lsin. Buning uchun avvalo, $|x^2-4|$ va $|x-(-2)|$ ifodalar orasidagi bog'lanishni topish zarur. Bu bog'lanishni topish uchun esa, ularning ikkalasini ham soddalashtiramiz:

$$|x^2-4|=|x-2||x+2| \text{ va } |x-(-2)|=|x+2|.$$

$|x-2|$ ko'paytuvchi sonlar o'qida chegaralanmagan. Shuning uchun ko'paytuvchini sodda holda baholash uchun -2 nuqtani o'z ichida saqlaydigan biror oraliqni ajratamiz. Masalan, $a=-2$ nuqtaning $\delta=1$ atrofi $(-3;-1)$ ni qaraylik. $\forall x\in(-3;-1)$ uchun $|x-2|<5$ tengsizlik o'rinli bo'ladi.

Shunday qilib,

$$|x^2-4|<5|x+2| \qquad (1)$$

tengsizlik o'rinli bo'ladi. $a=-2$ nuqtaning δ atrofi bo'lgan $(-2-\delta;\ -2+\delta)$ oraliq $(-3;-1)$ atrofdan chiqib ketmasligi kerak, buning uchun $\delta=\min\left(1;\ \frac{\varepsilon}{5}\right)$ deb olish yetarli.

b) Topilgan *δ ning «ishlash»ini* ko'rsatamiz.

Agar $0<|x-(-2)|<\frac{\varepsilon}{5}$ bo'lsa, bundan $5|x-(-2)|<\varepsilon$ bo'lishi va (1) ga muvofiq, $|x^2-4|<\varepsilon$ tengsizlik kelib chiqadi.

Shunday qilib,

$$\lim_{x\to-2} x^2=4.$$

2-misol. Ushbu

$$f(x)=\sin\frac{1}{x} \quad (x\neq 0)$$

funksiyaning $x\to 0$ da limitga ega emasligini ko'rsating.

Yechilishi. Nol nuqtaning atrofidan nolga intiluvchi va noldan farqli ikkita harxil

$$\{x'_n\}=\left\{\frac{1}{n\pi}\right\},\quad \{x''_n\}=\left\{\frac{2}{(4n+1)\pi}\right\}$$

ketma-ketliklarni olaylik. U holda, ularga mos ketma-ketliklar:

$$f(x'_n)=\sin\frac{1}{x'_n}=\sin n\pi=0,$$

$$f(x''_n)=\sin\frac{1}{x''_n}=\sin\frac{(4n+1)\pi}{2}=\sin\left(2n\pi+\frac{\pi}{2}\right)=1.$$

bo'lib, $\lim\limits_{x\to\infty} f(x'_n)=0$, $\lim\limits_{x\to\infty} f(x''_n)=1$ bo'ladi. Bu esa, $f(x)=\sin\frac{1}{x}$ funksiyaning $x=0$ nuqtada limiti mavjud emasligini isbotlaydi.

Maple tizimidan foydalanib misolni yechish:

> limit(sin(((1)/x)),x=0);

-1..1

.

3-misol. Ushbu

1) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}$, $(\beta\neq 0)$, 2) $\lim\limits_{x\to 0}\dfrac{\operatorname{tg}x-\sin x}{\sin^3 x}$,

3) $\lim\limits_{x\to\infty} x\sin\dfrac{\pi}{x}$, 4) $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\pi}{\cos x}-2x\operatorname{tg}x\right)$

limitlarni hisoblang.

Yechilishi. 1) Berilgan $\dfrac{\sin\alpha x}{\sin\beta x}$ kasr funksiyaning shaklini almashtiramiz:

$$\alpha\frac{\sin\alpha x}{\alpha x}\frac{1}{\beta\dfrac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\dfrac{\sin\alpha x}{\alpha x}}{\dfrac{\sin\beta x}{\beta x}}.$$

U holda(6.3.4) va (6.5.5) formulalarga asosan,

$$\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\frac{\alpha}{\beta}\lim_{x\to 0}\frac{\dfrac{\sin\alpha x}{\alpha x}}{\dfrac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\dfrac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}.$$

Maple tizimidan foydalanib misolni yechish:

> limit(sin(alpha*x)/sin(beta*x),x=0);

$\frac{a}{b}$

.

2) $$\frac{\operatorname{tg}x-\sin x}{\sin^3 x}=\frac{\sin x(1-\cos x)}{\cos x\sin^3 x}=\frac{2\sin^2\frac{x}{2}}{\cos x\sin^2 x}=\frac{1}{2}\frac{\dfrac{\sin\frac{x}{2}}{x/2}\dfrac{\sin\frac{x}{2}}{x/2}}{\cos x\dfrac{\sin x}{x}\dfrac{\sin x}{x}}$$

$\frac{x}{2}=y$ deb olib, (6.3.4) va (6.5.5) formulalarga asosan,

$$\lim_{x\to 0}\frac{tgx-\sin x}{\sin^3 x}=\frac{1}{2}.$$

Maple tizimidan foydalanib misolni yechish:

> limit((tan(x)-sin(x))/(sin(x))^3,x=0);

$\frac{1}{2}$

.

3) Bunda $\frac{1}{x}=y$ almashtirishni bajaramiz. $x\to\infty$ da $y\to 0$. $x\sin\frac{\pi}{x}=\pi\frac{\sin\pi y}{\pi y}$ va (6.5.5) formulani e'tiborga olsak,

$$\lim_{x\to\infty} x\sin\frac{\pi}{x}=\lim_{y\to 0}\pi\frac{\sin\pi y}{\pi y}=\pi$$

ekanligini topamiz.

Maple tizimidan foydalanib misolni yechish:

> limit(x*sin(Pi/x),x=infinity);

π

.

4) $\frac{\pi}{2}-x=y$ almashtirishni bajaramiz:

$$\frac{\pi}{\cos x}-2xtgx=\frac{\pi-2x\sin x}{\cos x}=\frac{\pi-2(\frac{\pi}{2}-y)\cos y}{\cos(\frac{\pi}{2}-y)}=\frac{2y\cos y}{\sin y}=\frac{2\cos y}{\dfrac{\sin y}{y}}.$$

(6.3.4) va (6.5.5) formulani e'tiborga olgan holda,

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{\cos x}-2x\operatorname{tg}x\right)=\lim_{y\to 0}\frac{2\cos y}{\dfrac{\sin y}{y}}=2.$$

Maple tizimidan foydalanib misolni yechish:

> limit(Pi/cos(x)-2*x*tan(x),x=Pi/2);

2

.

4-misol. Ushbu

$$1)\ \lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{x^2},\quad 2)\ \lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}}$$

limitlarni hisoblang.

Yechilishi. 1) $\left(\frac{x^2+4}{x^2-4}\right)^{x^2}=\frac{(1+\frac{4}{x^2})^{x^2}}{(1-\frac{4}{x^2})^{x^2}}$. Bu kasrning surat va maxrajiga (6.3.4) va (6.5.6) formulani qo'llasak, natijada

$$\lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{x^2}=\lim_{x\to\infty}\frac{(1+\frac{4}{x^2})^{x^2}}{(1-\frac{4}{x^2})^{x^2}}=\frac{\lim\limits_{x\to\infty}(1+\frac{4}{x^2})^{x^2}}{\lim\limits_{x\to\infty}(1-\frac{4}{x^2})^{x^2}}=\frac{e^4}{e^{-4}}=e^8$$

ekanligini topamiz.

Maple tizimidan foydalanib misolni yechish:

> limit(((x^2+4)/(x^2-4))^(x^2),x=infinity);

e^8.

2) $\cos x=1-2\sin^2\frac{x}{2}$ ekanligini e'tiborga olib, (6.5.5) formulaga asosan,

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}}=\lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{-\frac{1}{x^2}}=e^{\lambda},$$

bunda $\lambda=\lim\limits_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{-x^2}=\frac{1}{2}\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{1}{2}$.

Shunday qilib, izlanayotgan limit $e^{\frac{1}{2}}$ ga teng bo'lar ekan, ya'ni

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}}=e^{\frac{1}{2}}=\sqrt{e}.$$

Maple tizimidan foydalanib misolni yechish:

> limit((cos(x))^(-1/(x^2)),x=0);

$\sqrt{e}$.

Mustaqil yechish uchun misol va masalalar

δ ning qanday qiymatlarida $0<|x-x_0|<\delta$ ekanligidan $|f(x)-b|<\varepsilon$ tengsizlikning o'rinliligi kelib chiqadi?

1. $f(x)=3x-2;\ x_0=1;\ b=1;\ \varepsilon=0{,}001.$

2. $f(x)=x^2;\ x_0=2;\ b=4;\ \varepsilon=0{,}001.$

$x\to x_0$ da $f(x)$ funksiyaning cheksiz katta ekanligi ma'lum. $|f(x)|>E$ tengsizlik o'rinli bo'lishi uchun x qanday bo'lishi lozim.

3. $f(x)=\frac{2+3x}{x};\ x_0=0;\ E=10^3$. **4** $f(x)=\frac{x+2}{x-4};\ x_0=4;\ E=1000.$

Funksiya limitining Geyne ta'rifidan foydalanib, quyidagi limitlarni toping.

5. $\lim\limits_{x\to 1}\frac{2x+1}{5x+3}$. **6.** $\lim\limits_{x\to 0}x\cos\frac{1}{x}$.

7. $\lim\limits_{x\to 0}x^2\sin\frac{1}{x}$. **8.** $\lim\limits_{x\to 0}x\,\mathrm{arcctg}\frac{1}{x}$.

Funksiya limitining Geyne ta'rifidan foydalanib, quyidagi limitlarning mavjud emasligini isbotlang:

9. $\lim\limits_{x\to 2}\sin\frac{1}{x-2}$. **10.** $\lim\limits_{x\to\infty}\cos x$. **11.** $\lim\limits_{x\to 0}\cos\frac{1}{x}$.

Quyidagi limitlarni hisoblang.

12. $\lim\limits_{t\to 0}t(2-\frac{3}{t})$. **13.** $\lim\limits_{x\to 3}\frac{x^3-27}{x-3}$. **14.** $\lim\limits_{x\to -2}\frac{(x^2-x-6)^2}{x+2}$.

15. $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}$. **16.** $\lim\limits_{t\to 0}\frac{1-1/t^2}{1-\frac{1}{t}}$. **17.** $\lim\limits_{t\to 0}\frac{1-\frac{1}{t}}{1+\frac{1}{t}}$.

Quyida berilgan funksiyalarning bir tomonli limitlarini toping.

18. $f(x)=2^{\frac{1}{x-1}},\ x\to 1\pm 0.$ **19.** $f(x)=\frac{4}{(x-2)^3},\ x\to 2\pm 0.$

20. $f(x)=\frac{x^2-1}{|x-1|},\ x\to 1\pm 0.$ **21.** $f(x)=\frac{\sqrt{1-\cos 2x}}{x},\ x\to 0\pm 0.$

Quyidagi limitlarni hisoblang.

22. $\lim\limits_{x\to 4}\frac{x^2-16}{x^2+x-20}$. **23.** $\lim\limits_{x\to -1}\frac{7x^2+4x-3}{2x^2+3x+1}$.

24. $\lim\limits_{x\to -5}\dfrac{x^2-2x-35}{2x^2+11x+5}$.

25. $\lim\limits_{x\to 0}\dfrac{3x^2+x}{4x^2-5x+1}$.

26. $\lim\limits_{x\to -\infty}\dfrac{4x^2+7x-10}{2x^4-3x+7}$.

27. $\lim\limits_{x\to -\infty}\dfrac{5x^4-2x^3+3}{2x^2+5x-7}$.

28. $\lim\limits_{x\to 0}\dfrac{3-\sqrt{x^2+9}}{5x^2}$.

29. $\lim\limits_{x\to 9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$.

30. $\lim\limits_{x\to -1}\dfrac{\sqrt{5+x}-2}{\sqrt{8-x}-3}$.

31. $\lim\limits_{x\to 10}\dfrac{\sqrt{x-1}-3}{\sqrt{x+6}-4}$.

32. $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{4x^2}$.

33. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^3 x}{3x^2}$.

34. $\lim\limits_{x\to 0}\dfrac{\sin 7x+\sin 3x}{x\cos 2x}$.

35. $\lim\limits_{x\to 0}\dfrac{\sin 4x-\sin 2x}{3x}$.

36. $\lim\limits_{x\to 0}\sin 3x\,\mathrm{ctg}\,2x$.

37. $\lim\limits_{x\to 0}\dfrac{\arcsin^2 3x}{2x\sin 5x}$.

38. $\lim\limits_{x\to 0}\dfrac{\arcsin 8x}{\sin 5x}$.

39. $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}\,2x-\sin 2x}{x^2}$.

40. $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x+10}\right)^{-4x}$.

41. $\lim\limits_{x\to\infty}\left(\dfrac{2x}{2x-3}\right)^{5x}$.

42. $\lim\limits_{x\to\infty}\left(\dfrac{4x}{3+4x}\right)^{-3x}$.

43. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{5x}$.

44. $\lim\limits_{x\to\infty}\left(\dfrac{1-x}{2-x}\right)^{3x}$.

45. $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+8}\right)^{-2x}$.

Birinchi ajoyib limitga doir misollarni yeching.

46. $\lim\limits_{x\to 0}\dfrac{\sin 20x}{x}$. **47.** $\lim\limits_{x\to 0}\dfrac{\sin 9x}{\sin 6x}$. **48.** $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{2x^2}$.

49. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$. **50.** $\lim\limits_{x\to 0}\dfrac{tg 8x}{3x}$. **51.** $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

Quyidagi funksiyalarning qaysi biri cheksiz kichik bo‘ladi:

52. $f(x)=\dfrac{x^2-6x+9}{x^3-27},\ x\to 3$.

53. $f(x)=\dfrac{1}{1+4^x}$; *a)* $x\to+\infty$; *b)* $x\to-\infty$.

Quyidagi funksiyalardan qaysi biri cheksiz katta bo‘ladi?

54. $f(x)=\dfrac{4x^3-8x+4}{x^3-3x^2+3x-1},\ x\to 1$.

55. $f(x)=\dfrac{1}{x^3-6x^2+9x}-\dfrac{1}{x^2-2x-3},\ x\to 3$.

Quyidagi tasdiqlardan qaysilari $x\to+\infty$ da to‘g‘ri ekanligini ko‘rsating.

56. $20x^2+x\sin x=O(x^2)$

57. $x^2=O(20x^2+x\sin x)$

Mustaqil yechish uchun berilgan misollarning javoblari

1. $\delta\le\dfrac{0{,}001}{3}=\dfrac{1}{3000}$. **2** $\delta\le\sqrt{4{,}001}-2\approx 0{,}00025$. **3.** $\left(-\dfrac{2}{1003};0\right)\cup\left(0;\dfrac{2}{9997}\right)$

4. $\left(\dfrac{3996}{1001};4\right)\cup\left(4;\dfrac{4003}{999}\right)$. **5.** $\dfrac{3}{8}$. **6.** 0. **7.** 0. **8.** 0. **12.** -3.

13. 27. **14.** 0. **15.** $\dfrac{1}{4}$. **16.** Mavjud emas. **17.** -1.

18. $f(1-0)=0,\ f(1+0)=+\infty$. **19.** $f(0-0)=-\infty,\ f(0+0)=+\infty$.

20. $f(1-0)=-2,\ f(1+0)=2$. **21.** $f(0-0)=-\sqrt{2},\ f(0+0)=\sqrt{2}$. **22.** $\dfrac{8}{9}$. **23.** 10. **24.** $\dfrac{4}{3}$. **25.** 0. **26.** 0. **27.** ∞. **28.** $\dfrac{1}{30}$. **29.** $-\dfrac{6}{5}$. **30.** $-\dfrac{3}{2}$. **31.** $\dfrac{4}{3}$. **32.** 3.125. **33.** $\dfrac{1}{3}$. **34.** 10. **35.** $-\dfrac{2}{3}$. **36.** $\dfrac{3}{2}$. **37.** 0.9. **38.** $\dfrac{8}{3}$. **39.** 0. **40.** e^{20}. **41.** $e^{7,5}$. **42.** $e^{\frac{9}{4}}$. **43.** e^{-10}. **44.** e^{3}. **45.** $e^{1,5}$. **46.** 20. **47.** $\dfrac{9}{8}$. **48.** 4. **49.** $\dfrac{a}{b}$. **50.** $\dfrac{8}{3}$. **51.** $\cos a$.

52. Ha. **53.** *a)* Ha. *b)* Yo‘q. **54.** Ha. **55.** Ha.

6.2-amaliy mashg'ulot
Funksiyaning uzluksizligi

1-misol. Ushbu $f(x)=\sqrt{x+6}$ funksiyaning $a=3$ nuqtada uzluksiz ekanligini ko'rsating.

Yechilishi. Berilgan funksiya $a=3$ nuqtada aniqlangan. **a)** $\forall\varepsilon>0$ son olib, bu ε songa ko'ra, δ>0 sonni δ=3ε bo'lsin deb karalsa, u holda $|x-3|<\delta$ bo'lganda

$$|f(x)-f(3)|=\left|\sqrt{x+6}-3\right|=\frac{|x-3|}{\sqrt{x+6}+3}<\frac{|x-3|}{3}<\frac{\delta}{3}=\varepsilon$$

bo'ladi.

b) Endi topilgan δ ning «ishlashini» ko'rsatamiz. $|x-3|<\delta=3\varepsilon$ bo'lsin.

Bundan $\varepsilon>\frac{|x-3|}{3}=\frac{\left|\sqrt{x+6}-3\right|\left|\sqrt{x+6}+3\right|}{3}>\left|\sqrt{x+6}-3\right|$, demak, $|x-3|<\delta=3\varepsilon$ tengsizlikni qanoatlantiruvchi x ning qiymatlarida $\left|\sqrt{x+6}-3\right|<3$ tengsizlik bajariladi.

Bu esa, ta'rifga asosan, qaralayotgan funksiyaning $a=3$ nuqtada uzluksiz ekanligini bildiradi.

2-misol. Ushbu

$$f(x)=\begin{cases}\frac{1}{1+8^{\frac{1}{x}}}, x\neq 0,\\ 0, \quad x=0\end{cases}$$

funksiyaning $x=0$ nuqtada o'ngdan va chapdan uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiyaning $x\to 0$ da o'ng va chap limitlarini topamiz:

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}\frac{1}{1+8^{\frac{1}{x}}}=0=f(0+0)=f(0),$$

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{1}{1+8^{\frac{1}{x}}}=1=f(0-0)\neq f(0).$$

Demak, ta'rifga ko'ra, berilgan funksiya $x=0$ nuqtada o'ngdan uzluksiz bo'lib, chapdan uzluksiz emas.

3-misol. Ushbu

$$f(x)=\begin{cases}\frac{1}{7}(2x^2+5), -\infty<x\leq 1 \text{ bo'lganda;}\\ 5-4x,\ 1<x<3 \text{ bo'lganda;}\\ x-5,\ 3\leq x<\infty \text{ bo'lganda,}\end{cases}$$

funksiyani uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiya (-∞;∞) da aniqlangan va (-∞;1), (1;3), (3;∞) larda uzluksiz bo'lib, u faqat $x=1$ va $x=3$ nuqtalarda uzilishga ega bo'lishi mumkin. Berilgan funksiyaning $x=1$ nuqtadagi bir tomonli limitlarini hisoblaymiz:

$$f(1\text{-}0)=\lim_{x\to 1-0}\frac{1}{7}(2x^2+5)=1,\quad f(1+0)=\lim_{x\to 1+0}(5-4x)=1.$$

Ma'lumki, $x=1$ nuqtada berilgan funksiyaning kiymati birinchi analitik ifoda bilan aniqlanadi:

$$f(1)=\frac{1}{7}(2\cdot 1^2+5)=1.$$

Demak, $f(1\text{-}0)=f(1+0)=f(1)=1$ bulgani uchun, 8.1-teoremaga asosan, berilgan funksiya $x=1$ nuqtada uzluksiz bo'ladi.

Endi $f(x)$ funksiyaning $x=3$ nuqtada chap va o'ng limitlarini hisoblaymiz:

$$f(3\text{-}0)=\lim_{x\to 3-0}(5-4x)=-7,\quad f(3+0)=\lim_{x\to 3+0}(x-5)=-2.$$

Bu yerdan $f(3\text{-}0)\neq f(3+0)=f(3)$.

Demak, $f(x)$ funksiya $x=3$ nuqtada 1-tur uzilishga ega bo'lib, uning $x=3$ nuqtadagi sakrash kattaligi

$$\omega=|f(3+0)-f(3-0)|=|-2+7|=5$$

bo'ladi.

Mustaqil yechish uchun misol va masalalar

1. Ushbu 1) $f(x)=3x-2$, 2) $f(x)=x^3$ funksiyalar uchun uzluksizlikning "$\varepsilon-\delta$" ta'rifiga ko'ra, $a=1$ nuqta uchun quyidagi jadvalni to'ldiring.

1)

ε	2	0,5	0,01	0,001	0,0001
δ					

2)

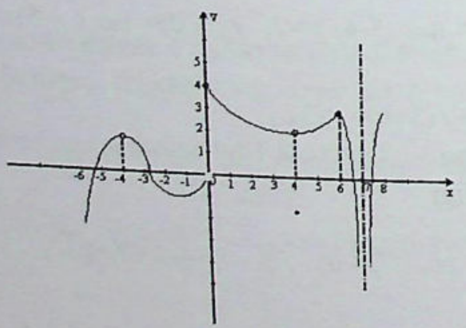
ε	2	0,5	0,01	0,001	0,0001
δ					

Koshi ta'rifdan foydalanib quyidagi funksiyalarning uzluksizligini ko'rsating.

2. $f(x)=x^2$. **3.** $f(x)=\sqrt{x}$. **4.** $f(x)=|x|$.

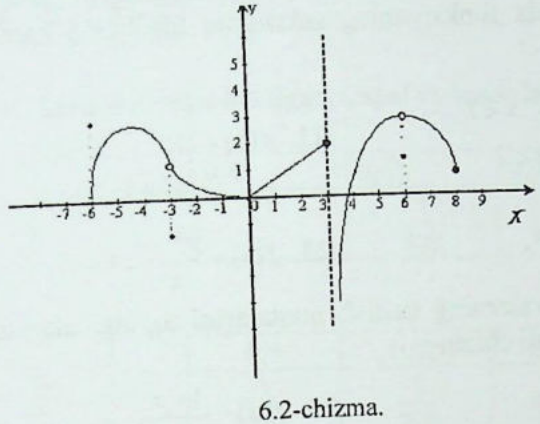
5. $f(x)=arctgx$. **6.** $f(x)=\begin{cases} x^2, & x - \text{рационал сон бўлганда}, \\ -x^2, & x - \text{иррационал сон бўлганда}. \end{cases}$

7. 6.1-chizmada f funksiyaning grafigi berilgan. 1) Qaysi nuqtalarda f funksiya uzilishga ega? 2) Har bir uzilish nuqtasida f funksiya chapdan uzluksizmi, o'ngdanmi yoki har ikala tomondan uzulishga egami, shuni aniqlang. 3) Agar uzluksizga ega bo'lsa, f funksiya qaysi quqtalarda yo'qotish mumkin bo'lgan uzluksizga ega, qaysi nuqtalarda birinchi tur uzilishiga.



6.1-chizma

8. 6.2-chizmada f funksiyaninggrafigi berilgan. Funksiyaning uzluksizlik integvallarini toping.



6.2-chizma.

Berilgan funksiya ko'rsatilgan nuqtada uzluksizmi, yo'qmi, shuni aniqlang. Agar uzluksiz bo'lmasa, uzulishining turini aniqlang.

9. $f(x)=x^2-3x+1, \quad x_0=3.$ **10.** $f(x)=\sqrt{x^2+4}, \quad x_0=2.$

11. $f(x)=\begin{cases} x^2+4, & x<2, \\ x^3, & x\ge 2, \end{cases} \quad x_0=2.$ **12.** $f(x)=\begin{cases} x^2+9, & x<2, \\ 7, & x=2, \\ x^3, & x>3, \end{cases} \quad x_0=3.$

13. $f(x)=x^2 signx$ funksiyaning grafigini chizing. Agar funksiya grafigi uzilishga ega bo'lsa, uzilish nuqtalarda uzilishning turini aniqlang.

14. $f(x)=\dfrac{x^2-9}{x-3}$. **15.** $f(x)=|x-1|$.

16. $f(x)=\begin{cases} x-1, & x<1, \\ 0, & x=1, \\ x^2, & x>1. \end{cases}$ **17.** $f(x)=\begin{cases} -1, & x<-1, \\ x^3, & -1\le x\le 1, \\ 1, & x>1. \end{cases}$

18-19-misollarda $f(x)$ funksiya $x=1$ nuqtadan tashqari R ning barcha nyqtalarida aniqlangan va uzluksiz. Agar mumkin bo'lsa, $f(1)$ qiymatni shunday tanlanki,natijada $f(x)$ funksiya butun R da uzluksiz bo'lsin.

18. $f(x)=\dfrac{x^2-1}{x-1}$. **19.** $f(x)=\dfrac{x-1}{|x-1|}$.

Quyidagi funksiyalarning uzilish nuqtalarini toping, turlarini aniqlang, 1-tur uzilish nuqtalarida funksiyaning sakrashini hisoblang hamda grafigini chizing.

20. $f(x)=\begin{cases} x^2+5, & x<2, \\ x^3, & x\ge 2. \end{cases}$ **21.** $f(x)=\begin{cases} x^2+5, & x<2, \\ 10, & x=2, \\ x^3+1, & x<2. \end{cases}$

22. $f(x)=(\operatorname{sign} x)^2$. **23.** $f(x)=\dfrac{|x|-x}{x^2}$.

Quyidagi funksiyalarning uzilish nuqtalarini toping, ularning turlarini aniqlang va grafiklarini chizing.

24. $f(x)=\dfrac{|x+3|}{x+3}$. **25.** $f(x)=\dfrac{1+x}{1+x^3}$.

26. $f(x)=\dfrac{|x-1|}{x^2-x^3}$. **27.** $f(x)=\dfrac{x^2-4}{x^2-5x+6}$.

28. $f(x)=\begin{cases} x^2+2, & x\le 0, \\ x-1, & x>0. \end{cases}$ **29.** $f(x)=\dfrac{1}{\ln x}$.

Ko'rsatilgan nuqtalarda berilgan funksiyalarni uzluksizlikka tekshiring:

30. $f(x)=2^{\frac{1}{x-5}}+1; x_1=5, x_2=6$. **31.** $f(x)=\dfrac{(x-1)^2}{|x-1|},\ x_1=1,\ x_2=2.$

32. $f(x)=6^{\frac{1}{x-1}}-3; x_1=1, x_2=2$. **33.** $f(x)=\dfrac{x-3}{x^2-9},\ x_1=3,\ x_2=-3.$

34. $f(x)=\dfrac{x+4}{x-3}; x_1=3, x_2=4$. **35.** $f(x)=\dfrac{x^2-4}{x-2},\ x_1=2,\ x_2=-2.$

Quyidagi funksiyalarning uzluksizlikka tekshiring va grafigini chizing:

36. $y=\lim\limits_{n\to\infty}(1-x)^{2n},\ |x|\le 1.$ **37** $y=\lim\limits_{n\to\infty}\dfrac{x^n-x^{-n}}{x^n+x^{-n}},\ x\ne 0.$

Quyidagi funksiyalar a va b ning qanday qiymatlarida uzluksiz bo'ladi?

39. $f(x)=\begin{cases} (x-2)^3, & x\le 0, \\ ax^2+b, & 0<x<1, \\ 4\sqrt{x}, & x\ge 1. \end{cases}$ **40.** $f(x)=\begin{cases} ax^2+1, & x>0, \\ -x, & x\le 0. \end{cases}$

41. $f(x)=\begin{cases} \dfrac{(x-1)^2}{x^2-1}, & |x|\ne 1, \\ a, & x=-1, \\ b, & x=1. \end{cases}$ **42.** $f(x)=\begin{cases} \dfrac{2^x-1}{x}, & x\ne 0, \\ a, & x=0. \end{cases}$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

1. 1)

ε	2	0,5	0,01	0,001	0,0001
δ	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{300}$	$\frac{1}{3000}$	$\frac{1}{30000}$

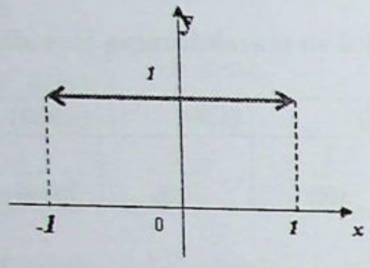
2)

ε	2	0,5	0,01	0,001	0,0001
δ	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{700}$	$\frac{1}{7000}$	$\frac{1}{70000}$

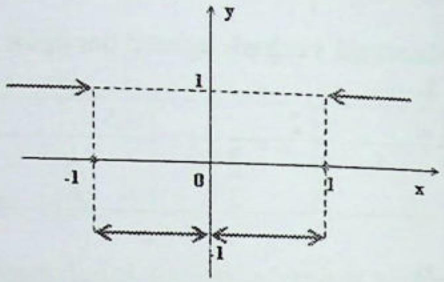
7. 1) $x=-4,\quad x=0,\quad x=4\quad x=7$ uzilish; 2) $x=-4$ ikki tomnlama uzilish, $x=0$ da o'ngdan uzluksiz, $x=4$ da har ikki tomondan uzilish, $x=7$ har ikki tomondan uzilish; 3) $x=4$ yo'qotilishi mumkin bo'lgan uzilish, $x=0$ birinchi tur tuzilish. **8.** $(-6;-3),(-3;3],(3;6),(6;8]$. **9.** Uzluksiz. 10. Uzluksiz. **11.** Uzluksiz. **12.** Yo'qotilishi mumkin bo'lgan uzulish. 14. $x=3$ da yo'qotilishi mumkin bo'lgan uzilish. **15.** Uzluksiz. **16.** Birinchi tur uzulish. **17**. Uzluksiz. **18.** $f(1)=2$. **19**. Mumkin emas. **20**. $x=2$ birinchi tur uzilish, $\Delta f(2)=-1$. **21**. $x=2$ birinchi tur uzilish, $\Delta f(2)=1$. **22**. $x=0$ yo'qotilishi mumkin bo'lgan nuqta. **23.** $x=0$ ikkinchi tur uzilish. **24**. $x=-3$ birinchi tur uzilish nuqtasi. **25.** $x=-1$ da yo'qotilishi mumkin bo'lgan uzilish. **26.** $x=0$ ikkinchi tur uzilish, $x=1$ birinchi tur uzilish. **27.** $x=2$ da yo'qotilishi mumkin bo'lgan uzilish, $x=3$ ikkinchi tur uzilish. **28.** $x=0$ birinchi tur uzilish. **29.** $x=0$ da yo'qotilishi mumkin bo'lgan nuqta, $x=1$ ikkinchi tur uzilish. **30.** $x_1=5$ ikkinchi tur uzilish, $x_2=6$ nuqtada uzluksiz. **31**. $x_1=1$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=2$ nuqtada uzluksiz. **32.** $x_1=1$ ikkinchi tur uzilish , $x_2=2$ nuqtada uzluksiz. **33.** $x_1=3$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-3$ ikkinchi tur uzilish. **34.** $x_1=3$ ikkinchi tur uzilish , $x_2=4$ nuqtada uzluksiz.

35. $x_1=2$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-2$ nuqtada uzluksiz. **36.** $x=-1, x=1$ yo'qotilishi mumkin bo'lgan uzilish nuktalari. Bu funksiyaning grafigi 6.3-chizmada tasvirlangan. **37.** $x=-1, x=1$ da birinchi tur uzilish, $x=0$ da esa, yo'qotilishi mumkin bo'lgan uzilish. Bu funksiyaning grafigi 6.4-

chizmada tasvirlangan. **39.** $a=12,\ b=8$. **40**. Shunday a son mavjud emas. **41.** Shunday a va b sonlar mavjud emas.



6.3-chizma.



6.4-chizma.

6-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

6.1-masala. Quyidagi funksiyaning limitlarini hisoblang.

6.1.1. $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$. **6.1.2.** $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x^2}-4}$.

6.1.3. $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{x-6}+2}{x+2}$. **6.1.4.** $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x^2-1}$.

2.6.3. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$. **6.1.6.** $\lim\limits_{x\to\infty}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$.

6.1.7. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8+3x+x^2}-2}{x+x^2}$. **6.1.8.** $\lim\limits_{x\to 0}\dfrac{\sqrt{1-2x+x^2}-(1+x)}{x}$.

6.1.9. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$. **6.1.10.** $\lim\limits_{x\to -8}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$.

6.1.11. $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$. **6.1.12.** $\lim\limits_{x\to 4}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[7]{x}}$.

6.1.13. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x^2}-1}$. **6.1.14.** $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$.

6.1.15. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{1+x}-\sqrt{2x}}$. **6.1.16.** $\lim\limits_{x\to 5}\dfrac{\sqrt{6-x}-1}{3-\sqrt{4+x}}$.

6.1.17. $\lim\limits_{x\to 1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}(m,n\in N)$. **6.1.18.** $\lim\limits_{x\to 16}\dfrac{\sqrt[4]{x^3}-8}{\sqrt{x}-4}$.

6.1.19. $\lim\limits_{x\to 6}\dfrac{\sqrt{x-2}-2}{x-6}$. **6.1.20.** $\lim\limits_{x\to 5}\dfrac{\sqrt{6-x}-1}{3-\sqrt{4+x}}$.

6.1.21. $\lim\limits_{x\to 1}\dfrac{\sqrt{1+8x}-3}{\sqrt{x}-1}$. **6.1.22.** $\lim\limits_{x\to 1}\dfrac{\sqrt{5+20x}-5}{\sqrt[3]{x^2}-1}$.

6.1.23. $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{2x-4}+2}{x+2}$. **6.1.24.** $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x^3-1}$.

6.1.25. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-\sqrt{1-2x}}{\sqrt[3]{1+2x}-\sqrt[3]{1-2x}}$.

6.1.26-misol. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+8}-2\sqrt{2}}{2-\sqrt{x+4}}$ limitni hisoblang.

Yechilishi ([2], 4-bo'lim; [3], 1-q., 75-85 betlar, [30], 2-bo'lim). Limitni hisoblash uchun kasr ifodani surati va maxrajining qo'shmalariga ko'paytiramiz:

$$\lim_{x\to 0}\frac{\sqrt{x+8}-2\sqrt{2}}{2-\sqrt{x+4}}=\lim_{x\to 0}\frac{\sqrt{x+8}-2\sqrt{2}}{2-\sqrt{x+4}}\cdot\frac{\sqrt{x+8}+2\sqrt{2}}{2+\sqrt{x+4}}\cdot\frac{2+\sqrt{x+4}}{\sqrt{x+8}+2\sqrt{2}}=$$

$$=\lim_{x\to 0}\frac{(x+8-8)\left(2+\sqrt{x+4}\right)}{(4-x-4)\left(\sqrt{x+8}+2\sqrt{2}\right)}=-\frac{\sqrt{2}}{2}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit((sqrt(x+8)-2*sqrt(2))/(2-sqrt(x+4)),x=0)=limit((sqrt(x+8)-2*sqrt(2))/(2-sqrt(x+4)),x=0);

$$\lim_{x\to 0}\frac{\sqrt{x+8}-2\sqrt{2}}{2-\sqrt{x+4}}=-\frac{\sqrt{2}}{2}.$$

6.2 – masala. Quyidagi funksiyaning limitlarini hisoblang.

6.2.1. $\lim\limits_{x\to 2}\dfrac{x^2+4x-5}{x^2-1}$.

6.2.2. $\lim\limits_{x\to 1}\dfrac{x^2+4x-5}{x^2-1}$.

6.2.3. $\lim\limits_{x\to -1}\dfrac{x^3-3x-2}{x+x^2}$.

6.2.4. $\lim\limits_{x\to 1}\dfrac{(2x^2-x-1)^2}{x^3+2x^2-x-2}$.

6.2.5. $\lim\limits_{x\to 1}\dfrac{x^2-7x+6}{x^2-1}$.

6.2.6. $\lim\limits_{x\to 0}\dfrac{x^7+5x^6+4x^3}{x^7+2x^3}$.

6.2.7. $\lim\limits_{x\to 7}\dfrac{2x^2-11x-21}{x^2-9x+14}$.

6.2.8. $\lim\limits_{x\to 1}\left(\dfrac{3}{1-x^3}+\dfrac{1}{x-1}\right)$.

6.2.9. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{2x^2-x-1}$.

6.2.10. $\lim\limits_{x\to -1}\dfrac{x^2+5x^2+7x+3}{x^3+4x^2+5x+2}$.

6.2.11. $\lim\limits_{x\to 1}\dfrac{x4-1}{2x^4-x^2-1}$.

6.2.12. $\lim\limits_{x\to 1}\dfrac{2x^2-x-1}{x^3+2x^2-x-2}$.

6.2.13. $\lim\limits_{x\to -1}\dfrac{x^3-3x-2}{x^2+2x+1}$.

6.2.14. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{x^3-x^2-x+1}$.

6.2.15. $\lim\limits_{x\to 2}\left(\dfrac{2}{2x-x^2}+\dfrac{1}{x^2-3x+2}\right)$.

6.2.16. $\lim\limits_{x\to -1}\dfrac{x^3-3x-2}{x^2-x-2}$.

6.2.17. $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^3-x^2-x+1}$.

6.2.18. $\lim\limits_{x\to 3}\dfrac{x^3-4x^2-3x+18}{x^3-5x^2+3x+9}$.

6.2.19. $\lim\limits_{x\to -3}\dfrac{x^2+4x-3}{x^3+4x^2+3x}$.

6.2.20. $\lim\limits_{x\to 2}\dfrac{x^3-3x-2}{x-2}$.

6.2.21. $\lim\limits_{x\to 2}\dfrac{x^2-4x+4}{x^2-4}$.

6.2.22. $\lim\limits_{x\to 1}\dfrac{x^2-10x+9}{x^2-1}$.

6.2.23. $\lim\limits_{x\to -1}\dfrac{x^3-3x-2}{x+x^2}$.

6.2.24. $\lim\limits_{x\to 1}\dfrac{(2x^2-x-1)^2}{x^3+2x^2-x-2}$.

6.2.25. $\lim\limits_{x\to\infty}\dfrac{x^2-11x+10}{x^2-1}$.

6.2.26-misol. $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x^3-27}$ limitini hisoblang.

Yechilishi ([2], 4-bo'lim; [3], 1-q., 75-85 betlar, [30], 2-bo'lim). Limitni hisoblash uchun kasr ifodaning surat va maxrajini ko'paytuvchilarga ajratamiz:

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^3-27}=\lim_{x\to 3}\frac{(x-3)(x-2)}{(x-3)(x^2+3x+9)}=\lim_{x\to 3}\frac{x-2}{x^2+3x+9}=\frac{1}{27}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit((x^2-5*x+6)/(x^3-27),x=3)=limit((x^2-5*x+6)/(x^3-27),x=3);

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^3-27}=\frac{1}{27}.$$

6.3 – masala. Quyidagi funksiyaning limitini hisoblang.

6.3.1. $\lim\limits_{x\to 0}\dfrac{4x^2-5x}{\sin 3x}$.

6.3.2. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{3x+1}}{\cos[\pi(x+1)/2]}$.

6.3.3. $\lim\limits_{x\to 0}\dfrac{4x}{\operatorname{tg}(\pi(2+x))}$.

6.3.4. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{\sqrt{2+x}-\sqrt{2}}$.

6.3.5. $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{4x^2}$.

6.3.6. $\lim\limits_{x\to 0}\dfrac{2^x-1}{\ln(1+2x)}$.

6.3.7. $\lim\limits_{x\to 0}\dfrac{\ln(1-7x)}{\sin(\pi(x+7))}$.

6.3.8. $\lim\limits_{x\to 0}\dfrac{\operatorname{arctg} 2x}{\sin(2\pi(x+10))}$.

6.3.9. $\lim\limits_{x\to 0}\dfrac{9\ln(1-2x)}{4\operatorname{arctg} 3x}$.

6.3.10. $\lim\limits_{x\to 0}\dfrac{\sin 9x}{x^2+5x}$.

6.3.11. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{\sin(\pi(x+2))}$.

6.3.12. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{x^2+\pi x}$.

6.3.13. $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{3\operatorname{arctg} x}$.

6.3.14. $\lim\limits_{x\to 0}\dfrac{1-\cos 10x}{e^{x^2}-1}$.

63.15. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\cos 7x-\cos 3x}$.

6.3.16. $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos x}{1-\cos x}$.

6.3.17. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\left(e^{3x}-1\right)^2}$.

6.3.18. $\lim\limits_{x\to 0}\dfrac{2x\sin x}{1-\cos x}$.

6.3.19. $\lim\limits_{x\to 0}\dfrac{\ln\left(x^2+1\right)}{1-\sqrt{x^2+1}}$.

6.3.20. $\lim\limits_{x\to 0}\dfrac{\sin[5(x+\pi)]}{e^{3x}-1}$.

6.3.21. $\lim\limits_{x\to 0}\dfrac{4x^2-5x}{tg\,3x}$.

6.3.22. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{3x+1}}{\cos[\pi(x+1)/2]}$.

6.3.23. $\lim\limits_{x\to 0}\dfrac{5x}{\mathrm{tg}\left(\pi(2+x)\right)}$.

6.3.24. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{x+4x^2}$.

6.3.25. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{4x^2}$.

6.3.26-misol. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\sin 7x-\sin 3x}$ limitni hisoblang.

Yechilishi ([2], 4-bo'lim; [3], 1-q., 75-85 betlar, [30], 2-bo'lim). Limitni hisoblashda quyidagi

$$1-\cos 2x=2\sin^2 x,\ \sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2},\ \lim_{x\to 0}\frac{\sin x}{x}=1$$

formulalardan foydalanamiz: $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\sin 7x-\sin 3x}=\lim\limits_{x\to 0}\dfrac{2\sin^2 x}{2\sin 2x\cos 5x}=0$ ekanligini topamiz.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit((1-cos(2*x))/(sin(7*x)-sin(3*x)),x=0)=limit((1-cos(2*x))/(sin(7*x)-sin(3*x)),x=0);

$$\lim_{x\to 0}\frac{1-\cos(2x)}{\sin(7x)-\sin(3x)}=0$$

6.4 – masala. Quyidagi funksiyaning limitini hisoblang.

6.4.1. $\lim\limits_{x\to 0}\left(\dfrac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$.

6.4.2. $\lim\limits_{x\to 1}\left(\dfrac{1+x}{3+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$.

6.4.3. $\lim\limits_{x\to 1}\left(\dfrac{1+x}{5+x}\right)^{\frac{1-x^2}{1-x}}$.

6.4.4. $\lim\limits_{x\to +\infty}\left(\dfrac{x+1}{2x-1}\right)^{x}$.

6.4.5. $\lim\limits_{x\to -\infty}\left(\dfrac{x+1}{2x-1}\right)^{x}$.

6.4.6. $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{x}\right)^{1+x}$.

6.4.7. $\lim\limits_{x\to 0}\left(\dfrac{\sin 4x}{x}\right)^{\frac{2}{x+2}}$.

6.4.8. $\lim\limits_{x\to 0}(\cos x)^{x+3}$.

6.4.9. $\lim\limits_{x\to 0}\left(\dfrac{2+x}{3-x}\right)^{x}$.

6.4.10. $\lim\limits_{x\to 0}\left(\dfrac{x^2+4}{x+2}\right)^{x^2+3}$.

6.4.11. $\lim\limits_{x\to 0}\left(\dfrac{x+2}{x+4}\right)^{\cos x}$.

6.4.12. $\lim\limits_{x\to 0}\left(\dfrac{\sin 6x}{2x}\right)^{2+x}$.

6.4.13. $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{\sin 3x}\right)^{x^2}$.

6.4.14. $\lim\limits_{x\to \infty}\left(\dfrac{x^2+8}{3x^2+10}\right)^{x+2}$.

6.4.15. $\lim\limits_{x\to 1}\left(\dfrac{11x+8}{22x+1}\right)^{\cos^2 x}$.

6.4.16. $\lim\limits_{x\to 0}\left(\dfrac{2^{2x}-1}{x}\right)^{x+1}$.

6.4.17. $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{2(x+5)}$.

6.4.18. $\lim\limits_{x\to 0}\left(e^x+x\right)^{\cos^4 x}$.

6.4.19. $\lim\limits_{x\to 0}\left(\dfrac{x^4+5}{x+10}\right)^{\frac{4}{x+2}}$.

6.4.20. $\lim\limits_{x\to \infty}\left(\dfrac{x^3+1}{x^3+8}\right)^{\frac{2}{x+1}}$.

6.4.21. $\lim\limits_{x\to 0}\left(\dfrac{1+x}{3+x}\right)^{\frac{1-\sqrt{x}}{1-x^2}}$.

6.4.22. $\lim\limits_{x\to 1}\left(\dfrac{2+x}{5+x}\right)^{\frac{1-\sqrt{x}}{1-x^3}}$.

6.4.23. $\lim\limits_{x\to 1}\left(\dfrac{1+x}{5+x}\right)^{\frac{1-x^3}{1-x}}$.

6.4.24. $\lim\limits_{x\to +\infty}\left(\dfrac{x+1}{7x-1}\right)^{2x}$.

6.4.25. $\lim\limits_{x\to -\infty}\left(\dfrac{x+1}{5x-1}\right)^{3x}$.

6.4.26-misol. $\lim\limits_{x\to \infty}\left(\dfrac{x^2+4}{x^2-4}\right)^{x^2}$ ni hisoblang.

Yechilishi ([2], 4-bo‘lim; [3], 1-q., 75-85 betlar, [30], 2-bo‘lim). $\left(\frac{x^2+4}{x^2-4}\right)^{x^2} = \frac{(1+\frac{4}{x^2})^{x^2}}{(1-\frac{4}{x^2})^{x^2}}$. Bu kasrning surat va maxrajiga (6.3.4) formulani qo‘llasak, natijada, (6.5.6) formulaga asosan,

$$\lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{x^2} = \lim_{x\to\infty}\frac{(1+\frac{4}{x^2})^{x^2}}{(1-\frac{4}{x^2})^{x^2}} = \frac{\lim\limits_{x\to\infty}(1+\frac{4}{x^2})^{x^2}}{\lim\limits_{x\to\infty}(1-\frac{4}{x^2})^{x^2}} = \frac{e^4}{e^{-4}} = e^8$$

ekanligini topamiz.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit(((x^2+4)/(x^2-4))^((x)^2),x=infinity)=limit(((x^2+4)/(x^2-4))^((x)^2),x=infinity);

$$\lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{(x^2)} = e^8$$

6.5 – masala. $y=f(x)$, funksiyaning $x=x_0$ nuqtadagi o‘ng va chap limitlari $(f(x_0+0), f(x_0-0))$ topilsin.

6.5.1. $f(x)=\text{arctg}\frac{1}{2-x}$, $x_0=2$.

6.5.2. $f(x)=\frac{1}{1+e^{\frac{1}{x}}}$, $x_0=0$.

6.5.3. $f(x)=\arccos x(x-1)$, $x_0=0$.

6.5.4. $f(x)=\frac{1}{1+e^{\frac{1}{x-5}}}$, $x_0=5$.

6.5.5. $f(x)=\frac{|\sin x|}{|x|}$, $x_0=0$.

6.5.6. $f(x)=\frac{\sqrt{1-\cos 2x}}{x}$, $x_0=0$.

6.5.7. $f(x)=e^{\frac{1}{2x}}$, $x_0=0$.

6.5.8. $f(x)=\frac{x}{(x-3)^3}$, $x_0=3$.

6.5.9. $f(x)=\frac{1}{1+2^{\frac{1}{2-x}}}$, $x_0=2$.

6.5.10. $f(x)=\frac{\sin x}{|x|}$, $x_0=0$.

6.5.11. $f(x)=\frac{(1+x)^{\frac{1}{x}}}{|x|}$, $x_0=0$.

6.5.12. $f(x)=\frac{x-|x|}{2x}$, $x_0=0$.

6.5.13. $f(x)=\frac{x^3-1}{|x-1|}$, $x_0=1$.

6.5.14. $f(x)=\begin{cases} x+1, & x\le 2, \\ -2x+1, & x>2 \end{cases}$, $x_0=2$.

6.5.15. $f(x)=\frac{1}{x-[x]}$, $x_0=-1$.

6.5.16. $f(x)=\text{sign}(\cos x)$, $x_0=\frac{\pi}{2}$.

6.5.17. $f(x)=\text{arctg}\frac{1}{1-x}$. $x_0=1$.

6.5.18. $f(x)=\text{arctg}(\text{tg}\, x)$, $x_0=\frac{\pi}{2}$.

6.5.19. $f(x)=\arccos(x-1)$, $x_0=0$.

6.5.20. $f(x)=\frac{\cos x}{3-2\sin x}$, $x_0=0$.

6.5.21. $f(x)=\text{arctg}\frac{1}{4-x}$, $x_0=4$.

6.5.22. $f(x)=\frac{1}{1+3^{\frac{1}{x}}}$, $x_0=0$.

6.5.23. $f(x)=\arccos x(x-2),\ x_0=0.$

6.5.24. $f(x)=\dfrac{1}{1+3^{\frac{1}{x-3}}},\ x_0=3.$

6.5.25. $f(x)=\dfrac{|tgx|}{|x|},\quad x_0=0.$

2.3.10.26-misol. $f(x)=\begin{cases}x^2+1, & x\le 1,\\ -2x+1, & x>1,\end{cases}$ funksiyaning $x=1$ nuqtadagi o'ng va chap limitlarini toping.

Yechilishi ([2], 4-bo'lim; [3], 1-q., 75-85 betlar, [30], 2-bo'lim). $x=1$ nuqtada funksiyaning o'ng va chap limitlarini topamiz:

$$\lim_{x\to 1-0}(x^2+1)=2=f(1-0),\ \lim_{x\to 1+0}(-2x+1)=-1=f(1+0).$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit(x^2+1,x=1,left)= limit(x^2+1,x=1,left);

Limit(-2*x+1,x=1,right)=limit(-2*x+1,x=1,right);

$$\lim_{x\to 1-} x^2+1=2\ ,\quad \lim_{x\to 1+} -2x+1=-1\ .$$

7-bob. FUNKSIYANING HOSILASI VA DIFFERENSIALI. HOSILANING TADBIQLARI

7.1-§. Funksiyaning hosilasi va differensiali

7.1. Funksiya hosilasining ta'riflari. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lsin. Bu oraliqdan $\forall x_0$ nuqta olib, unga $x_0+\Delta x\in(a,b)$ bo'lgan Δx ($\Delta x<0$ yoki $\Delta x>0$) orttirma beraylik, y holda $f(x)$ funksiya ham x_0 nuqtada $\Delta y=\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$ orttirmaga ega bo'ladi. Ushbu

$$\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

($\Delta x\neq 0$) nisbatni qaraymiz. Ravshanki, bu nisbat Δx ning funksiyasi bo'lib, u Δx ning noldan farqli qiymatlarida, jumladan, nol nuqtaning yetarli kichik $\dot U_\delta(0)$ atrofida aniqlangan, bunda $\Delta x=0$ nuqta-$\dot U_\delta(0)$ to'plamning limit nuqtasi.

7.1.1-ta'rif. Agar $\Delta x\to 0$ da $\dfrac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

mavjud va chekli bo'lsa, bu limit $f(x)$ funksiyaning x_0 *nuqtadagi hosilasi* deyiladi va

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\tag{7.1.2}$$

kabi belgilanadi.

Agar $x_0+\Delta x=x$ deb olinsa, unda $\Delta x=x-x_0$ va $\Delta x\to 0$ da $x\to x_0$, natijada (7.1.2) ning ko'rinishi

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

shaklda bo'ladi. Hosila quyidagi $y'(x_0)$, y'_x (*Lagranj*), $\dfrac{dy}{dx}$, $\dfrac{df}{dx}$ (*Leybnis*), Dy, Df (*Koshi*) belgilar yordamida ham yoziladi.

Berilgan nuqtada chekli hosilaga ega bo'lgan funksiya, shu nuqtada albatta uzluksiz bo'ladi, ya'ni funksiyaning nuqtada chekli hosilaga ega bo'lishi uchun, uning shu nuqtada uzluksiz bo'lishi zarur, lekin yetarli emas.

7.1.3-ta'rif. Agar $\Delta x \to +0$ $(\Delta x \to -0)$ da $\frac{\Delta y}{\Delta x}$ ning chekli limiti

$$\lim_{\Delta x \to +0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to +0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad \left(\lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to -0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right)$$

mavjud bo'lsa, bu limit $f(x)$ funksiyaning x_0 *nuqtadagi o'ng (chap) hosilasi* deb ataladi va u $f'(x_0+0)$ $(f'(x_0-0))$ kabi belgilanadi. Odatda funksiyaning o'ng va chap hosilalari bir tomonli hosilalar deb ham aytiladi.

7.1.4-eslatma. Agar $f(x)$ funksiya x_0 nuqtada hosilaga ega bo'lsa, funksiya shu nuqtada bir tomonli $f'(x_0+0)$, $f'(x_0-0)$ hosilalarga ham ega bo'lib, $f'(x_0+0) = f'(x_0-0) = f'(x_0)$ tengliklar o'rinli bo'ladi

7.1.5-eslatma. Agar $f(x)$ funksiya x_0 nuqtaning biror $U(x_0)$ atrofida uzluksiz, x_0 nuqtada bir tomonli $f'(x_0+0)$ va $f'(x_0-0)$ hosilalarga ega bo'lib, $f'(x_0+0)=f'(x_0-0)$ bo'lsa, $f(x)$ funksiya shu nuqtada $f'(x_0)$ hosilaga ega bo'ladi va $f'(x_0) = f'(x_0+0) = f'(x_0-0)$ tengliklar o'rinli.

7.1.6-eslatma. Agar $\Delta x \to 0$ da $\frac{\Delta y}{\Delta x}$ nisbat aniq ishorali cheksiz limitga ega, ya'ni

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \pm\infty$$

bo'lsa, u ham $f(x)$ funksiyaning x_0 *nuqtadagi hosilasi* deb yuritiladi. Bunday hosila *cheksiz hosila* deb ataladi.

7.2. Hosilaning geometrik ma'nosi. $f(x)$ funksiya $(a;b)$ oraliqda aniqlangan va uzluksiz bo'lib, $x_0 \in (a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lsin. U holda $f(x)$ funksiyaning grafigiga $M_0(x_0, f(x_0))$ nuqtada o'tkazilgan urinma mavjud bo'ladi. Ma'lumki, funksiyaning x_0 nuqtadagi $f'(x_0)$ hosilasi, shu urinmaning burchak koeffisiyentini ifodalaydi. Urinma chiziq tenglamasi

$$y - f(x_0) = f'(x_0)(x - x_0)$$

bo'lib, bunda $f'(x_0) = \operatorname{tg}\alpha$, egri chiziqning $M_0(x_0, f(x_0))$ nuqtasiga o'tkazilgan normalning tenglamasi esa,

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$$

ko'rinishda bo'ladi. $y_1 = f_1(x)$ va $y_2 = f_2(x)$ funksiyalar grafiklarining $M(x_1, y_1)$ kesishish nuqtasida o'tkazilgan urinmalar orasidagi φ burchak berilgan ikki egri chiziq orasidagi burchak bo'ladi va

$$\operatorname{tg}\varphi = \operatorname{tg}(\varphi_2 - \varphi_1) = \frac{f_2'(x_1) - f_1'(x_1)}{1 + f_1'(x_1) f_2'(x_1)} \qquad (7.2.1)$$

formuladan topiladi.

7.3. Hosilaning fizik ma'nosi. Moddiy nuqtaning to'g'ri chiziqli harakati $s = f(t)$ tenglama orqali ifodalangan bo'lsin, bunda t -vaqt, s -shu vaqt ichida o'tilgan yo'l (masofa). $s = f(t)$ funksiyaning t_0 nuqtadagi hosilasi $s = f(t)$ qonun bo'yicha harakat qilayotgan moddiy nuqtaning t_0 momentdagi oniy tezligini bildiradi, ya'ni

$$v = f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}.$$

Moddiy nuqtaning berilgan $t = t_0$ momentdagi a tezlanishi esa, v tezlikdan t vaqt bo'yicha olingan hosilaning $t = t_0$ dagi qiymatiga tengdir, ya'ni

$$a = v'(t_0) = \lim_{\Delta t \to 0} \frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t}.$$

7.4. Hosilani hisoblashning sodda qoidalari. 1. O'zgarmas sonning hosilasi nolga teng: $(C)' = 0$ (bunda C-o'zgarmas son).

2. O'zgarmas sonni hosila ishorasidan tashqariga chiqarish mumkin: $(Cf(x))' = C \cdot (f(x))'$ (bunda C-o'zgarmas son).

3. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x \in (a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilalarga ega bo'lsa, $f(x) \pm g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x) \qquad (7.4.1)$$

formula bo'yicha topiladi.

4. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x \in (a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lsa, $f(x) \cdot g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x) \qquad (7.4.2)$$

formula bo'yicha topiladi.

5. Agar $u_1(x),...,u_n(x)$ funksiyalarning har biri $x \in (a,b)$ nuqtada $u'_1(x),...,u'_n(x)$ hosilaga ega bo'lsa, $u_1(x) \cdot ... \cdot u_n(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[u_1(x) \cdot ... \cdot u_n(x)]' = u_1(x) \cdot ... \cdot u_n(x)[\frac{u'_1(x)}{u_1(x)} + ... + \frac{u'_n(x)}{un_1(x)}]$$

formula bo'yicha topiladi.

6. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x \in (a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lib, $g(x) \neq 0$ bo'lsa, $\frac{f(x)}{g(x)}$ funksiya ham x nuqtada hosilaga ega va u

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \qquad (7.4.3)$$

formula bo'yicha topiladi.

7. Agar $u_1(x),...,u_n(x)$ va $v_1(x),...,v_n(x)$ funksiyalarning har biri $x \in (a,b)$ nuqtada $u'_1(x),...,u'_n(x)$ va $v'_1(x),...,v'_n(x)$ hosilaga ega bo'lib, $v_i(x) \neq 0, i = \overline{1,n}$ bo'lsa, $\frac{u_1(x) \cdot ... \cdot u_n(x)}{v_1(x) \cdot ... \cdot v_n(x)}$ funksiya ham x nuqtada hosilaga ega va u

$$\left[\frac{u_1(x) \cdot ... \cdot u_n(x)}{v_1(x) \cdot ... \cdot v_n(x)}\right]' = \frac{u_1(x) \cdot ... \cdot u_n(x)}{v_1(x) \cdot ... \cdot v_n(x)}\left[\frac{u'_1(x)}{u_1(x)} + ... \frac{u'_n(x)}{u_n(x)} - \left(\frac{v'_1(x)}{v_1(x)} + ... \frac{v'_n(x)}{v_n(x)}\right)\right]$$

formula bo'yicha topiladi.

8. $f(x)$ funksiya **5.12.17–** teoremaning hamma shartlarini qanoatlantirsin. Agar $f(x)$ funksiya $x_0 \in (a,b)$ nuqtada $f'(x_0) \neq 0$ hosilaga ega bo'lsa, bu funksiyaga teskari $x = f^{-1}(y)$ funksiya ham x_0 nuqtaga mos bo'lgan y_0 $(y_0 = f(x_0))$ nuqtada hosilaga ega va bu hosila

$$[f^{-1}(y)]'\Big|_{y=y_0} = \frac{1}{f'(x_0)}$$

formula orqali topiladi.

9. $y = F(u)$funksiya (c,d) oraliqda aniqlangan, $u = f(x)$ funksiya esa, (a,b) intervalda aniqlangan bo'lib, bu funksiyalar yordamida $y = F(f(x))$ murakkab funksiya tuzilgan bo'lsin.

Agar $u = f(x)$ funksiya $x_0 \in (a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lib, $y = F(u)$ funksiya esa, x_0 nuqtaga mos u_0 $(u_0 = f(x_0))$ nuqtada $F'(u_0)$ hosilaga ega bo'lsa, $\Phi(x) = F(f(x))$ murakkab funksiya ham x_0 nuqtada hosilaga ega va bu hosila $\Phi'(x) = [F(f(x))]'\big|_{x=x_0} = F'(u_0) \cdot f'(x_0)$ formula orqali topiladi.

10. Oshkormas $F(x,y) = 0$ funksiya uchun

$$y'(x) = -\frac{F'_x}{F'_y}$$

formula o'rinli.

11. Parametrik tenglamasi bilan berilgan $\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}, (t_0 \le t \le t_1)$ funksiyaning hosilasi, quyidagi,

$$y'_x = \frac{\psi'(t)}{\phi'(t)} \text{ yoki } y'_x = \frac{y'_t(t)}{x'_t(t)} \qquad (7.4.4)$$

formula bo'yicha topiladi.

3. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilalarga ega bo'lsa, $f(x)\pm g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x)\pm g(x)]' = f'(x)\pm g'(x) \qquad (7.4.1)$$

formula bo'yicha topiladi.

4. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lsa, $f(x)\cdot g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x)\cdot g(x)]' = f'(x)\cdot g(x)+f(x)\cdot g'(x) \qquad (7.4.2)$$

formula bo'yicha topiladi.

5. Agar $u_1(x),...,u_n(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $u'_1(x),...,u'_n(x)$ hosilaga ega bo'lsa, $u_1(x)\cdot...\cdot u_n(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[u_1(x)\cdot...\cdot u_n(x)]' = u_1(x)\cdot...\cdot u_n(x)[\frac{u'_1(x)}{u_1(x)}+...+\frac{u'_n(x)}{un_1(x)}]$$

formula bo'yicha topiladi.

6. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lib, $g(x)\neq 0$ bo'lsa, $\frac{f(x)}{g(x)}$ funksiya ham x nuqtada hosilaga ega va u

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)} \qquad (7.4.3)$$

formula bo'yicha topiladi.

7. Agar $u_1(x),...,u_n(x)$ va $v_1(x),...,v_n(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $u'_1(x),...,u'_n(x)$ va $v'_1(x),...,v'_n(x)$ hosilaga ega bo'lib, $v_i(x)\neq 0, i=\overline{1,n}$ bo'lsa, $\frac{u_1(x)\cdot...\cdot u_n(x)}{v_1(x)\cdot...\cdot v_n(x)}$ funksiya ham x nuqtada hosilaga ega va u

$$\left[\frac{u_1(x)\cdot...\cdot u_n(x)}{v_1(x)\cdot...\cdot v_n(x)}\right]' = \frac{u_1(x)\cdot...\cdot u_n(x)}{v_1(x)\cdot...\cdot v_n(x)}\left[\frac{u'_1(x)}{u_1(x)}+...\frac{u'_n(x)}{u_n(x)}-\left(\frac{v'_1(x)}{v_1(x)}+...\frac{v'_n(x)}{v_n(x)}\right)\right]$$

formula bo'yicha topiladi.

8. $f(x)$ funksiya **5.12.17–** teoremaning hamma shartlarini qanoatlantirsin. Agar $f(x)$ funksiya $x_0\in(a,b)$ nuqtada $f'(x_0)\neq 0$ hosilaga ega bo'lsa, bu funksiyaga teskari $x=f^{-1}(y)$ funksiya ham x_0 nuqtaga mos bo'lgan y_0 $(y_0=f(x_0))$ nuqtada hosilaga ega va bu hosila

$$[f^{-1}(y)]'|_{y=y_0} = \frac{1}{f'(x_0)}$$

formula orqali topiladi.

9. $y=F(u)$funksiya (c,d) oraliqda aniqlangan, $u=f(x)$ funksiya esa, (a,b) intervalda aniqlangan bo'lib, bu funksiyalar yordamida $y=F(f(x))$ murakkab funksiya tuzilgan bo'lsin.

Agar $u=f(x)$ funksiya $x_0\in(a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lib, $y=F(u)$ funksiya esa, x_0 nuqtaga mos u_0 $(u_0=f(x_0))$ nuqtada $F'(u_0)$ hosilaga ega bo'lsa, $\Phi(x)=F(f(x))$ murakkab funksiya ham x_0 nuqtada hosilaga ega va bu hosila $\Phi'(x)=[F(f(x))]'|_{x=x_0}=F'(u_0)\cdot f'(x_0)$ formula orqali topiladi.

10. Oshkormas $F(x,y)=0$ funksiya uchun

$$y'(x) = -\frac{F'_x}{F'_y}$$

formula o'rinli.

11. Parametrik tenglamasi bilan berilgan $\begin{cases} x=\phi(t) \\ y=\psi(t) \end{cases}, (t_0\le t\le t_1)$ funksiyaning hosilasi, quyidagi,

$$y'_x = \frac{\psi'(t)}{\phi'(t)} \text{ yoki } y'_x = \frac{y'_t(t)}{x'_t(t)} \qquad (7.4.4)$$

formula bo'yicha topiladi.

7.5. Asosiy elementar funksiyalarning hosilalari jadvali

1. $y = C.$ $y' = 0.$ **2.** $y = x^{\alpha}.$ $y' = \alpha \cdot x^{\alpha-1},$ $\alpha \in R.$

3. $y = \sqrt{x}.$ $y' = \frac{1}{2\sqrt{x}}.$ **4.** $y = e^{x}.$ $y' = e^{x}.$

5. $y = a^{x}.$ $y' = a^{x}\ln a.$ **6.** $y = \ln x.$ $y' = \frac{1}{x}.$

7. $y = \log_a x.$ $y' = \frac{\log_a e}{x} = \frac{1}{x\ln a}.$ **8.** $y = \lg x.$ $y' = \frac{1}{x}\lg e = \frac{1}{x\ln 10}.$

9. $y = \sin x.$ $y' = \cos x.$ **10.** $y = \cos x.$ $y' = -\sin x.$

11. $y = \mathrm{tg}\, x.$ $y' = \frac{1}{\cos^2 x} = \sec^2 x.$

12. $y = \mathrm{ctg}\, x.$ $y' = -\frac{1}{\sin^2 x} = -\mathrm{cosec}^2 x.$

13. $y = \arcsin x.$ $y' = \frac{1}{\sqrt{1-x^2}}.$ **14.** $y = \arccos x.$ $y' = -\frac{1}{\sqrt{1-x^2}}.$

15. $y = \mathrm{arctg}\, x.$ $y' = \frac{1}{1+x^2}.$ **16.** $y = \mathrm{arcctg}\, x.$ $y' = -\frac{1}{1+x^2}.$

17. $y = \sec x.$ $y' = \frac{\sin x}{\cos^2 x} = \sin x \cdot \sec x^2 x.$

18. $y = \mathrm{cos}ex.$ $y' = -\frac{\cos x}{\sin^2 x} = -\cos x \cdot \mathrm{cos}ec^2 x.$

19. $y = \mathrm{sh}\, x.$ $y' = \mathrm{ch}\, x.$ **20.** $y = \mathrm{ch}\, x.$ $y' = \mathrm{sh}\, x.$

21. $y = \mathrm{th}\, x.$ $y' = \frac{1}{\mathrm{ch}^2 x}.$ **22.** $y = \mathrm{cth}\, x.$ $y' = \frac{-1}{\mathrm{sh}^2 x}.$

23. $y = \mathrm{Arsh}\, x.$ $y' = \frac{1}{\sqrt{1+x^2}}.$ **24.** $y = \mathrm{Arch}\, x.$ $y' = \frac{1}{\sqrt{x^2-1}}.$

25. $y = \mathrm{Arth}\, x.$ $y' = \frac{1}{1-x^2}.$ **26.** $y = \mathrm{Arcth}\, x.$ $y' = \frac{-1}{1-x^2}.$

27. $y = x^{x}.$ $y' = x^{x}(1+\ln x).$ **28.** $y = u^{v};$ $y' = u^{v}\left[\frac{v}{u}\cdot u' + v'\ln u\right].$

7.6. Funksiyaning differensiali. $y = f(x)$ funksiya $(a;b)$ oraliqda berilgan bo'lsin. $x_0 \in (a,b)$ nuqtani olib, unga Δx ($\Delta x < 0$ yoki $\Delta x > 0$)

orttirma beramiz $(x_0 + \Delta x \in (a,b))$. Natijada berilgan funksiya ham shu nuqtada orttirma oladi va u $\Delta y = f(x_0 + \Delta x) - f(x_0)$ kabi ifodalanadi.

7.6.1-ta'rif. Agar $y = f(x)$ funksiyaning $x_0 \in (a,b)$ nuqtadagi Δy orttirmasi ushbu

$$\Delta y = \Delta f(x_0) = A \cdot \Delta x + \alpha \cdot \Delta x \tag{7.6.2}$$

(bunda $A - \Delta x$ ga bog'liq bo'lmagan o'zgarmas son, $\alpha = \alpha(\Delta x)$ bo'lib, $\Delta x \to 0$ da $\alpha(\Delta x) \to 0$) ko'rinishda tasvirlansa, funksiya x_0 nuqtada *differensiallanuvchi* deyiladi. (7.6.2) munosabatni, quyidagicha,

$$\Delta y = A \cdot \Delta x + o(\Delta x)$$

yozish ham mumkin.

7.6.3-ta'rif. $f(x)$ funksiya Δy orttirmasining Δx ga nisbatan chiziqli bosh qismi $A\Delta x$ ga funksiyaning *differensiali* deyiladi va $dy = df(x_0)$ kabi belgilanadi.

Demak, $dy = df(x_0) = A\Delta x,\ \Delta x = dx$ ekanligini e'tiborga olsak, $dy = Adx$ bo'ladi (erkli o'zgaruvchi x ning orttirmasi Δx ni uning differensiali dx bilan almashtirish mumkin). (7.6.2) formulani ushbu

$$y(x_0 + \Delta x) = y(x_0) + dy(x_0) + \alpha(\Delta x)\Delta x$$

ko'rinishda ham yozish mumkin. Agar $dy(x_0) \neq 0$ bo'lsa, funksiyaning $x_0 + \Delta x$ nuqtadagi qiymatini

$$y(x_0 + \Delta x) \approx y(x_0) + dy(x_0)$$

taqribiy formula bilan hisoblash mumkin.

7.6.4-teorema. $f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lishi uchun uning shu nuqtada chekli $f'(x)$ hosilaga ega bo'lishi zarur va yetarli.

Agar $f(x)$ funksiya differensiallanuvchi bo'lsa,

$$dy = f'(x)dx \tag{7.6.5}$$

ekanligini ko'rish qiyin emas. Ma'lumki, differensiallanuvchi funksiyalar uchun dy bilan dx lar proporsional o'zgarib, $f'(x)$ proporsionallik koeffisiyentini ifodalaydi.

Ixtiyoriy differensiallanuvchi u va v funksiyalar uchun quyidagi

$$d(\alpha u \pm \beta v) = \alpha\, du \pm \beta\, dv,\ \alpha, \beta \in R;\ \ d(u \cdot v) = v\, du + u\, dv;$$

$$d\left(\frac{u}{v}\right) = \frac{v\, du - u\, dv}{v^2},\ v \neq 0$$

tengliklar o'rinli.

Funksiya differensialining (7.6.5) ifodasidan foydalanib, elementar funksiyalarning differensiallari jadvalini keltiramiz:

1. $d(x^{\mu}) = \mu x^{\mu-1} dx\ (x > 0)$. **2.** $d(a^x) = a^x \ln a\, dx,\ (a > 0,\ a \neq 1)$.

3. $d(\log_a x) = \frac{1}{x} \log_a e\, dx,\ (a > 0,\ a \neq 1)$. **4.** $d(\sin x) = \cos x dx$.

5. $d(\cos x) = -\sin x dx$. **6.** $d(e^x) = e^x dx$.

7. $d(tgx) = \frac{1}{\cos^2 x} dx,\ x \neq \frac{\pi}{2} + k\pi,\ k \in Z$.

8. $d(ctgx) = -\frac{1}{\sin^2 x} dx,\ x \neq k\pi,\ k \in Z$.

9. $d(\ln x) = \frac{1}{x} dx$.

10. $d(\arcsin x) = \frac{1}{\sqrt{1-x^2}} dx, -1 < x < 1$.

11. $d(\arccos x) = -\frac{dx}{\sqrt{1-x^2}},\ -1 < x < 1$.

12. $d(arctgx) = \frac{1}{1+x^2} dx$. **13.** $d(arcctgx) = -\frac{1}{1+x^2} dx$.

7.2.-§.Funksiyaning yuqori tartibli hosilasi va differensiali

7.7. Funksiyaning yuqori tartibli hosilasi. $f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin.

7.7.1-ta'rif. Agar $f(x)$ funksiya (a,b) oraliqning har bir $x \in (a,b)$ nuqtasida $f'(x)$ hosilaga ega bo'lib, bu $f'(x)$ funksiya $x_0 \in (a,b)$ nuqtada hosilaga ega bo'lsa, u $f(x)$ funksiyaning x_0 *nuqtadagi ikkinchi tartibli*

hosilasi deb ataladi va $y''_{x=x_0}$, $f''(x_0)$, $\left.\left(\frac{d^2y}{dx^2}\right)\right|_{x=x_0}$ belgilardan biri orqali belgilanadi.

$f(x)$ funksiya (a,b) oraliqning har bir $x \in (a,b)$ nuqtasida $(n-1)$ -tartibli $f^{(n-1)}(x)$ hosilaga ega bo'lsin. Bu $f^{(n-1)}(x)$ funksiyaning $x_0 \in (a,b)$ nuqtadagi hosilasi (agar u mavjud bo'lsa), $f(x)$ funksiyaning x_0 nuqtadagi *n-tartibli hosilasi* deb ataladi va u $y^{(n)}_{x=x_0}$, $f(x_0)^{(n)}$, $\left((\frac{d^n y}{dx^n})_{x=x_0}\right)$ belgilardan biri orqali belgilanadi. Odatda, $f(x)$ funksiyaning $f'(x), f''(x),...$ hosilalari uning yuqori tartibli hosilalari deyiladi.

Agar $s = s(t)$ -to'g'ri chiziq bo'ylab harakat qilayotgan material nuqtaning harakat qonunini ifodalasa, u holda $s''(t)$ -shu nuqtaning t vaqtdagi ichidagi tezlanishini ifodalaydi. Demak, ikkinchi tartibli hosilaning fizik ma'nosi -material nuqtaning tezlanishidan iborat ekan.

7.7.2-eslatma. $f(x)$ funksiyaning biror $x \in (a,b)$ nuqtadagi $f'(x)$ hosilasi mavjudligidan, uning shu nuqtadagi yuqori tartibli hosilalarga ega bo'lishi har doim ham kelib chiqavermaydi.

$f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda aniqlangan bo'lib, ular $x \in (a,b)$ nuqtada n-tartibli $f^{(n)}(x)$, $g^{(n)}(x)$ hosilalarga ega bo'lsin(buni quyidagicha tushunish lozim: $f(x)$ va $g(x)$ funksiyalar x nuqtani o'z ichiga olgan $(\alpha, \beta) \subset (a, b)$ oraliqda $f', f'',..., f^{(n-1)}$, $g', g'',..., g^{(n-1)}$ hosilalarga ega bo'lib, x nuqtada esa, $f^{(n)}(x)$, $g^{(n)}(x)$ hosilalarga ega). U holda

1) $[Cf(x)]^{(n)} = Cf^{(n)}(x)$, $C = const$; 2) $(f(x) \pm g(x))^{(n)} = f^{(n)}(x) \pm g^{(n)}(x)$;

3) $(f(x) \cdot g(x))^{(n)} = f^{(n)}(x) \cdot g(x) + C_n^1 f^{(n-1)}(x) \cdot g'(x) +$

$+ C_n^2 f^{(n-2)}(x) \cdot g''(x) + ... +$

$$+ C_n^k f^{(n-k)}(x) g^{(k)}(x) + ... + f(x) g^{(n)}(x), \qquad (7.7.3)$$

bunda $C_n^k = \frac{n(n-1)...(n-k+1)}{k!}$. (7.7.3)-formulaga *Leybnis formulasi* deyiladi.

Asosiy elementar funksiyalarning n-tartibli hosilalarini topish formulalari:

1. $y=x^m$ $y^{(n)}=m(m-1)(m-2)...(m-n+1)x^{m-n}$.

Agar m butun son va $n>m$ bo'lsa, $y^{(n)}(x)=0$ bo'ladi. Xususiy holda, $m=-1$ bo'lsa, $y=\frac{1}{x}$ funksiyaning n-tartibli hosilasi $y^{(n)}=\frac{(-1)^n n!}{x^{n+1}}$ bo'ladi.

2. $y=\ln x$, $y^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^n}$.

3. $y=\log_a x$, $y^{(n)}=\frac{(-1)^{n-1}(n-1)!}{\ln a}\frac{1}{x^n}$. **4.** $y=e^{bx}$, $y^{(n)}=b^n e^{bx}$.

5. $y=a^{bx}$, $y^{(n)}=b^n a^x \ln^n a$.

6. $y=\sin bx$, $y^{(n)}=b^n\sin(bx+n\frac{\pi}{2})$.

7. $y=\cos bx$, $y^{(n)}=b^n\cos(bx+n\frac{\pi}{2})$

8. $y=(ax+b)^\alpha$, $y^{(n)}=((ax+b)^\alpha)^{(n)}=a^n\alpha(\alpha-1)\cdots(\alpha-n+1)(ax+b)^{\alpha-n}$.(7.7.4)

7.8. Funksiyaning yuqori tartibli differensiali. $y=f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin. Agar $f(x)$ funksiya $x\in(a,b)$ nuqtada chekli $f'(x)$ hosilaga ega bo'lsa, funksiyaning differensiali ushbu $dy=f'(x)dx$ formula bo'yicha hisoblanadi.

Faraz qilaylik, $f(x)$ funksiya $x\in(a,b)$ nuqtada ikkinchi tartibli hosilaga ega bo'lsin. U holda, belgilangan dx lar uchun funksiyaning dy differensiali faqat x ning funksiyasi bo'ladi va uning differensialini hisoblash mumkin, bunda ham $dx=\Delta x$ deb olinadi.

7.8.1-ta'rif. $f(x)$ funksiya differensiali dy ning $x\in(a,b)$ nuqtadagi differensialiga berilgan $f(x)$ funksiyaning *ikkinchi tartibli differensiali* deb ataladi va u d^2y yoki $d^2f(x)$ kabi belgilanadi, ya'ni $d^2y=d(dy)$ yoki $d^2f(x)=d(df(x))$.

Funksiyaning ikkinchi tartibli differensiali uning ikkinchi tartibli hosilasi orqali quyidagicha yoziladi:

$$d^2y=y''dx^2, \qquad (7.8.2)$$

bunda $dx^2=dx\cdot dx=(dx)^2$.

$f(x)$ funksiya $x\in(a,b)$ nuqtada n-tartibli $f^{(n)}(x)$ hosilaga ega bo'lsin. Funksiyaning $(n-1)$ – tartibli differensiali $d^{n-1}y$ dan olingan differensial, berilgan $f(x)$ funksiyaning $x\in(a,b)$ nuqtadagi n*-tartibli differensiali* deb ataladi va u d^ny yoki $d^nf(x)$ kabi belgilanadi, ya'ni

$$d^ny=d(d^{n-1}y) \text{ yoki } d^nf(x)=d(d^{n-1}f(x)); \qquad d^ny=y^{(n)}dx^n. \qquad (7.8.3)$$

Erkli o'zgaruvchi x ning n -tartibli differensiali $n>1$ da, ta'rif bo'yicha, $d^nx=0$ deb olinadi.

$f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda berilgan bo'lib, ular $x\in(a,b)$ nuqtada differensialga ega bo'lsin. U holda quyidagi

1) $d^n[Cf(x)]=Cd^nf(x)$, $C=const$;

2) $d^n[f(x)\pm g(x)]=d^nf(x)\pm d^ng(x)$;

3) $d^n[f(x)\cdot g(x)]=d^n[f(x)]\cdot g(x)+C_n^1d^{n-1}[f(x)]\cdot d[g(x)]+....+$
$+C_n^kd^{n-k}[f(x)]\cdot d^k[g(x)]+....+f(x)\cdot d^n[g(x)]$,

formulalar o'rinli bo'ladi, bunda $C_n^k=\frac{n(n-1)...(n-k+1)}{k!}$.

$u=f(x)$ funksiya (a,b) oraliqda, $y=F(u)$ funksiya esa, (c,d) oraliqda berilgan bo'lib, ular yordamida $y=F(f(x))$ murakkab funksiya tuzilgan bo'lsin. $u=f(x)$ funksiya $x\in(a,b)$ nuqtada $f'(x)$, $F(u)$ funksiya esa, mos $u\in(c,d)$ nuqtada $F'(u)$ hosilaga ega deb, $y=F(f(x))$ funksiyaning differensialini hisoblaymiz:

$$dy=F'(f(x))\cdot f'(x)dx=F'(f(x))\cdot df(x).$$

7.8.4-eslatma. (7.8.2) va (7.8.3) formulalar, $n>1$ bo'lganda, faqat x - erkli o'zgaruvchi bo'lgan holda o'rinli.

x-erksiz o'zgaruvchi bo'lgan holda, ya'ni $y=y(x(t))$ murakkab funksiya uchun (7.8.2) formula ushbu:

$$d^2y=d(dy)=d(y'_xdx)=d(y'_x)\cdot dx+y'_xd(dx)=y''_{xx}dx^2+y'_xd^2x \qquad (7.8.5)$$

ko'rinishda bo'ladi.

Agar x-erkli o'zgaruvchi bo'lsa, $d^2x=0$ bo'ladi va (7.8.2) formula (7.8.5) formula bilan ustma-ust tushadi.

7.3-§. Funksiyaning hosila yordamida tekshirish

7.9. Nuqtada funksiyaning o'sishi (kamayishi). Funksiyaning lokal ekstremum qiymatlari. $y=f(x)$ funksiya biror belgilangan c nuqtaning atrofida aniqlangan bo'lsin.

7.9.1-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $x<c$ bo'lganda $f(x)<f(c)$ tengsizlik, $x>c$ bo'lganda esa, $f(x)>f(c)$ tengsizlik o'rinli bo'lsa, $y=f(x)$ *funksiya c nuqtada o'sadi* deyiladi.

7.9.2-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $x<c$ bo'lganda, $f(x)>f(c)$ tengsizlik, $x>c$ bo'lganda esa, $f(x)<f(c)$ tengsizlik bajarilsa, $y=f(x)$ *funksiya c nuqtada kamayadi* deyiladi.

7.9.3-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $f(c)$ qiymat, funksiyaning $U_\delta(c)$ atrofdagi qiymatlari ichida eng kattasi (eng kichigi) bo'lsa, $y=f(x)$ funksiya *c nuqtada lokal maksimum (lokal minimum) ga ega* deyiladi.

Odatda funksiyaning c nuqtadagi lokal maksimum va lokal minimum qiymatlari birgalikda lokal *ekstremum qiymatlari* deb yuritiladi.

7.9.4-teorema (funksiyaning nuqtada o'suvchi, kamayuvchi bo'lishining yetarli sharti). Agar $y=f(x)$ funksiya c nuqtada differensiallanuvchi bo'lib, uning bu nuqtadagi $f'(c)$ hosilasi musbat (manfiy) bo'lsa, u holda bu funksiya c nuqtada o'suvchi (kamayuvchi) bo'ladi.

7.9.5-eslatma. $y=f(x)$ funksiyaning c nuqtada o'suvchi (kamayuvchi) bo'lishi uchun uning shu nuqtadagi $f'(c)$ hosilasining musbat (manfiy) bo'lishi zaruriy shart bo'la olmaydi. Masalan, $y=x^2$ funksiya $x=0$ nuqtada o'suvchi, lekin uning $x=0$ nuqtadagi hosilasi $f'(0)=0$.

7.9.6-eslatma. Agar funksiya x_0 nuqtada o'suvchi bo'lsa, uning x_0 nuqtaning biror atrofida o'suvchi bo'lishi shart emas.

7.10. Differensial hisobning asosiy teoremalari

7.10.1-teorema (Ferma teoremasi). $y=f(x)$ funksiya biror X oraliqda aniqlangan bo'lib, bu oraliqning ichki c nuqtasida o'zining eng katta (eng kichik) qiymatiga erishsin. Agar c nuqtada funksiya chekli $f'(x)$ hosilaga ega bo'lsa, u holda $f'(c)=0$ bo'ladi.

Ferma teoremasi sodda geometrik ma'noga ega. $y=f(x)$ funksiya Ferma teoremasining shartlarini qanoatlantirganda, $f(x)$ funksiyaning grafigidagi $(c,f(c))$ nuqtaga o'tkazilgan urinma Ox o'qqa parallel bo'ladi.

Ferma teoremasining fizik ma'nosi quyidagicha: to'g'ri chiziq bo'ylab harakat qilayotgan zarrachaning qaytish momenti tezligi nolga teng.

7.10.2-eslatma. $f'(c)=0$ shart, funksiyaning c nuqtada lokal ekstremumga ega bo'lishi uchun yetarli shart bo'la olmaydi.

7.10.3-teorema (Roll teoremasi). $y=f(x)$ funksiya $[a,b]$ da aniqlangan bo'lib: 1) uzluksiz; 2) hech bo'lmaganda (a,b) da chekli hosilaga ega; 3) $[a,b]$ kesmaning chetlarida o'zaro teng ($f(a)=f(b)$) qiymatlarni qabul qilsa, u holda, kamida bitta shunday c $(a<c<b)$ nuqta topiladiki, $f'(c)=0$.

Roll teoremasining geometrik ma'nosi quyidagicha: $y=f(x)$ funksiya Roll teoremasining hamma shartlarini qanoatlantirganda, bu funksiyaning grafigida shunday $(c;f(c))$ nuqta topiladiki, bu nuqtada funksiya grafigiga o'tkazilgan urinma Ox o'qqa parallel bo'ladi.

7.10.4-eslatma. Roll teoremasida $y=f(x)$ funksiyadan, uning $[a,b]$ kesmada uzluksizligi, kesmaning ichki nuqtalarida esa, uning differensiallanuvchiligi, talab qilingan edi. Funksiya kesmaning ichki nuqtalarida differensiallanuvchiligidan, uning shu nuqtalarda uzluksizligi kelib chiqadi, shuning uchun Roll teoremasida funksiyaga qo'yilgan 1) shartning o'rniga, $f(x)$ ning a nuqtada o'ngdan, b nuqtada esa, chapdan uzluksiz bo'lishini talab qilish yetarli.

7.10.5-eslatma. Roll teoremasining barcha shartlari muhim. Agar teoremadagi $y=f(x)$ funksiyaga qo'yilgan shartlarning birortasi bajarilmasa, teoremaning tasdig'i o'rinli bo'lmasligi mumkin.

7.10.6-teorema (Lagranj teoremasi). Agar $f(x)$ funksiya: 1) $[a,b]$ kesmada aniqlangan va uzluksiz; 2) hyech bo'lmaganda (a,b) oraliqda chekli hosilaga ega bo'lsa, u holda shunday c $(a<c<b)$ nuqta topiladiki, bu nuqtada

$$\frac{f(b)-f(a)}{b-a}=f'(c) \qquad (7.10.7)$$

tenglik o'rinli bo'ladi.

Lagranj teoremasining geometrik ma'nosi quyidagicha: faraz qilaylik, $f(x)$ funksiya Lagranj teoremasining barcha shartlarini qanoatlantirsin.

$f(x)$ funksiya grafigining $A(a, f(a))$, $B(b, f(b))$ nuqtalarini to‘g‘ri chiziq orqali tutashtiramiz. $f'(x)$- $f(x)$ funksiya grafigining $(x, f(x))$ nuqtasida o‘tkazilgan urinmaning burchak koeffisiyentidir, ya’ni $\operatorname{tg}\alpha = f'(x)$. Shunday c $(a<c<b)$ nuqta topiladiki, $f(x)$ funksiya grafigiga $(c, f(c))$ nuqtada o‘tkazilgan urinma AB to‘g‘ri chiziqqa parallel bo‘ladi.

(7.10.7) formulani boshqacha ham yozish mumkin: $\forall x_0 \in [a;b]$ nuqtani olib, unga ixtiyoriy Δx orttirma beramiz $(x_0 + \Delta x \in [a;b])$. $[x_0, x_0 + \Delta]$ kesma uchun (7.10.7) Lagranj formulasini yozamiz:

$$f(x_0 + \Delta x) - f(x_0) = \Delta x \cdot f'(c),$$

bunda $\forall c \in (x_0, x_0 + \Delta x)$. Agar $c = x_0 + \theta\Delta x,\ \ 0 < \theta < 1$ deb belgilasak,

$$f(x_0 + \Delta x) - f(x_0) = \Delta x \cdot f'(x_0 + \theta\Delta x). \qquad (7.10.8)$$

Odatda, (7.10.8) formula, chekli orttirmalar haqidagi *Lagranj formulasi* deb yuritiladi.

7.10.9-eslatma. Agar (7.10.7) formulada $f(a) = f(b)$ deb olinsa, u holda $f'(c) = 0$ $(a<c<b)$ bo‘lib, Lagranj teoremasidan Roll teoremasining kelib chiqishini ko‘ramiz.

Lagranj teoremasidan kelib chiqadigan natijalar.

7.10.10-natija. Agar $f(x)$ funksiya (a, b) oraliqda differensiallanuvchi va bu oraliqda $f'(c) = 0$ bo‘lsa, u holda bu oraliqda $f(x)$ funksiya o‘zgarmas bo‘ladi.

7.10.11-natija. $f(x)$ va $g(x)$ funksiyalar biror (a, b) oraliqda uzluksiz, bu oraliqda differensiallanuvchi bo‘lib, $f'(x) = g'(x),\ \ x \in (a;b)$ bo‘lsa, u holda bu funksiyalarning biri ikkinchisidan o‘zgarmas songa farq qiladi, ya’ni $f(x) = g(x) + C$.

7.10.12-natija. $f(x)$ funksiya biror x_0 nuqtaning $U_\delta(x_0)$ atrofida uzluksiz va $\dot{U}_\delta(x_0)$ atrofda differensiallanuvchi bo‘lsin. Agar chekli $\lim_{x \to x_0} f'(x) = A$ mavjud bo‘lsa, u holda $f(x)$ funksiya x_0 nuqtada differensiallanuvchi va $f'(x_0) = A$ bo‘ladi.

7.10.13-teorema (Koshi teoremasi). $f(x)$ va $g(x)$ funksiyalar:

1) $[a,b]$ kesmada aniqlangan va uzluksiz;

2) (a,b) oraliqda chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lib, $\forall x \in (a,b)$ uchun $g'(x) \neq 0$ bo‘lsin.

U holda shunday $c(a<c<b)$ nuqta topiladiki,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \qquad (7.10.14)$$

tenglik o‘rinli bo‘ladi.

Odatda (7.10.14) formula – *Koshi formulasi* deb ataladi.

7.10.15-eslatma. (7.10.7) Lagranj formulasi, (7.10.14) Koshi formulasidan, $g(x) = x$ bo‘lganda kelib chiqadi.

7.10.16-eslatma. (7.10.14) formulada $b > a$ deb olish shart emas.

7.11. Funksiyaning monotonlik sharti. Kamaymovchi (o‘smovchi), o‘suvchi (kamayuvchi) funksiyalarning ta’riflarini yana bir marta eslatib o‘tamiz.

1^0. $x_1 < x_2$ shartni qanoatlantiradigan $\forall x_1, x_2 \in (a,b)$ uchun $f(x_1) \le f(x_2)$, $(f(x_1) \ge f(x_2))$ tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya (a, b) oraliqda *kamaymovchi (o‘smovchi)* deyiladi.

2^0. $x_1 < x_2$ shartni qanoatlantiradigan $\forall x_1, x_2 \in (a,b)$ uchun $f(x_1) < f(x_2)$, $(f(x_1) > f(x_2))$ tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya (a, b) *oraliqda o‘suvchi (kamayuvchi)* deyiladi. Ba’zan, o‘suvchi (kamayuvchi) iboralarii o‘rniga *qat’iy o‘suvchi (qat’iy kamayuvchi)* iboralari ham ishlatiladi.

7.11.1-teorema. (a, b) oraliqda differensiallanuvchi $f(x)$ funksiya shu oraliqda kamaymovchi (o‘smovchi) bo‘lishi uchun, bu funksiyaning $f'(x)$ hosilasi (a, b) oraliqda manfiy (musbat) bo‘lmasligi zarur va yetarli.

7.11.2-teorema. $f(x)$ funksiya (a, b) oraliqda differensiallanuvchi bo‘lsin. Agar $f(x)$ funksiyaning xosilasi $f'(x)$ funksiya (a, b) oraliqda musbat (manfiy) bo‘lsa, u holda $f(x)$ funksiya $(a; b)$ oraliqda o‘suvchi (kamayuvchi) bo‘ladi.

7.11.3-eslatma. $f(x)$ funksiyaning (a, b) oraliqda o‘suvchi (kamayuvchi) bo‘lishi uchun uning hosilasi $f'(x)$ ning musbat (manfiy) bo‘lishi zaruriy shart bo‘la olmaydi.

7.12. Funksiyaning monotonlik oraliqlarini aniqlash. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo‘lsin.

7.12.1-teorema. $f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lsin. Funksiyaning shu oraliqda o'suvchi (kamayuvchi) bo'lishi uchun

$$f'(x) \geq 0 \ \ (f'(x) \leq 0), \ x \in (a,b)$$

tengsizlikning bajarilishi zarur va yetarli.

7.12.2-teorema. $f(x)$ funksiya (a,b) oraliqda chekli hosilaga ega bo'lib, $f'(x) > 0$ $(f'(x) < 0)$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya (a,b) oraliqda qat'iy o'suvchi(qat'iy kamayuvchi) bo'ladi.

7.12.3-teorema. $f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lsin. Bu funksiyaning (a,b) oraliqda o'zgarmas bo'lishi uchun

$$f'(x) = 0, \ x \in (a,b)$$

bo'lishi zarur va yetarli.

7.12.4-natija. Agar $f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda aniqlangan, chekli $f'(x)$ va $g'(x)$ hosilalarga ega va $\forall x \in (a;b)$ oraliqda $f'(x)=g'(x)$ bo'lsa, (a,b) da bu funksiyalarning biri ikkinchisidan o'zgarmas songa farq qiladi, ya'ni $f(x)=g(x)+C$, $\forall x \in (a;b)$.

7.12.3 *-teoremaning geometrik ma'nosi quyidagicha:*

1) $f'(x) > 0$ $(\text{tg}\alpha > 0)$ shart funksiya grafigining har bir nuqtasiga o'tkazilgan urinma abssissalar o'qining musbat yo'nalishi bilan o'tkir burchak tashkil qilishini;

2) $f'(x) < 0$ shart esa,-usha o'rinma o'tmas burchak tashkil qilishini anglatadi.

7.12.5-eslatma. $f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lib, bu funksiyaning (a,b) oraliqda qat'iy o'suvchi (qat'iy kamayuchi) bo'lishidan, $f'(x)$ ning $\forall x \in (a,b)$ da musbat (manfiy) bo'lishi har doim ham kelib chiqavermaydi.

Funksiya hosilasining (a,b) oraliqda musbat (manfiy) bo'lishi funksiyaning qat'iy monoton bo'lishi uchun zaruriy shart bo'la olmaydi.

Funksiyani monotonlikka tekshirganda avvalo uning hosilasini topish (u mavjud bo'lgan joyda) kerak, so'ngra hosila musbat (manfiy) bo'ladigan oraliqlarni aniqlash kerak. Hosilasi musbat (manfiy) bo'lgan oraliqlarda funksiya monoton o'suvchi (kamayuvchi) bo'ladi.

7.13. Funksiyaning ekstremum qiymatlari. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lib, $x_0 \in (a,b)$ bo'lsin.

7.13.1-ta'rif. Agar $x_0 \in (a,b)$ nuqtaning shunday $U_\delta(x_0)$ atrofi mavjud bo'lib, $\forall x \in U_\delta(x_0)$ uchun $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada *lokal maksimumga (lokal minimumga)* ega deyiladi. $f(x_0)$ qiymat esa, $f(x)$ funksiyaning $U_\delta(x_0)$ atrofdagi *lokal maksimumi (lokal minimumi)* deyiladi.

7.13.2-ta'rif. Agar $x_0 \in (a,b)$ nuqtaning shunday $U_\delta(x_0)$ atrofi mavjud bo'lib, $\forall x \in U_\delta(x_0)$ uchun $f(x) < f(x_0)$ $(f(x) > f(x_0))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada qat'iy maksimumga (qat'iy minimumga) ega deyiladi. $f(x_0)$ qiymat esa, $f(x)$ funksiyaning $U_\delta(x_0)$ atrofdagi *qat'iy lokal maksimumi (qat'iy lokal minimumi)* deyiladi.

Funksiyaning maksimumi va minimumi, umumiy nom bilan, uning *ekstremumi* deyiladi.

Yuqorida keltirilgan ta'riflardagi x_0 nuqta, $f(x)$ funksiyaning *lokal maksimum (lokal minimum), qat'iy maksimum (qat'iy minimum) nuqtasi,* deb yuritiladi. Funksiyaning maksimum va minimum nuqtalari uning *ekstremum nuqtalari* deyiladi.

Funksiyaning $U_\delta(x_0)$ atrofdagi lokal maksimum (lokal minimum) qiymatlari

$$f(x_0) = \max_{x \in U_\delta(x_0)}\{f(x)\} \quad \left(f(x_0) = \min_{x \in U_\delta(x_0)}\{f(x)\} \right)$$ kabi belgilanadi.

7.13.3-teorema (funksiya ekstremumga ega bo'lishining zaruriy sharti). Agar $f(x)$ funksiya x_0 $(x_0 \in (a,b))$ nuqtada hosilaga ega bo'lib, u shu nuqtada ekstremumga ega bo'lsa, $f'(x_0) = 0$ bo'ladi.

Bu shart funksiya ekstremumga ega bo'lishi uchun yetarli shart bo'la olmaydi. Odatda funksiyaning hosilasi nolga aylanadigan nuqtalar *stasionar (turg'un, kritik) nuqtalari* deb ataladi.

Funksiyaning hosilasi nolga, cheksizga teng yoki uning hosilasi mavjud bo'lmagan nuqtalarda ham ekstremum mavjud bo'lishi mumkin. Odatda bunday nuqtalar *ekstremumga shubhali nuqtalar* deb ataladi.

a) Ekstremum mavjud bo'lishining birinchi yetarli sharti. $x_0 \in (a,b)$ nuqtaning $U_\delta^-(x_0) = \{x \in R: \ x_0 - \delta < x < x_0; \ \delta > 0\}$, $U_\delta^+(x_0) = \{x \in R: \ x_0 < x < x_0 + \delta; \ \delta > 0\}$ chap va o'ng atroflarini qaraymiz. Faraz qilaylik, $y = f(x)$ funksiya x_0

nuqtada uzluksiz bo'lib, $\mathring{U}_\delta(x_0)$ da chekli $f'(x)$ hosilaga ega bo'lsin (x_0 nuqtada hosila mavjud bo'lmasligi ham mumkin).

1. Agar $\forall x \in U_\delta^-(x_0)$ uchun, $f'(x) > 0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x) < 0$ bo'lsa, ya'ni $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini «+» dan «-» ga o'zgartirsa, $f(x)$ funksiya x_0 nuqtada *lokal maksimumga* erishadi.

2. Agar $\forall x \in U_\delta^-(x_0)$ uchun $f'(x) < 0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x) > 0$ bo'lsa, ya'ni $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini «-» dan «+» ga o'zgartirsa, $f(x)$ funksiya x_0 nuqtada *lokal minimumga* erishadi.

Agar $\forall x \in U_\delta^-(x_0)$ uchun $f'(x) > 0$, $\forall x \in U_\delta^-(x_0)$ uchun $f'(x) > 0$

yoki $\forall x \in U_\delta^-(x_0)$ uchun $f'(x) < 0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x) < 0$ bo'lsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.

$y = f(x)$ funksiyaga ekstremum qiymat beruvchi nuqtalarni birinchi tartibli hosila yordamida topish qoidasi:

1. $f'(x)$ hosila topiladi.
2. $y = f(x)$ funksiyaning kritik nuqtalari, ya'ni $f'(x)$ hosila nolga aylanadigan yoki uzilishga ega bo'lgan nuqtalar topiladi.
3. Topilgan kritik nuqtalar $f(x)$ funksiyaning aniqlanish sohasini oraliqlarga ajratadi, shu oraliqlarda $f'(x)$ hosilaning ishorasi tekshiriladi.
4. Funksiyaning ekstremum nuqtalardagi qiymatlari hisoblanadi.

b) Ekstremum mavjud bo'lishining ikkinchi yetarli sharti. x_0 nuqta $f(x)$ funksiyaning stasionar nuqtasi, ya'ni $f'(x_0) = 0$ bo'lsin. Agar $f(x)$ funksiyaning x_0 nuqtada ikkinchi tartibli hosilasi mavjud bo'lib, $f''(x_0) < 0$ $(f''(x_0) > 0)$ bo'lsa, $f(x)$ funksiya x_0 nuqtada *maksimumga (minimumga)* erishadi.

$y = f(x)$ funksiyaga ekstremum qiymat beruvchi nuqtalarni ikkinchi tartibli hosila yordamida topish qoidasi:

1. $f'(x)$ hosila topiladi.
2. Berilgan funksiyaning kritik nuqtalari, ya'ni $f'(x) = 0$ bo'ladigan nuqtalar topiladi.
3. $f''(x)$ ikkinchi tartibli hosila topiladi.
4. Ikkinchi tartibli hosilaning ishorasi har bir kritik nuqtada tekshiriladi. Bunda, agar ikkinchi tartibli hosila manfiy bo'lsa, u holda funksiya nuqtada maksimumga, musbat bo'lsa, minimumga ega bo'ladi. Agar ikkinchi tartibli hosila nolga teng bo'lsa, u holda funksiyaning ekstremumini birinchi yetarli shart bo'yicha tekshirish yoki yuqori tartibli hosilalardan foydalanib tekshirishga to'g'ri keladi (c) bandga q.).
5. Funksiyaning ekstremum nuqtalardagi qiymatlari hisoblanadi.

c) Ekstremum mavjud bo'lishining uchinchi yetarli sharti. $f(x)$ funksiyaning $x_0 \in (a,b)$ nuqtada $f'(x_0), f''(x_0), ..., f^{(n)}(x_0)$ hosilalari mavjud bo'lib, biror n>2 son uchun $f'(x_0) = f''(x_0) = ... = = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$ bo'lsin.

Agar: *a)* n juft son bo'lib $(n = 2m, m \in N)$, $f^{(n)}(x_0) = f^{(2m)}(x_0) < 0$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada maksimumga; $f^{(n)}(x_0) = f^{(2m)}(x_0) > 0$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada minimumga ega bo'ladi.

b) n toq son bo'lsa $(n = 2m+1, m \in N)$, $f(x)$ funksiya ekstremumga ega bo'lmaydi.

7.14. Funksiyaning eng katta va eng kichik qiymatlarini topish. $f(x)$ funksiya $[a; b]$ kesmada aniqlangan va uzluksiz bo'lsin. Veyershtrassning ikkinchi teoremasiga ko'ra, funksiyaning $[a; b]$ kesmada eng katta, hamda eng kichik qiymatlari mavjud bo'ladi va funksiya bu qiymatlarga kesmaning nuqtalarida erishadi.

Funksiyaning eng katta qiymati quyidagicha topiladi:

1) $f(x)$ funksiyaning $(a; b)$ oraliqdagi maksimum qiymatlari topiladi. - $\{\max f(x)\}$ $f(x)$ funksiyaning $(a; b)$ oraliqdagi hamma maksimum qiymatlaridan iborat to'plam bo'lsin.

2) $f(x)$ funksiyaning $[a; b]$ kesmaning chetlaridagi ya'ni $x = a, x = b$ nuqtalardagi $f(a)$ va $f(b)$ qiymatlari hisoblanadi. So'ngra $\{\max f(x)\}$ to'plamning barcha elementlari bilan $f(a)$ va $f(b)$ lar taqqoslanadi. Bu qiymatlar ichida eng kattasi, $f(x)$ funksiyaning $[a; b]$ kesmadagi eng katta qiymati bo'ladi. Xuddi shu usulda funksiyaning eng kichik qiymati ham topiladi.

Biror oraliqda uzluksiz bo'lgan funksiyaning eng katta va eng kichik qiymatlarini topish uchun:

1) bu oraliqda funksiyaning tegishli stasionar nuqtalarini topish, bu topilgan stasionar nuqtalarni ekstremumga tekshirish va funksiyaning bu nuqtalardagi qiymatlarini hisoblash;

2) funksiyaning oraliqning chetki nuqtalaridagi qiymatlarini topish;

3) topilgan qiymatlarni funksiyaning oraliqning ichidagi nuqtalarida qiymatlari bilan solishtirish kerak; bu qiymatlarning eng kichigi va eng kattasi, mos ravishda, funksiyaning qaralayotgan oraliqdagi eng kichik va eng katta qiymatlari bo'ladi.

7.15. Funksiya grafigining qavariqligi va botiqligi. $y=f(x)$ funksiya $(a;b)$oraliqda berilgan bo'lsin.

7.15.1-ta'rif. Agar $y=f(x)$ funksiyaning grafigi $(a;b)$ oraliqning ixtiyoriy nuqtasidan o'tkazilgan urinmadan yuqorida (pastda) yotsa, bu funksiyaning grafigi *qavariq (botiq)* deyiladi.

$y=f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lsin.

7.15.2-teorema. $f(x)$ funksiyaning grafigi (a,b) oraliqda qavariq (qatiy qavariq) bo'lishi uchun, uning $f'(x)$ hosilasining shu oraliqda kamayuvchi (qat'iy kamayuvchi) bo'lishi zarur va yetarli.

7.15.3-teorema. $f(x)$ funksiyaning (a,b) oraliqda botiq (qat'iy botiq) bo'lishi uchun, uning $f'(x)$ hosilasining shu oraliqda o'suvchi (qat'iy o'suvchi) bo'lishi zarur va yetarli.

$y=f(x)$ funksiya (a,b) oraliqda ikkinchi tartibli hosilaga ega bo'lsin.

7.15.4-teorema. $f(x)$ funksiyaning grafigi (a,b) oraliqda qavariq (botiq) bo'lishi uchun $f''(x)\geq 0$ $(f''(x)\leq 0)$ tengsizlikning bajarilishi zarur va yetarli.

7.16. Funksiya grafigining egilish nuqtalari. $f(x)$ funksiya x_0 nuqtaning biror $U_\delta(x_0)$ $(\delta>0)$ atrofida aniqlangan bo'lsin.

7.16.1-ta'rif. Agar $f(x)$ funksiya $U_\delta^-(x_0)$ oraliqda botiq (qavariq) bo'lib, $U_\delta^+(x_0)$ oraliqda esa, qavariq (botiq) bo'lsa, ya'ni $f(x)$ funksiya x_0 nuqtadan o'tishda o'z qavariqligining yo'nalishini o'zgartirsa, u holda, x_0 nuqta $f(x)$ funksiyaning *egilish nuqtasi* deyiladi, va bunda $(x_0,f(x_0))$ nuqta $-f(x)$ funksiya grafigining *egilish nuqtasi* deyiladi.

Agar $x_0\in(a,b)$ $-f(x)$ funksiya grafigi egilish nuqtasining abssissasi bo'lsa, bu nuqtada ikkinchi tartibli hosila mavjud bo'lishi ham, bo'lmasligi ham mumkin.

Funksiyaning ikkinchi tartibli hosilasi nolga aylanadigan yoki mavjud bo'lmaydigan nuqtalar *II -tur kritik nuqtalar* deyiladi. Bu nuqtalarda egilish mavjud bo'lishi ham, bo'lmasligi ham mumkin.

7.16.2-teorema (egilish nuqtasi bo'lishning zaruriy sharti). Agar $M(x_0,f(x_0))$ nuqta $f(x)$ funksiya grafigi uchun egilish nuqtasi bo'lib, $f(x)$

funksiya x_0 nuqtada ikkinchi tartibli hosilaga ega bo'lsa, u holda $f''(x_0)=0$ bo'ladi.

Bu shart funksiya grafigi egilish nuqtasiga ega bo'lishi uchun yetarli shart bo'la olmaydi.

7.16.3-teorema (egilish nuqtasi bo'lishning birinchi yetarli sharti). $f(x)$ funksiya x_0 nuqtaning biror atrofida ikkinchi tartibli hosilaga ega va $f''(x_0)=0$ bo'lsin. U holda, ko'rsatilgan atrofda ikkinchi tartibli $f''(x_0)$ hosila x_0 nuqtaning chap va o'ng atrofida har xil ishoraga ega bo'lsa, $f(x)$ funksiyaning grafigi $M(x_0,f(x_0))$ nuqtada egilishga ega bo'ladi.

7.16.4-teorema (egilish nuqtasi bo'lishning ikkinchi yetarli sharti). Agar $f(x)$ funksiya x_0 nuqtada chekli uchinchi tartibli hosilaga ega va bu nuqtada $f''(x_0)=0$, $f'''(x_0)\neq 0$ shartlarni qanoatlantirsa, u holda $f(x)$ funksiyaning grafigi $M(x_0,f(x_0))$ nuqtada egilishga ega bo'ladi.

7.16.5-teorema (egilish nuqtasi bo'lishning uchinchi yetarli sharti). $n\geq 2-$ biror juft son bo'lsin. Agar $f(x)$ funksiya x_0 nuqtaning biror atrofida n-tartibli hosilaga, x_0 nuqtaning o'zida esa $n+1$ -tartibli hosilaga ega bo'lib,

$$f^{(1)}(x_0)=f^{(2)}(x_0)=...=f^{(n)}(x_0)=0,\ f^{(n+1)}(x_0)\neq 0$$

shartlar bajarilsa, u holda $f(x)$ funksiyaning grafigi $M(x_0,f(x_0))$ nuqtada egilishga ega bo'ladi.

$y=f(x)$ *funksiya grafigining egilish nuqtalarini topish qoidasi:*

1.Funksiyaning ikkinchi tartibli hosilasi $f''(x)$ topiladi.

2. $y=f(x)$ funksiyaning I I tur kritik nuqtalari, ya'ni $f''(x)$ hosila nolga aylanadigan yoki uzilishga ega bo'lgan nuqtalar topiladi.

3. Topilgan kritik nuqtalar $f(x)$ funksiyaning aniqlanish sohasini oraliqlarga ajratadi. Bu oraliqlarda ikkinchi tartibli $f''(x)$ hosilaning ishorasi tekshiriladi. Agar bunda x_0 kritik nuqta, qavariqlik va botiqlik oraliqlarini ajratib tursa, u holda x_0 -funksiya grafigining egilish nuqtasi abssissasidan iborat bo'ladi;

4. Funksiyaning egilish nuqtalardagi qiymatlari hisoblanadi.

7.17. Funksiya grafigining asimptotalari

Oy va Ox o'qlarga parallel, hamda koordinata o'qlariga parallel bo'lmagan asimptotalarni qaraymiz.

Vertikal asimptotalar. $y = f(x)$ funksiya a nuqtaning biror $\varepsilon > 0$ atrofida aniqlangan, ya'ni $x \in \dot{U}_{\varepsilon}(a)$ bo'lsin.

7.17.1-ta'rif. Agar $\lim\limits_{x\to a-0} f(x)$, $\lim\limits_{x\to a+0} f(x)$ lardan biri yoki ularning ikkalasi ham cheksiz bo'lsa, $x = a$ to'g'ri chiziq $f(x)$ funksiya grafigining *vertikal* yoki *Oy o'qqa parallel asimptotasi* deyiladi .

Demak, $y = f(x)$ funksiya grafigining vertikal asimptotalarini izlash uchun funksiyaning qiymatini cheksizlikka aylantiradigan (cheksiz uzilishga ega bo'lgan) $x = a$ nuqtani topish kerak ekan. Bunda $x = a$ to'g'ri chiziq vertikal asimptota bo'ladi.

7.17.2-eslatma. Umuman olganda, $y = f(x)$ funksiyaning grafigi bir nechta vertikal asimptotalarga ega bo'lishi ham mumkin.

Gorizontal asimptotalar.

7.17.3-ta'rif. Agar $\lim\limits_{\substack{x\to+\infty \\ (x\to-\infty)}} f(x) = b \ (b \in R)$ bo'lsa, $y = b$ to'g'ri chiziq $x \to +\infty \ (x \to -\infty)$ da $y = f(x)$ funksiya grafigining *gorizontal yoki Ox o'qqa parallel asimptotasi* deyiladi.

Og'ma asimptotalar.

7.17.4-ta'rif. Shunday k va b chekli sonlar mavjud bo'lib, $x \to +\infty \ (x \to -\infty)$da $f(x)$ funksiya quyidagi $f(x) = kx + b + \alpha(x)$ ko'rinishda ifodalansa (bunda $\lim\limits_{x\to\pm\infty} \alpha(x) = 0$), $y = kx + b$ to'g'ri chiziq $y = f(x)$ funksiya grafigining *og'ma asimptotasi* deyiladi. Xususiy holda $k = 0$ bo'lsa, $y = b$ to'g'ri chiziq gorizontal asimptota bo'ladi.

7.17.5-teorema. $y = f(x)$ funksiyaning grafigi $x \to \pm\infty$ da $y = kx + b$ og'ma asimptotaga ega bo'lishi uchun,

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = k, \quad \lim_{x\to\pm\infty} [f(x) - kx] = b \qquad (7.17.6)$$

munosabatlarning bajarilishi zarur va yetarli.

(7.17.6) limitlarni hisoblashda quyidagi xususiy hollar bo'ladi:

1-hol. Argument x ning ishorasiga bog'liq bo'lmagan holda, ushbu $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to-\infty} \frac{f(x)}{x} = k$, $\lim\limits_{x\to+\infty} [f(x) - kx] = \lim\limits_{x\to-\infty} [f(x) - kx] = b$ ikkala limit ham mavjud va chekli . Bu holda $y = kx + b$ to'g'ri chiziq funksiya grafigining *ikki tomonlama og'ma asimptotasi* bo'ladi.

2-hol. Argument x ham musbat, ham manfiy ishorali cheksizlikka intilganda, ushbu

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k_1, \ \lim_{x\to-\infty} \frac{f(x)}{x} = k_2, \ \lim_{x\to+\infty} [f(x) - kx] = b_1, \ \lim_{x\to-\infty} [f(x) - kx] = b_2$$

limitlar mavjud, lekin ular o'zaro har xil (hech bo'lmaganda, $k_1 \neq k_2$, yoki $b_1 \neq b_2$). Bu holda $y_1 = k_1x + b_1$ va $y_2 = k_2x + b_2$ to'g'ri chiziqlar funksiya grafigining, mos ravishda, ikkita *bir tomonli (o'ng va chap) og'ma asimptotalari* bo'ladi.

3-hol. Faqat $x \to +\infty$ da $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = k$, $\lim\limits_{x\to+\infty} [f(x) - kx] = b$ ikkala limit ham mavjud. Bu holda $y = kx + b$ to'g'ri chiziq funksiya grafigining *faqat o'ng og'ma asimptotasi* bo'ladi.

4-hol. Faqat $x \to -\infty$ da $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = k$, $\lim\limits_{x\to-\infty} [f(x) - kx] = b$ ikkala limit ham mavjud. Bu holda $y = kx + b$ to'g'ri chiziq funksiya grafigining *faqat chap og'ma asimptotasi* bo'ladi.

Agar yuqoridagi hollarning barchasida $k = 0$ bo'lsa, $y = b$ to'g'ri chiziq gorizontal asimptota bo'ladi.

Funksiya grafigining asimtotalarga nisbatan joylashishini aniqlash uchun, har bir $x \to +\infty$, $x \to -\infty$ hollarda $f(x) - (kx + b)$ ayirmaning ishorasi tekshiriladi.

Agar ayirmaning ishorasi musbat (manfiy) bo'lsa, funksiya grafigi asimptotadan yuqori (past)da joylashgan bo'ladi. Agar ayirma ishorasini o'zgartirsa, u holda asimptota funksiya grafigini kesadi.

7.17.7-eslatma. Berilgan $y = f(x)$ funksiya uchun faqat $\lim\limits_{x\to\infty} \frac{f(x)}{x} = k$ mavjud bo'lib, $\lim\limits_{x\to\infty} [f(x) - kx]$ mavjud bo'lmasa (yoki cheksiz) bo'lsa, berilgan funksiyaning grafigi asimptotaga ega bo'lmaydi, lekin asimptotik yo'nalishga ega bo'ladi.

7.18. Funksiyani to'liq tekshirish va uning grafigini chizish. Funksiyaning o'zgarish xarakterini hosila yordamida o'rganish funksiya grafigini aniqroq yasashda muhim rol o'ynaydi.*Funksiyani to'liq tekshirish va uning grafigini yasashni quyidagi sxema bo'yicha olib borish maqsadga muvofiq bo'ladi:*

1. Funksiyaning aniqlanish sohasini topish.

2. Funksiyani uzluksizlikka tekshirish va uzilish nuqtalarini topish.
3. Funksiyaning juft, toqligi hamda davriyligini aniqlash.
4. Funksiya grafigining o'qlar bilan kesishish nuqtalarini topish.
5. Funksiyaning ishorasi saqlanadigan oraliqlarni aniqlash.
6. Funksiya grafigining asimptotalarini topish.
7. Funksiyaning monotonlik oraliqlarini topish va ekstremumga tekshirish.
8. Funksiya grafigining qavariqligi hamda botiqligini aniqlash, egilish nuqtalarini topish.
9. Funksiyaning grafigini chizish.

7.4-§. Lopital qoidalari. Teylor formulasi

7.19. Lopital qoidalari. Funksiyalarning limitini hisoblash jarayonida, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ ko'rinishidagi aniqmasliklarni ochish vaqtida, ba'zan qiyinchiliklarga duch kelinadi. Agar berilgan funksiyalarning hosilalari mavjud bo'lsa, ulardan foydalanganda, berilgan aniqmasliklarni ochish yengillashadi. Odatda hosilalardan foydalanib aniqmasliklarni ochish *Lopital qoidalari* deb ataladi.

1. **Lopitalning birinchi qoidasi** $\left(\frac{0}{0}\right)$. Agar $x\to a$ da $f(x)\to 0,\ g(x)\to 0$ bo'lsa, $\frac{f(x)}{g(x)}$ ning $x\to a$ dagi limiti $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikni ifodalaydi. Ba'zi hollarda, $x\to a$ da $\frac{f(x)}{g(x)}$ nisbatning limitini topishga qaraganda $\frac{f'(x)}{g'(x)}$ nisbatning limitini topish yengil bo'ladi.

7.19.1-teorema (Lopitalning birinchi qoidasi). $f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda aniqlangan, uzluksiz bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=0$;

2) (a,b) da chekli $f'(x)$va $g'(x)$ hosilalar mavjud va $\forall x\in(a,b)$, $g'(x)\neq 0$;

3) $\exists\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=A$ (A-chekli yoki cheksiz) bo'lsin.

U holda $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ ham mavjud va

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A \qquad (7.19.2)$$

tenglik o'rinli bo'ladi.

7.19.3-eslatma. Teoremaning 3) sharti bajarilmaganda ham, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ mavjud bo'lishi mumkin.

7.19.4-eslatma. 7.19.1-teoremada $f'(x)$va $g'(x)$ hosilalarning $x=a$ nuqtada uzluksizligi talab qilinsa, u holda $g'(a)\neq 0$ shartda **(7.19.2)** formulani quyidagicha yozish mumkin:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

7.19.5-eslatma. 7.19.1-teorema $a=+\infty$ yoki $a=-\infty$ bo'lgan hol uchun ham o'rinli, ya'ni $f(x)$ va $g(x)$ funksiyalar $c<x<\infty$ oraliqda aniqlangan va shu oraliqda differensiallanuvchi bo'lib, quyidagi shartlarni qanoatlantirsin:

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty} g(x)=0,\qquad g'(x)\neq 0,\ \forall x\in(c;+\infty).$$

U holda, $\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}$ mavjud bo'lsa, $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ ham mavjud bo'ladi va $\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}$ tenglik o'rinli bo'ladi.

7.19.6-eslatma. Agar $f'(x)$va $g'(x)$ funksiyalar **7.19.1**-teoremaning hamma shartlarini qanoatlantirsa, Lopital qoidasini takroriy qo'llash mumkin, ya'ni

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}=\ldots$$

2. Lopitalning ikkinchi qoidasi $\left(\frac{\infty}{\infty}\right)$. Agar $x \to a$ da $f(x) \to \infty$, $g(x) \to \infty$ bo'lsa, $\frac{f(x)}{g(x)}$ ning $x \to a$ dagi limiti $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmaslikni ifodalaydi. Bunday aniqmasliklarni ochishda ham $f(x)$ va $g(x)$ funksiyalarning hosilalaridan foydalanish muhim rol o'ynaydi.

7.19.7-teorema (Lopitalning ikkinchi qoidasi). $f(x)$ va $g(x)$ funksiyalar (a, b) oraliqda quyidagi shartlarni qanoatlantirsin:

1) $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$;

2) (a, b) da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $\forall x \in (a, b)$, $g'(x) \neq 0$;

3) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$ (A-chekli yoki cheksiz), u holda

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = A$$

tenglik o'rinli bo'ladi.

7.19.8-eslatma. Agar $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty$ bo'lsa, $f(x)\cdot g(x)$ ifoda $(0\cdot\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu ko'rinishdagi aniqmaslikni ochishda, uni

$$f(x)\cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \frac{g(x)}{\frac{1}{f(x)}}$$

kabi yozish orqali $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltirilib, Lopital qoidalari qo'llaniladi.

7.19.9-eslatma. Agar $\lim_{x\to a} f(x) = +\infty$, $\lim_{x\to a} g(x) = +\infty$ bo'lsa, $f(x) - g(x)$ ifoda $(\infty - \infty)$ ko'rinishdagi aniqmaslikni ifoda qiladi, uni ham quyidagi

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}}$$

kabi yozish orqali $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltiriladi va Lopital qoidalari qo'llaniladi.

7.19.10-eslatma. Agar $x \to a$ da $f(x)$ funksiya 1, 0 va ∞, $g(x)$ funksiya esa, mos ravishda ∞, 0 va 0 ga intilganda $(f(x))^{g(x)}$ – daraja – ko'rsatkichli ifoda $1^\infty, 0^0, \infty^0$ ko'rinishdagi aniqmasliklarni ifoda qiladi. Bu ko'rinishdagi aniqmasliklarni ochish uchun, avvalo berilgan ifoda logarifmlanadi:

$$\ln y = g(x)\cdot \ln f(x),$$

bu ifoda, $x \to a$ da $(0\cdot\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi, ya'ni yuqorida o'rganilgan holga keladi.

Shunday qilib: 1) $(0\cdot\infty)$ yoki $(\infty - \infty)$ ko'rinishdagi aniqmasliklar, algebraik almashtirishlar natijasida, $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltiriladi va ularga Lopital qoidasi qo'llaniladi.

2) $1^\infty, 0^0, \infty^0$ ko'rinishdagi aniqmasliklar logarifmlash natijasida, yoki $(f(x))^{g(x)} = e^{g(x)\cdot\ln(f(x))}$ shakl o'zgartirishlar orqali $(0\cdot\infty)$ ko'rinishdagi aniqmaslikka keltiriladi, so'ngra uni $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltirilib, Lopital qoidasi qo'llaniladi.

7.20. Teylor formulasi.

7.20.1-teorema (Teylor teoremasi). $f(x)$ funksiya biror a nuqtaning atrofida $n+1$ tartibgacha hosilalarga ega, x -funksiya argumentining a nuqtaning atrofidagi ixtiyoriy qiymati, p – ixtiyoriy musbat son bo'lsin. U holda, a va x nuqtalar orasida shunday ξ nuqta topiladiki, bu nuqtada

$$f(x) = (a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \quad (7.20.2)$$

formula o'rinli bo'ladi, bunda

$$R_{n+1}(x)=\left(\frac{x-a}{x-\xi}\right)^{p}\cdot\frac{(x-\xi)^{n+1}}{n!\,p}\cdot f^{(n+1)}(\xi). \qquad (7.20.3)$$

(7.20.2)-formula *Teylor formulasi* deyiladi, $R_{n+1}(x)$ ifoda esa, Teylor formulasining qoldiq hadi deyiladi. Odatda, (7.20.3) ko'rinishdagi qoldiq hadni umumiy ko'rinishdagi qoldiq had deb ham yuritiladi.

Teylor formulasidan kengroq foydalanish maqsadida, uning qoldiq hadining umumiy ko'rinishi (7.20.3) dan xususiy holda kelib chiqadigan quyidagi turli xil ko'rinishlarini keltiramiz:

$$R_{n+1}(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],$$

$$R_{n+1}(x)=\frac{(x-a)^{n+1}\cdot(1-\theta)^{n}}{n!}\cdot f^{(n+1)}[a+\theta(x-a)],$$

$$R_{n+1}(x)=o[(x-a)^{n}].$$

Qoldiq hadning bu ko'rinishlari, mos ravishda, qoldiq hadning *Lagranj, Koshi va Peano ko'rinishlari* deyiladi.

7.20.4-eslatma. Peano ko'rinishidagi qoldiq hadni keltirib chiqarishda **7.20.1**-teoremadagi $f(x)$ funksiyaga nisbatan qo'yilgan shartni «yengillashtirish» mumkin, ya'ni, $f(x)$ funksiya a nuqtaning atrofida $f'(x), f''(x),...., f^{(n)}(x)$ hosilalarga ega bo'lib, $f^{(n)}(x)$ hosila esa, a nuqtada uzluksiz bo'lsin, degan shart yetarli bo'ladi.

$f(x)$ funksiyaning (7.20.2) Teylor formulasida $a=0$ deb olinsa, ushbu

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^{2}+...+\frac{f^{(n)}(0)}{n!}x^{n}+R_{n+1}(x)$$

formula hosil bo'ladi. Odatda, bu formulaga Makloren formulasi deyiladi. Bu holda, $R_{n+1}(x)$ qoldiq had:

1) Lagranj ko'rinishida: $R_{n+1}(x)=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x)$;

2) Koshi ko'rinishida: $R_{n+1}(x)=\frac{x^{n+1}(1-\theta)^{n}}{n!}f^{(n+1)}(\theta x)$;

3) Peano ko'rinishida: $R_{n+1}(x)=o(x^{n})$ $(0<\theta<1)$ kabi yozilishi mumkin.

Ko'p hollarda, (7.20.2) Teylor formulasi quyidagi ko'rinishda yozilishi ham mumkin. Buning uchun (7.20.2) formulada $a=x_0$, $x-a=\Delta x$ va qoldiq had Lagranj ko'rinishida olinsa,

$$f(x_0+\Delta x)-f(x_0)=\frac{f'(x_0)}{1!}\Delta x+\frac{f''(x_0)}{2!}(\Delta x)^{2}+...+\frac{f^{(n)}(x_0)}{n!}(\Delta x)^{n}+ \qquad (7.20.5)$$
$$+\frac{f^{(n+1)}(x_0+\theta\Delta x)}{(n+1)!}\cdot(\Delta x)^{n+1} \quad (0<\theta<1)$$

formula hosil bo'ladi.

(7.20.5) Teylor formulasi chekli orttirmalar haqidagi Lagranj formulasining umumlashmasi hisoblanadi. Xususiy holda, (7.20.5) dan, $n=0$ deb olinsa, Lagranj formulasi kelib chiqadi.

Agar shunday o'zgarmas M son mavjud bo'lib, x argumentning $x_0=0$ nuqta atrofidagi barcha qiymatlarida, hamda $\forall n\in N$ uchun,

$$\left|f^{(n)}(x)\right|\le M$$

tengsizlik o'rinli bo'lsa, u holda ushbu

$$\left|R_{n+1}(x)\right|=\left|\frac{f^{(n+1)}(\theta x)x^{n+1}}{(n+1)!}\right|\le M\cdot\frac{|x|^{n+1}}{(n+1)!}$$

tengsizlik o'rinli bo'ladi.

x ning har bir belgilangan qiymatida $\lim\limits_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0$ bo'lishini e'tiborga olsak, u holda n ning yetarli katta qiymatlarida $R_{n+1}(x)$ ning yetarli kichik bo'lishiga ishonch hosil qilamiz. Bu holda $x_0=0$ nuqtaning atrofida $f(x)$ funksiyani

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^{2}+...+\frac{f^{(n)}(0)}{n!}x^{n}$$

ko'p had bilan taqribiy almashtirish mumkin bo'ladi:

$$f(x)\approx f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^{2}+...+\frac{f^{(n)}(0)}{n!}x^{n}.$$

Agar $f(x)$ funksiya $x_0=0$ nuqtaning atrofida istalgan tartibdagi hosilaga (cheksiz differensiallanuvchi) ega bo'lsa:

a) $f(x)$ – juft bo‘lganda, $\forall n\in N$ uchun $f(x)=\sum_{k=0}^{n}\frac{f^{2k}(0)}{(2k)!}x^{2k}+R_{2n+1}(x)$,

b) $f(x)$– toq bo‘lganda, $\forall n\in N$ uchun

$$f(x)=\sum_{k=0}^{n}\frac{f^{(2k+1)}}{(2k+1)!}x^{2k+1}+R_{2n+2}(x);$$

bo‘ladi.

7.21. Elementar funksiyalar uchun Makloren formulasi

Ushbu

$$f(x)=e^x,\ f(x)=\sin x,\ f(x)=shx,\ f(x)=\cos x,$$
$$f(x)=chx,\ f(x)=(1+x)^m,\ f(x)=\ln(1+x)$$

funksiyalar uchun Peano ko‘rinishidagi qoldiq hadli Makloren formulalari, mos ravishda quyidagicha bo‘ladi:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+o(x^n),$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-....+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+o(x^{2n+2}),$$

$$\text{sh}\,x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+....+\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2}),$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+....+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n+1}),$$

$$\text{ch}x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+....+\frac{x^{2n}}{(2n)!}+o(x^{2n+1}),$$

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+....+\frac{m(m-1)....(m-(n-1))}{n!}x^n+o(x^n),$$

yoki

$$(1+x)^m=\sum_{k=0}^{n}C_\alpha^k x^k+0(x^n),\ C_\alpha^0=1, C_\alpha^k=\frac{\alpha(\alpha-1)....(\alpha-(k-1))}{k!},\ k=1,2,...$$

Xususiy holda,

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n),\qquad \frac{1}{1-x}=\sum_{k=0}^{n}x^k+o(x^n),$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{(-1)^{n-1}x^n}{n}+o(x^n),$$

$$\ln(1-x)=-\sum_{k=1}^{n}\frac{x^k}{k}+o(x^n)\quad .$$

7.22. Hosilaning iqtisodiy masalalarni yechishga qo‘llanilishi

1. ***Xarajatlar funksiyasi*** $C(x)$ ma’lum bir x birlik mahsulotni ishlab chiqarish uchun zarur bo‘lgan xarajatlarni aniqlaydi. $P(x)=D(x)$-$C(x)$ ifoda foydani ko‘rsatadi, bunda $D(x)-x$ birlik mahsulotni ishlab chiqarishdagi daromad.

x birlik mahsulotni ishlab chiqarishda $\frac{C(x)}{x}$ ifoda ***o‘rtacha xarajatlar*** $A(x)$ ni ifodalaydi. Limit xarajatlar $M(x)=C'(x)$ bo‘ladi.

2. Ishlab chiqaruvchi uchun ishlab chiqarishning ***optimal qiymati*** debmahsulotning shunday x qiymatiga aytiladiki, bunda $P(x)$ foyda eng katta bo‘ladi.

1-masala. Xarajatlar funksiyasi $C(x)=100+\frac{1}{2}x^2$ ko‘rinishda bo‘lib x birlik mahsulotni ishlab chiqarishdagi daromad quyidagicha aniqlansin;

$$D(x)=\begin{cases}4000x,\ agar\ x\le 100\ bo'lsa\\ 4000\left(100+\sqrt{x-100}\right),agar\ x>100\ bo'lsa\end{cases}$$

Ishlab chiqaruvchi uchun ishlab chiqarish optimal bo‘lgan x_o qiymatni toping.

Yechilishi: Foyda funksiyasi quyidagi ko‘rinishda bo‘lsin;

$$P(x)=\begin{cases}-100+4000x-\frac{1}{2}x^2,agar\ x\le 100\ bo'lsa\\ 399900+400\sqrt{x-100}-\frac{1}{2}x^2,agar\ x>100\ bo'lsa\end{cases}$$

Foyda funksiyasining hosilasini topamiz;

$$P'(x)=\begin{cases}4000-2x,agar\ x\le 100\ bo'lsa,\\ \frac{2000}{\sqrt{x-100}}-x,agar\ x>100\ bo'lsa\end{cases}$$

x<o bo'lganda $P'(x) > 0$ ekanligini ko'rish qiyin emas,demak,$[o; 100]$kesmada foydaning eng katta qiymati $P(100) =$ 399900 dan iborat bo'ladi.Endi foydaning

$(100; +\infty)$ intervaldagi eng katta qiymatini topamiz. Faqat bitta x=200 kritik nuqta mavjud. Bunda100<x<200 oraliqda$P'(x) > 0$, va $x > 200$ da $P'(x) < 0$, ya'ni x=200 $P(x)$ funksiyaning$(100; +\infty)$ oraliqdagi maksimal qiymati bo'ladi.

Shunday qilib $x_{opt} = 200(bir)$ ekan.

2-masala. Ishchilar brigadasining mehnat unumdorligi

$$y = -t^2 + 8t + 20, 0 < t < 8,$$

funksiya yordamida ifodalangan, bunda t–smenadagi ish soati. Mehnat unumdorligining o'zgarish tezligini va nisbiy o'zgarish tezligini $t = 2$ va $t = 5$ vaqtlarda toping.

Yechilishi. Mehnat unumdorligining o'zgarish tezligi

$$y' = -2t + 8 \qquad (1)$$

nisbiy o'zgarish tezligi esa

$$T_y = (lny)' = \frac{y'}{y} = \frac{-2t + 8}{-t^2 + 8t + 20} \qquad (2)$$

bo'ladi.

Topilgan (1),(2) ifodalardan $t = 2$ va $t = 5$ soatlarda

$$y'(2) = -2 \cdot 2 + 8 = 4;\ T_y(2) = \frac{-2 \cdot 2 + 8}{-2^2 + 8 \cdot 2 + 20} == \frac{4}{-4 + 16 + 20} = \frac{1}{8}$$

$$y'(5) = -2 \cdot 5 + 8 = -2;\ T_y(5) = \frac{-2 \cdot 5 + 8}{-5^2 + 8 \cdot 5 + 20} == \frac{-2}{-25 + 40 + 20} = \frac{-2}{35}$$

qiymatlarni topamiz.

Topilgan qiymatlardagi musbat ishora mehnat unumdorligi va uning nisbiy o'zgarishi $t = 2$ soatda o'sayotganligi, manfiy ishoralar esa $t = 5$ da kamayotganligini ko'rsatadi.

3-masala. Talab funksiyasi $D(P) = D_0 \exp(-kP^2), k > 0$formula bilan berilgan. Bunda D_0 va k – berilgan qiymatlar. R ning qanday kiymatlarida talab elastik bo'ladi?

Yechilishi.$E = \lim_{\Delta P \to 0} \left[\frac{\Delta D}{D} : \frac{\Delta P}{P}\right] = P\frac{D'(P)}{D(P)}$ formulaga binoan $E(D)$ ni topamiz

$$E(D) = P\frac{D'(P)}{D(P)} = P\frac{[D_0 \exp(-kP^2)]}{D_0 \exp(-kP^2)} = P\frac{-2kPD_0 \exp(-kP^2)}{D_0 \exp(-kP^2)} = -2kP^2$$

Talab elastik bo'lishi uchun a) hol $|E(D)| > 1$ya'ni $2kP^2 > 1$bo'lishi kerak.

Bundan $P > \frac{1}{\sqrt{2k}}$ shartini topamiz.

7-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

7.1. Funksiyaning hosilasi. Egri chiziqqa urinma o'tkazish masalasi ([1], 1-t., 10-bo'lim; [3], 1-q., 120-121 betlar; [5], 1-t., 189-196 betlar; [12], 1-q., 182-185 betlar; [30], 3-bo'lim).

7.2. Hosilaning geometrik va fizik ma'nosi ([3], 1-q., 123-124 betlar; [5], 1-t., 186-188 betlar; [12], 1-q., 185-188 betlar; [9], 1-t., 6-bo'lim; [30], 3-bo'lim).

7.3. Funksiyaning uzluksiz bo'lishi va uning hosilaga ega bo'lishi orasidagi bog'lanish ([3], 1-q., 124-125 betlar; [12], 1-q., 188 bet; [9], 1-t., 6-bo'lim; [30], 3-bo'lim).

7.4. Funksiyaning differensiallanuvchilik sharti ([3], 1-q., 139-140 betlar; [5], 1-t., 211-214 betlar; [12], 1-q., 196-198 betlar).

7.5. Teskari funksiyaning hosilasi ([3], 1-q., 128-131 betlar; [5], 1-t., 196-198 betlar; [12], 1-q., 188-189 betlar; [1], 1-t., 10-bo'lim; [9],1-t., 6-bo'lim; [30], 7-bo'lim).

7.6. Murakkab funksiyaning hosilasi ([3], 1-q., 128 bet; [5], 1-t., 202-209 betlar; [12], 1-q., 189-190 betlar; [1], 1-t., 10-bo'lim; [9],1-t., 6-bo'lim; [30], 7-bo'lim).

7.7. Oshkormas funksiyaning hosilasi ([16], 156 bet; [30], 14-bo'lim).

7.8. Hosilani hisoblashning sodda qoidalari ([3], 1-q., 125-127 betlar; [5], 1-t., 199-202 betlar; [12], 1-q., 190-196 betlar; [1], 1-t., 10-bo'lim; [9], 1-t., 6-bo'lim; [30], 3-bo'lim).

7.9. Elementar funksiyalarning hosilalari ([3], 1-q., 131-132 betlar; [5], 1-t., 193-196 betlar; [12], 1-q., 190-196 betlar; [1], 1-t., 10-bo'lim; [9], 1-t., 3-bo'lim; [30], 3-bo'lim).

7.10. Funksiyaning differensiali ([3], 1-q., 139-141 betlar; [5], 1-t., 213-215 betlar; [12], 1-q., 198-199 betlar; [30], 3-bo'lim).

7.11. Differensiallashning sodda qoidalari ([3], 1-q., 141-143 betlar; [5], 1-t., 215-216 betlar; [12], 1-q., 200 bet; [30], 3-bo'lim).

7.12. Taqribiy hisoblashda differensialdan foydalanish ([3], 1-q., 143-145 betlar; [5], 1-t., 218-220 betlar; [12], 1-q., 201-202 betlar, [30], 3-bo'lim).

7. 13. Funksiyaning yuqori tartibli hosilalari ([3], 1-q., 145-146 betlar; [5], 1-t., 231-232 betlar; [12], 1-q., 202-207 betlar; [30], 3-bo'lim).

7. 14. Leybnis formulasi ([3], 1-q., 147-148 betlar; [5], 1-t., 236-238 betlar; [12], 1-q., 205-206 betlar).

7. 15. Funksiya yuqori tartibli differensiallarining invariant emasligi ([3], 1-q., 150-151 betlar; [5], 1-t., 242-243 betlar)

7. 16. Ferma teoremasi ([3], 1-q., 133 bet; [5], 1-t., 223-224 betlar; [12], 1-q., 210-211 betlar; [1], 1-t., 10-bo'lim; [30], 4-bo'lim).

7. 17. Roll teoremasi ([3], 1-q., 134 bet; [5], 1-t., 225-226 betlar; [12], 1-q., 211-212 betlar; [1], 1-t.,10-bo'lim; [9], 1-t., 6-bo'lim; [30], 3-bo'lim).

7. 18. Chekli ortirmalar haqidagi formula (Lagranj formulasi). Lagranj teoremasi ([3], 1-q., 134-135 betlar; [5], 1-t., 226-228 betlar; [12], 1-q., 212-213 betlar; [1], 1-t., 10-bo'lim; [9], 1-t., 6-bo'lim; [30], 4-bo'lim).

7.19. Koshi teoremasi ([3], 1-q., 135-136 betlar; [5], 1-t., 229-230 betlar; [12], 1-q., 213-214 betlar).

7.20. Lagranj teoremasidan kelib chiqadigan natijalar: funksiyaning o'zgarmaslik sharti, funksiyaning monotonlik sharti ([3], 1-q., 135 bet; [28], 168-171 betlar; [1], 1-t., 10-bo'lim; [30], 4-bo'lim).

7.21. Teylor formulasi. Qoldiq hadning har xil ko'rinishlari. Elementar funksiyalar uchun Makloren formulasi ([3], 1-q., 151-157 betlar; [5], 1-t., 246-257 betlar; [12], 1-q., 214-226 betlar; [9], 1-t., 7-bo'lim).

7.22. Funksiyaning monotonligi. Funksiyaning ekstremum qiymatlari ([3], 1-q., 158-164 betlar; [5], 1-t., 268-290 betlar; [12], 1-q., 227-236 betlar; [30], 4-bo'lim).

7.23. Funksiya grafigining qavariqlari va botiqligi ([3], 1-q., 165-168 betlar; [5], 1-t., 294-304 betlar; [30], 4-bo'lim).

7.24. Funksiya grafigining egilish nuqtaliri ([3], 1-q., 168-169 betlar; [5], 1-t., 294-304 betlar; [12], 1-q., 242-243 betlar; [30], 4-bo'lim).

7.25. Egilishning zaruriy va yetarli shartlari([5], 1-t., 294-304 betlar; [30], 4-bo'lim).)

7.26. Funksiya grafigining asimptotalari ([3], 1-q., 169-170 betlar; [5], 1-t., 308-311 betlar; [12], 1-q., 243-245 betlar; [30], 2-bo'lim).

7.27. Funksiya grafigini chizish ([5], 1-t., 305-306 betlar; [12], 1-q., 245-246 betlar; [30], 4-bo'lim).

7.28. Aniqmasliklarni ochishda Lopital qoidasi ([3], 1-q., 170-176 betlar; [5], 1-t., 314-322 betlar; [12], 1-q., 246-256 betlar; [1], 1-t., 10-bo'lim; [30], 4-bo'lim).

7.1-amaliy mashg‘ulot.
Funksiyaning hosilasi va differensiali

1-misol. Ushbu $y = \operatorname{tg} x$ funksiyaning hosilasini ta’rifdan foydalanib toping.

Yechilishi. Ma’lumki, $y = \operatorname{tg} x$ funksiya R ning $x = \frac{\pi}{2} + k\pi \; (k \in Z)$ nuqtalardan tashqari nuqtalarida aniqlangan. $\forall x \in D(y)$ nuqtani olib, unga $x + \Delta x \in D(y)$ bo‘ladigan $\Delta x(\Delta x < 0 \text{ yoki } \Delta x > 0)$ orttirma beraylik. Bunda, argumentning Δx orttirmasiga mos ravishda, berilgan funksiya ham

$$\Delta y = \operatorname{tg}(x + \Delta x) - \operatorname{tg} x = \frac{\sin(\Delta x)}{\cos(x + \Delta x)\cos x} \qquad (1)$$

orttirma oladi. (1) ning ikkala tomonini Δx ga bo‘lib,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\cos(x + \Delta x) \cdot \cos x} \cdot \frac{\sin \Delta x}{\Delta x}$$

nisbatni hosil qilamiz va uning $\Delta x \to 0$ dagi limitini hisoblaymiz:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\cos(x + \Delta x) \cdot \cos x} \cdot \frac{\sin \Delta x}{\Delta x} = \frac{1}{\cos^2 x}.$$

Demak, $y' = (tgx)' = \frac{1}{\cos^2 x} \quad x \neq \frac{\pi}{2} + k\pi, \, k \in Z.$

2-misol. Ushbu $y = |x - 2|$ funksiyaning $x = 2$ nuqtadagi o‘ng va chap hosilalarini toping.

Yechilishi. (7.1.2) formuladan,

$$\lim_{\Delta x \to 0+0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{\Delta x}{\Delta x} = 1,$$

$$\lim_{\Delta x \to 0-0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0-0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0-0} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0-0} \frac{-\Delta x}{\Delta x} = -1.$$

Demak, $f(x) = |x - 2|$ funksiyaning $x = 2$ nuqtadagi o‘ng hosilasi $f'(2 + 0) = 1$, chap hosilasi esa $f'(2 - 0) = -1$ ekan.

3-misol. Ushbu $y = \sqrt[4]{x}$ funksiyaning $x = 0$ nuqtadagi hosilasini toping.

Yechilishi. Ta’rifga ko‘ra,

$$\lim_{\Delta x \to 0+0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{\sqrt[4]{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0+0} \frac{1}{\sqrt[4]{\Delta x^3}} = +\infty$$

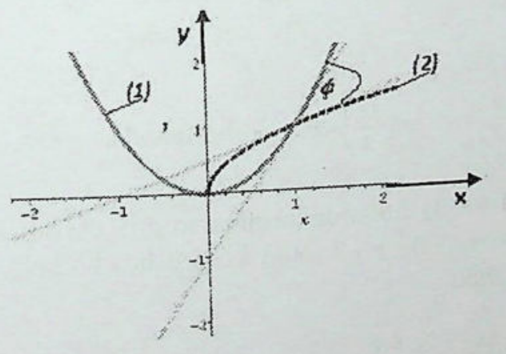
bo‘ladi. Demak, $f'(0) = +\infty$.

4-misol. Ushbu $f(x) = x^2$ va $f(x) = \sqrt{x}$ funksiyalar grafiklarining $M(1;1)$ kesishish nuqtasida o‘tkazilgan urinmalar orasidagi burchakni toping.

Yechilishi. Berilgan funksiyalarning $x = 1$ nuqtadagi hosilalarini topamiz:

$$f'(x) = 2x, \; f'(1) = 2, \; f'(x) = \frac{1}{2\sqrt{x}}, \; f'(1) = \frac{1}{2}.$$

(7.2.1) formulaga ko‘ra, $\operatorname{tg}\varphi = \frac{2 - \frac{1}{2}}{1 + 2 \cdot \frac{1}{2}} = \frac{3}{4}$ bo‘ladi. Bu yerdan $\varphi = \operatorname{arctg}\frac{3}{4}$ (7.1-chizma).



7.1-chizma.

(1) -$y = x^2$, (2) -$y = \sqrt{x}$

Maple tizimidan foydalanib misolni yechish:

> solve(x^2=sqrt(x),x);

1.0

> **a:=diff(x^2,x);b:=diff(sqrt(x),x);**

$a := 2x$

$$b := \frac{1}{2\sqrt{x}}$$

> **arctan(subs(x=1,(a-b)/(1+a*b)));**

$\mathrm{arctg}\frac{3}{4}$.

5-misol. Tenglamasi $x = 2\cos t - \cos 2t,\ y = 2\sin t - \sin 2t$ ko'rinishda bo'lgan $y = y(x)$ chiziqqa: 1) $t = \frac{\pi}{2}$; 2) $t = \pi$; 3) $t = \frac{3\pi}{2}$ nuqtalarda o'tkazilgan urinma chiziqlarning tenglamasini tuzing.

Yechilishi. $t = \frac{\pi}{2}$ bo'lganda, $x = 1,\ y = 2$; $t = \pi$ bo'lganda, $x = -3,\ y = 0$; $t = \frac{3\pi}{2}$ bo'lganda, $x = 1,\ y = -2$ bo'ladi. Endi berilgan $y = y(x)$ funksiyaning $y'_x(x)$ hosilasini topamiz:

$$y'_x = \frac{y'_t}{x'_t} = \frac{\sin\frac{3t}{2}\cdot\sin\frac{t}{2}}{\cos\frac{3t}{2}\cdot\sin\frac{t}{2}} = \mathrm{tg}\frac{3t}{2},$$

bundan, $y'_x\left(\frac{\pi}{2}\right) = -1$; $y'_x\left(\frac{3\pi}{2}\right) = 1$; $t = \pi$ bo'lganda esa, $y'_x(x)$ funksiya aniqlanmagan.

Demak, 1) va 3) hollarda urinma to'g'ri chiziqlar tenglamalari, mos ravishda, $y - 2 = -(x - 1)$, $y + 2 = x - 1$ ko'rinishda bo'ladi

6-misol. Ushbu

1) $y = 2x^5 - 5x^3 + 7x + 6$; 2) $y = \sqrt[3]{x} + \frac{1}{\sqrt{x}} - 0{,}2x^5$;

3) $y = \frac{3x^3 - 2x^2 + 3}{x^2 - x + 1}$; 4) $y = \frac{x^2 + \sqrt{x}}{x - 2\sqrt[3]{x}}$;

5) $y = \frac{\sin\varphi + \cos\varphi}{1 - \cos\varphi}$; 6) $y = 3e^x - \ln x$; 7) $y = \mathrm{e}^x(\mathrm{tg}\,x + \mathrm{ctg}\,x)$;

8) $y = \mathrm{arctg}\,x + x^2 + \mathrm{arcctg}\,x$; 9) $y = e^x \log_2 x$; 10) $y = \mathrm{sh}\,x\,\mathrm{ch}\,x$

funksiyalarning hosilalarini toping.

Yechilishi. Hosilani hisoblashning sodda qoidalari va elementar funksiyalarning hosilalari jadvaliga asosan topamiz:

1) Yig'indi (ayirma)ning hosilasini ifodalaydigan (7.4.1) formula va darajali funksiya hosilasi formulasidan foydalanamiz:

$$y' = (2x^5 - 5x^3 + 7x + 6)' = (2x^5)' - (5x^3)' + (7x)' + (6)' = 10x^4 - 15x^2 + 7;$$

Maple tizimidan foydalanib misolni yechish:

> **Diff((2*x^5)-(5*x^3)+7*x+6,x)=**

diff((2*x^5)-(5*x^3)+7*x+6,x);

$$\frac{d}{dx}(2x^5 - 5x^3 + 7x + 6) = 10x^4 - 15x^2 + 7.$$

2) Avvalo (7.4.1) formuladan va so'ngra darajali funksiya hosilasini topish formulasidan foydalanamiz:

$$\left(\sqrt[3]{x} + \frac{1}{\sqrt{x}} - 0{,}2x^5\right)' = \left(x^{1/3} + x^{-1/2} - 0{,}2x^5\right)' = \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{3}{2}} - 0{,}2\cdot 5x^4 =$$

$$= \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{2x\sqrt{x}} - x^4;$$

Maple tizimidan foydalanib misolni yechish:

> **Diff(x^(1/3)+1/sqrt(x)-0.2*x^5,x)=**

diff(x^(1/3)+1/sqrt(x)-0.2*x^5,x);

$$\frac{d}{dx}\left(x^{(1/3)} + \frac{1}{\sqrt{x}} - 0{,}2x^5\right) = \frac{1}{3x^{(2/3)}} - \frac{1}{2x^{(3/2)}} - 1.0x^4.$$

3) Ikki funksiyaning nisbati (kasrning) hosilasini ifodalaydigan formula va yuqorida foydalangan formulalarni qo'llaymiz:

$$y' = \frac{(3x^3-2x^2+3)'(x^2-x+1)-(3x^3-2x^2+3)(x^2-x+1)'}{(x^2-x+1)^2} =$$

$$= \frac{(9x^2-4x)(x^2-x+1)-(2x-1)\cdot(3x^3-2x^2+3)}{(x^2-x+1)^2} = \frac{3x^4-6x^3+11x^2-10x+3}{(x^2-x+1)^2};$$

Maple tizimidan foydalanib misolni yechish:

Diff(((3*x^3)-(2*x^2)+3)/((x^2)-

x+1),x)=diff(((3*x^3)-(2*x^2)+3)/((x^2)-x+1),x);

$$\frac{d}{dx}\left(\frac{3x^3-2x^2+3}{x^2-x+1}\right) = \frac{9x^2-4x}{x^2-x+1} - \frac{(3x^3-2x^2+3)(2x-1)}{(x^2-x+1)^2}$$

4) Ikki funksiya nisbati (kasrning) hosilasini ifodalaydigan (7.4.3) formula, shuningdek, yig'indi (ayirma) ning hosilasini topish (7.4.1) formula hamda jadvaldagi ildizning hosilasi formulasidan foydalanamiz:

$$y' = \frac{\left(x^2+\sqrt{x}\right)'\cdot\left(x-2\sqrt[3]{x}\right)-\left(x-2\sqrt[3]{x}\right)'\cdot\left(x^2+\sqrt{x}\right)}{\left(x-2\sqrt[3]{x}\right)^2} =$$

$$= \frac{\left(2x+\frac{1}{2\sqrt{x}}\right)\left(x-2\sqrt[3]{x}\right)-\left(1-\frac{2}{3}\cdot x^{-\frac{2}{3}}\right)\left(x^2+\sqrt{x}\right)}{\left(x-2\sqrt[3]{x}\right)^2} = \frac{x^2-\frac{10}{3}x\sqrt[3]{x}-\frac{1}{2}\sqrt{x}-\frac{1}{3\sqrt[6]{x}}}{\left(x-2\sqrt[3]{x}\right)^2};$$

Maple tizimidan foydalanib misolni yechish:

> diff(((x^2)+sqrt(x))/(x-2*x^(1/3)),x);

$$\frac{2x+\frac{1}{2\sqrt{x}}}{x-2x^{(1/3)}} - \frac{\left(x^2+\sqrt{x}\right)\left(1-\frac{2}{3x^{(2/3)}}\right)}{\left(x-2x^{(1/3)}\right)^2}$$

5) Ikki funksiya nisbati (kasrning) hosilasini ifodalaydigan (7.4.3) formula, yig'indi (ayirma) ning hosilasi uchun keltirilgan (7.4.1) formula hamda jadvaldagi sinus va kosuns funksiyalarning hosilalari formulalaridan foydalanamiz:

$$y' = \frac{(\sin\varphi+\cos\varphi)'(1-\cos\varphi)-(1-\cos\varphi)'\cdot(\sin\varphi+\cos\varphi)}{(1-\cos\varphi)^2} =$$

$$= \frac{(\cos\varphi-\sin\varphi)(1-\cos\varphi)-(\sin\varphi+\cos\varphi)\sin\varphi}{(1-\cos\varphi)^2} = \frac{\cos\varphi-\sin\varphi-1}{(1-\cos\varphi)^2};$$

Maple tizimidan foydalanib misolni yechish:

> diff((sin(φ)+cos(φ))/(1-cos(φ)),φx);

$$\frac{\cos(\varphi)-\sin(\varphi)}{1-\cos(\varphi)} - \frac{(\sin(\varphi)+\cos(\varphi))\sin(\varphi)}{(1-\cos(\varphi))^2}.$$

6) Avvalo (7.4.1) formuladan, so'ngra jadvaldagi ko'rsatkichli va logarifmik funksiyalar hosilalari formulalaridan foydalanamiz:

$$y' = (3e^x - \ln x)' = 3e^x - \frac{1}{x};$$

Maple tizimidan foydalanib misolni yechish:

> diff((3*exp(x)-ln(x)),x);

$$3e^x - \frac{1}{x}$$

7) Ko'paytmaning hosilasi to'g'risidagi (7.4.2) formula va jadvaldagi ko'rsatkichli va trigonometric funksiyalar hosilalari formulalaridan foydalanamiz:

$$y' = \left(e^x(\operatorname{tg}x+\operatorname{ctg}x)\right)' = e^x(\operatorname{tg}x+\operatorname{ctg}x)+e^x\left(\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}\right) = \frac{2e^x(\sin 2x-2\cos 2x)}{\sin^2 2x};$$

Maple tizimidan foydalanib misolni yechish:

> diff(exp(x)*(tan(x)+cot(x)),x);

$$e^x(\tan(x)+\cot(x)) + e^x(\tan(x)^2-\cot(x)^2).$$

8) Yig'indining hosilasini ifodalaydigan (7.4.1) formula, so'ngra jadvaldagi teskari trigonometrik funksiyalar va darajali funksiya hosilalari formulalaridan foydalanamiz:

$$y' = (\operatorname{arctg} x + x^2 + \operatorname{arcctg} x)' = \frac{1}{1+x^2} + 2x - \frac{1}{1+x^2} = 2x;$$

Maple tizimidan foydalanib misolni yechish:

> Diff(arctan(x)+x^2+arccot(x),x)= diff(arctan(x)+x^2+arccot(x),x);

$$\frac{d}{dx}(\arctan(x) + x^2 + \operatorname{arccot}(x)) = 2x$$

9) Ko'paytmaning hosilasi to'g'risidagi (7.4.2) formula va jadvaldagi ko'rsatkichli va logarifmik funksiya hosilalari formulalaridan foydalanamiz

$$y' = (e^x \cdot \log_2 x)' = e^x \cdot \log_2 x + e^x \cdot \frac{1}{x \ln 2} = e^x (\log_2 x + \frac{1}{x \ln 2});$$

Maple tizimidan foydalanib misolni yechish:

> diff(exp(x)*log[2](x),x);

$$\frac{e^x \ln(x)}{\ln(2)} + \frac{e^x}{x \ln(2)}$$

10) Ko'paytmaning hosilasi formulasi (7.4.2) va jadvaldagi giperbolik funksiyalar hosilalari formulalaridan foydalanamiz:

$$y' = (\operatorname{sh} x \cdot \operatorname{ch} x)' = \operatorname{ch}^2 x + \operatorname{sh}^2 x.$$

Maple tizimidan foydalanib misolni yechish:

> Diff(sinh(x)*cosh(x),x)=diff(sinh(x)*cosh(x),x);

$$\frac{d}{dx}(\sinh(x)\cosh(x)) = \cosh(x)^2 + \sinh(x)^2$$

7-misol. Ushbu $y^5 + y^3 + y - x = 0$ oshkormas shaklda berilgan $y = y(x)$ funksiyalarning hosilalari y'_x ni toping.

Yechilishi. Agar biror oraliqda differensiallanuvchi $y = y(x)$ funksiya $F(x, y) = 0$ tenglamani qanoatlantirsa, u holda uning hosilasi

$$\frac{d}{dx} F(x, y) = 0 \qquad (1)$$

tenglamadan topiladi. **1)**-holda, (1) tenglamaning ko'rinishi

$$\frac{d}{dx}(y^5 + y^3 + y - x) = 0$$

shaklda bo'ladi. Bundan, $5y^4 \cdot y'_x + 3y^2 \cdot y'_x + y'_x - 1 = 0$, $y'_x = \frac{1}{5y^4 + 3y^2 + 1}$.

Maple tizimidan foydalanib misolni yechish:

> z:=diff(y(x)^5+y(x)^3+y(x)-x,x);

$$z := 5\, y(x)^4 \left(\frac{d}{dx} y(x)\right) + 3\, y(x)^2 \left(\frac{d}{dx} y(x)\right) + \left(\frac{d}{dx} y(x)\right) - 1$$

> Q:=solve(z=0,diff(y(x),x));

$$Q := \frac{1}{5\, y(x)^4 + 3\, y(x)^2 + 1}$$

8-misol. Ushbu

$$1)\; y = 2\sqrt{x^3}(3\ln x - 2); \qquad 2)\; y = \arccos e^x$$

funksiyalarning differensiyalarini toping.

Yechilishi. Ushbu

$$d(\alpha u \pm \beta v) = \alpha\, du \pm \beta\, dv,\ \alpha, \beta \in R;\ d(u \cdot v) = v\, du + u\, dv;\ d\left(\frac{u}{v}\right) = \frac{v\, du - u\, dv}{v^2},\ v \neq 0$$

formulalardan foydalanib, berilgan funksiyalarning differensiallarini hisoblaymiz:

$$1)\; dy = d\left(2\sqrt{x^3}(3\ln x - 2)\right) = 2\sqrt{x^3}\, d(3\ln x - 2) + (3\ln x - 2)\, d(2\sqrt{x^3}) =$$

$$= 2\sqrt{x^3} \cdot \frac{3}{x} dx + (3\ln x - 2) \cdot \frac{3x^2}{\sqrt{x^3}} dx = 9\sqrt{x} \cdot \ln x\, dx;$$

$$2)\; dy = d(\arccos e^x) = -\frac{1}{\sqrt{1 - e^{2x}}} \cdot e^x \cdot dx.$$

Mustaqil yechish uchun misol va masalalar

1. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning $f'(x)$ hosilalarini toping.

1) $f(x)=x^3-2x$.   2) $f(x)=2^x\cos x$.   3) $f(x)=\cos 5x$.

4) $f(x)=\operatorname{tg}x+2x$.   5) $f(x)=\sqrt[3]{x}$.   6) $f(x)=\ln(4x-1)$.

2. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning $f'(x_0)$ hosilalarini toping:

1) $f(x)=(x-4)^3(x+3)$, $f'(4)$.   2) $f(x)=\dfrac{(x-2)^3\ln x}{\sin x}$, $f'(2)$.

3. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning hosilalari mavjudligini tekshiring.

1) $f(x)=|\ln x|$, $x_0=1$.   2) $f(x)=|(x-1)(x-2)|$, $x_0=1$, $x_0=2$.

3) $f(x)=|\sin x|$, $x_0=\pi$.   4) $f(x)=|\cos x|$.   5) $f(x)=|\pi^2-x^2|\cdot\sin^2 x$.

Hosilalar jadvali va sodda qoidalar yordamida quyidagi funksiyalarning hosilalarini toping.

Darajali funksiyalar

4. $y=3x^2+6x+3$.   **5.** $y=2\sqrt{x}-\dfrac{1}{x^3}+\sqrt{3}$.   **6.** $y=\dfrac{1-x^3}{1+x^3}$.

7. $y=(1+2x)^{30}$.   **8.** $y=\left(7x^2-\dfrac{4}{x}+6\right)^8$.   **9.** $y=\left(t^3-\dfrac{1}{t^3}+3\right)^4$.

10. $y=\sqrt{1-x^2}$.   **11.** $y=\dfrac{1+x}{\sqrt{1-x}}$.   **12.** $y=\dfrac{t^3}{(1-t)^2}$.

Trigonometrik funksiyalar

13. $y=x^2\cdot\sin x+2x\cos x-2\sin x$.   **14.** $y=\sin^3 x+\sin^{-2}x$.

15. $y=\cos^{-4}x$.   **16.** $y=\dfrac{\operatorname{tg}x}{x}$.   **17.** $y=\cos x-\dfrac{1}{3}\cos^3 x$.

Logarifmik funksiyalar.

18. $y=x^3\log_4 x$.   **19.** $y=x\ln x$.   **20.** $y=\sqrt{\ln x}$.

21. $y=\ln(1-2x^2)$.   **22.** $y=\log_2(x^2-4x)$.   **23.** $y=\ln^5(\sin x)$.

Ko'rsatkichli funksiyalar.

24. $y=3^x$.   **25.** $y=2^{\sin 2x}$.   **26.** $y=5^{\cos 4x}$.

27. $y=\dfrac{x^2}{4^x}$.   **28.** $y=e^x\cos x$.   **29.** $y=x^4-5^{2x}$.

Giperbolik funksiyalar.

30. $y=\operatorname{sh}^2 x$.   **31.** $y=\ln(\operatorname{ch}x)$.   **32.** $y=\operatorname{sh}^2 x+\operatorname{ch}^2 x$.

33. $y=\operatorname{th}^3 x$.   **34.** $y=x^2\operatorname{sh}x$.   **35.** $y=\dfrac{1+\operatorname{th}x}{1-\operatorname{th}x}$.

Teskari trigonometrik funksiyalar.

36. $y=x\arcsin x$.   **37.** $y=(\arccos x)^2$.   **38.** $y=\sqrt{x}\operatorname{arctg}x$.

39. $y=\arcsin\dfrac{3}{x}$.   **40.** $y=\dfrac{x^2}{\operatorname{arctg}x}$.   **41.** $y=\dfrac{\arccos x}{x}$.

Quyidagi funksiyalarning hosilalarini toping.

42. $y=\sqrt{x}+\ln x-\dfrac{1}{\sqrt{x}}$.   **43.** $y=\dfrac{x^4}{4}\left[(\ln x)^2-\dfrac{1}{2}\ln x+\dfrac{1}{8}\right]$.

44. $y=\operatorname{ctg}\pi x+\dfrac{\cos\pi x}{2\sin^3\pi x}$.   **45.** $y=e^{\sqrt[3]{x}}\left(\sqrt[3]{x^2}-2\sqrt[3]{x^2}+2\right)$.

46. $y=\dfrac{1}{\sqrt{2}}\ln\left(\sqrt{2}\cdot\operatorname{tg}x+\sqrt{1+2\operatorname{tg}^2 x}\right)$.   **47.** $y=\dfrac{1}{2a}\left[\ln\dfrac{\sqrt{a^2+x^2}}{a+x}-\dfrac{a}{a+x}\right]$.

48. $y=+\dfrac{\sin x}{2\cos^2 x}-\dfrac{1}{2}\ln\operatorname{tg}\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$.   **49.** $y=\ln\operatorname{tg}\dfrac{x}{2}-\operatorname{ctg}x\cdot\ln(1+\sin x)-x$.

50. Quyidagi funksiyalarning ko'rsatilgan nuqtalardagi o'ng va chap hosilalarini toping.

1) $f(x)=|x+3|$, $f'(-3+0)$, $f'(-3-0)$.

2) $y=f(x)=|x^2-5x+6|$, $x=2$, $x=3$.   3) $f(x)=|2^x-2|$, $x=1$.

51. Quyidagi funksiyalarning $x=0$ nuqtadagi o'ng va chap hosilalarini toping:

1) $f(x)=\begin{cases} x, & x\le 0,\\ \sqrt[3]{x^4}\ln x, & x>0;\end{cases}$ 2) $f(x)=\begin{cases} 1+e^{1/x}, & x<0,\\ \sqrt{1+\sqrt[3]{x^4}}, & x\ge 0.\end{cases}$

52. Funksiyalarning uzilish nuqtalaridagi o'ng va chap hosilalarini toping.

1) $f(x)=\sqrt{\frac{x^2+x^3}{x}}$; 2) $f(x)=\begin{cases}\frac{1}{1+e^{1/x}}, & x\neq 0,\\ 0, & x=0;\end{cases}$.

53. $f(x)=|x-x_0|\cdot\varphi(x)$ funksiya uchun $f'_+(x_0)$ va $f'_-(x_0)$ larni toping, bunda $\varphi(x)$ -berilgan x_0 nuqtada uzluksiz funksiya.

54. $y=\frac{1}{2}x^2-\ln x$ funksiya grafigiga abssissasi $x_0=2$ bo'lgan nuqtada o'tkazilgan urinmaning burchak koeffisiyentini toping.

55. $y=x^2-3x+2$ parabolaga abssissasi $x_0=2$ bo'lgan nuqtada o'tkazilgan urinmaning burchak koeffisiyentini toping.

56. $y=4\sin\frac{x}{3}$ funksiya grafigining $M(\frac{3\pi}{2},4)$ nuqtasidan o'tkazilgan urinma tenglamasini yozing.

57. $y=x^2+1$ egri chiziqqa o'tkazilgan urinma $y=2x+3$ to'g'ri chiziqqa parallel. Urinish nuqtasining ordinatasini toping.

58. $y=x^2-2x+1$ egri chiziqdagi qanday nuqtada unga o'tkazilgan urinma $y=-4(x+1)$ to'g'ri chiziqqa parallel bo'ladi?

59. $y=\frac{x}{1-x}$ funksiya grafigiga abssissasi $x_0=3$ bo'lgan nuqtadan o'tkazilgan urinmaning Ox o'qi bilan tashkil etgan burchagi α bo'lsa, $\mathrm{tg}\,2\alpha$ ni toping.

60. $y=\frac{x+2}{x-2}$ funksiya grafigiga qanday nuqtalarda o'tkazilgan urinma, Ox o'qining musbat yo'nalishi bilan 135° li burchak tashkil etadi?

61. $y=\sqrt[3]{x}$ funksiyaning grafigi qanday nuqtada abssissa o'qiga 30° li burchak ostida joylashgan bo'ladi?

62. $y=x^3+2x-1$ funksiya grafigiga qanday nuqtada o'tkazilgan urinma, $x+y=0$ to'g'ri chiziqqa perpendikulyar bo'ladi?

63. $y=x^4$ va $y^4=x$ funksiyalarning grafiklari qaysi nuqtalarda, qanday burchak ostida kesishishlarini aniqlang.

64. $y=\ln x$ chiziq Ox o'qni qanday burchak ostida kesadi?

65. $y=\sin x$ chiziq sinusoida Ox o'qni qanday burchak ostida kesadi?

66. a ning qanday qiymatida $y=a^x$ chiziq Oy o'qni 45° li burchak ostida kesadi.

67. Ushbu

1) $y=\sin x\sqrt{3}$; 2) $y=\frac{x}{1+x^2}$; 3) $y=\frac{x}{\sqrt{3+x^2}}$

funksiyalar Oy o'qni qanday burchak ostida kesadi?

68. $x=a\cos t,\ y=b\sin t$ ellipsga o'tkazilgan urinmaning Ox o'q bilan tashkil qilgan burchagini toping.

69. $(2;1)$ nuqtada $x=t^2-3t+4,\ y=t^2-4t+4$ chiziqqa o'tkazilgan urinmani toping.

70. $x=2t^3-9t^2+12t-1,\ y=t^2+t+1$ chiziqqa qanday nuqtada o'tkazilgan urinma Oy o'qqa parallel bo'lali.

71. $x=2t-t^2,\ y=3t+t^3$ chiziqqa: 1) $t=-1$; 2) $t=1$; 3) $t=\sqrt{2}$ nuqtalarda o'tkazilgan urinma to'g'ri chiziqlar tenglamasini yozing.

72. Quyidagi funksiyalarning grafiklari qaysi nuqtalarda qanday burchak ostida kesishishlarini aniqlang:

1) $f_1(x)=x-x^3$, $f_2(x)=5x$.

2) $f_1(x)=\sqrt{2}\sin x$, $f_2(x)=\sqrt{2}\cos x$. 3) $f_1(x)=\frac{1}{x}$, $f_2(x)=\sqrt{x}$.

4) $f_1(x)=\ln x$, $f_2(x)=\frac{x^2}{2e}$. 5) $f_1(x)=x^2-4x+4$, $f_2(x)=-x^2+6x-4$.

6) $f_1(x)=x^3$, $f_2(x)=\frac{1}{x^2}$. 7) $f_1(x)=4x^2+2x-8$, $f_2(x)=-x^3-x+10$.

73. Qanday nuqtalarda quyidagi $y=y(x)$ funksiyalar grafigiga o'tkazilgan urinmalar berilgan to'g'ri chiziqlarga parallel bo'ladi?

1) $y=x^2-7x+3$, $5x+y-3=0$. 2) $y=\frac{1}{3}\sin 3x+\frac{\sqrt{3}}{3}\cos 3x$, $y=-x$.

74. Qanday nuqtalarda quyidagi $y=y(x)$ unksiyalar grafigiga o'tkazilgan urinmalar berilgan to'g'ri chiziqlarga perpendikulyar bo'ladi?

1) $y=\ln x$, $2y+x+1=0$. 2) $y=x^3-3x+5$, $y=-x/9$.

75. $y=f(x)$ funksiya grafigiga berilgan nuqtada o'tkazilgan urinma tenglamasini yozing:

1) $y=\mathrm{arctg}\,2x$, $x=0$. 2) $y=e^x$, $x=1$.

3) $y=|x-1|\sqrt[3]{x+2}$, $x=6$. 4) $y=\sqrt{5-x^2}$, $x=1$.

76. $s=2\sin 3t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilayotgan nuqtaning $t=\frac{\pi}{9}$ paytdagi tezligini toping.

77. $s=\sin^2 t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilayotgan nuqtaning $t=\frac{\pi}{6}$ paytdagi tezligini toping.

78. $s=e^t+\cos t+5t$ qonuniyat bo'yicha harakatlanayotgan moddiy nuqtaning $t=0$ dagi tezligini toping.

79. Ikki moddiy nuqta, mos ravishda, $s_1=2{,}5t^2-6t+1$ va $s_2=0{,}5t^2+2t-3$ qonuniyatlar bo'yicha harakatlanmoqda. Qaysi vaqtda birinchi nuqtaning tezligi ikkinchisinikidan 3 marta katta bo'ladi?

80. Moddiy nuqta $s=\ln t+\frac{1}{16}t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilmoqda. Harakat boshlangandan qancha vaqt o'tgach, nuqtaning tezligi $\frac{1}{8}$ $м/c$ bo'ladi?

81. Massasi $m=1{,}5$ bo'lgan jism $s(t)=t^2+t+1$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilmoqda. Jismning harakati boshlangandan 5 sekund vaqt o'tgandagi kinetik energiyasini toping (m massa kilogrammlarda, s yo'l – metrlarda berilgan).

82. Absissalar o'qi bo'ylab ikkita nuqta, mos ravishda, $x=100+5t$ va $x=\frac{t^2}{2}$ qonuniyatlar bo'yicha harakat qilmoqda. Bu nuqtalar uchrashish paytida (momentida) bir-biridan qanday tezlikda uzoqlashadi? (x metrlar bilan o'lchanadi, t – sekundlar bilan).

83. G'ildirak shunday aylanadiki, uning burilish burchagi vaqtning kvadratiga proporsionaldir. Birinchi aylanish 8 sekund vaqt davomida amalga oshirildi. Harakat boshlangandan 64 sekund vaqt o'tgandagi burchak tezligini toping.

84. Ko'rsatilgan nuqtalarda quyidagi funksiyalarga teskari bo'lgan funksiyalarning hosilalarini toping:

1) $y=2x-\frac{\cos x}{2},\ y=-\frac{1}{2}$. 2) $y=2x^2-x^4,\ 0<x<1,\ y=\frac{3}{4}$.

85. Parametrik shaklda berilgan quiydagi $y=y(x)$ funksiyalarning y'_x hosilalarini toping:

1) $x=e^{-t},\ y=t^3,\ -\infty<t<+\infty$. 2) $x=a\cos t,\ y=b\sin t,\ 0<t<\pi$.

3) $x=\ln(1+t^2),\ y=t-\operatorname{arctg} t$. 4) $x=a(t-\sin t),\ y=a(1-\cos t)$.

86. Oshkormas shaklda berilgan quiydagi $y=y(x)$ funksiyalarning y'_x hosilalarini toping:

1) $x+\sqrt{xy}+y=a$. 2) $e^x\cdot\sin y-e^{-y}\cos x=0$. 3) $y^2=2px,\ y>0$.

4) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\ y>0$. 5) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ y>0$. 6) $e^y+xy=e,\ x=0,\ y=1$.

Quyidagi funksiyalarning differensialini toping.

87. $y=\ln x+x^2$. **88.** $y=e^{3x}+\sqrt{x}$. **89.** $y=\cos^2 x+3$.

90. $y=\operatorname{tg}4x+\frac{2}{x}$. **91.** $y=\log_3 x+\sin 5x$.

92. Quyidagi funksiyalarning differensialini toping.

1) $y=\ln\ln\left(\frac{x}{2}\right)$. 2) $y=\cos\frac{1}{\log_2 x}$. 3) $y=e^{\frac{\sqrt{1-x}}{1+x}}$. 4) $y=x^{x^2}$.

93. Funksiyaning orttirmasini uning differensiali bilan almashtirib, quyidagi $y=f(x)$ funksiyalarning ko'rsatilgan nuqtadagi taqribiy qiymatlarini toping:

1) $y=\sqrt[3]{x}$, $a)\ x=65$; $b)\ x=125{,}1324$. 2) $y=\sqrt[4]{x}$, $a)\ x=90$; $b)\ x=15{,}8$.

94. Funksiyaning orttirmasini uning differensiali bilan almashtirib, quyidagi ifodalarning taqribiy qiymatlarini toping:

1) $\sqrt[3]{1{,}02}$. 2) $\sin 29^0$. 3) $\cos 151^0$. 4) $\lg 11$.

96. Quyidagi $y=f(x)$ funksiyalarni berilgan nuqtalarda differensiallanuvchilikka tekshiring.

1) $y=\sqrt{x^3},\ x_0=0$. 2) $y=\sqrt{1-x},\ x_0=\frac{1}{2}$. 3) $y=3x^3+4x^2+5x-2$, $\forall x_0\in R$.

Mustaqil yechish uchun berilgan misollarning javoblari

1. 1) $3x^2-2x$. 2) $2^x(\ln 2\cos x-\sin x)$. 3) $-5\sin 5x$. 4) $\frac{1}{\cos^2 x}+2$. 5) $\frac{1}{3\sqrt[3]{x^2}}$. 6) $\frac{4}{4x-1}$. **2.** 1) 0. 2) 0. **3.** 1) Mavjud emas. 2) Mavjud emas. 3) Mavjud emas. 4) $x=\frac{2k-1}{2}\pi,\ k\in Z$ nuqtada hosilaga ega emas. 5) . Hamma joyda hosilaga ega. **4.** $6(x+1)$ **5.** $\frac{1}{\sqrt{x}}+\frac{3}{x^4}$. **6.** $-\frac{6x^2}{(1+x^3)^2}$. **7.** $60(1+2x)^{29}$.

8. $16\left(7x^2-\frac{4}{x}+6\right)^7\left(7x+\frac{2}{x^2}\right)$. **9.** $12\left(t^3-\frac{1}{t^3}+3\right)^3\left(t^2+\frac{1}{t^4}\right)$. **10.** $-\frac{x}{\sqrt{1-x^2}}$. **11.** $\frac{3-x}{2(1-x)^{3/2}}$.

12. $\frac{t^2(3-t)}{(1-t)^3}$. **13.** $x^2\cos x$. **14.** $\cos x\left(3\sin^2 x-2\operatorname{cosec}^3 x\right)$. **15.** $4\sin x\cdot\sec^5 x$.

16. $\frac{x-0{,}5\sin 2x}{x^2\cos^2 x}$. **17.** $-\sin^3 x$. **18.** $3x^2\log_4 x+\frac{x^2}{2\ln 2}$. **19.** $\ln x+1$. **20.** $\frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}$.

21. $-\frac{4x}{1-2x^2}$. **22.** $\frac{2(x-2)}{(x^2-4x)\ln 2}$. **23.** $5\ln^4(\sin x)\operatorname{ctg}x$. **24.** $3^x\ln 3$. **25.** $2^{1+\sin 2x}\cdot\cos 2x\cdot\ln 2$.

26. $-4\cdot 5^{\cos 4x}\cdot\sin 4x\cdot\ln 5$. **27.** $\frac{2(x-x^2\ln 2)}{4^x}$. **28.** $e^x(\cos x-\sin x)$. **29.** $4x^3-2\cdot 5^{2x}\ln 5$.

30. $\frac{1}{2}\operatorname{sh}2x$. **31.** $\operatorname{th}x$. **32.** $2\operatorname{sh}2x$. **33.** $3\operatorname{th}^2 x\cdot\frac{1}{ch^2 x}$. **34.** $x(2\operatorname{sh}x+x\operatorname{ch}x)$.

35. $\frac{2}{\operatorname{ch}^2 x\cdot(1-\operatorname{th}x)^2}$. **36.** $\arcsin x+\frac{x}{\sqrt{1-x^2}}$. **37.** $-\frac{2\arccos x}{\sqrt{1-x^2}}$. **38.** $\frac{1}{2\sqrt{x}}\operatorname{arctg}x+\frac{\sqrt{x}}{1+x^2}$.

39. $-\dfrac{3x^3}{\sqrt{x^2-9}}\operatorname{sign}x$. **40.** $\dfrac{2x\operatorname{arctg}x+2x^3\operatorname{arctg}x-x^2}{(1+x^2)\operatorname{arctg}^2 x}$. **41.** $-\dfrac{x+\sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}}$.

42. $\dfrac{x+2\sqrt{x}+1}{2x\sqrt{x}}$. **43.** $x^3\ln^2 x$. **44.** $-\dfrac{3\pi}{2\sin^4 \pi x}$. **45.** $\dfrac{1}{3}e^{\sqrt[3]{x}}$. **46.** $\dfrac{1}{\sqrt{1-\sin^4 x}}$.

47. $\dfrac{x^2}{(x^2+a^2)(x+a)^2}$. **48.** $\dfrac{1}{\cos^3 x}$. **49.** $\dfrac{\ln(1+\sin x)}{\sin^2 x}$. **50.** 1)

$f'_+(-3+0)=1$, $f'_-(-3-0)=-1$. 2) $f'_-(2)=f'_-(3)=-1$, $f'_+(2)=f'_+(3)=1$.

3) $f'_-(1)=\ln 4$, $f'_-(1)=-\ln 4$. 4) $f'(2k)=+\infty$, $f'(2k-1)=-\infty$, $k\in Z$. **51.**

1) $f'_-(0)=1$, $f'_+(0)=0$. 2) $f'(0)=0$.

52. 1) $f'_-(0)=-\dfrac{1}{2}$, $f'_+(0)=\dfrac{1}{2}$. 2) $f'_-(0)=-\infty$, $f'_+(0)=0$. **53.** $f'_+(x_0)=\varphi(x_0)$, $f'_-(x_0)=-\varphi(x_0)$.

54. k=3/2. **55.** k = 1. **56.** $y=4$. **57.** $y=2$. **58.** (-1; 4). 59. $\dfrac{8}{15}$. **60.** (4;3), (0;-1). **61.** $\left(\dfrac{1}{\sqrt[4]{27}};\dfrac{1}{\sqrt[4]{3}}\right)$. **62.** (-1;0), (1,-2). **63.** $\varphi=\operatorname{arctg}\dfrac{3}{8}$. **64.** 45^0. **65.** $x=2\pi n$ da 45^0, $x=(2n+1)\pi$ da -45^0 $(n\in Z)$. **66.** $a=e$. **67.** 1) 30^0. 2) 45^0. 3) 60^0. **68.** $\operatorname{tg}\alpha=-\dfrac{b}{a}\operatorname{ctg}t$.

69. $y=2x-3$. **70.** (4;3) ва (3;7). **71.** 1) va 3) hollarda, mos ravishda, $y+2=0$ va $y-\sqrt{2}=\dfrac{3}{2}(1+\sqrt{2})(x-2\sqrt{2}+2)$; 2) holda $t=1$ nuqtada $\dfrac{3(1-t^2)}{2(1-t)}$ funksiya aniqlanmagan. 72. 1) (0;0), $\varphi=\operatorname{arctg}(2/3)$. 2) $\left(\dfrac{\pi}{4}+\pi k;\,(-1)^k\right),(k\in Z),\varphi=\dfrac{\pi}{2}$.

3) (1; 1), $\varphi=\operatorname{arctg}3$. 4) $(\sqrt{e};\,1/2)$, $\varphi=0$. 5) (1;1) ва (4; 4), $\varphi=\operatorname{arctg}\dfrac{6}{7}$.

6) (1; 1), $\varphi=\dfrac{5\pi}{4}$. 7)(3; 34), $\varphi=0$. **73.** 1) (1;–3). 2) $\left(\dfrac{1}{3}\left(\dfrac{\pi}{3}-2k\pi\right);\,\dfrac{1}{\sqrt{3}}\right)$,

$\left(\dfrac{1}{3}\left(-\dfrac{2\pi}{3}-2m\pi\right);\,-\dfrac{1}{\sqrt{3}}\right)$, $k,m\in Z$. **74.** 1) $\left(\dfrac{1}{2};\,-\ln 2\right)$. 2) $M_1(-2;3)$, $M_2(2;7)$.

75. 1) $2x-y=0$. 2) $ex-y=0$. 3) $29x-12y-54=0$. 4) $x+2y-5=0$. **76.** $3\sqrt{3}$ (m/c).

77. $\dfrac{\sqrt{3}}{2}(m/c)$. **78.** 6 (m/s). **79.** 6 (s). **80.** 16 (s). **81.** 90,75 Jaul. **82.** $15\dfrac{\text{м}}{\text{сек}}$.

83. $4\pi\cdot$рад/сек. **84.** 1) $x'\left(-\dfrac{1}{2}\right)=\dfrac{1}{2}$. 2) $x'\left(\dfrac{3}{4}\right)=\dfrac{\sqrt{2}}{2}$. **85.** 1) $y'_x=-3t^2e^t$. 2) $y'_x=-\dfrac{b}{a}\operatorname{ctg}t$.

3) $y'_x=\dfrac{t}{2}$. 4) $y'_x=\operatorname{ctg}\dfrac{t}{2}$ $(t\neq 2k\pi, k\in Z)$. **86.** 1) $y'_x=\dfrac{2a-2x-y}{x+2y-2a}$. 2) $y'_x=\dfrac{e^x\sin y+e^{-y}\sin x}{e^x\cos y+e^{-y}\cos x}$.

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3) $y'_x=\dfrac{p}{y}$. 4) $y'_x=-\sqrt[3]{\dfrac{y}{x}}$, $|x|<a$. 5) $y'_x=\dfrac{b^2x}{a^2y}$, $|x|>a$. 6) $y'_x(0)=-\dfrac{1}{e}$. 7) $y'_x=-\dfrac{b^2x}{a^2y}$.

8) $y'_x=\dfrac{x+y}{x-y}$. 9) $y'_x(0)=\dfrac{1}{\sqrt{3}}$. 10) $y'_x=\dfrac{4y-2x-4}{8y-4x-3}$. **87.** $\left(\dfrac{1}{x}+2x\right)dx$. **88.** $\left(3e^{3x}+\dfrac{1}{2\sqrt{x}}\right)dx$

.**89.** $-\sin 2x\cdot dx$. **90.** $\left(\dfrac{4}{\cos^2 4x}-\dfrac{2}{x^2}\right)dx$. **91.** $\left(\dfrac{1}{x\ln 3}+5\cos 5x\right)dx$. **92.** 1) $\dfrac{dx}{x\ln\dfrac{x}{2}}$, $x>2$.

2) $\dfrac{\sin\left(\dfrac{1}{\log_2 x}\right)}{\left(x\log_2^2 x\right)\ln 2}dx$. 3) $-\dfrac{1}{\sqrt{1+x}\sqrt{1-x^2}}e^{\sqrt{\frac{1-x}{1+x}}}dx$. 4) $x^{1+x^2}(1+2\ln x)dx$.

93. 1) *a*) 4,0208; *b*) 5,00177. 2) *a*) 3,083; *b*) 1,9938.

94. 1) 1,007. 2) 0,4849. 3) $-0{,}8748$. 4) 1,043. **95.** 1) $\dfrac{12}{11}dx$. 2) $\dfrac{y_0^2-4x_0^3}{5y_0^4-2x_0y_0}dx$.

3) $-\dfrac{11}{20}dx$. 4) 0. 5) $\dfrac{1}{2}dx$. 6) $\dfrac{1}{8}dx$. **96.** 1) Differensiallanuvchi; 2) Differensiallanuvchi; 3) Differensiallanuvchi.

7.2-amaliy mashg'ulot.
Funksiyaning yuqori tartibli hosilasi va differensiali

1-misol. Ushbu 1). $y=\ln(x^2-3x+2)$; 2). $y=3^{5x}$ funksiyalarning n-tartibli hosilalarini toping.

Yechilishi. 1) $y'=\dfrac{2x-3}{x^2-3x+2}$. Hisoblashlarni soddalashtirish uchun oxirgi funksiyani quyidagicha shakl almashtiramiz:

$$y'=\frac{2x-3}{x^2-3x+2}=\frac{x-2+x-1}{(x-1)(x-2)}=\frac{1}{x-1}+\frac{1}{x-2}=(x-1)^{-1}+(x-2)^{-1}.$$

Bundan, (7.7.4) formulada $\alpha=-1$, $a=1$, $b=-1$, $b=-2$ deb olib,

$$\left(\frac{1}{x-1}\right)^{(n)}=(-1)(-2).....(-1-n+1)(x-1)^{-1-n},$$

$$\left(\frac{1}{x-2}\right)^{(n)}=(-1)(-2).....(-1-n+1)(x-2)^{-1-n},$$

bo'lishini topamiz. Demak,

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$$y^{(n)}=(\ln(x^2-3x+2))^{(n)}=(-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n}+\frac{1}{(x+2)^n}\right].$$

2) $y'=3^{5x}\cdot\ln 3\cdot 5,\quad y''=(3^{5x}\cdot 5\cdot\ln 3)'=3^{5x}\cdot 5^2\ln^2 3,$

$y^{(3)}=3^{5x}5^3\cdot\ln^3 3$ va hokazo $y^{(n)}=3^{5x}\cdot 5^n\cdot\ln^4 3$ deb yozish mumkin.

Bu formulaning to‘g‘riligini matematik induksiya usuli bilan ko‘rsatamiz:

Bu formula $n=1$ da o‘rinli. Faraz qilaylik, $n=k$ da ham o‘rinli bo‘lsin, ya’ni $y^{(k)}=3^{5x}\cdot 5^k\cdot\ln^k 3$. U holda, $(y^{(k)})'=(3^{5x}\cdot 5^k\cdot\ln^k 3)'=3^{5x}\cdot 5^{k+1}\ln^{k-1}3$.

Demak, formula $n=k+1$ bo‘lganda ham o‘rinli ekan. Bundan, $\forall n$ uchun o‘rinli ekanligi kelib chiqadi.

2-misol. x ni erkli o‘zgaruvchi deb, ushbu $y=xe^{x^2}$ funksiyaning uchinchi tartibli differensialini toping.

Yechilishi. 1-usul. Ikkinchi tartibli differensialning ta’rifiga ko‘ra,

$$d^2y=d(dy)=d(xde^{x^2}+e^{x^2}dx)=d(2x^2e^{x^2}dx+e^{x^2}dx)=$$
$$=2d(x^2e^{x^2})\cdot dx+d(e^{x^2})\cdot dx=$$
$$=2\left[d(e^{x^2})\cdot x^2+e^{x^2}d(x^2)\right]dx+d(e^{x^2})dx= \qquad (*)$$
$$=4e^{x^2}\cdot x^3dx^2+4xe^{x^2}dx^2+2xe^{x^2}dx^2=$$
$$=(6e^{x^2}x+4e^{x^2}\cdot x^3)dx^2=2e^{x^2}(3x+2x^3)dx^2$$

Uchinchi tartibli differensialning ta’rifiga ko‘ra,

$$d(d^2y)=d\left[2e^{x^2}(3x+2x^3)dx^2\right]=2d\left[e^{x^2}(3x+2x^3)dx^2\right]=$$
$$=2\left[e^{x^2}d(3x+2x^3)+(3x+2x^3)d(e^{x^2})\right]dx^2=$$
$$=2\left[e^{x^2}(3+6x^2)+(3x+2x^3)2xe^{x^2}\right]dx^3=2e^{x^2}(3+12x^2+4x^4)dx^3.$$

2-usul. Berilgan funksiyaning uchinchi tartibli hosilasini topamiz: (*)ni e’tiborga olib,

$$y'''=(xe^{x^2})'''=\left[2e^{x^2}3x+2x^3\right]''=$$
$$=2\left[(e^{x^2})'(3x+2x^3)+e^{x^2}(3x+2x^3)'\right]=2e^{x^2}(3+12x^2+4x^4).$$

$d^ny=d(d^{n-1}y)$ ёки $d^nf(x)=d(d^{n-1}f(x))$; $d^ny=y^{(n)}dx^n$ formulaga asosan $(n=3)$, $d^3y=2e^{x2}(3+12x^2+4x^4)dx^3$ bo‘lishini topamiz.

Maple tizimidan foydalanib misolni yechish:

```
> Diff(x*exp(x^2),x$3)=diff(x*exp(x^2),x$3);
```

$$\frac{d^3}{dx^3}\left(x\,e^{(x^2)}\right)=6\,e^{(x^2)}+24\,x^2\,e^{(x^2)}+8\,x^4\,e^{(x^2)}$$

Mustaqil yechish uchun berilgan misol va masalalar

Quyidagi misollarning ko‘rsatilgan tartibdagi hosilalarini toping.

1. $y=x(2x-1)^2(x+3)^3,\quad y^{(7)}=?$

2. $y=\dfrac{1}{1+x^3},\quad y''=?$

3. $y=1+10x+\dfrac{1}{x^{98}},\quad y''=?$

4. $y=\cos^2x,\quad y''=?$

5. $y=(1+x^2)\operatorname{arctg}x,\quad y''=?$

6. $y=\operatorname{arctg}(x+\sqrt{x^2+1}),\quad y''=?$

7. $y=\sqrt{1-x^2}\arcsin x,\quad y''=?$

8. $y=\ln(x+\sqrt{x^2+1}),\quad y''=?$

9. $y=\sqrt[5]{x^3},\quad y'''=?$

10. $y=x^5\ln x,\quad y'''=?$

Quyidagi funksiyalarning berilgan nuqtalardagi ko‘rsatilgan tartibdagi hosilalarini toping.

11. $y=x^6-4x^3+4,\quad y^{(IV)}(1)=?$

12. $y=\dfrac{x^5}{(x-1)^4},\quad y^{(II)}(5)=?$

13. $y=\dfrac{\arcsin x}{\sqrt{1+x^2}},\quad y^{(II)}(0)=?$

14. $y=\operatorname{arctg}x,\quad y^{(II)}(1)=?$

15. $y=e^{\sqrt{x}},\quad y^{(II)}(4)=?$

16. $y=\dfrac{1+x}{\sqrt{1-x}},\quad y^{(100)}(x)=?$

Quyidagi funksiyalarning ko‘rsatilgan tenglamalarni qanoatlantirishini isbotlang.

17. $y=e^x\sin x,\quad y''-2y'+2y=0;$

18. $y=e^{-x}\sin x,\quad y''+2y'+2y=0;$

19. $y=c_1\cos x+c_2\sin x,\quad y''+y=0$ (c_1 va c_2 -ixtiyoriy o‘zgarmas sonlar);

20. $y=c_1chx+c_2shx,\quad y''-y=0$ (c_1 va c_2 -ixtiyoriy o‘zgarmas sonlar);

Quyidagi parametrik shaklda berilgan $y=y(x)$ funksiyalarning ko'rsatilgan tartibdagi hosilalarini toping:

21. $x=at^2,\ y=bt^3;\ x''_{yy}=?$

22. $x=t^3+3t+1,\ y=t^3-3t+1;\ y''_{xx}=?$

23. $x=e^{\alpha t}\cos\beta t,\ y=e^{\alpha t}\sin\beta t;\ y''_{xx}=?$

24. $x=a(\cos t-\ln ctg\frac{t}{2}),\ y=a\sin t;\ y'''_{xxx}=?$

Quyidagi parametrik shaklda berilgan $y=y(x)$ funksiyalarning berilgan nuqtada ko'rsatilgan tartibdagi hosilalarini toping:

25. $x=\ln(1+\sin\varphi),\ y=\ln(1-\cos 2\varphi);\ (\ln(\frac{3}{2});\ln(\frac{1}{2}));\ y''_{xx}=?$

26. $x=\mathrm{ch}\,t\sin t+\mathrm{sh}\,t\cos t,\ y=\mathrm{ch}\,t\cos t-\mathrm{sh}\,t\sin t;\ (0;1);\ y''_{xx}=?$

Parametrik shaklda berilgan $y=y(x)$ funksiyalarning berilgan tenglamalarni qanoatlantirishini isbotlang:

27. $x=e^t\sin t,\ y=e^t\cos t;\ y''(t+y)^2=2(xy'-y).$

28. $x=\sin t,\ y=\sin kt;\ (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+k^2y=0.$

29. $x=\sin t,\ y=Ae^{\sqrt{2}t}+Be^{\sqrt{2}t};\ (1-x^2)y''-xy'-2y=0,\ -\frac{\pi}{2}<t<\frac{\pi}{2}$, A va B - ixtiyoriy o'zgarmas sonlar;

Quyidagi oshkormas shaklda berilgan $y=y(x)$ funksiyalarning x bo'yicha ko'rsatilgan tartibdagi hosilalarini toping:

30. $y^2=2px,\ y''_{xx}=?$ **31.** $e^{x-y}=x+y,\ y''_{xx}=?$

32. $\mathrm{arctg}\,y-y+x=0,\ y''_{xx}=?$ **33.** $e^x-e^y=y-x,\ y''_{xx}=?$

34. $x^2+5xy+y^2-2x+y-6=0,$ (1;1) nuqtadagi $y''_{xx}=?$

Quyidagi funksiyalarning $x=0$ nuqtada nechanchi tartibli hosilalarga ega ekanligini aniqlang va mavjud hosilalarning bu nuqtadagi qiymatini hisoblang:

35. $y=\begin{cases}1-\cos x, & x<0 \text{ bo'lganda,}\\ \ln(1+x)-x, & x\ge 0 \text{ bo'lganda.}\end{cases}$

36. $y=\begin{cases}shx-x, & x<0 \text{ bo'lganda,}\\ x-\sin x, & x\ge 0 \text{ bo'lganda.}\end{cases}$

37. $y=\begin{cases}shx, & x<0 \text{ bo'lganda,}\\ \sin x chx, & x\ge 0 \text{ bo'lganda.}\end{cases}$

x ni erkli o'zgaruvchi deb, quyidagi $y=y(x)$ funksiyalarning ko'rsatilgan tartibdagi differensiallarini toping:

38. $y=(x+1)^3(x-1)^2,\ d^2y=?$ **39.** $y=(x^3+2x^2+x+3)e^{-2x},\ d^2y=?$

40. $y=\sin^2 x,\ d^4y=?$ **41.** $y=x\cos 2x,\ d^{12}y=?$

42. $y=\mathrm{arctg}\left(\frac{b}{a}tgx\right);\ d^2y=?$ **43.** $y=\cos x chx;\ d^8y=?$

Agar du, d^2u, dv, dv^2 lar mavjud bo'lsa, quyidagi $y=y(x)$ funksiyalar uchun d^2y ni toping.

44. $y=\sqrt{u^2+v^2};\ d^2y=?$ **45.** $y=u^v;\ d^2y=?$

46. $y=\frac{2u+v}{u};\ d^2y=?$ **47.** $y=u\ln v;\ d^2y=?$

Quyidagi $y=y(x)$ funksiyalarning berilgan nuqtadagi ko'rsatilgan tartibdagi differensiallarini toping:

48. $y=xe^{x^2};\ d^2y|_{x=1}=?$ **49.** $y=\cos^2 x;\ d^3y\Big|_{x=\frac{\pi}{4}}=?$

50. $y=x\sqrt[3]{(x-5)^2};\ d^2y|_{x=-3}=?$ **51.** $y=\frac{1}{ax+b};\ d^ny|_{x=0}=?$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

1. 0. **2.** $\frac{6x(2x^3-1)}{(x^3+1)^3}$. **3.** $9702/x^{100}$. **4.** $-2\cos 2x$. **5.** $\frac{2x}{1+x^2}+2arctgx$.

6. $-x(1+x^2)^{-2}$. **7.** $-\frac{\arcsin x+x\sqrt{1-x^2}}{\sqrt{(1-x^2)^3}}$. **8.** $-x(x^2+1)^{-3/2}$. **9.** $\frac{42}{125}x^{-\frac{12}{5}}$.

10. $x^2(60\ln x+47)$. **11.** 360. **12.** $\frac{625}{1024}$. **13.** 0. **14.** $-\frac{1}{2}$.

15. $e^{2/32}$. **16.** $\frac{1\cdot3\cdot5\cdot...\cdot197(399-x)}{2^{100}(1-x)^{201/2}}$. **21.** $-\frac{2a}{9b^2t^4}$. **22.** $\frac{4t}{3(t^2+1)^3}$.

23. $\frac{(\alpha^2+\beta^2)\beta e^{-\alpha t}}{(\alpha\cos\beta t-\beta\sin\beta t)^3}$. **24.** $\frac{\sin t(1+3\sin^2 t)}{a^2\cos^7 t}$. **25.** -12. **26.** $-\frac{1}{2}$.

30. $-p^2/y^3$. **31.** $4(x+y)/(x+y+1)^3$. **32.** $-\frac{2(1+y^2)}{y^5}$.

33. $(e^x-e^y)(1-e^{x+y})/(1+e^y)^3$. **34.** $\frac{111}{256}$. **35.** $y'(0)=0$, $y''(0)$ mavjud emas.

36. $y'(0)=0$, $y''(0)=0$, $y'''(0)=1$, $y^{(IV)}(0)=0$, $y^{(V)}(0)$ mavjud emas.

37. $y'(0)=1$, $y''(0)=0$, $y'''(0)$ mavjud emas. **38.** $4(x+1)(5x^2-2x-1)dx^2$.

39. $2(2x^3-2x^2-3x+6)e^{-2x}dx^2$. **40.** $-8\cos 2x dx^4$. **41.** $4096(6\sin 2x+x\cos 2x)dx^{12}$.

42. $\frac{ab(a^2-b^2)\sin 2x}{(a^2\cos^2 x+b^2\sin^2 x)}dx^2$. **43.** $17\cos x\, ch x\, dx^8$.

44. $\frac{(u^2-v^2)(ud^2u+vd^2v)+(vdu-udv)^2}{(u^2-v^2)^{3/2}}$.

45. $u^v\left(\frac{v}{u}d^2u+\ln u d^2v+\frac{v(v-1)}{u^2}du^2+\frac{2(v\ln u+1)}{u}dudv+\ln^2 u dv^2\right)$.

46. $\frac{1}{u^3}\left(u^2d^2v-uvd^2u-2ududv+2vudu^2\right)$ **47.** $\ln v d^2u+\frac{2}{v}dudv+\frac{u}{v}d^2v-\frac{u}{v^2}dv^2$.

48. $10e dx^2$. **49.** $4dx^3$. **50.** $-\frac{5}{8}dx^2$. **51.** $\frac{(-1)^n a^n n!}{b^{n+1}}dx^n$.

7.3-amaliy mashg'ulot.
Funksiyani hosila yordamida tekshirish

1-misol. $f(x)=x^3-3x^2+4$ funksiyani to'liq tekshiring va grafigini chizing.

Yechilishi. Berilgan funksiya R da aniqlangan, uzluksiz va differensiallanuvchi. Funksiyaning grafigi asimptotalarga ega emas, $\lim_{x\to-\infty} f(x)=-\infty$, $\lim_{x\to+\infty} f(x)=+\infty$.

Berilgan funksiyani ushbu $f(x)=(x+1)(x-2)^2$ ko'rinishda tasvirlaymiz. $(x+1)(x-2)^2=0$ tenglamadan $x_1=-1$, $x_2=2$ nuqtalar $f(x)=0$ tenglamaning ildizlari ekanligini topamiz.

Demak, funksiyaning grafigi $(-1;0)$, $(2;0)$ nuqtalarda abssissa o'qi bilan, $(0;4)$ nuqtada esa, ordinata o'qi bilan kesishadi.

Funksiyani ekstremumga tekshiramiz: $f'(x)=3x^2-6x=3x(x-2)$. Bundan birinchi tartibli hosila $x_1=0$, $x_2=2$ nuqtalarda nolga aylanishi kelib chiqadi, ya'ni bu nuqtalar-kritik nuqtalar bo'ladi. $x\in(-\infty;0)$ bo'lganda $f'(x)>0$, $x\in(0;2)$ bo'lganda $f'(x)<0$, $x\in(2;+\infty)$ bo'lganda esa, $f'(x)>0$ bo'ladi.

Shunday qilib, funksiyaning birinchi tartibli $f'(x)$ hosilasi $x_1=0$ nuqtadan o'tishda o'z ishorasini «+» dan «-» ga, $x_2=2$ nuqtadan o'tishda esa «-» dan «+» ga o'zgartiradi. Demak, ekstremum mavjud bo'lishining birinchi yetarli shartiga asosan, funksiya $x_1=0$ nuqtada maksimum $\left(f_{max}(0)=4\right)$ ga, $x_2=2$ nuqtada esa minimum $\left(f_{min}(2)=0\right)$ ga ega bo'ladi. Funksiya grafigini chizishda qulaylik uchun, funksiya to'g'risida yuqorida olingan ma'lumotlar yordamida quyidagi jadvalni tuzamiz:

1-jadval.

x	$(-\infty;0)$	0	$(0;2)$	2	$(2;+\infty)$
sign y'	+	0	-	0	+
$y=f(x)$ funksiyaning o'zgarishi	↗	$f_{max}(0)=4$	↘	$f_{min}(2)=0$	↗

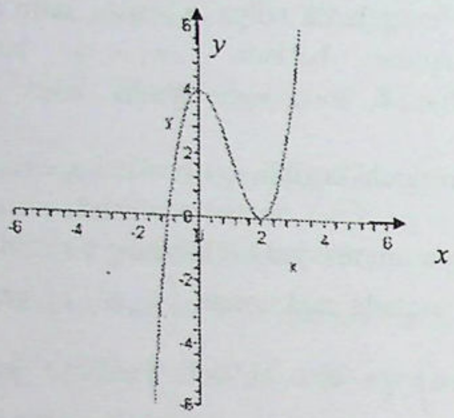
Funksiyaning ikkinchi tartibli hosilasini topamiz: $f''(x)=6(x-1)$.

Funksiyaning ikkinchi tartibli hosilasi $x=1$ nuqtada nolga aylanadi, ya'ni $x=1$ nuqta II-tur kritik nuqta bo'ladi. $x\in(-\infty;1)$ da $f''(x)<0$, $x\in(1;+\infty)$ da esa $f''(x)>0$. Demak, 7.15.4-teoremaga asosan, $(-\infty;1)$ oraliqda funksiyaning grafigi qavariq, $(1;+\infty)$ oraliqda botiq, $x=1$ nuqta funksiyaning grafigi egilish nuqtasining abssissasi bo'lib, $(1;2)$ nuqta egilish nuqtasi bo'ladi.

2-jadval.

x	$(-\infty;1)$	1	$(1;+\infty)$
sign f''	-	0	+
$y=f(x)$ funksiya grafigi kavariqligining yo'nalishi	∩	$(1;2)$	∪

Yuqoridagi mulohazalar va 1-,-2-jadvallar yordamida funksiyaning grafigini (1-chizma) chizamiz.



1-chizma.

2-misol. $f(x)=\frac{(x-1)^2}{(x+1)^3}$ funksiyani to'liq tekshiring va grafigini chizing.

Yechilishi. Berilgan funksiya R ning $x=-1$ nuqtadan tashqari barcha nuqtalarida aniqlangan. Ravshanki, funksiya grafigi mos ravishda $(1;0)$ va $(0;1)$ nuqtalarda abssissa va ordinata o'qlarini kesadi. $x<-1$ da funksiya manfiy, $x>-1$ da esa musbat. Funksiya $x=-1$ nuqtada II-tur uzilishga ega va $\lim_{x\to-1-0}f(x)=-\infty$, $\lim_{x\to-1+0}f(x)=+\infty$ bo'lgani uchun $x=-1$ to'g'ri chiziq funksiya grafigi uchun vertikal asimptota, $y=0$ to'g'ri chiziq esa, gorizontal asimptota.

Berilgan funksiya davriy ham emas, juft ham emas va toq ham emas.

Funksiyani ekstremumga tekshirish uchun uning birinchi tartibli hosilasini topamiz:

$$f'(x)=-\frac{x^2-6x+5}{(x+1)^4}=-\frac{(x-1)(x-5)}{(x+1)^4}.$$

Birinchi tartibli $f'(x)$ hosila $x_1=1$ va $x_2=5$ nuqtalarda nolga aylanadi, ya'ni bu nuqtalar kritik nuqtalar bo'ladi. $x\in(-\infty;1)$ da $f'(x)<0$, $x\in(1;5)$ da $f'(x)>0$, $x\in(5;+\infty)$ da esa $f'(x)<0$ bo'ladi. Demak, $f'(x)$ hosila $x_1=1$ nuqtadan o'tishda o'z ishorasini «-» dan «+» ga o'zgartiradi, $x_2=5$ nuqtadan o'tishda

esa, «+» dan «-» ga o'zgartiradi. Shunday qilib, funksiya $x_1=1$ nuqtada minimum $f_{min}(1)=0$ qiymatga, $x_2=5$ nuqtada esa, maksimum $f_{max}(5)=\frac{2}{27}$ qiymatga ega bo'ladi.

1-jadval

x	$(-\infty;1)$	1	$(1;5)$	5	$(5;\infty)$	$x<-1$	-1	$x>-1$
$signf'$	-	0	+	0	-	-	∞	-
$y=f(x)$- funksiyaning o'zgarishi	↘	0	↗	$\frac{2}{27}$	↘	Manfiy	∞	Musbat

Funksiyaning qavariqlik, botiqlik oraliqlari va egilish nuqtalarini topamiz:

$$f''(x)=\frac{2(x^2-10x+13)}{(x+1)^5}.$$

Ikkinchi tartibli $f''(x)$ hosila $x_1=5+\sqrt{12}$, $x_2=5-\sqrt{12}$ nuqtalarda nulga aylanadi, ya'ni bu nuqtalar II-tur kritik nuqta bo'ladi. $x\in(-\infty;5-\sqrt{12})$ da $f''(x)>0, x\in(5-\sqrt{12};5+\sqrt{12})$da $f''(x)<0, x\in(5+\sqrt{12};+\infty)$ da esa $f''(x)>0$.

Demak, $(-\infty;5-\sqrt{12})$ oraliqda funksiya grafigi qavariq $(5-\sqrt{12};5+\sqrt{12})$ oraliqda botiq, $(5+\sqrt{12};+\infty)$ oraliqda qavariq.

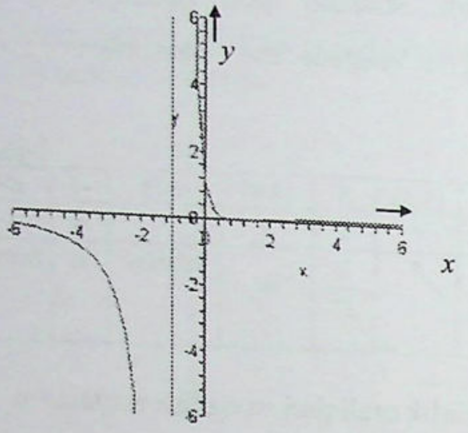
Shunday qilib, $x_1=5+2\sqrt{3}$, $x_2=5-2\sqrt{3}$ nuqtalar funksiya grafigi egilish nuqtalarining abssissalari bo'lib, $(5+2\sqrt{3};\ 0{,}0659)$ va

$(5-2\sqrt{3};\ 0{,}0196)$ nuqtalar egilish nuqtasi bo'ladi.

2-jadval.

x	$(-\infty;5-2\sqrt{3})$	$5-2\sqrt{3}$	$(5-2\sqrt{3},5+2\sqrt{3})$	$5+2\sqrt{3}$	$(5+2\sqrt{3};+\infty)$
sign f''	+	0	-	0	+
$f(x)$funksiya grafigi kavariqligining yo'nalishi	∪	0,0196	∩	0,0659	∪

Yuqoridagi mulohazalar va 1-, 2-jadvallarga binoan berilgan funksiyaning grafigini (2 -chizma) chizamiz.



2-chizma.

Mustaqil yechish uchun berilgan misol va masalalar

Quyidagi funksiyalarni monotonlikka tekshiring.

1. $y=3x-x^2$. **2.** $y=\dfrac{\sqrt{x}}{x+100}\ (x\geq 0)$. **3.** $y=x+\sin x$.

4. $y=x^2-\ln x^2$. **5.** $y=x^2e^{-x}$. **6.** $y=\dfrac{1-x+x^2}{1+x+x^2}$.

7. Quyidagi funksiyalarning o‘suvchi va kamayuvchi bo‘lish oraliqlarini toping:

1) $y=\dfrac{\sin x+\cos x}{1+|\cos x|}$. 2) $y=(x-2)^5(2x+1)^4$.

3) $y=\sqrt[3]{(2x-a)(a-x)^2}$. 4) $y=\dfrac{2x}{1+2x}$.

5) $y=x-e^x$. 6) $y=x-2\sin x\ (0\leq x\leq 2\pi)$.

8. Ushbu 1) $y=\dfrac{a^2-1}{3}x^3+(a-1)x^2+2x$; 2) $y=ax+3\sin x+4\cos x$ funksiyalar a ning qanday qiymatlarida o‘suvchi bo‘ladi.

Quyidagi funksiyalarni ekstremumga tekshiring.

9. $y=2+x-x^2$. **10.** $y=(x-1)^3$

11. $y=\dfrac{3}{4}x^4+x^3-9x^2+7$. **12.** $y=x^4e^{-x^2}$.

13. $y=2\sin x+\cos 2x$. **14.** $y=\dfrac{3x^2+4x+4}{x^2+x+1}$.

15. $y=\sin x+\dfrac{1}{2}\sin 2x$. **16.** $y=(x^2-2x)\ln x-\dfrac{3}{2}x^2+4x$.

Quyidagi funksiyalarning ko‘rsatilgan oraliqlarda eng katta va eng kichik qiymatlarini toping.

17. $y=2x^3-3x^{25}-12x+1,\ x\in[-2;2,5]$. **18.** $y=x+\sqrt{x},\ x\in[0;4]$.

19. $y=x^3-3x^2+1,\ x\in[-1;4]$. **20.** $y=\mathrm{arctg}\,x-\dfrac{1}{2}\ln x,\ x\in\left[\dfrac{1}{\sqrt{3}},\sqrt{3}\right]$.

21. $y=2\sin x+\sin 2x,\ x\in\left[0;\dfrac{3}{2}\pi\right]$. **22.** $y=x-2\ln 2,\ x\in[1;e]$.

23. $y=-\dfrac{1}{3}x^3-\dfrac{1}{6}x$ funksiyaning $x\in[-1;1]$ kesmadagi eng katta va eng kichik qiymatlari yig‘indisini hisoblang.

Quyidagi funksiyalar grafigining qavariqlik va botiqlik oraliqlarini toping.

24. $y=x^4+x^3-18x^2+24x-12$. **25.** $y=x+x^{5/3}$.

26. $y=x+\sin x$. **27.** $y=2-|x^5-1|$

28. $y=3x^4-4x^3+1$. **29** $y=x^\alpha,\alpha>1,x>0$.

Quyidagi funksiyalar grafigining egilish nuqtalarini toping.

30. $y=x+36x^2-2x^3-x^4$. **31.** $y=1+x^2-\dfrac{x^4}{2}$.

32. $y=3x^4-8x^3+6x^2-12$. **33.** $y=\dfrac{x+1}{x^2+1}$.

34. a parametrning qanday qiymatlarida $f(x)=ax^3+e^x$ funksiya egilish nuqtasiga ega bo‘ladi.

Quyidagi funksiyalarni to‘liq tekshiring va ularning grafigini chizing.

35. $y=\dfrac{x^4}{(1+x)^3}$. **36.** $y=\sqrt[3]{x^2}-\sqrt[3]{x^2-4}$. **37.** $y=x^2\ln(x+2)$.

38. $y=x^3e^{-4x}$. **39.** $y=\dfrac{1}{3}\sqrt{(2x+1)^3}+4\sqrt{x}$. **40.** $y=\dfrac{x^2\sqrt{x^2-1}}{2x^2-1}$.

41. $y=|x|\sqrt{1-x^2}$. **42.** $y=(x^2-2)e^{-2x}$.

Mustaqil yechish uchun berilgan misollarning javoblari

1. $\left(-\infty;\frac{1}{2}\right)$ da funksiya o‘suvchi, $\left(\frac{1}{2};\infty\right)$ da esa funksiya kamayuvchi.

2. $(0;100)$ da funksiya o‘suvchi, $(100;\infty)$ da esa, funksiya kamayuvchi. **3.** R da funksiya o‘suvchi. 4. $(-\infty;-1)\cup(0;1)$ da funksiya kamayuvchi, $(-1;0)\cup(1;\infty)$ da esa, funksiya o‘suvchi. **5.** $(-\infty;0)\cup(2;\infty)$ da funksiya kamayuvchi, $(0;2)$ da funksiya o‘suvchi. **6.** $(-\infty;-1)\cup(1;+\infty)$ da funksiya o‘suvchi, $(-1;1)$ da esa, funksiya kamayuvchi. **7.** 1) $\left(-\frac{\pi}{2}+2k\pi;\frac{\pi}{2}+2\pi k\right), k\in Z$ da funksiya o‘suvchi, $\left(\frac{\pi}{2}+2k\pi;\frac{3\pi}{2}+2k\pi\right), k\in Z$ da esa funksiya kamayuvchi. 2) $\left(-\infty;-\frac{1}{2}\right)$ da o‘suvchi; $\left(-\frac{1}{2},\frac{11}{18}\right)$ da kamayuvchi, $\left(\frac{11}{18};\infty\right)$ da o‘suvchi. 3) $\left(-\infty;-\frac{2}{3}\right)$da o‘suvchi; $\left(\frac{2}{3}a,a\right)$ da kamayuvchi, $(a;\infty)$ da o‘suvchi. 4) $(-\infty;-1)$da kamayuvchi; $(-1,1)$ da o‘suvchi; $(1;\infty)$ da kamayuvchi. 5) $(-\infty;0)$ da o‘suvchi; $(0,\infty)$ da kamayuvchi. 6) $\left(-\infty;\frac{\pi}{3}\right)$ kamayuvchi; $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ da o‘suvchi ; $\left(\frac{5\pi}{3},2\pi\right)$da kamayuvchi. **8.** 1) $a\le 3,\ a\ge 1$; 2) $a\ge 5$.**9.** $y_{max}\left(\frac{1}{2}\right)=2\frac{1}{4}$.

10. Ekstremumga ega emas. **11.** $y_{min}(-2)=-9,\quad y_{min}(3)=-40{,}5,\quad y_{max}(0)=7$.

12. $y_{max}(\pm\sqrt{2})=4e^{-2},\ y_{min}(0)=0$.

13. $y_{max}\left(\frac{\pi}{6}\right)=\frac{3}{2},\quad y_{max}\left(\frac{5\pi}{6}\right)=\frac{3}{2},\quad y_{min}\left(\frac{\pi}{2}\right)=1,\ y_{max}\left(\frac{\pi}{2}\right)=1,\ y_{min}\left(\frac{3\pi}{2}\right)=-3$.

14. $y_{max}(0)=4,\ y_{max}(-2)=\frac{2}{3}$. **15.** $y_{min}\left(2\pi k-\frac{\pi}{3}\right)=-\frac{3\sqrt{3}}{4}, k\in Z$, $y_{max}\left(2\pi k+\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{4},\ k\in Z$. **16.** $y_{max}(1)=2\frac{1}{2},\quad y_{min}(e)=\frac{e(4-e)}{2}$.

17. $y_{eng\ katta}(-1)=8,\ y_{eng\ kichik}(2)=19$.

18. $y_{eng\ katta}(4)=6,\ y_{eng\ kichik}(0)=0$. **19.** $y_{eng\ katta}(4)=17,\ y_{eng\ kichik}(2)=y_{eng\ kichik}(-1)=-3$.

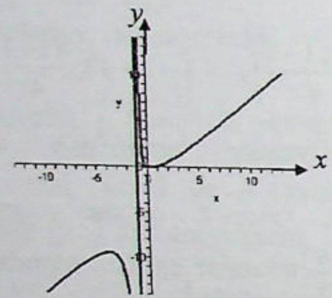
20. $y_{eng\ katta}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}+0{,}25\cdot\ln 3,\quad y_{eng\ kichik}(\sqrt{3})=\frac{\pi}{6}-0{,}25\cdot\ln 3$.

21. $y_{eng\ katta}\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2},\ y_{eng\ kichik}\left(\frac{3\pi}{2}\right)=-2$. **22.** $y_{eng\ katta}(1)=1,\ y_{eng\ kichik}(2)=2(1-\ln 2)$.

16.37.Eng kattasi yo‘q, $y_{eng\ kichik}(0)=1$.**23.** 0. **24.** $(-\infty;-2)\cup\left(\frac{3}{2};\infty\right)$ da qavariq; $\left(-2;\frac{3}{2}\right)$da botiq. **25.** $(-\infty;0)$da botiq; $(0;\infty)$ da qavariq. **26.** $(2\pi k,(2k+1)\pi), k\in Z$ da botiq; $((2k+1)\pi,(2k+2)\pi), k\in Z$ da qavariq. **27.** $(-\infty;0)\cup(1;\infty)$ da botiq; $(0;1)$ da qavariq. **28.** $(-\infty;0)\cup\left(\frac{2}{3};+\infty\right)$ da qavariq; $\left(0;\frac{2}{3}\right)$ da botiq. **29.** Qavariq. **30.** $(-3;294)$. **31.** $\left(\frac{1}{\sqrt{3}};\frac{23}{18}\right),\left(-\frac{1}{\sqrt{3}};\frac{23}{18}\right)$. **32.** $\left(\frac{1}{3};12\frac{11}{27}\right),(1;13)$. **33.** $\left(-2-\sqrt{3};\frac{-\sqrt{3}-1}{4}\right)$, $\left(-2+\sqrt{3};\frac{\sqrt{3}+1}{4}\right),(1;1)$. **34.** $a\in\left(-\infty;-\frac{e}{6}\right),\ a\in(0;+\infty)$. **35.** Funksiyaning aniqlanish sohasi: $(-\infty;-1)\cup(-1;+\infty)$. $x=-1$- vertikal asimptota, $y=x-3$ og‘ma asimptota. $y_{min}(0)=0,\ y_{max}(-4)=-\frac{256}{27}$. $\left(-6;-\frac{3296}{125}\right)$ va $\left(2;\frac{16}{27}\right)$nuqtalar egilish nuqtalari (1-chizma). **36.** R da aniqlangan, juft funksiya. Grafik Oy o‘qiga nisbatan simmetrik, $y=0$ -gorizontal asimptota. $y_{min}(0)=\sqrt[3]{4},\ y_{max}(\pm\sqrt{2})=2\sqrt[3]{2}$. $(2;\sqrt[3]{4}),(-2;\sqrt[3]{4})$-egilish nuqtalari (2-chizma). **37.** Funksiya $(-2;+\infty)$ oraliqda aniqlangan. $x=-2$ vertikal asimptota. $y_{min}(0)=0,\ y_{max}(-0{,}73)\approx 0{,}12$. $(-0{,}37;0{,}075)$– egilish nuqtasi (3-chizma). **38.** Funksiya R da aniqlangan, $x\to+\infty$ da $y=0$- gorizontal asimptota. $y_{max}\left(\frac{3}{4}\right)=\left(\frac{3}{4e}\right)^3$. Egilish nuqtalari: $(0;0)$, $\left(\frac{3-\sqrt{3}}{4};\left(\frac{3-\sqrt{3}}{4}\right)^3e^{\sqrt{3}-3}\right)$, $\left(\frac{3+\sqrt{3}}{4};\left(\frac{3+\sqrt{3}}{4}\right)^3e^{-\sqrt{3}-3}\right)$ (4-chizma). **39.** Funksiya $x\ge 0$da aniqlangan, ordinata o‘qi bilan esa $\left(0;\frac{1}{3}\right)$ nuqtada kesishadi; funksiya qa’tiy o‘suvchi; $\left(\frac{\sqrt{5}+1}{2};\approx 8\right)$-egilish nuqtasi (5-chizma). **40.** Funksiya $|x|\ge 1$da aniqlangan; ordinata o‘qiga nisbatan simmetrik; ordinata o‘qi bilan kesishish nuqtalari: $(1;0),(-1;0)$; $x\to+\infty$ da $y=\frac{x}{2}$ va $x\to-\infty$da $y=-\frac{x}{2}$ asimptotalari;

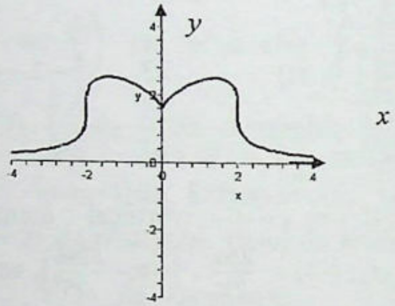
$(-\infty;-1)$ da kamayuvchi $(1;+\infty)$ da o'suvchi (6-chizma). **41.** Funksiya $|x|\le 1$ da aniqlangan; ordinata o'qiga nisbatan simmetrik; o'qlar bilan kesishish nuqtalari: $(-1;0);(0;0),(1;0)$; $y_{min}(0)=0$, $y_{max}\left(\pm\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$ (7-chizma). **42.** Funksiya R da aniqlangan, koordinata o'qlari bilan kesishish nuqtalari: $(-\sqrt{2};0),(\sqrt{2};0),(0;-2)$; $x\to+\infty$ da $y=0$ asimptota; $y_{min}(-1)\approx-7{,}4$, $y_{max}(2)\approx 0{,}04$; funksiyaning egilish nuqtalarining abssissalari: $x=1-\sqrt{10}/2\approx-0{,}6$, $x=1+\sqrt{10}/2\approx 2{,}6$. (8-chizma).

35.



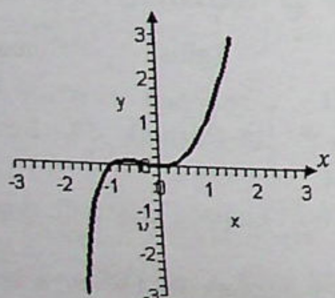
1-chizma.

36.



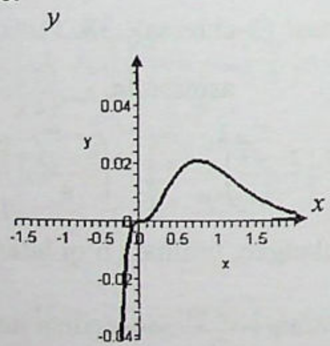
2-chizma.

37.



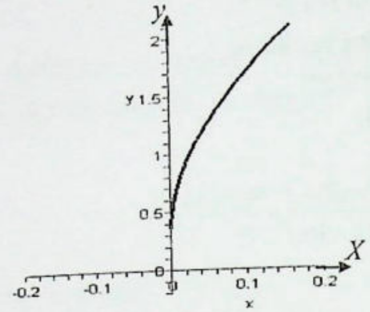
3-chizma .

38.



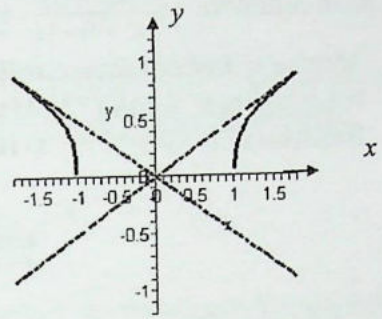
4-chizma.

39.



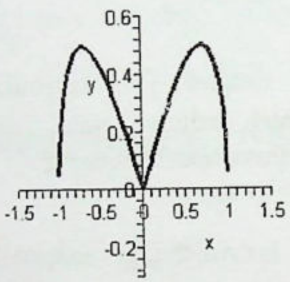
5-chizma.

40.



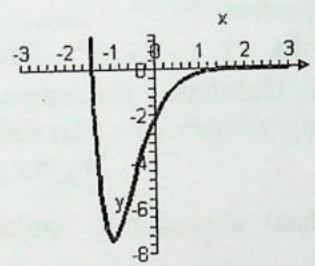
6-chizma.

41.



7-chizma.

42.



8-chizma.

7.4-amaliy mashg'ulot.
Lopital qoidalari. Teylor formulasi

1-misol. $\lim\limits_{x\to 2}\frac{\ln(x^2-3)}{x^2+3x-10}$ limitni hisoblang.

Yechilishi. Ravshanki, $f(x)=\ln(x^2-3)$, $g(x)=x^2+3x-10$ funksiyalar $x\to 2$ da $f(x)=\ln(x^2-3)\to 0$, $g(x)=x^2+3x-10\to 0$. $x=2$ nuqtaning $x=\pm\sqrt{3}$ nuqtalarni o'z ichida saqlamaydigan ixtiyoriy kichik atrofida $f'(x)=\frac{2x}{x^2-3}$, $g'(x)=2x+3$ hosilalar ega va $g'(x)=2x+3\ne 0$ $\left(x>-\frac{3}{2}\right)$, $\lim\limits_{x\to 2}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to 2}\frac{\frac{2x}{x^2-3}}{2x+3}=\lim\limits_{x\to 2}\frac{2x}{(x^2-3)(2x+3)}=\frac{4}{7}$ mavjud.

Demak, berilgan limitni hisoblashga Lopitalning birinchi qoidasini qo'llash mumkin: $\lim\limits_{x\to 2}\frac{\ln(x^2-3)}{x^2+3x-10}=\lim\limits_{x\to 2}\frac{2x}{(x^2-3)(2x+3)}=\frac{4}{7}$.

Marlye tizimidan foydalanib misolni yechish:

> Limit(ln(x^2-3)/(x^2+3*x-10),x=2)= limit(ln(x^2-3)/(x^2+3*x-10),x=2);

$$\lim_{x\to 2}\left(\frac{\ln(x^2-3)}{x^2+3x-10}\right)=\frac{4}{7}.$$

2-misol. Ushbu

$$1)\ \lim_{x\to 0+0}(1+x)^{\ln x};\quad 2)\ \lim_{x\to 0}(\mathrm{ctgx})^{\sin x};\quad 3)\ \lim_{x\to +0}[\arcsin(\ln x)]^{tgx}$$

limitlarni hisoblang

Yechilishi. 1) $y=(1+x)^{\ln x}$ funksiya $x=0$ nuqtada (1^{∞}) ko'rinishdagi aniqmaslikni ifodalaydi. Bu aniqmaslikni ochish uchun avvalo, berilgan funksiyani logarifmlab, $(0\cdot\infty)$ ko'rinishdagi aniqmaslikka keltiramiz:

$$\lim_{x\to 0+0}\ln y=\lim_{x\to 0+0}\ln x\ln(1+x).$$

Buni shakl almashtirish natijasida $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltiramiz va Lopitalning birinchi qoidasini qo'llaymiz:

$$\lim_{x\to 0+0}\ln y=\lim_{x\to 0+0}\frac{\ln(1+x)}{\frac{1}{\ln x}}\overset{\left(\frac{0}{0}\right)}{=}-\lim_{x\to 0+0}\frac{\frac{1}{1+x}}{\frac{1}{x}\cdot\frac{1}{\ln^2 x}}=-\lim_{x\to 0+0}\frac{x\cdot\ln^2 x}{1+x}\overset{(0\cdot\infty)}{=}=\lim_{x\to 0+0}\frac{1}{1+x}\cdot x\ln^2 x=$$

$$=\lim_{x\to 0+0}\frac{\ln^2 x}{\frac{1}{x}}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to 0+0}\frac{2\cdot\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}}=-2\lim_{x\to 0+0}x\ln x\overset{(0\cdot\infty)}{=}-2\lim_{x\to 0+0}\frac{\ln x}{\frac{1}{x}}\overset{\left(\frac{\infty}{\infty}\right)}{=}2\lim_{x\to 0+0}x^2=0.$$

Demak, $\lim\limits_{x\to 0+0}y=e^0=1$.

Marlye tizimidan foydalanib misolni yechish:

> Limit((1+x)^(ln(x)),x=0,right)= limit((1+x)^(ln(x)),x=0,right);

$$\lim_{x\to 0+}(1+x)^{\ln(x)}=1.$$

2) Berilgan $y=(\mathrm{ctg}\,x)^{\sin x}$ funksiya $x=0$ nuqtada (∞^0) ko'rinishdagi aniqmaslikni ifodalaydi. $y=(\mathrm{ctg}\,x)^{\sin x}=e^{\sin x\cdot\ln(\mathrm{ctg}\,x)}$ ayniyatdan foydalanib,

$$\lim_{x\to 0}(\mathrm{ctg}\,x)^{\sin x}=\lim_{x\to 0}e^{\sin x\ln\mathrm{ctg}\,x}=e^{\lim\limits_{x\to 0}\sin x\ln\mathrm{ctg}\,x}.\ (1)$$

bo'lishini topamiz. So'ngra, (1) ning o'ng tomonidagi ko'rsatkichli funksiyaning limitini hisoblaymiz:

$$\lim_{x\to 0}\sin x\ln\mathrm{ctg}\,x=\lim_{x\to 0}\frac{\ln\mathrm{ctg}\,x}{\frac{1}{\sin x}}=\lim_{x\to 0}\frac{\frac{1}{\mathrm{ctg}\,x}\cdot\left(-\frac{1}{\sin^2 x}\right)}{-\frac{1}{\sin^2 x}\cdot\cos x}=\lim_{x\to 0}\frac{1}{\mathrm{ctg}\,x\cdot\cos x}=\lim_{x\to 0}\frac{\sin x}{\cos^2 x}=0.$$

Buni hisobga olsak, (1) dan

$$\lim_{x\to 0}(\mathrm{ctg}\,x)^{\sin x}=e^0=1$$

bo'ladi.

Marlye tizimidan foydalanib misolni yechish:

> Limit((ctan(x))^(sin(x)),x=0)= limit((ctan(x))^(sin(x)),x=0);

$$\lim_{x\to 0}(c\tan(x))^{\sin(x)}=1.$$

3) Bu holda berilgan funksiya $x=0$ nuqtada (0^0) ko'rinishdagi aniqmaslikni ifodalaydi. Endi

$$[\arcsin x]^{tg\,x}=e^{tg\,x\cdot\ln(\arcsin x)}$$

ayniyatga asosan, $\lim\limits_{x\to 0+0}(\arcsin x)^{tgx}=e^{\lim\limits_{x\to 0+0}tgx\cdot\ln(\arcsin x)}$ bo'ladi. Oxirgi tenglikning o'ng tomonidagi ko'rsatkichli funksiyaning limitini hisoblaymiz:

$$\lim_{x\to 0+0} \operatorname{tg} x\cdot \ln(\arcsin x) \overset{(0\cdot\infty)}{=} = \lim_{x\to 0+0} \frac{\ln(\arcsin x)}{\frac{1}{\operatorname{tg} x}} \overset{\left(\frac{\infty}{\infty}\right)}{=} = \lim_{x\to 0+0} \frac{\frac{1}{\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\operatorname{tg}^2 x}\cdot\frac{1}{\cos^2 x}} =$$

$$= \lim_{x\to 0+0} \frac{1}{\sqrt{1-x^2}}\cdot \lim_{x\to 0+0} \frac{\sin^2 x}{\arcsin x} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to 0+0} \frac{\sin 2x}{\frac{1}{\sqrt{1-x^2}}} = 0.$$

Shunday qilib, $\lim\limits_{x\to 0+0} (\arcsin x)^{\sin x} = e^0 = 1$.

Marlye tizimidan foydalanib misolni yechish:

> Limit((arcsin((x)))^tan(x),x=0,right)= limit((arcsin((x)))^tan(x),x=0,right);

$\lim\limits_{x\to 0+} [\arcsin(x)]^{\tan(x)} = 1$.

Mustaqil yechish uchun berilgan misol va masalalarning

Lopital qoidalaridan foydalanib, quyidagi funksiyalarning limitini hisoblang

1. $\lim\limits_{x\to 1}\frac{3x^2+5x-8}{4x^2+3x-7}$. **2.** $\lim\limits_{x\to 4}\frac{\ln(x^2-15)}{3x^2-10x-8}$. **3.** $\lim\limits_{x\to 0}\frac{e^{ax}-e^{-2ax}}{\ln(1+x)}$.

4. $\lim\limits_{x\to -1}\frac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}$. **5.** $\lim\limits_{x\to 0}\frac{\ln\sin 2x}{\ln\sin x}$. **6.** $\lim\limits_{x\to 0}\frac{\ln x}{\ln\sin x}$.

7. $\lim\limits_{x\to 0}\frac{e^{1/x^2}-1}{\operatorname{arctg} x^2-\pi}$. **8.** $\lim\limits_{x\to 0}\frac{\ln\cos 2x}{\ln\cos 5x}$. **9.** $\lim\limits_{x\to 1}\frac{a^{\ln x}-x}{\ln x}$.

10. $\lim\limits_{x\to 1}\frac{\operatorname{tg} x-x}{x-\sin x}$. **11.** $\lim\limits_{x\to 0}\frac{x\cdot\operatorname{ctg} x-1}{x^2}$. **12.** $\lim\limits_{x\to 0}\frac{\sin ax-\sin bx}{\operatorname{sh} ax-\operatorname{sh} bx}$.

13. $\lim\limits_{x\to 0}\frac{3\operatorname{tg} 4x-12\operatorname{tg} x}{3\sin 4x-12\sin x}$. **14.** $\lim\limits_{x\to 1}\frac{x^\alpha-1}{x^\beta-1}$, $\beta\neq 0$. **15.** $\lim\limits_{x\to -\infty}\frac{x^3}{e^x}$.

16. Quyidagi limitlarni Lopital qoidasi bo'yicha hisoblash mumkin emasligini ko'rsating va ularning limitini hisoblang:

1). $\lim\limits_{x\to 0}\frac{x^3\sin(1/x)}{\sin^2 x}$; 2). $\lim\limits_{x\to\infty}\frac{x+\cos x}{x-\cos x}$.

17. $\lim\limits_{x\to\infty}\frac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ limitni hisoblashga Lopital qoidasini qo'llash mumkinmi, agar limit mavjud bo'lsa, uni hisoblang.

Quyidagi ko'phadlarni $x-x_0$ ning manfiy bo'lmagan darajalari bo'yicha Teylor formulasiga yoying:

18. $P(x)=1+3x+5x^2-2x^3$, $x_0=-1$.

19. $P(x)=x^4-4x^3+7x^2-5x+3$, $x_0=2$.

20. $P(x)=x^9+3x^4+2x-2$, $x_0=1$.

Quyidagi funksiyalarni x ning manfiy bo'lmagan darajalari bo'yicha ko'rsatilgan tartibgacha Makloren formulasiga yoying:

21. $f(x)=\frac{(1+x)^{100}}{(1-2x)^{40}\cdot(1+2x)^{60}}$, $o(x^2)$ hadgacha.

22. $f(x)=e^{\sqrt{1+2x}}$, $o(x^2)$ hadgacha .

23. $f(x)=xe^x$, $o(x^3)$ hadgacha.

24. $f(x)=\frac{e^x+e^{-x}}{2}$, $o(x^4)$ hadgacha.

Teylor formulasidan foydalanib, quyidagi limitlarni toping.

25. $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+3x}-\sqrt{1+2x}}{x^2}$. **26.** $\lim\limits_{x\to 0}\frac{e^x-(1+x)}{x}$.

27. $\lim\limits_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}$. **28.** $\lim\limits_{x\to\infty} x^2\left(e^{-\frac{1}{x^2}}-1\right)$.

Mustaqil yechish uchun berilgan misollarning javoblari

1. 1. **2.** $\frac{4}{7}$. **3.** $3a$. **4.** $\frac{4}{9}$. **5.** 1. **6.** 1. **7.** 0. **8.** $\frac{4}{25}$. **9.** $\ln a-1$. **10.** 2. **11.** $-\frac{1}{3}$. **12.** 1. **13.** -2. **14.** $\frac{\alpha}{\beta}$. **15.** 0. **17.** Lopital qoidasini qo'llash mumkin emas, limit mavjud emas. **18.** $5-13(x+1)+11(x+1)^2-2(x+1)^3$.

19. $5+7(x-2)+7(x-2)^2+4(x-2)^3+(x-2)^4$.

20. $2-3(x-1)+(x-1)^2+15(x-1)^3+25(x-1)^4+\frac{7}{3}(x-1)^5+7(x-1)^6+(x-1)^7$.

21. $1+60x+195x^2+o(x^2)$. **22.** $e+ex+o(x^2)$. **23.** $x+\frac{x^2}{1!}+\frac{x^3}{2!}+o(x^3)$.

24. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^4)$. **25.** $-\frac{1}{2}$ **26.** $\frac{1}{2}$ **27.** $\frac{1}{24}$ **28.** -1

7-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

7.1-masala. Quyida berilgan funksiyaning hosilasini hisoblang.

7.1.1. $y=\frac{2(3x^3+4x^2-x-2)}{15\sqrt{1+x}}$.

7.1.2. $y=\frac{(1+x^8)\sqrt{1+x^2}}{12\cdot x^{12}}$.

7.1.3. $y=\frac{(x^2-6)\sqrt{(4+x^2)^3}}{120x^5}$.

7.1.4. $y=\frac{\sqrt{2x+3}(x-2)}{x^2}$.

7.1.5. $y=\frac{1}{3}(3x-2x^3)$, $x_0=1$.

7.1.6. $y=(x\sqrt{x+1})/(x^2+x+1)$

7.1.7. $y=\frac{(x+3)\sqrt{2x-1}}{2x+7}$.

7.1.8. $y=\frac{(2x^2-1)\sqrt{1+x^2}}{3x^3}$.

7.1.9. $y=\frac{(x^2-8)\cdot\sqrt{x^2-8}}{6x^3}$.

7.1.10. $y=(x^2-2)\sqrt{4+x^2}$.

7.1.11. $y=\frac{\sqrt{x-1}\cdot(3x+2)}{4x^2}$.

7.1.12. $y=(1-x^2)\sqrt[5]{x^3+\frac{1}{x}}$.

7.1.13. $y=\frac{x-1}{(x^2+5)\sqrt{(x^2+5)}}$.

7.1.14. $y=3\frac{\sqrt[3]{x^2+5}}{x+1}$.

7.1.15. $y=\frac{x+7}{6\sqrt{x^2+2x+7}}$.

7.1.16. $y=\frac{2+x^2}{2\sqrt{1-x^4}}$.

7.1.17. $y=\frac{3x+\sqrt{x}}{\sqrt{x^2+2}}$.

7.1.18. $y=\frac{1}{(x+2)\sqrt{x^2+4x+5}}$.

7.1.19. $y=\frac{\sqrt{(1+x^2)^3}}{3x^3}$.

7.1.20. $y=\frac{2x^2-x-1}{3\sqrt{2+4x}}$.

7.1.21. $y=\frac{x^2}{(x^2+6)\sqrt{(x^2+4)}}$.

7.1.22. $y=\frac{\sqrt[5]{x^2+4}}{x+2}$.

7.1.23. $y=\frac{x+9}{\sqrt{x^2+2x}}$.

7.1.24. $y=\frac{3+x^3}{\sqrt{1-x^4}}$.

7.1.25. $y=\frac{3x^2+\sqrt{x}}{\sqrt{x^3+4}}$.

7.1.26-misol. $y=\frac{x^2+\sqrt{x}}{x-2\sqrt[3]{x}}$ funksiyaning hosilasini hisoblang.

Yechilishi. 7.4 bandining hosilani hisoblashning 1-6 sodda qoidalari va hosilalar jadvaldan foydalanib, topamiz:

$$y'=\frac{(x^2+\sqrt{x})'\cdot(x-2\sqrt[3]{x})-(x-2\sqrt[3]{x})'\cdot(x^2+\sqrt{x})}{(x-2\sqrt[3]{x})^2}=$$

$$=\frac{\left(2x+\frac{1}{2\sqrt{x}}\right)(x-2\sqrt[3]{x})-\left(1-\frac{2}{3}\cdot x^{-\frac{2}{3}}\right)(x^2+\sqrt{x})}{(x-2\sqrt[3]{x})^2}=\frac{x^2-\frac{10}{3}x\sqrt[3]{x}-\frac{1}{2}\sqrt{x}-\frac{1}{3\sqrt[6]{x}}}{(x-2\sqrt[3]{x})^2}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

```
> Diff((x^2+sqrt(x))/(x-2*x^(1/3)),x)=diff((x^2+sqrt(x))/(x-
2*x^(1/3)),x);
```

$$\frac{d}{dx}\left(\frac{x^2+\sqrt{x}}{x-2x^{(1/3)}}\right)=\frac{2x+\frac{1}{2\sqrt{x}}}{x-2x^{(1/3)}}-\frac{(x^2+\sqrt{x})\left(1-\frac{2}{3x^{(2/3)}}\right)}{(x-2x^{(1/3)})^2}.$$

7.2-masala. Quyidagi funksiyaning hosilasini toping.

7.2.1. $y=\frac{1}{\sqrt{2}}\cdot\text{arctg}\frac{3x-1}{\sqrt{2}}+\frac{1}{3}\cdot\frac{3x-1}{3x^2-2x+1}$.

7.2.2. $y=\ln\left(4x-1+\sqrt{16x^2-8x+2}+\sqrt{16x^2-8x+2}\cdot\text{arctg}(4x-1)\right)$

7.2.3. $y=\frac{x+2}{x^2+4x+6}+\frac{1}{\sqrt{2}}\cdot\text{arctg}\frac{x+2}{\sqrt{2}}$.

7.2.4. $y=5x-\ln\left(1+\sqrt{1+e^{10x}}\right)-e^{-5x}\arcsin(x^2+1)$

7.2.5. $y=\sqrt{x^2-8x+17}\cdot\text{arctg}(x-4)-\ln\left(x-4+\sqrt{x^2-8x+17}\right)$.

7.2.6. $y=\ln\left(e^{5x}+\sqrt{e^{10x}-1}\right)+\arcsin x(e^{-3x})$

7.2.7. $y=\ln\left(2x-3+\sqrt{4x^2-12x10}\right)-\sqrt{4x^2-12x+10}\cdot\text{arctg}(2x-3)$.

7.2.8. $y = \arcsin e^{-4x} + \ln\left(e^{4x} + \sqrt{e^{2x} - 1}\right)$

7.2.9. $y = \ln\left(5x + \sqrt{25x^2 + 1}\right) - \sqrt{25x^2 + 1} \cdot \operatorname{arctg} 5x.$

7.2.10. $y = (3x + 1)^4 \arcsin\dfrac{1}{3x + 1} + \left(3x^2 + 2x + 1\right)\sqrt{9x^2 + 6x}, \quad 3x + 1 > 0.$

7.2.11. $y = \dfrac{1}{\sqrt{2}} \cdot \operatorname{arctg}\dfrac{2x + 1}{\sqrt{2}} + \dfrac{2x + 1}{4x^2 + 4x + 3}.$

7.2.12. $y = \ln\left(e^{3x} + \sqrt{e^{6x} - 1}\right) + \arcsin e^{-3x}.$

7.2.13. $y = \arcsin e^{-2x} + \ln\left(e^{2x} + \sqrt{e^{2x} + 2}\right).$

7.2.14. $y = \dfrac{x \arcsin x}{\sqrt{1 - x^2}} + \ln\sqrt{1 - x^2}.$

7.2.15. $y = x^3 \arcsin x + \dfrac{x^2 + 2}{3}\sqrt{1 - x^2}.$

7.2.16. $y = 3\arcsin\dfrac{3}{4x + 1} + 2\sqrt{4x^2 + 2x - 2}, \ 4x + 1 > 0.$

7.2.17. $y = \sqrt{1 + x^2} \cdot \operatorname{arctg} x - \ln\left(x + \sqrt{1 + x^2}\right)$

7.2.18. $y = 2\arcsin\dfrac{2}{3x + 4} + \sqrt{9x^2 + 24x + 12}, \ 3x + 4 > 0.$

7.2.19. $y = \sqrt{1 - 3x - 2x^2} + \dfrac{3}{2\sqrt{2}} \cdot \arcsin\dfrac{4x + 3}{\sqrt{17}}$

7.2.20. $y = \ln\dfrac{\sqrt{x^2 - x + 1}}{x} + \sqrt{3} \cdot \operatorname{arctg} x.$

7.2.21. $y = \dfrac{1}{\sqrt{2}} \cdot \operatorname{arcctg}\dfrac{3x - 1}{\sqrt{2}} + \dfrac{1}{3} \cdot \dfrac{3x^2 - 1}{3x^4 - 2x^2 + 1}.$

7.2.22. $y = \lg\left(8x - 1 + \sqrt{16x^2 - 8x + 2} + \sqrt{16x^2 - 8x + 2} \cdot \operatorname{arcctg}(4x - 1)\right)$

7.2.23. $y = \dfrac{x + 3}{x^2 + 4x} + \dfrac{1}{\sqrt{2}} \cdot \operatorname{arcctg}\dfrac{x + 2}{\sqrt{2}}.$

7.2.24. $y = 5x - \lg\left(1 + \sqrt{1 + e^{10x}}\right) - e^{-6x} \arccos(x + 1).$

7.2.25. $y = \sqrt{x^2 - 8x + 17} \cdot \operatorname{arcctg}(x - 4) - \lg\left(x - 4 + \sqrt{x^2 - 8x + 17}\right).$

7.2.26-misol. $y = \dfrac{1}{12}\ln\dfrac{x^4 - x^2 + 1}{\left(x^2 + 1\right)^2} - \dfrac{1}{2\sqrt{3}}\operatorname{arctg}\dfrac{\sqrt{3}}{2x^2 - 1}$

funksiyaning hosilasini hisoblang.

Yechilishi ([2], 6-bo‘lim; [3], 1-q., 128-132 betlar; [9], 1-t., 6-bo‘lim; [30], 3-bo‘lim). Berilgan funksiyaning hosilasini topishda, avvalo, ifodaning shaklini o‘zgartirib, so‘ngra murakkab, logarifmik, arktangens va kasr funksiyalarning hosilalarini topish formulalaridan foydalanamiz:

$$y' = \left(\frac{1}{12}\ln\left(x^4 - x^2 + 1\right) - \frac{1}{6}\ln\left(x^2 + 1\right) - \frac{1}{2\sqrt{3}} arctg\frac{\sqrt{3}}{2x^2 - 1}\right)' = \frac{2x^3 - x}{6\left(x^4 - x^2 + 1\right)} -$$

$$- \frac{x}{3\left(x^2 + 1\right)} - \frac{1}{2\sqrt{3}} \frac{1}{1 + \left(\dfrac{\sqrt{3}}{2x^2 - 1}\right)^2}\left(\frac{\sqrt{3}}{2x^2 - 1}\right)' = \frac{2x^3 - x}{6\left(x^4 - x^2 + 1\right)} - \frac{x}{3\left(x^2 + 1\right)} +$$

$$+ \frac{x}{2\left(x^4 - x^2 + 1\right)} = \frac{x^3}{\left(x^4 - x^2 + 1\right)\left(x^2 + 1\right)}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Diff((1/12)*ln(x^4-x^2+1)-(1/6)*ln(x^2+1)-(1/(2*sqrt(3)))*arctan(sqrt(3)/(2*x^2-1)),x)=diff((1/12)*ln(x^4-x^2+1)-(1/6)*ln(x^2+1)-(1/(2*sqrt(3)))*arctan(sqrt(3)/(2*x^2-1)),x);

$$\frac{d}{dx}\left(\frac{1}{12}\ln(x^4 - x^2 + 1) - \frac{1}{6}\ln(x^2 + 1) - \frac{1}{6}\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2x^2 - 1}\right)\right) =$$

$$\frac{4x^3 - 2x}{12(x^4 - x^2 + 1)} - \frac{x}{3(x^2 + 1)} + \frac{2x}{(2x^2 - 1)^2\left(1 + \dfrac{3}{(2x^2 - 1)^2}\right)}.$$

7.3-masala. Quyidagi, tenglamasi parametrik ko‘rinishda berilgan, funksiyaning hosilasini hisoblang.

7.3.1. $\begin{cases} x = \cos 2t, \\ y = 2\sin^2 t. \end{cases}$

7.3.2. $\begin{cases} x = e^t \cos t, \\ y = e^t \sin t. \end{cases}$

7.3.3. $\begin{cases} x = \sqrt{1-t^2}, \\ y = \dfrac{1}{t}. \end{cases}$

7.3.4. $\begin{cases} x = \mathrm{sh}^2 t, \\ y = \dfrac{1}{\mathrm{ch}^2 t}. \end{cases}$

7.3.5. $\begin{cases} x = t + \sin t, \\ y = 2 - \cos t. \end{cases}$

7.3.6. $\begin{cases} x = \dfrac{1}{t}, \\ y = \dfrac{1}{1+t^2}. \end{cases}$

7.3.7. $\begin{cases} x = \sqrt{t}, \\ y = \dfrac{1}{\sqrt{1-t}}. \end{cases}$

7.3.8. $\begin{cases} x = \dfrac{1}{t}, \\ y = \dfrac{1}{1+t^2}. \end{cases}$

7.3.9. $\begin{cases} x = \sqrt{t}, \\ y = \sqrt[3]{t-1}. \end{cases}$

7.3.10. $\begin{cases} x = \sqrt{t^3 - 1}, \\ y = \ln t. \end{cases}$

7.3.11. $\begin{cases} x = \sqrt{t-1}, \\ y = \dfrac{t}{\sqrt{t-1}}. \end{cases}$

7.3.12. $\begin{cases} x = \sqrt{t^3 - 1}, \\ y = \ln t. \end{cases}$

7.3.13. $\begin{cases} x = \sqrt{t-1}, \\ y = \dfrac{1}{\sqrt{t}}. \end{cases}$

7.3.14. $\begin{cases} x = \sqrt{t-3}, \\ y = \ln(t-2). \end{cases}$

7.3.15. $\begin{cases} x = t + \sin t, \\ y = 2 + \cos t. \end{cases}$

7.3.16. $\begin{cases} x = \mathrm{sh}\, t, \\ y = \mathrm{th}^2 t. \end{cases}$

7.3.17. $\begin{cases} x = \cos^2 t, \\ y = \mathrm{tg}^2 t. \end{cases}$

7.3.18. $\begin{cases} x = \sin t, \\ y = \ln \cos t. \end{cases}$

7.3.19. $\begin{cases} x = \mathrm{tg}\, t, \\ y = \dfrac{1}{\sin 2t}. \end{cases}$

7.3.20. $\begin{cases} x = \sqrt{t^3 - 1} \\ y = \ln t. \end{cases}$

7.3.21. $\begin{cases} x = \sin 2t, \\ y = 2\sin^2 t. \end{cases}$

7.3.22. $\begin{cases} x = e^{3t} \cos 2t, \\ y = e^{3t} \sin 2t. \end{cases}$

7.3.23. $\begin{cases} x = \sqrt{1-t^4}, \\ y = \dfrac{1}{t^2}. \end{cases}$

7.3.24. $\begin{cases} x = \mathrm{sh}^2 2t, \\ y = \dfrac{1}{\mathrm{ch}^2 2t}. \end{cases}$

7.3.25. $\begin{cases} x = t^2 + \sin 2t, \\ y = 4 - \cos 2t. \end{cases}$

7.3.26-misol. Ushbu, tenglamasi,

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

parametric ko'rinishda berilgan funksiyaning hosilasini hisoblang.

Yechilishi ([30], 3-bo'lim). (7.4.4) formuladan foydalanib, funksiyaning hosilasini topamiz:

$$x_t' = a(1-\cos t), \quad y_t' = a\sin t, \quad y'_x = \frac{y_t'}{x_t'} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> s:=diff(a*(1-cos(t)),t)/diff(a*(t-sin(t)),t);

$$s := \frac{\sin(t)}{1-\cos(t)}.$$

7.4-masala. Quyida berilgan $y = f(x)$ funksiyaning $y^{(n)}$ hosilasini toping.

7.4.1. $y = \sin 2x + \cos(x+1)$.

7.4.2. $y = \sqrt[5]{e^{7x+1}}$.

7.4.3. $y = \dfrac{4x+7}{2x+3}$.

7.4.4. $y = \lg(5x+2)$.

7.4.5. $y = 2^{3x}$.

7.4.6. $y = \dfrac{x}{2(3x+2)}$.

7.4.7. $y = \dfrac{2x+5}{13(3x+1)}$.

7.4.8. $y = 4^{3x+5}$.

7.4.9. $y = \sin(x+1) + \cos 2x$.

7.4.10. $y = \sqrt[3]{e^{2x+1}}$.

7.4.11. $y = \dfrac{4x+15}{5x+1}$.

7.4.12. $y = \lg(3x+1)$.

7.4.13. $y = 7^{5x}$.

7.4.14. $y = \dfrac{x}{9(4x+9)}$.

7.4.15. $y = 5^{2x+3}$.

7.4.16. $y = \dfrac{1+x^2}{1-x^2}$.

7.4.17. $y = x\cos^2 x$.

7.4.18. $y = \frac{x^2}{\sqrt{1-2x}}$.

7.4.19. $y = e^{ax}\cos(bx+c)$.

7.4.20. $y = \frac{11+12x}{6x+5}$.

7.4.21. $y = \sin 3x + \sin(x+21)$.

7.4.22. $y = \sqrt[5]{e^{7x+1}}$.

7.4.23. $y = \frac{2x+7}{3x+5}$.

7.4.24. $y = \ln(4x+3)$.

7.4.25. $y = 7^{5x}$.

7.4.26-misol. $y = \sqrt{e^{3x+1}}$ funksiyaning n-tartibli $y^{(n)}$ hosilasini hisoblang.

Yechilishi. $y^{(n)} = (e^{bx})^{(n)} = b^n e^{bx}$ formulaga asosan, berilgan funksiyaning hosilasini topamiz:

$$y^{(n)} = \left(\sqrt{e^{3x+1}}\right)^{(n)} = \left(e^{\frac{3}{2}x+\frac{1}{2}}\right)^{(n)} = \left(\frac{3}{2}\right)^n \sqrt{e^{3x-1}}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Diff(sqrt(exp(3*x+1)),x$n)=diff(sqrt(exp(3*x+1)),x$n);

$$\frac{d^n}{dx^n}\left(\sqrt{e^{(3x+1)}}\right) = \frac{d^n}{dx^n}\left(\sqrt{e^{(3x+1)}}\right)$$

7.5-masala. Quyidagi funksiyani to'la tekshiring va grafigini chizing.

7.5.1. $y = 1 - \sqrt[3]{x^2 - 2x}$.

7.5.2. $y = 2x - 3\sqrt[3]{x^2}$.

7.5.3. $y = 12\sqrt[3]{6(x-2)^2}/(x^2+8)$.

7.5.4. $y = x(x-1)^3$

7.5.5. $y = (x+2)^2(x-1)^2$.

7.5.6. $y = \frac{20x^2}{(x-1)^2}$.

7.5.7. $y = \sqrt[3]{x(x+2)}$.

7.5.8. $y = \frac{(x-1)^2}{(x+1)^3}$.

7.5.9. $y = x^2\sqrt{x+1}$.

7.5.10. $y = \frac{x}{(1-x^2)^2}$.

7.5.11. $y = -x^3 + 4x - 3$.

7.5.12. $y = x = (x+1)^{3/2}$.

7.5.13. $y = \frac{8x}{\sqrt{x^2-4}}$.

7.5.14. $y = \frac{x^5-8}{x^4}$.

7.5.15. $y = 3\sqrt[3]{(x-3)^2} - 2x + 6$.

7.5.16. $y = 4x + 8 - 6\sqrt[3]{(x+2)^2}$.

7.5.17. $y = \sqrt[3]{x(x+2)}$.

7.5.18. $y = \sqrt[3]{x^2 4x + 3}$.

7.5.19. $y = 6x - 6 - 9\sqrt[3]{(x-1)^2}$.

7.5.20. $y = \left(\frac{x+1}{x-1}\right)^4$.

7.5.21. $y = x - \sqrt[3]{x^2}$.

7.5.22. $y = 2x - 6\sqrt[3]{x^2}$.

7.5.23. $y = \sqrt[3]{6(x-2)^2}$.

7.5.24. $y = x^2(x-1)^2$

7.5.25. $y = (x+2)(x-3)^3$.

7.5.26-misol. Ushbu $f(x) = \frac{x^3}{6(3-x)^2}$ funksiyani to'liq tekshiring va uning grafigini chizing.

Yechilishi ([2], 6-bo'lim; [9], 1-t., 7-bo'lim; [30], 4-bo'lim).

1. Funksiyaning aniqlanish sohasi: $D(f) = (-\infty; 3) \cup (3; +\infty)$.

2. $x = 3$ nuqta -funksiyaning 2-tur uzilish nuqtasi:

$$\lim_{x\to 3\pm 0} f(x) = \lim_{x\to 3\pm 0}\frac{x^3}{6(3-x)^2} = +\infty.$$

3. Shuningdek, funksiya davriy ham emas, juft ham emas, toq ham emas, chunki

$$f(-x) = \frac{(-x^3)}{6(3+x)^2} = \frac{x^3}{4(3+x)^2} \neq \begin{cases} f(x), \\ -f(x). \end{cases}$$

4. Funksiyaning koordinatalar o'qlari bilan kesishishi: Oy o'qi bilan $x=0$ da $y=0$ bo'ladi; Ox o'qi bilan $y=0$ bo'lganda $x=0$ bo'ladi. Shunday qilib, bitta $O(0;0)$ nuqtada kesishadi.

5. Funksiyaning ishorasi saqlanadigan oraliqlarni aniqlaymiz, aniqlanish sohasini nuqtalar yordamida funksiya nolga teng bo'ladigan oraliqlarga ajratamiz. Bu oraliqlarning har birida funksiyaning ishorasini tekshiramiz natijada quyidagi jadvalni tuzamiz:

X	$(-\infty;0)$	0	$(0;3)$	3	$(3;+\infty)$
sign y	–	0	+	∞	+
$y=f(x)$ funksiya grafigining joylanishi	Ox o'qdan pastda		Ox o'qdan yuqorida		Ox o'qdan yuqorida

6. Funksiya grafigining asimptotalarini topamiz:

a) *Oy* o'qqa parallel to'g'ri chiziqlar – vertikal asimptotalar bo'ladi.

$$\lim_{x\to 3}\frac{x^3}{6(3-x)^2}=+\infty$$

bo'lgani uchun $x=3$ to'g'ri chiziq vertikal asimptota.

b) *Ox* o'qqa parallel to'g'ri chiziqlar-gorizontal asimptotalar bo'ladi. Funksiyaning grafigi gorizontal asimptotaga ega emas.

g) *Ox* va *Oy* o'qlarga parallel bo'lmagan to'g'ri chiziqlar og'ma asimptotalar bo'ladi, ya'ni $y=kx+b$ og'ma asimptotaning formulasidan k va b larni hisoblaymiz:

$$k=\lim_{x\to+\infty}\frac{x^3}{6x(3-x)^2}=\frac{1}{6}\lim_{x\to+\infty}\frac{1}{6\left(\frac{3}{x}-1\right)^2}=\frac{1}{6}.$$

$$b=\lim_{x\to\infty}\left[\frac{x^3}{6(3-x)^2}-\frac{1}{6}x\right]=\lim_{x\to\infty}\frac{x^3-x(3-x)^2}{6(3-x)^2}=\frac{1}{6}\lim_{x\to\infty}\frac{6x^2-9x}{(3-x)^2}=1.$$

Demak, $y=\frac{1}{6}x+1$ -to'g'ri chiziq og'ma asimptota bo'ladi.

7. Funksiyaning monotonlik oraliqlari va ekstremum qiymatlarini topamiz:

$$y'=\frac{x^2(x-9)}{6(3-x)^3}.$$

a) $x=0$, $x=9$ nuqtalarda $y'=0$ bo'ladi. b) $x=3$ nuqtada $y'=\infty$ bo'ladi.

$$y'\geq 0,\ \frac{x^2(x-9)}{6(x-9)^3}\geq 0\ \Rightarrow (x-3)(x-9)\geq 0\ \Rightarrow\ (-\infty;3)\cup[9;\infty)$$

$$y'\leq 0,\ \Rightarrow\begin{cases}(x-3)(x+9)\leq 0\Rightarrow & x\in(-3;9]\\ x\neq 3\end{cases}.$$

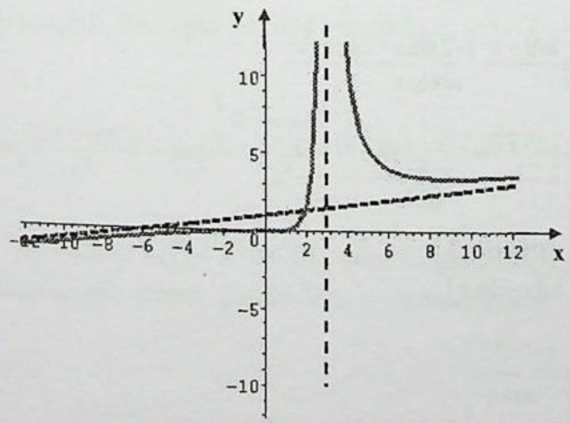
$y_{\min}(9)=\frac{27}{8}$. $A\left(9;\frac{27}{8}\right)$-berilgan funksiya grafigining minimum nuqtasi bo'ladi.

X	$(-\infty;0)$	0	$(0;3)$	3	$(3;9)$	+	$(9;\ \infty)$
sign y''	+	0	+	∞	-	0	+
funksiyaning o'zgarishi	↗	0	↗	∞	↘	$\frac{27}{8}$	↗

8. Funksiyaning qavariqlik va botiqlik oraliqlarini topamiz, buning uchun ikkinchi tartibli hosilani hisoblaymiz: $y''=\frac{9x}{(x-3)^4}$.

$x=0$ va $x=3$ nuqtalar berilgan funksiyaning 2-tur kritik nuqtalari bo'ladi. $x=0$ bo'lganda $y''(0)=0$, $x=3$ bo'lganda esa $y''(3)=\infty$ bo'ladi. Endi jadval tuzamiz:

X	$(-\infty;0)$	0	$(0;3)$	3	$(3;+\infty)$
sign y''	-	0	+	∞	+
Funksiya grafiginig qavariqlik yo'nalishi	↑	0	↓	∞	↓



1-chizma.

Funksiyaning grafigi 1-chizmada tasvirlangan.

7.6-masala. Teylor formulasidan foydalanib, quyidagi limitni hisoblang.

7.6.1. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}$. **7.6.2.** $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}$.

7.6.3. $\lim\limits_{x\to 0}\dfrac{\cos x-1+\dfrac{x^2}{2}}{x^4}$. **7.6.4.** $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-\sqrt{1+x^2}-x\cos x}{\ln^3(1-x)}$.

7.6.5. $\lim\limits_{x\to 0}\dfrac{arctgx-\arcsin x}{x^2}$. **7.6.6.** $\lim\limits_{x\to 0}\dfrac{tgx-x}{\sin x-x}$.

7.6.7. $\lim\limits_{x\to 0}\dfrac{arctgx-\arcsin x}{tgx-\sin x}$. **7.6.8.** $\lim\limits_{x\to 0}\dfrac{2\arcsin x-\arcsin 2x}{x^3}$.

7.6.9. $\lim\limits_{x\to 0}\dfrac{\sqrt[5]{1+2x}-1}{\sqrt[4]{1+x}-\sqrt{1-x}}$. **7.6.10.** $\lim\limits_{x\to 0}\dfrac{1+x\cos x-\sqrt{1+2x}}{\ln(1+x)-x}$.

7.6.11. $\lim\limits_{x\to 0}\dfrac{e^x-\sqrt{1+2x}}{\ln\cos x}$. **7.6.12.** $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x^2}-xctgx}{x\cdot\sin x}$.

7.6.13. $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}$. **7.6.14.** $\lim\limits_{x\to 0}\dfrac{(1+x)^x-1}{x^2}$.

7.6.15. $\lim\limits_{x\to 0}\dfrac{e^x-\sqrt[3]{1+3x+\dfrac{9}{2}x^2}}{x^3}$.

7.6.16. $\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)-2\sin x+2x\cos x^2}{arctg\,x^3}$.

7.6.17. $\lim\limits_{x\to 0}\dfrac{x\sqrt{1+\sin x}-\frac{1}{2}\ln(1+x^2)-x}{tg^3x}$.

7.6.18. $\lim\limits_{x\to 1}\dfrac{\sin(\sin\pi x)}{\ln(1+\ln x)}$.

7.6.19. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1e^{\pi x-2x^2}}{\cos x}$.

7.6.20. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x\cdot tgx}\right)$.

7.6.21. $\lim\limits_{x\to 0}\dfrac{\ln(1+2x)-2x}{x^2}$.

7.6.22. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1-2x}{x^2}$.

7.6.23. $\lim\limits_{x\to 0}\dfrac{\cos 2x-1+2x^2}{x^4}$.

7.6.24. $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-\sqrt{1+x^2}-x\cos x}{\ln^3(1-x)}$.

7.6.25. $\lim\limits_{x\to 0}\dfrac{arctg2x-\arcsin 2x}{x^2}$.

7.6.26-misol. Teylor formulasi va Lopital qoidalaridan foydalanib, limitni toping:

$$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}.$$

Yechilishi ([2], 6-bo'lim; [3], 1-q., 171-176 betlar; [9], 1-t., 6-bo'lim; [30], 4-bo'lim). *1-usul.* $\operatorname{tg}x=x+\dfrac{x^3}{3}+o(x^4)$, $\sin x=x-\dfrac{x^3}{6}+o(x^4)$ yoyilmalardan foydalanib, berilgan limitni topamiz:

$$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{2}x^3+o(x^4)}{x^3}=\lim_{x\to 0}\left[\frac{1}{2}+\frac{o(x^3)}{x^3}\right]=\frac{1}{2}.$$

2-usul. $f(x)=tgx-\sin x$, $g(x)=x^3$ bo'lib, ular Lopital birinchi qoidasining hamma shartlarini qanoatlantiradi, jumladan, $x=0$ nuqtaning ixtiyoriy kichik atrofida

$$f'(x)=\frac{1}{\cos^2 x}-\cos x,\quad g'(x)=3x^2$$

hosilalar mavjud bo'lib, $g'(x)=3x^2\neq 0$ $(x\neq 0)$. Lekin, $f'(x)$, $g'(x)$, $f''(x)$ va $g''(x)$ funksiyalar ham o'z navbatida $x=0$ nuqtaning kichik atrofida 3.9-teoremaning barcha shartlarini qanoatlantiradi. Shuning uchun, berilgan limitni hisoblashga Lopitalning birinchi qoidasini uch marta qo'llaymiz:

$$\lim_{x\to 0}\frac{tgx-\sin x}{x^3}\overset{\left(\frac{0}{0}\right)}{=}=\lim_{x\to 0}\frac{(tgx-\sin x)'}{(x^3)'}=\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-\cos x}{3x^2}\overset{\left(\frac{0}{0}\right)}{=}$$

$$=\lim_{x\to 0}\frac{\frac{2}{\cos^3 x}\sin x+\sin x}{6x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\frac{-\frac{6}{\cos^4 x}\sin x+\frac{2}{\cos^2 x}+\cos x}{6}=\frac{1}{2}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Limit((tan(x)-sin(x))/(x^3),x=0)=limit((tan(x)-sin(x))/(x^3),x=0);

$$\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{x^3}=\frac{1}{2}.$$

8-bob. ANIQMAS VA ANIQ INTEGRALLAR

8.1-§. Aniqmas integralning ta'rifi. Aniqmas integralning xossalari. Integrallash usullari

Biz 7-bobda berilgan funksiyaning hosilasini topish masalasini o'rgandik. Usbu bobda unga teskari masalani qaraymiz, ya'ni funksiyaning hosilasi berilganda, uning o'zini topish masalasini o'rganamiz. Mexanika va fizikada bu masala quyidagicha ifoda qilinadi: moddiy nuqtaning tezligiga ko'ra, uning harakat qonunini topish masalasi. Bunday masalalarni yechish, matematik nuqtai nazardan boshlang'ich funksiya va aniqmas integral tushunchalariga olib keladi.

8.1. Boshlang'ich funksiya tushunchasi. $f(x)$ va $F(x)$ funksiyalar biror (a,b) (chekli yoki cheksiz) oraliqda aniqlangan bo'lsin.

8.1.1-ta'rif. Agar $F(x)$ funksiya (a,b) oraliqda differensiallanuvchi bo'lib, $\forall x\in(a,b)$ lar uchun $F'(x)=f(x)$ tenglik o'rinli bo'lsa, $F(x)$ funksiya (a,b) oraliqda $f(x)$ funksiyaning *boshlang'ich funksiyasi* deyiladi.

8.1.2-eslatma. Agar $f(x)$ va $F(x)$ funksiyalar $[a,b]$ kesmada aniqlangan, $F(x)$ funksiya $[a,b]$ kesmada differensiallanuvchi, $\forall x\in(a,b)$ uchun $F'(x)=f(x)$ bo'lib, a va b nuqtalarda esa, $F'(a+0)=f(a)$, $F'(b-0)=f(b)$ tenglik o'rinli bo'lsa, u holda $F(x)$ funksiya $[a,b]$ kesmada $f(x)$ funksiyaning *boshlang'ich funksiyasi* deyiladi.

Misollar. 1) $f(x)=x^6$ bo'lsin. Bu funksiyaning R dagi boshlang'ich funksiyasi $F(x)=\frac{x^7}{7}$ bo'ladi, chunki

$$F'(x)=\frac{7x^6}{7}=x^6.$$

2) $y=\sin x$ funksiyaning R dagi boshlang'ich funksiyasi $F(x)=-\cos x$ bo'ladi.

8.1.3-lemma. $G(x)$ va $F(x)$ funksiyalarning har biri (a,b) oraliqda differensiallanuvchi bo'lib, ularning har biri bitta $f(x)$ funksiyaning boshlang'ich funksiyalari bo'lsa, bu $G(x)$ va $F(x)$ funksiyalar (a,b) oraliqda bir-biridan o'zgarmas songa farq qiladi, ya'ni $G(x)=F(x)+C$, $x\in(a,b)$.

8.2. Aniqmas integral tushunchasi. Agar $f(x)$ funksiyaning biror boshlang'ich funksiyasi $F(x)$ ma'lum bo'lsa, uning boshqa istalgan boshlang'ich funksiyasi ushbu

$$F(x)+C \quad (C=const)$$

formula bilan topiladi .

8.2.1-ta'rif. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lsin. Bu funksiyaning shu oraliqdagi barcha boshlang'ich funksiyalarining umumiy ko'rinishiga, $f(x)$ funksiyaning *aniqmas integrali* deyiladi va

$$\int f(x)dx$$

kabi belgilanadi, bunda $\int$ – integral belgisi, $f(x)$ integral ostidagi *funksiya*, $f(x)dx$ esa *integral ostidagi ifoda* deyiladi.

Agar $F(x)$ funksiya (a,b) oraliqda $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsa, u holda $f(x)$ funksiyaning aniqmas integrali

$$\int f(x)dx = F(x)+C$$

kabi yoziladi. Bunda C-ixtiyoriy o'zgarmas son. Ba'zi hollarda $\int f(x)dx$-ni $f(x)$ funksiyaning boshlang'ich funksiyalari to'plami emas, balki bu to'plamning ixtiyoriy elementi deb qarash mumkin.

Integral ostidagi ifodani, ya'ni $f(x)dx$-ni $F'(x)dx$ shaklida yoki

$$f(x)dx = F'(x)dx = d(F(x))$$

deb yozish mumkin. Berilgan funksiyaning aniqmas integralini topish amali, uni differensiallash amaliga teskari amal bo'lib, u integrallash deyiladi.

8.3. Aniqmas integralning xossalari. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lsin.

1⁰. Agar $F(x)$ funksiya (a,b) oraliqda differensiallanuvchi bo'lsa,

$$\int dF(x) = F(x)+C \text{ yoki } \int F'(x)dx = F(x)+C \qquad (8.3.1)$$

bo'ladi. (8.3.1) tenglik 8.2.1-ta'rifdan kelib chikadi.

2⁰. $f(x)$ funksiya (a,b) oraliqda boshlang'ich funksiyaga ega bo'lsin. U holda $\forall x \in (a,b)$ uchun

$$d\left(\int f(x)dx\right) = f(x)dx$$

tenglik o'rinli bo'ladi.

Isboti. $F(x)$ funksiya $f(x)$ uchun ixtiyoriy boshlang'ich funksiya bo'lsin. U holda aniqmas integralning ta'rifiga ko'ra, ushbu

$$\int f(x)dx = F(x)+C$$

tenglik o'rinli bo'ladi. Bundan

$$d\left(\int f(x)dx\right) = d(F(x)+C) = dF(x)+dC = dF(x), dC = 0, dF(x) = F'(x)dx = f(x)dx.$$

Demak $d\left(\int f(x)dx\right) = f(x)dx$ tenglik o'rinli bo'ladi. ■

3⁰. Agar $f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda boshlang'ich funksiyalarga ega bo'lsa, u holda $f(x)+g(x)$ funksiya ham (a,b) oraliqda boshlang'ich funksiyaga ega va

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx \qquad (8.3.2)$$

tenglik o'rinli bo'ladi.

Isboti. $F_1(x)$ va $F_2(x)$ funksiyalar mos ravishda (a,b) oraliqda $f(x)$ va $g(x)$ funksiyalarning boshlang'ich funksiyalari, ya'ni

$$F_1'(x) = f(x), \qquad F_2'(x) = g(x)$$

bo'lsin. $F(x) = F_1(x)+F_2(x)$ deb belgilasak, $F(x)$ funksiya $f(x)+g(x)$ uchun boshlang'ich funksiya bo'ladi. Haqiqatdan ham

$$F'(x) = F_1'(x)+F_2'(x) = f(x)+g(x), \quad x \in (a,b).$$

Shuning uchun

$$\int (f(x)+g(x))dx = F(x)+C = F_1(x)+F_2(x)+C \qquad (8.3.3)$$

Ikkinchi tamondan $F_1(x)$ va $F_2(x)$ funksiyalar mos ravishda $f(x)$ va $g(x)$ funksiyalarning boshlang'ich funksiyalari bo'lgani uchun

$$\int f(x)dx = F_1(x)+C, \quad \int g(x)dx = F_2(x)+C_2,$$

(bunda C_1 va C_2 ixtiyoriy o'zgarmas sonlar) tengliklar o'rinli. Bu tengliklarni mos ravishda qo'shsak

$$\int f(x)dx + \int g(x)dx = (F_1(x)+F_2(x)) + (C_1+C_2) = F_1(x)+F_2(x)+C \quad (8.3.4)$$

(8.3.3) va (8.3.4) dan (8.3.2) kelib chiqadi.■

4⁰. Agar $f(x)$ funksiya (a,b) oraliqda boshlang'ich funksiyaga ega va $k \in R\backslash\{0\}$ bo'lsa, u holda $k\,f(x)$ funksiya ham (a,b) oraliqda boshlang'ich funksiyaga ega va

$$\int kf(x)dx = k\int f(x)dx$$

tenglik o'rinli bo'ladi.

8.4. Elementar funksiyalarning aniqmas integrallari jadvali

1. $\int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1}+C, \ \mu \neq -1.$
2. $\int \frac{dx}{x} = \ln|x|+C, \ x \neq 0.$
3. $\int a^x dx = \frac{a^x}{\ln a}+C, \ a>0, \ a \neq 1$, xususiy holda $\int e^x dx = e^x + C$.
4. $\int Sinx dx = -Cosx + C$.
5. $\int Cosx dx = Sinx + C$.
6. $\int \frac{dx}{Cos^2 x} = tgx + C$.
7. $\int \frac{dx}{Sin^2 x} = -Ctgx + C$
8. $\int Shx dx = Chx + C$.
9. $\int Chx dx = Shx + C$.
10. $\int \frac{1}{Ch^2 x}dx = thx + C$.

11. $\int \frac{1}{Sh^2 x}dx = -Chx + C$.
12. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}arctg\frac{x}{a} + C = -\frac{1}{a}arcCtgx + C$.
13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$.
14. $\int \frac{dx}{\sqrt{a^2-x^2}} = arcSin\frac{x}{a} + C = -arcCos\frac{x}{a} + C, \ |x| < |a|.$
15. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C, \quad |x| < |a|.$

8.5. O'zgaruvchini almashtarib integrallash usuli. O'zgaruvchini almashtirish (o'rniga qo'yish) usuli aniqmas integrallarni hisoblashning eng muhim usuli bo'lib, unda hisoblash talab qilingan integral, hisoblash uchun qulay (oson) bo'lgan integralga almashtiriladi, u esa, quyidagi teoremaga asoslanadi.

8.5.1-teorema. $f(x)$ va $\varphi(t)$ funksiyalar mos ravishda X va $T=\{t\}$ oraliqlarda aniqlangan bo'lib, $E(\varphi)\subset X$ bo'lsin. Agar $f(x)$ funksiya X oraliqda $F(x)$-boshlang'ich funksiyaga ega, yani

$$\int f(x)dx = F(x)+C, \quad (8.5.2)$$

$x=\varphi(t)$ funksiya esa, T oraliqda differensiyalanuvchi bo'lsa, u holda $F(\varphi(t))\varphi'(t)$ funksiya T da $F(\varphi(t))$ - boshlang'ich funksiyaga ega va

$$\int f(\varphi(t))\varphi'(t)dt = \int f(x)dx\Big|_{x=\varphi(t)} \quad (8.5.3)$$

formula o'rinli bo'ladi.

Isboti. $f(x)$ va $F(x)$ funksiyalar X oraliqda aniqlangan va $E(\varphi)\subset X$ bo'lgani uchun $f(\varphi(t))$ va $F(\varphi(t))$ murakkab funksiyalar manoga ega bo'ladi. Shartga ko'ra, $F'(x)=f(x)$, $x\in X$. U holda, murakkab funksiyaning hosilasini topish formulasiga asosan, hamda $F'(x)=f(x)$ ekanligini e'tiborga olgan holda

$$\left(F(\varphi(t))\right)'_t = F'_x(x)\big|_{x=\varphi(t)}\varphi'(t) = f(\varphi(t))\varphi'(t), \ t\in T$$

ni hosil qilamiz. Bundan, $F(\varphi(t))$funksiya, $f(\varphi(t))\varphi'(t)$funksiya uchun boshlang'ich funksiya ekanligi kelib chiqadi. Aniqmas integralning tarifiga ko'ra:

$$\int f(\varphi(t))\varphi'(t)dt = F[\varphi(t)] + C. \qquad (8.5.4)$$

(8.5.2) formulaga x ning o'rniga $x=\varphi(t)$ ni qo'yib

$$\int f(x)dx\Big|_{x=\varphi(t)} = F[\varphi(t)] + C \qquad (8.5.5)$$

ni hosil qilamiz. (8.5.4) va (8.5.5) formulalarning o'ng tomonlari o'zaro teng, u holda, chap tomonlari ham o'zaro teng, yani (8.5.3) formula o'rinli bo'ladi. ■

(8.5.3) formulaga, o'zgaruvchilarni almashtirish (o'rniga qo'yish) usuli deb ataladi. Albatta, integrallashning bu usuli hamma integrallarni hisoblash uchun ham qo'l kelavermaydi. Integrallarni hisoblaganda, o'zgaruvchilarni almashtirish usulini qo'llashda, almashtirishlarni to'g'ri tanlash hisoblovchining mahoratga bog'liq. Masalan, $\int \frac{\varphi'(x)}{\varphi(x)}dx$ ko'rinishidagi integralni hisoblashda $(\varphi(x)\neq 0)$, albatta $t=\varphi(x)$ almashtirishni olish kerak:

$$\int \frac{\varphi'(x)}{\varphi(x)}dx = \int \frac{d\varphi(x)}{\varphi(x)} = \int \frac{dt}{t} = \ln|t| + C = \ln|\varphi(x)| + C. \qquad (8.5.6)$$

Bazi hollarda, integrallarni hisoblashda, o'zgaruvchilarni almashtirish usulini qo'llash uchun, avvalo, integral ostidagi funksiyaning shaklini o'zgartirish maqsadga muvofiq bo'ladi. Masalan, $\int \frac{1}{\sin x}dx$ – integralni hisoblashda, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ tenglikni etiborga olib, integral ostidagi ifodaning shaklini o'zgartirib, (8.5.6) formulani etiborga olgan holda,

$$\int \frac{1}{\sin x}dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = \int \frac{1}{2tg\frac{x}{2}\cdot\cos^2\frac{x}{2}}dx =$$

$$= \int \frac{\left(tg\frac{x}{2}\right)'}{tg\frac{x}{2}}dx = \ln|t| + C = \ln\left|tg\frac{x}{2}\right| + C$$

bo'lishini topimiz.

(8.5.5) formulaning quydagi xususiy hollarini qaraymiz:

a) $F(x)$ funksiya, $f(x)$ funksiya uchun boshlang'ich funksiya bo'lsin, ya'ni $\int f(x)dx = F(x) + C$. U holda

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C, \quad a\neq 0, \qquad (8.5.7)$$

bunda $\varphi(x) = ax+b$, $f(ax+b)dx = \frac{1}{a}f(ax+b)d(ax+b)$.

b) $\int \frac{dt}{t} = \ln|t|$ni e'tiborga olgan holda

$$\int \frac{d\varphi(x)}{\varphi(x)} = \int \frac{\varphi'(x)dx}{\varphi(x)} = \ln|\varphi(x)| + C, \ \varphi(x)\neq 0. \qquad (8.5.8)$$

c) $\int t^\alpha dt = \frac{t^{\alpha+1}}{\alpha+1} + C \quad \alpha\neq -1, \ t>0$, ni e'tiborga olgan holda

$$\int (\varphi(x))^\alpha \varphi'(x)dx = \int (\varphi(x))^\alpha d\varphi(x) = \frac{(\varphi(x))^{\alpha+1}}{\alpha+1} + C, \qquad (8.5.9)$$

$\varphi(x)>0, \ \alpha\neq -1$.

$$J = \int f(x)dx \qquad (8.5.10)$$

integralni hisoblash talab qilingan bo'lsin. $x=\varphi(t)$ - qat'iy manoton va differensiallanuvchi funksiya bo'lsin, $t\in(\alpha,\beta)$. U holda bu funksiyaga teskari

$$t = \omega(x) \qquad (8.5.11)$$

funksiya mavjud bo'ladi. (8.5.10) dagi integral ostidagi ifodani, ya'ni $x=\varphi(t)$ almashtirish natijasida uni

$$f(x)dx = f(\varphi(t))\varphi'(t)dt$$

shaklga keltiriladi. Quyidagi

$$u(t) = f(\varphi(t))\varphi'(t) \qquad (8.5.12)$$

belgilashni olaylik. $\phi(t)$ funksiya $u(t)$ funksiya uchun boshlang'ich funksiya bo'lsin, ya'ni

$$\int u(t)dt = \phi(t) + C \tag{8.5.13}$$

Demak, (8.5.10)- (8.5.13) lardan

$$J = \int f(x)dx = \int u(t)dt = \phi(t) + C = \phi(\omega(x)) + C \tag{8.5.14}$$

(8.5.14) *o'zgaruvchilarni almashtirish formulasi* deyiladi. Bu formula yordamida integral ostidagi ifoda (8.5.12) ko'rinishiga keltiriladi. Bunda $f(x)$funksiya uchun boshlang'ich funksiya mavjud.

8.6. Bo'laklab integrallash usuli

8.6.1-teorema. Agar $u(x)$ va $v(x)$ funksiyalar (a,b) oraliqda aniqlangan va differensiallanuvchi bo'lib, (a,b) oraliqda $\int vdu$ integral mavjud bo'lsa, u holda shu oraliqda $\int udv$ integral ham mavjud va

$$\int udv = uv - \int vdu \tag{8.6.2}$$

tenglik o'rinli bo'ladi.

Isboti. Shartga ko'ra $u(x)$ va $v(x)$ funksiyalar (a,b) oraliqda differensiallanuvchi. U holda ushbu

$$d(uv) = vdu + udv$$

tenglik o'rinli bo'ladi. Bundan $udv = d(uv) - vdu$. Bu tenglikning ikkala tamonidan olingan integral mavjud va

$$\int udv = \int d(uv) - \int vdu = uv - \int vdu$$

tenglik o'rinli bo'ladi, chunki aniqmas integralning 1^0 –xossasiga asosan $\int d(uv) = uv + C$, $\int vdu$-integral esa, teoremaning shartiga ko'ra, mavjud. U holda aniqmas integralning 3^0 –xossasiga asosan $\int udv$ integral ham mavjud, bunda $\int d(u,v) = uv + C$ $(C = const)$ $\int vdu$ integral tarkibiga kiradi. ■

Bo'laklab integrallash usuli bilan hisoblanadigan integrallarning ko'proq qismini, shartli ravishda, quyidagi uch guruhga ajratish mumkin.

1. Birinchi guruhga integral ostidagi funksiya tarkibida ko'paytuvchi sifatida

$$\ln x,\ arc\sin x,\ arc\cos x,\ arc\,tgx,\ (arc\sin x)^2,$$
$$(arc\cos x)^2,\ (arc\,tgx)^2,\ \ln\varphi(x),...$$

funksiyalardan biri qatnashgan integrallar kiradi.

8.6.3-eslatma. Integral ostidagi funksiya tarkibida ko'paytuvchi sifatida $(arc\,tgx)^2, (arc\cos x)^2,...$ funksiyalar qatnashsa, integralni hisoblashda (8.6.2) formula ikki marta qo'llaniladi.

2. Ikkinchi guruhga $\int p(x)\cos(kx)dx$, $\int p(x)\sin(kx)dx$, $\int p(x)a^{kx}dx$ ko'rinishdagi integrallar kiradi, bunda $p(x)$– o'zgarmas koeffisiyentli n – darajali ko'phad, k - o'zgarmas son.

8.6.4-eslatma. Ikkinchi guruhdagi integrallarni hisoblashda $p(x)$ ko'phadning darajasi qancha bo'lsa, shuncha marta (8.6.2) formula qo'llaniladi, bunda *u(x)* sifatida $p(x)$ ni olish maqsadga muvofiq bo'ladi. (8.6.2) formula har bir qo'llanganda, $p(x)$ ko'phadning darajasi bittaga kamayadi.

3. Uchinchi guruhga, quyidagi

$$\int e^{\alpha x}\sin bx\,dx,\ \int e^{\alpha x}\cos bx\,dx,\ \int \sin(\ln x)dx,\ \int\cos(\ln x)dx,....$$

integrallar kiradi.

Endi, yuqorida keltirilgan uchta guruhning birortasiga ham kirmaydigan, lekin (8.6.2) bo'laklab integrallash formulasi yordamida hisoblanadigan integrallarning ba'zilarini qaraymiz:

$$x^k\ln^m x,\ x^k Sinbx,\ x^k Cosbx,\ x^k e^{\alpha x}$$

va hokazolar.

Misol.

$$\int x^2\ln x dx = \begin{bmatrix} u = \ln x, & du = \frac{1}{x}dx \\ dv = x^2dx, & v = \frac{x^3}{3} \end{bmatrix} = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2dx = \frac{1}{3}x^3\ln x - \frac{x^3}{9} + C$$

8.7. Ratsional funksiyalarni integrallash. Ikkita algebraik ko'phadning nisbatiga, yani

$$f(x)=\frac{P_m(x)}{Q_n(x)} \qquad (8.7.1)$$

ifodaga *ratsional funksiya* yoki *ratsional kasr* deyiladi. Bunda,

$$P_m(x)=b_0+b_1x+...+b_mx^m,\ Q_n(x)=a_0+a_1x+...+a_nx^n\ (b_m,a_n\neq 0, m\geq 0, n\geq 1)$$

qo'phadlar *haqiqiy koeffitsiyentli ko'phadlar* deb faraz qilinadi.

Agar $m<n$ bo'lsa, (8.7.1) ni *to'g'ri kasr*, $m\geq n$ bo'lganda esa, (8.7.1) *noto'g'ri kasr* deyiladi.

Agar (8.7.1) kasr noto'g'ri kasr bo'lsa, u holda kasrning suratini maxrajiga (ko'phadni ko'p hadga bo'lish qoidasiga ko'ra) bo'lish bilan,

$$f(x)=w(x)+\frac{P_r(x)}{Q_n(x)}\ (r<n) \qquad (8.7.2)$$

ko'rinishga keltiriladi, bunda $w(x)$–biror ko'phad.

Faraz qilaylik, $f(x)$ funksiya ratsional funksiya bo'lib, uning integralini hisoblash talab qilingan bo'lsin. $f(x)$ funksiya butun ratsional funksiya, yani

$$f(x)=a_0+ax+...+a_nx^n$$

bo'lganda, uning integral

$$\int f(x)dx=\int\left(a_0+a_1x+...+a_nx^n\right)dx=a_0x+a_1\cdot\frac{x^2}{2}+...+a_n\cdot\frac{x^n}{n}+C \qquad (8.7.3)$$

kabi hisoblanadi.

Agar $f(x)$–kasr ratsional funksiya, yani

$$f(x)=\frac{b_0+b_1x+...+b_mx^m}{a_0+a_1x+...+a_nx^n}\quad m\in N, n\in N$$

bo'lib, $m\geq n$ bo'lganda uni (8.7.2) ko'rinishga keltiriladi va uning integrali:

$$\int f(x)dx=\int w(x)dx+\int\frac{P_k(x)}{Q_n(x)}dx\ (k<n) \qquad (8.7.4)$$

bo'ladi. $w(x)$–ko'phadning integralini hisoblash hyech qanday qiyinchilik tug'dirmaydi (8.7.3) ga qarang).

Demak, $f(x)=\dfrac{P_m(x)}{Q_n(x)}$ $(m>n)$ ratsional funksiyani integrallash, to'g'ri kasrni integrallashga keltirilar ekan.

To'g'ri kasrni integrallash uchun kerak bo'ladigan algebra kursida o'rganilgan bazi tushunchalar va tasdiqlarni isbotsiz keltiramiz. Ushbu

$$P_m(x)=b_0+b_1x+b_2x^2+...+b_mx^m, b_k\in R, k=0,1,2,...,m, m\in N$$

ko'p had berilgan bo'lsin, bunda $a_k(k=0,1,2,...,m)$ ko'phadning koeffitsiyentlari, m esa, ko'phadning *darajasi* deyiladi.

Agar biror $\alpha\in R$ son $P_m(\alpha)=0$ tenglikni qanoatlantirsa, α son $P_m(x)$ ko'phadning *ildizi* deyiladi.

Bezu teoremasiga asosan, $P_m(x)$ ko'phad $x-\alpha$ ga qoldiqsiz bo'linadi:

$$P_m(x)=(x-\alpha)Q(x), Q(\alpha)\neq 0,$$

bunda, $Q(x)$–$(n-1)$ darajali ko'phad. Agar $P_m(x)$ ko'phad $(x-\alpha)^k (k\in N)$ ga qoldiqsiz bo'linsa, u holda α songa $P_m(x)$ ko'phadning k–*karrali ildizi* deyiladi. Bu holda $P_m(x)$ ko'phad

$$P_m(x)=(x-\alpha)^kQ_1(x), Q_1(\alpha)\neq 0$$

ko'rinishida ifodalanadi, bunda $Q_1(x)$–$(m-k)$ darajali ko'phad. Agar $z=\alpha+i\beta$ kompleks son $P_m(x)$ ko'phadning ildizi bo'lsa, $\bar{z}=\alpha-i\beta$ son ham $P_m(x)$ ko'phadning ildizi bo'ladi. U holda $P_m(x)$ ko'phadning ifodasida ushbu

$$(x-z)(x-\bar{z})=[x-(\alpha+i\beta)]\cdot[x-(\alpha-i\beta)]=x^2+px+q$$
$$(p=-2\alpha, q=\alpha^2+\beta^2)$$

kvadrat uch had ko'paytuvchi bo'lib qatnashadi.

Faraz qilaylik, $\alpha_1,\alpha_2,...,\alpha_k$–haqiqiy sonlar $Q_n(x)$ ko'phadning mos ravishda $\lambda_1,\lambda_2,...,\lambda_k$ karrali ildizlari, $z_1,z_2,...,z_s$ – kompleks sonlar esa, $Q_n(x)$ ko'phadning mos ravishda $\gamma_1,\gamma_2,...,\gamma_s$ – karrali ildizlari bo'lsin.

8.7.5-teorema. Har qanday $n-$ darajali $Q_n(x)=a_0+a_1x+...+a_nx^n$ $(a_m \in R, v=0,1,2,..b, a_n \neq 0)$ ko‘phad quyidagi

$$Q_n(x)=(x-\alpha_1)^{\lambda_1}(x-\alpha_2)^{\lambda_2}...(x-\alpha_k)^{\lambda_k}\cdot\left(x^2+p_1x+q_1\right)^{\gamma_1}\times$$
$$\times\left(x^2+p_2x+q_2\right)^{\gamma_2}...\left(x^2+p_sx+q_s\right)^{\gamma_s} \qquad (8.7.6)$$

ko‘rinishda ifodalanadi, bunda $\lambda_1+\lambda_2+...+\lambda_k+2(\gamma_1+\gamma_2+...+\gamma_s)=n$ bo‘lib, $x^2+p_ix+q_i\,(i=1,2,...,s)$ kvadrat uch had haqiqiy ildizga ega emas.

8.7.7-teorema. Agar $\frac{P_n(x)}{Q_s(x)}$ to‘g‘ri ratsional kasrning maxraji $Q_s(x)$ ko‘phad, quyidagi $Q_s(x)=(x-\alpha)^m Q(x)(m \in N)$ ko‘rinishda tasvirlangan bo‘lib, $Q(x)$ ko‘phad $(x-\alpha)$ga bo‘linmasa, u holda $\frac{P_n(x)}{Q_s(x)}$ to‘g‘ri kasr

$$\frac{P_n(x)}{Q_s(x)}=\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+...+\frac{A_m}{(x-\alpha)^m}+\frac{P(x)}{Q(x)}$$

ko‘rinishida tasvirlanadi, bunda $A_i \in R, i=1,2,...,m;$ $P(x)-$ko‘phad

8.7.8-teorema. Agar $\frac{P_n(x)}{Q_s(x)}$ to‘g‘ri ratsional kasrning maxraji $Q_s(x)$ ko‘phad, ushbu $Q_s(x)=\left(x^2+px+q\right)^m Q(x)(m \in N)$ ko‘rinishda tasvirlangan bo‘lib (x^2+px+q kvadratik uch had haqiqiy ildizlarga ega emas), $Q(x)$- ko‘phad x^2+px+q ga bo‘linmasa, u holda $\frac{P_n(x)}{Q_s(x)}$ to‘g‘ri kasr

$$\frac{P_n(x)}{Q_s(x)}=\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+...+\frac{B_mx+C_m}{\left(x^2+px+q\right)^m}+\frac{P(x)}{Q(x)}$$

ko‘rinishida tasvirlanadi, bunda $B_i \in R, C_i \in R, i=1,2,...,m; P(x)-$ ko‘phad

8.7.9-teorema. Agar $\frac{P_m(x)}{Q_n(x)}$ to‘g‘ri ratsional kasrning maxraji $Q_n(x)$ (8.7.6) shaklida tasvirlangan bo‘lsa, u holda bu kasrni quyidagi kasrlar yig‘indisi shaklida tasvirlash mumkin bo‘ladi:

$$\frac{P_m(x)}{Q_n(x)}=\frac{B_1^{(1)}}{(x-\alpha_1)}+\frac{B_2^{(1)}}{(x-\alpha_1)^2}+...+\frac{B_{\lambda_1}^{(1)}}{(x-\alpha_1)^{\lambda_1}}+...+\frac{B_1^{(k)}}{(x-\alpha_k)}+\frac{B_2^{(k)}}{(x-\alpha_k)^2}+...+$$
$$+\frac{B_{\lambda_k}^{(k)}}{(x-\alpha_k)^{\lambda_k}}+\frac{M_1^{(1)}x+N_1^{(1)}}{x^2+p_1x+q_1}+\frac{M_2^{(1)}x+N_2^{(1)}}{\left(x^2+p_1x+q_1\right)^2}+...+\frac{M_\gamma^{(1)}x+N_\gamma^{(1)}}{\left(x^2+p_1x+q_1\right)}+...+$$

$$+\frac{M_1^{(s)}x+N_1^{(s)}}{\left(x^2+p_sx+q_s\right)}+\frac{M_2^{(s)}x+N_2^{(s)}}{\left(x^2+p_sx+q_s\right)^2}+...+\frac{M_{\gamma_s}^{(s)}x+N_{\gamma_s}^{(s)}}{\left(x^2+p_sx+q_s\right)^{\gamma_s}} \qquad (8.7.10)$$

bunda $B_1^{(1)}, B_2^{(1)},..., B_{\lambda_k}^{(k)}, M_1^{(1)}, N_1^{(1)},..., M_{\gamma_s}^{(s)}, N_{\gamma_s}^{(s)}-$ lar nomalum haqiqiy o‘zgarmas sonlar, ularning tarkibidagi bazilari nolga teng bo‘lishi mumkin. Bu nomalum o‘zgarmas sonlarni topish uchun (8.7.10) ni umumiy maxrajga keltirib (umumiy maxraj $Q_n(x)$), ikki ko‘p hadning tengligi haqidagi teoremaga asosan, (8.7.10) ning o‘ng tomonidagi suratda hosil bo‘lgan ko‘phad bilan $P_m(x)$ ko‘phaddagi x ning bir xil darajalari oldidagi koeffisiyentlarni tenglashtirish natijasida, nomalum koeffisiyentlarga nisbatan chiziqli algebraik tenglamalar sistemasi hosil bo‘ladi. Bu sistemadan nomalum koeffisiyentlarni topib, topilgan qiymatlarni (8.7.10) tenglikga keltirib qo‘yiladi. Kasrning suratidagi nomalum koeffisiyentlarni topishning bu usuli, odatda nomalum koeffisiyentlar usuli deb yuritiladi.

8.8. Sodda kasrlarni integralash. $f(x)=\frac{P_m(x)}{Q_n(x)}$ kasr ratsional funksiyaning integralini hisoblash,

$$w(x)=c_0x^k+c_1x^{k-1}+...+c_{n-1}x+c_n$$

ko‘rinishidgi ko‘phadni integrallashga hamda *sodda kasrlar* deb ataluvchi quyidagi

$$\text{I. } \frac{B}{x-\alpha}, \text{ II. } \frac{B}{(x-\alpha)^r}, \text{ III. } \frac{Mx+N}{x^2+px+q}, \text{ IV. } \frac{Mx+N}{\left(x^2+px+q\right)^k}, \qquad (8.8.1)$$

($r>1, B, M, N, P, q-$haqiqiy sonlar, $q-\frac{p^2}{4}>0$) ko‘rinishdagi sodda kasrlarni integrallashga keltiriladi. Bu sodda kasrlarning integrallari quyidagicha hisoblanadi: I va II ko‘rinishidagi kasrlar $t=x-\alpha$ almashtirish yordamida integrallanadi:

$$\int\frac{Bdx}{x-\alpha}=B\int\frac{dt}{t}=B\ln|t|+C=B\ln|x-\alpha|+C, \qquad (8.8.2)$$

$$\int\frac{B}{(x-\alpha)^r}dx=B\int\frac{dt}{t^r}=-\frac{B}{(r-1)}\cdot\frac{1}{t^{r-1}}+C=-\frac{B}{(r-1)(x-\alpha)^{r-1}}+C. \qquad (8.8.3)$$

III ko'rinishdagi kasrni integrallash uchun $x^2+px+q-$ kvadrat uch hadni ushbu

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\left(q-\frac{p^2}{4}>0\right)$$

ko'rinishga keltirib, $t=x+\frac{p}{2}$ almashtirish va $a=+\sqrt{q-\frac{p^2}{4}}$ belgilash yordamida

$$\int\frac{(Mx+N)}{x^2+px+q}dx=\int\frac{Mt+\left(N-\frac{MP}{2}\right)}{t^2+a^2}dt=\frac{M}{2}\int\frac{d\left(t^2+a^2\right)}{t^2+a^2}+$$

$$+\left(N-\frac{MP}{2}\right)\cdot\frac{1}{a}\int\frac{d\left(\frac{t}{a}\right)}{\left(\frac{t}{a}\right)^2+1}=\frac{M}{2}\ln\left(t^2+a^2\right)+\frac{2N-MP}{2a}arctg\frac{t}{a}+C=$$

$$=\frac{M}{2}\ln\left(x^2+px+q\right)+\frac{2N-MP}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q^2-\frac{p^2}{4}}}+C \qquad (8.8.4)$$

munosabatga ega bo'lamiz.

IV ko'rinishidagi kasrni integrallashda yuqoridagi $t=x+\frac{p}{2}$, $a=\sqrt{q-\frac{p^2}{4}}$ – belgilashlar yordamida

$$\int\frac{Mx+N}{\left(x^2+px+q\right)^r}dx=\int\frac{Mt+\left(N-\frac{MP}{2}\right)}{\left(t^2+a^2\right)^r}dt=\frac{M}{2}\int\frac{d\left(t^2+a^2\right)}{\left(t^2+a^2\right)^r}+$$

$$+\left(N-\frac{MP}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^r}=-\frac{1}{(r-1)}\cdot\frac{1}{\left(t^2+a^2\right)^{r-1}}+\left(N-\frac{MP}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^r}$$

ni hosil qilamiz. Keyingi integral rekkurent formula orqali hisoblanadi:

$$J_{r+1}=\frac{t}{2na^2\left(t^2+a^2\right)^r}+\frac{2n-1}{2na^2}J_r$$

bunda $r=1$ bo'lganida

$$J_1=\int\frac{dt}{a^2+t^2}=\frac{1}{a}arctg\frac{t}{a}+C$$

Shunday qilib, har qanday haqiqiy koeffisiyentli, haqiqiy o'zgaruvchili ratsional (ratsional kasr) funksiyaning boshlang'ich funksiyasi-logarifm, arktangens va ratsional funksiya orqali ifoda qilinar ekan.

8.2-§. Ba'zi irratsional va tarkibida □rigonometric funksiyalar qatnashgan ifodalarni integrallash

8.9. Ba'zi irratsional ifodalarni integrallash. Agar integral ostidagi funksiya irratsional bo'lsa, bazi hollarda uning integralini hisoblash masalasi almashtirish natijasida ratsional funksiyaning integralini hisoblashga olib kelinadi. Bu usulga integral ostidagi ifodani *ratsionallashtirish usuli* deyiladi. Biz bu paragrafda integral ostida irratsional ifodalar qatnashgan integrallarning bazi turlarini ratsionallashtirish usullarini keltiramiz.

8.9.1. $\int R\left(x,x^\alpha,x^\beta,...,x^\nu\right)dx$ **ko'rinishdagi ifodalarni integrallash.** Bunda $R\left(x,x^\alpha,x^\beta,...,x^\nu\right)$- o'z argumentlarining ratsional funksiyasi, $\alpha=\frac{m_1}{n_1}$, $\beta=\frac{m_2}{n_2}....\nu=\frac{m_s}{n_s}$ - ratsional sonlar. Bu integralni hisoblash, $x=t^k(k-n_1,n_2,....$ larning eng kichik umumiy bo'linuvchisi) almashtirish yordamida ratsional funksiyaning integralini hisoblashga keltiriladi, ya'ni

$$\int R\left(x,x^\alpha,x^\beta,...,x^\nu\right)dx=\int R_1(t)kt^{k-1}dt,$$

bunda $R_1(t)$-ko'phad.

8.9.2 $\int R[x,(ax+b)^\alpha,(ax+b)^\beta,...,(ax+b)^\nu]dx$,

$$\int R[x,\left(\frac{ax+b}{cx+d}\right)^\alpha,\left(\frac{ax+b}{cx+d}\right)^\beta,...,\left(\frac{ax+b}{cx+d}\right)^\nu]dx \qquad (a,\ b,\ c,\ d\in R,\ ad-cb\neq 0)$$

ko'rinishdagi ifodalarni integrallash. 8.9.2-bandagi ko'rinishidagi integrallar, mos ravishda, $ax+b=t^k$ yoki $\frac{ax+b}{cx+d}=t^k$ almashtirish yordamida ratsionallashtiriladi, bunda k - $\alpha, \beta, ..., \nu$ ratsional sonlarning eng kichik umumiy bo'linuvchisi. Demak, ushbu

$$\int R[x,(ax+b)^{\alpha},(ax+b)^{\beta},...,(ax+b)^{\nu}]dx = \frac{k}{a}\int R_1(t)t^{k-1}dt,$$

$$\int R[x,\left(\frac{ax+b}{cx+d}\right)^{\alpha},\left(\frac{ax+b}{cx+d}\right)^{\beta},...,\left(\frac{ax+b}{cx+d}\right)^{\nu}]dx = \int R_1(t)\varphi(t)dt$$

tengliklar o'rinli bo'ladi, bunda $R_1(t)$, $\varphi(t)$- kasr ratsional funksiyalar.

8.9.3. $\int R\left(x,\sqrt{a^2-x^2}\right)dx$, $\int R\left(x,\sqrt{a^2+x^2}\right)dx$, $\int R\left(x,\sqrt{x^2-a^2}\right)dx$ **ko'rinishdagi ifodalarni integrallash**. Quyidagi

$$\int R\left(x,\sqrt{a^2-x^2}\right)dx,\ \int R\left(x,\sqrt{a^2+x^2}\right)dx,\ \int R\left(x,\sqrt{x^2-a^2}\right)dx$$

ko'rinishdagi integrallar, mos ravishda $x=a\sin t$, $x=atgt$, $x=a\sec t$, $a\in R$, $a\neq 0$, almashtirishlar natijasida ratsionallashtirilib hisoblanadi.

8.9.4. $R\left(x,\sqrt{ax^2+bx+c}\right)dx$ $\left(a\neq 0,\ b^2-4ac\neq 0\right)$ **ko'rinishdagi ifodalarni integrallash.** Quyidagi

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx \qquad (8.9.5)$$

integralni hisoblash, undagi a,b,c koeffisiyentlarga bog'liq, uchta almashtirish yordamida ratsional funksiyaning integralini hisoblashga keltiriladi:

1-hol. $a>0$ bo'lganda, (8.9.5) integralda $\sqrt{ax^2+bx+c}=\pm t\pm\sqrt{a}x$ almashtirish bajariladi.

2-hol. $c>0$ bo'lganda, (8.9.5) integralda $\sqrt{ax^2+bx+c}=\pm xt\pm\sqrt{c}$ almashtirish bajariladi.

3-hol. $a\neq 0$, $b^2-4ac>0$ bo'lganda esa, (8.9.5) integralda $\sqrt{ax^2+bx+c}=t(x-x_1)$ yoki $\sqrt{ax^2+bx+c}=t(x-x_2)$ almashtirish bajariladi. Bunda, x_1 va x_2 lar kvadratik uch hadning ildizlari.

Odatda, yuqorida keltirilgan uchta almashtirishlar - *Eyler almashtirishlari* deb aytiladi.

8.9.6. Binomial differensiallarni integrallash.

8.9.7-ta'rif. Ushbu $x^m\left(a+bx^n\right)^p dx$-ko'rinishdagi ifodaga, *binomial differensial* deyiladi, bunda a,b - haqiqiy sonlar, m,n,p - lar esa, ratsional sonlar.

$$\int x^m\left(a+bx^n\right)^p dx \qquad (8.9.8)$$

ko'rinishdagi integralni hisoblash, quyidagi uchta holda, ratsional funksiyani integrallashga keltiriladi:

1-hol. p- butun son. Bu holda m,n kasr sonlar maxrajining eng kichik umumiy bo'linuvchisini λ orqali belgilab, (8.9.8) integralda $x=t^{\lambda}$ almashtirish bajarilsa, integral ostidagi ifoda ratsional ifodaga aylanib, (8.9.8) integral ratsional funksiyani integrallashga keltiriladi.

2-hol. $\frac{m+1}{n}$- butun son. Bu holda (8.9.8) integralda $a+bx^n=t^s$, (s son - p kasrning maxraji) almashtirish bajarilsa, integral ostidagi ifoda ratsional ifodaga aylanib, (8.9.8) integralni hisoblash ratsional funksiyani integrallashga keltiriladi.

3-hol. $\frac{m+1}{n}+p$-butun son bo'lsin. Bu holda (8.9.8) integralda, $t^s=ax^{-n}+b$ (s son - p kasrning maxraji) almashtirish bajarilishi natijasida ratsional funksiya integralini hisoblashga keltiriladi.

8.10. Tarkibida trigonometrik funksiyalar qatnashgan ifodalarni integrallash. 8.10.1. $\int R(\sin x,\cos x)dx$ **ko'rinishdagi integrallarni hisoblash**. Ushbu

$$\int R(\sin x,\cos x)dx \qquad (8.10.2)$$

integralni qaraymiz.

1) $R(\sin x,\cos x)$ - $\sin x$ va $\cos x$ larning ratsional funksiyasi bo'lsin. Bu holda (8.10.2) integralda $t=tg\frac{x}{2}$ $(-\pi<x<\pi)$ *universal* almashtirish olinib, uni hisoblash, t ga nisbatan ratsional funksiyaning integralini hisoblashga keltiriladi. Haqiqatan ham, quyidagi

$$\sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t}{1+t^2},\ dx=\frac{2dt}{1+t^2} \qquad (8.10.3)$$

munosabatlarni e'tiborga olsak, (8.10.2) integral,

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt$$

ko'rinishga keladi.

2) $R(-\sin x; \cos x) = -R(\sin x; \cos x)$ bo'lsa, u holda, $z = \cos x$, $x \in (0; \pi)$ almashtirish bajarilsa, (8.10.2) integral ostidagi ifoda, z ning ratsional funksiyasiga keltiriladi.

3) $R(\sin x; -\cos x) = -R(\sin x; \cos x)$ bo'lsa, u holda, $z = \sin x$, $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ almashtirish bajarilsa, (8.10.2) integral ostidagi ifoda, z ning ratsional funksiyasiga keltiriladi.

4) $R(-\sin x; -\cos x) = R(\sin x; \cos x)$ bo'lsa, u holda, $z = tgx$, $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ yoki $z = \cos 2x$ almashtirishlardan biri bajarilsa, (8.10.2) integral ostidagi ifoda, z ning ratsional funksiyasiga keltiriladi.

8.10.5. $\int \sin\alpha x \cos\beta x\, dx$, $\int \sin\alpha x \sin\beta x\, dx$, $\int \cos\alpha x \cos\beta x\, dx$ **ko'rinishidagi integrallarni hisoblash.** Bu integrallar ostidagi ifodalarda, quyidagi

$$\sin\alpha x \cos\beta x = \frac{1}{2}[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x],$$
$$\sin\alpha x \sin\beta x = \frac{1}{2}[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x],$$
$$\cos\alpha x \cos\beta x = \frac{1}{2}[\cos(\alpha+\beta)x + \cos(\alpha-\beta)x]$$

formulalardan foydalanib, ularni integrallash mumkin.

8.10.6. $\int \sin^m x \cos^n x dx$ $(n, m \in Z)$ **ko'rinishdagi integrallarni hisoblash.**

I. n, m - lar manfiy bo'lmagan juft sonlar bo'lgan hol. Bu holda, darajani pasaytirish, ya'ni

$$\sin^2 x = \frac{1}{2}(1-\cos 2x),\ \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

formulalar qo'llaniladi.

II. n, m - lar natural sonlar bo'lib, hech bo'lmaganda ularning birortasi toq bo'lgan hol. Bu holda, misol uchun, m toq bo'lganda, $\sin x = t$; n toq bo'lganda esa, $\cos x = t$ almashtirish olinadi va $1-\cos^2 x = \sin^2 x$ yoki $1-\sin^2 x = \cos^2 x$ formulalarning biridan foydalanishga to'g'ri keladi.

III. n, m - larning ikkalasi ham butun manfiy bo'lib, $|n|$ va $|m|$ sonlar juft yoki toq bo'lgan hol. Bu holda $tgx = t$ yoki $ctgx = t$ almashtirish olinib,

$$1 + tg^2 x = \frac{1}{\cos^2 x},\ 1 + ctg^2 x = \frac{1}{\sin^2 x}$$

formulalarning biridan foydalaniladi.

IV. n, m - butun manfiy sonlar bo'lib, $|n|$ va $|m|$ sonlarning biri toq bo'lgan hol. U holda, agar $|m|$ - toq bo'lganda, $\sin x = t$, $|n|$- toq bo'lsa esa, $\cos x = t$ almashtirish olish maqsadga muvofik bo'ladi. Ba'zi hollarda $|n|$ va $|m|$ larning darajalari katta bo'lganda integral ostidagi funksiyaning suratida 1 ni $\sin^2 x + \cos^2 x$ yig'indi bilan almashtirish qulay bo'ladi.

V. n - juft son, m - esa butun manfiy son bo'lgan hol. Bu holda $\sin^2 x = 1-\cos^2 x$ formuladan foydalanib, integral

$$\int \frac{dx}{\cos^\alpha x},\ \alpha \in N$$

ko'rinishdagi integralga keltiriladi.

m- juft son, n- esa, butun manfiy bo'lganda $\cos^2 x = 1-\sin^2 x$ formuladan foydalanib, hisoblanishi kerak bo'lgan integral,

$$\int \frac{dx}{\sin^\alpha x},\ \alpha \in N$$

ko'rinishdagi integralga keltiriladi.

VI. n - toq son, m - esa, butun manfiy son bo'lsa, bu holda $\cos x = t$ almashtirish olib, $\sin^2 x = 1-\cos^2 x$ formuladan foydalanish kerak.

m - toq, n - esa, butun manfiy son bo'lganda $\sin x = t$ almashtirish olib, $\cos^2 x = 1-\sin^2 x$ formuladan foydalanish kerak.

Ba'zi hollarda, ya'ni $|n|$ va $|m|$ darajalar yetarli katta bo'lgan integral ostidagi funksiyaning suratidagi 1 ni $\sin^2 x+\cos^2 x$ ga almashtirish qulay bo'ladi.

8.10.7. Misol. Quyidagi integralni hisoblang:
$\int \sin^4 x\cos^2 x dx$.

Yechilishi. Bu yerda $n=4,\ m=2$ bo'lgani uchun, 1 - holga asosan, $\sin^2 x=\frac{1}{2}(1-\cos 2x)$ formuladan foydalanib, integralni hisoblaymiz:

$$\int \sin^4 x\cos^2 x dx=\frac{1}{4}\int \sin^2 x\cdot\sin^2 2x dx=\frac{1}{4}\int\frac{1-\cos 2x}{2}\cdot\frac{1-\cos 4x}{2}dx=$$

$$=\frac{1}{16}\int(1-\cos 4x-\cos 2x+\cos 2x\cos 4x)dx=$$

$$=\frac{1}{16}x-\frac{1}{64}\sin 4x-\frac{1}{32}\sin 2x+\frac{1}{16}\int(\cos 2x+\cos 6x)dx=$$

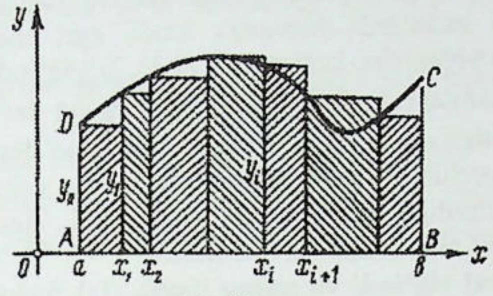
$$=\frac{1}{16}x-\frac{1}{64}\sin 2x-\frac{1}{64}\sin 4x+\frac{1}{192}\sin 6x+C.$$

8.3-§. Aniq integral. Aniq integralning ta'rifi. Darbu yig'indilari. Aniq integralning mavjudligi. Integrallanuvchi funksiyalarning sinflari

8.11. Egri chiziqli trapesiyaning yuzi. Ko'pgina muhim fizik, mexanik va geometrik masalalarning matematik ifodasi aniq integral tushunchasiga olib kelinadi. Masalan, Oy o'q bo'ylab harakat qilayotgan material nuqtaning tezligi ma'lum bo'lganda, uning biror vaqt oralig'idagi o'tgan yo'lini topish, egri chiziqli trapesiyaning (ya'ni $y=f(x)$ funksiya grafigi, $x=a, x=b$ vertikal to'g'ri chiziqlar va Ox o'q bilan chegaralangan figura) yuzini topish kabi masalalar, shular jumlasiga kiradi.

Ana shu masalalardan biri bo'lgan *egri chiziqli trapesiyaning* yuzini topish masalasini qaraymiz:

$f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lsin. Qulaylik uchun $\forall x\in[a,b]$ da $f(x)\geq 0$ deb faraz qilamiz. $y=f(x)$ funksiya grafigi, abssissalar (Ox) o'qi hamda ikkita $x=a$ va $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan figurani (sohani) T orqali belgilaymiz (8.1-chizma):



8.1-chizma.

$$T=\{(x,y)\in R^2: 0\leq y\leq f(x), a\leq x\leq b\}. \tag{8.11.1}$$

Odatda, T figura *egri chiziqli trapesiya* deb yuritiladi.

T figuraning $S=S(T)$ yuzini topish uchun $[a,b]$ kesmani ixtiyoriy $a=x_0<x_1<x_2<...<x_n=b$ nuqtalar yordamida, $[x_{k-1},x_k],(k=1,2,...n)$ bo'laklarga bo'lamiz. Natijada, T trapesiya

$$T_k=\{(x,y)\in R^2: 0\leq y\leq f(x), x_{k-1}\leq x\leq x_k\}$$

kichik egri chiziqli trapesiyachalarga bo'linadi.

Agar $x_k-x_{k-1}=\Delta x_k$ deb belgilasak, u holda, T_k -trapesiyachaning yuzini taqribiy ravishda to'g'ri to'rtburchakning yuzi $f(\xi_k)\Delta x_k$ ga almashtiramiz ya'ni $S_k=S(T_k)\approx f(\xi_k)\Delta x_k$ bo'ladi, bunda ξ_k, $[x_{k-1},x_k]$ kesmadan olingan ixtiyoriy nuqta.

Demak, egri chiziqli trapesiya yuzi taxminan

$$S\approx\sum_{k=1}^{n}f(\xi_k)\Delta x_k \tag{8.11.2}$$

bo'ladi.

Agar $[x_{k-1},x_k]$ kesmaning uzunligini kamaytirib (buning uchun bo'linish nuqtalari soni n ni ortirib) borsak, natijada, Δx_k nolga intila boradi. U holda (8.11.2) yig'indi miqdorning egri chiziqli trapesiyaning S yuziga teng bo'linishining aniqlik darajasi ortadi.

Biz, egri chiziqli trapesiya yuzining qattiy ta'rifiga ega emasligimizni e'tirof etamiz va yuqorida keltirilgan mulohazalarimizda bu yuzani ichki intuktiv his etganimizga suyandik, xolos (figura yuzi tushunchasining aniq ta'rifini keyingi bandlarda beramiz). Endi, egri chiziqli trapesiya yuzi tushunchasini quyidagicha keltiramiz. (8.11.2) yig'indining $[a,b]$ kesmani bo'lish va ξ_k nuqtani tanlash qonuniga bog'liq bo'lmagan, Δx_k nolga intilgandagi limitiga *egri chiziqli trapesiyaning yuzi* deymiz. Umuman, ko'p masalalrning yechimi yuqorida keltirilgan yig'indilarga o'xshash yig'indilarning limitini topishga bog'liq bo'ladi. Bunday yig'indilarning limiti aniq integral tushunchasiga olib keladi.

8.12. Integral yig'indi va uning limiti. $f(x)$ funksiya $[a,b]$ $([a,b]\subset R)$ kesmada aniqlngan bo'lsin. $[a,b]$ kesmani ushbu

$$a=x_0<x_1<x_2<...<x_n=b$$

munosobatda bo'lgan ixtiyoriy chekli sondagi $x_0,x_1,...,x_n$ nuqtalar yordamida bo'laklrga bo'lamiz. Bu bo'linishni $[a,b]$ kesmaning P *bo'linishi* deb ataymiz va uni $P=\{x_k\}_{k=1}^{n}=\{a=x_0<x_1<x_2<...<x_n=b\}$ kabi belgilaymiz, bunda x_k $(k=0,1,2,...,n)$ nuqtlar P bo'linishning bo'luvchi nuqtalri, $[x_k,x_{k+1}]$ kesmani esa, P bo'linish orolig'i, $\lambda_p=\max\{\Delta x_k\}$ ni P bo'linishning *diametri* deb ataymiz.

Agar $[a,b]$ kesmaning ixtiyoriy bo'linishidagi bo'linishlar oralig'ining uzunliklari bir xil bo'lsa, u holda bunday bo'linish $[a,b]$ kesmaning *regulyar bo'linishi* deyiladi.

Bo'linishlar to'plamini $\Omega=\{P\}$ deb belgilaymiz. $\forall P\in\Omega$ ni va bu bo'linishning har bir $[x_{k-1},x_k],(k=1,2,...n-1)$ bo'lagidan ixtiyoriy ξ_k $(x_{k-1}\le\xi_k\le x_k$ nuqtani tanlab, $f(x)$ funksiyaning bu nuqtadagi $f(\xi_k)$ qiymatini $\Delta x_k=x_k-x_{k-1}$ ga ko'paytirib, ushbu

$$\sigma_P(f)=\sum_{k=0}^{n-1}f(\xi_k)\Delta x_k \qquad (8.12.1)$$

yig'indini tuzamiz.

8.12.2-ta'rif. Ushbu

$$\sigma_P(f)=\sum_{k=0}^{n-1}f(\xi_k)\Delta x_k \qquad (8.12.3)$$

yig'indi, $f(x)$ funksiyaning $[a,b]$ kesmadagi *integral (Riman) yig'indisi* deyiladi.

8.13. Aniq integralning ta'rifi. $f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo'lsin.

8.13.1-ta'rif. Agar $\forall\varepsilon>0$ olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilib, $[a,b]$ kesmaning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'linishi uchun tuzilgan $\sigma_P(f)$ yig'indi ixtiyoriy tanlangan ξ_k nuqtalarda:

$$|\sigma_P(f)-I|<\varepsilon \qquad (8.13.2)$$

tengsizlikni qanoatlantirsa, u holda I son σ_P integral yig'indining $\lambda_P\to 0$ dagi *limiti* deyiladi va u

$$\mathrm{I}=\lim_{\lambda_P\to 0}\sigma_P(f)=\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1}f(\xi_k)\Delta x_k$$

kabi yoziladi.

8.13.3-ta'rif. Agar $\lambda_P\to 0$ da $f(x)$ funksiyaning (8.12.3) Riman integral yig'indisi chekli limitga ega bo'lsa, u holda $f(x)$ funksiya $[a,b]$ kesmada *integrallanuvchi* deyiladi.

Agar $\lambda_P\to 0$ da yig'indining limiti mavjud bo'lmasa yoki cheksiz bo'lsa, u holda $f(x)$ funksiya $[a,b]$ kesmada *integrallanmaydi* deyiladi.

$\sigma_P(f)$-integral yig'indining limiti I ga $f(x)$ funksiyaning $[a,b]$ kesma bo'yicha olingan *aniq integrali* deyiladi va u

$$I=\int_a^b f(x)dx \qquad (8.13.4)$$

kabi belgilanadi (integral a dan b gacha yef iks de iks deb o'qiladi), bunda $f(x)$ funksiya integral ostidagi funksiya, a son integralning *quyi*, b son esa, *yuqori chegarasi* deb ataladi.

8.13.5-eslatma. $f(x)$ funksiyaning $[a,b]$ kesmada Riman ma'nosida integrallanuvchi bo'linishi uchun uning ixtiyoriy Riman integral yig'indisi

$$\sigma_P(f)=\int_a^b f(x)dx+\alpha_P(f) \qquad (8.13.6)$$

shartni qanoatlantirishi zarur va yetarli. Bunda $\lambda_P\to 0$ da $\alpha_P(f)\to 0$.

8.14. Nyuton-Leybnis formulasi[1]. Biz bu bandda differensial hisobi bilan integral hisobini bog'lovchi integral hisobining *asosiy formulasi* deb ataluvchi formulani aniq integralning ta'rifidan foydalanib isbot qilamiz. Bu formula matematik adabiyotlarda differensial va integral hisobini yaratishda ko'p hissalarini qo'shgan buyuk olimlar Nyuton va Leybnislar nomlari bilan ataladi.

8.14.1-teorema. $f(x)$ funksiya $[a,b]$ kesmada Riman ma'nosida integrallanuvchi, $F(x)$ funksiya esa, $[a,b]$ kesmada uzluksiz va kesmaning ichki nuqtalarida differensiallanuvchi hamda

$$F'(x)=f(x),\ a<x<b \tag{8.14.2}$$

tenglik o'rinli bo'lsin, u holda ushbu [1]

$$\int_a^b f(x)dx = F(b)-F(a) \tag{8.14.3}$$

tenglik o'rinli bo'ladi.

Isboti. $[a,b]$ kesmaning ixtiyoriy $P=\{a=x_0<x_1<...<x_n=b\}$ bo'linishini olib, uning ixtiyoriy qismiy oralig'i $[x_{k-1},x_k]$ uchun $F(x)$ funksiyaga ushbu $f(b)-f(a)=f'(c)(b-a)$ Lagraj formulasini qo'laymiz:

$$F(x_k)-F(x_{k-1})=F'(\xi_k)\Delta x_k\ ,\ (x_{k-1}\le\xi_k\le x_k)$$

(8.14.2) tenglikni e'tiborga olib, keyingi tenglikni ushbu

$$F(x_k)-F(x_{k-1})=f(\xi_k)\Delta x_k \tag{8.14.4}$$

ko'rinishda yozib olamiz. (8.14.4) tenglikning ikkala tomonidan k bo'yicha 1 dan n gacha yig'indi olamiz va ba'zi hisoblashlarni bajargandan keyin

$$F(b)-F(a)=\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})]=\sum_{k=1}^{n}f(\xi_k)\Delta x_k \tag{8.14.5}$$

[1] Алимов Ш.О., Ашуров Р.Р. Математик тахлил, 1-қ.. Т. 2008.

tenglikni hosil qilamiz. Bu tenglikning chap tomoni $[a,b]$ kesmani bo'lishga, ya'ni n ga bog'liq emas, o'ng tomoni esa $f(x)$ ning $[a,b]$ kesma bo'yicha olingan integral yig'indini ifodalaydi. Bu holda (8.14.5) tenglikda limitga o'tsak, natijada isbot qilinishi kerak bo'lgan (8.14.3) formulani hosil qilamiz. ■

8.14.6-eslatma. Ko'p hollarda integral hisobining asosiy formulasi qisqacha

$$\int_a^b f(x)dx = F(x)\Big|_{x=a}^{x=b} = F(x)\Big|_a^b = F(b)-f(a) \tag{8.14.7}$$

ko'rinishda yoziladi.

8.14.8-misol. $f(x)=C=const$ funksiya ixtiyoriy $[a,b]$ kesmada integrallanuvchi bo'ladi.

Haqiqatdan ham, $[a,b]$ kesmanimg ixtiyoriy $P=\{x_k\}_{k=1}^{n}$ bo'linishi va $\forall\xi_k\in[x_{k-1},x_k]$ nuqtada $f(\xi_k)=C$ bo'ladi. U holda integral yig'indi

$$\sigma_P(f,\{\xi_k\})=C\cdot\Delta x_1+C\cdot\Delta x_2+...+C\cdot\Delta x_k=C(b-a)$$

bo'ladi. Bu tenglikdan

$$\lim_{\lambda_P\to 0}\sigma_P(f,\{\xi_k\})=C(b-a)$$

bo'ldi. Shunday qilib aniq integralning ta'rifiga ko'ra,

$$\lim_{\lambda_P\to 0}\sigma_P(f,\{\xi_k\})=\int_a^b Cdx=C(b-a). \tag{8.14.9}$$

Xususiy holda, $f(x)=1$ bo'lganda, quyidagiga ega bo'lamiz:

$$\int_a^b 1.dx=\int_a^b dx=b-a.$$

8.14.10-teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada Riman ma'nosida integrallnuvchi bo'lsa, u holda $f(x)$ funksiya $[a,b]$ kesmada chegaralangan bo'ladi.

8.14.11-teorema. $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lib, $c\in[a,b]$ bo'lsin. U holda ushbu

$$g(x)=\begin{cases} f(x), & agar \ \ x\neq c \ \ bo'lganda, \\ \mu, \ agar \ \ x=c \ \ bo'lganda, \mu - ixtiyoriy \ \ haqiqiy \ \ son \end{cases}$$

funksiya ham $[a,b]$ kesmada integrallanuvchi va

$$\int_a^b g(x)dx=\int_a^b f(x)dx \qquad (8.14.12)$$

tenglik o'rinli bo'ladi.

8.14.13-eslatma. Agar $[a,b]$ kesmada integralanuvchi funksiyaning shu kesmaining ixtiyoriy chekli sondagi nuqtalaridagi qiymati o'zgartirilsa, u holda yangi hosil bo'lgan funksiya shu kesmada integrallanuvchi bo'ladi va bu funksiyalardan olingan integralning qiymatlari bir-biriga teng bo'ladi.

Dirixle funksiyasi sanoqli sondagi nuqtalar (ratsional nuqtalar) da noldan farqli bo'lib, uning integrallanuvchi emas. Bundan ko'rinadiki, agar integrallanuvchi funksiyaning sanoqli sondagi nuqtalarda qiymatini o'zgartirganda hosil bo'lgan funksiya integrallanuvchi bo'lmasligi ham mumkin ekan.

8.15. Darbu yig'indilari. $f(x)$ funksiya $[a,b]$ kesmada aniqlangan va chegaralangan bo'lsin. $[a,b]$ kesmaning ixtiyoriy $P=\{a=x_0<x_1<...<x_n=b\}$ bo'linishini qaraymiz. $f(x)$ funksiyaning $[x_{k-1},x_k]$ kesmadagi aniq quyi va aniq yuqori chegaralarini mos ravishda m_k va M_k deb belgilaymiz (shartga ko'ra $f(x)$ chegaralangan bo'lganligi uchun uning quyi va yuqori chegarasi mavjud). Ushbu

$$S_P(f)=M_1\Delta x_1+M_1\Delta x_1+...+M_n\Delta x_n=\sum_{k=1}^{n}M_k\Delta x_k,$$

$$s_P(f)=m_1\Delta x_1+m_1\Delta x_1+...+m_n\Delta x_n=\sum_{k=1}^{n}m_k\Delta x_k, \ \Delta x=x_k-x_{k-1}$$

yig'indilarni mos ravishda P bo'linish bo'yicha $f(x)$ funksiyaning *yuqori va quyi Darbu yig'indilari* deyiladi.

Darbu yig'indilarining xossalari. Darbu yig'indilari quyidagi xossalarga ega:

1^0. $[a,b]$ kesmaning bo'luvchi nuqtalariga yangi bo'linish nuqtasi qo'shish bilan Darbuning quyi yig'indisi kamaymaydi, yuqori yig'indisi esa ortmaydi.

2^0. $[a,b]$ kesmaning $\forall P_1,P_2\in\Omega$ bo'linishlariga nisbatan mos Darbuning har qanday quyi yig'indisi, har qanday yuqori yig'indisidan oshmaydi, ya'ni $s_{P_2}(f)\le S_{P_2}(f)$ tengsizlik o'rinli bo'ladi.

8.15.1-ta'rif. $\{s_p(f)\}$ to'plamning aniq yuqori chegarasi J_* ga $f(x)$ funksiyaning $[a,b]$ oraliqdagi *quyi integrali (quyi Riman integrali)* deb ataladi va u

$$J_*=\underline{\int_a^b} f(x)dx$$

kabi belgilanadi.

$\{S_p(f)\}$ to'plamning aniq quyi chegarasi J^* ga $f(x)$ funksiyaning $[a,b]$ oraliqdagi yuqori integrali (yuqori Riman integrali) deb ataladi va u

$$J^*=\overline{\int_a^b} f(x)dx$$

kabi belgilanadi.

Demak,

$$J_*=\underline{\int_a^b} f(x)dx=\sup_P\{s_P(f)\}, \ J^*=\overline{\int_a^b} f(x)dx=\inf_P\{S_P(f)\}.$$

Shuni ta'kidlaymizki, $[a,b]$ kesmada chegaralangan har qanday funksiyaning quyi va yuqori integrallari mavjud.

8.15.2-ta'rif. Agar $f(x)$ funksiyaning $[a,b]$ kesmadagi quyi va yuqori integrallari bir-biriga teng bo'lsa, u holda $f(x)$ funksiya $[a,b]$ da *Darbu ma'nosida integrallanuvchi* deyiladi, ularning umumiy qiymati

$$J=\underline{\int_a^b} f(x)dx=\overline{\int_a^b} f(x)dx$$

ga $f(x)$ funksiyaning $[a,b]$ kesmadagi Darbu ma'nosidagi aniq integrali deyiladi.

8.16. Aniq integralning mavjudlik sharti. Aniq integralning mavjud yoki mavjud emasligi, ya'ni funksiyaning integrallanuvchi bo'linishi yoki bo'lmasligini 8.13.1-ta'rif bo'yicha aniqlash mumkin. Lekin ko'pchilik hollarda integral yig'indining chekli limitga ega yoki ega emasligini ko'rsatish, shuningdek yuqori va quyi integrallarni topish har doim oson bo'lavermaydi. Bundan tashqari integral yig'indining limiti tushunchasi, o'ziga xos murakkab xarakterga ega bo'lgan tushunchadir.

8.16.1-teorema (Aniq integralning mavjudlik sharti). $[a,b]$ oraliqda chegaralangan $f(x)$ funksiya integrallanuvchi bo'linishi uchun $\forall\varepsilon>0$ son olinganda shunday $\delta=\delta(\varepsilon)>0$ son topilib, $[a,b]$ oraliqning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'linishiga nisbatan Darbu yig'indilari

$$S_P(f)-s_P(f)<\varepsilon \qquad (8.16.2)$$

tengsizlikni qanoatlantirishi zarur va yetarli.

Funksiyaning integrallanuvchi bo'lishligining zaruriy va yetarli shartini quyidagicha ham ifodalash mumkin:

8.16.3-teorema. $f(x)$ funksiyaning $[a,b]$ oraliqda integrallanuvchi bo'linishi uchun $\forall\varepsilon>0$ son olinganda ham $\exists\delta>0$ son topilib, $[a,b]$ oraliqning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'linishida

$$\sum_{k=0}^{n-1}\omega_k\Delta x_k<\varepsilon \qquad (8.8.5)$$

shartning bajarilishi zarur va yetarlidir.

8.17. Integrallanuvchi funksiyalarning sinflari. Bu 8.9-bandda $[a,b]$ kesmada uzluksiz, ba'zi uzilishga ega va monoton bo'lgan funksiyalarning integrallanuvchi bo'linishini keltiramiz. $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lsin.

8.17.1-teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'ladi.

8.17.2-teorema.Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va bu oraliqning chekli sondagi nuqtalarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, u holda u shu oraliqda integrallanuvchi bo'ladi.

8.17.3-teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada monoton va chegaralangan bo'lsa, $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'ladi.

8.4-§. Aniq integralning xossalari. Aniq integralni hisoblash

8.18. Aniq integralning asosiy xossalari. 1. *Tenglik bilan ifoda qilinadigan xossalari.* Aniq integralning ta'rifidan kelib chiqadigan quyidagi xossalarni keltiramiz:

1^0. $\int_a^a f(x)dx=0$ deb kelishib olamiz.

Bu xossani uzunligi nolga teng bo'lgan kesma bo'yicha olingan aniq integral deb qaraymiz.

2^0. $a<b$ bo'lganda $\int_b^a f(x)dx=-\int_a^b f(x)dx$ deb olamiz.

Buni $a<b$ bo'lganda, $[a,b]$ kesmada yo'nalish a dan b ga qarab yo'nalgandagi aniq integral tushunchasini yo'nalish $b$dan a ga qarab yo'nalganda $[a,b]$ kesma bo'yicha olingan aniq integral tushunchasiga tatbig'i deb tushunish kerak.

3^0. $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo'lsin. U holda ixtiyoriy λ va μ haqiqiy sonlar uchun $\lambda f(x)\pm\mu g(x)$ funksiya ham $[a,b]$ kesmada integrallanuvchi va

$$\int_a^b[\lambda f(x)\pm\mu g(x)]dx=\lambda\int_a^b f(x)dx+\mu\int_a^b g(x)\,dx$$

tenglik o'rinli bo'ladi.

Odatda 3^0 – xossani Riman integralining *chiziqlilik* xossasi deb yuritiladi.

Natija. Agar $f_1(x), f_2(x),....,f_n(x)$ funksiyalar $[a,b]$ da integrallanuvchi bo'lsa, u holda, $c_1f_1(x)+...+c_nf_n(x)$ $(c_k=const,\ k=1,2,....,n)$ funksiya ham integrallanuvchi va ushbu

$$\int_a^b[c_1f_1(x)+...+c_nf_n(x)]dx=c_1\int_a^b f_1(x)dx+...+c_n\int_a^b f_n(x)dx$$

tenglik o'rinli bo'ladi.

4^0. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integralllanuvchi bo'lsa, u holda $f(x)\cdot g(x)$ funksiya ham shu oraliqda integralllanuvchi bo'ladi.

Natija. Agar $f(x)$ funksiya $[a,b]$ kesmada integralllanuvchi bo'lsa, u holda $\forall n \in N$ uchun $[f(x)]^n$ funksiya ham shu oraliqda integralllanuvchi bo'ladi.

5^0. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi va $|f(x)|$ funksiyaning $[a,b]$ dagi aniq quyi chegarasi musbat bo'lsa, u holda $\frac{1}{f(x)}$ funksiya ham $[a,b]$ da integrallanuvchi bo'ladi.

Natija. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, va $|g|$ funksiyaning aniq quyi chegarasi musbat bo'lsa, u holda $\frac{f(x)}{g(x)}$ ham $[a,b]$ da integrallanuvchi bo'ladi.

6^0. Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda, u istalgan $[\alpha,\beta]\subset[a,b]$ oraliqda ham integrallanuvchi bo'ladi.

7^0. Agar $f(x)$ funksiya $[a,c]$ va $[c,b]$ kesmalarda integrallanuvchi bo'lsa, u holda bu funksiya $[a,b]$kesmada ham integrallanuvchi va

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik o'rinli bo'ladi.

2. Tengsizliklar bilan ifoda qilinadigan xossalar

8^0. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi va $f(x)\geq 0$ $\forall x\in[a,b]$ uchun bo'lsa, u holda

$$\int_a^b f(x)dx \geq 0$$

bo'ladi.

Natija. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi va $\forall x\in[a,b]$ uchun $f(x)\geq g(x)$ bo'lsa, u holda

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

tengsizlik o'rinli bo'ladi.

9^0. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi va $a<b$ bo'lsa, u holda $|f(x)|$ funksiya ham shu oraliqda integrallanuvchi va

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

tengsizlik o'rinli bo'ladi.

3. O'rta qiymat haqidagi teoremalar.

10^0. Agar $f(x)$ funksiya $[a,b]\,(a<b)$ oraliqda integrallanuvchi va $\forall x\in[a,b]$ lar uchun $m\leq f(x)\leq M$ bo'lsa, u holda

$$m(b-a)\leq \int_a^b f(x)dx \leq M(b-a)$$

tengsizlik o'rinli bo'ladi, bunda $m = \inf\limits_{x\in[a,b]} \{f(x)\}, M = \sup\limits_{x\in[a,b]} \{f(x)\}$.

11^0. **Birinchi o'rta qiymat haqidagi teorema.** Agar $f(x)$ funksiya $[a,b]$ $(a\leq b)$ oraliqda integrallanuvchi va $\forall x\in[a,b]$ lar uchun $m\leq f(x)\leq M$ bo'lsa, u holda

$$\int_a^b f(x)\,dx = \mu(b-a),$$

tenglik o'rinli bo'ladi, bunda $m\leq\mu\leq M$.

Natija. Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda bu oraliqda shunday c $(c\in[a,b])$ nuqta topiladiki.

$$\int_a^b f(x)dx = f(c)(b-a)$$

tenglik o'rinli bo'ladi.

12^0. **Umumlashgan o'rta qiymat haqidagi teorema**. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi, m va M lar $f(x)$ ning $[a,b]$ dagi aniq chegaralari bo'lib, hamda $g(x)\geq 0$ yoki $g(x)\leq 0$, $x\in[a,b]$ bo'lsa, u holda shunday $\mu(m\leq\mu\leq M)$ son majud bo'lib,

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

tenglik o'rinli bo'ladi.

Natija. Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz, $g(x)$ esa, $[a,b]$ da integrallanuvchi bo'lsa, hamda $g(x)$ funksiya o'z ishorasini o'zgartirmasa, u holda shunday $c(c \in [a,b])$ nuqta topiladiki

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \qquad (8.18.1)$$

tenglik o'rinli bo'ladi.

Odatda (8.18.1) formulaga umumlashgan o'rta qiymat haqidagi formula deb yuritiladi.

8.19. Chegaralari o'zgaruvchi bo'lgan aniq integrallar. $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. U holda aniq integralning 6^0 -xossasiga asosan, u ixtiyoriy $[a,x](a \le x \le b)$ oraliqda ham integrallanuvchi bo'ladi, ya'ni $\int_a^x f(t)dt$ – integral ma'noga ega bo'ladi. Ushbu

$$F(x) = \int_a^x f(t)dt \qquad (8.19.1)$$

funksiyani qaraymiz. Bu $F(x) = \int_a^x f(t)dt$ funksiya $[a,b]$ oraliqda aniqlangan va u yuqori *chegaraning funksiyas*i deyiladi. Uning quyidagi asosiy xossalarini keltiramiz.

8.19.2. Teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $F(x) = \int_a^x f(t)dt$ funksiya $[a,b]$ oraliqda uzluksiz bo'ladi.

8.19.3. Teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi, u $x_0 \in [a,b]$ nuqtada uzluksiz bo'lsa, u holda $F(x) = \int_a^x f(t)dt$ funksiya x_0 nuqtada differensiallanuvchi va $F'(x_0) = f(x_0)$ bo'ladi.

8.19.4. Teorema. Agar $f(x)$ funksiya biror oraliqda uzluksiz bo'lsa, u holda shu oraliqda bu funksiya uchun boshlang'ich funksiya mavjud bo'ladi.

Bu teoremadan quyidagi xulosani keltirib chiqarish mumkin: chegarasi o'zgaruvchi bo'lgan integral, integrallanuvchi funksiyani uzluksiz funksiyaga, uzluksiz funksiyani esa, differensiallanuvchi funksiyaga o'tkazar ekan. Ma'lumki, differensiallash amali qandaydir ma'noda funksiyaning xususiyatini «yomonlashtiradi» Masalan, uzluksiz

funksiyaning hosilasi (agar u mavjud bo'lsa), uzilishga ega bo'linishi ham mumkin.

7.19.5. Eslatma. (8.19.1) formuladan, quyi chegara bo'yicha aniq integralni differensiallash formulasini keltirib chiqish mumkin: f funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin, u holda, shu oraliqda $G(x) = \int_x^b f(t)dt,\ a \le x \le b$ funksiya aniqlangan bo'ladi.

$$\int_a^b f(t)dt = \int_a^x f(t)dt + \int_x^b f(t)dt$$

ayniyatdan

$$G(x) = \int_a^b f(t)dt - F(x) \qquad (8.19.6)$$

Agar $f(x)$ funksiya $x \in [a,b]$ nuqtada uzluksiz bo'lsa, u holda **8.19.3**– teoremaga asosan, $F(x)$ funksiya x nuqtada differensiallanuvchi bo'ladi. Bu holda (8.19.6) dan $G(x)$ funksiya ham differensiallanuvchi bo'ladi va $\frac{d}{dx}G(t) = -\frac{d}{dx}F(x)$.

Shunday qilib, $\frac{d}{dx}\int_x^b f(x)dt = -f(x)$.

8.20. Aniq integralni hisoblash usullari. Biz yuqoridagi 8.3.3– ta'rifda funksiyaning $[a,b]$ oraliqdagi aniq integralini yig'indining chekli limiti sifatida ta'riflangan edik. Lekin integral yig'indining limiti tushunchasi murakkab xarakterga ega bo'lib, uni hisoblashda ancha noqulayliklar va qiyinchiliklarga duch kelinadi. Shuning uchun, aniq integralni ta'rif bo'yicha hisoblash usulidan boshqa soddaroq, qulayroq usullarni topish zarruriyati tug'iladi. Bu usullarni quyida keltirib o'tamiz.

8.20.1. Integral hisobining asosiy formulasi (N'yuton-Leybnis formulasi). Funksiyalarning aniq integrallarini hisoblashda eng qulay va keng qo'llaniladigan formula, integral hisobining asosiy formulasi, ya'ni *N'yuton Leybnis formulasi* bo'lib hisoblanadi. Bu formulani biz yuqorida 8.4-bandda aniq integralning ta'rifidan foydalanib topgan edik. Bu formulani aniq integralning xossalaridan foydalanib keltirib chiqarsa ham bo'ladi.

Haqiqatan ham, agar $f(x)$ funksiya $[a,b]$ da uzluksiz bo'lsa, u holda u shu oraliqda boshlang'ich funksiyaga ega bo'ladi. Yuqoridagi 8.14.1–

teoremaga asosan, $F(x)=\int_a^x f(t)dt$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladi. $\Phi(x)$ funksiya, $f(x)$ ning $[a,b]$ dagi ixtiyoriy boshlang'ich funksiyasi bo'lsin. Ma'lumki, $F(x)$ va $\Phi(x)$ boshlang'ich funksiyalar biri ikkinchisidan o'zgarmas songa farq qiladi, ya'ni

$$\int_a^x f(x)dt=\Phi(x)+C,\ a\le x\le b$$

Bunda, $x=a$ deb olib, $0=\Phi(a)+C$, $C=-\Phi(a)$ ekanligini topamiz, so'ngra $x=b$ deb

$$\int_a^b f(x)dx=\Phi(b)-\Phi(a)=\Phi(x)\Big|_a^b \qquad (8.20.2)$$

formulani topamiz. Bu (8.20.2) formulaga ham *N'yuton – Leybnis* formulasi deb ataladi.

8.20.3-misol. Ushbu $\int_a^b x^m dx,\ m\neq -1$ integralni Nyuton – Leybnis formulasi orqali hisoblang.

Yechilishi. Ma'lumki, integral ostidagi $f(x)=x^m$ funksiyaning boshlang'ich funksiyasi, $F(x)=\frac{x^{m+1}}{m+1}$ dan iborat. Nyuton – Leybnis formulasiga asosan,

$$\int_a^b x^m dx=\frac{x^{m+1}}{m+1}\Big|_a^b=\frac{b^{m+1}-a^{m+1}}{m+1},\ m\neq -1$$

bo'ladi. Xususiy holda, $m=-1$ bo'lganda,

$$\int_a^b \frac{dx}{x}=\ln|x|\Big|_a^b=\ln|b|-\ln|a|\ .\blacksquare$$

Shunday qilib, uzluksiz $f(x)$ funksiyaning aniq integralini hisoblash masalasi, integral ostidagi integrallanuvchi funksiyaning boshlang'ich

funksiyasini topish masalasiga keltirilar ekan. Lekin, har qanday integrallanuvchi funksiyaning boshlang'ich funksiyasini topish oson bo'lavermaydi. Shuning uchun, aniq integralni hisoblashda, boshqa usullardan foydalanishga to'g'ri keladi.

8.20.4. Aniq integrallarda o'zgaruvchilarni almashtirish usuli.

8.20.5-teorema. $x=g(t)$ funksiya $[\alpha,\beta]$ oraliqda aniqlangan uzluksiz differensiallanuvchi va uning qiymatlar to'plami $[a,b]$ oraliqdan iborat bo'lib, $g(\alpha)=a$, $g(\beta)=b$ bo'lsin. Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda

$$\int_a^b f(x)dx=\int_\alpha^\beta f[g(t)]g'(t)dt \qquad (8.20.6)$$

formula o'rinli bo'ladi. Bu (8.20.6) formulaga *aniq integrallarda o'zgaruvchilarni almashtirish* formulasi deyiladi.

8.20.7-misol. Ushbu

$$\int_0^{\frac{\sqrt{2}}{2}}\sqrt{\frac{1-x}{1+x}}dt$$

integralni hisoblang.

Yechilishi. $x=\cos t$ almashtirishni olamiz. Bu almashtirishning to'g'riligini ko'rsatish uchun, 8.20.5–teoremaning shartlarini tekshirib ko'ramiz: $x=g(t)=\cos t$ funksiya R da uzluksiz, yangi t o'zgaruvchi, $t\in[\frac{\pi}{4};\frac{\pi}{2}]$ kesmada o'zgarganda, eski $x=g(t)$ o'zgaruvchi, $\left[0,\frac{\sqrt{2}}{2}\right]$ kesmada o'zgaradi, ya'ni $g\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, $g\left(\frac{\pi}{2}\right)=0$; $g'(t)=-\sin t$, $[\frac{\pi}{4};\frac{\pi}{2}]$ kesmada uzluksiz.

Demak, 8.20.5– teoremaning hamma shartlari bajariladi.

Shunday qilib, $x=\cos t,\ dx=-\sin t\,dt,\ g\left(\frac{\pi}{2}\right)=0$, ya'ni

$$\beta=\frac{\pi}{2},\ g\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2},\ \ \alpha=\frac{\pi}{4},\ \frac{1-x}{1+x}=\frac{1-\cos t}{1+\cos t}=tg^2\frac{t}{2}$$

Bularni e'tiborga olgan holda,

$$\int_0^{\frac{\sqrt{2}}{2}} \sqrt{\frac{1-x}{1+x}}dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} tg\frac{t}{2}\cdot(-\sin t)dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\cos t)dt =$$

$$=[t-\sin t]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2}-1-\frac{\pi}{4}+\frac{\sqrt{2}}{2} = \frac{\pi}{4}-1+\frac{\sqrt{2}}{2}. \blacksquare$$

8.20.8. Aniq integrallarni bo'laklab integrallash usuli

8.20.9-teorema. $u(x)$ va $v(x)$ funksiyalar $[a,b]$ kesmada differensiallanuvchi funksiyalar bo'lsin. U holda,

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]\big|_a^b - \int_a^b v(x)u'(x)dx \qquad (8.20.10)$$

formula o'rinli bo'ladi.

8.20.11-eslatma. $v'(x)dx = dv,\ u'(x)dx = du$ ekanligini e'tiborga olsak, u holda (8.20.12) formulani

$$\int_a^b udv = [uv]\big|_a^b - \int_a^b vdu \qquad (8.20.12)$$

ko'rinishda yozish ham mumkin.

Odatda (8.20.10) va (8.20.12) formulalarni aniq integrallarda *bo'laklab integrallash* formulalari deb yuritiladi.

8.20.13-misol. Ushbu

$$\int_0^{\sqrt{3}} x\, arctgx\, dx$$

integralni hisoblang.

Yechilishi. Integral ostidagi funksiyalar, 8.20.9– teoremaning barcha shartlarini qanoatlantirishiga ishonch hosil qilish qiyin emas. $u = arctgx,\ dv = xdx$ deb olib, (8.20.12) formulaga asosan,

$$\int_0^{\sqrt{3}} x\, arctg\, x\, dx = [du = \frac{dx}{1+x^2},\ v = \frac{x^2}{2}] =$$

$$= \frac{x^2}{2}arctgx\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2dx}{1+x^2}] = \frac{\pi}{3}\cdot\frac{3}{2} - \frac{1}{2}(x - arctgx)\Big|_0^{\sqrt{3}} =$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \blacksquare$$

8.21. Aniq integrallarni taqribiy hisoblash. Biz yuqorida integral ostidagi funksiyaning boshlang'ich funksiyasi ma'lum bo'lsa, aniq integralni N'yuton – Leybnis formulasi yordamida hisoblash mumkinligini ko'rdik. Lekin, integral ostidagi funksiya murakkab bo'lganda, uning boshlang'ich funksiyasini topish masalasi oson hal bo'lavermaydi. Bunday hollarda aniq integralni taqribiy hisoblash usullaridan foydalanib hisoblashga to'g'ri keladi. Bu usullar integral ostidagi funksiyani ko'phad bilan taqribiy almashtirishga, ya'ni berilgan funksiyani ko'p had bilan yaqinlashtirish haqidagi Veyershtrass teoremasiga asoslanadi.

Ushbu

$$\int_a^b f(x)dx \qquad (8.21.1)$$

integralni hisoblash talab qilingan bo'lsin, bunda $f(x)$ funksiya $[a,b]$ kesmada uzluksiz, deb faraz qilinadi. Biz, yuqorida, aniq integralni hisoblash usullarini ko'rib o'tdik, lekin ba'zi fizik, mexanik masalalarni yechishda, integral ostidagi funksiyaning boshlang'ich funksiyasini elementar funksiyalar orqali ifodalab bo'lmaydigan integrallar uchraydi. Bunday integrallarni taqribiy hisoblashga to'g'ri keladi. Integrallarni taqribiy hisoblash uchun bir nechta formulalar mavjud bo'lib, biz quyida ularning ba'zilari bilan tanishamiz.

8.21.2. To'g'ri to'rtburchaklar usuli. $[a,b]$ kesmaning, ixtiyoriy

$$P = \{a = x_0 < x_2 < ... < x_{2n} = b\}$$

regulyar bo'linishini qaraymiz. $[x_{2k-2}, x_{2k}]$ kesmaning o'rtasidagi nuqtani x_{2k-1} orqali belgilaymiz (8.2-chizma). To'g'ri to'rtburchak usuli, (8.21.1) integralni, mos ravishda, balandliklari $f(x_{2k-1})$, asoslari esa, $x_{2k} - x_{2k-2} = \frac{b-a}{n}$ ga teng bo'lgan, to'g'ri to'rtburchaklar yuzlarining,

$$\frac{b-a}{n}[f(x_1) + f(x_3) + ... + f(x_{2n-1})]$$

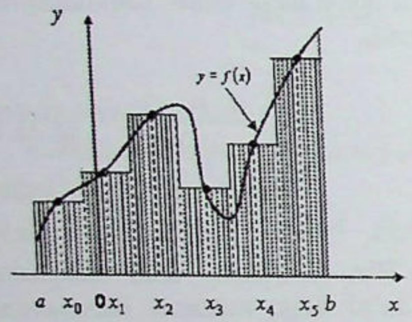
yig'indisiga taqribiy almashtirishdan iborat, ya'ni

$$\int_a^b f(x)dx \approx \frac{b-a}{n}[f(x_1)+f(x_3)+...+f(x_{2n-1})] \quad . \quad (8.21.3)$$

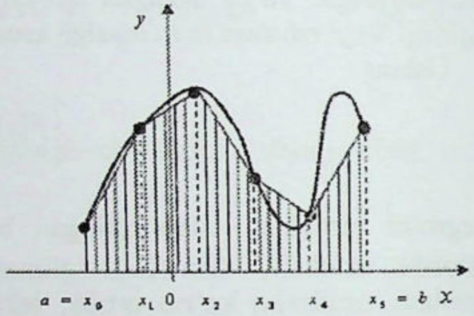
(8.21.2) formulaga, *to'g'ri to'rtburchak formulasi* deyiladi. Bu formulani, «qo'shimcha» had yordamida, ushbu

$$\int_a^b f(x)dx = \frac{b-a}{n}[f(x_1)+f(x_3)+....+f(x_{2n-1})]+R_n \quad (8.21.4)$$

ko'rinishda yozish mumkin, bunda R_n – qoldiq had.



8.2-chizma.



8.3-chizma.

Bu almashtirishda qo'yilgan xatoni hisoblash uchun berilgan $f(x)$ funksiyaga ba'zi bir talablarni qo'yishga to'g'ri keladi.

Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz ikkinchi tartibli hosilaga ega bo'lsa, u holda shu kesmada shunday η nuqta topiladiki, (8.21.4) formuladagi qoldiq had uchun,

$$R_n = \frac{(b-a)^3}{24n^2} f''(\eta)$$

tenlik o'rinli bo'ladi.

8.21.5. Trapesiyalar usuli. (8.21.1) integralni hisoblash talab qilingan bo'lsin. $[a,b]$ kesmaning ixtiyoriy $P = \{a = x_0 < x_2 < ... < x_n = b\}$ regulyar bo'linishini olamiz va $y = f(x)$ funksiyaning $y_0 = f(x_0)$, $y_1 = f(x_1)$, ..., $y_n = f(x_n)$qiymatlarini hisoblaymiz. Trapesiyalar usuli, (8.21.1) integralni,

$$\frac{b-a}{2n}\{[f(x_0)+f(x_1)]+[f(x_1)+f(x_2)]+...+[f(x_{n-1})+f(x_n)]\} =$$
$$= \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right\}$$

yig'indiga, yoki asoslari, mos ravishda, $f(x_{k-1})$ va $f(x_k)$ larga, balandliklari $x_k - x_{k-1} = \frac{b-a}{n}$ ga teng bo'lgan, trapesiyalar yuzlarining yig'indisiga (8.3-chizma) taqribiy almashtirishdan iborat, ya'ni

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n} f(x_k)\right\} . \quad (8.21.6)$$

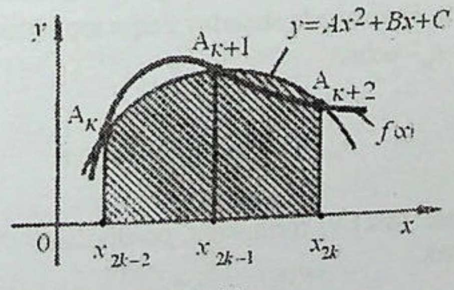
(8.21.5) formulani, qoldiq had yordamida,

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n} f(x_k)\right\}+R_{tr}$$

ko'rinishda yozish mumkin, bunda R_{tr} - qoldiq had. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz ikkinchi tartibli hosilaga ega bo'lsa, u holda, $[a,b]$ kesmada shunday η nuqta topiladiki, R_{tr} uchun,

$$R_{tr} = -\frac{(b-a)^3}{12n^2} f''(\eta), \; a \le \eta \le b,$$

tenglik o'rinli bo'ladi.



8.4-chizma.

8.21.7. Parabolalar usuli. (8.21.1) integralni hisoblash uchun, $[a,b]$ kesmaning ixtiyoriy $P=\{a=x_0<x_2<...<x_{2n}=b\}$ regulyar bo‘linishini olamiz. $[x_{2k-2}, x_{2k}]$ kesmaning o‘rtasidagi nuqtani x_{2k-1} orqali belgilaymiz:

$$x_{2k-1}=\frac{x_{2k-2}+x_{2k}}{2},\ k=\overline{1,n}.$$

Parabolalar usulida berilgan (8.21.1) integral (8.4-chizma.), $f(x)$ funksiya grafigining, abssissalari x_{2k-2}, x_{2k-1} va x_{2k} bo‘lgan nuqtalardan o‘tuvchi parabolalar ostida joylashgan trapesiyalar yuzlarining

$$\frac{b-a}{6n}\{[f(x_0)+4f(x_1)+f(x_2)]+[f(x_2)+4f(x_3)+f(x_4)]+...+$$
$$+[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})]\}=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})+4\sum_{k=0}^{n-1}f(x_{2k+1})\}$$

yig‘indisiga taqribiy almashtiriladi, ya’ni

$$\int_a^b f(x)dx\approx\frac{b-a}{6n}\{[f(x_0)+4f(x_1)+f(x_2)]+[f(x_2)+4f(x_3)+f(x_4)]+$$
$$+...+[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})]=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})+4\sum_{k=0}^{n-1}f(x_{2k+1})\}$$

yoki

$$\int_a^b f(x)dx=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})\}+4\sum_{k=0}^{n-1}f(x_{2k+1})+R_{pa},\qquad(8.21.9)$$

bunda R_{pa}- qoldiq had.

Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz to‘rtinchi tartibli hosilaga ega bo‘lsa, u holda $[a,b]$ kesmada shunday nuqta topiladiki, (7.15.9) formulalardagi had R_{pa}- uchun,

$$R_{pa}=-\frac{(b-a)^5}{2880n^4}f^{(4)}(\eta),\ a\le\eta\le b,$$

tenglik o‘rinli bo‘ladi. (8.21.9) formulaga, *parabolalar (Simpson) formulasi* deyiladi (8.8-chizma).

8.21.10-misol. Ushbu

$$\int_1^9\sqrt{5x+4}dx$$

integralni, integrallash oralig‘ini, 8 ta o‘zaro teng bo‘lakka bo‘lib: 1) Nyuton – Leybnis; 2) to‘g‘ri to‘rtburchaklar; 3) trapesiyalar va 4) parabolalar formulalari yordamida hisoblang, so‘ngra taqribiy hisoblash formulasida qo‘yilgan absolyut va nisbiy xatolarni foizlarda toping.

Yechilishi. 1) Berilgan integralni, Nyuton – Leybnis formulasiga asosan, hisoblaymiz:

$$J=\int_1^9\sqrt{5x+4}dx=\frac{1}{5}\int_1^9(5x+4)^{1/2}d(5x+4)=\frac{2}{15}(5x+4)^{\frac{3}{2}}\Big|_1^9=\frac{2}{15}(343-27)=\frac{632}{15}=42\frac{2}{15}.$$

2) To‘g‘ri to‘rtburchaklar formulasi bo‘yicha hisoblaymiz: $[1;9]$ kesmaning, $P=\{1=x_0<x_2<...<x_8=9\}$ regulyar bo‘linishini qaraymiz. $y=f(x)=\sqrt{5x+4}$ funksiyaning, bu kesmalarning o‘rtalaridagi qiymatlarini hisoblaymiz: $x_{2k-1}=\dfrac{x_{2k-2}+x_{2k}}{2}$ $(k=\overline{1,8})$;

$x_1=\frac{3}{2}$, $\quad f(x_1)=f(\frac{3}{2})=\sqrt{\frac{23}{2}}=3{,}3911$,

$x_3=5/2$, $\quad f(x_3)=f(5/2)=\sqrt{\frac{33}{2}}=4{,}0620$,

$x_5=7/2$, $\quad f(x_5)=f(7/2)=\sqrt{\frac{43}{2}}=4{,}6368$,

$x_7=9/2$, $\quad f(x_7)=f(9/2)=\sqrt{\frac{53}{2}}=5{,}1478$,

$x_9=11/2$, $\quad f(x_9)=f(11/2)=\sqrt{\frac{63}{2}}=5{,}6124$

$x_{11}=13/2$, $\quad f(x_{11})=f(13/2)=\sqrt{\frac{73}{2}}=6{,}0415$,

$x_{13}=15/2$, $\quad f(x_{13})=f(15/2)=\sqrt{\frac{83}{2}}=6{,}4420$,

$x_{15}=17/2$, $\quad f(x_{15})=f(17/2)=\sqrt{\frac{93}{2}}=6{,}8190$.

Har bir kesmaning uzunligi $h=\frac{9-1}{8}=1$. Funksiyaning topilgan qiymatlarini (7.15.2) to'g'ri turtburchaklar formulasiga keltirib qo'yib, hisoblaymiz:

$$J_1=\int_1^9\sqrt{5x+4}dx=\frac{632}{15}\approx\sqrt{\frac{23}{2}}+\sqrt{\frac{33}{2}}+\sqrt{\frac{43}{2}}+\sqrt{\frac{53}{2}}+\sqrt{\frac{63}{2}}+\sqrt{\frac{73}{2}}+\sqrt{\frac{83}{2}}+\sqrt{\frac{93}{2}}=42{,}1526.$$

Taqribiy hisoblashlardagi absolyut xato:

$$\Delta=|J-J_1|=\left|\frac{632}{15}-42{,}1526\right|=|42{,}1333-42{,}1526|=0{,}0193.$$

Nisbiy xato (foizlarda): $\delta=\frac{\Delta}{J}100\%=\frac{15\cdot 0{,}0193\cdot 100}{632}\approx 0{,}0458\%$.

3) Trapesiyalar formulasi bo'yicha hisoblaymiz:

$a=x_0=1,\quad b=x_8=9.$

$$\begin{array}{ll}
a=x_0=1 & f(a)=f(1)=3,\\
x_1=2 & f(x_1)=f(2)=\sqrt{14}=3{,}7416\\
x_2=3 & f(x_2)=f(3)=\sqrt{19}=4{,}3588\\
x_3=4 & f(x_3)=f(4)=\sqrt{24}=4{,}8989\\
x_4=5 & f(x_4)=f(5)=\sqrt{29}=5{,}3851\\
x_5=6 & f(x_5)=f(6)=\sqrt{34}=5{,}8309\\
x_6=7 & f(x_6)=f(7)=\sqrt{39}=6{,}2449\\
x_7=8 & f(x_7)=f(8)=\sqrt{44}=6{,}6332\\
b=x_8=9 & f(x_8)=f(9)=7
\end{array}\quad \left.\right\} 37{,}0934$$

Funksiyaning topilgan qiymatlarini (8.21.5) formulaga keltirib qo'yib, hisoblaymiz:

$$J_2=\frac{632}{15}\approx\frac{1}{2}[3+7+2(\sqrt{14}+\sqrt{19}+\sqrt{24}+\sqrt{29}+\sqrt{34}+\sqrt{39}+\sqrt{44})]=5+37{,}0924=42{,}0934$$

Absolyut xato: $\Delta=|J-J_2|=\left|\frac{632}{15}-47{,}0934\right|=42{,}1333-42{,}0934=0{,}0399.$

Nisbiy xato (foizlarda): $\delta=\frac{\Delta}{J}100\%=\frac{15*0{,}399*100\%}{632}\approx 0{,}0946\%.$

4) Parabolalar formulasi bo'yicha hisoblaymiz:

$x_2=2,\quad f(x_2)=f(2)=\sqrt{14},$

$x_4=3,\quad f(x_4)=f(3)=\sqrt{19},$

$x_6=4,\quad f(x_6)=f(4)=\sqrt{24},$

$x_8=5,\quad f(x_8)=f(5)=\sqrt{29},$

$x_{10}=6,\quad f(x_{10})=f(6)=\sqrt{34},$

$x_{12}=7,\quad f(x_{12})=f(7)=\sqrt{39},$

$x_{14}=8,\quad f(x_{14})=f(8)=\sqrt{44},$

Topilgan qiymatlarni (8.21.7) formulaga keltirib qo'yib, hisoblaymiz:

$$J_3=\frac{632}{15}\approx\frac{1}{2}[\,3+7+2(\sqrt{14}+\sqrt{19}+\sqrt{24}+\sqrt{29}+\sqrt{34}+\sqrt{39}+\sqrt{44})+$$
$$+4\left(\sqrt{\frac{23}{2}}+\sqrt{\frac{33}{2}}+\sqrt{\frac{43}{2}}+\sqrt{\frac{53}{2}}+\sqrt{\frac{63}{2}}+\sqrt{\frac{73}{2}}+\sqrt{\frac{83}{2}}+\sqrt{\frac{93}{2}}\right)]=\frac{1}{6}[10+2\cdot 37{,}0934+4\cdot 42{,}1526]=$$
$$=\frac{10+74{,}1868+168{,}6104}{6}=42{,}1328.$$

Absolyut xato: $\Delta=|J-J_3|=42{,}1333-42{,}1328=0{,}0005.$

Nisbiy xato (foizlarda): $\delta=\frac{\Delta}{J}100\%=\frac{15\cdot 0{,}0005\cdot 100}{632}\approx 0{,}0111\%.$ ■

8.5-§. Aniq integralning tadbiqlari

8.22. Aniq integralning geometriyaga va mexanikaga qo'llanilishi

8.22.1. Dekart koordinatalar sistemasida berilgan tekis shaklning yuzini aniq integral yordamida hisoblash.

8.22.2-ta'rif. Tekislikning L oddiy (karrali nuqtalarga ega bo'lmagan) yopiq egri chiziq bilan chegaralangan qismi – *tekis shakl (figura)* deyiladi. Bunda L - tekis shaklning *chegarasi*.

1. $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lib, $\forall x\in[a;b]$ da $f(x)\geq 0$ bo'lsin. Yuqoridan $f(x)$ funksiyaning grafigi, yon tomonlardan $x=a$ va $x=b$ to'g'ri chiziqlar, pastdan esa Ox o'q bilan chegaralangan shaklning (odatda bunday shakl, *egri chiziqli trapesiya,* deb yuritiladi) yuzi,

$$S=\int_a^b f(x)dx$$

formula bo'yicha hisoblanadi.

2. Agar $[a;b]$ kesmada aniqlangan, uzluksiz $f(x)$ funksiya manfiy, ya'ni $f(x)<0$ bo'lsa, u holda, asosi $[a;b]$ kesmadan iborat bo'lib, quyidan $y=f(x)$, yuqoridan $y=0$ funksiyaning grafigi bilan chegaralangan trapesiyaning yuzi quyidagicha bo'ladi:

$$S=\left|\int_a^b f(x)dx\right|=-\int_a^b f(x)dx.$$

3. $f(x)$, $g(x)$ funksiyalar $[a;b]$ kesmada aniqlangan uzluksiz, $f(x)\geq 0$, $g(x)\geq 0$ va $\forall x\in[a;b]$ uchun, $f(x)\geq g(x)$ bo'lsin. U holda, $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ chiziqlar bilan chegaralangan sohaning yuzi,

$$S=\int_a^b [f(x)-g(x)]dx$$

formula orqali topiladi.

4. Agar (S)– soha, ushbu $(S)=\{(x,y): c\leq y\leq d,\ x_1(y)\leq x\leq x_2(y)\}$ ko'rinishda bo'lsa, u holda uning S yuzi, quyidagi,

$$S=\int_c^d [x_2(y)-x_1(y)]dy$$

formula orqali topiladi.

8.22.3.Tenglamasi parametrik shaklda berilgan egri chiziq bilan chegaralangan sohaning yuzini aniq integral yordamida hisoblash

1. Agar $x=x(t)$, $y=y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ da $x(t)\geq 0$, $y(t)\geq 0$ va $x(t)$ funksiya uzluksiz manfiy bo'lmagan $x'(t)$ hosilaga ega bo'lsa, u holda izlanayotgan sohaning yuzi

$$S=\int_\alpha^\beta y(t)\cdot x'(t)dt$$

formula bo'yicha topiladi.

2. $x=x(t)$, $y=y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ da $x(t)\geq 0$, $y(t)\geq 0$ va $y(t)$ funksiya, uzluksiz, manfiy bo'lmagan $y'(t)$ hosilaga ega bo'lsa, u holda, izlanayotgan sohaning yuzi,

$$S=\int_\alpha^\beta x(t)\cdot y'(t)dt$$

formula orqali topiladi.

8.22.4. Tenglamasi qutb koordinatalar sistemasida berilgan egri chiziq bilan chegaralangan sohaning yuzini aniq integral yordamida hisoblash. Faraz qilaylik, $r=r(\varphi)$ funksiya $[\alpha;\beta]$ oraliqda aniqlangan $(0<\beta-\alpha\leq 2\pi)$, uzluksiz va $\forall\varphi\in[\alpha;\beta]$ da $r(\varphi)\geq 0$ bo'lsin. U holda, $r=r(\varphi)$ funksiyaning grafigi hamda $\overrightarrow{OA}$ va $\overrightarrow{OB}$ radius vektorlar bilan chegaralangan soha – egri chiziqli sektorni qaraymiz. Qaralayotgan sektorning yuzi,

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi$$

formula bo'yicha hisoblanadi.

8.22.5. Dekart koordinatalar sistemasida berilgan chiziqning yoyi uzunligini aniq integral yordamida hisoblash. $\breve{AB}$ yoy, fazoda $x=x(t)$, $y=y(t)$, $z=z(t)$, $t\in[\alpha,\beta]$, tenglamalar sistemasi orqali aniqlangan bo'lsin. Bunda, $x(t),y(t),z(t)$ funksiyalar, $[\alpha;\beta]$ kesmada uzluksiz differensiallanuvchi funksiyalardir.

1. Fazoda $\breve{AB}$ yoyning uzunligi ,

$$l=\int_\alpha^\beta\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

formula bo'yicha hisoblanadi.

2. $\breve{AB}$ yoy tekislikda berilganda, uning uzunligi,

$$l=\int_\alpha^\beta\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

formula orqali hisoblanadi.

3.Agar egri chiziqning ($\breve{AB}$-yoyning) tenglamasi oshkor, ya'ni $y=f(x)$, $x\in[a,b]$ ko'rinishda berilsa, ($f(x)$–uzluksiz differensiallanuvchi), uning yoy uzunligi,

$$l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

formula orqali topiladi.

8.22.6. Qutb koordinatalar sistemasida berilgan chiziqning yoyi uzunligini aniq integral yordamida hisoblash. Agar $\breve{AB}$ egri chiziq yoyi qutb koordinatalar sistemasida $r=r(\varphi)$ $(\varphi_1\le\varphi\le\varphi_2)$ tenglama bilan berilgan bo'lib, $r(\varphi)$ funksiya $[\varphi_1,\varphi_2]$ da uzluksiz va uzluksiz hosilaga ega bo'lsa, u holda, egri chiziqning *yoy uzunligi*,

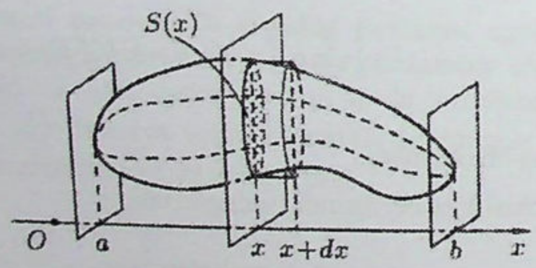
$$l=\breve{AB}=\int_{\varphi_1}^{\varphi_2}\sqrt{[r(\varphi)]^2+[r'(\varphi)]^2}\,d\varphi$$

formula bo'yicha hisoblanadi.

8.22.7. Dekart koordinatalar sistemasida berilgan aylanma jismning hajmini aniq integral yordamida hisoblash.

1. Bizga biror T jism berilgan bo'lib, uning Oy o'qqa parallel bo'lgan kesimlarining yuzasi ma'lum bo'lsin. Bu yuza, x o'zgaruvchining funksiyasi bo'ladi, uni $S=S(x)$ orqali belgilaymiz (8.5-chizma).

Agar $S=S(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa,



8.5-chizma.

T jismning V hajmi, ushbu

$$V=\int_a^b S(x)dx$$

formula bo'yicha hisoblanadi.

2. $y=f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lib, $\forall x\in[a,b]$ uchun $f(x)\ge 0$ bo'lsin. Yuqoridan $y=f(x)$ funksiya grafigi, yon tomonlardan $x=a$, $x=b$ vertikal to'g'ri chiziqlar, pastdan Ox o'qdagi $[a,b]$ kesma bilan chegaralangan shaklni Ox o'q atrofida aylanishidan hosil bo'lgan aylanma T jismning hajmi,

$$V_x=\pi\int_a^b f^2(x)dx \qquad (8.22.8)$$

formula bo'yicha topiladi.

Agar D egri chiziqli trapesiya, yuqoridan $f(x)$, pastdan $g(x)$ uzluksiz egri chiziqlar bilan, yon tomonlardan esa, $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, uning Ox o'q atrofida aylanishidan hosil bo'lgan aylanma T jismning hajmi,

$$V_x=\pi\int_a^b\left[f^2(x)-g^2(x)\right]dx \qquad (8.22.9)$$

formula bo'yicha topiladi.

3. $y=y(x)$ funksiya, $[\alpha,\beta]$ kesmada $x=x(t)$, $y=y(t)$ parametrik tenglamalari bilan berilgan bo'lsin. Bu funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ kesmada $y(t)\ge 0$ va $x(t)$ funksiya, uzluksiz, manfiy bo'lmagan $x'(t)$ hosilaga ega, hamda $a=x(\alpha)$, $b=y(\beta)$ bo'lsa, u holda, T aylanma jismning hajmi,

$$V=\pi\int_\alpha^\beta y^2(t)\cdot x'(t)dt \qquad (8.22.10)$$

formula bo'yicha topiladi.

4. Agar $x(t)$ funksiya $[\alpha,\beta]$ kesmada kamayuvchi va $a=x(\alpha)$, $b=y(\beta)$ bo'lsa, u holda, yuqoridagi shartlar bajarilganda, T aylanma jismning hajmi,

$$V=-\pi\int_\alpha^\beta y^2(t)\cdot x'(t)dt$$

formula bo'yicha topiladi.

5. $Oy\ (x=0)$ o'q atrofida aylantirishdan hosil bo'lgan Φ aylanma jismning hajmi, yuqoridagi **(8.22.8)**, **(8.22.9)**, **(8.22.10)** formulalarga o'xshash quyidagi

$$V_y=\pi\int_c^d g^2(y)dy,\ V_y=\pi\int_c^d\left[f^2(y)-g^2(y)\right]dy,\ V=\pi\int_\alpha^\beta x^2(t)\cdot y'(t)dt$$

formulalar bo'yicha topiladi.

8.22.11. Qutb koordinatalar sistemasida berilgan aylanma jismning hajmini aniq integral yordamida hisoblash.

1. Agar $\breve{AB}$ egri chiziqning tenglamasi qutb koordinatalar sistemasida $r=r(\varphi)$, $0\le\alpha\le\varphi\le\beta\le2\pi$, ko'rinishda berilgan bo'lib, $[\alpha,\beta]$ kesmada $r(\varphi)$ - uzluksiz bo'lsa, u holda, qutb nuri atrofida aylanishdan hosil bo'lgan $T=\{(r,\varphi):\alpha\le\varphi\le\beta,\ 0\le r\le r(\varphi)\}$ aylanma sektorning hajmi,

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi d\varphi$$

formula bo'yicha topiladi.

2. Agar $\breve{AB}$ egri chiziq tenglamasi qutb koordinatalar sistemasida $r=r(\varphi)$, $-\frac{\pi}{2}\le\alpha\le\varphi\le\beta\le\frac{\pi}{2}$, ko'rinishda berilgan bo'lib, $[\alpha,\beta]$ kesmada $r(\varphi)$ uzluksiz bo'lsa, u holda, $\varphi=\frac{\pi}{2}$ qutb nuri atrofida aylanishdan hosil bo'lgan $T=\{(r,\varphi):\alpha\le\varphi\le\beta,\ 0\le r\le r(\varphi)\}$ aylanma sektorning hajmi,

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\cos\varphi\, d\varphi$$

formula bo'yicha topiladi.

8.22.12. Dekart koordinatalar sistemasida berilgan aylanma sirtning yuzini aniq integral yordamida hisoblash.

1. $y=f(x)$ funksiya $[a;b]$ kesmada aniqlangan uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo'lib, $\forall x\in[a;b]$ uchun, $f(x)\ge0$ bo'lsin. Bu funksiya grafigining $A(a;f(a))$ va $B(b;f(b))$ nuqtalar orasidagi $\breve{AB}$ yoyini Ox o'q atrofida aylantirish natijasida hosil bo'lgan sirtning yuzi,

$$Q_x=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}dx$$

formula orqali hisoblanadi.

2. Agar $\breve{AB}$ egri chiziq, $x=\varphi(t)$, $y=\psi(t)$, $\alpha\le t\le\beta$, tenglama bilan berilganda, Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzini

$$Q_x=2\pi\int_\alpha^\beta\psi(t)\sqrt{(\varphi'(t))^2+(\psi'(t))^2}dt$$

formula orqali hisoblanadi.

8.22.13. Qutb koordinatalar sistemasida berilgan aylanma sirtning yuzini aniq integral yordamida hisoblash. Agar $\breve{AB}$ egri chiziqning tenglamasi, qutb koordinatalar sistemasida, $r=r(\varphi)$, $\alpha\le\varphi\le\beta$, ko'rinishda berilgan bo'lib, $r(\varphi)$ - $[\alpha,\beta]$ kesmada uzluksiz, uzluksiz hosilaga ega bo'lsa, u holda, (4.3.40) formula,

$$Q=2\pi\int_\alpha^\beta r(\varphi)\sin\varphi\sqrt{(r(\varphi))^2+(r'(\varphi))^2}d\varphi$$

ko'rinishda bo'ladi.

8.22.14. Aniq integralning mexanika masalalariga tadbiqlari

Silliq egri chiziqning statik momenti va og'irlik markazi

1. Zichligi $\rho=\rho(x)$ bo'lgan $y=f(x), x\in[a;b]$ tekis egri chiziqning massasi

$$m=\int_a^b\rho(x)\sqrt{1+y'^2}dx$$

formulalar bo'yicha hisoblanadi.

2. Tekis egri chiziqning koordinatalar boshiga nisbatan inersiya momenti ushbu

$$I_0=\int_a^b\rho(x)(x^2+y^2)\sqrt{1+y'^2}dx$$

formula bo'yicha, uning Ox va Oy koordinatalar o'qlariga nisbatan inersiya momentlari, mos ravishda,

$$I_x = \int_a^b \rho(x) y^2 \sqrt{1+y'^2}\,dx, \quad I_y = \int_a^b \rho(x) x^2 \sqrt{1+y'^2}\,dx$$

formulalar bo‘yicha hisoblanadi.

3. Tekis egri chiziqning koordinatalar boshiga nisbatan statik momentini

$$M_0 = \int_a^b \rho(x)\sqrt{x^2+y^2}\sqrt{1+y'^2}\,dx$$

formula bo‘yicha, Ox va Oy koordinatalar o‘qlariga nisbatan statik momentlari esa, mos ravishda,

$$M_x = \int_a^b \rho(x) y\sqrt{1+y'^2}\,dx, \quad M_y = \int_a^b \rho(x) x\sqrt{1+y'^2}\,dx$$

formulalar bo‘yicha hisoblanadi.

4. Tekis egri chiziqning $C(x_M, y_M)$ - massalar markazi

$$x_M = \frac{\int_a^b \rho(x) x\sqrt{1+y'^2}\,dx}{\int_a^b \rho(x)\sqrt{1+y'^2}\,dx}; \quad y_M = \frac{\int_a^b \rho(x) y\sqrt{1+y'^2}\,dx}{\int_a^b \rho(x)\sqrt{1+y'^2}\,dx}.$$

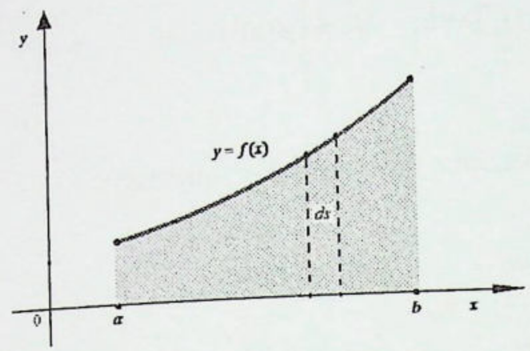
formulalar bo‘yicha hisoblanadi.

8.22.15. Teorema (Guldinning birinchi teoremasi). (AB) egri chiziqni, uni kesib o‘tmaydigan, o‘q atrofida aylantirish natijasida hosil bo‘lgan aylanish sirtining yuzi, uning og‘irlik markazi chizgan aylana uzunligining shu egri chiziqning l yoy uzunligiga ko‘paytmasiga teng:

$$S = 2\pi\, y_M l = 2\pi \int_0^l y\,ds.$$

Tekis figuraning statik momentlari va og‘irlik markazi

Текис фигура, юқоридан $y = f(x)$, пастдан $y = 0$ функциялар, ён томонларидан эса, $x = a$ ва $x = b$ вертикал чизиқлар билан чегараланган бўлсин. m массали текис фигуранинг ҳар бир нуқтасида чизиқли зичлиги $\rho = \rho(x)$ x ўзгарувчининг узлуксиз функцияси



8. 6-chizma.

бўлсин (8.6-chizma). Shaklda ds yuza elementini ajratamiz.

1. Soha bu qismining massasi $dm = \rho(x)ds$. Aniq integralning geometrik talqinidan foydalanib, massa uchun,

$$m = \int_a^b \rho(x) f(x)\,dx$$

formulani hosil qilamiz.

2. Tekis sohaning koordinatalar boshiga nisbatan inersiya momenti

$$I_0 = \int_a^b \rho(x)(x^2 + f^2(x)) f(x)\,dx$$

formula bo‘yicha, Ox va Oy koordinatalar o‘qlariga nisbatan inersiya momentlari, mos ravishda

$$I_x = \int_a^b \rho(x) f^3(x)\,dx; \quad I_y = \int_a^b \rho(x) x^2 f(x)\,dx$$

formulalar bo‘yicha hisoblanadi.

3. Ta’rifga binoan, tekis sohaning koordinatalar boshiga nisbatan *statik momenti*

$$M_0 = \int_a^b \rho(x)\sqrt{x^2 + f^2(x)}\,dx$$

formula bo‘yicha, Ox va Oy koordinatalar o‘qlariga nisbatan statik momentlari, mos ravishda

$$M_x=\int_a^b \rho(x)f^2(x)dx;\quad M_y=\int_a^b \rho(x)xf(x)dx$$

formulalar bo'yicha hisoblanadi.

4. $C(x_M, y_M)$ massalar markazining koordinatalari, quyidagi:

$$x_M=\frac{\int_a^b x\rho(x)f(x)dx}{\int_a^b \rho(x)f(x)dx};\quad y_M=\frac{\int_a^b \rho(x)f^2(x)dx}{\int_a^b \rho(x)f(x)dx}.$$

formulalar bo'yicha hisoblanadi.

8.22.16. Teorema (Guldinning ikkinchi teoremasi). Tekis figurani, uni kesmaydigan o'q atrofida aylantirish natijasida hosil bo'lgan jismning hajmi, uning og'irlik markazi chizgan aylana uzunligini tekis figuraning yuziga ko'paytmasiga teng, ya'ni $V=2\pi\eta\cdot S$.

5. Qutb koordinatalar sistemasida sektor, $0\le r\le r(\varphi)$, $\alpha\le\varphi\le\beta$, tengsizliklar orqali berilgan bo'lsin, bunda $0<\beta-\alpha\le 2\pi$, $r(\varphi)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz. Massa sektorda tekis tarqalgan bo'lsin, ya'ni uning ρ zichligi o'zgarmas bo'lsin. Bu yerda ham, umumiylikni buzmaslik uchun, $\rho=1$ deb olamiz. U holda, Ox va Oy o'qlarga nisbatan statik momentlar va og'irlik markazining koordinatalari, quyidagi

$$M_x=\frac{1}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi\, d\varphi,\quad M_y=\frac{1}{3}\int_\alpha^\beta r^3(\varphi)\cos\varphi\, d\varphi,$$

$$x_M=\frac{M_y}{S},\ y_M=\frac{M_x}{S},$$

formulalar orqali topiladi, bunda S - sektorning yuzi, ya'ni

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi.$$

8-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

8.1 Boshlangich funksiya tushunchasi. Boshlangich funksiya to'g'risidagi teoremalar ([3],1-q., 177-178 betlar; [5], 2-t., 11-14 betlar; [10], 1-q., 291-292 betlar; [12], 1-q., 248-249 betlar; [9], 1-t., 9- bo'lim, [30], 4- bo'lim).

8.2. Aniqmas integralning ta'rifi va uning xossalari ([3],1-q., 179-180 betlar; [5], 2-t., 11-19 betlar; [10], 1-q., 291-293 betlar; [12], 1-q.,250-251 betlar; [9], 1-t., 9- bo'lim, [30], 4- bo'lim).

8.3. Aniqmas integrallar uchun jadvallar ([3],1-q., 182-183 betlar; [5], 2-t., 11-19 betlar; [10], 1-q., 294-295 betlar; [12], 1-q.,251-253 betlar; [9], 1-t., 9- bo'lim, [30], 4- bo'lim).

8.4. Aniqmas integrallarda o'zgaruvchilarni almashtirish va bo'laklab integrallash usullari ([3],1-q., 185-189 betlar; [5], 2-t., 23-27 betlar; [10], 1-q., 297-300 betlar; [12], 1-q.,254-255 betlar; [9], 1-t., 9- bo'lim, [30], 8- bo'lim).

8.5. Ratsional kasr funksiyalarni sodda kasrlarga ajratish ([3],1-q., 192-199 betlar; [5], 2-t., 36-47 betlar; [10], 1-q., 311-317 betlar; [12], 1-q.,258-264 betlar, [9], 1-t., 9- bo'lim, [30], 8- bo'lim).

8.6. Sodda kasrlarni integrallash. Ratsional funksiyalarni integrallash ([3],1-q., 190-192 betlar; [5], 2-t., 36-47 betlar; [10], 1-q., 311-317 betlar; [12], 1-q.,264-267 betlar, [9], 1-t., 9- bo'lim, [30], 8- bo'lim).

8.7. Trigonometrik funksiyalar qatnashgan ifodalarni integrallash ([3],1-q., 200-203 betlar; [5], 2-t., 74-83 betlar; [10], 1-q., 321-327 betlar; [12], 1-q.,277-278 betlar; [9], 1-t., 9- bo'lim, [30], 8- bo'lim).

8.8. Binomial differensiallarni integrallash ([3],1-q., 207-210 betlar; [5], 2-t., 51-54 betlar; [10], 1-q., 327-317 betlar; [10], 1-q., 321-327 betlar; [12], 1-q.,274-276 betlar).

8.9. Irratsional ifodalarni integrallash ([3],1-q., 203-207 betlar; [5], 2-t., 50-72 betlar; [10], 1-q., 321-327 betlar; [12], 1-q.,267-276 betlar; [30], 8- bo'lim).

8.10. Aniq integral tushunchasi va uning geometrik ma'nosi ([3],1-q., 211-212 betlar; [5], 2-t., 94-106 betlar; [10], 1-q., 300-334 betlar; [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.11. Aniq integralning xossalari ([3],1-q., 230-238 betlar; [5], 2-t., 108-117 betlar; [10], 1-q., 347-356 betlar; [12], 1-q.,304-312 betlar, [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.12. Integrallanuvchi funksiyalarning sinflari ([3],1-q., 227-230 betlar; [5], 2-t., 101-103 betlar; [10], 1-q., 341-346 betlar; [12], 1-q.,300-304 betlar, [1], 1-t., 11- bo'lim, [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.13. O'rta qiymat haqidagi teoremalar ([3],1-q., 238-240 betlar; [5], 2-t., 113-119 betlar; [10], 1-q., 347-350 betlar; [12], 1-q.,312-314 betlar, [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.14. Nyuton – Leybnis formulasi ([3],1-q., 246-247 betlar; [5], 2-t., 123-125 betlar; [10], 1-q., 359-361 betlar; [12], 1-q.,319-320 betlar, [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.15. Aniq integrallarda o'zgaruvchilarni almashtirish va bo'laklab integrallash usullari ([3],1-q., 247-251 betlar; [5], 2-t., 130-137 betlar; [10], 1-q., 361-363 betlar; [12], 1-q.,320-323 betlar; [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.16. Tekis figuralarning yuzini hisoblash ([3],1-q., 261-270 betlar; [5], 2-t., 186-192 betlar; [10], 1-q., 406-416 betlar; [12], 1-q.,353-361 betlar, [9], 1-t., 9- bo'lim, [30], 5- bo'lim).

8.17. Yoy uzunligi ta'rifi va uni hisoblash ([3],1-q., 271-282 betlar; [5], 2-t., 169-185 betlar; [10], 1-q., 391-405 betlar; [30], 6- bo'lim).

8.18. Aylanma sirtning yuzi va uni hisoblash ([3],1-q., 283-288 betlar; [5], 2-t., 214-222 betlar; [12], 1-q.,361-364 betlar [30], 6- bo'lim).

8.19. Aniq integrallarni taqribiy hisoblash usullari ([3],1-q., 252-260 betlar; [5], 2-t., 153-164 betlar; [10], 1-q., 422-441 betlar; [12], 1-q.,324-328 betlar, [30], 8- bo'lim).

8.20. Aniq integrallarning mexanika va fizika masalalarini yechishga tadbiqlari, ([3],1-q., 288-293 betlar; [5], 2-t., 225-239 betlar; [12], 1-q.,365-369 betlar [30], 6- bo'lim).

8.1-amaliy mashg'ulot.
Boshlang'ich funksiya va aniqmas integral

1-misol. Berilgan oraliqda berilgan $F(x)$ funksiya berilgan $f(x)$ funksiyaning boshlang'ich funksiyasi ekanligini ko'rsating va aniqmas integralini yozing:

1), $F(x)=\frac{x^4}{4}$, $f(x)=x^3$, $X=(-\infty,\infty)=R$;

2) $F(x)=\frac{\sin ax}{a}$, $f(x)=\cos ax\ (a=const)$, $X=(-\infty,\infty)=R$;

3) $F(x)=\sqrt{1-x^2}$, $f(x)=-\frac{x}{\sqrt{1-x^2}}$, $X=(-1;1)$;

4) $F(x)=\frac{2}{3}\sqrt{x^3}$, $f(x)=\sqrt{x}$, $X=(0;\infty)$;

5) $F(x)=\sqrt{x}-\cos(x+1)$, $f(x)=\frac{1}{2\sqrt{x}}+\sin(x+1)$, $X\in(0;+\infty)$.

Yechilishi. 1) $f(x)=x^3$ funksiyaning $X=(-\infty,\infty)=R$ oraliqdagi boshlang'ich funksiyasi $F(x)=\frac{x^4}{4}$ bo'ladi, chunki $F'(x)=\left(\frac{x^4}{4}\right)'=x^3$. Demak, $\int x^3dx=\frac{x^4}{4}+C$.

2) $f(x)=\cos ax\ (a=const)$ funksiyaning R dagi boshlang'ich funksiyasi $F(x)=\frac{\sin ax}{a}$ bo'ladi, chunki $F'(x)=\left(\frac{\sin ax}{a}\right)'=\cos ax=f(x)$. Demak, $\int\cos ax dx=\frac{\sin ax}{a}+C$.

3) $f(x)=-\frac{x}{\sqrt{1-x^2}}$ funksiyaning $X=(-1;1)$ oraliqdagi boshlang'ich funksiyasi $F(x)=\sqrt{1-x^2}$ bo'ladi, chunki $F'(x)=(\sqrt{1-x^2})'=-\frac{x}{\sqrt{1-x^2}}=f(x)$.

Demak, $\int\frac{-x}{\sqrt{1-x^2}}dx=\sqrt{1-x^2}+C$.

4) $f(x)=\sqrt{x}$ funksiyaning $X=(0;\infty)$ oraliqdagi boshlang'ich funksiyasi $F(x)=\frac{2}{3}\sqrt{x^3}$ bo'ladi, chunki $F'(x)=\left(\frac{2}{3}\sqrt{x^3}\right)'=\sqrt{x}=f(x)$ Demak, $\int\sqrt{x}dx=\frac{2}{3}\sqrt{x^3}+C$.

5) $f(x)=\frac{1}{2\sqrt{x}}+\sin(x+1)$ funksiyaning $X\in(0;+\infty)$ oraliqdagi boshlang'ich funksiyasi $F(x)=\sqrt{x}-\cos(x+1)$ dan iborat bo'ladi, chunki

$$F'(x)=\left(\sqrt{x}-\cos(x+1)\right)'=\frac{1}{2\sqrt{x}}+\sin(x+1)=f(x).$$

Demak, $\int\left(\frac{1}{2\sqrt{x}}+\sin(x+1)\right)dx=\sqrt{x}-\cos(x+1)+C$. ■

2– misol. $\int x^2(x+2)(x-3)dx$ integralni hisoblang.

Yechilishi. Aniqmas integralning 3-xossasi, hamda jadvaldagi 3 – formulaga asosan:

$$\int x^2(x+2)(x-3)dx=\int\left(x^4-x^3-6x^2\right)dx=\frac{x^5}{5}-\frac{x^4}{4}-2x^3+C$$ bo'ladi

Tekshirish. Topilgan boshlang'ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko'ramiz:

$$\left(\frac{x^5}{5}-\frac{x^4}{4}-2x^3+C\right)' = x^4-x^3-6x^2+0 = x^2\left(x^2-x-6\right) = x^2(x+2)(x-3), \quad x\in R.$$

Demak, berilgan funksiyaning aniqmas integrali to'g'ri topilgan ekan.

Misolni Maple tizimidan foydalanib yechish:

> Int((x)^2*(x+2)*(x-3),x)=int((x)^2*(x+2)*(x-3),x);

$$\int x^2(x+2)(x-3)dx = \frac{x^5}{5}-\frac{x^4}{4}-2x^3. \blacksquare$$

3–misol. $\int \frac{2^x+5^x}{10^x}dx$ integrallarni hisoblang

Yechilishi. Aniqmas integralning 3–xossasi va integrallar jadvalning 5-formulasidan foydalanib,

$$\int \frac{2^x+5^x}{10^x}dx = \int \frac{1}{5^x}dx + \int \frac{1}{2^x}dx = \int 5^{-x}dx + \int 2^{-x}dx = -\frac{2^{-x}}{\ln 2}-\frac{5^{-x}}{\ln 5}+C, \quad x\in R,$$

bo'lishini topamiz.

Tekshirish. Topilgan boshlang'ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko'ramiz:

$$\left(-\frac{2^{-x}}{\ln 2}-\frac{5^{-x}}{\ln 5}+C\right)' = \frac{1}{2^x}+\frac{1}{5^x} = \frac{2^x+5^x}{10^x}.$$

Demak, topilgan boshlang'ich funksiyaning hosilasi integral ostidagi funksiyaga teng ekan.$\blacksquare$

Mustaqil yechish uchun misollar

Quyidagi integrallarni hisoblang:

1. $\int 4x^7dx.$ **2.** $\int \frac{dx}{x^5}.$

3. $\int (5x^3-2x^2+3x-8)dx.$ **4.** $\int \frac{3x^4+5x^3-6x\sqrt[4]{x}+4}{x}dx.$

5. $\int \frac{2x^4-5x\sqrt[3]{x}+7\sqrt{x}}{x\sqrt{x}}dx.$ **6.** $\int \left(x^{3/2}-\frac{2}{x^{3/2}}\right)^2dx.$

Quyidagi integrallarni hisoblang:

7. $\int (x-1)(x+2)dx.$ **8.** $\int x^2(x+1)(5x-3)dx.$

9. $\int \sqrt[3]{x^2}(8\sqrt[3]{x}-1)dx.$ **10.** $\int \frac{\left(\sqrt{x}-1\right)^3}{x\sqrt{x}}dx.$

Kuyidagi integrallarni hisoblang:

11. $\int 8^x dx.$ **12.** $\int 5^{3x}\cdot e^x dx.$

13. $\int 5^{x-2}dx.$ **14.** $\int \frac{32^x-2^x}{4^x}dx.$

Quyidagi integrallarni hisoblang:

15. $\int 8\cos x\,dx.$ **16.** $\int \frac{\sin x}{9}dx.$

17. $\int \frac{1}{5\sin^2 x}dx.$ **18.** $\int \left(\frac{2}{\cos^2 x}-\frac{5}{\sin^2 x}\right)dx.$

Quyidagi integrallarni hisoblang:

19. $\int \frac{dx}{\cos^2 x\sin^2 x}.$ **20.** $\int \frac{1-\cos 2x}{6\sin x}dx.$

43. $\int \frac{dx}{\sin^2 x+\cos 2x}.$ **44.** $\int \frac{1-4ctg^2 x}{\cos^2 x}dx.$

Quyidagi integrallarni hisoblang:

21. $\int \frac{1+\sqrt{4-x^2}}{\sqrt{4-x^2}}dx.$ **22.** $\int \sqrt{\frac{4+x^2}{16-x4}}dx.$

23. $\int \frac{\sqrt{4+x^2}-\sqrt{4-x^2}}{\sqrt{16-x^4}}dx.$ **24.** $\int \sqrt{\frac{3+x^2}{x^4-9}}dx.$

Quyidagi integrallarni hisoblang:

25. $\int \frac{3+x^2}{1+x^2}dx.$ **26.** $\int \frac{1-2x^2}{x^2(1-x^2)}dx.$

27. $\int \frac{(2x^2+5)dx}{x^2(x^2+5)}.$ **28.** $\int \frac{x^2-7}{9-x^2}dx.$

Quyidagi integrallarni hisoblang:

29. $\int \frac{dx}{ch^2x\,sh^2x}.$ **30.** $\int \frac{ch2x}{ch^2x\,sh^2x}.$

Mustaqil yechish uchun misollarning javoblari

1. $\frac{1}{2}x^8+C.$ **2.** $-\frac{1}{4x^4}+C.$

3. $\frac{5}{4}x^3-\frac{2}{3}x^3+\frac{3}{2}x^2-8x+C.$ **4.** $\frac{3}{4}x^4+\frac{5}{3}x^3-\frac{24}{5}x^{\frac{5}{4}}+4\ln|x|+C.$

5. $\frac{x^3}{3}+\frac{x^2}{2}-2x+C.$ **6.** $x^5+\frac{x^4}{2}-x^3+C.$ **7.** $4x^2-\frac{3}{5}x^{5/3}+C.$

8. $x-6\sqrt{x}+3\ln|x|+\frac{2}{\sqrt{x}}+C.$ **9.** $\frac{8^x}{\ln 8}+C.$

10. $\frac{125^x\cdot e^x}{3\ln 5+1}+C.$ **11.** $\frac{5^{x-2}}{\ln 5}+C.$ **12.** $\frac{16^x+3}{2^x\cdot\ln 2}+C.$

13. $8\sin x+C.$ **14.** $-\frac{1}{9}\cos x+C.$ **15.** $-\frac{1}{5}ctgx+C.$

16. $2tgx+5tgx+C.$ **17.** $tgx-ctgx+C.$ **18.** $-\frac{1}{3}\cos x+C.$

19. $tgx+C.$ **20.** $tgx+4ctgx+C.$ **21.** $\arcsin\frac{x}{2}+x+C.$

22. $\arcsin\frac{x}{4}+C.$ **23.** $\arcsin\frac{x}{2}-\ln\left|x+\sqrt{x^2+4}\right|+C.$

24. $\ln\left|x+\sqrt{x^2-3}\right|+C.$ **25.** $x+2arctgx+C.$ **26.** $\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|-\frac{1}{x}+C.$

27. $\frac{1}{\sqrt{5}}arctg\frac{x}{\sqrt{5}}-\frac{1}{x}+C.$ **28.** $\frac{1}{3}\ln\left|\frac{3+x}{3-x}\right|-x+C.$ **29.** $-cthx-thx+C.$

30. $thx-cthx+C.$

8.2-amaliy mashg'ulot.
Integrallarni hisoblashda o'zgaruvchilarni almashtirish va bo'laklab integrallash

1– misol. Quyidagi integrallarni hisoblang:

$$1)\int\frac{x^2dx}{3-x^6};\quad 2)\int\frac{\cos xdx}{\sqrt{1+3\sin x}};\quad 3)\int tg^3\varphi\, d\varphi;\quad 4)\int\frac{\sqrt{2+\ln x}}{x},\ x>0;$$

$$5)\int x\sqrt{a-x}\,dx;\quad 6)\int\frac{dt}{t^2\sqrt{a+t^2}};\quad 7)\int\frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}\,dx.$$

Yechilishi. 1) $\int\frac{x^2dx}{3-x^6}$ integralni hisoblashda $t=x^3$ almashtirishni olish qulay. Bu almashtirishni bajarib, integrallar jadvalining 15 – formulasiga asosan, quyidagiga ega bo'lamiz:

$$dt=3x^2dx,\quad x^2dx=\frac{1}{3}dt,$$

$$\int\frac{x^2dx}{3-x^6}=\frac{1}{3}\int\frac{dt}{3-t^2}=-\frac{1}{3}\int\frac{dt}{t^2-(\sqrt{3})^2}=\left(-\frac{1}{6\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C\right)\Bigg|_{t=x^3}=-\frac{1}{6\sqrt{3}}\ln\left|\frac{x^3-\sqrt{3}}{x^3+\sqrt{3}}\right|+C.$$

Misolni Maple tizimidan foydalanib, yechish:

> f:=(x^2)/(3-x^6);

$$f:=\frac{x^2}{3-x^6}$$

> Int(f,x);

$$\int\frac{x^2}{3-x^6}dx$$

> int(f,x);

$$\frac{1}{9}\sqrt{3}\ \mathrm{arctanh}\left(\frac{x^3\sqrt{3}}{3}\right)$$

> Int(f,x)=int(f,x);

$$\int\frac{x^2}{3-x^6}dx=\frac{1}{9}\sqrt{3}\ \mathrm{arctanh}\left(\frac{x^3\sqrt{3}}{3}\right)$$

2) $\int\frac{\cos xdx}{\sqrt{1+3\sin x}}$ integralni hisoblashda $t=1+3\sin x$ almashtirishni olish maqsadga muvofiq bo'ladi. Bundan

$$dt=3\cos xdx,\quad \cos xdx=\frac{1}{3}dt,$$

$$\int\frac{\cos x\,dx}{\sqrt{1+3\sin x}}=\frac{1}{3}\int\frac{dt}{\sqrt{t}}=\frac{1}{3}\int t^{-\frac{1}{2}}dt=\frac{1}{3}\left(2t^{\frac{1}{2}}+C\right)\Bigg|_{t=1+3\sin x}=\frac{2}{3}\sqrt{1+3\sin x}+C.$$

Misolni Maple tizimidan foydalanib yechish:

> f:=cos(x)/sqrt(1+3*sin(x));

$$f:=\frac{\cos(x)}{\sqrt{1+3\sin(x)}}$$

> Int(f,x);

$$\int\frac{\cos(x)}{\sqrt{1+3\sin(x)}}dx$$

> int(f,x);

$$\frac{2}{3}\sqrt{1+3\sin(x)}$$

> Int(f,x)=int(f,x);

$$\int\frac{\cos(x)}{\sqrt{1+3\sin(x)}}dx=\frac{2}{3}\sqrt{1+3\sin(x)}$$

3) $\int tg^3\varphi\,d\varphi$ integralni hisoblash uchun $\varphi=arctgt,\ t=tg\varphi$ almashtirish va

$$d\varphi=\frac{1}{1+t^2}dt,\ \int tg^3\varphi\,d\varphi=\int\frac{t^3}{1+t^2}dt=\int\left(t-\frac{t}{1+t^2}\right)dt=\frac{t^2}{2}-\frac{1}{2}\int\frac{d(1+t^2)}{1+t^2}=$$
$$=\left(\frac{t^2}{2}-\frac{1}{2}\ln\left|1+t^2\right|+C\right)_{t=tg\varphi}=\frac{tg^2\varphi}{2}-\frac{1}{2}\ln\left|1+tg^2\varphi\right|+C=\frac{tg^2\varphi}{2}+\ln\left|\cos\varphi\right|+C.$$
bo'ladi.

4) $\int\frac{\sqrt{1+\ln x}}{x}dx$ integralni hisoblashda $t=1+\ln x$ almashtirish olamiz. Natijada

$$dt=\frac{dx}{x},\quad \int\frac{\sqrt{1+\ln x}}{x}dx=\int\sqrt{t}\,dt=\int t^{\frac{1}{2}}dt=\left(\frac{2}{3}t^{\frac{3}{2}}t+C\right)\Bigg|_{t=1+\ln x}=\frac{2}{3}\sqrt{(1+\ln x)^3}+C.$$

bo'ladi.

Misolni Maple tizimidan foydalanib yechish:

> f:=sqrt(2+ln(x))/x;

\>

$$f:=\frac{\sqrt{2+\ln(x)}}{x}$$

> Int(f,x);

$$\int\frac{\sqrt{2+\ln(x)}}{x}dx$$

> int(f,x);

$$\frac{2}{3}(2+\ln(x))^{(3/2)}$$

> Int(f,x)=int(f,x);

\>

$$\int\frac{\sqrt{2+\ln(x)}}{x}dx=\frac{2}{3}(2+\ln(x))^{(3/2)}$$

5) $\int x\sqrt{a-x}\,dx$ - integralni hisoblashda $t=\sqrt{a-x}$ almashtirishni olish qulay bo'ladi. Bundan, $t^2=a-x$, $dt=-\frac{1}{2\sqrt{a-x}}dx$, $x=a-t^2$, $dx=-2t\,dt$,

$$\int x\sqrt{a-x}dx=-2\int\left(a-t^2\right)t^2dt=-2\int\left(at^2-t^4\right)dt=$$
$$=\left(-2a\frac{t^3}{3}+2\frac{t^5}{5}+C\right)\Bigg|_{t=\sqrt{a-x}}=\frac{2}{15}\left(3x^2-ax-2a^2\right)+C$$

Misolni Maple tizimidan foydalanib yechish:

> Int(x*sqrt(a-x),x)=int(x*sqrt(a-x),x);

$$\int x\sqrt{a-x}\,dx=-\frac{2(2a+3x)(a-x)^{(3/2)}}{15}$$

6) $\int\frac{dt}{t^2\sqrt{a+t^2}}$ integralda $x=\frac{1}{t}$ almashtirish olib, uni quyidagicha hisoblaymiz:

$$\int\frac{dt}{t^2\sqrt{a+t^2}}=-\int\frac{x\,dx}{\sqrt{a+x^2}}=-\frac{1}{2}\int\left(a+x^2\right)^{-\frac{1}{2}}d\left(a+x^2\right)=$$
$$=\left(-\frac{1}{2}\cdot2\sqrt{a+x^2}+C\right)\Bigg|_{x=\frac{1}{t^2}}=-\sqrt{a+\frac{1}{t^2}}+C$$

Misolni Maple tizimidan foydalanib yechish:

> Int(1/(t^2*sqrt(a+t^2)),t)=int(1/(t^2*sqrt(a+t^2)),t);

$$\int\frac{1}{t^2\sqrt{a+t^2}}dt=-\frac{\sqrt{a+t^2}}{t\,a}$$

7) $\int\frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx$ integralda avvalo, integral ostidagi ifodaning shaklini o'zgartiramiz:

$$\int\frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx=\int\frac{1+\frac{1}{x^2}}{\sqrt{x^2-7+\frac{1}{x^2}}}dx=$$
$$=\int\frac{d\left(x-\frac{1}{x}\right)}{\sqrt{\left(x-\frac{1}{x}\right)^2-5}}dx=\int\frac{dt}{\sqrt{t^2-5}},$$

bunda $t=x-\frac{1}{x}$. Endi integrallar jadvalining 17-formulasiga asosan, integralni hisoblaymiz. Natijada,

$$\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx = \left(\ln\left|t+\sqrt{t^2-5}\right|+C.\right)\Bigg|_{t=x-\frac{1}{x}} = \ln\left|x-\frac{1}{x}+\sqrt{x^2-7+\frac{1}{x^2}}\right|+C.$$

bo'ladi.

Misolni Maple tizimidan foydalanib yechish:

> Int((x^2+1)/sqrt(x^6-7*x^4+x^2),x)=int((x^2+1)/sqrt(x^6-7*x^4+x^2),x);

$$\int \frac{x^2+1}{\sqrt{x^6-7\,x^4+x^2}}\,dx =$$

$$\frac{1}{2}\,\frac{x\sqrt{x^4-7\,x^2+1}\left(\ln\left(-\frac{7}{2}+x^2+\sqrt{x^4-7\,x^2+1}\right)+\operatorname{arctanh}\left(\frac{-2+7\,x^2}{2\sqrt{x^4-7\,x^2+1}}\right)\right)}{\sqrt{x^6-7\,x^4+x^2}}$$

2- misol. Ushbu

$$\int x^2 3^x dx$$

integralni hisoblang.

Yechilishi. $u=x^2, dv=3^x dx$ deb belgilab,

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \tag{1}$$

formulaga asosan, topamiz

$$du = 2xdx,\ v=\frac{3^x}{\ln 3},$$
$$\int x^2\cdot 3^x dx = \frac{x^2\cdot 3^x}{\ln 3} - \frac{2}{\ln 3}\int x\cdot 3^x dx. \tag{2}$$

Oxirgi integralga yana (1) formulani kullaymiz: $u=x, dv=3^x dx$ deb belgilab, (1) formulaga asosan,

$$du=dx,\ v=\frac{3^x}{\ln 3}$$
$$\int x\cdot 3^x dx = \frac{x\cdot 3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x dx = \frac{x\cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2}.$$

ekanligini olamiz. Oxirgi natijani (2) ga keltirib quysak, natijada

$$\int x^2\cdot 3^x = \frac{1}{\ln 3}\cdot x^2 3^x - \frac{2x3^x}{(\ln 3)^2} + \frac{2\cdot 3^x}{(\ln 3)^3} = \frac{3^x}{\ln 3}\left(x^2 - \frac{2x}{\ln 3} + \frac{2}{(\ln 3)^2}\right)+C.$$

bo'lishini topamiz.

Misolni Maple tizimidan foydalanib yechish:

> restart:with(student):J=Int(((x)^2)*(3)^x,x);

$$J=\int x^2 3^x\,dx$$

> J=intparts(Int(((x)^2)*(3)^x,x),x^2);

$$J=\frac{x^2\cdot 3^x}{\ln 3} - \frac{2}{\ln 3}\int x\cdot 3^x dx$$

> intparts(%,x);

$$J=\frac{x^2 3^x}{\ln(3)} - \frac{2x3^x}{(\ln 3)^2} + \int \frac{2\,3^x}{(\ln 3)^2}dx$$

> value(%);

$$J=\frac{x^2 3^x}{\ln(3)} - \frac{2\,x\,3^x}{\ln(3)^2} + \frac{2\,3^x}{\ln(3)^3}$$

3- misol. Ushbu

$$\int x^2 arctgx\,dx$$

integralni hisoblang.

Yechilishi. $u=arctgx,\ dv=x^2dx$ deb belgilaylik. U holda, $du=\frac{1}{1+x^2}dx,\ v=\frac{x^3}{3}$, formulalarga asosan, integralning qiymati

$$\int x^2 arctgx\,dx = \frac{1}{3}x^3 arctgx - \frac{1}{3}\int\frac{x^3}{1+x^2}dx =$$
$$=\frac{1}{3}x^3 arctgx - \frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx = \frac{1}{3}x^3 arctgx - \frac{x^2}{6}+\frac{1}{6}\int\frac{d(1+x^2)}{1+x^2} =$$
$$=\frac{1}{3}x^3 arctgx - \frac{x^2}{6}+\frac{1}{6}\ln\left(1+x^2\right)+C$$

bo'ladi.

Mustaqil yechish uchun misollar

Quyidagi integrallarni hisoblang:

1. $\int x\sqrt{x^2-7}\,dx.$ **2.** $\int \frac{x^2}{\sqrt[5]{3-x^3}}dx.$ **3.** $\int \frac{e^{3x}\,dx}{(e^{3x}-4)^3}.$

4. $\int \frac{xdx}{x^2+8}.$ **5.** $\int \frac{2x+1}{x^2+x-5}dx.$ **6.** $\int \frac{2x-3}{8+3x-x^2}dx.$

Quyidagi integrallarni hisoblang:

7. $\int \frac{e^x dx}{2e^x+7}.$ **8.** $\int \frac{e^{\frac{5}{x}}dx}{x^2}dx.$ **9.** $\int \frac{1}{x\ln x}dx.$

10. $\int \frac{\sqrt[7]{\ln^5 x}}{x}dx.$ **11.** $\int x^2 e^{1-x^3}\,dx.$ **12.** $\int \frac{5^x\,dx}{9+5^x}.$

Quyidagi integrallarni hisoblang:

13. $\int tgxdx.$ **14.** $\int ctgxdx.$ **15.** $\int \sin x\cos^5 x\,dx.$

16. $\int \frac{\cos x}{\sqrt[5]{\sin x}}dx.$ **17.** $\int \frac{\sin x}{\cos^2 x\sqrt{\cos x}}dx.$ **18.** $\int \frac{\sqrt[7]{128\,tgx}}{\cos^2 x}dx.$

Quyidagi integrallarni hisoblang:

19. $\int \frac{1-4\arcsin x}{\sqrt{1-x^2}}dx.$ **20.** $\int \frac{\sqrt[4]{arctgx}}{1+x^2}dx.$ **21.** $\int \frac{dx}{(x^2+1)\,arctgx}.$

22. $\int e^{5\cos x-1}\sin xdx.$ **23.** $\int \frac{\sin xdx}{\cos^2 x-5}.$ **24.** $\int \frac{\cos\frac{2}{x^3}}{x^4}dx.$

Quyidagi integrallarni hisoblang:

25. $\int ch^3 x\,shxdx.$ **26.** $\int \frac{shx}{ch^2 x}dx.$ **27.** $\int sh^7 x\,chx\,dx.$

28. $\int \frac{chx}{sh^2 x}dx.$ **29.** $\int \frac{cthx}{sh^2 x}dx.$ **30.** $\int \frac{thx}{ch^2 x}dx.$

Integrallarni hisoblang:

31. $\int x\sin x\,dx$ **32.** $\int (x+3)\cos xdx.$

33. $\int (1-4x)\sin xdx.$ **34.** $\int (2x+4)e^x\,dx.$

35. $\int (x+5)6^x\,dx.$ **36.** $\int x^2\cos x\,dx.$

37. $\int (x^2+3)\sin x\,dx.$ **38.** $\int (2x-x^2)e^{-x}\,dx.$

Integrallarni hisoblang:

39. $\int \ln x\,dx.$ **40.** $\int \ln(1+x^2)dx.$

41. $\int x^2\ln(x+4)dx.$ **42.** $\int xarctgx\,dx.$

Quyidagi integlarni hisoblang:

43. $\int e^x\cos xdx.$ **44.** $\int e^x\sin xdx.$ **45.** $\int 2^x\cos xdx.$

Mustaqil yechish uchun misollarning javoblari

1. $\frac{1}{3}\sqrt{(x^2-7)^3}+C.$ **2.** $-\frac{5}{12}\sqrt[5]{(3-3^3)^4}+C.$ **3.** $-\frac{1}{6}\frac{1}{(e^{3x}-4)^2}+C.$

4. $\frac{1}{2}\ln|x^2+8|+C.$ **5.** $\ln|x^2+x-5|+C.$ **6.** $-\ln|8+3x-x^2|+C.$

7. $\frac{1}{2}\ln|2e^x+7|+C.$ **8.** $-\frac{1}{5}e^{5/x}+C.$ **9.** $\ln|\ln x|+C.$

10. $\frac{7}{12}(\ln x)^{12/7}+C.$ **11.** $-\frac{1}{3}e^{1-x^3}+C.$ **12.** $\frac{\ln|9+5^x|}{\ln 5}+C.$

13. $-\ln|\cos x|+C.$ **14.** $\ln|\sin x|+C.$ **15.** $-\frac{\cos^6 x}{6}+C.$

16. $\frac{5}{6}\sqrt[5]{\sin^6 x}+C.$ **17.** $\frac{2}{3\cos x\sqrt{\cos x}}+C.$ **18.** $\frac{7}{4}tgx\,\sqrt[7]{tgx}+C.$

19. $\arcsin x-2\arcsin^2 x+C.$ **20.** $\frac{4}{5}\sqrt[4]{arctg^5 x}+C.$ **21.** $\ln|arctgx|+C.$

22. $-\frac{1}{5}e^{5\cos x-1}+C.$ **23.** $\frac{1}{2\sqrt{5}}\ln\left|\frac{\cos x+\sqrt{5}}{\cos x-\sqrt{5}}\right|+C.$ **24.** $-\frac{1}{6}\sin\frac{2}{x^3}+C.$

25. $\frac{1}{4}ch^4 x+C.$ **26.** $-\frac{1}{chx}+C.$ **27.** $\frac{sh^8 x}{8}+C.$

28. $-\frac{1}{shx}+C.$ **29.** $-0{,}5cgh^2 x+C.$ **30.** $0{,}5th^2 x+C.$

31. $-x\cos x+\sin x+C.$ **32.** $\sin x-(x+3)\cos x+C.$

33. $(4x-1)\cos x-4\sin x+C.$ **34.** $2e^x(1+x)+C.$ **35.** $\frac{x\ln 6+5\ln 6-1}{\ln^2 6}6^x+C.$

36. $x^2\sin x-2\sin x+2x\cos x+C.$ **37.** $2x\sin x-(x^2+1)\cos x+C.$ **38.** $e^{-x}x^2+C.$

39. $x\ln x-x+C.$ **40.** $x\ln(1+x^2)-2x+2arctgx+C.$

41. $\frac{1}{3}(4+x)^3\ln(x+4)-\frac{352}{9}-\frac{16}{3}x+\frac{2}{3}x^2-\frac{1}{9}x^3-4\ln(x+4)(x+4)^2+16(4+x)+$
$16(4+x)\ln(x+4)+C.$

42. $\frac{x^2}{2}arctgx-\frac{1}{2}x+\frac{1}{2}arctgx+C.$ **43.** $\frac{1}{2}e^x\cos x+\frac{1}{2}e^x\sin x+C.$

44. $\frac{1}{2}e^x\sin x-\frac{1}{2}e^x\cos x+C.$ **45.** $\left(2^{1+x}tg\frac{x}{2}-2^x tg^2\frac{x}{2}\ln 2+2^x\ln 2\right)\frac{\cos^2\frac{x}{2}}{1+\ln^2 2}+C.$

8.3-amaliy mashg‘ulot.
Ratsional funksiyalarni integrallash

1 – misol. Ushbu

$$\int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}dx$$

integralni hisoblang

Yechilishi. Integral ostidagi kasr – noto‘g‘ri kasr bo‘lganligi uchun, $P_6(x)=x^6-2x^4+3x^3-9x^2+4$ ko‘phadni $Q_5(x)=x^5-5x^3+4x$ ko‘phadga bo‘lib, $w(x)=x$ bo‘linma va $P_4(x)=3x^4+3x^3-13x^3+4$ qoldiqni topamiz, ya’ni

$$\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}=x+\frac{3x^4+3x^3-13x^2+4}{x^5-5x^3+4x}.$$

Ravshanki, $Q_5(x)=x^5-5x^3+4x$ ko‘phad $x=1$ haqiqiy ildizga ega, $Q_5(x)=x^5-5x^3+4x$ ko‘phadni $x-1$ *ga* bo‘lib,

$$Q_5(x)=x(x-1)(x+1)(x-2)(x+2)$$

ko‘rinishga keltiramiz.

8.7.7-teoremaga asosan, $\frac{P_4(x)}{Q_5(x)}$ kasrni ko‘rinishdagi sodda kasrlar yig‘indisi sifatida tasvirlanadi, ya’ni

$$\frac{3x^4+3x^3-13x^2+4}{x(x-1)(x+1)(x-2)(x+2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+1}+\frac{D}{x-2}+\frac{E}{x+2}.$$

Oxirgi tenglikni umumiy maxrajga keltirib, ushbu

$$3x^4+3x^3-13x^2+4=A(x^2-1)(x^2-4)+Bx\cdot(x+1)(x^2-4)+$$
$$+C\cdot x(x-1)(x^2-4)+D\cdot x(x^2-1)(x+2)+E\cdot x(x^2-1)(x-2)$$

tenglikni hosil qilamiz. Tenglikning o‘ng tomonidagi qavslarni ochib, ko‘phadlarning o‘zaro tengligi haqidagi xossadan foydalanib, x ning bir xil darajalari oldidagi koeffisiyentlarni tenglashtiramiz:

$$\left.\begin{array}{l|l} x^4 & 3=A+B+C+D+E \\ x^3 & 3=B+C+2D-2E \\ x^2 & -13=-5A-4B-4C-D-E \\ x & 0=-4B+4C-2D+2E \\ x^0 & 4=4A \end{array}\right\} \qquad (*)$$

Natijada, A,B,C,D,E noma’lumlarga nisbatan beshta chiziqli algebraik tenglamalar sistemasi hosil qilindi. Bu sistemani yechib, noma’lum noma’lum koeffisiyentlarni topamiz: bunda, $A=1$, (*) sistemaning ikkinchisi va to‘rtinchisini birga yechib, $B=C-1$ eqanligini, birinchi va uchinchisini birga yechib, $D=-E$ eqanligini topamiz. Bularni e’tiborga olib, to‘rtinchi tenglamadan, $C=\frac{3}{2}$, $B=\frac{1}{2}$ ekanligini topamiz. Shunday qilib, noma’lum koeffisiyentlarning hammasi topildi:

$$A=1,\ B=\frac{1}{2},\ \ C=\frac{3}{2},\ \ D=1,\ \ E=-1.$$

Demak,

$$\int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}dx=\int\left(x+\frac{3x^4+3x^3-13x^3+4}{x(x-1)(x+1)(x-2)(x+2)}\right)dx=$$
$$=\int x\,dx+\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x-1}+\frac{3}{2}\int\frac{dx}{x+1}+\int\frac{dx}{x-2}-\int\frac{dx}{x+2}=\frac{x^2}{2}+\ln|x|+$$
$$+\frac{1}{2}\ln|x-1|+\frac{3}{2}\ln|x+1|+\ln|x-2|-\ln|x+2|+C=\frac{x^2}{2}+\ln\left|\frac{x(x-2)(x+1)\sqrt{|x^2-1|}}{x+2}\right|+C.$$

2- **misol**. Quyidagi:

$$1)\ \frac{1}{x^4(1+x^2)};\qquad 2)\ \frac{2x^9+3x^4}{(x^{10}+1)^2}$$

integrallarni hisoblang

Yechilishi. $1)\int\frac{1}{x^4(1+x^2)}dx=\int\frac{1-x^4+x^4}{x^4(1+x^2)}dx=\int\frac{(1-x^2)(1+x^2)+x^4}{x^4(1+x^2)}dx=$

$$=\int\frac{1-x^2}{x^4}dx+\int\frac{dx}{1+x^2}=-\frac{1}{3x^3}+\frac{1}{x}+arctgx+C$$

$$2)\int\frac{2x^9+3x^4}{\left(x^{10}+1\right)^2}dx=\frac{1}{5}\int\frac{d\left(x^{10}+1\right)}{\left(x^{10}+1\right)^2}+\frac{3}{5}\int\frac{dx^5}{\left((x^5)^2+1\right)^2}=\frac{1}{5}\int\frac{du}{u^2}\Bigg|_{u=x^{10}+1}+\frac{3}{5}\int\frac{du}{\left(u^2+1\right)^2}\Bigg|_{u=x^5}=$$

$$=-\frac{1}{x\left(x^{10}+1\right)}+\frac{x^5}{2\left(x^5+1\right)}+\frac{1}{2}arctgx^5+C.$$

Mustaqil yechish uchun misollar

Integrallarni hisoblang:

1. $\int\frac{dx}{x(2+x)}$. **2.** $\int\frac{dx}{x(5+2x)}$. **3.** $\int\frac{dx}{x\left(\frac{3}{2}+\frac{5}{7}x\right)}$.

4. $\int\frac{dx}{x(a+bx)}$. **5.** $\int\frac{dx}{(x+1)(x-3)}$. **6.** $\int\frac{dx}{(2x+3)(x+4)}$.

Integrallarni hisoblang:

7. $\int\frac{dx}{x(4+x^2)}$. **8.** $\int\frac{dx}{x(3+4x)^2}$. **9.** $\int\frac{dx}{x(a+bx)^2}$.

10. $\int\frac{dx}{(2+x)(x-3)^2}$. **11.** $\int\frac{dx}{(3-4x)(5-2x)^2}$. **12.** $\int\frac{dx}{(a+bx)(c+fx)^2}$.

Integrallarni hisoblang:

13. $\int\frac{dx}{x(a+bx)(c+fx)}$. **14.** $\int\frac{dx}{x(x^2+3x+5)}$. **15.** $\int\frac{dx}{(x-3)(3x^2+4x+2)}$.

Mustaqil yechish uchun misollarning javoblari

1. $-\frac{1}{2}\ln\left|\frac{2}{x}+1\right|+C.$ **2.** $-\frac{1}{5}\ln\left|\frac{5}{x}+2\right|+C.$ **3.** $-\frac{2}{3}\ln\left|\frac{3}{2x}+\frac{5}{7}\right|+C.$

4. $-\frac{1}{a}\ln\left|\frac{a}{x}+b\right|+C.$ **5.** $-\frac{1}{4}\ln\left|\frac{x+1}{x-3}\right|+C.$ **6.** $\frac{1}{5}\ln\left|\frac{2x+3}{x+4}\right|+C.$

7. $\frac{1}{4}\left(\frac{1}{4+x}-\frac{1}{4}\ln\left|\frac{4}{x}+1\right|\right)+C.$ **8.** $\frac{1}{3}[\frac{1}{3+4x}-\frac{1}{3}\ln\left|\frac{3}{x}+4\right|]+C.$

9. $\frac{1}{a}[\frac{1}{a+bx}-\frac{1}{a}\ln\left|\frac{a}{x}+b\right|]+C.$ **10.** $\frac{-1}{5(x-3)}+\frac{1}{25}\ln\left|\frac{x+2}{x-3}\right|+C.$

11. $\frac{1}{14(2x-5)}-\frac{1}{49}\ln\left|\frac{3-4x}{5-2x}\right|+C.$ **12.** $\frac{1}{(bc-af)(c+fx)}+\frac{b}{(af-bc)^2}\ln\left|\frac{a+bx}{c+fx}\right|+C.$

13. $\frac{1}{ac}\ln|x|+\frac{b}{a(af-bc)}\ln|a+bx|-\frac{c}{c(af-bc)}\ln|c+fx|+C.$

14. $\frac{1}{10}\ln\frac{x^2}{x^2+3x+5}-\frac{3}{5\sqrt{11}}arctg\frac{2x+3}{\sqrt{11}}+C.$

15. $\frac{1}{82}\ln\left|\frac{(x-3)^2}{3x^2+4x+2}\right|-\frac{11}{41\sqrt{2}}arctg\frac{3x+2}{\sqrt{2}}+C.$

8.4-amaliy mashg'ulot.
Tarkibida trigonometrik funksiyalar qatnashgan ifodalarni integrallash

1-misol. Ushbu $\int\frac{dx}{\sin x+2\cos x+6}$ integralni hisoblang.

Yechilishi. $t=tg\frac{x}{2}$ universal almashtirish olib va (8.10.3) formulalarga asosan, uni t ga nisbatan ratsional funksiyaning integralini hisoblashga keltiramiz:

$$\int\frac{dx}{\sin x+2\cos x+6}=2\int\frac{dt}{4t^2+2t+8}=\frac{1}{2}\int\frac{dt}{\left(t+\frac{1}{4}\right)^2+\frac{31}{16}}=\frac{2}{\sqrt{31}}arctg\frac{4}{\sqrt{31}}(t+\frac{1}{4})+C=$$

$$=\frac{2}{\sqrt{31}}arctg\frac{4}{\sqrt{31}}(tg\frac{x}{2}+\frac{1}{4})+C.\blacksquare$$

2-misol. Ushbu $\int\frac{dx}{\sin^4x\cos x}$ integralni hisoblang.

Yechilishi. Agar $\frac{1}{\sin^4x\cos x}$ ifodada $\cos x$ ni $-\cos x$ ga almashtirsak, u holda, uning ishorasi, qarama - qarshi ishoraga o'zgaradi. Shuning uchun, $t=\sin x,\ x\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ almashtirishni olish qulay bo'ladi. Bunda,

$$x=\arcsin t,\ dx=\frac{1}{\sqrt{1-t^2}}dt,\ \cos x=\sqrt{1-\sin^2x}=\sqrt{1-t^2},$$

va

$$\int\frac{dx}{\sin^4x\cos x}=\int\frac{1}{t^4\sqrt{1-t^2}}\frac{1}{\sqrt{1-t^2}}dt=\int\frac{dt}{t^4(1-t^2)}=$$

$$= \int \frac{dt}{t^4} + \int \frac{dt}{t^2} + \int \frac{dt}{1-t^2} = -\frac{1}{3t^3} - \frac{1}{t} + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C = -\frac{1}{3\sin x^3} - \frac{1}{\sin x} + \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$

3-misol. Ushbu $\int \frac{dx}{\sin^3 x\cos^2 x}$ integralni hisoblang.

Yechilishi. Integral ostidagi ifodada $\sin x$ ni $-\sin x$ ga almashtirgan-da, u o'z ishorasini teskarisiga almashtiradi. Bu holda 2) almashtirishni, ya'ni $t = \cos x,\ x \in (0;\pi)$ deb olish qulay bo'ladi. Bunda,

$$x = \arccos t,\ dx = -\frac{1}{\sqrt{1-t^2}}dt,\ \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-t^2},$$

va

$$\sin^3 x = \sqrt{(1-\cos^2 x)^3} = \sqrt{(1-t^2)^3}$$

$$\int \frac{dx}{\sin^3 x\cos^2 x} = \int \frac{1}{\sqrt{(1-t^2)^3}}\frac{1}{t^2\sqrt{1-t^2}}dt = \int \frac{1}{t^2(1-t^2)}dt = \int \frac{1}{t^2}dt + \int \frac{1}{1-t^2}dt =$$

$$= -\frac{1}{t} + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C = -\frac{1}{\cos x} + \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C.$$

4-misol. Ushbu $\int \frac{\sin x dx}{2\sin x + 3\cos x}$ integralni hisoblang.

Yechilishi. Integral ostidagi ifoda, $\sin x$ va $\cos x$ larni ,mos ravishda, $-\sin x$ va $-\cos x$ larga almashtirganda, o'z ishorasini o'zgartirmaydi. Bu holda $t = tgx,\ x \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ almashtirish bajarilib, berilgan integral ostidagi ifoda t ning ratsional funksiyasiga keltiriladi: $x = arctgt,\ dx = \frac{dt}{1+t^2}$ va

$$\int \frac{\sin x dx}{2\sin x + 3\cos x} = \int \frac{tgx dx}{2tgx + 3} = \int \frac{tdt}{(2t+3)(1+t^2)}.$$

$\frac{t}{(2t+3)(1+t^2)}$ to'g'ri ratsional kasrni, noma'lum koeffisiyentli (8.7.6) ko'rinishdagi sodda kasrlar yig'indisi shaklida tasvirlaymiz:

$$\frac{t}{(2t+3)(1+t^2)} = \frac{A}{2t+3} + \frac{Bt+C}{1+t^2}$$

Undagi noma'lum koeffisiyentlarni topish uchun, yuqorida ko'rsatilgan noma'lum koeffisiyentlar usulidan foydalanib, chiziqli algebraik tenglamalar sistemasini yechamiz:

$$t = A(1+t^2) + (Bt+C)(2t+3) \Rightarrow \begin{cases} t^2 \big| & A + 2B = 0, \\ t \big| & 3B + 2C = 1, \\ t^0 \big| & A + 3C = 0, \end{cases} \Rightarrow A = -\frac{6}{13}, B = \frac{3}{13}, C = \frac{2}{13},$$

$$\int \frac{\sin x dx}{2\sin x + 3\cos x} = \int [\frac{A}{2t+3} + \frac{Bt+C}{1+t^2}]dt = -\frac{6}{13}\int \frac{dt}{2t+3} + \frac{1}{13}\int \frac{3t+2}{1+t^2}dt =$$

$$= -\frac{3}{13}\ln|2t+3| + \frac{3}{26}\ln|1+t^2| + \frac{2}{13}arctgt + C = -\frac{3}{13}\ln|2\sin x + 3\cos| + \frac{2}{13}x + C.$$

Mustaqil yechish uchun misollar

Integrallarni hisoblang.

1. $\int \cos^2 x dx.$ **2.** $\int \cos^3 pxdx.$ **3.** $\int \sin^3 pxdx.$

4. $\int \cos^4 pxdx.$ **5.** $\int \sin^5 xdx.$ **6.** $\int tg^3 dx.$

7. $\int \cos 5x\cos 9x dx.$ **8.** $\int \sin 5x\sin 3x dx.$ **9.** $\int \cos 4x\cos x\, dx.$

10. $\int \sin 7x\cos 3x\, dx.$ **11.** $\int \sin 3x\cos 5x\, dx.$ **12.** $\int \cos px\cos qx\, dx.$

13. $\int \frac{dx}{\sin x}.$ **14.** $\int \frac{dx}{\sin^3 x}.$ **15.** $\int \frac{dx}{\sin^4 x}.$

16. $\int \frac{dx}{\sin^5 x}.$ **17.** $\int \frac{dx}{\sin^6 x}.$ **18.** $\int \frac{dx}{\cos x}.$

19. $\int \frac{\sin^2 x}{\cos x}dx.$ **20.** $\int \frac{\sin^3 x}{\cos x}dx.$ **21.** $\int \frac{\sin^4 x}{\cos x}dx.$

22. $\int \frac{\sin x}{\cos^2 x}dx.$ **23.** $\int \frac{\sin^2 x}{\cos^2 x}dx.$ **24.** $\int \frac{\sin^3 x}{\cos^2 x}dx.$

25. $\int \frac{sh^2 x}{chx}dx.$ **26.** $\int \frac{ch^3 x}{chx}dx.$ **27.** $\int \frac{sh^4 x}{chx}dx.$

28. $\int \frac{sh^2 x}{ch^2 x}dx.$ **29.** $\int \frac{sh^3 x}{ch^2 x}dx.$ **30.** $\int \frac{sh^2 x}{ch^3 x}dx.$

31. $\int \sin^2 x\cos^2 dx.$ **32.** $\int \sin^2 x\cos^3 x dx.$ **33.** $\int \sin^2 x\cos^4 x dx.$

34. $\int \sin^3 x\cos x dx.$ **35.** $\int \frac{dx}{3+5\cos x}.$ **36.** $\int \frac{dx}{10+8\cos x}.$

37. $\int \frac{dx}{5+4\cos x}.$ **38.** $\int \frac{dx}{13+5\sin x}.$

Mustaqil yechish uchun misollarning javoblari

1. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$. **2.** $\frac{1}{p}\sin px-\frac{1}{3p}\sin^3 px+C$. **3.** $\frac{1}{3p}\cos^3 px-\frac{1}{p}\cos px+C$.

4. $\frac{3}{8}x+\frac{1}{4p}\sin 4px+C$. **5.** $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$. **6.** 6) $\frac{1}{2}tg^2x+\ln|\cos x|+C$.

7. $\frac{1}{8}\sin 4x+\frac{1}{28}\sin 4x+C$. **8.** $\frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+C$. **9.** $\frac{1}{6}\sin 3x+\frac{1}{10}\sin 5x+C$.

10. $-\frac{1}{8}\cos 4x-\frac{1}{20}\cos 10x+C$. **11.** $-\frac{1}{16}\cos 8x+\frac{1}{4}\cos 2x+C$.

12. $\frac{\sin(p+q)x}{2(p+q)}+\frac{\sin(p-q)x}{2(p-q)}+C,\ (p^2\neq q^2)$ **13.** $\ln\left|tg\frac{x}{2}\right|+C$.

14. $-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+C$. **15.** $-\frac{\cos x}{3\sin^3 x}-\frac{3}{2}ctgx+C=-\frac{1}{3ctg^3x}-ctgx+C$.

16. $-\frac{\cos x}{4\sin^4 x}-\frac{3}{8}\frac{\cos x}{\sin^2 x}+\frac{3}{8}\ln\left|tg\frac{x}{2}\right|+C$. **17.** $-\frac{1}{5}ctg^5x-\frac{2}{3}ctg^3x-ctgx+C$.

18. $\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$. **19.** $-\sin x+\ln\left|tg\left(\frac{\pi}{2}+\frac{x}{2}\right)\right|+C$.

20. $\frac{1}{2}\cos^2 x-\ln|\cos x|+C$. **21.** $\frac{1}{3}\sin^3 x-\sin x+\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$.

22. $\frac{1}{\cos x}+C$. **23.** $tgx-x+C$. **24.** $\cos x+\frac{1}{\cos x}+C$. **25.** $shx-arctg(shx)+C$.

26. $\frac{1}{2}ch^2x-\ln chx+C$. **27.** $\frac{1}{3}sh^3x-shx+arctg(shx)+C$. **28.** $x-thx+C$.

29. $-\frac{3}{2}x+\frac{1}{4}sh2x+thx+C$. **30.** $-\frac{shx}{2ch^2x}+\frac{1}{2}arctg(shx)+C$.

31. $\frac{1}{8}x-\frac{1}{32}\sin 4x+C$. **32.** $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$.

33. $\frac{x}{16}+\frac{1}{64}\sin 2x-\frac{1}{64}\sin 4x-\frac{1}{192}\sin 6x+C$. **34.** $\frac{1}{4}\sin^4 x+C$. **35.** $\frac{1}{4}\ln\left|\frac{tg\frac{x}{2}+2}{tg\frac{x}{2}-2}\right|+C$.

36. $\frac{1}{3}arctg\left(\frac{tg\frac{x}{2}}{3}\right)+C$. **37.** $\frac{2}{3}arctg\left(\frac{tg\frac{x}{2}}{3}\right)+C$. **38.** $\frac{1}{6}arctg\left(\frac{13tg\frac{x}{2}+5}{12}\right)+C$.

8.5-amaliy mashg'ulot.
Ba'zi irratsional ifodalarni integrallash

1-misol. Ushbu $\int\frac{x+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$ integralni hisoblang.

Yechilishi. Integral ostidagi funksiya, $x_1=x,\ x_2=x^{\frac{1}{3}},\ x_3=x^{\frac{1}{6}}$ o'zgaruvchilarga nisbatan ratsional funksiya. Berilgan integral 4.1- banddagi integral ko'rinishida bo'lib, $\alpha=\frac{1}{3},\ \beta=\frac{1}{6},\ n_1=3,\ n_2=6,\ k=6$. Uni hisoblash, $x=t^6$ almashtirish yordamida ratsional funksiyaning integralini hisoblashga keltiriladi: $dx=6t^5dt$,

$$\int\frac{x+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx=\int\frac{t^6+t}{t^6(1+t^2)}6t^5dt=6\int\frac{t^5+1}{1+t^2}dt=6\int(t^3+1)dt+6\int\frac{t+1}{1+t^2}dt=$$

$$=\frac{3}{2}t^4+3t^3+6arctgt+3\ln(1+t)+C=\frac{3}{2}\sqrt[3]{x^2}+3\sqrt{x}+6arctg\sqrt[6]{x}+3\ln\left(1+\sqrt[3]{x}\right)+C$$

2-misol. Ushbu $\int x^2\sqrt{4-x^2}dx$ integralni hisoblang.

Yechilishi. Berilgan integral

$$\int R\left(x,\sqrt{a^2-x^2}\right)dx,\ \int R\left(x,\sqrt{a^2+x^2}\right)dx,\ \int R\left(x,\sqrt{x^2-a^2}\right)dx$$

ko'rinishdagi ifodalarning birinchi integrali ko'rinishida bo'lganligi uchun, $x=2\sin t$ almashtirishni bajarib, hisoblaymiz:

$$\int x^2\sqrt{4-x^2}dx=\left|x=2\sin t,\ dx=2\cos tdt,\ \sqrt{4-4\sin^2 t}=2\cos t\right|=$$

$$=16\int\sin^2 t\cos^2 tdt=4\int\sin^2 2tdt=2\int(1-\cos 4t)dt=2t-\frac{1}{2}\sin 4t+C.$$

$$\sin 4t=2\sqrt{1-\sin^2 t}\sin t\left(1-2\sin^2 t\right),-\frac{\pi}{2}\le t\le\frac{\pi}{2},\ \sin\left(\arcsin\frac{x}{2}\right)=\frac{x}{2},\ -2\le x\le 2$$

formulalarni e'tiborga olgan holda, eski o'zgaruvchiga qaytib, integralni hisoblaymiz:

$$\int x^2\sqrt{4-x^2}dx=2t-\frac{1}{2}\sin 4t+C=2\arcsin\frac{x}{2}+\frac{x}{4}\left(x^2-2\right)\sqrt{4-x^2}+C.$$

3-misol. Ushbu $\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}$ integralni hisoblang.

Yechilishi. $1+x-x^2$ kvadrat uchhad kompleks ildizga ega va $a<0,\ c>0$ bo'lgani uchun *2-hol* asosan $\sqrt{1+x-x^2}=tx-1$ almashtirishni bajaramiz:

$$1+x-x^2=t^2x^2-2tx+1;\ 1-x=t^2x-2t;\ \ x=\frac{1+2t}{1+t^2},$$

$$dx=\frac{2(1-t-t^2)}{(1+t^2)^2}dt,\ \ \sqrt{1+x-x^2}=\frac{t^2+t-1}{t^2+1}$$

Shunday qilib

$$\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}=\int\frac{2(1-t-t^2)}{(1+t^2)^2\left(1+\frac{1+2t}{t^2+1}\right)\cdot\left(\frac{t^2+t-1}{t^2+1}\right)}dt=$$

$$=-2\int\frac{dt}{(t+1)^2+1}=-2\,arctg(t+1)+C=-2arctg\frac{\sqrt{1+x-x^2}+x+1}{x}+C.$$

Mustaqil yechish uchun misollar

Quyidagi integrallarni hisoblang.

1. $\int\frac{dx}{\sqrt{x}+\sqrt[4]{x}}$. **2.** $\int\frac{dx}{\sqrt[3]{x}+\sqrt{x}}$. **3.** $\int\frac{dx}{1+\sqrt{x+1}}$.

4. $\int\frac{1+\sqrt{x+1}}{2+\sqrt{x+1}}dx$. **5.** $\int\frac{dx}{\sqrt[3]{x}+2\sqrt[4]{x}+\sqrt{x}}$. **6.** $\int\frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx$..

Quyidagi integrallarni (Eyler almashtirishlaridan foydalanib) hisoblang.

7. $\int\frac{dx}{1+\sqrt{x^2+2x+2}}$. **8.** $\int\frac{dx}{x\sqrt{4x^2+4x+3}}$.

9. $\int\frac{dx}{(1+x)\sqrt{1+x+x^2}}$. **10.** $\int\frac{dx}{x+\sqrt{x^2-x+1}}$.

Quyidagi integrallarni, binomial differensiallarni integrallash usulidan foydalanib,hisoblang.

11. $\int\sqrt[3]{x}(1+\sqrt{x})^2dx$. **12.** $\int\frac{1}{\sqrt[6]{x}}(\sqrt[3]{x}-1)^2dx$.

13. $\int\frac{dx}{\sqrt[4]{x^3}(\sqrt[8]{x}-1)^5}$. **14.** $\int\frac{dx}{x(1+\sqrt[3]{x})^2}$.

Quyidagi integrallarni, $\int R(x,\sqrt{x^2+a^2})dx$, $\int R(x,\sqrt{a^2-x^2})dx$ ifodalarni integrallash usulidan foydalanib, hisoblang.

15. $\int\sqrt{9-x^2}dx$. **16.** $\int\sqrt{9-16x^2}dx$.

17. $\int\sqrt{a^2-b^2x^2}dx$. **18.** $\int\frac{dx}{\sqrt{(4-x^2)^3}}$.

Mustaqil yechish uchun misollarning javoblari

1. $2\sqrt{x}-4\sqrt[4]{x}+4\ln\left|\sqrt[4]{x}+1\right|+C$. **2.** $6[\frac{1}{3}.\sqrt{x}-\frac{1}{2}\sqrt[3]{x}+\sqrt[6]{x}-\ln\left|1+\sqrt[6]{x}\right|]+C$.

3. $2\sqrt{x+1}-2\ln\left|\sqrt{x+1}+1\right|+C$. **4.** $x+1-2\sqrt{x+1}+2\ln\left|\sqrt{x+1}+2\right|+C$.

5. $2\sqrt{x}-3\sqrt[3]{x}-8\sqrt[4]{x}+6\sqrt[6]{x}+48\sqrt[12]{x}++3\ln\left(1+\sqrt[12]{x}\right)+\frac{33}{2}\ln\left(\sqrt[6]{x}-\sqrt[12]{x}+2\right)-$

$-\frac{171}{\sqrt{7}}arctg\frac{2\sqrt[12]{x}-1}{\sqrt{7}}+C$. **6.** $6t-3t^2+\frac{3}{2}t^4+\frac{6}{5}t^5-\frac{6}{7}t^7+3\ln(1+t^2)-6arctgt+C,\ t=\sqrt[6]{x+1}$.

7. $\ln\left(x+1+\sqrt{x^2+2x+2}\right)+\frac{2}{x+2+\sqrt{x^2+2x+2}}+C$. **7.** $\frac{1}{\sqrt{3}}\ln\frac{2x+\sqrt{4x^2+4x+3}-\sqrt{3}}{2x+\sqrt{4x^2+4x+3}+\sqrt{3}}+C$.

9. $-2arctg\frac{1+x+\sqrt{1+x-x^2}}{x}+C$. **10.** $2\ln|t|-\frac{1}{2}\ln|t-1|+\frac{3}{t+1}-\frac{3}{2}\ln|t+1|+C$, bunda $t=\frac{\sqrt{x^2-x+1}+1}{x}$. **11.** $\frac{3}{7}x^{7/3}+\frac{24}{11}x^{11/6}+3x^{4/3}+C$. **12.** $6\sqrt[6]{x}+\frac{9}{2}\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|-\frac{3\sqrt[6]{x}}{\sqrt[3]{x-1}}+C$.

13. $\frac{-8}{3(\sqrt[8]{x}-1)^3}-\frac{2}{(\sqrt[8]{x}-1)^4}+C$. **14.** $3\left(\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+\frac{1}{1+\sqrt[3]{x}}\right)+C$.

15. $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\arcsin\frac{x}{3}+C$. **16.** $x\frac{\sqrt{9-16x^2}}{2}+\frac{9}{8}\arcsin\frac{3x}{4}+C$.

17. $\frac{x\sqrt{a^2-b^2x^2}}{2}+\frac{a^2}{2^b}\arcsin\frac{bx}{a}+C$. **18.** $\frac{x}{4\sqrt{4-x^2}}+C$.

8.6-amaliy mashg'ulot.
Aniqmas integralning fizika masalalarini yechishga tadbiqlari

Harakat boshlangandan o'tgan t vaqt ichida moddiy nuqta $s(t)$ yo'l o'tgan bo'lsin, u holda, $v(t)$ oniy tezlik, $s(t)$ funksiyaning hosilasiga teng, ya'ni $v(t)=s'(t)$. Amaliyotda teskari masala ham uchraydi: moddiy nuqtaning $v(t)$ harakat tezligi berilganda, uning bosib o'tgan $s(t)$ yo'lini toping. Amaliyotdagi bunday masala, $s(t)$ funksiyaning *boshlang'ich funksiyasi* tushunchasiga olib keladi.

1-masala. Moddiy nuqtaning tekis to'g'ri chiziqli harakat tezligi $v(t)=t^2(t+2)(t-3)$ qonun bo'yicha o'zgaradi. Uning harakat qonunini toping.

Yechilishi. Ma'lumki, moddiy nuqtaning tekis to'g'ri chiziqli harakat tezligi $s(t)$ yo'ldan t vaqt bo'yicha olingan hosilaga, ya'ni $v(t)=\frac{ds}{dt}=t^2(t+2)(t-3)$ ga teng. Bu yerdan $ds = t^2(t+2)(t-3)dt$ ga ega bo'lamiz. Moddiy nuqtaning harakat qonunni aniqmas integralning 3^0 – xossasi, hamda jadvaldagi 3 – formulaga asosan, integrallab topamiz:

$$\int ds=\int t^2(t+2)(t-3)dt=\int\left(t^4-t^3-6t^2\right)dx=\frac{t^5}{5}-\frac{t^4}{4}-2t^3+C,$$

ya'ni, $s(t)=\frac{t^5}{5}-\frac{t^4}{4}-2t^3+C$.

Tekshirish. Topilgan boshlang'ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko'ramiz:

$$s'(t)=\left(\frac{t^5}{5}-\frac{t^4}{4}-2t^3+C\right)'=$$
$$=t^4-t^3-6t^2+0=t^2\left(t^2-t-6\right)=t^2(t+2)(t-3),\ t\in R.$$

Demak, Moddiy nuqtaning harakat qonuni to'g'ri topilgan ekan. ■

2-masala. Moddiy nuqta $a(t)=6t-12$ tezlanish bilan to'g'ri chiziqli harakat qiladi. $t=0$ paytda (sanoq boshida) boshlang'ich tezlik $v_0=9(m/sek)$ ga, sanoq boshidan masofasi $s_0=10(m)$ ga teng. Quyidagilarni toping: 1) moddiy nuqtaning tezligi va harakat qonunini; 2) $t=2(sek)$ dagi tezlanish, tezlik va yo'lni; 3) tezlik eng kichik bo'lgan paytini.

Yechilishi. 1) Tezlikni topamiz: $a(t)=\frac{dv}{dt}=6t-12$ yoki $dv(t)=(6t-12)dt$. Integrallab topamiz: $\int dv(t)=\int(6t-12)dt;\ v(t)=3t^2-12t+C_1$. Boshlang'ich $t=0,\ v_0=9$ shartlardan foydalanib, quiydagiga ega bo'lamiz: $9=3\cdot0^2-12\cdot0+C_1$ ya'ni $C_1=9$. Demak, $v(t)=3t^2-12t+9$.

Nuqtaning harakat qonunini topamiz; $v(t)=\frac{ds}{dt}=3t^2-12t+9$ yoki $ds(t)=\left(3t^2-12t+9\right)dt$. Integrallab topamiz:

$$\int ds(t)=\int\left(3t^2-12t+9\right)dt+C_2;\ s(t)=t^3-6t^2+9t+C_2.$$

Boshlang'ich $t=0,\ s_0=10$ shartlardan foydalanib, quiydagiga ega bo'lamiz: $10=0^3-6\cdot0^2+9\cdot0+C_2$ ya'ni $C_2=10$.

Shunday qilib, moddiy nuqtaning harakat qonuni: $s(t)=t^3-6t^2+9t+10$.

2) $t=2(sek)$ bo'lganda, a, v va s larni topamiz:

$a(2)=6\cdot2-12=0;\ v(2)=3\cdot2^2-12\cdot2+9=-3(m/sek);$

$s(2)=2^3-6\cdot2^2+9\cdot2+10=12(m).$

3) Tezlik o'zgarishini aniqlovchi $v(t)$ funksiyani ekstremumga tekshiramiz:

$v(t)=3t^2-12t+9;\ v'(t)=6t-12;\ 6t-12=0,\ t=2;\ v''(2)=6>0.$

Demak, $t=2(sek)$ bo'lganda tezlik eng kichik bo'ladi.

Mustaqil yechish uchun masalalar

1. Moddiy nuqtaning tekis to'g'ri chiziqli harakat tezligi $v(t)=3t^2-2t$ qonun bo'yicha o'zgaradi. Uning harakat qonunini toping.

2. Moddiy nuqtaning tekis to'g'ri chiziqli harakat tezligi $v(t)=\sin t+\frac{8}{\pi^2}t$ qonun bo'yicha o'zgaradi. Agar nuqta $t=\frac{\pi}{2}(sek)$ ichida $3(m)$ yurgan bo'lsa, uning harakat qonuni $s(t)$ni toping.

3. Moddiy nuqta $a(t)=12t^2+6t$ tezlanish bilan to'g'ri chiziqli harakat qiladi. Agar $t=1(sek)$ vaqt momentda uning tezligi $v=8(m/sek)$, bosib o'tilgan yo'li $s=6(m)$ bo'lsa, bu nuqtaning harakat qonunini toping.

4. Moddiy nuqta $a(t)=24t^2+8$ tezlanish bilan to'g'ri chiziqli harakat qiladi. Agar $t=1(sek)$ vaqt momentda uning tezligi $v=10(m/sek)$, bosib o'tilgan yo'li $s=12(m)$ bo'lsa, bu nuqtaning harakat qonunini toping.

5. Moddiy nuqta $a(t)=-6t-18$ tezlanish bilan to'g'ri chiziqli harakat qiladi. $t=0$ paytda (sanoq boshida) boshlang'ich tezlik $v_0=24(m/sek)$ ga, sanoq boshidan masofasi $s_0=15(m)$ ga teng. Quyidagilarni toping: 1) moddiy nuqtaning tezligi va harakat qonunini; 2) $t=2(sek)$ dagi tezlanish, tezlik va yo'lni; 3) tezlik eng katta bo'ladigan vaqtni.

6. Moddiy nuqtaning tekis to'g'ri chiziqli harakatida bosib o'tilgan yo'lning ifodasini, $t=0$ vaqt momentida s_0 yo'l o'tilgan deb hisoblab, toping.

7. Quyidagi ifodalarni: 1) moddiy nuqtaning tekis tezlanuvchan to'g'ri chiziqli harakati tezligini, $t=0$ vaqt momentida tezlik $v=v_0$ deb hisoblab; 2) moddiy nuqtaning tekis tezlanuvchan to'g'ri chiziqli harakatda bosib o'tgan yo'l ifodasini, moddiy nuqtaning boshlang'ich vaqt momentidagi tezligi $v=v_0$ va ungacha $s=s_0$ yo'l bosib o'tilgan deb hisoblab, toping.

8. Moddiy nuqtaning tekis aylanma harakati φ burchak siljishi ifodasini, nuqtaning boshlang'ich vaqt momentidagi burchak holati φ_0 deb hisoblab, toping.

9. Quyidagi ifodalarni: 1) moddiy nuqtaning tekis tezlanuvchan aylanma harakati burchak tezligini, nuqtaning boshlang'ich vaqt momentidagi burchak tezligi φ_0 deb hisoblab; 2) moddiy nuqtaning tekis tezlanuvchan aylanma harakati burchak siljishini, boshlang'ich vaqt momentida nuqta ω_0 burchak tezligiga va φ_0 burchak siljishiga ega deb hisoblab, toping.

10. Ox o'q bo'ylab harakatlanayotgan moddiy nuqtaga ta'sir qilayotgan barcha kuchlarning yig'indisi quyidagicha ifodalanadi: $\vec{F}=Bt\,\vec{i}+t^2\,\vec{j}$. Moddiy nuqta impulsining o'zgarish qonunini toping. Boshlang'ich vaqt momentida moddiy nuqta impulsining qiymati p_0 ga teng deb hisoblansin.

11. Ox o'q bo'ylab harakatlanayotgan m massali moddiy nuqtaga ta'sir qilayotgan barcha kuchlarning teng ta'sir etuvchisi quyidagicha ifodalanadi: $F_x=F_0\cos\Omega t$ bunda F_0,Ω- musbat o'zgarmas miqdorlar. Moddiy nuqta tezligining o'zgarish qonunini toping. Boshlang'ich vaqt momentida moddiy nuqta tinch holatda, deb hisoblansin.

12. Absolyut (mutloq) qattiq jism uning massalar markazidan o'tib, Oz o'q bilan ustma-ust tushadigan qo'zg'almas o'q atrofida aylanmoqda. Jismga ta'sir qiluvchi tashqi kuchlarning aylanish o'qiga nisbatan kuch

momenti $M=8t^3\vec{k}$ ifoda orqali yoziladi.Qattiq jism impuls momentining vaqt bo'yicha o'zgarish qonunini toping.

13. Massasi m va radiusi r bo'lgan bir jinsli disk Oz o'q bilan ustma-ust tushuvchi va uning massalar markazidan o'tuvchi qo'zg'almas o'q atrofida ω_0 burchak tezligi bilan tekis aylanmoqda. Boshlang'ich vaqt momentida diskka tashqi kuchlarning $\vec{M}=(6t^2+2)\vec{k}$ momenti ta'sir qiladi.Disk aylanishining burchak tezligi o'zgarishi qonunini toping.

14. Massasi m va radiusi r bo'lgan bir jinsli obruch (gimnastika xalqasi), Oz o'q bilan ustma-ust tushuvchi va obruch markazidan uning tekisligiga perpendikulyar ravishda o'tuvchi qo'zg'olmas o'q atrofida, obruchga qo'yilgan va aylanish o'qiga proyeksiyasi $M_z=At+Bt^2$ bo'lgan kuch momenti ta'siri ostida, aylanmoqda. Obruch nuqtalari chiziqli tezligining o'zgarish qonunini toping.

Mustaqil yechish uchun masalalarning javoblari

1. $s=t^3-t^2+C$. **2.** $s(t)=-\cos t+\frac{4}{\pi^2}t^2+2$. **3.** $s=t^4+t^3+t+3$.

4. $s=2t^4+4t^2-6t+12$. **5. 1)** $v(t)=-3t^2+18t+24,\ s(t)=-t^3+9t^2+24t+15$;

2) $a(2)=6(m/sek^2),\ v(2)=48(m/sek),\ s(2)=91(m)$; **3)** $v_{max}=v(3)=51(m/sek)$.

6. s_0+vt. 6-7 *masalalar uchun ko'rsatma.* To'g'ri chiziqli harakatda moddiy nuqtaning tezligi $v=\frac{ds}{dt}$, bu yerdan, $ds=vdt$ va $s=\int vdt$, uning tezlanishi esa, $a=\frac{dv}{dt}$ (tangensial tezlanish to'ladir), buyerdan, $dv=adt$ va $v=\int adt$. **7.** $v_0+at;\ s_0+v_0t+\frac{at^2}{2}$. **8.** $\varphi_0+\omega t$. 8-9-*masalalar uchun ko'rsatma.* Moddiy nuqta qo'zg'olmas o'q atrofida aylanma harakat sodir etganda, uning burchak tezligi $\omega=\frac{d\varphi}{dt}$, bu yerdan, $d\varphi=\omega dt$ va $\varphi=\int\omega dt$, burchak tezlanishi esa, $\beta=\frac{d\omega}{dt}$, bu yerdan, $d\omega=\beta dt$ va $\omega=\int\beta dt$.

9. $\omega_0+\beta t;\ \varphi_0+\omega_0 t+\frac{\beta t^2}{2}$.

10. $\left(p_0+\frac{Bt^2}{2}\right)\vec{i}$. *10-11- masalalar uchun ko'rsatma.* Impulsning o'zgarish qonuniga binoan, $\frac{d\vec{p}}{dt}=\vec{F}$ yoki Ox o'qqa proyeksiyalaganda, $\frac{dp_x}{dt}=F_x$, bundan, $p_x=\int F_x dt$ va $p_x=p$. 10- masalaning shartiga binoan, $t=0$ bo'lganda $p=p_0$, ya'ni $p_x=p_0$ va $C=p_0$. 11- masalani yechishda $p_x=mv_x$, ya'ni $v=v_x=\frac{1}{m}\int F_x dt$ ekanligini hisobga olish lozim bo'ladi. Shartga ko'ra, $t=0$ bo'lganda $v=0$ bo'lganligidan, $C=0$. **11.** $v=\frac{F_0}{m\Omega}\sin\Omega t$.

12. $\vec{L}=2t^4\vec{k}$. *12-14- masalalar uchun ko'rsatma.* Impuls momentining o'zgarish qonuniga muvofiq, absolyut qattiq jism qo'zg'almas o'q atrofida aylanganda, $\frac{d\vec{L}}{dt}=\vec{M}$, bunda $\vec{L}$- jism impulsining aylanish o'qiga nisbatan momenti; $\vec{M}$- jismga ta'sir etuvchi barcha tashqi kuchlarning shu o'qqa nisbatan kuch momenti. Modomiki, mazkur holda $\vec{L}$ qoridagi ekan, yu $\vec{M}$ || ifodani, skalyar ko'rinishda yozish mumkin, bu yerdan, $L=\int Mdt$ yoki $L_z=\int M_z dt$. 12- masalada $t=0$ vaqt momentida jism tinch holatda, ya'ni $\vec{\omega}=0$, mos ravishda, $L=J\vec{\omega}=0$ bo'ladi. Bu shartlarda, $C=0$. 13- masalada diskning inersiya momenti $J=\frac{1}{2}mr^2$ va, demak, $L=J\omega=\frac{1}{2}mr^2\omega$, bundan, $\omega=\frac{2}{mr^2}\int Mdt$. Masalaning shartiga binoan, $t=0$ vaqt momentida $\vec{\omega}=\vec{\omega}_0$. Integrallash amali bajarilgandan so'ng olingan ifodaga bu qiymatlarni keltirib qo'yib, $C=\omega_0$ ekanligini olamiz. 14- masalada obruchning inersiya momenti $J=mr^2$, demak, $L_z=J\omega_z=mr^2\frac{v}{r}=mrv$, ya'ni, $v=\frac{1}{mr}\int M_z dt$. Modomiki, obruch kuchning M momenti ta'siri ostida aylana boshlar ekan, $t=0$ vaqt momentida uning tezligi $v=0$ bo'ladi va buyerdan $C=0$. $2t^4k$.

13. $\left(\omega_0+\frac{4t(t^2+1)}{mR^2}\right)\vec{k}$. **14.** $\frac{1}{mR}\left(\frac{At^2}{2}+\frac{Bt^3}{3}\right)$.

8.7-amaliy mashg'ulot.
Aniq integralni hisoblash. Nyuton-Leybnis formulasi

1–misol. Ushbu

$$\int_0^{\ln 3}\frac{e^x\sqrt{e^x+1}}{e^x+5}dx$$

integralni hisoblang.

Yechilishi. $\sqrt{e^x+1}=t$ almashtirishni bajaramiz: $x_1=0$ bo'lganda, $t_1=\sqrt{2}$, $x_2=\ln 3$ bo'lganda, $t_2=2$. Demak, x o'zgaruvchi, $[0;\ln 3]$ kesmada o'zgarganda, yangi t o'zgaruvchi $[\sqrt{2};2]$ kesmada o'zgaradi. $x=\ln(t^2-1)$ funksiya, $t=\sqrt{e^x+1}$ funksiyaga teskari, $[\sqrt{2};2]$ kesmada monoton, uzluksiz va $x_t'=\frac{2t}{t^2-1}$.

Shunday qilib, teoremaning hamma shartlari o'rinli, shuning uchun, (8.20.6) formulaga asosan,

$$\int_0^{\ln 3}\frac{e^x\sqrt{e^x+1}}{e^x+5}dx=\int_{\sqrt{2}}^{2}\frac{(t^2-1)\cdot t\cdot 2tdt}{(t^2+4)(t^2-1)}=\int_{\sqrt{2}}^{2}\frac{2t^2dt}{t^2+4}=$$

$$=2\int_{\sqrt{2}}^{2}\left(1-\frac{4}{t^2+4}\right)dt=2[t-2arctg\frac{t}{2}]_{\sqrt{2}}^{2}=2[2-2\cdot\frac{\sqrt{\pi}}{4}-\sqrt{2}+2arctg\frac{\sqrt{2}}{2}]=$$

$$=4-\sqrt{\pi}-2\sqrt{2}+2arctg\frac{\sqrt{2}}{2}.$$

2–misol. Ushbu $\int_0^{\sqrt{3}}x\,arctgx\,dx$ integralni hisoblang.

Yechilishi. Integral ostidagi funksiyalar, teoremaning barcha shartlarini qanoatlantirishiga ishonch hosil qilish qiyin emas. $u=arctgx$, $dv=xdx$ deb olib, $\int_a^b udv=uv\Big|_a^b-\int_a^b vdu$ formulaga asosan,

$$\int_0^{\sqrt{3}}x\,arctg\,x\,dx=[du=\frac{dx}{1+x^2},\ v=\frac{x^2}{2}]=$$

$$=\frac{x^2}{2}arctgx\Big|_0^{\sqrt{3}}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2dx}{1+x^2}]=\frac{\pi}{3}\cdot\frac{3}{2}-\frac{1}{2}(x-arctgx)\Big|_0^{\sqrt{3}}=$$

$$=\frac{\pi}{2}-\frac{1}{2}\left(\sqrt{3}-\frac{\pi}{3}\right)=\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\pi}{6}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$$

Misolni Marlye tizimidan foydalanib, yechish:

> int(x*arctan(x),x=0..sqrt(3));

$$\frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$$

Mustaqil yechish uchun misollar

Quyidagi integrallarni, Nyuton – Leybnis formulasiga asosan, hisoblang.

1. $\int_0^1 (2x-3)dx$. **2.** $\int_1^0 5x^4 dx$. **3.** $\int_1^4 2\sqrt{x}dx$.

4. $\int_1^5 2\sqrt{x-1}dx$. **5.** $\int_{-2}^0 (x+1)(x-2)dx$. **6.** $\int_1^2 \left(3t+\frac{4}{t^2}\right)dt$.

7. $\int_0^\pi (1+\cos)dx$. **8.** $\int_0^1 \left(x^2+\sqrt{x}\right)dx$. **9.** $\int_0^1 \left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right)dx$

Quyidagi integrallarga, Nyuton–Leybnis formulasini formal ravishda qo‘llaganda, noto‘g‘ri natijaga kelinishini izohlang.

10. $\int_0^{dx} \frac{dx}{(2+tg^2x)\cos^2 x}$. **11.** $\int_{-1}^1 \frac{d}{dx}\left(arctg\frac{1}{x}\right)dx$. **12.** $\int_{-1}^1 \frac{dx}{x}$.

Quyidagi integrallarni hisoblang.

13. $\int_0^2 3^x dx$. **14.** $\int_0^1 \frac{x^2 dx}{1+x^6}$. **15.** $\int_{e^2}^{e^3} \frac{dx}{x\ln x}$.

16. $\int_{-\pi}^{\pi} \sin^2 x dx$. **17.** $\int_0^1 \frac{dx}{4x^2+4x+5}$. **18.** $\int_2^3 \frac{dx}{x^2-2x-8}$.

Quyidagi aniq integrallarni, o‘zgartiruvchilarni almashtirish yordamida, hisoblang.

19. $\int_0^3 \sqrt{y+1}dy$. **20.** $\int_{-1}^0 \sqrt{y+1}\,dx$. **21.** $\int_0^\pi 3\cos^2 x\sin dx$.

22. $\int_{2\pi}^{3\pi} \cos^2 x\sin x dx$. **23.** $\int_{-1}^1 \frac{5x}{(4+x^2)^2}dx$. **24.** $\int_0^1 \frac{5x}{(4+x^2)^2}dx$.

25. $\int_0^{\frac{\pi}{6}} (1-\cos 3t)\sin 3t dt$. **26.** $\int_{\frac{\pi}{6}}^{\pi/3} (1-\cos 3t)\sin 3t dt$. **27.** $\int_0^{2\pi} \frac{\cos t}{\sqrt{4+3\sin t}}dt$.

28. Ushbu $\int_1^7 (x^2-6x+13)dx$ integralda, $x^2-6x+13=t$ almashtirishni olish mumkinmi? Javobingizni sharhlang.

29. Ushbu $\int_0^1 \sqrt{1-x^2}dx$ integralda, $x=\sin t$ almashtirishni olish mumkinmi? Javobingizni sharhlang.

Bo‘laklab integrallash formulasi yordamida, aniq integrallarni hisoblang.

30. $\int_0^4 xe^{-x}dx$. **31.** $\int_1^2 x\ln x dx$. **32.** $\int_0^{\pi/2} t^2\sin 2t dt$.

33. $\int_0^1 x\,arctg(x^2)dx$. **34.** $\int_0^{\pi/3} x\,tg^2 x dx$. **35.** $\int_1^2 x^3\ln x dx$.

Mustaqil yechish uchun misollarning javoblari

1. -2. **2.** -1. **3.** $7/3$. **4.** $8/3$. **5.** $\frac{2}{3}$. **6.** $13/2$. **7.** π. **8.** 1. **9.** $-\frac{4}{15}$. **10.** $\frac{1}{\sqrt{2}}arctg\left(\frac{tgx}{\sqrt{2}}\right)$ funksiya, integral ostidagi funksiya uchun boshlang‘ich funksiya bo‘lib, u, $0\le x\le 2\pi$ da uzilishga ega. **11.** $arctg\frac{1}{x}$ funksiya, $x=0$ nuqtada uzilishga ega. **12.** Integral ostidagi $\frac{1}{x}$ funksiya va uning $\ln|x|$ boshlang‘ich funksiyasi, $[-1;1]$ kesmada uzilishga ega. **13.** $\frac{8}{\ln 3}$. **14.** $\frac{\pi}{12}$. **15.** $\ln 1{,}5$. **16.** π. **17.** $\frac{1}{4}arctg(4/7)$. **18.** $\frac{1}{6}\ln(2/5)$. **19.** $14/3$. **20.** $2/3$. **21.** 3) 2. **22.** 2. **23.** 0. **24.** $1/8$. **25.** $1/6$. **26.** $1/2$. **27.** 0. **28.** Yo‘q. **29.** Mumkin. **30.** $1-5e^{-4}$. **31.** $\ln 4-\frac{3}{4}$. **32.** $\frac{\pi^2-4}{8}$. **33.** $\frac{\pi}{8}-\frac{1}{4}\ln 2$. **34.** $\frac{\pi\sqrt{3}}{3}-\ln 2-\frac{\pi^2}{18}$. **35.** $4\ln 2-\frac{15}{16}$.

8.8-amaliy mashg‘ulot.
Aniq integralning fizika masalarini yechishga tadbiqlari

1. To‘g‘ri chiziqli harakat. Jism to‘g‘ri chiziqli harakat qilganda, uning t vaqt davomida o‘zgarmas v tezlikda bosib o‘tgan yo‘li s, ushbu $s=\mathrm{v}t$ formula bo‘yicha aniqlanadi.

Moddiy nuqta t vaqtning funksiyasi bo‘lgan $\mathrm{v}=\mathrm{v}(t)$ orqali ifodalanadigan o‘zgaruvchan tezlik bo‘yicha to‘g‘ri chiziqli harakatlanayotgan bo‘lsin. Nuqtaning t_1 vaqtdan t_2 vaqtgacha bo‘lgan vaqt oralig‘ida o‘tgan yo‘lini topish talab qilinadi. Vaqtning elementar $[t,t+\Delta t]$ oralig‘ini qaraymiz. Shu vaqt davomida nuqta $ds=\mathrm{v}(t)dt$ yo‘lni bosib o‘tadi.

Bu ifodani $[t_1,t_2]$ kesmada integrallab, nuqtaning shu vaqt oralig'ida o'tgan yo'lini topamiz:

$$s=\int_{t_1}^{t_2} v(t)dt. \qquad (1)$$

Nuqtaning harakat trayektoriyasi ushbu

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases}$$

parametrik shakldagi tenglamalar bilan berilgan bo'lsin, bu yerda $x(t), y(t)$ lar $[t_1,t_2]$ kesmada uzluksiz diferentsiallanuvchi funksiyalar.

$[t_1,t_2]$ vaqt oralig'ida o'tilgan yo'lni topish uchun, parametrik shaklda berilgan egri chiziqning uzunligini topish formulasidan foydalanamiz:

$$s=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)}dt.$$

1-misol. Jismning harakat tezligi $v=(2t^2+t)\ (sm/sek)$ tenglama orqali berilgan. Jismning harakat boshlangandan so'ng 6 sekund vaqt davomida o'tgan yo'lini aniqlang.

Yechilishi. (1) formulaga asosan,

$$s=\int_0^6 (2t^2+t)dt=\left(\frac{2}{3}t^3+\frac{1}{2}t^2\right)\Big|_0^6=\frac{2}{3}6^3+\frac{1}{2}\cdot 6^2=162(sm).\blacksquare$$

2. O'zgaruvchan kuchning bajargan ishi. Moddiy nuqta s yo'lning to'g'ri chiziqli qismida o'zgarmas F kuch ta'siri ostida harakat qilayotgan bo'lsin. U holda kuchning yo'lning shu qismida bajargan A ishi,

$$A=F\cdot s$$

formula bo'yicha hisoblanadi.

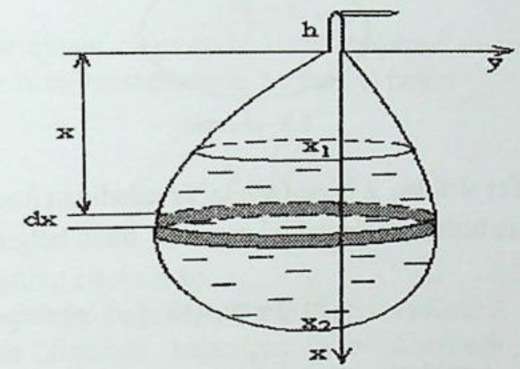
3. Suyuqlikni idishdan chiqarib olishda bajarilgan ish. Solishtirma og'irligi γ bo'lgan suyuqlikni rezervuardan h balandlikka chiqarib olish uchun bajarilishi zarur bo'lgan ishni hisoblash uchun, koordinatalar

sistemasini 8.7 -chizmada ko'rsatilgandek kiritamiz (Ox o'qini izlanayotgan miqdorning ortishi tomonga yo'naltirish qulay).

Faraz qilaylik, dP - suyuqlikning dx balandlikdagi qatlamining og'irligi bo'lsin. U holda, $dP=\gamma dV=\gamma S(x)dx$,

bunda $S(x)$ qatlamning yuzi.

Og'irligi dP bo'lgan suyuqlik qatlamini $x+h$ balandlikka ko'tarish uchun sarflangan ish, ya'ni elementar ish - dA bo'lsin. Og'irligi dP bo'lgan suyuqlik qatlamini $x+h$ balandlikka



8.7–chizma.

ko'tarish lozim bo'lganligidan,

$$dA=dP(x+h)=\gamma(x+h)s(x)dx,$$

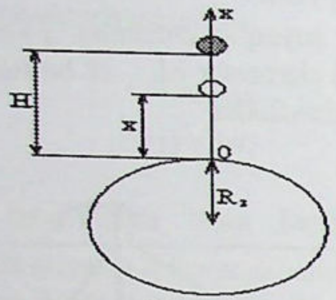
bo'ladi. U holda, $A=\gamma\int_{x_1}^{x_2}(x+h)s(x)dx$, bunda $s(x)$- rezervuarning koordinatalar boshidan x masofada joylashgan kesimi yuzi, x_1,x_2- suyuqlikning yuqori va quyi chegaralariga mos qiymatlar.

4. Yo'ldoshni ko'tarishda bajarilgan ish

m massali yo'ldoshni Yer yuzidan H balandlikka ko'tarishda bajariladigan ishni hisoblash uchun,

$$F(r)=G\frac{Mm}{r^2}$$

gravitasiya qonunidan foydalanamiz, bunda G- gravitasion doimiylik, M-Yerning massasi, r-Yer markazidan yo'ldoshgacha bo'lgan masofa. Koordinatalar sistemasini 8.8-chizmada ko'rsatilgandek kiritamiz.



8.8 –chizma.

Yo'ldoshga Yer sirtidan x masofada ta'sir qiladigan kuchni $F(x)$ orqaii, x balandlikda erkin tushish tezlanishini esa, $a(x)$ bilan belgilaymiz. U holda,

$$F(x) = m \cdot a(x).$$

Agar $x = 0$ (yo'ldosh Yer sirtida joylashgan holda) bo'lsa, $a(x) = 9{,}8\left(m/sek^2\right) = g$, demak, $F(0) = ma(0) = mg$. Ikkinchi tomondan, gravitasiya qonunidan foydalanib, $F(x) = G\dfrac{mM}{(R_{yer}+x)^2}$, $F(0) = G\dfrac{mM}{R_{yer}{}^2}$ munosabatlarni hosil qilamiz.

Demak, $ma(x) = G\dfrac{mM}{(R_{yer}+x)^2}$, $mg = G\dfrac{mM}{R_{yer}^2}$.

Oxirgi formuladan $M = \dfrac{gR_{yer}^2}{G}$ ekanligini topamiz. U holda (tortishish kuchi)

$$F(x) = G\frac{m}{(R_{yer}+x)^2} \cdot \frac{gR_{yer}^2}{G} = \frac{mgR_{yer}^2}{(R_{yer}+x)^2}$$

bo'ladi. x balandlikda $F(x)$ tortishish kuchini bartaraf etuvchi elementar dA- ish bo'lsin. U holda $dA = F(x)dx = \dfrac{mgR_{yer}^2}{(R_{yer}+x)^2}$.

Shunday qilib, yo'ldoshni Yer sirtidan H baladlikka ko'tarishda bajarilgan ish,

$$A = \int_0^H \frac{mgR_{yer}^2}{(R_{yer}+x)^2}dx$$

formula bo'yicha hisoblanadi.

5. Prujinani cho'zishda bajarilgan ish. Prujinani (uning qattiqligi koeffisiyenti k ga teng) s birlik uzunlikka cho'zish uchun sarf qilinadigan ishni hisoblash uchun, Guk qonunidan foydalanamiz:

$$F = k \cdot \Delta x,$$

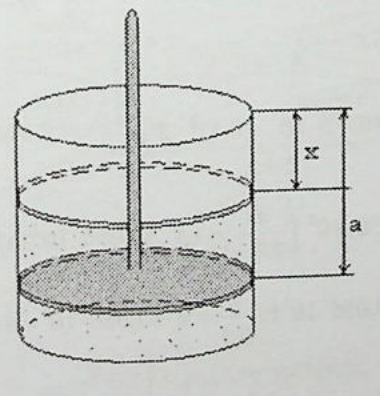
bunda F - prujinani cho'zuvchi kuch. Prujinani dx uzunlikka cho'zish uchun, elementar ishni bajarish zarur bo'ladi. U holda

$$A = \int_0^s F(x)dx,$$

bunda s - prujinaning cho'zilishi.

6. Gazni siqishda bajarilgan ish. Gazni radiusi R va balandligi H bo'lgan silindrda siqishda bajarilgan ishni hisoblash uchun, gazning $P_1V_1 = P_2V_2$ holat tenglamasidan foydalanamiz.

Faraz qilaylik, jarayonning boshida silindrdagi bosim P_0 ga teng bo'lsin. Ma'lumki, silindrnig hajmi $V_0 = \pi R^2 H$. $P(x)$ deb, porshenning boshlang'ich holatdan x masofaga siljitganda silindrdagi gazning bosimini belgilaymiz (8.9-chizma).



8.9- chizma.

Bunda gazli silindrning hajmi, $V(x)=\pi R^2(H-x)$ bo'ladi. U holda, 8.9-chizmaning holati tenglamasidan,

$$P_0V_0=P(x)V(x)\Rightarrow P_0\pi R^2H=P(x)\pi R^2(H-x).$$

Oxirgi tenglamadan gazning bosim kuchi $P(x)$ ni topamiz: $P(x)=\dfrac{P_0H}{H-x}$.

Ma'lumki, porshen dx masofaga siljiganda sarf qilinadigan elementar ish,

$$dA=P(x)dx=\frac{P_0H}{H-x}dx$$

bo'ladi. Natijada, silindrik rezervuarda gazni siqish vaqtida bajarilgan ish ushbu

$$A=\int_0^a\frac{P_0H}{H-x}dx=P_0H\int_0^a\frac{dx}{H-x}dx$$

formula bo'yicha hisoblanadi, bunda a - porshenning siljish masofasi.

3-masala. Silindr $103\cdot3\cdot10^3(Pa)$ atmosfera bosimidagi gazga to'ldirilgan. Gazni ideal deb hisoblab, gazni silindrning ichida $1{,}5(m)$ ga siljigan porshen yordamida izotermik siqish jarayonida bajarilgan ish miqdorini aniqlang (silindr asosining radiusi $0{,}4(m)$, baladligi $2(m)$ deb oling).

Yechilishi. Bu misolni yechishda $A=\int_0^a\frac{P_0\pi R^2H}{H-x}dx=P_0\pi R^2H\int_0^a\frac{dx}{H-x}$ formuladan foydalanamiz:

$$A=\pi\int_0^{1,5}\frac{103\cdot3\cdot10^3\cdot(0{,}4)^2\cdot2}{2-x}dx=$$

$$=\pi33{,}056\cdot10^3\int_0^{1,5}\frac{dx}{2-x}=-\pi\cdot33{,}056\cdot10^3\ln|2-x|\,|_0^{1,5}=$$

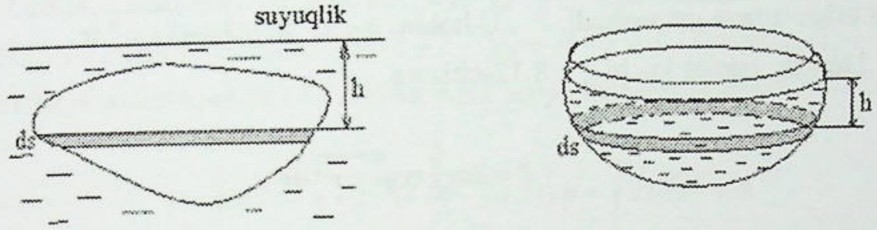
$$=\pi\cdot33{,}056\cdot10^3\ln\frac{2}{0{,}5}=\pi\cdot33{,}056\cdot10^3\cdot\ln4\quad(J).$$ ■

4.2.34. Suyuqlikning bosimi

Paskal qonuniga binoan suyuqlikning ds yuza elementiga bosim kuchi

$$dP=\gamma hds \qquad (4.2.35)$$

formula bo'yicha topiladi, bunda γ - suyuqlikning solishtirma og'irligi, h -

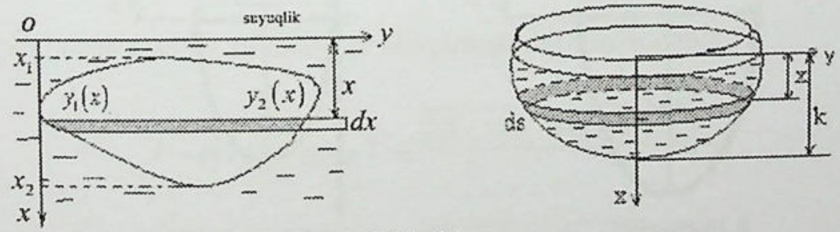


8.10-chizma.

suyuqlik ustunining ds element ustidagi balandligi (8.10-chizma).

7. Suyuqlikka vertikal cho'ktirilgan plastinkaga ta'sir bosimi. Berilgan shakldagi plastinkaga ta'sir bosimini aniqlash uchun, koordinatalar sistemasini tanlaymiz. Ox o'qni vertikal pastga yo'naltiramiz (bunda Ox o'qni izlanayotgan miqdorning o'sishi tomoniga yo'naltirish qulay va bundan buyon ana shunday qilamiz), Oy o'q jismning sirti bilan usma-ust tushadi (8.11 -chizma). U holda

$$h=x,\quad ds=(y_2(x)-y_1(x))dx,\quad dP=\gamma x(y_2(x)-y_1(x))dx.$$



8.11-chizma.

Demak,

$$P=\gamma\int_{x_1}^{x_2}x(y_2(x)-y_1(x))dx,$$

bunda x_1, x_2 - x o‘zgaruvchining plastikka yuqori va pastki chegaralariga mos keluvchi qiymatlari, $y_1(x), y_2(x)$ - plastinkaning, mos ravishda, chap va o‘ng chegaralarini ifodalovchi chiziqlar to‘plamlaridir.

8. Suyuqlikning idish devorlariga bosimi. Solishtirma og‘irligi γ bo‘lgan suyuqlikninng idish devorlariga bosimi kuchini aniqlash uchun koordinatalar sistemasini tanlaymiz (8.12-chizma). Ox o‘qini, yuqoridagidek, vertikal yuqoridan pastga yo‘naltirsak, Oy o‘q esa, sirtning yuzi bilan ustma-ust tushadi. U holda, $h = x$, $ds = 2\pi y\sqrt{1+y'^2}dx$.

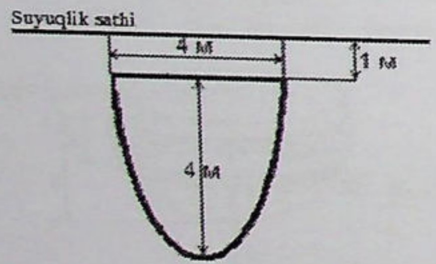
Demak, bosim kuchi 8.12-chizma.

$$P = 2\pi\gamma\int_2^k xy\sqrt{1+y'^2}dx$$

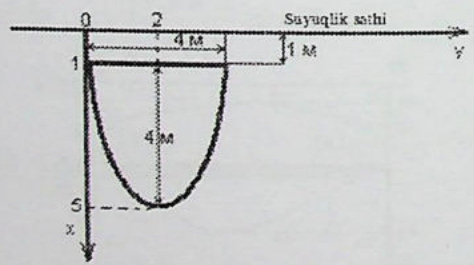
formula bo‘yicha hisoblanadi, bunda k- idishdagi suyuqlikning balandligi, $y = y(x)$- Ox o‘q atrofida aylanishidan idishning sirtida hosil bo‘ladigan chiziqning tenglamasidir. Endi yuqorida yozilgan formulalarini namoyish qiladigan misollarni qaraymiz.

4-masala. Suvning kesimi parabola shaklida bo‘lgan platinada ta’siri bosimini toping. Suvning solishtirma og‘irligi $1(t/m^3)$ (8.13-chizma).

Koordinatalar sistemasini 8.14-chizmadagi kabi kiritamiz. $y_1(x)$ va $y_2(x)$ larning ifodalarini topish uchun, parabolaning tenglamasini tuzamiz.



8.13-chizma



8.14-chizma.

Tarmoqlari Ox o‘qining musbat yo‘nalishiga qarama-qarshi yo‘nalgan parabolaning tenglamasi $(y-y_0)^2 = -2p(x-x_0)$ ko‘rinishga ega bo‘ladi, bunda (x_0, y_0) - parabola uchining koordinatalaridir. Chizmadan ko‘rinadiki,

parabolaning uchi (5;2) nuqtada joylashgan. P ning qiymatini aniqlaymiz. Masalaning shartiga ko‘ra, parabola $(1;0)$ nuqtadan o‘tganligidan, bu nuqtaning koordinatalari parabola tenglamasini qanoatlantiradi:

$$(0-2)^2 = -2p(1-5),\ 4 = -2p(-4),\ p = \frac{1}{2}.$$

Demak, parabola tenglamasi, $(y-2)^2 = 5-x$ ko‘rinishini oladi. Bu yerdan $y-2 = \pm\sqrt{5-x}$, $y_1 = -\sqrt{5-x}+2$; $y_2 = \sqrt{5-x}+2$. Suvning $(\gamma = 1)$ plastinkaga bosim kuchini (4.2.39) formula bo‘yicha hisoblaymiz:

$$P = \int_1^5 x(\sqrt{5-x}+2-(-\sqrt{5-x}+2))dx = \int_1^5 x2\sqrt{5-x}dx.$$

Avvalo, $\int x\sqrt{5-x}dx$ aniqmas integralni hisoblaymiz. Buning uchun, $\sqrt{5-x} = t$ almashtirishni kiritamiz. U holda, $5-x = t^2, x = 5-t^2, dx = -2tdt$ ifodalarni hosil qilamiz.

Demak,

$$\int x\sqrt{5-x}dx = \int(5-t^2)t(-2tdt) = 2\int(t^4-5t^2)dt = 2(\frac{t^5}{5} - 5\frac{t^3}{3}) + C =$$

$$= 4[\frac{\sqrt{(5-x)^5}}{5} - \frac{5\sqrt{(5-x)^3}}{3}] + C.$$

Endi bosim kuchini ifodalovchi aniq integralni hisoblaymiz:

$$P = 2\int_1^5 x\sqrt{5-x}dx = 2\cdot 4[\frac{\sqrt{(5-x)^5}}{5} - \frac{5\sqrt{(5-x)^3}}{3}]|_1^5 =$$

$$= 8[\frac{\sqrt{(5-5)^5}}{5} - \frac{5\sqrt{(5-5)^3}}{3} - (\frac{\sqrt{(5-1)^5}}{5} - \frac{5\sqrt{(5-1)^3}}{3})] = \frac{832}{15}m. \blacksquare$$

Mustaqil yechish uchun masalalar

1. Bo'shliqda pastga tushayotgan jismning tezligi $v=9{,}8t\ (m/sek)$ formula bo'yicha aniqlanadi. Jism tushash boshlangandan 10 sekund vaqt o'tganda qancha yo'l bosib o'tishini aniqlang.

2. Jismning harakat tezligi $v=\left(3t^2-2t\right)\ (sm/sek)$ formula bo'yicha aniqlanadi. Jism harakat boshlangandan 4 sekund vaqt o'tganda qancha yo'l bosib o'tadi?

3. Jismning harakat tezligi $v=\sqrt{5t+4}\ (m/sek)$ formula bo'yicha aniqlanadi. Jism harakat boshlangandan 9 sek vaqt o'tganda qancha yo'l bosib o'tadi?

4. Jismning harakat tezligi $v=(4t-\frac{6}{t^2})\ (sm/sek)$.Uning uchinchi sekundda bosib o'tgan yo'lini aniqlang.

5. Ushbu: a) $v(t)=t^2$; b) $v(t)=t^{-2}$; v) $v(t)=t^{-1}$ tezliklar bilan to'g'ri chiziq bo'ylab harakatlanayotgan material nuqtaning[1;27] vaqt oralig'ida o'tgan yo'li topilsin?

6. Moddiy nuqta to'g'ri chiziq bo'ylab harakatlanayapti. Uning tezlanishi $\bar{a}=(A+Bt+Ct^2)\bar{i}$ qonun bo'yicha o'zgaradi, bu yerda $A=8\left(m/sek^2\right)$; $B=5\left(m/sek^3\right)$; $C=3\ \left(m/sek^4\right)$. Harakat boshlangandan so'ng 0,4 sekundda moddiy nuqta qanday tezlikka erishadi. Bu vaqtda u qancha yo'l yuradi?

7. Moddiy nuqtaning tezlanishi $\bar{a}=At\bar{i}-B\bar{j}$ qonuniyat bo'yicha o'zgaradi, bu yerda $A=4\left(m/sek^3\right)$; $B=5\left(m/sek^2\right)$. Agar $t=0$ vaqtda $r_0=0,\ v_0=0$ bo'lsa, moddiy nuqta $t=1$ sekund vaqt momentida koordinatalar boshidan qancha masofada bo'ladi?

8. Ikkita moddiy nuqta koordinata boshidan bir xil sanoq sistemasida harakatlanayapti. Ularning tezllik vektorlari $\bar{v}_1=(2t\bar{i}-6t^2\bar{k})(m/sek)$; $\bar{v}_2=(4{,}5t^2\bar{i}-4t\bar{j}+2t\bar{k})(m/sek)$ qonuniyat bo'yicha o'zgaradi, bu yerda $\bar{i}$, $\bar{j}$, $\bar{k}$ - mos ravishda Ox, Oy, Oz o'qlarining birlik vektorlari, ya'ni ortlari. Harakat boshlangandan so'ng ikki sekunddagi moddiy nuqtalar orasidagi masofani toping.

9. Ox o'qi bo'ylab to'g'ri chizliqli harakatlanayotgan m massali moddiy nuqtaga $t=0$ vaqt momentida $\bar{F}=Ae^{-kt}\bar{i}\ (N)$ formula bilan ifodalanuvchi kuch ta'sir qila boshladi, bu yerda $A,k-$o'zgarmaslar. Agar harakat hamma vaqt rel'yativ bo'lmasa, harakat boshlangandan keyin bir sekundda moddiy nuqtaning tezlanishi qanday o'zgaradi?

10. Qo'zg'almas o'qi atrofida aylanayotgan jismga aylanish o'qiga nisbatan tashqi kuchlarning natijaviy $\bar{M}=(Ct^2+D)\bar{k}\ (kg\cdot m^2/sek)$ momenti ta'sir qila boshladi, bu yerda $C,D-$o'zgarmaslar. Aylanish o'qiga nisbatan t_1 dan t_2 vaqt oralig'ida jismning impul's o'zgarishini toping.

11. Massasi $m=1{,}1$ (kg), radiusi $R=0{,}2$ (m) bo'lgan sport chambaragiga (xalqaga) $t=0$ vaqt momentida tashqi kuchlarning aylaninsh o'qiga nisbatan $\bar{M}=(6t+2)\bar{k}\ (kg\cdot m^2/sek)$formula bilan ifodalanuvchi natijaviy momenti ta'sir qila boshladi. Kuch momentining ta'siri boshlangandan so'ng 2 sekunddagi chambarak aylanishining burchak tezligi o'zgarishini toping.

12. m massali raketani Yer sirtidan h balandlikka chiqarish uchun bajariladigan ishni toping.

13. Asosi kvadratdan iborat, balandligi H, asosining tomoni a bo'lgan piramida solishtirma og'irligi γ bo'lgan materialdan yasalgan. Piramidani yasashda og'irlik kuchini bartaraf etish uchun bajarilgan ishni toping.

14. Solishtirma og'irligi γ bo'lgan R radiusli shar chuqurligi $H>R$ bo'lgan basseynning tubida yotibdi. Sharni suvdan chiqarib olish uchun bajariladigan ishni toping.

15. Asosining yuzi $S=4000\left(sm^2\right)$, balandligi $H=50(sm)$ bo'lgan yog'ochdan yasalgan silindr shaklidagi suzgich suv yuzida suzib yuribdi. Agar yog'ochning solishtirma og'irligi $\gamma=0{,}8\left(G/sm^3\right)$ bo'lsa: a) suzgichni suvdan chiqarib olish uchun; b) suzgichni suvga to'liq cho'ktirish uchun bajariladigan ishni toping.

16. Uzunligi $0{,}5(m)$ va kesimining radiusi $4(mm)$ bo'lgan misdan yasalgan sterjenni $2(mm)$ ga cho'zish uchun bajariladigan ishni hisoblang.

17. Ikkita e_0 va e elektrik zaryadlar Ox o'qining mos ravishda, $x_0=0$ va $x_1=a$ nuqtalarida joylashgan. Ikkinchi zaryadning $x_2=b\ (b>a)$ nuqtaga ko'chirishda bajarilgan ishni toping.

18. Vintli prujinaning siqilishi unga ta'sir qiluvchi kuchga proporsionaldir. Agar prujinani $1(sm)$ ga siqish uchun $1(kG)$ kuch zarurligi ma'lum bo'lsa, uni $10(sm)$ ga siqish uchun bajariladigan ishni toping.

19. Porshen tagidagi silindrda $V_0=0{,}1\left(m^3\right)$ hajmdagi havo $P_0=10330\left(kG/m^2\right)$ bosim ostida joylashgan. Temperatura o'zgarmas bo'lgan holda havoning hajmini ikki marta kamaytirish uchun bajariladigan ishni toping.

20. Diametri $D=20(dm)$ va uzunligi $L=1(m)$ bo'lgan harakatlanuvchi porshenli silindr $P_0=10(kg/sm^3)$ bosim ostida bug' bilan to'ldirilgan. Agar porshen silindrning ichida $l=80(sm)$ ga siljisa, adiabatik siqish uchun bajariladigan ishni toping.

21. Koordinatalari vaqt bilan $x=B+Ct-Dt^2$, bu yerda $B=7(m)$, $C=-4(m/sek)$, $D=3(m/sek^2)$, qonunga ko'ra o'zgaruvchi, massasi $m=1$ kg bo'lgan moddiy nuqta biror kuch ta'sirida to'g'ri chiziq bo'yicha harakatlanmoqda. Ushbu kuch ta'sirida dastlabki $1{,}5(sek)$ da qancha ish bajariladi? Bu vaqt momentida harakatlanayotgan moddiy nuqtaning P quvvati qanday o'zgaradi?

22. Moddiy nuqta konservativ kuchlar ta'sirida koordinatasi (x_1,y_1,z_1) bo'lgan nuqtadan koordinatasi (x_2,y_2,z_2) bo'lgan nuqtaga ko'chirildi. Ox o'qi bo'ylab kuchning F_x tashkil etuvchisi koordinatalarining qiymati $F_x=\frac{B}{x}+C$ bu yerda $B=2(H\cdot m)$; $C=0{,}5(H)$, qonun bo'yicha o'zgaradi. Agar $x_1=2m$, $x_2=3m$ bo'lsa, moddiy nuqtani konservativ kuchlar ta'sirida ko'chirishda qancha ish bajariladi?

23. Nuqtaviy zaryadni zaryadlangan sfera markaziga nisbatan r_1 masofadan r_2 masofaga siljitish davomida R radiusli tekis zaryadlangan sferik sirt hosil qilgan elektrostatik maydon kuchlari bajargan ishni toping.

24. Nuqtaviy zaryadni zaryadlangan moddiy ipga nisbatan r_1 masofadan r_2 masofaga ko'chirish davomida cheksiz uzun tekis zaryadlangan moddiy ipda hosil bo'lgan elektrostatik maydon kuchlari bajargan ishni toping.

Mustaqil yechish uchun masalalarning javoblari

1. 490*м*. **2.** 48*м*. **3.** $\frac{134}{3}$*м*. **4.** 9 *см*. **5.** a) 3,75; b) 0,5; v) $\ln 2$. **6.** 3,82 (m/sek), 0,71 (m). *Ko'rsatma.* Moddiy nuqtaning tezligi va tezlanishi ta'riflariga asoslanib, mos ravishda ular uchun $\bar{v}=\frac{d\bar{s}}{dt}$ va $\bar{a}=\frac{d\bar{v}}{dt}$ formulalarni hosil qilamiz, bu yerda $\bar{s}$ – moddiy nuqtaning biror $[t_1;t_2]$ vaqt oralig'ida bosib o'tgan yo'li. Ma'lumki, tezlikning miqdori $\bar{v}=\frac{d\bar{s}}{dt}$ vektor - funksiyaning moduli orqali topiladi. $v=\frac{ds}{dt}$ formulaga ko'ra moddiy nuqtaning biror $[t_1;t_2]$ vaqt oralig'idagi elementar siljishi $ds=vdt$ bo'lib, uning biror $[t_1;t_2]$ vaqt oralig'ida bosib o'tgan yo'li uchun

$$s=\int_{t_1}^{t_2} vdt \tag{1}$$

formulani olamiz. Bundan moddiy nuqtaning koordinata o'qlaridagi proeksiyalari quyidagicha bo'ladi:

$$x_2-x_1=\int_{t_1}^{t_2} v_x dt;\quad y_2-y_1=\int_{t_1}^{t_2} v_y dt;\quad z_2-z_1=\int_{t_1}^{t_2} v_z dt, \tag{2}$$

bu yerda, v_x,v_y,v_z – mos ravishda tezlik vektori $\bar{v}$ ning Ox,Oy,Oz koordinata o'qlaridagi proeksiyalari. Moddiy nuqtaning $[t_1;t_2]$ vaqt oralig'idagi tezlanish miqdori $a(t)=\frac{dv(t)}{dt}$ bo'lgani uchun, uning elementar tezligi $dv=adt$ bo'lib, uning bu vaqt oralig'ida tezligining o'zgarishi $v_2-v_1=\int_{t_1}^{t_2} adt$ bo'ladi va uning koordinata o'qlaridagi proeksiyalari quyidagi formulalar orqali topiladi:

$$v_{x_2}-v_{x_1}=\int_{t_1}^{t_2} a_x dt;\quad v_{y_2}-v_{y_1}=\int_{t_1}^{t_2} a_y dt;\quad v_{z_2}-v_{z_1}=\int_{t_1}^{t_2} a_z dt, \tag{3}$$

bu yerda, a_x,a_y,a_z – mos ravishda tezlanish vektori $\bar{a}$ ning Ox,Oy,Oz koordinata o'qlaridagi proeksiyalari.

Xuddi yuqoridagiga o'xshash, biror φ burchak ostida harakatlanayotgan moddiy nuqtaning burchak tezligi va burchak o'zgarishi uchun quyidagi formulalar o'rinli:

$$\omega_2-\omega_1=\int_{t}^{t_2} \beta dt;\qquad \varphi_2-\varphi_1=\int_{t_1}^{t_2} \omega dt \tag{4}$$

bu yerda, $\omega -$ moddiy nuqtaning burchak tezligi, $\beta -$ moddiy nuqtaning burchak tezlanishi. **7.** 2 (m).*Ko'rsatma.* (1), (2) va (3) formulalardan foydalaniladi.

8. 22,98 (m). *Ko'rsatma.* (1), (2) va (3) formulalardan foydalaniladi.

9. $\Delta v = v_2 - v_1 = \frac{A}{km}\left(1 - e^{-k}\right)\vec{i}\,(m/sek)$. *Ko'rsatma.* Moddiy nuqta impulsining o'zgarish qonuni $\vec{p}(t) = m\vec{v}t$, bu yerda $m = const$, ya'ni $m\frac{d\vec{v}(t)}{dt} = \vec{F}(t)$ yoki $\vec{v} \parallel \vec{F}$ bo'lgani uchun $m\frac{dv(t)}{dt} = F(t)$ bo'lib, $\Delta v = v_2 - v_1 = \frac{1}{m}\int\limits_{t_1}^{t_2} Ae^{-kt}dt$, bu yerda $t_1 = 0,\ t_2 = 0$ sek. **10.** $\Delta L = L_2 - L_1 = \left[\frac{c}{3}\left(t_2^3 - t_1^3\right) + D(t_2 - t_1)\vec{k}\right]\left(kg \cdot m^2/sek\right)$.

Ko'rsatma. Moddiy nuqta impuls momenti $\vec{L}$ ning o'zgarish qonuniga asosan $\frac{d\vec{L}(t)}{dt} = \vec{M}(t)$, bu yerda $\vec{M} -$ impul's momenti $\vec{L}$ ga nisbatan aniqlangan, shu o'q bo'yicha jismga ta'sir etuvchi tashqi kuchlarning natijaviy momenti. $\vec{M}$ va $\vec{L}$ yo'nalishlari ustma-ust tushganligi uchun quyidagi $\frac{dL(t)}{dt} = M(t)$ skalyar formadan foydalanishimiz mumkin, bundan elementar impul's momenti $dL(t) = M(t)dt$ bo'lib, $L_2 - L_1 = \int\limits_{t_1}^{t_2} M(t)dt$ bo'ladi.

11. $250(rad/sek)$. *Ko'rsatma.* Moddiy nuqta impuls momenti $L = J\omega$ ekanligini e'tiborga oling, bu yerda $J -$ chambarakning aylaninsh o'qiga nisbatan inersiya momenti. Bu holda $J = mR^2$, ya'ni $\omega_2 - \omega_1 = \frac{1}{mR^2}\int\limits_{t_1}^{t_2} M(t)\,dt$.

12. $mgR\frac{h}{R+h}$. **13.** $\frac{1}{12}\gamma a^2 H^2$. **14.** $\frac{4}{3}\pi R^3(R + (\gamma - 1)H)$. **15.** $32(kGm), 2(kGm)$.

16. $7{,}68\pi(Nm)$. **17.** $e_0 e(\frac{1}{a} - \frac{1}{b})$. **18.** $0{,}5(kGm)$. **19.** $1033\ln 2(kGm)$.

20. $\frac{\pi \cdot 10^3}{0{,}4}(5^{0,4} - 1)$. **21.** $4{,}5(J), 30(Vt)$. *Ko'rsatma.* O'zgaruvchan kuchning bajargan ishi quyidagi formulalardan biri bilan topiladi: $A = \int\limits_{x_1}^{x_2} F_x dx;\quad A = \int\limits_{t_1}^{t_2} F_x v_x dt$. Moddiy nuqtaning oniy P quvvati $P = \frac{dA(t)}{dt}$ formula bilan aniqlanadi. **22.** $1{,}31(J)$. *Ko'rsatma.* Bu yerda moddiy nuqtani konservativ kuchlar ta'sirida, yo'ldan bog'liq bo'lmagan holda, ya'ni

moddiy nuqta (x_1, y_1, z_1) nuqtadan (x_2, y_2, z_2) nuqtaga qanday yo'l bilan ko'chishiga bog'liq bo'lmaganda bajarilgan ishni topish kerak. **23.** $A = \frac{q\sigma R}{\varepsilon_0 \varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$. *27-29 masalalar uchun ko'rsatma.* Har uchta misolda ham elektrostatik maydon kuchlari $\vec{F} = q\vec{E}$ (bu yerda $\vec{E} -$ maydon kuchlanganligi) vektor- funksiya orqali ifodalanadi, bundan $\vec{F} \parallel \vec{E}$ bo'lgani uchun bajarilgan ish $A = \int\limits_{r_1}^{r_2} qEdr$ formula orqali topiladi. **24.** $A = \frac{q\lambda}{2\pi\varepsilon_0\varepsilon}\ln\frac{r_2}{r_1}$.

8.9-amaliy mashg'ulot.
Aniq integralning geometrik masalarini yechishga tadbiqlari

1-misol. $x = y^2$ va $x - y = 2$ chiziqlar bilan chegaralangan sohaning yuzini: *a)* x ga nisbatan; *b)* y ga nisbatan integrallash yordamida hisoblang.

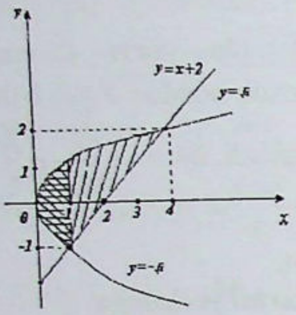
Yechilishi. Avvalo, berilgan chiziqlarning $(1;-1)$, $(4;2)$ nuqtalarda kesishishiga ishonch hosil qilish mumkin.

a) x bo'yicha integrallash uchun, tasavvur qilinadigan to'g'ri to'rtburchaklarni vertikal joylashtiramiz va tenglamalarni y ga nisbatan yechamiz: $x = y^2$ tenglamani $y$ga nisbatan yechib, $y = \pm\sqrt{x}$ bo'lishini olamiz, bunda $y = \sqrt{x}$- parabolaning yuqori yarmidan, $y = -\sqrt{x}$ esa, parabolaning quyi yarmidan iborat. $x - y = 2$ to'g'ri chiziq tenglamasini, $y = x - 2$ shaklida yozamiz (8.15- chizma). Qaralayotgan sohaning yuqori chegarasi, $y = \sqrt{x}$ egri chiziqdan iborat. Uning quyi chegarasi esa, ikkita, har xil tenglamalar orqali ifodalanadi: $x = 0$ dan $x = 1$ gacha o'zgarganda, $y = -\sqrt{x}$ egri chiziq, $x = 1$ dan $x = 4$ gacha o'zgarganda esa, $y = x - 2$ to'g'ri chiziq.

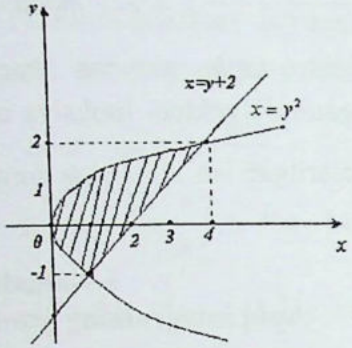
Shunday qilib, $x = y^2$ va $x - y = 2$ chiziqlar bilan chegaralangan sohaning yuzi, $S = \int\limits_a^b [f(x) - g(x)]dx$ formula orqali topiladi

$$S = \int\limits_0^1 [\sqrt{x} - (-\sqrt{x})]dx + \int\limits_1^4 [\sqrt{x} - (x - 2)]dx = 2\int\limits_0^1 \sqrt{x}dx + \int\limits_1^4 [\sqrt{x} - x + 2]dx =$$

$$=[\frac{4}{3}x^{3/2}]\Big|_0^1+[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x]\Big|_1^4=\frac{9}{2}.$$



8.15-chizma.



8.16-chizma.

b) y bo'yicha integrallash uchun, biz tasavvur qilinadigan to'g'ri to'rt burchaklarni gorizontal joylashtiramiz (8.16-chizma). Bunda, o'ngdan chegaralovchi to'g'ri chiziq $x=y+2$ va chapdan chegaralovchi egri chiziq esa, $x=y^2$ bolib, y o'zgaruvchi -1 dan 2 gacha o'zgarar ekan.

Shunday qilib, $x=y^2$ va $x-y=2$ chiziqlar bilan chegaralangan sohaning yuzi, (4.3.5) formula orqali topiladi

$$S=\int_{-1}^{2}[y+2-y^2]dy=[\frac{1}{2}y^2+2y-\frac{1}{3}y^3]\Big|_{-1}^{2}=\frac{9}{2}.\blacksquare$$

2-misol. Ushbu $\left.\begin{array}{l}x=a(t-\sin t),\\ y=a(1-\cos t)\end{array}\right\}$ $(0\le t\le 2\pi)$ egri chiziq va Ox o'q bilan chegaralangan sohaning yuzini toping.

Yechilishi. Sikloidaning bir arkasi, t ning, 0 dan 2π gacha o'zgarishi natijasida chiziladi, chunki $y(0)=y(2\pi)=0$, t ning qolgan qiymatlarida, $y>0$, $x(0)=0$ va $x(2\pi)=2\pi a$.

Shunday qilib, izlanayotgan sohaning yuzi, $S=\int_{\alpha}^{\beta}y(t)\cdot x'(t)dt$ formula orqali topiladi. Bunda,

$$x=a(t-\sin t),\ y=a(1-\cos t),\ dx=a(1-\cos t)dt$$

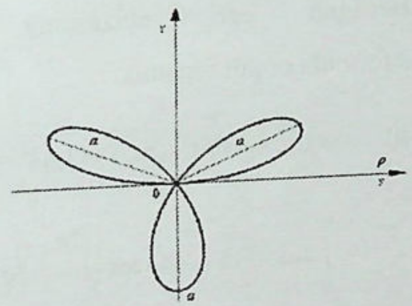
ekanligini e'tiborga olsak, u holda (x, $[0;2\pi a]$ oraliqda o'zgarganda, t, $[0;2\pi]$ oraliqda o'zgaradi),

$$S=a\int_0^{2\pi}(1-\cos t)\cdot a(1-\cos t)dt=a^2\int_0^{2\pi}(1-\cos t)^2dt=3\pi a^2(\text{kv. bir.}).$$

Demak, izlanayotgan yuza, radiusi a ga teng bo'lgan doira yuzining uchlanganiga teng ekan.■

3-misol. Ushbu $r=a\sin 3\varphi,\ a>0$, egri chiziq bilan chegaralangan sohaning yuzini toping.

Yechilishi. Berilgan egri chiziq, har biri egri chiziqli sektordan iborat bo'lgan sohani (8.17-chizma) chegaralaydi. Ulardan birinchisini, ya'ni $(S_1)=\left\{(r;\varphi):0\le\varphi\le\frac{\pi}{3},0\le r\le a\sin 3\varphi\right\}$ sektorni qaraymiz, uning yuzi,



8.17-chizma. $\rho=a\sin 3\varphi$, $a>0$.

$S=\frac{1}{2}\int_{\alpha}^{\beta}r^2(\varphi)d\varphi$ formula yordamida hisoblanadi:

$$S_1=\frac{1}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\varphi\, d\varphi=\frac{\pi a^2}{12}.$$

Shunday qilib, izlanayotgan sohaning yuzi, $S=\frac{3}{12}\cdot\pi a^2=\frac{\pi a^2}{4}$ (kv. bir.

4-misol. Ushbu $y=\frac{4}{5}x^{5/4}$, $0\le x\le 9$ egri chiziqning yoy uzunligini toping.

Yechilishi. Berilgan egri chiziqning yoy uzunligini, $l=\int_a^b\sqrt{1+[f'(x)]^2}dx$ formuladan foydalanib, topamiz:

$$y'_x=\frac{4}{5}\cdot\frac{5}{4}\cdot x^{\frac{1}{4}}=x^{\frac{1}{4}};\ S=\int_0^9\sqrt{1+\left(y'_x\right)^2}dx=\int_0^9\sqrt{1+x^{\frac{1}{2}}}dx.$$

Oxirgi integralda, $\sqrt{1+\sqrt{x}}=t$ almashtirishni bajarib,

$$l=4\int_1^2 t^2\left(t^2-1\right)dt=\frac{232}{15}$$

ekanligini topamiz.■

5-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$, egri chiziqning yoy uzunligini toping.

Yechilishi. Berilgan egri chiziqning yoy uzunligini, $l=\int_\alpha^\beta\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}dt$ formula orqali topamiz:

$$x'_t=a(1-\cos t),\ y'_t=a\sin t,\ \sqrt{(x'_t)^2+(y'_t)^2}=2a\sin\frac{t}{2}dt,$$

$$l=\int_0^{2\pi}2a\sin\frac{t}{2}dt=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=8a.■$$

6-misol. Ushbu $r=a(1-\sin\varphi),\ -\frac{\pi}{2}\le\varphi\le-\frac{\pi}{6}$ tenglama bilan berilgan chiziq yoyning uzunligini toping.

Yechilishi. Berilgan chiziqning, $r=a(1-\sin\varphi)$ tenglamasidan,

$r'_\varphi=-a\cos\varphi,\ \sqrt{r^2+(r'_\varphi)^2}=\sqrt{a^2(1-\sin\varphi)^2+a^2\cos^2\varphi}=$

$=a\sqrt{1-2\sin\varphi+\sin^2\varphi+\cos^2\varphi}=a\sqrt{2}\sqrt{1-\sin\varphi}$

ekanligini topamiz. Endi, izlanayotgan chiziq yoyining uzunligi, $l=\breve{AB}=\int_{\varphi_1}^{\varphi_2}\sqrt{[r(\varphi)]^2+[r'(\varphi)]^2}d\varphi$ formula yordamida topiladi:

$$l=\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}a\sqrt{2}\cdot\sqrt{1-\sin\varphi}d\varphi=a\sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}\sqrt{1-\sin\varphi}\,d\varphi=2a.■$$

7-misol. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoid sirti bilan chegaralangan jismning hajmini toping.

Yechilishi. Dastlab berilgan tenglama bo'yicha ellipsoidni yasaymiz.

Ellipsoidni Oxz tekislikka parallel bo'lgan, $y\in[-b;b]$ kesmada o'zgaruvchi, $y=p$ tekisliklar bilan kesamiz. Kesimda ellips hosil bo'ladi:

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1-\frac{p^2}{b^2},\ 1-\frac{p^2}{b^2}>0,\ \ \frac{x^2}{a_1^{\ 2}}+\frac{z^2}{c_1^{\ 2}}=1$$

Bunda ellipsning yarim o'qlari, $a_1=\frac{a}{b}\sqrt{b^2-p^2},\ c_1=\frac{c}{b}\sqrt{b^2-p^2}$. Bu kesimlarning yuzlari, p ga bog'liq bo'lgan, ellips bilan chegaralangan yuzaga teng bo'ladi:

$$S(p)=\pi\,a_1b_1=\frac{\pi\,ac}{b^2}\left(b^2-p^2\right).$$

$S(p)$ kesimlarning yuzasini $V=\int_a^b S(x)dx$ formulaga keltirib quyib, V jismning hajmini topamiz:

$$V=\int_{-b}^{b}\frac{\pi\,ac}{b^2}\left(b^2-p^2\right)dp=2\frac{\pi\,ac}{b^2}\int_0^b\left(b^2-p^2\right)dp=$$

$$=2\frac{\pi\,ac}{b^2}\left(b^2p-\frac{p^3}{3}\right)\Bigg|_0^b=\frac{4}{3}\pi\,acb.■$$

8-misol. Quyidagi, $y^2=2px,\ y=0,\ x=a$ chiziqlar bilan chegaralangan shaklni, $Ox\ (y=0)$o'q atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini toping.

Yechilishi. D soha, yuqoridan $y^2=2px$ uzluksiz funksiya bilan, yon tomonlardan, $x=a$ va $x=0$ to'g'ri chiziqlar, pastdan esa, $Ox\ (y=0)$ o'q bilan chegaralangan. Endi, D egri chiziqli sohani Ox $(y=0)$ o'q atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini, $V_x=\pi\int_a^b f^2(x)dx$ formula bo'yicha hisoblaymiz:

$$V_x=\pi\int_0^a y^2dx=\pi\int_0^a 2pxdx=2p\pi\frac{x^2}{2}\bigg|_0^a=\pi pa^2.\blacksquare$$

9-misol. Quyidagi
$x=a(t-\sin t),\ y=a(1-\cos t),\ (1\le t\le 2\pi),\ y=0$
chiziqlar bilan chegaralangan shaklni Ox $(y=0)$ o'q atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini toping.

Yechilishi. Aylanma jismning hajmini $V=\pi\int_\alpha^\beta y^2(t)\cdot x'(t)dt$ formula bo'yicha topamiz:

$$V_x=\pi\int_0^{2\pi}a^2(1-\cos t)^2a(1-\cos t)dt=\pi a^3\int_0^{2\pi}(1-\cos t)^3dt=$$
$$=8\pi a^3\int_0^{2\pi}\left(\sin^2\frac{t}{2}\right)^3dt=5\pi a^3.\blacksquare$$

10-misol. Ushbu $r=a(1+\cos\varphi)$ kardiodaning qutb o'qi atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini toping.

Yechilishi. $r=a(1+\cos\varphi)$ kardiodaning qutb o'qi atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini $V=\frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi d\varphi$ formulaga ko'ra, topamiz:

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi d\varphi=\frac{2\pi}{3}\int_0^\pi a^3(1-\cos\varphi)^3\sin\varphi d\varphi==\frac{\pi}{6}a^3(1-\cos\varphi)^4=\frac{8\pi}{3}a^3.\blacksquare$$

11-misol. Ushbu $y^2=2px$ $(0\le x\le x_0)$ chiziqning: *a*) Ox o'q, *b*) Oy o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzini toping.

Yechilishi. *a*) Berilgan chiziqning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzi, $Q_x=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}dx$ formulaga asosan topiladi, bunda, $f(x)=\sqrt{2px},\ f'(x)=\frac{p}{\sqrt{2px}},\ \sqrt{1+(f'(x))^2}=\sqrt{1+\frac{p}{2x}}$.

Shunday qilib, berilgan chiziqning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzi:

$$Q_x=2\pi\int_0^{x_0}f(x)\sqrt{1+(f'(x))^2}dx=2\pi\sqrt{p}\int_0^{x_0}\sqrt{p+2x}dx=$$
$$=\frac{2\pi\sqrt{p}}{3}[\sqrt{(p+2x_0)^3}-\sqrt{p^3}]=\frac{2\pi}{3}[(p+2x_0)\sqrt{p^2+2px_0}-p^2].$$

b) Berilgan chiziqning Ox o'qqa nisbatan simmetrikligini e'tiborga olib, izlanayotgan sirtning yuzini, $Q_y=4\pi\int_0^{\sqrt{2px_0}}x(y)\sqrt{1+[x'(y)]^2}dy$ formula orqali topamiz:

$$Q_y=\frac{2\pi}{p}\int_0^{\sqrt{2px_0}}y^2\sqrt{1+\frac{y^2}{p^2}}dy=\frac{2\pi}{p^2}\int_0^{\sqrt{2px_0}}y^2\sqrt{p^2+y^2}dy=$$
$$=[u=y,\ dv=y\sqrt{p^2+y^2}dy;\ du=dy;\ v=\frac{1}{3}(p^2+y^2)^{\frac{3}{2}}]=$$
$$=\frac{2\pi}{p^2}\left\{\frac{y(p^2+y^2)^{3/2}}{4}-\frac{p^2}{8}[y\sqrt{p^2+y^2}+p^2\ln(y+\sqrt{p^2+y^2})]\right\}\bigg|_0^{\sqrt{2px_0}}=$$
$$=\frac{\pi}{4}[(p+4x_0)\sqrt{2x_0(p+2x_0)}-p^2\ln\frac{\sqrt{2x_0}+\sqrt{p+2x_0}}{\sqrt{p}}].\blacksquare$$

12-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)\ (0\le t\le 2\pi)$, chiziqning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzini toping.

Yechilishi. Izlanayotgan sirtning yuzi, $Q_x=2\pi\int_\alpha^\beta\psi(t)\sqrt{(\varphi'(t))^2+(\psi'(t))^2}dt$ formula orqali hisoblanadi:

$$x'_t=a(1-\cos t),\ y'_t=a\sin t,\ ds=\sqrt{(x'_t)^2+(y'_t)^2}dt=$$
$$=\sqrt{a^2(1-\cos t)^2+a^2\sin t}dt=a\sqrt{2}\sqrt{1-\cos t}\cdot dt=2a\sin\frac{t}{2}dt,$$
$$Q=2\pi\int_0^{2\pi}a(1-\cos t)\cdot 2a\cdot\sin\frac{t}{2}dt=2\pi\int_0^{2\pi}4a^2\sin^3\frac{t}{2}dt=$$

$$=16\pi a^2\int_0^{\pi}\sin^3 v dv=\frac{64}{3}\pi a^2.\blacksquare$$

13-misol. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ tenglama bilan berilgan astroidaning birinchi koordinata burchagida joylashgan yoy qismi og'irlik markazining koordinatalarini topiing.

Yechilishi. Simmetriya shartidan foydalanib $x_M=y_M$ ekanligini aniqlaymiz. Shundan so'ng astroida yoyining chorak qismi $l=\frac{3}{2}a$ ekanligini tasdiqlaymiz. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ ifodani differensiallab, quyidagini hosil qilamiz $\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y'=0$. Bundan

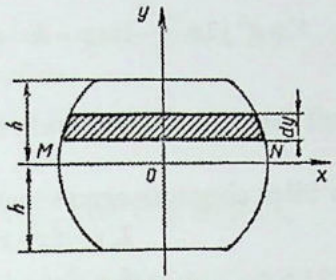
$$y'=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}},\ 1+y'^2=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}},\ \sqrt{1+y'^2}=\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Shuning uchun,

$$x_M=y_M=\frac{2}{3a}\int_0^a x\cdot a^{\frac{1}{3}}x^{-\frac{1}{3}}dx=\frac{2a^{\frac{1}{3}}}{3a}\int_0^a x^{\frac{2}{3}}dx=\frac{2}{5}a^{-\frac{2}{3}}x^{\frac{5}{3}}\Big|_0^a=\tfrac{2}{5}a.\blacksquare$$

14-misol. Yog'och ko'priklarni yasashda ko'p hollarda ikki qismga bo'lingan doiraviy bolor (g'o'la) lar bilan ish ko'riladi (8.18-chizma). Shunday kesimning gorizontal o'rta chiziqqa nisbatan inersiya momentini aniqlang.

Yechilishi. Koordinata sistemasi chizmada ko'rsatilgan tarzda o'rganamiz. U holda $dI_x=y^2dS$, bu yerda $dS=MN\cdot dy=2xdy=2\sqrt{R^2-y^2}dy$. Bundan



8.18-chizma.

$$I_x=2\int_{-h}^{h}y^2\sqrt{R^2-y^2}dy=4\int_0^h y^2\sqrt{R^2-y^2}dy.$$

$y=R\sin t$ belgilash kiritib, integralni hisoblaymiz:

$$I_x=\frac{R^4}{2}\arcsin\frac{h}{R}+\frac{h}{R}(2h^2-R^2)\sqrt{R^2-h^2}.$$

Xususiy holda, ya'ni $h=R$ bo'lganda doiraning diametrlardan biriga nisbatan inersiya momenti: $I_x=\frac{\pi R^4}{4}.\blacksquare$

15-misol. Tomoni a bo'lgan kvadrat uning uchidan o'tgan va diagonali bilan $\frac{\pi}{4}<\varphi<\frac{\pi}{2}$ burchak hosil qilgan o'q atrofida aylanadi. Aylanuvchi jismning hajmi va sirtining yuzasi topilsin.

Yechilishi. Guldenning birinchi teoremasiga asosan, aylanuvchi jismning yuzasi aylanuvchi shaklning perimetri bilan shu shaklning og'irlik markazi chizgan aylana uzunligining ko'paytmasiga teng.

Bizning misolda perimetr $4a$ ga teng, og'irlik markazi kvadratning og'irlik markazi bilan ustma – ust tushadi. Bu markaz aylanish o'qidan $d=\frac{a\sqrt{2}}{2}\sin\varphi$ masofada bo'ladi. Shuning uchun

$$S_{T.B.}=4a\cdot 2\pi\frac{a\sqrt{2}}{2}\sin\varphi=4\pi a^2\sqrt{2}\sin\varphi.$$

Xuddi shu usul bilan aylanuvchi jismning hajmini ham topish mumkin. Guldenning ikkinchi teoremasiga asosan u aylanuvchi shaklning yuzasi bilan shu shakl og'irlik markazi chizgan aylana uzunligining ko'paytmasiga teng. Ma'lumki, kvadratning yuzasi a^2 ga teng. Shuning uchun

$$V = a^2 \cdot 2\pi \frac{a\sqrt{2}}{2} \sin\varphi = \pi a^3 \sqrt{2}\sin\varphi. ■$$

Mustaqil yechish uchun misol va masalalar

Quyidagi chiziqlar bilan chegaralangan sohaning yuzini toping.

1. $y = x,\ y = 2 - x^2.$ **2.** $y = 6x - x^2 - 7,\ y = x - 3.$

3. $y = \sin x,\ y = 0,\ 0 \le x \le \pi.$ **4.** $y = x^2 + 1,\ x + y = 3.$

Quyidagi, tenglamasi parametrik shaklda berilgan, chiziqlar bilan chegaralangan sohaning yuzini toping.

5. $x = a(t - \sin t),\ y = a(1 - \cos t)\ (0 \le t \le 2\pi),\ y = 0.$

6. $x = a\cos^3 t,\ y = a\sin^3 t.$

Quyidagi, tenglamalari qutb koordinatalar sistemasida berilgan, chiziqlar bilan chegaralangan sohaning yuzini toping.

7. $r^2 = a^2\cos 4\varphi$ (lemniskata) .

8. $r = a(1 + \cos\varphi)$ (kardoida) .

Quyidagi berilgan chiziqlarning ko'rsatilgan kesmalardagi yoy uzunliklarini toping.

9. $y^2 = x^3\ 0 \le x \le 1\ (y \ge 0).$

10. $y = \ln x,\ \frac{3}{4} \le x \le \frac{12}{5}.$

11. $x = a(t - \sin t),\ y = a(1 - \cos t)\ (0 \le t \le 2\pi).$

12. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),\ \ (0 \le t \le 2\pi).$

Quyida berilgan chiziqlarning ko'rsatilgan kesmalardagi yoy uzunliklarini toping.

13. $r = \cos^3\frac{\varphi}{3},\ 0 \le \varphi \le \frac{\pi}{2}.$ **14.** $r = \varphi^2,\ 0 \le \varphi \le \pi.$

Quyidagi chiziqlar bilan chegaralangan figuralarni ko'rsatilgan o'q atrofida aylantirishdan hosil bo'lgan aylanma jismlarning hajmlarini toping.

15. $y = \sin x,\ 0 \le x \le \pi$: *a)* *Ox* o'q atrofida. *b)* *Oy* o'q atrofida.

16. $y = 2x - x^2,\ y = 0$: *a)* *Ox* o'q atrofida. *b)* *Oy* o'q atrofida.

17. $y = \sin 2x,\ 0 \le x \le \frac{\pi}{2},\ y = 0$: *Ox* o'q atrofida.

18. $y = b\sqrt[3]{\left(\frac{x}{a}\right)^2},\ 0 \le x \le a$ *Ox* o'q atrofida.

19. $x = a(t - \sin t),\ y = a(1 - \cos t)\ \ (0 \le t \le 2\pi)$: *a)* *Oy* o'q atrofida; *b)* $y = 2a$ to'g'ri chiziq atrofida.

20. $x = a\sin^3 t,\ y = b\cos^3 t\ \ (0 \le t \le 2\pi)$: *a)* *Ox* o'q atrofida; *b)* *Oy* o'q atrofida.

21. $0 \le r \le a\sqrt[3]{\cos 3\varphi},\ |\varphi| \le \frac{\pi}{6}$, qutb o'qi atrofida.

22. $0 \le r \le 2a\frac{\sin^2\varphi}{\cos\varphi},\ 0 \le \varphi \le \frac{\pi}{3}$, qutb o'qi atrofida.

Quyidagi sirtlar bilan chegaralangan jismning hajmini toping:

23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ |z| = H,\ \ H > 0.$

24. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = \frac{c}{a}x,\ z = 0,\ \ z = 0.$

Quyidagi chiziqlarni ko'rsatilgan o'q atrofida aylantirishda hosil bo'lgan aylanma sirtlarining yuzlarini hisoblang.

25. $y = tgx\ \ \left(0 \le x \le \frac{\pi}{4}\right)$, *Ox* o'q atrofida.

26. $y = e^{-x}\ \ (0 \le x \le a)$, *Ox* o'q atrofida.

27. $x = a(3\cos t - \cos 3t),\ y = a(3\sin t - \sin 3t),\ 0 \le t \le \frac{\pi}{2}$, *a)* *Ox* o'q atrofida; *b)* *Oy* o'q atrofida.

28. $x = e^t\sin t,\ y = e^t\cos t,\ \ 0 \le t \le \frac{\pi}{2}$, *Ox* o'q atrofida .

Quyidagi silindrik chiziqlarni ko'rsatilgan o'q atrofida aylanti-rishda hosil bo'lgan sirtlarning yuzlarini hisoblang.

29. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$, *Ox* o'q atrofida.

30. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.\ a > b$, *Oy* o'q atrofida.

Quyida berilgan chiziqlarning M_x va M_y statik momentlarini toping.

31. $\frac{x}{a} + \frac{y}{b} = 1,\ x \ge 0,\ y \ge 0.$ **32.** $x^2 + y^2 = 4,\ y \ge 0.$

33. $y = chx,\ \ 0 \le x \le 1.$ **34.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, y \ge 0,\ \ a > b.$

35. $y=\cos x$, $\left(-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right)$, kosinusoidaning Ox o‘qqa nisbatan statik momentini toping.

36. $y=\sin x$, $0\le x\le\pi$, sinusoidaning Ox o‘qqa nisbatan statik momentini toping.

Tenglamalari parametrik shaklda berilgan quyidagi chiziqning M_x va M_y statik momentlarini toping.

37. $x=a\sin t,\ y=b\cos t,\ 0\le t\le\frac{\pi}{2}, a>b\ (\rho=1)$.

38. $x=a\sin^3 t,\ y=a\cos^3 t,\ \ 0\le t\le\frac{\pi}{2}\ \ (\rho=1)$.

Qutb koordinatalar sistemasida berilgan chiziqning M_x va M_y statik momentlarini toping.

39. $r=2a\cos\varphi,\ 0\le\varphi\le\frac{\pi}{2}\ (\rho=1)$.

40. $\eta=a(1+\cos\varphi),\ -\pi\le\varphi\le\pi\ (\rho=1)$.

Quyidagi berilgan chiziqlar og‘irlik markazining x_M va y_M koordinatalarini toping.

41. $x^2+y^2=4,\ y\ge0$.

42. $y=ach(x/a),\ |x|\le b$.

43. $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$.

44. $r=a(1+\cos\varphi),\ 0\le\varphi\le\pi$.

Quyidagi berilgan chiziqlarning, Ox o‘qqa nisbatan, I_x inersiya momentini toping.

45. $y=\sqrt{r^2-x^2}, -r\le x\le r$.

46. $y=e^x, 0\le x\le 0{,}5$.

47. $x=R\cos\varphi,\ y=R\sin\varphi,\ 0\le\varphi\le\alpha\le 2\pi$.

Quyidagi berilgan chiziqning, koordinatalar o‘qlariga nisbatan, I_x va I_y inersiya momentlarini toping.

48. $y=chx,\ 0\le x\le 1$.

49. $x=a\cos^3 t, y=a\sin^3 t, 0\le t\le\frac{\pi}{2}$.

Quyida berilgan chiziqlar bilan chegaralangan tekis shakllarning M_x va M_y statik momentlarini toping.

50. $x+y=1, x=0,\ y=0$.

51. $y=\cos x, |x|\le\frac{\pi}{2}, y=0$.

Quyidagi misollarda berilgan egri chiziqlar bilan chegaralangan sohaning grafigini chizing va og‘irlik markazini toping.

52. $y=6x-x^2, y=x$.

53. $x^2=4y,\ x-2y+4=0$.

Quyidagi qutb koordinatalar sistemasida berilgan egri chiziqlar bilan chegaralangan sohaning og‘irlik markazi koordinatalarini toping.

54. $r=a\sin 2\varphi,\ 0\le\varphi\le\frac{\pi}{2}$.

55. $r=a\,(1+\cos\varphi)$ kardoida.

56. Gulden teoremasidan foydalanib, R radiusli yarim aylananing og‘irlik markazi koordinatalarini aniqlang.

57. Gulden teoremasidan foydalanib, shar sirtning yuzini hisoblang.

58. Gulden teoremasidan foydalanib, doiraviy konusning hajmi va yon sirtining yuzini hisoblang.

59. $y=chx,\ 0\le x\le 1$, chiziq yoyining Ox va Oy o‘qlariga nisbatan simmetrik momentlari va inersiya momentlarini toping.

60. $x=a\cos t,\ \ y=a\sin t$ aylana birinchi choragida joylashgan yoyi massalari markazining koordanatalarini toping.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{9}{2}$. **2.** $\frac{9}{2}$. **3.** 2. **4.** $4\frac{1}{2}$. **5.** $3\pi a^2$. **6.** $\frac{3}{8}\pi a^2$. **7.** a^2.

8. $\frac{3}{2}\pi a^2$. **9.** $\frac{8}{27}\left(\frac{13\sqrt{13}}{8}-1\right)$. **10.** $\frac{27}{20}+\ln 2$. **11.** $8a$. **12.** $2\pi a^2$.

13. $\frac{9}{8}(2\pi+3\sqrt{3})$. **14.** $\frac{1}{3}[\sqrt{(\pi^2+4)^3}-8]$. **15.** $a)\ 0{,}5\pi^2$. $b)\ 2\pi^2$.

16. $a)\ \frac{16}{15}\pi$. $b)\ \frac{8}{3}\pi$. **17.** $0{,}25\pi^2$. **18.** $\frac{3}{7}\pi ab^2$. **19.** $a)\ \frac{32}{15}\pi p^3$. $b)\ \frac{4}{3}\pi p^3$.

20. 21. $\pi a^3/24$. **22.** $\pi a^3(51-64\ln 2)/4$. **23.** $2\pi abH$. **24.** $\frac{2}{3}abc$.

25. $\pi[(\sqrt{5}-\sqrt{2})+\ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}]$.

26. $\pi\left(\sqrt{2}-e^{-a}\sqrt{1+e^{2a}}-\ln\dfrac{e^{-a}+\sqrt{1+e^{2a}}}{1+\sqrt{2}}\right)$. **27.** *a)* $9\pi^2a^2$. *b)* $24\pi a^2$.

28. $\dfrac{2\pi\sqrt{2}}{5}\left(e^{\pi}-2\right)$. **29.** $2\pi\,[b^2+ab\dfrac{\arcsin\varepsilon}{\varepsilon}]$, $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$.

30. $2\pi\,[a^2+\dfrac{b^2}{2\varepsilon}\ln\dfrac{1+\varepsilon}{1-\varepsilon}]$, $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$. **31.** $M_x=\dfrac{b\sqrt{a^2+b^2}}{2}$, $M_y=\dfrac{a\sqrt{a^2+b^2}}{2}$.

32. $M_x=4$, $M_y=0$. **33.** $M_x=0{,}25\,(2+sh2)$, $M_y=sh1-ch1+1$.

34. $M_x=b\left(b+\dfrac{a}{\varepsilon}\arcsin\varepsilon\right)$, $M_y=0$, $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$. **35.** $M_x=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$.

36. $\sqrt{2}+\ln\left(1+\sqrt{2}\right)$.

37. $M_x=\dfrac{ab}{2\varepsilon}\left(\varepsilon\sqrt{1-\varepsilon^2}+\arcsin\varepsilon\right)$, $M_y=\dfrac{a^2}{2}\left(1+\dfrac{1-\varepsilon^2}{2\varepsilon}\ln\dfrac{1+\varepsilon}{1-\varepsilon}\right)$, $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$.

38. $M_x=M_y=\dfrac{3}{5}a^2$. **39.** $M_x=2a^2$, $M_y=\pi a^2$. **40.** $M_x=0$, $M_y=\dfrac{32a^2}{5}$.

41. $x_M=0$, $y_M=\dfrac{4}{\pi}$. **42.** $x_M=0; y_M=\dfrac{ash(2b/a)+2b}{4sh(b/a)}$.

43. $x_M=\pi a;\ y_M=4a/3$. **44.** $x_M=y_M=4a/5$. **45.** $\dfrac{\pi r^3}{2}$.

46. $\dfrac{1}{3}[\sqrt{(1+e)^3}-2\sqrt{2}]$. **47.** $\dfrac{1}{4}(2\alpha-\sin 2\alpha)R^3$.

48. $I_x=sh1+\dfrac{1}{3}sh^3 1$, $I_y=3sh1-2ch1$. **49.** $I_x=I_y=\dfrac{3}{8}a^3$.

50. $M_x=\dfrac{1}{6}$, $M_y=\dfrac{1}{6}$. **51.** $M_x=\dfrac{\pi}{4}$, $M_y=0$. **52.** $x_M=\dfrac{5}{2}$; $y_M=5$.

53. $x_M=1$, $y_M=\dfrac{8}{5}$. **54.** $\xi_M=\eta_M=\dfrac{128a}{105\pi}$.

55. $\xi_M=\dfrac{5a}{6}$; $\eta_M=0$. $M_x=\dfrac{bh^2}{6}$; $I_x=\dfrac{bh^3}{12}$. **56.** $x_M=0$, $y_M=\dfrac{4R}{3\pi}$.

57. $S=4\pi R^2$.

58. $V=\dfrac{1}{3}\pi R^2H$, $S=\pi RL$. **59.** $\dfrac{1}{4}(2+sh2); sh1-ch1+1; sh1+\dfrac{1}{3}sh^3 1; 3sh1-2ch1$.

60. $C(\frac{2a}{\pi};\frac{2a}{\pi})$.

8-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

8.1 – masala. Quyidagi aniqmas integralni hisoblang.

8.1.1. $\int\left(5-x^3\right)^4dx$. **8.1.2.** $\int x^3(6-x)^2dx$.

8.1.3. $\int(1-x)(1-2x)(1-3x)dx$. **8.1.4.** $\int\left(\dfrac{2-x}{x}\right)^3dx$.

8.1.5. $\int\left(\dfrac{a}{x}+\dfrac{a^2}{x^2}+\dfrac{a^3}{x^3}\right)dx$. **8.1.6.** $\int\dfrac{x^2+2}{\sqrt[3]{x}}dx$

8.1.7. $\int\dfrac{\sqrt[3]{x}+3\sqrt[3]{x^2+3}}{\sqrt[5]{x}}dx$. **8.1.8.** $\int\dfrac{(1-x)^2}{x^2\sqrt[3]{x}}dx$.

8.1.9. $\int\left(1-\dfrac{1}{x^3}\right)\sqrt{x\sqrt{x}}\,dx$. **8.1.10.** $\int\dfrac{\left(\sqrt{3x}-\sqrt[3]{5x}\right)^2dx}{\sqrt{x}}$.

8.1.11. $\int\dfrac{\sqrt{x^5+x^{-5}+3}}{x^2}dx$. **8.1.12.** $\int\dfrac{x^2}{1+x^2}dx$.

8.1.13. $\int\dfrac{x^2}{1-x^2}dx$. **8.1.14.** $\int\dfrac{x^2+4}{x^2-1}dx$.

8.1.15. $\int\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$. **8.1.16.** $\int\left(3^x+5^x\right)^2dx$.

8.1.14. $\int\dfrac{2^{x+1}-5^{x-1}}{10^x}dx$. **8.1.18.** $\int(2\sin 2x+3\cos 3x)dx$

8.1.19. $\int ctg^2xdx$. **8.1.20.** $\int tg^2xdx$.

8.1.21. $\int\left(3-x^3\right)^3dx$. **8.1.22.** $\int x^3(5-2x)^2dx$.

8.1.23. $\int(1+x)(1-2x)(1+3x)dx$. **8.1.24.** $\int\left(\dfrac{2-3x}{2x}\right)^2dx$.

8.1.25. $\int\left(\dfrac{2}{x}+\dfrac{4}{x^2}+\dfrac{8}{x^3}\right)dx$.

8.1.26-misol. $\int\dfrac{1+2x^2}{x^2\left(1+x^2\right)}dx$ aniqmas integralni hisoblang.

Yechilishi ([3], 1-q., 177-184 betlar; [9], 1-t., 9- bo'lim; [30], 8-bo'lim). Integral ostidagi ifodani shakl almashtirib va

$$\int x^{\mu}dx=\frac{x^{\mu+1}}{\mu+1}+C,\ \mu\neq-1,\ \int\frac{dx}{a^2+x^2}=\frac{1}{a}arctg\frac{x}{a}+C=-\frac{1}{a}arcctgx+C$$

formulalardan foydalanib, aniqmas integralni hisoblaymiz:

$$\int\frac{1+2x^2}{x^2(1+x^2)}dx=\int\frac{1+x^2+x^2}{x^2(1+x^2)}dx=\int(\frac{1}{x^2}+\frac{1}{1+x^2})dx=arctgx-\frac{1}{x}+C.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

>

Int((1+2*x^2)/((x^2)*(1+x^2)),x)=int((1+2*x^2)/((x^2)*(1+x^2)),x);

$$\int\frac{1+2x^2}{x^2(x^2+1)}dx=-\frac{1}{x}+\arctan(x)$$

8.2 – masala. O'zgaruvchilarni almashtirish usulidan foydalanib, quyidagi aniqmas integralni hisoblang.

8.2.1. $\int x\sqrt{x-5}\,dx.$ **8.2.2.** $\int\frac{dx}{1+e^x}.$

8.2.3. $\int\frac{x^2+3}{\sqrt{(2x-5)^3}}dx.$ **8.2.4.** $\int\frac{\sin x}{\sqrt{\cos x}}dx.$

8.2.5. $\int\frac{dx}{(\arccos x)^5\sqrt{1-x^2}}.$ **8.2.6.** $\int\frac{x^2+1}{\sqrt[3]{x^3+3x+1}}dx.$

8.2.4. $\int\sqrt[3]{1+3\sin x}\cos x dx.$ **8.2.8.** $\int\frac{dx}{a^2\sin^2x+b^2\cos^2x}.$

8.2.9. $\int\frac{\sqrt[3]{1+\ln x}}{x}dx.$ **8.2.10.** $\int\frac{dx}{x\ln x}.$

8.2.11. $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx.$ **8.2.12.** $\int\frac{1+\ln x}{3+x\ln x}dx.$

8.2.13. $\int\frac{\sin 2x}{1+\sin^2x}dx.$ **8.2.14.** $\int\frac{xdx}{4+x^4}.$

8.2.15. $\int\frac{x^3dx}{x^8-2}.$ **8.2.16.** $\int\frac{dx}{(1+x)\sqrt{x}}.$

8.2.14. $\int\frac{xdx}{(x^2-1)^{3/2}}.$ **8.2.18.** $\int\frac{x^2dx}{(8x^3+27)^{2/3}}.$

8.2.19. $\int\frac{e^xdx}{2+e^x}.$ **8.2.20.** $\int\frac{dx}{e^x+e^{-x}}.$

8.2.21. $\int\frac{dx}{x\ln x\cdot\ln(\ln x)}.$ **8.2.2.** $\int\frac{e^xdx}{1+e^x}.$

8.2.23. $\int\frac{3}{\sqrt{(3x-5)^3}}dx.$ **8.2.24.** $\int\frac{\sin x}{\sqrt{\cos^3x}}dx.$

8.2.25. $\int\frac{dx}{(\arcsin x)^5\sqrt{1-x^2}}.$

8.2.26-misol. Quyidagi $J=\int\frac{dx}{\sqrt{x^2+\alpha}},\ \alpha\neq0$ aniqmas integralni o'zgaruvchilarni almashtirish usulidan foydalanib, hisoblang.

Yechilishi ([3], 1-q., 185-187 betlar; [9], 1-t., 9- bo'lim; [30], 8-bo'lim). $x+\sqrt{x^2+\alpha}=t(x)$ deb belgilasak,

$$dt=\left(x+\sqrt{x^2+\alpha}\right)'dx=\left(1+\frac{x}{\sqrt{x^2+\alpha}}\right)dx=\frac{t(x)}{\sqrt{x^2+\alpha}}dx,\ \frac{dx}{\sqrt{x^2+\alpha}}=\frac{dt(x)}{t(x)}$$

ni hosil qilamiz. Natijada

$$J=\int\frac{dt(x)}{t(x)}=\ln|t(x)|+C=\ln\left|x+\sqrt{x^2+\alpha}\right|+C.$$

Shunday qilib,

$$\int\frac{dx}{\sqrt{x^2+\alpha}}=\ln\left|x+\sqrt{x^2+\alpha}\right|+C.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> **Int(1/sqrt(x^2+a),x)=int(1/sqrt(x^2+a),x);**

$$\int\frac{1}{\sqrt{x^2+a}}dx=\ln(x+\sqrt{x^2+a})$$

8.3 – masala. Bo'laklab integrallash usulidan foydalanib, quyidagi aniqmas integralni hisoblang.

8.3.1. $\int x\cos 2x dx.$ **8.3.2.** $\int x\ln 3x dx.$

8.3.3. $\int xe^{-x}dx.$ **8.3.4.** $\int x3^xdx.$

8.3.5. $\int x^n\ln x dx\,(n\neq-1).$ **8.3.6.** $\int x\,arctgx dx.$

8.3.4. $\int\ln x dx.$ **8.3.8.** $\int x^3e^{-x^2}dx.$

8.3.9. $\int\sqrt{x}\ln^2x dx.$ **8.3.10.** $\int\arccos x dx.$

8.3.11. $\int (\arcsin x)^2 dx.$

8.3.12. $\int \frac{x^2}{(1+x^2)^2}.$

8.3.13. $\int \frac{dx}{(a^2+x^2)^2}.$

8.3.14. $\int e^{\sqrt{x}} dx.$

8.3.15. $\int x \sin\sqrt{x} dx$

8.3.16. $\int e^{ax}\cos bx dx.$

8.3.14. $\int e^{ax}\sin bx dx.$

8.3.18. $\int e^{2x}\sin^2 x dx.$

8.3.19. $\int \frac{\ln\cos x}{\cos^2 x} dx.$

8.3.20. $\int \sqrt{2-x^2} dx.$

8.3.21. $\int x\sin 2x dx.$

8.3.22. $\int x \lg 3x dx.$

8.3.23. $\int x e^{-2x} dx.$

8.3.24. $\int x 5^{2x} dx.$

8.3.25. $\int x^2 \ln 3x dx$.

8.3.26-misol. Quyidagi $\int x^2 \ln x dx$ aniqmas integralni, bo'laklab integrallash usulidan foydalanib, hisoblang.

Yechilishi ([3], 1-q., 187-190 betlar; [9], 1-t., 9- bo'lim; [30], 8- bo'lim).

$$\int x^2 \ln x dx = \begin{bmatrix} u=\ln x, & du=\frac{1}{x}dx \\ dv = x^2 dx, & v=\frac{x^3}{3} \end{bmatrix} = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{x^3}{9} + C.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Int(x^2*ln(x),x)=int(x^2*ln(x),x);

$$\int x^2 \ln(x)\, dx = \frac{1}{3}x^3\ln(x) - \frac{x^3}{9}$$

8.4- masala. Nyuton – Leybnis formulasidan foydalanib, quyidagi aniq integralni hisoblang.

8.4.1. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^3 x} dx.$

8.4.2. $\int_0^2 \frac{dx}{\sqrt{16-x^2}}.$

8.4.3. $\int_0^{\pi} \frac{dx}{1+2\sin x}.$

8.4.4. $\int_{-2}^{-1} \frac{dx}{(11+5x)^3}.$

8.4.5. $\int_{-3}^{-2} \frac{dx}{x^2-1}.$

8.4.6. $\int_{-\pi}^{\pi} \sin^2\frac{x}{2} dx$

8.4.7. $\int_0^{\frac{\pi}{6}} \frac{x^2}{x^2+1} dx.$

8.4.8. $\int_e^{e^2} \frac{dx}{x\ln x}.$

8.4.9. $\int_{\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\sin(1/x)}{x^2} dx.$

8.4.10. $\int_0^1 \frac{e^x}{1+e^{2x}} dx.$

8.4.11. $\int_0^1 \left(\sqrt{x}+\sqrt[3]{x^2}\right) dx.$

8.4.12. $\int_2^9 \sqrt[3]{(x-1)^2} dx.$

8.4.13. $\int_0^1 \frac{x^2 dx}{1+x^6}.$

8.4.14. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} tg^4 x dx.$

8.4.15. $\int_0^1 \frac{dx}{4x^2+4x+5}.$

8.4.16. $\int_{-2}^{-1} \frac{x+1}{x^2(x-1)} dx.$

8.4.17. $\int_2^3 \frac{dx}{x^2-2x-8}.$

8.4.18. $\int_0^1 \frac{dx}{4x^2+4x+5}.$

8.4.19. $\int_3^4 \frac{x^2+3}{x-2} dx.$

8.4.20. $\int_0^1 \frac{x^2+3x}{(x+1)(x^2+1)} dx.$

8.4.21. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos 3x}{\sin^3 3x} dx.$

8.4.22. $\int_0^2 \frac{dx}{\sqrt{9-x^2}}.$

8.4.23. $\int_0^{\pi} \frac{dx}{1+2\sin 2x}.$

8.4.24. $\int_{-2}^{-1} \frac{x dx}{(4+5x)^3}.$

8.4.25. $\int_3^2 \frac{dx}{x^2-1}.$

8.4.26-misol. $\int_1^e \frac{dx}{x(1+\ln^2 x)}$ integralni hisoblang.

Yechilishi ([2], 7- bo'lim; [3], 1-q., 246-247 betlar; [9], 1-t., 9- bo'lim; [30], 8- bo'lim). Integral ostidagi ifodaning shaklini almashtiramiz va

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C = -\frac{1}{a}\operatorname{arcctg}\frac{x}{a} + C$$

formulaga asosan, boshlang'ich funksiyasini topib, so'ngra (4.12) Nyuton – Leybnis formulasidan foydalanib, aniq integralni hisoblaymiz:

$$\int_1^e \frac{dx}{x(1+\ln^2 x)} = \int_1^e \frac{d(\ln x)}{1+\ln^2 x} = \operatorname{arctg}(\ln x)\Big|_1^e = \frac{\pi}{4}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> Int(1/(x*(1+(ln(x))^2)), x=1..exp(1))= int(1/(x*(1+(ln(x))^2)), x=1..exp(1));

$$\int_1^e \frac{1}{x(1+\ln(x)^2)}\,dx = \frac{\pi}{4}$$

8.5- masala. Tenglamalari Dekart koordinatalar sistemasida, parametrik shaklda, qutb koordinatalar sistemasida berilgan quyidagi egri chiziqlarning yoy uzunligini hisoblang.

8.5.1. $y=\ln x,\ \sqrt{3}\le x\le\sqrt{15}$.

8.5.2. $y=\frac{x^2}{4}-\frac{\ln x}{2},\ 1\le x\le 2.$

8.5.3. $y=\sqrt{1-x^2}+\arcsin x,\ 0\le x\le 7{,}9.$

8.5.4. $y=\ln\frac{5}{2x},\ \sqrt{3}\le x\le\sqrt{8}.$

8.5.5. $y=2+\arcsin\sqrt{x}+\sqrt{x-x^2},\ \ 1/4\le x\le 1.$

8.5.6. $y=\ln(x^2-1),\ 2\le x\le 3.$

8.5.7. $\begin{cases} x=5(t-\sin t), \\ y=5(1-\cos t),\ 0\le t\le\pi. \end{cases}$

8.5.8. $\begin{cases} x=3(2\cos t-\cos 2t) \\ y=3(2\sin t-\sin 2t,)\ 0\le t\le 2\pi \end{cases}$

8.5.9. $\begin{cases} x=4(\cos t+t\sin t) \\ y=4(\sin t-t\cos t),\ \ 0\le t\le 2. \end{cases}$

8.5.10. $\begin{cases} x=e^t(\cos t+\sin t), \\ y=e^t(\cos t-\sin t),\ 0\le t\le\pi. \end{cases}$

8.5.11. $\begin{cases} x=3(t-\sin t), \\ y=3(1-\cos t),\ \ \pi\le t\le 2\pi. \end{cases}$

8.5.12. $\begin{cases} x=3(\cos t+t\sin t), \\ y=3(\sin t-t\cos t),\ \ \ 0\le t\le\pi/3 \end{cases}$

8.5.13. $\begin{cases} x=6\cos^3 t, \\ y=6\sin^3 t,\ 0\le t\le\pi/3 \end{cases}$

8.5.14. $\begin{cases} x=6(\cos t+t\sin t), \\ y=6(\sin t-t\cos t),\ 0\le t\le\pi. \end{cases}$

8.5.15. $\begin{cases} x=8\cos^3 t, \\ y=8\sin^3 t,\ \ 0\le t\le\pi/6. \end{cases}$

8.5.16. $\begin{cases} x=4(t-\sin t), \\ y=4(1-\cos t),\ \ \pi/2\le t\le 2\pi/3. \end{cases}$

8.5.17. $\rho=3\varphi,\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$.

8.5.18. $\rho=\sqrt{2\varphi},\ \ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}.$

8.5.19. $\rho=2\varphi,\ \ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}.$

8.5.20. $\rho=5\varphi,\ \ -\pi/2\le\varphi\le\pi/2.$

8.5.21. $y=\lg x,\ \sqrt{3}\le x\le\sqrt{15}$.

8.5.22. $y=\frac{x^2}{4},\ 1\le x\le 2.$

8.5.23. $y=\sqrt{1-x^2},\ 0\le x\le 7{,}9.$

8.5.24. $y=\ln\frac{2}{x},\ \sqrt{3}\le x\le\sqrt{8}.$

8.5.25. $y=2+\arcsin\sqrt{x},\ \ 1/4\le x\le 1.$

8.5.26-misol. Ushbu $\begin{cases} x=20(t-\sin t), \\ y=20(1-\cos t) \end{cases}$ $(0\le t\le\pi)$ parametric tenglamalar bilan berilgan egri chiziqning (sikloidaning) yoy uzunligini toping.

Yechilishi ([2], 7- bo'lim; [3], 1-q., 271-282 betlar; [9], 1-t., 10-bo'lim; [30], 6-bo'lim). Avval $x=20\ (t-\sin t),\ \ y=20\ (1-\cos t)$ funksiyalarning hosilalarini hisoblaymiz: $x'(t)=20(1-\cos t),\ y'(t)=20\sin t$. Unda

$$x'^2(t)+y'^2(t)=20'^2(1-\cos t)^2+20^2\sin^2 t=400(1-\cos t)$$

bo'lib, $\sqrt{x'^2(t)+y'^2(t)}=20\sqrt{2(1-\cos t)}$. $l=\int_\alpha^\beta\sqrt{(x'(t))^2+(y'(t))^2dt}$ formulaga ko'ra izlanayotgan egri chiziqning uzunligi,

$$l=\int_0^{2\pi}20\sqrt{2(1-\cos t)}dt$$

bo'ladi. Bu ifodaaning o'ng tomonidagi integralni hisoblaymiz:

$$l=\int_0^{2\pi}20\sqrt{2(1-\cos t)}dt=20\int_0^{2\pi}\sqrt{4\cdot\sin^2\frac{t}{2}}dt=40\int_0^{2\pi}\sin\frac{t}{2}dt=$$

$$=80\int_0^{2\pi}\sin\frac{t}{2}d\left(\frac{t}{2}\right)=-80\cdot\cos\frac{t}{2}\Big|_0^{2\pi}=160.$$

9-bob. XOSMAS INTEGRALLAR

9.1-§. Birinchi tur (chegarasi cheksiz) xosmas integrallar

Biz yuqorida aniq integral (Riman integrali) tushunchasini kesmada chegaralangan funksiyalar uchun kiritgan edik. Agar integrallash oralig'i cheksiz, integral ostidagi funksiya chegaralanmagan bo'lganda, yuqorida berilgan aniq integral (Riman integrali) ta'rifi o'z kuchini yo'qotadi, chunki cheksiz oraliq bo'lganda, integrallash oralig'ini chekli uzunlikka ega bo'lgan n ta bo'lakga bo'lib bo'lmaydi, oraliq chekli bo'lib integral ostidagi funksiya chegaralanmagan holda esa, integral yig'indisi chekli limitga ega bo'lmaydi. Biz kelgusida Riman integral tushunchasini ikki yo'nalish bo'yicha umumlashtiramiz:

1) integrallash chegarasi cheksiz, ya'ni $[a,+\infty),(-\infty;a],(-\infty,\infty)$ hollar uchun;

2) chekli oraliqda integral ostidagi funksiya chegaralanmagan hol uchun.

9.1. Chegarasi cheksiz xosmas integral tushunchasi. $f(x)$ funksiya $[a,\infty)$ (a–biror haqiqiy son) aniqlangan bo'lib, bu oraliqning istalgan $[a,A]$ $(a\le A)$ qismida Riman ma'nosida integrallanuvchi bo'lsin, ya'ni $\int_a^A f(x)dx$ integral mavjud bo'lsin. Bu aniq integralni $F(A)=\int_a^A f(x)dx$, deb belgilaymiz. Unda, $F(A)$ funksiyaning $A\to+\infty$ dagi limitining mavjudlik masalasining qo'yilishi tabiiy:

$$\lim_{A\to+\infty}F(A)=\lim_{A\to+\infty}\int_a^A f(x)dx. \quad (9.1.1)$$

9.1.2-ta'rif. Agar (9.1.1) limit mavjud bo'lsa, bu limitga $f(x)$ funksiyadan $[a,\infty)$ oraliq bo'yicha olingan *birinchi tur xosmas integrali* deyiladi va u

$$\int_a^{+\infty}f(x)dx \quad (9.1.3)$$

kabi belgilanadi. Bu holda (9.1.3) xosmas integral yaqinlashuvchi deb ataladi. Demak,

$$\int_a^{+\infty} f(x)dx = \lim_{A\to+\infty} F(t) = \lim_{A\to+\infty}\int_a^A f(x)dx. \qquad (9.1.4)$$

Agar (9.1.4) limit cheksiz yoki mavjud bo'lmasa, (9.1.3) xosmas integral uzoqlashuvchi deyiladi.

Funksiyaning $(-\infty,a]$ va $(-\infty,+\infty)$ oraliqlar bo'yicha xosmas integrallari ham yuqoridagi kabi ta'riflanadi:

$$\int_{-\infty}^a f(x) = \lim_{A'\to-\infty}\int_{A'}^a f(x)dx \ (A'<a), \qquad (9.1.5)$$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{A\to\infty \\ A'\to-\infty}}\int_{A'}^A f(x)dx \qquad (9.1.6)$$

Agar biror a haqiqiy son uchun $\int_{-\infty}^a f(x)dx$ va $\int_a^{+\infty} f(x)dx$ xosmas integrallarning har bir yaqinlashuvchi bo'lsa, u hola $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integral ham yaqinlashuvchi va

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)dx$$

tenglik o'rinli bo'ladi.

Ta'kidlaymizki, agar $\int_a^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi va $\forall b>a$ bo'lgan son bo'lsa, uholda $\int_b^{+\infty} f(x)dx$ xosmas integral ham yaqinlashuvchi va

$$\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$$

tenglik o'rinli bo'ladi. Bu tasdiqning isboti xosmas integralning ta'rifidan bevosita kelib chiqadi.

9.2. Yaqinlashuvchi xosmas integrallarning sodda xossalari. Biz bundan keyin (9.1.4) ko'rinishdagi xosmas integral bilan ish ko'ramiz. Bu

integral uchun chiqarilgan xulosa va tasdiqlar, osonlikcha (9.1.5) va (9.1.6) ko'rinishdagi integrallarga ko'chiriladi.

Xosmas integrallarning sodda xossalari

1^0. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_A^{\infty} f(x)dx \,(A>a)$ integral ham yaqinlashuvchi bo'ladi va aksincha $\int_A^{\infty} f(x)dx$ yaqinlashuvchi bo'lsa, $\int_a^{\infty} f(x)dx$ integral ham yaqinlashuvchi hamda

$$\int_a^{\infty} f(x)dx = \int_a^A f(x)dx + \int_A^{\infty} f(x)dx$$

tenglik o'rinli bo'ladi.

2^0. $\int_a^{\infty} f(x)dx$ integral yaqinlashuvchi bo'lganda, $\lim_{A\to\infty}\int_A^{\infty} f(x)dx = 0$ bo'ladi.

3^0. $\int_a^{\infty} f(x)dx$ integralning yaqinlashuvchiligidan, $\int_a^{\infty} kf(x)dx \ (k = const)$ integralning ham yaqinlashuvchiligi kelib chiqadi va $\int_a^{\infty} k\, f(x)dx = k\int_a^{\infty} f(x)dx$ tenglik o'rinli bo'ladi.

4^0. Agar $\int_a^{\infty} f(x)dx$ va $\int_a^{\infty} g(x)dx$ integrallar yaqinlashuvchi bo'lsa, u holda $\int_a^{\infty}[f(x)) \pm g(x)]dx$ integral ham yaqinlashuvchi va

$$\int_a^{\infty}[f(x) \pm g(x)]dx = \int_a^{\infty} f(x)dx \pm \int_a^{\infty} g(x)dx$$

tenglik o'rinli bo'ladi.

9.3. Chegaralari cheksiz xosmas integralning yaqinlashuvchiligi

9.3.1 Manfiy bo'lmagan funksiya xosmas integralining yaqinlashuvchiligi. Agar $f(x)\geq 0$ bo'lsa, u holda

$$F(A) = \int_a^A f(x)dx \qquad (9.3.2)$$

integral A ning o'suvchi funksiyasini ifodalaydi. Bu funksiyaning $A \to \infty$ da chekli limitga ega bo'lishi, monoton funksiyaning chekli limitga ega bo'lishi haqidagi teoremaga asosan, quyidagicha ifoda qilinadi:

9.3.3-teorema. (9.1.3) xosmas integralning ($f(x) \geq 0$ bo'lgan holda) mavjud bo'lishi, ya'ni yaqinlashuvchi bo'lishi uchun

$$\int_a^A f(x)dx \leq M \quad (M = const)$$

shartning bajarilishi zarur va yetarli.

Agar bu shart bajarilmasa, u holda (9.1.3) integralning qiymati $+\infty$ ga teng bo'ladi (xosmas intgegral uzoqlashuvchi bo'ladi).

9.3.4-teorema. $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan hamda $\forall x \in [a,+\infty)$ lar uchun $f(x) \leq g(x)$ tengsizlik o'rinli bo'lib, $\int_a^{+\infty} g(x)dx$ integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashuvchi bo'ladi, agar $\int_a^{+\infty} f(x)dx$ integral uzoqlashuvchi bo'lsa, $\int_a^{+\infty} g(x)dx$ integral ham uzoqlashuvchi bo'ladi.

Amaliyotda ko'p ishlatiladigan, 9.3.4– teoremadan natija sifatida kelib chiqadigan quyidagi teoremani keltiramiz:

9.3.5-teorema. $f(x)$ va $g(x)$ manfiy bo'lmagan funksiyalar $[a,+\infty)$ da berilgan bo'lib,

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = k \quad (0 \leq k \leq +\infty)$$

bo'lsin.

Agar $k < +\infty$ va $\int_a^{+\infty} g(x)dx$ integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashuvchi bo'ladi.

Agar $k > 0$ bo'lib, $\int_a^{+\infty} g(x)dx$ integral uzoqlashuvchi bo'lsa, uholda $\int_a^{+\infty} f(x)dx$ integral ham uzoqlashuvchi bo'ladi.

Shunday qilib, $0 < k < +\infty$ bo'lganda, bir vaqtda ikkala integral bir vaqtda yaqinlashuvchi va bir vaqtda uzoqlashuvchi bo'ladi. Taqqoslash uchun $g(x)$ funksiyaning o'rniga aniq funksiyani tanlash bilan xususiy holda $\int_a^{\infty} f(x)dx$ integralning yaqinlashishi yoki uzoqlashishi uchun ushbu alomatni olish mumkin:

1^0. Agar x ning yetarlicha katta qiymatlarida $(x > x_0 > a)$ $f(x) = \frac{\varphi(x)}{x^{\alpha}}$ bo'lsa, u holda $\forall x > x_0$ lar uchun $\varphi(x) \leq c < +\infty$ va $\alpha > 1$ bo'lganda $\int_a^{\infty} f(x)dx$ integral yaqinlashuvchi, $\varphi(x) \geq c < 0$ va $\alpha \leq 1$ bo'lganda esa, $\int_a^{\infty} f(x)dx$ integral uzoqlashuvchi bo'ladi.

2^0. Agar $x \to +\infty$ da $f(x)$ funksiya $\frac{1}{x}$ ga nisbatan α ($\alpha > 0$) tartibli cheksiz kichik bo'lsa, u holda $\int_a^{\infty} f(x)\,dx$ integral $\alpha > 1$ bo'lgada yaqinlashuvchi $\alpha \leq 1$ bo'lganda esa, uzoqlashuvchi bo'ladi.

9.4. Ixtiyoriy funksiya xosmas integralining yaqinlashuvchiligi. $\int_a^{\infty} f(x)dx$ xosmas integralning yaqinlashuvchiligi 9.1.2-ta'rifga ko'ra , $A \to \infty$ da $F(A) = \int_a^A f(x)dx$ funksiyaning chekli limitga ega bo'lishlik shartidan iborat bo'ladi. Bu funksiyaga funksiyaning chekli limitga ega bo'lishi haqidagi Bolsano-Koshi teoremasini qo'llab quyidagi teoremaga kelamiz:

9.4.1-teorema (Koshi kriteriysi). $\int_a^{\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo'lishi uchun, $\forall \varepsilon > 0$ son olinganda ham, shunday $A_0 (A_0 > a)$ soni topilib, $A > A_0$, $A' > A$ bo'lgan $\forall A, A'$ lar uchun

$$\left|F(A') - F(A)\right| = \left|\int_a^{A'} f(x)dx - \int_a^{A} f(x)dx\right| = \left|\int_A^{A'} f(x)dx\right| < \varepsilon \qquad (9.4.2)$$

tengsizlikning bajarilishi zarur va yetarli

Bu kriteriydan foydalanib, quyidagi tasdiqni osonlikcha isbotlash mumkin:

Agar $\int_a^{\infty}|f(x)|\,dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^{\infty} f(x)dx$ integral ham yaqinlashuvchi bo'ladi.

(9.4.2) ni qisqacha quyidagicha ifodalash mumkin:

$$\forall\varepsilon>0\ \exists A_0(\varepsilon)\in(a,+\infty):\ \forall A',\ A\in(A_0(\varepsilon),+\infty)\rightarrow\left|\int_A^{A'} f(x)dx\right|<\varepsilon. \qquad (9.4.2^1)$$

9.4.3-eslatma. Agar $(9.4.2^1)$ shart bajarilmasa, ya'ni

$$\exists\varepsilon_0>0:\ \forall A_0(a,+\infty)\ \exists A, A'\in(A_0,+\infty):\ \left|\int_A^{A'} f(x)dx\right|\geq\varepsilon_0 \qquad (9.4.4)$$

bo'lsa, u holda $\int_a^{+\infty} f(x)dx$ integral uzoqlashuvchi bo'ladi.

9.4.7-eslatma. $\int_a^{\infty} f(x)dx$ integralning yaqinlashuvchiligidan, umuman olganda $\int_a^{\infty}|f(x)|dx$ integralning ham yaqinlashuvchiligi kelib chiqmaydi. Bu holat, absolyut va shartli yaqinlashuvchi xosmas integrallar tushunchasini kiritish zaruriyatini keltirib chiqaradi.

$f(x)$ funksiya $[a,+\infty)$ oraliqda beriglan bo'lsin.

9.4.8-ta'rif. Agar $\int_a^{\infty}|f(x)|dx$ integralning yaqinlashuvchi bo'lsa, u holda, $\int_a^{\infty} f(x)dx$ integral absolyut yaqinlashuvchi deyiladi, $f(x)$ esa, $[a,+\infty)$ oraliqda *absalyut integralanuvchi funksiya* deyiladi.

9.4.9-ta'rif. Agar $\int_a^{\infty} f(x)dx$ integral yaqinlashuvchi bo'lib, $\int_a^{+\infty}|f(x)|dx$ integral uzoqlashuvchi bo'lsa, u holda $\int_a^{\infty} f(x)dx$ integral *shartli yaqinlashuvchi* deyiladi.

9.4.10-teorema. Agar $\int_a^{\infty}|f(x)|dx$ integralning yaqinlashuvchi bo'lsa, u holda, $\int_a^{\infty} f(x)dx$ integral albatta yaqinlashuvchi bo'ladi.

9.4.11-teorema (Dirixle alomati) $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz va uning shu oraliqdagi boshlang'ich $F(x)$ $(F'(x)=f(x))$ funksiyasi chegaralangan;

2) $g(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz $g'(x)$ hosilaga ega;

3) $g(x)$ funksiya $[a,+\infty)$ oraliqda kamayuvchi va $\lim\limits_{x\to+\infty} g(x)=0$, u holda $\int_a^{\infty} f(x)g(x)dx$ integral yaqinlashuvchi bo'ladi.

9.4.12 – teorema (Abel alomati). $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda aniqlangan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya uzluksiz va

$$\int_a^{+\infty} f(x)dx$$

integral yaqinlashuvchi;

2) $g(x)$ funksiya uzluksiz hosilaga ega;

3) $g(x)$ $[a,+\infty)$da chegaralangan va monoton funksiya bo'lsa, u holda, $\int_a^{+\infty} f(x)g(x)dx$ integral yaqinlashuvchi bo'ladi.

9.2-§. Chegaralanmagan funksiyaning xosmas integrali

9.5. Chegaralanmagan funksiyaning xosmas integrali tushunchasi. $f(x)$ funksiya $[a,b)$ oraliqda berilgan bo'lib, u $[a,b-\eta]$ $(0<\eta<b-a)$ oraliqda xos ma'noda (Riman ma'nosida) integrallanuvchi, ya'ni $F(\eta)=\int_a^{b-\eta} f(x)dx$ integral mavjud bo'lsin. $[b-\eta,b]$ oraliqda integrallanuvchi bo'lmasin, ya'ni $\forall\eta>0$ uchun $f(x)$ funksiya chegaralanmagan bo'lsin. Bu holda, b nuqta maxsus nuqta deyiladi.

9.5.1-ta'rif. Agar $\eta \to 0$ da $F(\eta)$ funksiyaning (chekli yoki cheksiz) limiti mavjud bo'lsa, bu limit $f(x)$ funksiyadan olingan *ikkinchi tur xosmas integrali* deyiladi va u

$$\int_a^b f(x)dx = \lim_{\eta \to 0} \int_a^{b-\eta} f(x)dx \qquad (9.5.2)$$

kabi belgilanadi. Agar bu limit chekli bo'lsa, u holda $\int_a^b f(x)dx$ integral yaqinlashuvchi, $f(x)$ funksiya esa, $[a,b)$ oraliqda xosmas ma'noda integrallanuvchi deyiladi, aks holda integrallanmovchi deyiladi.

9.5.3-eslatma. Agar $\eta \to 0$ da $F(\eta)$ funksiyaning limiti mavjud bo'lmaganda ham $\int_a^b f(x)dx$ integral uzoqlashuvchi bo'ladi deb kelishib olamiz. Shuningdek, a nuqta $f(x)$ funksiyaning maxsus nuqtasi bo'lganda ham $(a,b]$ oraliq bo'yicha olingan xosmas integral yuqoridagidek ta'riflanadi. a va b nuqtalar bir vaqtda berilgan funksiyaning maxsus nuqtalari bo'lganda, (a,b) oraliq bo'yicha olingan xosmas integral quyidagicha ta'riflanadi:

$$\int_a^b f(x)dx = \lim_{\substack{\varepsilon \to 0 \\ \eta \to 0}} \int_{a+\varepsilon}^{b-\eta} f(x)dx .$$

9.5.4-misol. Ushbu

$$J_1 = \int_a^b \frac{dx}{(b-x)^p}, \qquad J_2 = \int_a^b \frac{dx}{(x-a)^p}, \ (p>0)$$

integrallarni yaqinlashuvchilikka tekshiring.

Yechilishi. 1) Ravshanki, $f(x) = \frac{1}{(b-x)^p}$ $(p>0)$ funksiya uchun b nuqta maxsus nuqta, u $[a,b-\eta]$ $(\eta<0)$ oraliqda xos ma'noda integrallanuvchi:

$$\int_a^{b-\eta} \frac{dx}{(b-x)^p} = \begin{cases} -\dfrac{(b-x)^{1-p}}{1-p}\Big|_a^{b-\eta} = \dfrac{(b-a)^{1-p} - \eta^{1-p}}{1-p}, & p \neq 1, \\ -\ln(b-x)\Big|_a^{b-\eta} = \ln\dfrac{b-a}{\eta}, & p = 1. \end{cases}$$

Bundan, $p<1$ da $\exists \lim_{\eta \to 0} \int_a^{b-\eta} \frac{dx}{(b-x)^p} = \frac{(b-a)^{1-p}}{1-p}$, $p \geq 1$ bo'lganida esa, $\lim_{\eta \to 0} \int_a^{b-\eta} \frac{dx}{(b-x)^p}$ mavjud emas. Demak, qaralayotgan J_1 xosmas integral $p<1$ bo'lganda yaqinlashuvchi, $p \geq 1$ da esa, uzoqlashuvchi.

Xuddi shunday J_2 xosmas integralning $p<1$ bo'lganda yaqinlashsvchi, $p \geq 1$ bo'lganida esa, uzoqlashuvchi yekanligini ko'rsatish mumkin. ■

Yuqoridagi ta'rif va mulohazalardan chegaralanmagan funksiyaning xosmas integrali ham, 9.1-9.4 bandlarda batafsil o'rganilgan chegarasi cheksiz (birinchi tur) xosmas integrali kabi ekanligini ko'ramiz.

Shuni e'tiborga olib, chegaralanmagan funksiyaning xosmas integrallari haqidagi tushunchalar, tasdiqlar va teoremalarni keltirib o'tish bilan chegaralanamiz.

9.6. Ikkinchi tur xosmas integral yaqinlashishi Koshi kriteriysi. $f(x)$ funksiya $[a,b)$ oraliqda berilgan bo'lib, b nuqta, $f(x)$ funksiya uchun maxsus nuqta bo'lsin.

$$\int_a^b f(x)dx \qquad (9.6.2)$$

xosmas integral yaqinlashuvchi bo'lishi uchun, $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ topilib, $0 < \eta < \delta$ va $0 < \eta' < \delta$ uchun

$$\left| \int_{b-\eta}^{b-\eta'} f(x)dx \right| < \varepsilon$$

tengsizlik bajarilishi zarur va yetarli.

9.7. Yaqinlashuvchi xosmas integrallning sodda xossalari

1^0. Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_c^b f(x)dx$ $(a<c<b)$ integral ham yaqinlashuvchi bo'ladi va aksincha. Bunda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik o'rinli bo'ladi.

2^0. Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^b cf(x)dx$ integral ham $(c = const)$ yaqinlashuvchi,

$$\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$$

bo‘ladi.

3^0. Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo‘lib, $\forall x \in [a,b]$ uchun $f(x) \geq 0$ bo‘lsa, u holda $\int_a^b f(x)dx \geq 0$ bo‘ladi.

4^0. Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda $\int_a^b (f(x) \pm g(x))dx$ integral ham yaqinlashuvchi bo‘lib, ushbu

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

tengliklar o‘rinli bo‘ladi.

5^0. Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo‘lib, $\forall x \in [a,b]$ uchun $f(x) \leq g(x)$ tengsizlik bajarilsa, u holda

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

tengsizlik ham o‘rinli bo‘ladi.

9.8. Manfiy bo‘lmagan funksiyalarning xosmas integrallari. $f(x)$ funksiya $[a,b)$ oraliqda aniqlangan (b nuqta $f(x)$ funksiyaning maxsus nuqtasi) bo‘lib, $\forall x \in [a,b)$ uchun $f(x) \geq 0$ bo‘lsin.

9.8.1-teorema. Ushbu $\int_a^b f(x)dx$ xosmas integral yaqinlashuvchi bo‘lishi uchun $\forall t \in (a,b)$ da

$$F(t) = \int_a^t f(x)dx \leq L,\ (L = const) \tag{9.8.2}$$

tengsizlikning bajarilishi zarur va yetarlidir.

9.8.3-natija. Agar $F(t) = \int_a^t f(x)dx$ $(\forall t \in (a,b))$ yuqoridan chegaralanmagan bo‘lsa, u holda $\int_a^b f(x)dx$ xosmas integral uzoqlashuvchi bo‘ladi.

9.8.4-teorema. Agar $\forall x \in (a,b)$ lar uchun $0 \leq f(x) \leq g(x)$ tengsizlik o‘rinli bo‘lib, $\int_a^b g(x)dx$ yaqinlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham yaqinlashuvchi bo‘ladi, $\int_a^b f(x)dx$ uzoqlashuvchi bo‘lsa, $\int_a^b g(x)dx$ ham uzoqlashuvchi bo‘ladi.

9.8.5-teorema. $f(x) \geq 0$, $(g(x) \geq 0)$, $x \in [a,b)$ uchun

$$\lim_{x \to b-0} \frac{f(x)}{g(x)} = k$$

bo‘lsin.

Agar $k < +\infty$ bo‘lib, $\int_a^b g(x)dx$ yaqinlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $k > 0$ bo‘lib, $\int_a^b g(x)dx$ uzoqlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham uzoqlashuvchi bo‘ladi.

9.8.6-natija. 9.8.5-teoremaning shartlarida $0 < k < +\infty$ bo‘lsa, u holda $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo‘ladi.

9.8.7-natija. Agar x o‘zgaruvchining b nuqtaga yetarlicha yaqin qiymatlarida

$$f(x) = \frac{\varphi(x)}{(b-x)^\alpha}\ ,\ (\alpha > 0)$$

ko‘rinishda tasvirlangan bo‘lsin. U holda:

1) $\varphi(x)\le C<+\infty$ va $\alpha<1$ bo'lsa, $\int_a^b f(x)dx$ integral yaqinlashuvchi bo'ladi.

2) $\varphi(x)\ge C>0$ va $\alpha\ge 1$ bo'lsa, $\int_a^b f(x)dx$ integral uzoqlashuvchi bo'ladi.

9.8.8-misol. Ushbu

$$\int_0^1 \frac{\sin^2 x}{\sqrt[3]{1-x}}dx$$

xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. $f(x)=\dfrac{\sin^2 x}{\sqrt[3]{1-x}}$, $\forall x\in[0,1)$ uchun $\varphi(x)=\sin^2 x\le 1$, $\alpha=\dfrac{1}{3}$. 9.8.7-natijaga asosan berilgan integral yaqinlashuvchi.

9.9. Xosmas integrallarni hisoblash. $f(x)$ funksiya $[a,b)$ oraliqda aniqlangan bo'lib, b nuqta uning maxsus nuqtasi bo'lsin.

9.9.1-teorema (Nyuton – Leybnis formulasi). Agar $f(x)$ funksiya $[a,b)$ oraliqda uzluksiz bo'lib, $F(x)$ esa, uning boshlang'ich funksiyasi ($F'(x)=f(x)$) bo'lsa, u holda

$$\int_a^b f(x)dx=F(x)\Big|_a^{b-0}=F(b-0)-F(a) \qquad (9.9.2)$$

tenglik o'rinli bo'ladi, bunda $F(b-0)=\lim\limits_{t\to b-0}F(t)$.

(9.9.2) formulaga *Nyuton – Leybnis formulasi* deyiladi.

9.9.3-teorema (O'zgaruvchilarni almashtirish formulasi). $f(x)$ funksiya $[a,b)$ oraliqda uzluksiz, $\varphi(t)$ funksiya esa, $[\alpha,\beta)$ da uzluksiz differensiallanuvchi bo'lib, $a=\varphi(\alpha)\le\varphi(t)<\lim\limits_{t\to b-0}\varphi(t)=b$ bo'lib, $\int_a^b f(x)dx$, $\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$ integrallarning biri yaqinlashuvchi bo'lsa, ikkinchisi ham yaqinlashuvchi va

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

tenglik o'rinli bo'ladi.

9.9.4-teorema (Bo'laklab integrallash formulasi). Agar $u=u(x)$ va $v=v(x)$ funksiyalar $[a,b)$ oraliqda differensiallanuvchi, $\lim\limits_{t\to b-0}(uv)$ mavjud hamda $\int_a^b udv$, $\int_a^b vdu$ integrallaning birortasi yaqinlashuvchi bo'lsa,

$$\int_a^b udv=(uv)\Big|_a^b-\int_a^b vdu \qquad (9.9.5)$$

tenglik o'rinli bo'ladi, bunda $(uv)\Big|_a^b=\lim\limits_{x\to b-0}u(x)v(x)-u(a)v(a)$.

(9.9.5) formulaga xosmas untegrallarda *bo'laklab integrallash* formulasi deyiladi.

9.9.6-natija. Agar $\int_a^b uv'dx$ yoki $\int_a^b vu'dx$ integral yaqinlashuvchi bo'lib, $\lim\limits_{x\to b-0}u(x)v(x)$ mavjud va chekli bo'lsa, u holda (9.9.5) formula o'rinli bo'ladi.

9.10. Xosmas integralning absolyut va shartli yaqinlashuvchanligi. $f(x)$ funksiya $[a,b)$ oraliqda aniqlangan bo'lib, b nuqta uning maxsus nuqtasi bo'lsin (bunda $\forall x\in(a,b)$ uchun $f(x)\ge 0$ bo'lishi shart emas).

9.10.1-ta'rif. Agar $\tilde{J}=\int_a^b|f(x)|dx$ xosmas integral yaqinlashuvchi bo'lsa, u holda $J=\int_a^b f(x)dx$ xosmas integral absolyut yaqinlashuvchi deyiladi. Bu holda, $f(x)$ funksiya $[a,b)$ oraliqda absolyut integrallanuvchi deyiladi.

9.10.2-ta'rif. Agar J integral yaqinlashuvchi bo'lib, $\tilde{J}$ integral uzoqlashuvchi bo'lsa, u holda J integral shartli yaqinlashuvchi deyiladi.

9.11. Chegaralanmagan funksiyaning xosmas integralining bosh qiymati. $f(x)$ funksiya $[a,b]$ kesmaning biror ichki c nuqtasida maxsuslikka ega bo'lganda ham bu funksiyadan olingan ikkinchi tur xosmas integrallar o'rganiladi. $f(x)$ funksiya ixtiyoriy $\varepsilon>0$ uchun $[a,c-\varepsilon]$ va $[c+\varepsilon,b]$ kesmalarda integrallanuvchi bo'lsa, u holda $f(x)$ funksiyadan olingan xosmas integral ikki xosmas integrallarning yig'indisi sifatida aniqlanadi:

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

Bunday shartda aniqlangan ikkinchi tur xosmas untegrallar uchun Koshi ma'nosida yoki bosh qiymat ma'nosidagi integral tushunchasini kiritish mumkin:

$$v\cdot p\int_a^b f(x)dx=\lim_{\varepsilon\to 0+0}\left(\int_a^{c-\varepsilon}f(x)dx+\int_{c+\varepsilon}^b f(x)dx\right). \qquad (9.11.1)$$

9.11.2-misol. Ushbu

$$\int_a^b\frac{dx}{x-c},\ (a<c<b)$$

integralning bosh qiymatini hisoblang.

Yechilishi. Avvalo, $\int_a^b\frac{dx}{x-c}$ integralni xosmas integral ma'nosida qaraylik:

$$\int_a^b\frac{dx}{x-c}=\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\left[-\int_a^{c-\varepsilon_1}\frac{dx}{c-x}+\int_{c+\varepsilon_2}^b\frac{dx}{x-c}\right]==\ln\frac{b-c}{c-a}+\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\ln\frac{\varepsilon_1}{\varepsilon_2}. \qquad (9.11.3)$$

Keyingi ifodaning limiti ε_1 va ε_2 larning nolga intilish tartibiga bog'liq.

Berilgan integral xosmas integral ma'nosida mavjud emas. Odatda bu integralni maxsus (singulyar) integral deb ataladi. Agar berilgan integralni (9.11.1) ta'rif bo'yicha qarasak, ya'ni $\varepsilon_1=\varepsilon_2$ shartda qarasak, u holda u bosh qiymat ma'nosida mavjud bo'ladi. (9.11.3) da $\varepsilon_1=\varepsilon_2$ shartni e'tiborga olib limitga o'tsak, natijada

$$v\cdot p\int_a^b\frac{dx}{x-c}=\lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon}\frac{dx}{x-c}+\int_{c+\varepsilon}^b\frac{dx}{x-c}\right)=\ln\frac{b-c}{c-a} \qquad (9.11.4)$$

ega bo'lamiz. Demak, $\int_a^b\frac{dx}{x-c}$ integral xosmas integral ma'nosida mavjud bo'lmasdan, bosh qiymat ma'nosida mavjud va

$$v\cdot p\int_a^b\frac{dx}{x-c}=\ln\frac{b-c}{c-a}.\blacksquare$$

9.12. Chegaralanmagan funksiya xosmas integralining ba'zi bir tatbiqlari. *9.12.1. Yuzani xosmas integral yordamida hisoblash.* $f(x)$ funksiya $[a,b)$ oraliqda aniqlangan, uzluksiz va $\forall x\in(a,b)$ lar uchun $f(x)\ge 0$ bo'lsin. U holda, $D=\{(x,y): a\le x\le b, 0\le y\le f(x)\}$ sohaning yuzi ushbu

$$D=\int_a^b f(x)dx \qquad (9.12.2)$$

xosmas integral orqali ifoda qilinadi.

9.12.3-misol. $y=x^{-\frac{2}{3}}$, $y=0$, $x=-1$, $x=1$ chiziqlar bilan chegaralangan shaklning yuzi hisoblansin.

Yechilishi. Talab qilingan shaklning yuzini (9.12.2) formula orqali hisoblaymiz:

$$D=\int_{-1}^1 x^{-\frac{2}{3}}dx=\lim_{\substack{\varepsilon\to 0+0\\ \eta\to 0-0}}\left(\int_{-1}^{\varepsilon}x^{-\frac{2}{3}}dx+\int_{\eta}^1 x^{-\frac{2}{3}}dx\right)=$$

$$=\lim_{\substack{\varepsilon\to 0+0\\ \eta\to 0-0}}\left(3\left(\varepsilon^{\frac{1}{3}}+1\right)+3\left(1-\eta^{\frac{1}{3}}\right)\right)=6\ \text{(kv. birlik)}.\blacksquare$$

9.13. Aylanma jism hajmini xosmas integral yordamida hisoblash. Ushbu $D=\{(x,y): a\le x\le b, 0\le y\le f(x)\}$ egri chiziqli trapetsiyani Ox va Oy o'qlari atrofida aylantirish natijasida hosil bo'lgan aylanma jismning hajmi, mos ravishda

$$V_x=\pi\int_a^b f^2(x)dx,\ V_y=2\pi\int_a^b|xy|dx \qquad (9.13.1)$$

xosmas integral orqali hisoblanadi.

9.13.2-misol. $y=\frac{1}{\sqrt{x-1}}$, $x\in(1,2]$ chiziq bilan chegaralangan shaklni Ox o'qi atrofida aylantirish natijasida hosil bo'lgan aylanma jism hajmi topilsin.

Yechilishi. (9.13.1) formulada asosan, quyidagiga ega bo'lamiz:

$$V_x=\pi\int_1^2\frac{dx}{\sqrt{x-1}}=\lim_{\varepsilon\to 0}\pi\int_{1+\varepsilon}^2\frac{dx}{\sqrt{x-1}}=\pi\lim_{\varepsilon\to 0}2\sqrt{x-1}\,|_{1+\varepsilon}^2=2\pi\ \text{(kb. birlik)}.\blacksquare$$

9.14. Aylanma sirt yuzini xosmas integral yordamida hisoblash. $f(x)$ funksiya $[a,b]$ kesma aniqlangan, uzluksiz $f'(x)$ hosilaga ega bo'lib, u $f(x)\geq 0$ bo'lsin. $f(x)$ funksiya grafigini Ox va Oy o'qlari atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzi mos ravishda quyidagi formulalar orqali topiladi:

$$S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx,\quad S=2\pi\int_c^d x(y)\sqrt{1+[x'(y)]^2}\,dy.$$

9.3-§. Eyler integrallari

9.15. Eyler integrallari. Beta funksiya va uning xossalari. Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad (a>0,b>0) \tag{9.15.1}$$

xosmas integralga Beta funksiya yoki 1-tur Eyler integrali deyiladi va $B(a,b)$ kabi belgilanadi, ya'ni

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx. \tag{9.15.2}$$

Integral ostidagi funksiya uchun:

1) $a<1,\ \ b\geq 1$ bo'lganda $x=0$ nuqta;
2) $a\geq 1,\ \ b<1$ bo'lganda $x=1$ nuqta;
3) $a<1,\ \ b<1$ bo'lganda $x=0$ va $x=1$ nuqtalar maxsus nuqtalar bo'ladi.

Demak, (9.15.1) integral parametrga bog'liq bo'lgan xosmas integraldir. Beta funksiya quyidagi xossalarga ega:

1⁰. (9.15.2) integral $M=\{(a,b)\in R^2:\ \ a\in(0;+\infty),\ \ b\in(0;+\infty)\}$ to'plamda yaqinlashuvchi.

2⁰. (9.15.2) integral $M_0=\{(a,b)\in R^2:\ \ a\in[a_0;+\infty),\ \ b\in[b_0;+\infty)\}$, $a_0>0,\ \ b_0>0$ to'plamda tekis yaqinlashuvchi, lekin M to'plamda esa, notekis yaqinlashuvchi.

3⁰. $B(a,b)$ funksiya M to'plamda uzluksiz funksiyadir.

4⁰. $\forall(a,b)\in M$ uchun $B(a,b)=B(b,a)$ (simmetrik) bo'ladi.

5⁰. $B(a,b)$ funksiya, quyidagicha ham ifodalanadi:

$$B(a,b)=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

6⁰. $\forall(a,b)\in M_1=\{(a,b)\in R^2:\ \ a\in(0;+\infty),\ \ b\in(1;+\infty)\}$ uchun

$$B(a,b)=\frac{b-1}{a+b-1}B(a,b-1).$$

7⁰. $b=n$ bo'lganda,

$$B(a,n)=\frac{n-1}{a+n-1}\frac{n-2}{a+n-2}\cdots\frac{1}{n+1}B(a,1),\ \ B(a,1)=\frac{1}{a}.$$

8⁰. $B(m,n)=\dfrac{(n-1)!(m-1)!}{(n+m-1)!}\quad (m,n\in N)$.

9⁰. $B(a,1-a)=\dfrac{\pi}{\sin a\pi}(0<a<1)$, xususiy holda $a=\dfrac{1}{2}$ bo'lganda,

$$B=\left(\frac{1}{2},\frac{1}{2}\right)=\pi.$$

9.16. Gamma funksiya va uning xossalari. Ushbu

$$\int_0^{+\infty}x^{a-1}e^{-x}dx \qquad (a>0) \tag{9.16.1}$$

xosmas integral Gamma funksiya yoki 2-tur Eyler integrali deyiladi va $\Gamma(a)$ kabi belgilanadi, ya'ni

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx \qquad (a>0). \tag{9.16.2}$$

Integral ostidagi funksiya uchun:

1) $a<1$ bo'lganda $x=0$ nuqta maxsus nuqta;
2) $a>0$ bo'lganda (9.16.1) integral yaqinlashuvchi;
3) $a\leq 0$ bo'lganda (9.16.1) integral uzoqlashuvchi;

Gamma funksiya quyidagi xossalarga ega:

1⁰.

$$\Gamma(a)=\lim_{n\to\infty} n^a \frac{1\cdot 2\cdot 3\cdots(n-1)}{a(a+1)\cdots(a+n+1)}. \qquad (9.16.3)$$

(9.16.3) formulaga *Eyler – Gauss formulasi* deyiladi.

2⁰. (9.16.2) integral $\forall a\in[a_0,b_0]\ (0<a_0<b_0<+\infty)$ oralikda tekis yakinlashuvchi, $a\in(0;+\infty)$ da esa, notekis yaqinlashuvchi.

3⁰. $\Gamma(a)$ funksiya $(0,\infty)$ oraliqda uzluksiz va barcha tartibdagi uzluksiz hosilalarga ega, ya'ni

$$\Gamma^{(n)}(a)=\int_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx \qquad (n\in N).$$

4⁰. $\Gamma(a+1)=a\Gamma(a) \quad (a>0)$.

5⁰. $\Gamma(n+1)=n!$.

6⁰. $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

7⁰. $\Gamma(a)\Gamma(1-a)=B(a,1-a)=\dfrac{\pi}{\sin a\pi}, \quad 0<a<1$.

8⁰. $\Gamma\left(n+\dfrac{1}{2}\right)=\dfrac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad n\in N$.

9⁰. Lejandr formulasi: $\Gamma(a)\Gamma\left(a+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a)$.

9.16.4-misol. Ushbu

$$\int_0^{\frac{\pi}{2}} \sin^{a-1}\varphi\cos^{b-1}\varphi d\varphi \quad (a,b>0)$$

integralni hisoblang.

Yechilishi. Bu integralda, $x=\sin\varphi$ almashtirishni bajarib, uni

$$\int_0^1 x^{a-1}\left(1-x^2\right)^{\frac{b}{2}-1}dx$$

ko'rinishga keltiramiz. Bu integralda $x^2=y$ almashtirishni olib, natijada

$$\frac{1}{2}\int_0^1 y^{\frac{a}{2}-1}(1-y)^{\frac{b}{2}-1}dy=\frac{1}{2}B\left(\frac{a}{2},\frac{b}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

ega bo'lamiz.

Demak, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{a-1}\varphi\cos^{b-1}\varphi d\varphi=\frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}$.

9.16.4-misol. Ushbu

$$I=\int_0^1\sqrt{\frac{1-x}{x}}\frac{dx}{(x+2)^2}$$

integralni Eyler integrallari yordamida hisoblang.

Yechilishi. Berilgan integralda $\dfrac{3x}{2+x}=t$ almashtirish bajaramiz va $x=\dfrac{2t}{3-t}$, $\dfrac{dx}{(x+2)^2}=\dfrac{dt}{6}$ ekanligini e'tiborga olib,

$$I=\frac{1}{2\sqrt{6}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt$$

ifodani hosil kilamiz. Beta funksiyaning ta'rifi va Gamma funksiyaning 4⁰-, 6⁰-, 7⁰ – xossalaridan foydalanib, $\Gamma(2)=1$ ekanligini e'tiborga olgan holda,

$$I=\frac{1}{2\sqrt{6}}B\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\pi}{4\sqrt{6}}$$

bo'lishini topamiz.

9.16.5-misol. Ushbu

$$I=\int_0^{\frac{\pi}{2}}\frac{tg^{\alpha}x}{(\sin x+\cos x)^2}dx, \quad 0<\alpha<1$$

integralni hisoblang.

Gamma funksiya quyidagi xossalarga ega:

1⁰.

$$\Gamma(a)=\lim_{n\to\infty} n^a \frac{1\cdot 2\cdot 3\cdots(n-1)}{a(a+1)\cdots(a+n+1)}. \qquad (9.16.3)$$

(9.16.3) formulaga *Eyler – Gauss formulasi* deyiladi.

2⁰. (9.16.2) integral $\forall a\in[a_0,b_0]\ (0<a_0<b_0<+\infty)$ oralikda tekis yakinlashuvchi, $a\in(0;+\infty)$ da esa, notekis yaqinlashuvchi.

3⁰. $\Gamma(a)$ funksiya $(0,\infty)$ oraliqda uzluksiz va barcha tartibdagi uzluksiz hosilalarga ega, ya'ni

$$\Gamma^{(n)}(a)=\int_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx \qquad (n\in N).$$

4⁰. $\Gamma(a+1)=a\Gamma(a) \quad (a>0)$.

5⁰. $\Gamma(n+1)=n!$.

6⁰. $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

7⁰. $\Gamma(a)\Gamma(1-a)=B(a,1-a)=\dfrac{\pi}{\sin a\pi}, \quad 0<a<1$.

8⁰. $\Gamma\left(n+\dfrac{1}{2}\right)=\dfrac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad n\in N$.

9⁰. Lejandr formulasi: $\Gamma(a)\Gamma\left(a+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a)$.

9.16.4-misol. Ushbu

$$\int_0^{\frac{\pi}{2}} \sin^{a-1}\varphi\cos^{b-1}\varphi d\varphi \quad (a,b>0)$$

integralni hisoblang.

Yechilishi. Bu integralda, $x=\sin\varphi$ almashtirishni bajarib, uni

$$\int_0^1 x^{a-1}\left(1-x^2\right)^{\frac{b}{2}-1}dx$$

ko'rinishga keltiramiz. Bu integralda $x^2=y$ almashtirishni olib, natijada

$$\frac{1}{2}\int_0^1 y^{\frac{b}{2}-1}(1-y)^{\frac{b}{2}-1}dy=\frac{1}{2}B\left(\frac{a}{2},\frac{b}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

ega bo'lamiz.

Demak, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{a-1}\varphi\cos^{b-1}\varphi d\varphi=\frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}$.

9.16.4-misol. Ushbu

$$I=\int_0^1\sqrt{\frac{1-x}{x}}\frac{dx}{(x+2)^2}$$

integralni Eyler integrallari yordamida hisoblang.

Yechilishi. Berilgan integralda $\dfrac{3x}{2+x}=t$ almashtirish bajaramiz va $x=\dfrac{2t}{3-t},\ \dfrac{dx}{(x+2)^2}=\dfrac{dt}{6}$ ekanligini e'tiborga olib,

$$I=\frac{1}{2\sqrt{6}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt$$

ifodani hosil kilamiz. Beta funksiyaning ta'rifi va Gamma funksiyaning 4⁰-, 6⁰-, 7⁰ – xossalaridan foydalanib, $\Gamma(2)=1$ ekanligini e'tiborga olgan holda,

$$I=\frac{1}{2\sqrt{6}}B\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\pi}{4\sqrt{6}}$$

bo'lishini topamiz.

9.16.5-misol. Ushbu

$$I=\int_0^{\frac{\pi}{2}}\frac{tg^{\alpha}x}{(\sin x+\cos x)^2}dx, \quad 0<\alpha<1$$

integralni hisoblang.

Yechilishi. Berilgan integralda $tgx = t \quad (t > 0)$ almashtirish bajarib,

$$I = \int_0^{\frac{\pi}{2}} \frac{tg^\alpha x\, dx}{\cos^2 x\,(1+tgx)^2} = \int_0^\infty \frac{t^\alpha dt}{(1+t)^2}$$

munosabatga ega bo‘lamiz.

$0 < \alpha < 1$ shartda berilgan integral yaqinlashuvchi. So‘ngra keyingi hosil bo‘lgan integralga Beta funksiyaning 5^0-xossasini, Gamma funksiyaning 4^0- va 7^0- xossalarini qo‘llash natijasida

$$I = B(1+\alpha,\quad 1-\alpha) = \alpha\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi\alpha}{\sin\alpha\pi}$$

bo‘lishini topamiz.

9-bob bo‘yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

9.1. Chegarasi cheksiz xosmas integrallarning ta’rifi ([3], 1-q.,294-295 betlar; [12], 2-q., 197-199 betlar; [5], 2-t., 552- 554 betlar; [9], 1-t., 10- bo‘lim, [30], 8- bo‘lim).

9.2. Chegarasa cheksiz xosmas integrallarning yaqinlashuvchiligi ([3], 1-q.,294-295 betlar; [12], 2-q., 197-199 betlar; [5], 2-t., 552bet; [9], 1-t., 10-bo‘lim, [30], 8- bo‘lim).

9.3. Yaqinlashuvchi chegarasi cheksiz xosmas integralning xossalari. ([3], 1-q.,296-300 betlar; [12], 2-q., 201-205 bet; [5], 2-t., 559 bet; [9], 1-t., 10- bo‘lim, [30], 8- bo‘lim).

9.4. Chegarasi cheksiz xosmas integral bilan sonli qatorlar orasidagi bog‘lanish ([5], 2-t., 538-560 betlar; [9], 1-t., 10- bo‘lim).

9.5. Mafiy bo‘lmagan funksiya xosmas integralning yaqinlashish sharti. Taqqoslash teoremalari ([3], 1-q.,301-305 betlar; [12], 2-q., 205-209 betlar; [5], 2-t., 559-561betlar; [9], 1-t., 10- bo‘lim, [30], 8- bo‘lim).

9.6. Ixtiyoriy funksiya xosmas integralning yaqinlashuvchiligi. (Koshi kriteriyasi) ([3], 1-q.,300-301 betlar; [12], 2-q., 209-210 betlar; [5], 2-t., 561-563 betlar).

9.7. Absalyut va shartli yaqinlashuvchi xosmas integrallar ([3], 1-q.,308-309 betlar; [12], 2-q., 211 bet; [5], 2-t., 563 bet., [9], 1-t., 10- bo‘lim, [30], 8-bo‘lim).

9.8. Chegarasi cheksiz xosmas integral yaqinlashishining yetarli shartlari (Dirixle va Abel alomatlari) ([3], 1-q., 310-313 betlar; [12], 2-q., 211-213 betlar; [10], 2-q., 376-378 betlar; [5], 2-t., 563-565 betlar).

9.9. Chegaralanmagan funksiyaning xosmas integrallari ([3], 1-q.,323-325 betlar; [12], 2-q., 222 bet; [9], 1-t., 10- bo‘lim, [30], 8- bo‘lim).

9.10. Chegaralanmagan xosmas integralning yaqinlashuvchiligi. ([3], 1-q., 323-325 betlar; [12], 2-q., 423-424; [5], 2-t., 598-599 betlar; [9], 1-t., 10-bo‘lim, [30], 8- bo‘lim).

9.11. Manfiy bo‘lmagan chegaralanmagan funksiya xosmas integralning yaqinlashuvchiligi to‘g‘risidagi taqqoslash teoremalari ([3], 1-q., 329-330 betlar; [12], 2-q., 230-233 betlar; [9], 1-t., 10- bo‘lim; [30], 8- bo‘lim).

9.12. Ixtiyoriy chegaralanmagan funksiya xosmas integrallarning yaqinlashuvchiligi. Koshi kriteriyasi ([3], 1-q., 327-329 betlar; [12], 2-q., 233-234 betlar).

9.13. Absalyut va shartli yaqinlashuvchi xosmas integrallar tushunchasi. ([3], 1-q., 308-309, 331-332 betlar; [12], 2-q., 234 bet; [9], 1-t., 10- bo'lim; [30], 8- bo'lim).

9.14. Xosmas integrallarning hisoblash usullari ([3], 1-q.,332-334 betlar; [12], 2-q., 213-216, 235-236 betlar; [9], 1-t., 10- bo'lim; [30], 8- bo'lim).

9.15. Chegaralanmagan funksiya xosmas integrallari uchun Direxle va Abel alomatlari ([3], 1-q.,334-336 betlar; [28], 400-403 betlar).

9.16. Xosmas integrallar yordamida yuzani hisoblash ([20], 194-195 betlar).

9.17. Xosmas integrallar yordamida aylanma sirt yuzini hisoblash. ([20], 195-196 betlar).

9.18. Xosmas integrallarni yaqinlashishgan tekshirishda funksiyaning bosh qismini ajratish usullaridan foydalanish. ([3], 1-q.,313-315, 334-335 betlar; [10], 2-q., 382-384 betlar; [20], 211-215 betlar).

9.19. Aylanma jismning hajmini xosmas integral yordamida hisoblash. ([20], 195 bet).

9.1-amaliy mashg'ulot.
Chegaralari cheksiz xosmas integrallar

1- misol. Ushbu

$$I=\int_0^{+\infty}\frac{dx}{4+x^2}$$

xosmas integralning yaqinlashuvchiligini ko'rsating va qiymatini toping.

Yechilishi. 9.1.2-ta'rifga ko'ra,

$$I=\int_0^{+\infty}\frac{dx}{4+x^2}=\lim_{t\to\infty}\int_0^t\frac{dx}{4+x^2}=\lim_{t\to\infty}F(t)$$

bo'ladi. Bunda

$$F(t)=\int_0^t\frac{dx}{4+x^2}=\frac{1}{2}arctg\frac{x}{2}\bigg|_0^t=\frac{1}{2}arctg\frac{t}{2}.$$

Endi $t\to\infty$ da $F(t)$ funksiyaning limitini topamiz:

$$I=\lim_{t\to\infty}F(t)=\lim_{t\to\infty}\frac{1}{2}arctg\frac{t}{2}=\frac{\pi}{4}$$

limit mavjud va chekli.

Demak, ta'rifga ko'ra, berilgan xosmas integral yaqinlashuvchi va uning qiymati $I=\frac{\pi}{4}$.

2-misol. Ushbu

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+14}$$

xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. (9.1.6) formulaga ko'ra,

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+14}=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\int_\tau^t\frac{dx}{(x+3)^2+5}=\lim_{\substack{\tau\to-\infty\\ \tau\to+\infty}}\left(\int_\tau^{-3}\frac{dx}{(x+3)^2+5}+\right.$$

$$\left.+\int_{-3}^t\frac{dx}{(x+3)^2+5}\right)=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\left(\frac{1}{\sqrt5}arctg\frac{x+3}{\sqrt5}\bigg|_\tau^{-3}+\frac{1}{\sqrt5}arctg\frac{x+3}{\sqrt5}\bigg|_{-3}^t\right)=$$

$$=\lim_{t\to+\infty}\frac{1}{\sqrt5}arctg\frac{t+3}{\sqrt5}-\lim_{\tau\to-\infty}\frac{1}{\sqrt5}arctg\frac{\tau+3}{\sqrt5}=\frac{1}{\sqrt5}\frac{\pi}{2}+\frac{\pi}{2\sqrt5}=\frac{\pi}{\sqrt5},$$

ya'ni chekli. Demak, berilgan xosmas integral yaqinlashuvchi bo'ladi.

3-misol. Ushbu

$$\int_0^{+\infty}2^x x dx$$

integralni yaqinlashuvchilikka tekshiring.

Yechilishi. $F(t)$ fuksiyasini tuzamiz:

$$F(t)=\int_0^t 2^x x dx=\frac{2^t}{\ln 2}(t-\frac{1}{\ln 2})+\frac{1}{(\ln 2)^2}.$$

Bu funksiya $t\to\infty$ da yuqoridan chegaralanmagan. Shuning uchun, **9.8.3-** natijaga ko'ra, berilgan integral uzoqlashuvchi bo'ladi.

4-misol. Ushbu

$$\int_1^{+\infty}\frac{arctgx}{4+x^2}dx$$

integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Ravshanki, $\forall x\geq 1$ uchun $arctgx\leq\frac{\pi}{2}$ va $\frac{arctgx}{4+x^2}\leq\frac{\pi}{2}\frac{1}{4+x^2}$ tengsizliklar o'rinli. Unda $\int_1^{+\infty}\frac{dx}{4+x^2}$ integralning yaqinlashuvchiligini e'tiborga olsak, 9.3.4- teoremaga asosan, berilgan integral yaqinlashuvchi bo'ladi.

5-misol. Ushbu

$$\int_1^{+\infty}\frac{cos^2 3x}{\sqrt[5]{x^7+1}}dx$$

integralning yaqinlashuvchiligini ko'rsating.

Yechilishi. Ravshanki, barch $x\in[1;+\infty)$ lar uchun $f(x)=\frac{cos^2 3x}{\sqrt[5]{x^7+1}}=\frac{\varphi(x)}{x^{\frac{7}{5}}}$, bunda

$$\varphi(x)=\frac{\cos^2 3x}{\sqrt[5]{\frac{1}{x^7}+1}}\leq C<+\infty,\quad \lambda=\frac{7}{5}>1$$

bo'lgani uchun, teoremaning *a*) bandiga asosan, berilgan xosmas integral yaqinlashuvchi bo'ladi.

6-misol Ushbu $\int_a^{\infty}\frac{dx}{x^\lambda}$ (a va λ - ixtiyoriy haqiqiy o'zgarmas sonlar) xosmas integralni yaqinlashishga tekshiramiz.

Yechilishi. $f(x)=\frac{1}{x^\lambda}$ funksiya $\forall A>0$ uchun $[a,A]$ oraliqda integrallanuvchi bo'lib,

$$F(A)=\int_a^A f(x)dx=\int_0^A\frac{dx}{x^\lambda}=\begin{cases}\left.\frac{x^{1-\lambda}}{1-\lambda}\right|_a^A=\frac{A^{1-\lambda}-a^{1-\lambda}}{1-\lambda},\lambda\neq 1,\\ \ln x\Big|_a^A=\ln A-\ln a,\lambda=1\end{cases}$$

bo'ladi, agar $\lambda>1$ bo'lganda, $A\to+\infty$ da $F(A)$ funksiyaning limiti mavjud va u $\frac{a^{1-\lambda}}{\lambda-1}$ ga, $\lambda\leq 1$ bo'lganda esa, ko'rsatilgan limit mavjud emas. Shunday qilib, $\lambda>1$ da $\int_a^{+\infty}\frac{dx}{x^\lambda}$ – xosmas integral yaqinlashuvchi; $\lambda\leq 1$ bo'lganda esa, $\int_a^{\infty}\frac{dx}{x^\lambda}$ – xosmas integral uzoqlashuvchi bo'ladi.■

7-misol. Ushbu $\int_1^{\infty}\frac{\sqrt{x}}{1+x}dx$ integralni qaraylik $1\leq x<+\infty$ uchun $\frac{1}{2\sqrt{x}}=\frac{\sqrt{x}}{x+x}\leq\frac{\sqrt{x}}{1+x}$ bo'lib, $\int_0^{+\infty}\frac{dx}{2\sqrt{x}}$ uzoqlashuvchi $\left(\alpha=\frac{1}{2}<1\right)$ bo'lgani uchun yuqoridagi keltirilgan alomatga ko'ra berilgan integral uzoqlashuvchi bo'ladi.

8-misol. Ushbu

$$J(\alpha)=\int_1^{\infty}\frac{\sin^2 x}{x^\alpha}dx \tag{9.4.6}$$

integralni yaqinlashishga tekshiring.

Yechilishi. a) $\alpha>1$ bo'lsin, u holda $\frac{\sin^2 x}{x^\alpha}\leq\frac{1}{x^\alpha}$. $\int_0^{\infty}\frac{1}{x^\alpha}dx$ integral yakinlashuvchi bo'lgani uchun J integral yaqinlashuvchi bo'ladi.

b) $\alpha\leq 1$ bo'lsin. Bu holda berilgan integralning uzoqlashuvchi ekanligini ko'rsatamiz: buning uchun (9.4.4) shartning bajarilishni ko'rsatamiz. $\delta>1$ uchun shunday $n(n\in N)$ ni tanlaymizki $\pi n>\delta$ bo'lsin va $A_\delta=n\pi$, $A_\delta^1=2\pi n$ deb olsak, u holada

$$\left|\int_{A_\delta}^{A_\delta^1}\frac{\sin^2 x}{x^\alpha}dx\right|=\int_{A_\delta}^{A_\delta^1}\frac{\sin^2 x}{x^\alpha}dx=\int_{\pi}^{2\pi n}\frac{\sin^2 x}{x^\alpha}dx\geq\int_{\pi n}^{2\pi n}\frac{\sin^2 x}{x}dx\geq$$

$$\geq\frac{1}{2\pi n}\int_{\pi n}^{2\pi n}\frac{1-\cos 2x}{2}dx=\frac{1}{4\pi n}\cdot\pi n=\frac{1}{4}=\varepsilon_0$$

Shunday qilib (9.4.4) shart bajariladi.

Demak, $\alpha\leq 1$ bo'lganda (9.4.6) integral uzoqlashuvchi bo'ladi.■

10-misol. Ushbu $\int_0^{+\infty}\sin x^2 dx$ Freneli integralini yaqinlashishga tekshiring.

Yechilishi. $\int\limits_1^{\infty}\sin x^2dx=\int\limits_1^{\infty}\frac{1}{x}\cdot x\sin x^2dx$, bunda ushbu $f(x)=x\cdot\sin x^2$, $g(x)=\frac{1}{x}$ belgilash olinsa, u holda $f(x)$ va $g(x)$ funksiyalar-ning **9.4.11** – teoremaning hamma shartlarini qanoatlantirishni ko‘rish qiyin emas. Shuning uchun Freneli integrali yaqinlashuvchi bo‘ladi.■

11-misol. Ushbu $J=\int\limits_0^{+\infty}\left(e^x+x\right)\cos e^{2x}dx$ xosmas integralni yaqinlashuvchilikka va absolyut yaqinlashuvchilikka tekshiring.

Yechilishi. $e^{2x}=t$ almashtirishni olamiz:

$$x=\frac{1}{2}\ln t,\ dx=\frac{dt}{2t},\ J=\frac{1}{2}\int\limits_1^{+\infty}\frac{\cos t}{\sqrt{t}}dt+\frac{1}{4}\int\limits_1^{+\infty}\frac{\ln t}{t}\cos t\,dt \qquad (*)$$

(*) dagi ikkala integral Dirixle alomatiga ko‘ra yaqinlashuvchi, chunki $\cos t$ funksiya chegaralangan boshlang‘ich funksiyaga ega, ya’ni $\left|\int\limits_1^{\xi}\cos t dt\right|\le 2$; $\frac{1}{\sqrt{t}}$ va $\frac{\ln t}{t}$ funksiyalar esa, $t\to+\infty$ da monoton bo‘lib, nolga intiladi. $f(t)=\frac{1}{2}\left(\frac{1}{\sqrt{t}}+\frac{\ln t}{2t}\right)$ deb olib, $J_1=\int\limits_1^{+\infty}f(t)|\cos t|dt$ integralning uzoqlashuvchanligini ko‘rsatamiz.

Haqiqatan ham, $t>1$ bo‘lganda

$$f(t)\ge\frac{1}{2t},\ |\cos t|\ge\cos^2 t=\frac{1+\cos 2t}{2},\ f(t)\cdot|\cos t|\ge\frac{1+\cos 2t}{4t}$$

$\int\limits_1^{+\infty}\frac{1+\cos 2t}{4t}dt$ integral uzoqlashuvchi, chunki $\int\limits_0^{+\infty}\frac{\cos 2t}{4t}dt$ integral Direxle alomatiga ko‘ra yaqinlashuvchi, $\int\limits_1^{+\infty}\frac{dt}{4t}$ integral esa, uzoqlashuvchi.

Demak, 9.4.14 – teoremaga asosan J_1 integral uzoqlashuvchi. Shunday qilib, berilgan J integral shartli yaqinlashuvchi bo‘ladi.■

Mustaqil yechish uchun misollar

Quyidagi xosmas integrallarning yaqinlashuvchi ekanligini ko‘rsating va qiymatini toping.

1. $\int\limits_1^{+\infty}\frac{dx}{\sqrt[3]{x^5}}$. **2.** $\int\limits_0^{+\infty}e^{-5x}dx$. **3.** $\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}$.

4. $\int\limits_0^{+\infty}xe^{-x^2}dx$. **5** $\int\limits_1^{+\infty}\frac{arctgx}{1+x^2}dx$.. **6.** $\int\limits_1^{+\infty}\frac{dx}{(x+2)\ln^2(x+2)}$.

7. $\int\limits_0^{+\infty}\frac{2xdx}{(x^2+1)^3}$. **8.** $\int\limits_{-\infty}^{0}\frac{dx}{(x+1)^3}$. **9.** $\int\limits_{-\infty}^{+\infty}\frac{dx}{(x^2+x+1)^2}$.

Quyidagi xosmas integrallarning uzoqlashuvchi ekanligini isbotlang.

10. $\int\limits_1^{+\infty}\frac{dx}{\sqrt[3]{x}}$. **11.** $\int\limits_0^{+\infty}\frac{xdx}{x^2+5}$. **12.** $\int\limits_0^{+\infty}\cos xdx$.

13. $\int\limits_1^{+\infty}\frac{dx}{\sqrt{16+x^2}}$. **14.** $\int\limits_e^{+\infty}\frac{dx}{(x+1)\ln(x+1)}$. **15.** $\int\limits_1^{+\infty}4^xdx$.

Quyidagi xosmas integrallarni hisoblang.

16. $\int\limits_{\sqrt{2}}^{+\infty}\frac{xdx}{(x^2+1)^3}$. **17.** $\int\limits_1^{+\infty}\frac{dx}{(1+x)\sqrt{x}}$. **18.** $\int\limits_0^{+\infty}\frac{dx}{e^x+\sqrt{e^x}}$. **19.** $\int\limits_{\sqrt{2}}^{+\infty}\frac{dx}{(x-1)\sqrt{x^2-2}}$.

Quyidagi funksiyalarning grafiklari va abssissalar o‘qi bilan chegaralangan shakllarning yuzini hisoblang.

20. $f(x)=\frac{1}{4+x^2},\quad -\infty<x<+\infty$. **21.** $f(x)=x^2e^{-x^3},\quad 0\le x<+\infty$.

22. $f(x)=\frac{\sqrt{x}}{(1+x)^2},\quad 1\le x<+\infty$. **23.** $f(x)=\frac{1}{\sqrt{1+e^x}},\quad 0\le x<+\infty$.

Quyidagi integrallarning yaqinlashuvchiligini isbotlang.

24. $\int\limits_0^{+\infty}\frac{x^3}{x^5+1}dx$. **25.** $\int\limits_0^{+\infty}\frac{x}{\sqrt[3]{1+x^7}}dx$. **26.** $\int\limits_2^{+\infty}(\cos\frac{x}{2}-1)dx$.

Quyidagi integrallarning uzoqlashuvchiligini isbotlang.

27. $\int\limits_0^{+\infty}\frac{x^3+1}{x^4}dx$. **28.** $\int\limits_0^{+\infty}\frac{xdx}{\sqrt[3]{x^5+2}}$. **29.** $\int\limits_0^{+\infty}\frac{\sin^2 x}{x}dx$.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{3}{2}$. **2.** $\frac{1}{5}$. **3.** π . **4.** $\frac{1}{2}$. **5.** $\frac{3\pi^2}{32}$. **6.** $\frac{1}{\ln 3}$. **7.** $\frac{1}{2}$. **8.** $-\frac{1}{2}$. **9.** $\frac{4\pi}{3\sqrt{3}}$. **16.** $\frac{1}{36}$. **17.** $\frac{\pi}{2}$. **18.** $2(1-\ln 2)$. **19.** $\frac{3\pi}{4}$. **20.** $\frac{\pi}{2}$. **21.** $\frac{1}{3}$. **22.** $\frac{1}{2}+\frac{\pi}{4}$. **23.** $2\ln(1+\sqrt{2})$.

9.2-amaliy mashg'ulot.
Chegaralanmagan funksiyaning xosmas integrallari

1-misol. Ushbu

$$I=\int_0^1 \frac{dx}{x^\alpha}$$

integralni yaqinlashishga tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=0$ nuqta maxsus nuqtadan iborat. Ushbu

$$F(\mu)=\int_\mu^1 \frac{dx}{x^\alpha}\ (0<\mu<1)$$

integralni qaraymiz:

$$F(\mu)=\int_\mu^1 \frac{dx}{x^\alpha}=\begin{cases}\frac{1}{1-\alpha}(1-\mu^{1-\alpha}), & \alpha\neq 1 \text{ бўлганда},\\ -\ln|\mu|, & \alpha=1\text{, бўлганда.}\end{cases}$$

Agar $\alpha<1$ bo'lsa,

$$\lim_{\mu\to 0} F(\mu)=\frac{1}{1-\alpha}$$

bo'ladi.

Agar $\alpha\geq 1$ bo'lsa,

$$\lim_{\mu\to 0} F(\mu)=\infty$$

bo'ladi.

Shunday qilib, yuqoridagi ta'rifga asosan, $\alpha<1$ bo'lganda I integral yaqinlashuvchi, $\alpha\geq 1$ bo'lganda esa I integral uzoqlashuvchi bo'ladi.

2-misol. Ushbu

$$\int_0^2 \frac{dx}{\ln x}$$

integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=1$ nuqta maxsus nuqtadan iborat. U holda, yuqoridagi ta'rifga asosan, quyidagiga ega bo'lamiz:

$$\int_0^2\frac{dx}{\ln x}=\int_0^1\frac{dx}{\ln x}+\int_1^2\frac{dx}{\ln x}=\lim_{\substack{\varepsilon\to 0\\ \mu\to 0}}\left(\int_0^{1-\varepsilon}\frac{dx}{\ln x}+\int_{1+\mu}^2\frac{dx}{\ln x}\right)=\lim_{\substack{\varepsilon\to 0\\ \mu\to 0}}[I_1(\varepsilon)+I_2(\mu)] \quad . \qquad (1)$$

$\varepsilon>0, \mu>0$ va $0<x<2$ bo'lsin. U holda

$$\frac{1}{\ln x}=\frac{1}{\ln(1+(x-1))}=[\ln(1+(x-1))]^{-1}=$$

$$=\left[(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\ldots\right]^{-1}=\frac{1}{x-1}+\frac{1}{2}+o(x-1),$$

$$I_1(\varepsilon)=\int_0^{1-\varepsilon}\frac{dx}{\ln x}=\ln|x-1|\Big|_0^{1-\varepsilon}+\frac{1}{2}(1-\varepsilon)+o(\varepsilon^2)=\ln\varepsilon-\frac{\varepsilon}{2}+o(\varepsilon^2)+C_1, \qquad (2)$$

$$I_2(\mu)=\int_{1-\mu}^2\frac{dx}{\ln x}=\ln|x-1|\Big|_{1+\mu}^2+\frac{1-\mu}{2}+o(\mu^2)=-\ln\mu-\frac{\mu}{2}+o(\mu^2)+C_2. \qquad (3)$$

(2) va (3) ni (1) ga keltirib qo'yamiz:

$$\int_0^2\frac{dx}{\ln x}=\lim_{\substack{\varepsilon\to 0\\ \mu\to 0}}\left[\ln\frac{\varepsilon}{\mu}-\frac{1}{2}(\varepsilon+\mu)+o(\mu^2)+o(\varepsilon^2)+C_1+C_2\right].$$

Oxirgi tenglikda o'rta qavs ichidagi ifodaning limiti mavjud emas.

Demak, berilgan 2-tur xosmas integral uzoqlashuvchi bo'ladi.

3–misol. Ushbu

$$\int_0^1\frac{arctg^2 x}{\sqrt{1-x^4}}dx$$

integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=1$ nuqta maxsus nuqta bo'ladi. Ravshanki, $\forall x\in[0;1)$da

$$0\leq\frac{arctg^2x}{\sqrt{1-x^4}}=\frac{arctg^2x}{\sqrt{1+x^2}\sqrt{1-x^2}}\leq\frac{\frac{\pi^2}{16}}{\sqrt{1-x^2}}$$

tengsizlik o'rinli. Ushbu

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_0^{1-0} = \frac{\pi}{2}$$

xosmas integral yaqinlashuvchi. Demak, 9.8.4-teoremaga binoan,

$$\int_0^1 \frac{arctg^2 x}{\sqrt{1-x^4}}dx$$

integral ham yaqinlashuvchi bo‘ladi.

4–misol. Ushbu

$$\int_0^1 \frac{\sqrt{1+x^2}}{1-x}dx$$

integralning uzoqlashuvchiligini ko‘rsating.

Yechilishi. Ma’lumki, $\forall x \in [0;1)$ da $\sqrt{1+x^2} \geq 1$ bo‘ladi. Demak, $[0;1)$ dagi barcha x lar uchun

$$\frac{\sqrt{1+x^2}}{1-x} > \frac{1}{1-x}$$

bo‘ladi. Ma’lumki, $\int_0^1 \frac{dx}{1-x}$ xosmas integral uzoqlashuvchi. 9.8.4-taqqoslash teoremaga binoan,

$$\int_0^1 \frac{\sqrt{1+x^2}}{1-x}dx$$

integral ham uzoqlashuvchi bo‘ladi.

5–misol. Ushbu

$$J = \int_0^1 \frac{\left(\sqrt[8]{x}+2\right)^2}{\sqrt{x}}dx$$

xosmas integralni hisoblang.

Yechilishi. Xosmas integralning chiziqlilik xossasidan foydalanib, berilgan integralni quyidagicha yozib olamiz:

$$\int_0^1 \frac{\left(\sqrt[8]{x}+2\right)^2}{\sqrt{x}}dx = \int_0^1 \frac{dx}{\sqrt[4]{x}} + 4\int_0^1 \frac{dx}{\sqrt[8]{x^3}} + 4\int_0^1 \frac{dx}{\sqrt{x}}.$$

$\frac{1}{\sqrt[4]{x}}$, $\frac{1}{\sqrt[8]{x^3}}$, $\frac{1}{\sqrt{x}}$ funksiyalar $(0,1]$ da aniqlangan bo‘lib, ularning boshlang‘ich funksiyalari quyidagicha $\frac{4}{3}\sqrt[4]{x^3}$, $\frac{8}{5}\sqrt[8]{x^5}$, $2\sqrt{x}$ bo‘ladi. Nyuton – Leybnis formulasiga asosan:

$$\int_0^1 \frac{dx}{\sqrt[4]{x}} = \frac{4}{3}\sqrt[4]{x^3}\,\Big|_0^1 = \frac{4}{3},\ \int_0^1 \frac{dx}{\sqrt[8]{x^3}} = \frac{8}{5}\sqrt[8]{x^5}\,\Big|_0^1 = \frac{8}{5},\ \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\Big|_0^1 = 2$$

Demak, $J = 15\frac{11}{15}$.■

6-misol. Ushbu

$$J = \int_0^2 \frac{dx}{(4-x)\sqrt{2-x}}$$

xosmas integralni hisoblang.

Yechilishi. Berilgan xosmas integralni hisoblash uchun $2-x=t^2$, $t>0$ almashtirishni bajaramiz. Bundan $x=2-t^2$, $dx=-2tdt$, $\alpha=\sqrt{2}$, $\beta=0$ bo‘ladi.

Demak, quyidagiga ega bo‘lamiz:

$$\int_0^2 \frac{dx}{(4-x)\sqrt{2-x}} = -2\int_{\sqrt{2}}^0 \frac{tdt}{t(t^2+2)} = 2\int_0^{\sqrt{2}} \frac{dt}{t^2+2} = \frac{\sqrt{2}}{4}\pi.■$$

7-misol. Ushbu

$$J = \int_0^{+\infty} xe^{-x}dx$$

xosmas integralni hisoblang.

Yechilishi. (9.9.5) formulaga asosan,

$$J = \int_0^{+\infty} x(-e^{-x})dx = -xe^{-x}\,\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}dx.$$

$x=0$ da $xe^{-x}=0$, $\lim\limits_{x\to+\infty} xe^{-x}=0$, u holda $J = \int_0^{+\infty} e^{-x}dx = -e^{-x}\,\Big|_0^{+\infty} = 1$.■

Mustaqil yechish uchun misollar

Quyidagi xosmas integrallarning yaqinlashuvchiligini ko‘rsating va qiymatini toping.

1. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$. **2.** $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. **3.** $\int_1^e \frac{dx}{x\sqrt{\ln x}}$. **4.** $\int_0^4 \frac{dx}{x+\sqrt{x}}$.

5. $\int_1^2 \frac{xdx}{\sqrt{x-1}}$. **6.** $\int_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}}dx$. **7.** $\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}dx$. **8.** $\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$.

9. $\int_0^3 \frac{x^2dx}{\sqrt{9-x^2}}$. **10.** $\int_{-1}^1 \frac{\arccos x}{\sqrt{1-x^2}}dx$. **11.** $\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx$. **12.** $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx$.

Quyidagi xosmas integrallarning uzoqlashuvchi ekanligini isbotlang.

13. $\int_{-1}^e \frac{dx}{x}$. **14.** $\int_0^e \frac{dx}{e^x-1}$. **15.** $\int_{-3}^3 \frac{xdx}{x^2-1}$. **16.** $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin^3 x}}dx$.

17. $\int_0^{\frac{1}{2}} \frac{dx}{x\ln x}$. **18.** $\int_{-1}^1 \frac{e^{\frac{1}{x}}}{x^3}dx$. **19.** $\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx$. **20.** $\int_0^{\frac{\pi}{2}} tgx\,dx$.

Quyidagi xosmas integrallarni hisoblang.

21. $\int_0^1 \frac{2-\sqrt[3]{x}-x^3}{\sqrt[5]{x^3}}dx$. **22.** $\int_{\sqrt{2}}^2 \frac{dx}{(x-1)\sqrt{x^2-2}}$. **23** $\int_0^1 \frac{dx}{\sqrt{x\sqrt{x}}}$.

24. $\int_0^{\frac{\pi}{2}} \sqrt{tgx}dx$. **25.** $\int_0^{\frac{\pi}{2}} \ln\cos x dx$. **26.** $\int_0^{\pi} x\ln\sin x dx$. **27.** $\int_0^{\frac{\pi}{4}} \sqrt{ctgx}dx$.

28. $\int_{-1}^1 \frac{dx}{(16-x^2)\sqrt{1-x^2}}$. **29.** $\int_a^b \frac{xdx}{\sqrt{(x-a)(b-x)}}$. **30.** $\int_0^1 \frac{x^3\arcsin x}{\sqrt{1-x^2}}dx$.

Quyidagi limitlarni hisoblang

31. $\lim\limits_{x\to+\infty} \frac{\int_0^x \sqrt{1+t^{10}}dt}{x^6}$. **32.** $\lim\limits_{x\to+\infty} \frac{\int_0^x t^{-1}e^{-t}dt}{\ln\frac{1}{x}}$.

Quyidagi funksiyaning grafigi va abssissalar o‘qi bilan chegaralangan shaklning yuzini toping.

33. $y=\frac{-x}{\sqrt{x+1}}$, $x\in(-1;0]$. **34.** $y=\frac{1}{\sqrt{2-5x}}$, $x\in[0;0,4)$.

35. $y=\frac{x}{\sqrt{(x-2)(5-x)}}$, $x\in(2;5)$. **36.** $y=\frac{1}{\sqrt{1-x}}$, $x\in[0;1)$.

Quyida berilgan chiziq va uning asimptotalari bilan chegaralangan shaklning yuzini toping.

37. $y^2=\frac{8-4x}{x}$. **38.** $y^2(x+1)=x^2$, $x<0$.

Misollarning javoblari

1. $\frac{3}{2}$. **2.** $\frac{\pi}{2}$. **3.** 2. **4.** $2\ln 3$. **5.** $\frac{8}{3}$. **6.** $\frac{10}{7}$. **7.** $\frac{\pi^2}{8}$. **8.** $\frac{1}{\ln 2}$. **9.** $\frac{9\pi}{4}$. **10.** $\frac{\pi^2}{2}$.
11. $-2e^{-1}$ **12.** 2. **21.** $\frac{625}{187}$. **22.** $\frac{\pi}{2}$. **23.** $\frac{4}{3}$. **24.** $\frac{\pi}{\sqrt{2}}$. **25.** $-\frac{\pi\ln 2}{2}$. **26.** $-\frac{\pi^2\ln 2}{2}$.
27. $\frac{1}{2\sqrt{2}}\left(\pi+\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$. **28.** $\frac{\pi}{4\sqrt{15}}$ **29.** $\frac{\pi(a+b)}{2}$. **30.** $\frac{7}{9}$. **31.** $\frac{1}{6}$. **32.** 1. **33.** 0 .
33. $\frac{4}{3}$. **34.** $\frac{2\sqrt{2}}{5}$. **35.** $\frac{7\pi}{2}$. **36.** 2. **37.** 4π. **38.** $\frac{8}{3}$.

9-bob bo‘yicha amaliy mashg‘ulotlarni mustahkamlash uchun nazorat topshiriqlari

9.1-masala. Quyidagi xosmas integrallarning yaqinlashuvchi ekanini ko‘rsating va qiymatini toping.

9.1.1. $\int_1^{\infty} \frac{dx}{\sqrt[3]{x^5}}$. **9.1.2.** $\int_1^{\infty} \frac{x^2dx}{\sqrt[4]{x^3-1}}$.

9.1.3. $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$. **9.1.4.** $\int_1^{\infty} \frac{dx}{(x+2)\ln^2(x+2)}$.

9.1.5. $\int_{-\infty}^{-2} \frac{dx}{(x+1)^3}$. **9.1.6.** $\int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)^2}$.

9.1.7. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$. **9.1.8.** $\int_1^{\infty} \frac{x^4dx}{(x^5+1)^4}$.

9.1.9. $\int_{a^2}^{\infty} \frac{dx}{x\sqrt{1+x^2}}$. **9.1.10.** $\int_0^{\infty} e^{-\sqrt{x}}dx$.

9.1.11. $\int\limits_0^1 \frac{dx}{\sqrt{1-x^2}}$. **9.1.12.** $\int\limits_0^{\infty} \frac{x^2+1}{x^4+1}dx$.

9.1.13. $\int\limits_{-\infty}^{0} xe^x dx$. **9.1.14.** $\int\limits_{-2}^{2} \frac{x\,dx}{\sqrt{4-x^2}}$.

9.1.15. $\int\limits_0^1 x^3 \frac{\arcsin x}{\sqrt{1-x^2}}dx$. **9.1.16.** $\int\limits_0^{\frac{\pi}{2}} \sqrt{tgx}dx$.

9.1.17. $\int\limits_0^{\infty} e^{-x}\sin x dx$. **9.1.18.** $\int\limits_1^{e} \frac{dx}{x\sqrt{\ln x}}$.

9.1.19. $\int\limits_{-1}^{1} \frac{3x^2+2}{\sqrt[3]{x^2}}dx$. **9.1.20.** $\int\limits_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}dx$.

9.1.21. $\int\limits_1^{\infty} \frac{x+1}{\sqrt[3]{x^5}}dx$. **9.1.22.** $\int\limits_1^{\infty} \frac{x^2+2}{\sqrt[4]{x^7}}dx$.

9.1.23. $\int\limits_1^{\infty} \frac{dx}{(1+x^2)x^2}$. **9.1.24.** $\int\limits_1^{\infty} \frac{xdx}{(x^2+2)\ln^3(x^2+2)}$.

9.1.25. $\int\limits_{-\infty}^{-4} \frac{dx}{(x+2)^4}$. **9.1.26.** $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2+2x+3}$.

Yechilishi ([2], 7-bo'lim, [9], 1-t., 10-bo'lim, [30], 11.8-bo'lim). Xosmas integralning ta'rifiga ko'ra, hisoblaymiz:

$$\int\limits_{-\infty}^{\infty} \frac{dx}{x^2+2x+3} = \int\limits_{-\infty}^{0} \frac{dx}{x^2+2x+3} + \int\limits_{0}^{\infty} \frac{dx}{x^2+2x+3} =$$

$$= \lim_{A\to-\infty}\int\limits_{A}^{0} \frac{dx}{x^2+2x+3} + \lim_{B\to+\infty}\int\limits_{0}^{B} \frac{dx}{x^2+2x+3} =$$

$$= \lim_{A\to-\infty}\left(\frac{1}{\sqrt{2}}arctg\frac{x+1}{\sqrt{2}}\right)\Big|_A^0 + \lim_{B\to+\infty}\left(\frac{1}{\sqrt{2}}arctg\frac{x+1}{\sqrt{2}}\right)\Big|_0^B =$$

$$= \frac{1}{\sqrt{2}}arctg\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\lim_{A\to-\infty}arctg\frac{A+1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\lim_{B\to+\infty}arctg\frac{B+1}{\sqrt{2}} -$$

$$-\frac{1}{\sqrt{2}}arctg\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) + \frac{1}{\sqrt{2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}.$$

Shunday qilib, berilgan xosmas integral yaqinlashuvchi va uning qiymati $\frac{\pi}{\sqrt{2}}$ teng ekan.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> int(1/(x^2+2*x+3), x=-infinity..infinity);

$$\frac{\pi\sqrt{2}}{2}.$$

9.1.27. $\int\limits_{-1}^{1} \frac{x+1}{\sqrt[5]{x^3}}dx$.

Yechilishi ([2], 7-bo'lim, [9], 1-t., 10-bo'lim, [30], 11.8-bo'lim). 2-tur xosmas integralning ta'rifiga ko'ra, integralni hisoblaymiz:

$$\int\limits_{-1}^{1} \frac{x+1}{\sqrt[5]{x^3}}dx = \int\limits_{-1}^{0} \frac{x+1}{\sqrt[5]{x^3}}dx + \int\limits_{0}^{1} \frac{x+1}{\sqrt[5]{x^3}}dx =$$

$$\lim_{\varepsilon\to0-}\int\limits_{-1}^{\varepsilon} \frac{x+1}{\sqrt[5]{x^3}}dx + \lim_{\mu\to0+}\int\limits_{\mu}^{1} \frac{x+1}{\sqrt[5]{x^3}}dx = \lim_{\varepsilon\to0-}\int\limits_{-1}^{\varepsilon}\left(x^{\frac{2}{5}} + x^{-\frac{3}{5}}\right)dx +$$

$$+\lim_{\mu\to0+}\int\limits_{\mu}^{1}\left(x^{\frac{2}{5}} + x^{-\frac{3}{5}}\right)dx = \lim_{\varepsilon\to0-}\int\limits_{\varepsilon}^{1}\left(x^{\frac{2}{5}} + x^{-\frac{3}{5}}\right)\Big|_{-1}^{\varepsilon} + \lim_{\mu\to0+}\left(\frac{5}{7}\cdot x^{\frac{7}{5}} + \frac{5}{2}x^{\frac{2}{5}}\right)\Big|_{\mu}^{1} =$$

$$= \lim_{\varepsilon\to0-}[\left(\frac{5}{7}\varepsilon^{\frac{7}{5}} + \varepsilon^{\frac{2}{5}}\right) - \left(\frac{5}{7}\cdot(-1)^{\frac{7}{5}} + \frac{5}{2}(-1)^{\frac{2}{5}}\right)] +$$

$$+\lim_{\mu\to0+}[\left(\frac{5}{7}\cdot + \frac{5}{2}\right) - \left(\frac{5}{7}\mu^{\frac{7}{5}} + \frac{5}{2}\mu^{\frac{2}{5}}\right)] = \frac{10}{7}.$$

Demak, berilgan xosmas integral yaqinlashuvchi va uning qiymati $\frac{10}{7}$ ga teng.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> int((x+1)/(x)^(3/5), x=-1..1);

$$-\frac{25\,(-1)^{(2/5)}}{14} + \frac{45}{14}.$$

9.2.-masala. Quyidagi xosmas integrallarni yaqinlashuvchiligiga tekshiring.

9.2.1. $\int\limits_0^{\infty} \frac{x^3+7}{x^5-x^2+2}dx.$

9.2.2. $\int\limits_0^{\infty} \frac{xdx}{\sqrt[3]{x^5+2}}.$

9.2.3. $\int\limits_0^{\infty} \frac{x^2dx}{x^4-x^2+1}.$

9.2.4. $\int\limits_0^{\infty} \frac{xdx}{\sqrt[3]{1+x^7}}.$

9.2.5. $\int\limits_{-3/2}^{+\infty} \frac{x+3}{x^2\sqrt{2x+3}}dx..$

9.2.6. $\int\limits_1^{\infty} \frac{\ln xdx}{x\sqrt{x^2-1}}$

9.2.7. $\int\limits_0^{+\infty} \frac{x^m}{1+x^n}(n \geq 0).$

9.2.8. $\int\limits_0^{1} \frac{\ln x}{1-x^2}dx.$

9.2.9. $\int\limits_0^{\infty} \frac{x^{13}}{\left(x^5+x^3+1\right)^3}dx.$

9.2.10. $\int\limits_2^{\infty} \frac{xdx}{\sqrt{x^4+1}}.$

9.2.11. $\int\limits_0^{\infty} \sqrt{x}e^{-x}dx.$

9.2.12. $\int\limits_2^{\infty} \frac{dx}{x^2+\sqrt[3]{x^4+1}}.$

9.2.13. $\int\limits_1^{2} \frac{dx}{\ln x}.$

9.2.14. $\int\limits_0^{\frac{\pi}{4}} \frac{\sin x}{x\sqrt{x}}dx.$

9.2.15. $\int\limits_1^{2} \frac{dx}{x\ln^p x}.$

9.2.16. $\int\limits_0^{\pi} \frac{1-\cos x}{x^k}dx.$

9.2.17. $\int\limits_0^{\frac{\pi}{2}} tg\,x\,dx.$

9.2.18. $\int\limits_0^{1} \frac{e^{\frac{1}{x}}}{x^3}dx.$

9.2.19. $\int\limits_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin^3 x}}dx.$

9.2.20. $\int\limits_0^{e} \frac{dx}{e^x-1}.$

9.2.21. $\int\limits_0^{\infty} \frac{x^2+7}{x^4-4x^2+3}dx.$

9.2.22. $\int\limits_1^{\infty} \frac{x^2dx}{\sqrt[3]{x^7+5}}.$

9.2.23. $\int\limits_0^{\infty} \frac{x^2dx}{x^4-x^2+1}.$

9.2.24. $\int\limits_1^{\infty} \frac{xdx}{\sqrt[5]{1+x^9}}.$

9.2.25. $\int\limits_{-3}^{+\infty} \frac{x+2}{\sqrt{x+3}}dx.$

9.2.26. $\int\limits_1^{+\infty} \frac{1+\arcsin 1/x}{1+x\sqrt{x}}dx.$

Yechilishi. Berilgan integralni quyidagicha shakl almashtiramiz:

$$J=\int\limits_1^{+\infty} \frac{1+\arcsin 1/x}{1+x\sqrt{x}}dx=\int\limits_1^{+\infty} \frac{dx}{1+x\sqrt{x}}+\int\limits_1^{\infty} \frac{\arcsin 1/x}{1+x\sqrt{x}}dx=J_1+J_2.$$

J_1 – integralda, integral ostidagi $f(x)=\frac{1}{1+x\sqrt{x}}$ funksiya $\forall x \in [1;+\infty)$ uchun $\frac{1}{1+x\sqrt{x}} \leq \frac{1}{x\sqrt{x}}=g(x)$. tengsizlikni qanoatlantiradi. $\int\limits_1^{\infty} \frac{dx}{x\sqrt{x}}=\int\limits_1^{\infty} \frac{dx}{x^{3/2}}$ -integral yaqinlashuvchi, chunki $\lambda=\frac{3}{2}>1$.U holda taqqoslash teoremasiga asosan J_1 – integral yaqinlashuvchi J_2 – integralda $\left|\arcsin\frac{1}{x}\right| \leq \frac{1}{x}$ tengsizlikni e'tiborga olsak, u holda $\forall x \in [1;+\infty)$ uchun $\frac{\left|\arcsin\frac{1}{x}\right|}{1+x\sqrt{x}} \leq \frac{\frac{1}{x}}{x\cdot\sqrt{x}}=\frac{1}{x^{5/2}}$ tengsizlik o'rinli, bunda $\lambda=\frac{5}{2}>1$.

Demak, **9.3.4**-taqqoslash teoremasiga asosan J_2 – integral ham yaqinlashuvchi. Shunday qilib J – integral yaqinlashuvchi.

9.3-masala. Quyidagi xosmas integrallarni hisoblang:

9.3.1. $\int\limits_0^{2} \frac{dx}{x\sqrt{\ln x}}.$

9.3.2. $\int\limits_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}.$

9.3.3. $\int\limits_{-\infty}^{-2} \frac{dx}{x\sqrt{x^2-1}}.$

9.3.4.. $\int\limits_{a^2}^{\infty} \frac{dx}{x\sqrt{1+x^2}}.$

9.3.5. $\int\limits_0^{\infty} e^{-ax}\cos bx\,dx.$

9.3.6. $\int\limits_0^{\infty} \frac{dx}{1+x^3}.$

9.3.7. $\int\limits_0^{1} \frac{x^m}{\sqrt{1-x^2}}dx.$

9.3.8. $\int\limits_0^{1} \frac{x^m}{\sqrt{1-x^2}}.$

9.3.9. $\int\limits_0^{\frac{\pi}{2}} \ln\sin x\,dx.$

9.3.10. $\int\limits_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}.$

9.3.11. $\int\limits_0^2 \frac{dx}{\sqrt{x^2-4x+5}}$.

9.3.12. $\int\limits_0^\infty \frac{\sqrt{x}}{(1+x)^2}dx$.

9.3.13. $\int\limits_0^\infty \frac{x\cdot \ln x}{\left(1+x^2\right)^2}dx$.

9.3.14. $\int\limits_{-\infty}^\infty \frac{dx}{\left(x^3+x+1\right)^3}$

9.3.15. $\int\limits_{-1}^1 \frac{3x^2+2}{\sqrt[3]{x^2}}dx$.

9.3.16. $\int\limits_0^1 \frac{(x+1)}{\sqrt[3]{(x-1)^2}}dx$.

9.3.17. $\int\limits_{-2}^2 \frac{x^2dx}{\sqrt{4-x^2}}$.

9.3.18. $\int\limits_0^1 \frac{x^3\arcsin x}{\sqrt{1-x^2}}dx$.

9.3.19. $\int\limits_{-0,5}^{-0,25} \frac{dx}{x\sqrt{2x+1}}$.

9.3.20. $\int\limits_1^2 \frac{dx}{x\sqrt{\ln^3 x}}$.

9.3.21. $\int\limits_1^\infty \frac{xdx}{(2+x)\sqrt{x}}$.

9.3.22. $\int\limits_{-\infty}^{-2} \frac{xdx}{e^{-x^2}}$.

9.3.23. $\int\limits_1^\infty \frac{dx}{x\sqrt{3+x^2}}$.

9.3.24. $\int\limits_0^\infty e^{-ax}\sin bx\,dx$.

9.3.25. $\int\limits_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}(a<b)(a,b\in R)$.

9.3.26. $\int\limits_{1/2}^2 \frac{x-1}{\sqrt{x-0,5}}$.

Yechilishi ([2], 7-bo'lim, [9], 1-t., 10-bo'lim, [30], 11.8-bo'lim). Berilgan integralni quyidagicha yozib olamiz:

$$J=\int\limits_{1/2}^2 \frac{x-1}{\sqrt{x-0,5}}dx=\int\limits_{1/2}^2 \frac{x-0.5}{\sqrt{x-0,5}}dx-\frac{1}{2}\int\limits_{1/2}^2 \frac{dx}{\sqrt{x-0,5}}=$$

$$=\int\limits_{1/2}^2 \sqrt{x-0,5}dx-\frac{1}{2}\int\limits_{1/2}^2 \frac{dx}{\sqrt{x-0,5}}=J_1-0,5\cdot J_2.$$

J_1 integralda integral ostidagi funksiya $[1;2]$oraliqda uzluksiz bo'lgani uchun J_1 integralni Nyuton - Leybisi formulasi bo'yicha yechiladi.

$$J_1=\int\limits_{1/2}^2 \sqrt{x-0,5}dx=\frac{2}{3}\left(x-\frac{1}{2}\right)^{3/2}\Big|_{1/2}^2=\sqrt{\frac{3}{2}}.$$

J_2 – integral xosmas integral, chunki $x=1/2$ maxsus nuqta. J_2 – integralni chegaralangan funksiya xosmas integralining ta'rifi bo'yicha hisoblaymiz:

$$J_2=\lim_{\eta\to 0}\int\limits_{1/2+\eta}^2 \frac{dx}{\sqrt{x-\frac{1}{2}}}=\lim_{\eta\to 0}2\cdot\sqrt{x-\frac{1}{2}}\Big|_{1/2+\eta}^2=2\left(\sqrt{\frac{3}{2}}-\lim_{\eta\to 0}\sqrt{(1/2+\eta)-\frac{1}{2}}\right)=\sqrt{6}.$$

Shunday qilib $J=J_1-0,5J_2=\frac{\sqrt{6}}{2}-\frac{\sqrt{6}}{2}=0.$

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