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OLIY MATEMATIKADAN MISOL VA MASALALAR TO'PLAMI

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OLIY MATEMATIKADAN MISOL VA MASALALAR TO'PLAMI

KO'P ARGUMENTLI FUNKSIYALAR, QATORLAR, DIFFERENSIAL TENGLAMALAR, KARRALI INTEGRALLAR, EGRI CHIZIQLI INTEGRALLAR, SIRT INTEGRALLAR

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Yaxshiboyev M.U., Muxiddinov S. R., Xamroyev A. *Oliy matematikadan misol va masalalar to'plami:* ko'p argumentli funksiyalar, qatorlar, differensial tenglamalar, karrali integrallar, egri chiziqli integrallar, sirt integrallar. 2 – qism, 2-nashr. Samarqand. 2013, –132 bet.

O'quv–uslubiy qo'llanmada ko'p argumentli funksiyalar limitlarini hisoblash. ko'p o'zgaruvchili funksiyaning uzluksizligi, ko'p o'zgaruvchili funksiyaning xususiy hosilalari va differensiallari, ko'p o'zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differensiallari, ko'p o'zgaruvchili murakkab va oshkormas funksiyalarni differensiallash, ko'p o'zgaruvchili funksiyaning ekstremumlari. Sonli qatorlar, musbat xadli qatorlar, ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchiligi, funksional ketma-ketliklar va qatorlar, darajali qator, uning yaqinlashish radiusi va intervali, Teylor qatori, funksiyalarni darajali qatorlarga yoyish, funksiyalarni furye (trigonometrik) qatoriga yoyish, furye integrali, o'zgaruvchilari ajraladigan birinchi tartibli differensial tenglamalar, Koshi masalasi, differensial tenglamaning turlari va yechish usullari, to'liq differensial teglama, Klero va Lagranj teglamalari, yuqori tartibli differensial tenglamalar, Koshi masalasi, tartibi pasayadigan differensial tenglamalar, bir jinsli bo'lgan chiziqli differensial tenglamalar, bir jinsli bo'lmagan chiziqli differensial tenglamalar, differensial tenglamalar sistemasi, operasion hisob, asl va tasvir funksiya, asllar o'ramasi, operasion usullarini differensial tenglamalar va ularning sistemalarini yechishga tatbiq etish, ikki karrali integrallarni hisoblash, uch karrali integrallarni hisoblash, karrali integrallarda o'zgaruvchilarni almashtirish, qutb, silindrik va sferik koordinat sistemalariga o'tish usuli, birinchi va ikkinchi tur egri chiziqli integrallar, birincni va ikkinchi tur sirt integrallari mavzularni keltirilgan. Har bir mavzuda tegishli ta'riflar tushunchaiar va tasdiqlar hamda mavzuni o'zlashtirish uchun misollar keltirilgan.

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TATU Samarqand filiali, 2013

MUNDARIJA

1- amaliy mashg'ulot.

KO'P ARGUMENTLI FUNKSIYALAR LIMITLARINI HISOBLASH. KO'P O'ZGARUVCHILI FUNKSIYANING UZLUKSIZLIGI

1.1. Ko'p o'zgaruvchili funksiyaning limiti. $u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

1- ta'rif (Geyne ta'rifi). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan va A ga intiluvchi har qanday $\{M_n\}$ $(M_n \neq A, n=1,2...)$ ketma-ketlik olinganda ham, funksiyaning unga mos kelgan $\{f(M_n)\}$ qiymatlari ketma-ketligi hamma vaqt, yagona B (chekli yoki cheksiz) limitga intilsa, shu B ga $f(M)$ *funksiyaning A nuqtadagi (yoki $M\to A$ dagi) limiti* deyiladi va u

$$\lim_{M\to A} f(M)=B \text{ yoki } \lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \cdots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=B \text{ yoki } M\to A \text{ da } f(M)\to B$$

kabi belgilanadi.

2.-ta'rif (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun, $\exists\delta>0$ bo'lib, $0<\rho(M;A)<\delta$ tengsizliklarni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda

$$|f(M)-B|<\varepsilon$$

tengsizlik bajarilsa, shu B songa $f(x)$ *funksiyaning A nuqtadagi ($M\to A$ dagi) limiti* deyiladi.

$u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda aniklangan bo'lib, ∞ esa, $\{M\}$ to'plamning limit nuqtasi bo'lsin.

3- ta'rif (Geyne ta'rifi). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan har qanday $\{M_n\}$ ketma-ketlik uchun $M\to\infty$ da funksiyaning unga mos kelgan $\{f(M_n)\}$ qiymatlari ketma-ketligi hamma vaqt yagona B songa intilsa, shu B songa $f(M)$ funksiyaning *$M\to\infty$ dagi limiti* deyiladi va $\lim_{M\to\infty} f(M)=B$ kabi belgilanadi.

4-ta'rif. Agar $\forall\varepsilon>0$ son uchun, shunday $\exists E>0$ bo'lib, $\rho(M,O)>E$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda $|f(M)-B|<\varepsilon$ tengsizlik bajarilsa, B son $f(M)$ *funksiyaning $M\to\infty$ dagi limiti* deyiladi va $\lim_{M\to\infty} f(M)=B$ yoki $\lim_{\substack{x_1\to\infty\\ x_2\to\infty\\ \cdots\\ x_m\to\infty}} f(x_1,x_2,....,x_m)=B$ kabi belgilanadi.

5-ta'rif (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun $\exists\delta>0$ bo'lib, $0<\rho(M,A)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda $|f(M)|>\varepsilon$ $(f(M)>\varepsilon; f(M)<-\varepsilon)$ bo'lsa, $f(M)$ *funksiyaning A nuqtadagi ($M\to A$ dagi) limiti* $+\infty$ $(-\infty)$ deyiladi.

Biz yukorida $u=f(M)=f(x_1,x_2,...,x_m)$ funksiyaning $A=A(a_1,a_2,...,a_m)$ nuqtadagi limiti $\lim_{M\to A} f(M)=B$ yoki $\lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \cdots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=B$ bilan tanishdik.

Demak, funksiyaning limiti, uning argumentlari $x_1,x_2,...,x_m$ larning bir yo'la, mos ravishda, $a_1,a_2,...,a_m$ sonlarga intilgandagi limitidan iborat ekan. Biz bundan buyon, bu limitni, *karrali limit* deb ataymiz.

Ko'p o'zgaruvchili funksiyalargagina xos bo'lgan, boshqa ko'rinishdagi, limit tushunchasini kiritamiz. $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A=A(a_1,a_2,...,a_m)$ nuqta- $\{M\}$ to'plamning limit nuqtasi bo'lsin. Berilgan funksiyaning $x_1\to a_1$ (qolgan barcha argumentlarini tayinlab) dagi limiti $\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)$ ni karaylik, bu limit $x_2,x_3,...,x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)=\varphi_1(x_2,x_3,...,x_m).$$

Endi $\varphi_1(x_2,...,x_m)$ funksiyaning $x_2\to a_2$ (qolgan barcha argumentlarni belgilab) dagi limitini qaraymiz, bu $\lim_{x_2\to a_2}\varphi_1(x_2,x_3,...,x_m)$ limit $x_3,x_4,...,x_m$ o'zgaruvchilarga bog'lik bo'ladi: $\lim_{x_2\to a_2}\varphi_1(x_2,x_3,...,x_m)=\varphi_2(x_3,x_4,....,x_m)$.

Xuddi shunday, birin ketin, $x_3\to a_3, x_4\to a_4,...,x_m\to a_m$ da limitga o'tib, $\lim_{x_m\to a_m}\lim_{x_{m-1}\to a_{m-1}}.....\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)$ ni hosil qilamiz.Bu limitga $f(x_1,x_2,...,x_m)$ funksiyaning *takroriy limiti* deyiladi.

Xuddi shunday, $f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,a_{i_k}$ argumentlari, mos ravishda, $a_{i_1},a_{i_2},...,a_{i_k}$ larga intilgandagi $\lim_{x_{i_k}\to a_{i_k}}....\lim_{x_{i_1}\to a_{i_1}} f(x_1,x_2,...,x_m)$ takroriy limitni ham qarash mumkin.

Ravshanki, $f(x_1,x_2,...,x_m)$ funksiyaning $x_1,x_2,...,x_m$ argumentlari, mos ravishda, $a_{i_1},a_{i_2},...,a_{i_k}$ sonlarga, turli tartibda intilganda, funksiyaning turli takroriy limitlari hosil bo'ladi.

1.2. Uzluksiz funksiyaning ta'riflari. $u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A=A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi va $A\in\{M\}$ bo'lsin.

6-ta'rif. Agar $M\to A$ da $u=f(M)$ funksiyaning limiti mavjud bo'lib,

$$\lim_{M\to A} f(M)=f(A) \text{ yoki } \lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \cdots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=f(a_1,a_2,...,a_m)$$

bo'lsa, u holda $f(M)$ funksiya *A nuqtada uzluksiz* deb ataladi, $A=\lim_{M\to A} M$ bo'lgani uchun, funksiyaning uzluksizlik shartini,

$$\lim_{M\to A} f(M)=f\left(\lim_{M\to A} M\right) \qquad (20.1)$$

ko'rinishda ham yozish mumkin.

$\{M\}$ to'plamning funksiya uzluksizligi shartini qanoatlantirmaydigan nuqtalari funksiyaning *uzilish nuqtalari* deyiladi.

7-ta'rif (Geyne ta'rifi). Agar $\{M\}\subset R^m$ to'plamning nuqtalaridan tuzilgan, $A\in\{M\}$ ga intiluvchi har qanday $\{M_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(M_n)\}$ ketma-ketlik, hamma vaqt $f(A)$ ga teng bo'lsa, $f(M)$ funksiya A *nuqtada uzluksiz* deb ataladi.

8-ta'rif. (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun, shunday $\delta>0$ topilsaki, $\rho(M,A)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda,

$$|f(M)-f(A)|<\varepsilon$$

tengsizlik bajarilsa, $f(M)$ funksiya A *nuqtada uzluksiz* deb ataladi.

Agar $f(M)$ funksiya $\{M\}$ to'plamning har bir nuqtasida uzluksiz bo'lsa, u holda $f(M)$ funksiya $\{M\}$ *to'plamda uzluksiz* deyiladi.

1-misol. Ushbu $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$ funksiya, (x,y) nuqta $(0,0)$ nuqtaga intilganda, limitga ega emasligini ko'rsating.

Yechilishi. Ravshanki, $y=kx^2,\ x\neq 0$ chiziq bo'ylab o'zgarmas qiymat qabul qilinadi, ya'ni $|f(x,y)|=\left(\dfrac{2x^2y}{x^4+y^2}\right)_{y=kx^2}=\dfrac{2x^2(kx^2)}{x^4+(kx^2)^2}=\dfrac{2k}{1+k^2}$. Demak,

$$\lim_{(x,y)\to(x_0,y_0)}\lim=\lim_{(x,y)\to(0,0)}[f(x,y)]_{y=kx^2}=\frac{2k}{1+k^2}$$

Bu limit $y=kx^2$ chiziq bo'ylab intilish yo'liga bog'liq ravishda o'zgaradi. Agar (x,y) nuqta $(0,0)$ nuqtaga $y=x^2$ parabola bo'ylab intilsa, ya'ni $k=1$ bo'lsa, limit 1 ga teng bo'ladi. Agar (x,y) nuqta $(0,0)$ nuqtaga Ox o'q bo'ylab intilsa, ya'ni $k=0$ bo'lsa, limit 0 ga teng bo'ladi. Bu esa, yuqoridagi ikki yo'l qoidasiga binoan, $f(x,y)$ funksiya, (x,y) nuqta $(0,0)$ nuqtaga intilganda, limitga ega emasligini anglatadi.

2-misol. Quyidagi $f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$

funksiyani $O(0,0)$, $A(1,-1)$ nuqtalarda har bir o'zgaruvchi bo'yicha xususiy va ikkala o'zgaruvchi bo'yicha birgalikda uzluksizlikka tekshiring.

Yechilishi. Funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda har bir o'zgaruvchi bo'yicha xususiy uzluksizligini ko'rsatamiz: $y\neq 0$ va $x\to x_0\neq 0$ bo'lsa,

$$\lim_{x\to x_0}f(x,y)=\lim_{x\to x_0}\frac{x^4+y^4}{x^2+y^2}=\frac{x_0^4+y^4}{x_0^2+y^2}=f(x_0,y),$$

$y=0$ va $x\to x_0\neq 0$ bo'lsa,

$$\lim_{x\to x_0}f(x,0)=\lim_{x\to x_0}\frac{x^4+0}{x^2+0}=\lim_{x\to x_0}x^2=x_0^2=f(x_0,0),$$

$y=0$ va $x\to 0$ bo'lsa,

$$\lim_{x\to 0}f(x,0)=\lim_{x\to 0}\frac{x^4+0}{x^2+0}=\lim_{x\to 0}x^2=0=f(0,0),$$

$x\neq 0$ va $y\to y_0\neq 0$ bo'lsa,

$$\lim_{y\to y_0}f(x,y)=\lim_{y\to y_0}\frac{x^4+y^4}{x^2+y^2}=\frac{x^4+y_0^4}{x^2+y_0^2}=f(x,y_0),$$

$x=0$ va $y\to y_0\neq 0$ bo'lsa,

$$\lim_{y\to y_0}f(0,y)=\lim_{y\to y_0}\frac{0+y^4}{0+y^2}=y_0^2=f(0,y_0),$$

$x=0$ va $y\to 0$ bo'lsa,

$$\lim_{y\to 0}f(0,y)=\lim_{x\to 0}\frac{0+y^4}{0+y^2}=\lim_{y\to 0}y^2=0=f(0,0)$$

Ravshanki, yuqoridagidek, funksiya $A(1,-1)$ nuqtada har bir o'zgaruvchi bo'yicha xususiy uzluksiz ekanligini ko'rsatish qiyin emas.

Berilgan funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda ikkala o'zgaruvchi bo'yicha ham bir yo'la uzluksiz ekanligini ko'rsatamiz. Agar o'zgaruvchilar, $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ deyilsa,

$$\lim_{\substack{x\to 0\\ y\to 0}}f(x,y)=\lim_{\rho\to 0}f(\rho\cos\varphi,\ \rho\sin\varphi)=\lim_{\rho\to 0}\rho^2(\cos^4\varphi+\sin^4\varphi)=0,$$

$$\lim_{\substack{x\to 1\\ y\to -1}}f(x,y)=\lim_{\substack{x\to 1\\ y\to -1}}\frac{x^4+y^4}{x^2+y^2}=\frac{1^4+(-1)^4}{1^2+(-1)^2}=1=f(1;-1)$$

bo'ladi. Bundan, berilgan funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda ikkala o'zgaruvchi bo'yicha ham bir yo'la uzluksiz ekanligi kelib chiqadi.

Mustaqil yechish uchun misollar

Quyidagi limitlarni hisoblang:

1.1. $\lim\limits_{\substack{x\to\pi\\ y\to\ln 2}}e^y\cos x.$ **1.2.** $\lim\limits_{\substack{x\to 1\\ y\to 1}}\dfrac{x-y}{x^2-y^2}.$

1.3. $\lim\limits_{\substack{x\to 1\\ y\to -1\\ z\to e}} \ln|x+y+z|$.

1.4. $\lim\limits_{\substack{x\to 0\\ y\to a}} \dfrac{\sin xy}{x}$.

1.5. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{tg\,2xy}{x^2 y}$.

1.6. $\lim\limits_{\substack{x\to 0\\ y\to 3}} \left(1+xy^2\right)^{\frac{y}{x^2y+xy^2}}$.

1.7. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{ax+by}{x^2+xy+y^2}$.

1.8. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{x^4+y^4}$.

1.9. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{|x^3|+|y^3|}$.

1.10. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} (x+y)^{-(x^2+y^2)}$.

1.11. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(x^2+y^2\right)^{|x|}$.

1.12. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 1}} f(x,y)$ni hisoblang, bunda

$$f(x,y)=\begin{cases} \dfrac{x^2 y}{\sqrt{1+x^2 y}-1}, & x^2 y\neq 0 \text{ } bo'lg\,anda,\\ 2, & x^2 y = 0 \text{ } bo'lg\,anda. \end{cases}$$

Quyidagi karrali limitlarni hisoblang.

1.13. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$.

1.14. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sin\left(x^4y^2\right)}{\left(x^2+y^2\right)^2}$.

1.15. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\left(x^2+y^2\right)}{1-\cos\left(x^2+y^2\right)}$.

1.16. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{e^{\frac{1}{x^4+y^4}}}{x^4+y^4}$.

1.17. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(1+x^2y^2\right)^{\frac{1}{x^2+y^2}}$.

1.18. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{(x+y)^2}{\left(x^2+y^4\right)^2}$.

1.19. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \left(x^2+y^2\right)\sin\dfrac{1}{x^2+y^2}$.

1.20. Ushbu $\lim\limits_{\substack{x\to 1\\ y\to 1}} f(x,y)$ limitni hisoblang, bunda

$$f(x,y)=\begin{cases} \dfrac{x^2+2xy-3y^2}{x^3-y^3}, & x\neq y \text{ } bo'lg\,anda,\\ 4/3\,, & x=y \text{ } bo'lg\,anda. \end{cases}$$

Quyidagi karrali limitlarning mavjud emasligini isbotlang.

1.21. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x}{x+y}$.

1.22. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x-y}{x+y}$.

1.23. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2-y^2}{x^2+y^2}$.

1.24. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{\ln(x+y)}{y}$.

1.25. Ushbu $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x^4+y^2}, & (x,y)\neq(0,0) \text{ } bo'lg\,anda,\\ 0\,, & (x,y)=(0,0) \text{ } bo'lg\,anda. \end{cases}$

funksiyaning (0, 0) nuqtada karrali limiti mavjud emasligini ko'rsating.

1.26. Ushbu

$$f(x,y)=\begin{cases} \left(1+\dfrac{1}{x+y}\right)^{x+y}, & x+y\neq 0 \text{ } bo'lg\,anda,\\ 1\,, & x+y=0 \text{ } bo'lg\,anda \end{cases}$$

funksiyaning $x\to\infty,\ y\to\infty$ dagi karrali limiti mavjud emasligini isbotlang.

Quyidagi $\lim\limits_{x\to x^0}\lim\limits_{y\to y^0} f(x,y)$ va $\lim\limits_{y\to y^0}\lim\limits_{x\to x^0} f(x,y)$ takroriy limitlarni hisoblang.

1.27. $f(x,y)=\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$, $x^0=0,\ y^0=0$. **1.28.**

$f(x,y)=\dfrac{\sin(x+y)}{2x+3y}$, $x^0=0,\ y^0=0$.

1.29. $f(x,y)=\dfrac{\cos x-\cos y}{x^2+y^2}$, $x^0=0,\ y^0=0$.

1.30. $f(x,y)=\dfrac{x^2+y^2}{x^2+y^4}$, $x^0=\infty,\ y^0=\infty$.

1.31. $f(x,y)=\dfrac{x^y}{1+x^y}$, $x^0=\infty,\ y^0=0$.

1.32. $f(x,y)=\sin\dfrac{\pi x}{2x+y}$, $x^0=\infty,\ y^0=\infty$.

1.33. $f(x,y)=\dfrac{1}{xy}tg\dfrac{xy}{1+xy}$, $x^0=0,\ y^0=\infty$.

1.34. $f(x,y)=\log_x(x+y)$, $x^0=1,\ y^0=0$.

1.35. $f(x,y)=\dfrac{\sin 3x+tg\,2y}{6x+3y}tg\dfrac{xy}{1+xy}$, $x^0=0,\ y^0=0$.

Quyidagi berilgan funksiyalarning (x^0, y^0) nuqtada karrali va takroriy limitlari mavjudmi?

1.36. $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$, $x^0=0$, $y^0=0$.

1.37. $f(x,y)=\log_x(x+y)$, $x^0=1$, $y^0=0$.

1.38. $f(x,y)=\dfrac{\sin x+\sin y}{x+y}$, $x^0=0$, $y^0=0$.

Quyidagi funksiyalarning uzilish nuqtalarini toping.

1.39. $u=\dfrac{1}{\sqrt{x^2+y^2}}$. **1.40.** $u=\dfrac{1}{x^2+y^2}$.

1.41. $u=\ln(9-x^2-y^2)$ **1.42.** $u=\dfrac{xy}{x+y}$.

1.43. $u=\sin\dfrac{x}{y}$. **1.44.** $u=\sin\dfrac{1}{xy}$.

Mustaqil yechish uchun misollarning javoblari

1.1. -2. **1.2.** 0,5. **1.3.** 1. **1.4.** a. **1.5.** 2. **1.6.** e^3. **1.7.** 0. **1.8.** 0. **1.9.** 0. **1.10.** 0. **1.11.** 0. **1.12.** 2. **1.13.** 0. **1.14.** 0. **1.15.** 0. **1.16.** 0. **1.17.** 1. **1.18.** 0. **1.19.** 1. **1.20.** $\frac{4}{3}$. **1.27.** 1, 1. **1.28.** $\frac{1}{2}$, $\frac{1}{3}$. **1.29.** $\frac{1}{2}$ va $-\frac{1}{2}$. **1.30.** 0 va 1. **1.31.** $\frac{1}{2}$ va 1. **1.32.** 0 va 1. **1.33.** 0 va 1. **1.34.** 1 va ∞. **1.35.** $\frac{1}{2}$ va $-\frac{2}{3}$. **1.36.** Karrali limit mavjud emas, $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)=-1$ va $\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=1$. **1.37.** Karrali limit mavjud emas, $\lim\limits_{x\to1}\lim\limits_{y\to0} f(x,y)=1$ va $\lim\limits_{y\to0}\lim\limits_{x\to1} f(x,y)=\infty$. **1.38.** $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)$ $=\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=1$. **1.39.** $O(0;0)$. **1.40.** $0(0;0)$. **1.41.** $x^2+y^2=9-$ aylananing hamma nuqtalari. **1.42.** $x+y=0$ chiziqning hamma nuqtalari. **1.42.** $y=0$ to'g'ri chiziqning hamma nuqtalari. **1.43.** Koordinata o'qlarining hamma nuqtalari.

K

2- amaliy mashg'ulot.

KO'P O'ZGARUVCHILI FUNKSIYANING XUSUSIY HOSILALARI VA DIFFERENSIALLARI

2.1. Ko'p o'zgaruvchili funksiyaning xususiy hosilalari. $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya ochiq $\{M\}$ ($\{M\}\subset R^m$) to'plamda aniqlangan bo'lsin. Bu to'plamdan $M(x_1,x_2,...,x_m)$ nuqtani olamiz va funksiyaning x_k argumentiga Δx_k orttirma beramiz (qolgan argumentlarini o'zgarmas, deb hisoblaymiz). Natijada, funksiya ham $\Delta_{x_k}u$ orttirma oladi. Ushbu

$$\frac{\Delta_{x_k}u}{\Delta x_k}=\frac{f(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)-f(x_1,x_2,...,x_m)}{\Delta x_k} \tag{2.1}$$

nisbatni qaraymiz, bunda $M(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)\in\{M\}$.

1–ta'rif. Agar $\Delta x_k\to0$ da (2.1) nisbatning limiti mavjud va chekli bo'lsa, bu limit $f(x_1,x_2,...,x_m)$ funksiyaning $M(x_1,x_2,...,x_m)$ nuqtadagi x_k argumenti bo'yicha *xususiy hosilasi* deyiladi va

$$\frac{\partial f(x_1,...,x_m)}{\partial x_k},\ \frac{\partial f}{\partial x_k},\ U'_{x_k},\ f'_{x_k}(x_1,x_2,...,x_m),\ f'_{x_k}$$

kabi belgilarning biri orqali yoziladi. Ta'rifga ko'ra,

$$\frac{\partial u}{\partial x_k}=\lim_{\Delta x_k\to0}\frac{\Delta_{x_k}u}{\Delta x_k}$$

ko'rinishda yozish mumkin.

2.2.Ko'p o'zgaruvchili funksiyaning differensiali. $u=f(M)$ funksiya $\{M\}$ ($\{M\}\subset R^m$) to'plamda berilgan bo'lib, bu funksiya $M(x_1,x_2,...,x_m)\in\{M\}$ nuqtada differensiallanuvchi bo'lsin. U holda $u=f(M)$ funksiyaning Δu to'liq orttirmasi uchun

$$\Delta u=A_1\Delta x_1+...+A_m\Delta x_m+\alpha_1\Delta x_1+...+\alpha_m\Delta x_m$$

formula o'rinli.

21.3–ta'rif. $u=f(M)$ funksiya Δu orttirmasining $\Delta x_1,\Delta x_2,...,\Delta x_m$ larga nisbatan chiziqli bosh qismi, $u=f(M)$ funksiyaning M nuqtadagi *differensiali (to'liq differensiali)* deb ataladi va y, du, df yoki $df(x_1,x_2,...,x_m)$ kabi belgilanadi.

Demak,

$$du=df=df(x_1,x_2,...,x_m)=A_1\Delta x_1+A_2\Delta x_2+...+A_m\Delta x_m \tag{2.2}$$

1-teorema. Agar $u=f(M)$ funksiya $M_0(x_1^0,...,x_m^0)$ nuqtaning biror atrofida barcha argumentlari bo'yicha xususiy hosilalarga ega bo'lib, bu hosilalar M_0 nuqtada uzluksiz bo'lsa, u holda, berilgan funksiya M_0 nuqtada differensiallanuvchi bo'ladi.

1-teoremani e'tiborga olsak, u holda, (2.2) funksiya differensiali ni quyidagi,

$$du=\frac{\partial u}{\partial x_1}\Delta x_1+\frac{\partial u}{\partial x_2}\Delta x_2+...+\frac{\partial u}{\partial x_m}\Delta x_m \tag{2.3}$$

ko'rinishda ham yozish mumkin. x_i $(i=1,2,...,m)$ o'zgaruvchining differensialini dx_i $(i=1,2,...,m)$ deb, ixtiyoriy $(x_1,x_2,...,x_m)$ larga bog'liq bo'lmagan son tushuniladi. Bu sonni, bundan keyin, Δx_i $(i=1,2,...,m)$ ga teng deb olishga kelishib olamiz, ya'ni $dx_i=\Delta x_i$ $(i=1,2,...,m)$. Bu kelishuvni e'tiborga olsak, (2.3) ni quyidagi,

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \dots + \frac{\partial u}{\partial x_m}dx_m \qquad (2.4)$$

ko'rinishda yozish mumkin. (2.4) ga ko'p o'zgaruvchili funksiyaning to'liq differensialini topish formulasi deyiladi.

1-misol. $f(x,y,z) = \mathrm{arctg}(xyz)$ funksiyaning $f_x'(x,y,z)$, $f_y'(x,y,z)$ va $f_z'(x,y,z)$ xususiy hosilalarini toping.

Yechilishi. Ushbu $(\mathrm{arctg}u)' = \frac{u'}{1+u^2}$ formulaga asosan, $f_x'(x,y,z)$, $f_y'(x,y,z)$ va $f_z'(x,y,z)$ xususiy hosilalarni topamiz:

$$f_x'(x,y,z) = (\mathrm{arctg}(xyz))_x' = \frac{yz}{1+(xyz)^2},$$

$$f_y'(x,y,z) = \frac{xz}{1+(xyz)^2}, \quad f_z'(x,y,z) = \frac{xy}{1+(xyz)^2}.$$

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning xususiy hosilalarini toping.

2.1. $u = x^2 + y^2 + 3x^2y^3$. **2.2.** $u = \frac{x(x-y)}{y^2}$.

2.3. $u = xyz + \frac{x}{yz}$. **2.4.** $u = \sin(xy + yz)$.

2.5. $u = tg(x+y) \cdot e^{x/y}$. **2.6.** $u = \sin\frac{x}{y} \cdot \cos\frac{y}{x}$.

2.7. $u = e^x(\cos y + x\sin y)$. **2.8.** $u = x^y$.

2.9. $u = \left(\frac{y}{x}\right)^z$. **2.10.** $u = \ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

2.11. $u = \arcsin\sqrt{\frac{x^2-y^2}{x^2+y^2}}$. **2.12.** $u = (1+\sin^2 x)^{\ln y}$.

2.13. $u = x^y y^z z^x$. **2.14.** $g(r,\theta) = r\cos\theta + r\sin\theta$.

2.15. $f(R_1,R_2,R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. **2.16.** $P(n,R,T,V) = \frac{nRT}{V}$.

Quyidagi funksiyalarning berilgan nuqtadagi xususiy hosilalarini toping.

2.17. $u = \frac{x}{y^2}$, (1;1) . **2.18.** $u = \ln\left(1+\frac{x}{y}\right)$, (1;2) .

2.19. $u = xye^{\sin\pi xy}$, (1;1) . **2.20.** $u = (2x+y)^{2x+y}$, (1;−1) .

2.21. Ushbu $u = \sqrt[3]{xy}$ funksiyaning $O(0;0)$ nuqtadagi xususiy hosilalarini toping. Bu funksiya $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

Quyidagi berilgan $u(x,y)$ funksiyalar $O(0;0)$ nuqtada xususiy hosilalarga egami; $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

2.22. $u = \sqrt{x^2+y^4}$. **2.23.** $u = \sqrt{x^4+y^4}$.

2.24. $u = \sqrt[5]{xy}$ **2.25.** $u = \sqrt[3]{x^2y^2}$.

2.26. $u = \begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2 \neq 0 \text{ bo'lganda}, \\ 0, & x^2+y^2 = 0 \text{ bo'lganda}. \end{cases}$

2.27. $u = \begin{cases} \frac{x^4+y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \text{ bo'lganda}, \\ 0, & x^2+y^2 = 0 \text{ bo'lganda}. \end{cases}$

2.28. $u(x,y)$ funksiya:

a) $u = \frac{x}{\sqrt{x^2+y^2}}$; *b)* $u = \ln(x^2+xy+y^2)$

ko'rinishlarda bo'lganda, $\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ ifodani hisoblang.

2.29. $u(x,y,z)$ funksiya:

a) $u = (x-y)(y-z)(z-x)$; *b)* $u = x + \frac{x-y}{y-z}$

ko'rinishlarda bo'lganda, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ ifodani hisoblang.

2.30-2.35- misollarda $u = f(x_1, x_2, \dots, x_m)$ funksiya uchun quyidagi tasdiqlarning qaysi biri to'g'ri, qaysi biri noto'g'ri?

2.30. $f(x_1, x_2, \dots, x_m)$ funksiya biror nuqtada hamma argumentlari bo'yicha xususiy hosilalarga ega bo'lsa, u shu nuqtada uzluksiz bo'ladi.

2.31. Agar funksiya R^m fazoning har bir nuqtasida hamma argumentlari bo'yicha xususiy hosilalarga ega bo'lsa, u R^m da uzluksiz bo'ladi.

2.32. Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u shu nuqtada hamma argumentlari bo'yicha xususiy hosilalarga ega bo'ladi.

2.33. Agar funksiyaning biror nuqtada hamma argumentlari bo'yicha xususiy hosilalari mavjud bo'lsa, u shu nuqtada differensiallanuvchi bo'ladi.

2.34. Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u holda, shu nuqtada funksiyaning hamma argumentlari bo'yicha uzluksiz xususiy hosilalari mavjud bo'ladi.

2.35. Agar funksiyaning biror nuqtada uzluksiz xususiy hosilalari mavjud bo'lsa, u holda, funksiya shu nuqtada differensiallanuvchi bo'ladi.

2.36. Agar $f(x,y)-Oxy$ tekislikdagi G ochiq sohada aniqlangan, uning f_x va f_y xususiy hosilalari G da chegaralangan bo'lsa, u holda, $f(x,y)$ G da uzluksizligini isbotlang.

Quyidagi berilgan funksiyalarning differensialini toping.

2.37. $u=2x^4-3x^2y^2+x^3y.$ **2.38.** $u=(y^3+2x^2+3)^4.$

2.39. $u=\frac{y}{x}+\frac{x}{y}.$ **2.40.** $u=-\frac{x}{\sqrt{x^2+y^2}}.$

2.41. $u=a^{-\frac{y}{x}}.$ **2.42.** $u=\ln\left(x+\sqrt{x^2+y^2}\right).$

2.43. $u=\ln\sin\frac{x+1}{\sqrt{y}}.$ **2.44.** $u=arctg\frac{x+y}{x-y}.$

2.45. $u=(1+xy)^y.$

Quyidagi funksiyalarning berilgan nuqtalardagi differensialini toping.

2.46. $u=\frac{x^2-y^2}{x^2+y^2}$. a) $(1;1)$; b) $(0;1)$. **2.47.** $u=\sqrt{xy+\frac{x}{y}}$, $(2;1)$

2.48 $u=\cos(xy+xz)$, $M(x,y,z)$ va $N\left(1;\frac{\pi}{6};\frac{\pi}{6}\right)$ nuqtalarda.

2.49. $u=e^{xy}\cdot M(x,y)$ va $O(0;0)$ nuqtada.

2.50. $u=x^y$, $M(x,y)$ va $M_0(2;3)$nuqtalarda.

2.51. $u=x\ln(xy)$, $M(x,y)$ va $M_0(-1;-1)$ nuqtalarda.

2.52. $u=\frac{x}{x^2+y^2+z^2}$, $M(1,0,1)$. **2.53.** $u=arctg\frac{xy}{z^2}$, $M(3,2,1)$.

2.54. $u=\left(xy+\frac{x}{y}\right)^z$, $M(1,1,1)$.

Quyidagi berilgan $f(u)$ funksiyani differensiallanuvchi va uning f_u hosilalani aniq deb faraz qilib, $f(u)$ funksiya uchun f_x, f_y toping.

2.55. $u=x^2+e^y$. **2.56.** $u=\sqrt[3]{x^3+xy^2}$. **2.57.** $u=arctg(x+\ln y)$.

Quyidagi berilgan $f(u)$, $f(u,v)$, $f(u,v,w)$ funksiyalarni differensiallanuvchi va ularning f_u, f_v, f_w hosilalari aniq deb faraz qilib, quyidagi φ funksiyaning differensialini toping.

2.58. $\varphi=f(u)$, $u=xy+\frac{y^2}{x}$.

2.59. 2) $\varphi=f(u,v)$, $u=\frac{y}{x+y}$, $v=x^2-y^3$.

2.60. $\varphi=f(u,v,w)$, $u=x^2+y^2+z^2$, $v=x+y+z$, $w=xyz$.

2.61. Agar $W=\sin(xy+\pi)$, $x=e^t$ va $y=\ln(t+1)$ bo'lsa, $t=0$ da $\frac{dw}{dt}$ hosilani hisoblang.

2.66. Agar $W=\sin(2x-y)$, $x=r+\sin s$, $y=rs$ bo'lsa, $r=\pi$ va $s=0$ bo'lganda, mos ravishda, $\frac{\partial w}{\partial t}$ va $\frac{\partial w}{\partial s}$ xususiy hosilalarni toping.

2.67. Ushbu $W(x,y,z)=xy+yz+xz$ funksiyaning $x=\cos t$, $y=\sin t$, $z=\cos 2t$ egri chiziqdagi t bo'yicha hosilasining $t=1$ dagi qiymatini toping.

2.68. $w=f(z,\sigma)$, $r=\sqrt{x^2+y^2}$, $\sigma=arctg\frac{y}{x}$ bo'lsin. U holda, $\frac{\partial w}{\partial x}$ va $\frac{\partial w}{\partial y}$ larni toping va javobingizni r va σ orqali ifodalang.

Quyidagi misollarda: *a)* zanjir qoidasidan foydalanib; *b)* bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t = t_0$ nuqtadagi qiymatini toping.

2.69. $W = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t_0 = \pi$.

2.70. $W = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$; $t_0 = 3$.

2.71. $W = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = arctgt$, $z = e^t$; $t_0 = 1$.

2.72. Agar $W = (x+y+z)^2$, $x = r - s$, $y = \cos(r+s)$, $z = \sin(r+s)$ bo'lsa, $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1,-1)}$ toping.

2.73. Agar $W = x^2 + \frac{y}{x}$, $x = u - 2v + 1$, $y = 2u + v - 2$ bo'lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0;0)}$ ni toping.

2.74. Agar $W = arctgx$ va $x = e^u + \ln v$ bo'lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(\ln 2;1)}$, $\left.\frac{\partial W}{\partial v}\right|_{(u,v)=(\ln 2;1)}$ larni toping.

2.75. Agar a va b - o'zgarmas sonlar, $w = u^3 + thu + \cos u$ va $u = ax + by$ bo'lsa, $a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$ munosabat o'rinli ekanligini ko'sating.

2.76. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = yf(x^2 - y^2)$ funksiya, $y^2\frac{\partial\varphi}{\partial x} + xy\frac{\partial\varphi}{\partial y} = x\varphi$ tenglamani qanoatlantirishini isbotlang.

2.77. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = xy + xf\left(\frac{y}{x}\right)$ funksiya, $x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} = xy + \varphi$ tenglamani qanoatlantirishini isbotlang.

2.78. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = \sin x + f(\sin y - \sin x)$ funksiya, $\cos y\frac{\partial\varphi}{\partial x} + \cos x\frac{\partial\varphi}{\partial y} = xy + \varphi$ tenglamani qanoatlantirishini isbotlang.

2.79. Agar $f(u,v)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y,z) = f\left(\frac{x}{y}, x^2 + y - z^2\right)$ funksiya, $2xz\frac{\partial\varphi}{\partial x} + 2yz\frac{\partial\varphi}{\partial y} + (2x^2 + y)\frac{\partial\varphi}{\partial z} = 0$ tenglamani qanoatlantirishini isbotlang.

2.80. Agar $w = f(s) - s$ ning differensiallanuvchi funksiyasi, $s = y + 5x$ bo'lsa, u holda $\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 0$ munasabat bajarilishini ko'rsating.

2.81. Agar a va b - o'zgarmas sonlar, $w = u^3 + thu + \cos u$ va $u = ax + by$ bo'lsa, $a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$ munasabat o'rinli ekanligini ko'sating.

2.82. Agar $f(u,v,w)$ diferensiallanuvchi funksiya va $u = x - y$, $v = y - z$ hamda $w = z - x$ bo'lsa, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$ ekanligini ko'rsating.

2.83. Faraz qilaylik, $W = f(x,y)$ diferensiallanuvchi funksiyada $x = r\cos\theta$ va $y = r\sin\theta$ qutb koordinatalariga o'tish amalga oshirilgan (qutb almashtirishlari bajarilgan) bo'lsin. U holda

a) $\frac{\partial W}{\partial r} = f_x' \cos\theta + f_y' \sin\theta$, $\frac{1}{r}\frac{\partial W}{\partial\theta} = -f_x' \sin\theta + f_y' \cos\theta$,

ekanligini ko'rsating;

b) *a)* banddagi tenglamalarni f_x' va f_y' larga nisbatan yechib, ularni $\frac{\partial W}{\partial r}$ va $\frac{\partial W}{\partial\theta}$ lar orqali ifodalang;

c) $(f_x')^2 + (f_y')^2 = \left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial\theta}\right)^2$

ekanligini ko'rsating.

2.84. f va $g - x$ va y ning shunday funksiyalardan iboratki, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ va $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$ munosabat o'rinli bo'lsin. Faraz qilaylik, $\frac{\partial f}{\partial x} = 0$, $f(1,2) = g(1,2) = 5$ va $f(0,0) = 4$ bo'lsin. U holda, $f(x,y)$ va $g(x,y)$ larni toping.

2.85. Birinchi tartibli xususiy hosilalardan $\frac{\partial w}{\partial x}=1+e^x\cos y$ va $\frac{\partial w}{\partial y}=2y-e^x\sin y$ hamda $(\ln 2,0)$ nuqtadagi qiymatini $2+\ln 2$ ga teng bo'lgan $(f(\ln 2,0)=2+\ln 2)$, $w=f(x,y)$ funksiyani toping.

2.86. Agar $u=f(x,y,z)$ funksiya biror E sohada differensiallanuvchi bo'lib, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=pu$ tenglamani qanoatlantirsa, u holda, uning p- darajali bir jinsli funksiya bo'lishini isbotlang.

2.87. Agar $u=f(x,y,z)$ funksiya biror E sohada ikki marta differensiallanuvchi bo'lsa, u holda,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2 u=p(p-1)u$$

tenglikning o'rinli ekanligini isbotlang.

2.88. Ushbu $u=x^y y^x$ funksiya $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(x+y+\ln u)u$ tenglamani qanoatlantirishini isbotlang.

2.89. Ushbu $u=\frac{x-y}{z-t}+\frac{t-x}{y-z}$ funksiya $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=0$ tenglamani qanoatlantirishni isbotlang.

Quyidagi funksiyaning M_0 nuqtada $\vec{M_0M}$ yo'nalish bo'yicha hosilasini toping.

2.90. $f(x,y)=5x+10x^2y+y^5$, $M_0(1,2)$, $M(5,-1)$.

2.91. $f(x,y)=xy^2z^3$, $M_0(3,2,1)$, $M(7,5,1)$.

2.92. $f(x,y,z)=\arcsin\frac{z}{\sqrt{x^2+y^2}}$, $M_0(1,1,1)$, $M(1,5,4)$.

2.93. Ushbu $f(x,y)=3x^4+y^3+xy$ funksiyaning $M_0(1,2)$ nuqtada, Ox o'q bilan 135° burchak tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

2.94. Ushbu $f(x,y)=arctg\frac{y}{x}$ funksiyaning $x^2+y^2=2x$ aylananing $M_0\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ nuqtasiga o'tkazilgan tashqi normalning yo'nalishi bo'yicha hosilasini toping.

2.95. Quyidagi $f(x,y,z)=\ln(e^x+e^y+e^z)$ funksiyaning $M_0(0,0,0)$ nuqtada, Ox, Oy, Oz koordinatalar o'qlari bilan, mos ravishda, $\frac{\pi}{3},\frac{\pi}{4}$ va $\frac{\pi}{3}$ burchaklarni tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

2.96-2.97- misollarda berilgan $f(x,y)$ funksiyaning P_0 nuqtada kamayish va o'sish yo'nalishlarini toping va har bir yo'nalish bo'yicha hosilasini toping. Shuningdek, $f(x,y)$ funksiyaning P_0 nuqtada $\vec{v}$ vektor yo'nalishidagi hosilasini toping.

2.96. $f(x,y)=\cos x\cos y$, $P_0\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, $\vec{v}=3\vec{i}+4\vec{j}$.

2.97. $f(x,y,z)=\ln(2x+3y+6z)$, $P_0(-1,-1,1)$, $\vec{v}=2\vec{i}+3\vec{j}+6\vec{k}$.

Quyidagi skalyar maydonning berilgan nuqtadagi gradiyentini toping.

2.98. $u(x,y)=x^2-2xy+3y-1$, $gradu\,|_{M(x,y)}=?$.

2.99. $u(x,y)=5x^2y-3xy^3+y^4$, $gradu\,|_{M(x,y)}=?$

2.100. $u=x^2+y^2$, $gradu|_{(3;2)}=?$

2.101. $u=\sqrt{4+x^2+y^2}$. $gradu|_{(2;1)}=?$

2.102. $u=\mathrm{arctg}\frac{y}{x}$. $gradu|_{(1;1)}=?$

2.103. $u=arctg\frac{x}{y}$ skalyar maydonning $(1;1)$ va $(-1;-1)$ nuqtalardagi gradiyentlari orasidagi burchakni toping.

2.104. $z_1=\sqrt{x^2+y^2}$, $z_2=x-3y+\sqrt{3xy}$ funksiyalarning $(3,4)$ nuqtadagi gradiyentlari orasidagi burchakni toping.

2.105. $grad(\varphi\psi)=\varphi\, grad\psi+\psi\, grad\varphi$ tenglikni isbotlang.

2.106. $z=\varphi(u,v)$, $u=\psi(x,y)$, $v=\zeta(x,y)$ funksiyalar berilganda, $grad\,z=\frac{\partial\varphi}{\partial u}gradu+\frac{\partial\varphi}{\partial v}gradv$ tenglikning to'g'riligini ko'rsating.

2.107-2.111 misollarda funksiyaning orttirmasini uning differensialiga almashtirib, quyida berilgan ifodalarni taqribiy hisoblang.

2.107. $(1{,}02)^{4{,}05}$. **2.108.** $\sqrt{8{,}04^2+6{,}03^2}$. **2.109.** $(1{,}02)^3\cdot(0{,}97)^2$.

2.110. $\sin 32^0\cos 59^0$. **2.111.** $\ln(0{,}9^3+0{,}99^3)$ **2.112.** $\sqrt{2{,}03^2+5e^{0{,}02}}$.

Mustaqil yechish uchun misollarning javoblari

2.1. $u_x'=2x+6xy^3,\ u_y'=3y^2+9x^2+y^2$. **2.2.** $u_x'=\dfrac{2x-y}{y^2},\ u_y'=\dfrac{xy-2x^2}{y^3}$.

2.3. $u_x'=yz+\dfrac{1}{yz},\ u_y'=xz-\dfrac{x}{y^2z},\ u_z'=xy-\dfrac{x}{yz^2}$. **2.4.** $u_x'=y\cdot\cos(xy+yz)$,

$u_y'=(x+z)\cos(xy+yz),\ \ u_z'=y\cdot\cos(xy+yz)$. **2.5.** $u_x'=\dfrac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)\cdot e^{x/y}\cdot\dfrac{1}{y}$,

$u_y'=\dfrac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)e^{x/y}\left(-\dfrac{x}{y^2}\right)$. **2.6.** $u_x'=\dfrac{1}{y}\cos\dfrac{x}{y}\cos\dfrac{y}{x}+\dfrac{y}{x^2}\sin\dfrac{x}{y}\cdot\sin\dfrac{y}{x}$,

$u_y'=-\dfrac{x}{y^2}\cos\dfrac{x}{y}\cos\dfrac{y}{x}-\dfrac{1}{x}\sin\dfrac{x}{y}\sin\dfrac{y}{x}$. **2.7.** $u_x'=e^x(x\sin y+\sin y+\cos y)$,

$u_y'=e^x(x\cos y-\sin y)$. **2.8.** $u_x'=yx^{y-1},\ u_y'=x^y\cdot\ln x$. **2.9.** $u_x'=z\left(\dfrac{y}{x}\right)^{z-1}\cdot\left(-\dfrac{y}{x^2}\right)=-\dfrac{z}{x}\left(\dfrac{y}{x}\right)^z$,

$u_y'=\dfrac{z}{y}\left(\dfrac{y}{x}\right)^z,\ u_z'=\left(\dfrac{y}{x}\right)^z\cdot\ln\dfrac{y}{x}$. **2.10.** $u_x'=-\dfrac{2}{\sqrt{x^2+y^2}},\ u_y'=\dfrac{2x}{y\sqrt{x^2+y^2}}$. **2.11.**

$u_x'=\dfrac{xy^2\sqrt{2x^2-2y^2}}{|y|(x^4-y^4)},\ u_y'=\dfrac{yx^2\sqrt{2x^2+2y^2}}{|y|(y^4-x^4)}$. **2.12.** $u_x'=\sin 2x\ln g(1+\sin^2 x)^{\ln y-1}$,

$u_y'=\dfrac{1}{y}(1+\sin^2 x)^{\ln y}\cdot\ln(1+\sin^2 x)$

. **2.13.** $u_x'=x^{y-1}y^{z+1}z^x+x^y y^z z^x\cdot\ln z$, $u_y'=x^y y^z z^x\ln x+x^y y^{z-1}z^{x+1}$,

$u_z'=x^y y^z z^x\ln y+x^{y+1}y^z z^{x-1}$. **2.14.** $\dfrac{\partial g}{\partial r}=\cos\theta+\sin\theta,\ \dfrac{\partial g}{\partial\theta}=-r\sin\theta+r\cos\theta$. **2.15.**

$\dfrac{\partial f}{\partial R_1}=-\dfrac{1}{R_1^{\,2}};\ \dfrac{\partial f}{\partial R_2}=-\dfrac{1}{R_2^{\,2}};\ \dfrac{\partial f}{\partial R_3}=-\dfrac{1}{R_3^{\,2}}$. **2.16.** $\dfrac{\partial P}{\partial n}=\dfrac{RT}{V};\ \dfrac{\partial p}{\partial R}=\dfrac{nT}{V}$;

$\dfrac{\partial P}{\partial T}=\dfrac{nR}{V};\ \dfrac{\partial P}{\partial V}=-\dfrac{nRT}{V^2}$. **2.17.** $u_x'(1;1)=1,\ \ u_y'(1;1)=-2$. **2.18.** $u_x'(1;2)=\dfrac{1}{3},\ \ u_y'(1;2)=-\dfrac{1}{6}$.

2.19. $u_x'(1;1)=1-\pi,\ u_y'(1;1)=1-\pi$. **2.20.** $u_x'(1;-1)=2,\ u_y'(1;-1)=1$. **2.21.**
$u_x'(0;0)=0,\ \ u_y'=(0;0)=0$. Funksiya $O(0;\,0)$ nuqtada differensiallanuvchi emas. **2.22.**
$u_x'(0;0),\ u_y'(0;0)$lar mavjud emas, $u(x,y)$ funksiya $0(0;0)$ nuqtada differensiallanuvchi. **2.23.** $u_x'(0;0)=u_y'(0;0)=0;\ \ u(x,y)$ funksiya $O(0;\,0)$ nuqtada differensiallanuvchi. **2.24.** $u_x'(0{,}0)=u_y'(0{,}0)=0;\ \ u(x,y)$ funksiya $0(0;0)$ nuqtada diferensiallanuvchi emas. **2.25.** $u_x'(0{,}0)=u_y'(0{,}0)=0;\ u(x,y)$ funksiya $O(0;\,0)$ nuqtada differensiallanuvchi. **2.26.** $u_x'(0{,}0)=u_y'(0{,}0);\ u(x,y)$ funksiya $O(0;\,0)$ nuqtada differensiallanuvchi. **2.27.** $u_x'(0{,}0)=u_y'(0{,}0)$; $u(x,y)$ funksiya $O(0;\,0)$ nuqtada differensiallanuvchi. **2.28.** *a)* 0; *b)* 2. **2.29.**

a) 0, *b)* 1. **2.30.** Noto'g'ri. **2.31.** Noto'g'ri ($n>1$ bo'lganda). **2.32.** To'g'ri. **2.33.** Noto'g'ri ($n>1$ bo'lganda). **2.34.** Noto'g'ri. **2.35.** To'g'ri. **2.37.** $(8x^3-6xy^2+3x^2y)dx+(x^3-6x^2y)dy$. **2.38.**
$4(y^3+2x^2y+3)\cdot(4xydx+(3y^2+2x^2)dy)$. **2.39.** $\dfrac{x^2-y^2}{xy}\left(\dfrac{dx}{x}-\dfrac{dy}{y}\right)$. **2.40.**

$y(x^2+y^2)^{-3/2}(ydx-xdy)$ **2.41.** $a^{-y/x}\dfrac{\ln a}{x^2}(ydx-xdy)$. **2.42.** $\dfrac{1}{\sqrt{y}}\left(dx+\dfrac{ydy}{x+\sqrt{x^2+y^2}}\right)$. **2.43.**

$\dfrac{1}{\sqrt{y}}ctg\dfrac{x+1}{\sqrt{y}}\left(dx-\dfrac{x+1}{2y}\right)dy$. **2.44.** $\dfrac{xdy-ydx}{x^2+y^2}$. **2.45.**

$(1+xy)^{y-1}(y^2dx+(xy+(1+xy)\ln(1+xy))dy)$. **2.46.** *a)* $dx-dy$, *b)* 0. **2.47.** $\dfrac{1}{2}dx$ **2.48.**

$du|_N=-\sin x(y+z)\cdot[(y+z)dx+xdy+xdz],\ du|_N=-\dfrac{\sqrt{3}}{2}\left(\dfrac{\pi}{3}dx+dy+dz\right)$.

2.49. $du|_M=e^{xy}(ydx+xdy),\ \ du|_O=0$.]**2.50.** $du|_M=x^y\left(\dfrac{y}{x}dx+\ln x\,dy\right)$,

$du|_{M_0}=12dx+8\ln 2\,dy$. **2.51.** $du|_M=(1+\ln xy)dx+\dfrac{x}{y}dy,\ du|_{M_0}=dx+dy$. **2.52.** $du|_M=-\dfrac{1}{2}dz$.

2.53. $du|_M=\dfrac{1}{37}(2dx+3dy-12dz)$. **2.54.** $du|_M=(2dx+\ln 4dz)$.

2.55. $f_x'=2xf_u',\ f_y'=e^yf_u'$. **2.56.** $f_x'=\dfrac{3x^2+y^2}{3\sqrt[3]{(x^3+xy^2)^2}}f_u',\ f_y'=\dfrac{2xy}{3\sqrt[3]{(x^3+xy^2)^2}}f_u'$.

2.57. $f_x'=\dfrac{1}{1+(x+\ln y)^2}f_u',\ f_y'=\dfrac{1}{y(1+(x+\ln y)^2)}f_u'$. **2.58.** $d\varphi=\left(y-\dfrac{y^2}{x^2}\right)f_u'dx+\left(x+\dfrac{2y}{x}\right)f_u'dy$.

2.59. $d\varphi=\left(2xf_v'-\dfrac{y}{(x+y)^2}f_u'\right)dx+\left(\dfrac{x}{(x+y)^2}f_u'-3y^2f_v'\right)dy$.

2.60. $d\varphi=(2xf_u'+f_v'+yzf_w')dx+(2yf_u'+f_v'+xzf_w')dy+(2zf_u'+f_v'+yxf_w')dz$.

2.61. $\left.\dfrac{dw}{dt}\right|_{t=0}=-1$. **2.66.** $\left.\dfrac{\partial w}{\partial r}\right|_{(r,s)=(\pi,0)}=2;\ \left.\dfrac{\partial w}{\partial s}\right|_{(r,s)=(\pi,0)}=2-\pi$.

2.67. $\left.\dfrac{dw}{dt}\right|_{t=1}=-(\sin 1+\cos 2)\sin 1+(\cos 1+\cos 2)\cos 1+2(\sin 1+\cos 1)\sin 2$.

2.68. $\dfrac{\partial w}{\partial x}=\cos\sigma\dfrac{\partial w}{\partial r}-\dfrac{\sin\sigma}{r}\cdot\dfrac{\partial w}{\partial\sigma};\quad \dfrac{\partial w}{\partial y}=\sin\sigma\dfrac{\partial w}{\partial r}+\dfrac{\cos\sigma}{r}\dfrac{\partial w}{\partial\sigma}$.

2.69. $\dfrac{dW}{dt}=0,\ \left.\dfrac{dW}{dt}\right|_{t_0=\pi}=0$. **2.70.** $\dfrac{dW}{dt}=1,\ \left.\dfrac{dW}{dt}\right|_{t_0=3}=1$. **2.71.** $\dfrac{dW}{dt}=arctgt+1,\ \left.\dfrac{dW}{dt}\right|_{t_0=1}=\pi+1$.

2.72. $\left.\dfrac{\partial W}{\partial r}\right|_{(r,s)=(1;\,-1)}=12$. **2.73.** $\left.\dfrac{\partial W}{\partial u}\right|_{(u,v)=(0;\,0)}=-7$.

2.74. $\left.\dfrac{\partial W}{\partial u}\right|_{(u,v)=(\ln 2;1)}=2,\ \left.\dfrac{\partial W}{\partial v}\right|_{(u,v)=(\ln 2;1)}=1$. **2.84.** $f(x,y)=\dfrac{y}{2}+4;\ g(x,y)=\dfrac{x}{2}+\dfrac{9}{2}$.

2.85. $w=f(x,y)=x+y^2+e^x\cos y$. **2.90.** -18. **2.91.** $\frac{52}{5}$. **2.92.** $\frac{1}{5}$. **2.93.** $-\frac{\sqrt{2}}{2}$. **2.94.** $\frac{\sqrt{3}}{2}$. **2.95.** $\frac{2+\sqrt{2}}{6}$. **2.96.** $\vec{u}=-\frac{\sqrt{2}}{2}\vec{i}-\frac{\sqrt{2}}{2}\vec{j}$ yo'nalishda o'sadi, $-\vec{u}=-\frac{\sqrt{2}}{2}\vec{i}+\frac{\sqrt{2}}{2}\vec{j}$ yo'nalishda kamayadi. $f'\left(P_0;\vec{u}\right)=\frac{\sqrt{2}}{2}$; $f'\left(P_0;-\vec{u}\right)=-\frac{\sqrt{2}}{2}$; $f'\left(P_0;\vec{u}_1\right)=-\frac{7}{10}$, $\vec{u}_1=\frac{\vec{v}}{|\vec{v}|}$.

2.97. $\vec{u}=\frac{2}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{6}{7}\vec{k}$ yo'nalishda o'sadi; $-\vec{u}=-\frac{2}{7}\vec{i}-\frac{3}{7}\vec{j}-\frac{6}{7}\vec{k}$ yo'nalishda kamayadi; $f'\left(P_0;\vec{u}\right)=7$; $f'\left(P_0;-\vec{u}\right)=-7$; $f'\left(P_0;\vec{u}_1\right)=7$; $\vec{u}_1=\frac{\vec{v}}{|\vec{v}|}$. **2.98.** $grad\,u\,|_{M(x,y)}=2(x-y)\vec{i}+(3-2x)\vec{j}$ **2.99.** $grad\,u\,|_{M(x,y)}=\left(10xy-3y^3\right)\vec{i}+\left(4y^3-9xy^2\right)\vec{j}$ **2.100.** $grad\,u|_{(3;\,2)}=6\vec{i}+4\vec{j}$. **2.101.** $grad\,u|_{(2;\,1)}=\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}$. **2.102.** $grad\,u|_{(0;\,1)}=-\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}$. **2.102.** $\varphi=\pi$. **2.107.** 1,08. **2.108.** 10,05. **2.109.** 1,00. **2.110.** $-0{,}03$. **2.111.** 0,273. **2.112.** 3,037.

3- amaliy mashg'ulot.

KO'P O'ZGARUVCHILI FUNKSIYANING YUQORI TARTIBLI XUSUSIY HOSILALARI VA DIFFERENSIALLARI

3.1. Yuqori tartibli xususiy hosilalar. $u=f(x_1,x_2,...,x_m)=f(M)$ funksiya $\{M\}\subset R^m$ ochiq to'plamda berilgan bo'lib, uning har bir $M(x_1,x_2,...,x_m)$ nuqtasida $f'_{x_1},f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsin. Bu xususiy hosilalar, o'z navbatida, $x_1,x_2,...,x_m$ o'zgaruvchilarning funksiyasi sifatida, $\{M\}$ to'plamda aniqlangan bo'lsin. $\frac{\partial u}{\partial x_i}$ $(i=1,2,...,m)$ funksiya ham, biror $M\in\{M\}$ nuqtada x_k argument bo'yicha xususiy hosilaga ega bo'lishi mumkin. Bu x_k argument bo'yicha xususiy hosila-berilgan $u=f(x_1,x_2,...,x_m)$ funksiyaning *ikkinchi tartibli xususiy hosilasi* deyiladi va u $\frac{\partial^2 u}{\partial x_k\partial x_i}$, $f^{(2)}_{x_i x_k}$, $u^{(2)}_{x_i x_k}$ $(i=1,2,...,m;\ k=1,2,...,m)$ kabi belgilanadi, bunda $i\neq k$ bo'lsa, u holda $\frac{\partial^2 u}{\partial x_k\partial x_i}$ xususiy hosilaga, *aralash xususiy hosila* deyiladi, $k=i$ bo'lganda $\frac{\partial^2 u}{\partial x_k\partial x_k}=f''$ deb yozish o'rniga, $\frac{\partial^2 u}{\partial x_k^2}=f''_{x_k^2}$ kabi yoziladi. Xuddi shunday, $f(x_1,x_2,...,x_m)$ funksiyaning uchinchi, to'rtinchi, va xokazo, tartibli xususiy hosilalarining ta'rifi beriladi. $f(x_1,x_2,...,x_m)$ funksiya $x_{i_1},x_{i_2},...,x_{i_n}$ argumentlari bo'yicha $(n-1)$- tartibli xususiy hosilalarga ega bo'lsin. Bu $(n-1)$ tartibli xususiy hosilalar ham, $M(x_1,x_2,...,x_m)\in\{M\}$ nuqtada x_{i_n} argumenti bo'yicha xususiy hosilaga

ega bo'lsin. Bu hosila, $u=f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,x_{i_n}$ argumentlar bo'yicha M nuqtadagi *n-tartibli xususiy hosilasi* deyiladi. Shunday qilib, $x_{i_1},x_{i_2},...,x_{i_n}$ argumentlar bo'yicha n-tartibli xususiy hosilani,

$$\frac{\partial^n u}{\partial x_{i_n},...,\partial x_{i_2}\partial x_{i_1}}=\frac{\partial}{\partial x_{i_n}}\left(\frac{\partial^{n-1}u}{\partial x_{i_{n-1}}...\partial x_{i_2}\partial x_{i_1}}\right)$$

kabi yozish mumkin. Agar $i_1,i_2,...,i_n$ indekslarning hammasi birdaniga bir-biriga teng bo'lmasa, u holda $\frac{\partial^n u}{\partial x_{i_n}...\partial x_{i_2}\partial x_{i_1}}$ xususiy hosila *n-tartibli aralash xususiy hosila* deyiladi.

3.2. Yukori tartibli differensiallar. x va y erkli o'zgaruvchilarga bog'liq bo'lgan $u=f(x,y)$ funksiyaning ikkinchi va uchinchi tartibli to'liq differensiallarini, quyidagi

$$d^2u=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2u=\frac{\partial^2 u}{\partial x^2}dx^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+\frac{\partial^2 u}{\partial y^2}dy^2=$$

$$=u''_{x^2}dx^2+2u''_{xy}dxdy+u''_{y^2}dy^2, \qquad (22.3)$$

$$d^3u=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^3u=\frac{\partial^3 u}{\partial x^3}dx^3+3\frac{\partial^3 u}{\partial x^2\partial y}dx^2dy+3\frac{\partial^3 u}{\partial x\partial y^2}dxdy^2+\frac{\partial^3 u}{\partial y^3}dy^3=$$

$$=u'''_{x^3}dx^3+3u'''_{xxy}dx^2dy+3u'''_{xyy}dxdy^2+u'''_{y^3}dy^3.$$

ko'rinishlarda yozish mumkin.

$f(M)$ funksiyaning

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+...+\frac{\partial u}{\partial x_m}dx_m$$

differensialini, simvolik ravishda (u ni formal ravishda qavsdan tashkariga chiqarib), quyidagicha

$$du=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}dx_m\right)u$$

yozamiz. Unda funksiyaning ikkinchi tartibli differensial,

$$d^2u=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}dx_m\right)^2u$$

kabi yozilishi mumkin. Bunda, simvolik ravishda. qavs ichidagi yig'indi kvadratga ko'tarilib, so'ngra u ga «*ko'paytiriladi*», bunda daraja ko'rsatkichlari xususiy hosilalarning tartibi, deb qaraladi. Xuddi shunday simvolik ravishda funksiyaning n-tartibli differensiali

$$d^nu=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}dx_m\right)^nu$$

kabi yoziladi.

3.26. $u=\ln\left(x^x y^y z^z\right),\ d^4u=?$ **3.27.** $u=e^{ax+by},\ d^nu=?$

3.28. $u=X(y)\cdot Y(y),\ d^nu=?.$ **3.29.** $u=\sin x\, chy,\ \Delta u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}-?$

Mustaqil yechish uchun misollarning javoblari

3.1. $u'''_{yxx}=-2y\sin xy-xy^2\cos xy,\ u'''_{yyx}=-2x\sin xy-x^2y\cos xy.$ **3.2.** $\frac{\partial^8u}{\partial x^4\partial y^4}=24(\cos y+\cos x)$ **3.3.** $\frac{\partial^{10}u}{\partial x^4\partial y^6}=-2^6\sin x\cdot\cos 2y.$ **3.4.** $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}=m!n!.$

3.5. $\frac{\partial^{10}u}{\partial x\partial y^9}=10!\left(2xtgx+\frac{c^2+y}{\cos^2 x}\right)$ **3.6.** $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}=\frac{2(-1)^m(n+m-1)!(nx+my)}{(x-y)^{m+n+1}},$ $\frac{\partial^nu}{\partial y^n}=\frac{2xn!}{(x-y)^{n+1}}$ **.3.7.** $\frac{\partial^4u}{\partial x\partial y\partial\xi\partial\eta}=-\frac{6}{r^4}+\frac{48(x-\xi)^2(y-\eta)^2}{r^8},r=\sqrt{(x-\xi)^2+(y-\eta)^2}$

3.8. $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}=e^{x+y}[x^2+y^2+2(mx+ny)+m(m-1)+n(n-1)].$ **3.9.** $\sin\frac{n\pi}{2}.$

3.10. $\frac{\partial^2u}{\partial x^2}=0,\ \frac{\partial^2u}{\partial x\partial y}=1,\ \frac{\partial^2u}{\partial y^2}=2.$ **3.11.** $\frac{\partial^2u}{\partial x^2}=2,\ \frac{\partial^2u}{\partial x\partial y}=-2,\ \frac{\partial^2u}{\partial y^2}=0.$ **3.12.** $\frac{\partial^2u}{\partial x^2}=2,\ \frac{\partial^2u}{\partial x\partial y}=0,\ \frac{\partial^2u}{\partial y^2}=-1.$ **3.13.** $\frac{\partial^2u}{\partial x^2}=-\frac{\pi^3}{16},\ \frac{\partial^2u}{\partial x\partial y}=\frac{\pi^2}{8},\ \frac{\partial^2u}{\partial y^2}=-\frac{\pi}{4}.$ **3.14.** $\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial y^2}=\frac{1}{2},\frac{\partial^2u}{\partial x\partial y}=0.$ **3.15.** $\frac{\partial^2u}{\partial x^2}=0,\ \frac{\partial^2u}{\partial y^2}=2,\ \frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=-1$ **.3.16.** $\frac{\partial^2u}{\partial x^2}=-30,\ \frac{\partial^2u}{\partial y^2}=0,\ \frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=1.$ **3.17.** $f_{xy}(0,0)=-1;\ \ f_{yx}(0,0)=1.$

3.18. $d^2u=e^{-1}(dx^2+dy^2).$ **3.19.** $2dx\ d^2u=-2dxdy.$ **3.20.** $u=-2(dx^2-\pi dxdy).$ **3.21.** $d^2u=-dx^2+4dxdy-2dy^2.$**3.22.** $d^2u=-2\sqrt{3}\,dxdy+\ln^2 2\ dy^2.$**3.23.** $d^2u=e[dx+dy+dz]^2+2[dxdy+dydz+dzdx].$**3.24.** $d^3u=6(dx^3-3dx^2dy+3dxdy^2+dy^3).$**3.25.** $d^3u=-8(xdx+ydy)^3\cos(x^2+y^2)-12(xdx+ydy)(dx^2+dy^2)\sin(x^2+y^2).$

3.26. $d^4u=2\left(\frac{dx^4}{x^3}+\frac{dy^4}{y^3}+\frac{dz^4}{z^3}\right).$ **3.27.** $d^nu=e^{ax+by}(adx+bdy)^n.$ **3.28.** $d^nu=\sum_{k=0}^{n}C_n^kX^{(n-k)}(x)\,Y^{(k)}(y)dx^{n-k}dy^k.$ **3.29.** $\Delta u=0.$

4- amaliy mashg'ulot.

KO'P O'ZGARUVCHILI MURAKKAB VA OSHKORMAS FUNKSIYALARNI DIFFERENSIALLASH

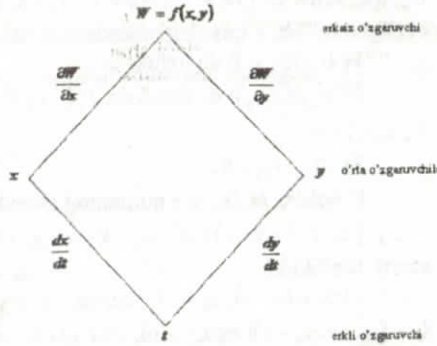
4.1. Ko'p o'zgaruvchili murakkab funksiyaning hosilasi. *Agar* $W=f(x,y)$ diferensiallanuvchi funksiya, x va y lar esa, t erkli o'zgaruvchining

diferensiallanuvchi funksiyalari bo'lsa, u holda, W *funksiya ham* t erkli o'zgaruvchining diferensiallanuvchi funksiyasi bo'ladi va

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}$$

formula o'rinli.

Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab, har bir yo'l bo'yicha pastga qarab harakat qilinib, yo'lda uchragan hosilalar ko'paytirilib, so'ngra ular qo'shiladi: $\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}.$



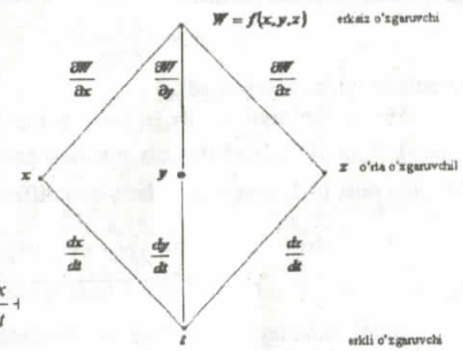
Uch o'zgaruvchili murakkab funksiya uchun zanjir qoidasi: Agar $W=f(x,y,z)$ diferensiallanuvchi funksiya, x, y va z lar esa, t erkli o'zgaruvchining diferensiallanuvchi funksiyalari bo'lsa, u holda, W funksiya ham t erkli o'zgaruvchining diferensiallanuvchi funksiyasi bo'ladi va

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}+\frac{\partial W}{\partial z}\frac{dz}{dt}$$

formula o'rinli.

Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab har bir yo'l bo'yicha pastga qarab harakat qilinib, yo'lda uchragan hosilalar ko'paytirilib, so'ngra ular qo'shiladi:

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+$$



Ikkita erkli o'zgaruvchi va uchta o'rta o'zgaruvchilar uchun zanjir qoidasi: Agar $W=f(x,y,z)$ $x=g(r,s)$, $y=h(r,s)$ va $z=k(r,s)$ diferensiallanuvchi funksiyalar bo'lsa, u holda, W funksiya ham r va s erkli o'zgaruvchilarga nisbatan xususiy hosilalarga ega bo'ladi va ular uchun,

$$\frac{\partial W}{\partial r}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial r},\quad \frac{\partial W}{\partial s}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial s}$$

formulalar o'rinli.

4.2. Oshkormas funksiyaning hosilasi.

1- teorema. $F(x,y)$ funksiya $M_0(x_0,y_0)\in R^2$ nuqtaning biror $U_{n,k}((x_0,y_0))=\{(x,y)\in R^2 : x_0-h<x<x_0+h,\ y_0-k<y<y_0+k\}\ (h>0,\ k>0)$ atrofida berilgan bo'lib, u quyidagi shartlarni qanoatlantirsin:

1) $U_{n,k}((x_0,y_0))$ da uzluksiz;

2) $U_{n,k}((x_0,y_0))$ da uzluksiz $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalarga ega va $F'_y(x_0,y_0)\neq 0$;

3) $F(x_0,y_0)=0$.

U holda, $M_0(x_0,y_0)$ nuqtaning shunday

$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2 : x_0-\delta<x<x_0+\delta,\ y_0-\varepsilon<y<y_0+\varepsilon\}\ (0<\delta<h,\ 0<\varepsilon<k)$

atrofi topiladiki:

a) $\forall x\in(x_0-\delta,\ x_0+\delta)$ uchun $F(x,y)=0$ tenglama yagona, y yechimga $(y\in(y_0-\varepsilon,\ y_0+\varepsilon))$ ega, ya'ni, $F(x,y)=0$ tenglama yordamida, $x\to y: F(x,y)=0$ oshkormas funksiya aniqlanadi;

b) $x=x_0$ bo'lganda, $y=y_0$ unga mos keldi;

c) $x\to y: F(x,y)=0$ oshkormas ko'rinishda aniqlangan funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz bo'ladi;

d) oshkormas ko'rinishdagi funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz hosilaga ega bo'ladi va uning hosilasi

$$y'_x=-\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)} \qquad (4.1)$$

formula bo'yicha hisoblanadi.

$F(x,y)$ funksiya, $U_{\delta,\varepsilon}((x_0,y_0))$ atrofda uzluksiz ikkinchi tartibli $F''_{x^2}(x,y)$, $F''_{xy}(x,y)$, $F''_{y^2}(x,y)$ hususiy hosilalarga ega bo'lsin. y ning x ga bog'liqligini e'tiborga olib, (4.1) tenglikni x bo'yicha differensiallab, quyidagini topamiz:

$$y''=\frac{2F'_xF'_yF''_{xy}-F'^2_yF''_{x^2}-F'^2_xF''_{y^2}}{(F'_y)^3}.$$

Xuddi shunday, oshkormas ko'rinishdagi funksiyaning uchinchi va hokazo tartibdagi hosilalari topiladi.

Mustaqil yechish uchun misollar

4.1. Ushbu $z=x^2+xy$, $x=1-t^2$, $y=t^4$ murakkab funksiyaning xususiy hosilasini toping.

4.2. Ushbu $z=e^{x-3y}$, $x=\sin t$, $y=t^2$ murakkab funksiyaning xususiy hosilasini toping.

4.3. Ushbu $z=e^xy^2$, $x=u^2-v^2$, $y=u\cdot v$ murakkab funksiyaning xususiy hosilasini toping.

4.4. Ushbu $z=x^2+y^2$, $x=u+v$, $y=u-v$ murakkab funksiyaning xususiy hosilasini toping.

Quyidagi misollarda: *a)* zanjir qoidasidan foydalanib; *b)* bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t=t_0$ nuqtadagi qiymatini toping.

4.5. $W=x^2+y^2$, $x=\cos t$, $y=\sin t$; $t_0=\pi$.

4.6. $W=\frac{x}{z}+\frac{y}{z}$, $x=\cos^2 t$, $y=\sin^2 t$, $z=\frac{1}{t}$; $t_0=3$.

4.7. $W=2ye^x-\ln z$, $x=\ln(t^2+1)$, $y=arctgt$, $z=e^t$; $t_0=1$.

4.8. Agar $W=(x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$, $z=\sin(r+s)$ bo'lsa, $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1;\,-1)}$ toping.

4.9. Agar $W=x^2+\frac{y}{x}$, $x=u-2v+1$, $y=2u+v-2$ bo'lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0;\,0)}$ ni toping.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning $\frac{dy}{dx}$ hosilalarini hisoblang.

4.10. $x^3y-y^3x-16=0$ **4.11.** $x^5+y^5-2xy-3=0$.

4.12. $x^2+y^2+\ln(x^2+y^2)-1=0$ **4.13.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$.

J4.10. $\frac{dy}{dx}=\frac{3x^2y-y^3}{3xy^2-x^3}$. **4.11.** $\frac{dy}{dx}=\frac{2y-5x^4}{5y^4-2x}$. **4.12.** $\frac{dy}{dx}=-\frac{x}{y}$. **4.13.** $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning berilgan A nuqtada f'_x, f'_y xususiy hosilalarini hisoblang.

4.15. $u^3-2u^2x+uxy-2=0$, $A(1;1)$. **4.14.** $u^3+3uxy+1=0$, $A(0;1)$.

Quyidagi oshkormas ko'rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping.

4.16. $x^3-2y^2+xy=0$, $(1;1)$. **4.17.** $x^2+xy+y^2-7=0$, $(1;2)$.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning birinchi va ikkinchi tartibli hosilalarini hisoblang.

4.18. $x^2-y^2-4=0$. **4.19.** $1+xy-\ln(e^{xy}+e^{-xy})=0$.

4.20. $2\cos(x-2y)-2y+x=0$. **4.21.** $x^3+y^3-3xy=0$.

Quyidagi oshkormas ko'rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping.

4.22. $z^3-xy+yz+y^3-2=0$, $(1;1;1)$.

4.23. $\sin(x+y)+\sin(y+z)+\sin(x+z)=0$, $(\pi;\pi;\pi)$.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning birinchi tartibli xususiy hosilalarini va to'liq differensiallarini hisoblang.

4.24. $x^2+y^2+z^2-6x=0$. **4.25.** $x^2+y^2+z^2-2xz=a^2$.

4.26. $z^2-xy=0$. **4.27.** $z^3+3x^2z-2xy=0$.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning birinchi va ikkichi tartibli to'liq differensiallarini hisoblang.

4.28. $x^2+y^2+z^2-2z=0$. **4.29.** $z^3-3xyz=a^3$. **4.30.** 3) $x-z\ln\frac{x}{y}=0$.

Mustaqil yechish uchun misollarning javoblari

4.1. $\frac{dz}{dt}=-6t^5+8t^3-4t$. **4.2.** $\frac{dz}{dt}=e^{\sin t-3t^2}(\cos t-6t)$.

4.3. $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=2uv^2(u^2+1)e^{u^2-v^2}$, $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=2e^{u^2-v^2}u^2v(1-v^2)$.

4.4. $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=4u$, $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=4v$.

4.4. $\frac{dW}{dt}=0$, $\left.\frac{dW}{dt}\right|_{t_0=\pi}=0$. **4.6.** $\frac{dW}{dt}=1$, $\left.\frac{dW}{dt}\right|_{t_0=3}=1$. **4.7.** $\frac{dW}{dt}=arctgt+1$, $\left.\frac{dW}{dt}\right|_{t_0=1}=\pi+1$. **4.8.** $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1;-1)}=12$. **4.9.** $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0;0)}=-7$. **4.10.** $\frac{dy}{dx}=\frac{3x^2y-y^3}{3xy^2-x^3}$. **4.11.** $\frac{dy}{dx}=\frac{2y-5x^4}{5y^4-2x}$. **4.12.** $\frac{dy}{dx}=-\frac{x}{y}$. **4.13.** $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$. **4.14.** $f_x'(1;1)=\frac{6}{5}$, $f_y'(1;1)=-\frac{2}{5}$. **4.15.** $f_x'(0;1)=1$, $f_y'(0;1)=0$.

4.16. $\frac{4}{3}$. **4.17.** $-\frac{4}{5}$. **4.18.** $y_x'=\frac{x}{y}$, $y_{x^2}''=\frac{y^2-x^2}{y^3}$. **4.19.** $y_x'=-\frac{x}{y}$, $y_{x^2}''=\frac{2y}{x^2}$. **4.20.** $y_x'=\frac{1}{2}$, $y_{x^2}''=0$. **4.21.** $y_x'=\frac{x^2-y}{x-y^2}$, $y_{x^2}''=\frac{(x^2-y)(y^2-x)+2x(y^2-x)^2+2y(x^2-y)^2}{(x-y^2)^3}$.

4.22. $\left.\frac{\partial z}{\partial x}\right|_{(1;1;1)}=\frac{1}{4}$, $\left.\frac{\partial z}{\partial y}\right|_{(1;1;1)}=-\frac{3}{4}$. **4.23.** $\left.\frac{\partial z}{\partial x}\right|_{(\pi;\pi;\pi)}=-1$, $\left.\frac{\partial z}{\partial y}\right|_{(\pi;\pi;\pi)}=-1$.

4.24. $\frac{\partial z}{\partial x}=\frac{3-x}{z}$, $\frac{\partial z}{\partial y}=\frac{-y}{z}$, $dz=\frac{1}{x}[(3-x)dx-ydy]$.

4.25. $\frac{\partial z}{\partial x}=1$, $\frac{\partial z}{\partial y}=\frac{y}{x-z}$, $dz=dx+\frac{y}{x-z}dy$. **4.26.** $\frac{\partial z}{\partial x}=\frac{y}{2z}$, $\frac{\partial z}{\partial y}=\frac{x}{2z}$, $dz=\frac{ydx+xdy}{2z}$.

4.27. $\frac{\partial z}{\partial x}=\frac{2y-6xz}{3(x^2+y^2)}$, $\frac{\partial z}{\partial y}=\frac{2x}{3(x^2+y^2)}$,
$dz=\frac{(2y-6xz)dx+2xdy}{3(x^2+y^2)}$. **4.28.** $dz=\frac{xdx+ydy}{1-z}$, $d^2z=\frac{1-z+x^2}{(1-z)^2}dx^2+$
$+\frac{2xy}{(1-z)^3}dxdy+\frac{1-z+y^2}{(1-z)^2}dy^2$. **4.29.** $dz=\frac{yzdx+xzdy}{z^2-xy}$,
$d^2z=\frac{-2xy^3z}{(z-xy)^3}dx^2+2\frac{z(z^4-2xyz^2-x^2y^2)}{(z-xy)^3}dxdy+\frac{2x^3yz}{(z-xy)^3}dy^2$.

4.30. $dz=\frac{yzdx+z^2dy}{y(x+z)}$, $d^2z=-\frac{z^2(ydx-xdy)}{(z-xy)^3}$.

5- amaliy mashg'ulot.

KO'P O'ZGARUVCHILI FUNKSIYANING EKSTREMUMLARI

$u=f(x,y)$ funksiya $M_0(x_0,y_0)$ nuqtaning biror $U_\delta(M_0)=\{(x,y)\in R^2:\rho(M,M_0)<\delta\}$ $(\delta>0)$ atrofida aniqlangan, u barcha birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo'lib, M_0 nuqta $u=f(M)$ funksiyaning stasionar nuqtasi, ya'ni

$$f_x'(M_0)=0,\ f_y'(M_0)=0$$

bo'lsin. $a_{11}=f_{x^2}''(M_0)$, $a_{12}=a_{21}=f_{xy}''(M_0)$, $a_{22}=f_{y^2}''(M_0)$ deb belgilaymiz.

1[0]. Agar $\begin{vmatrix}a_{11}a_{12}\\a_{21}a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}^2>0$ va $a_{11}>0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada minimumga erishadi.

2[0]. Agar $a_{11}a_{22}-a_{12}^2>0$ va $a_{11}<0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada maksimumga erishadi.

3[0]. Agar $a_{11}a_{22}-a_{12}^2<0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4[0]. Agar $a_{11}a_{22}-a_{12}^2=0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin.

1-teorema (shartsiz ekstremumning yetarli sharti). $f(M)$ funksiya $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqtaning biror atrofida ikkinchi tartibli uzluksiz xususiy hosilalarga ega va $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqta - $f(M)$ funksiyaning stasionar nuqtasi bo'lsin. U holda:

1) agar

$$B(dx_1,dx_2,...,dx_n)=\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f(M_0)}{\partial x_k\partial x_j}dx_k dx_j$$

kvadratik forma, ya'ni $f(M)$ funksiyaning $M_0(x_1^0, x_2^0, ..., x_n^0)$ nuqtadagi ikkinchi tartibli differensiali $B(dx_1, dx_2, ..., dx_n) = d^2 f(M_0)$ musbat (manfiy) aniqlangan bo'lsa, $M_0(x_1^0, x_2^0, ..., x_n^0)$ nuqta - $f(M)$ funksiyaning minimum (maksimum) nuqtasi bo'ladi.

2) agar $B(dx_1, dx_2, ..., dx_n)$ kvadratik forma aniqlanmagan bo'lsa (ham musbat, ham manfiy qiymatlar qabul qilsa), $M_0(x_1^0, x_2^0, ..., x_n^0)$ nuqta - $f(M)$ funksiyaning ekstremum nuqtasi bo'lmaydi.

1-misol. $z = x^3 - y^3 - 3xy$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiyadan x va y o'zgaruvchilar bo'yicha xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x} = 3x^2 - 3y, \quad \frac{\partial z}{\partial y} = -3y^2 - 3x.$$

So'ngra stasionar nuqtalarni aniqlaymiz:

$$\begin{cases} \dfrac{\partial z}{\partial x} = 0, \\ \dfrac{\partial z}{\partial y} = 0, \end{cases} \Rightarrow \begin{cases} x^2 - y = 0, \\ y^2 + x = 0. \end{cases}$$

Bu sistemaning yechimlari $(0, 0)$ va $(-1, 1)$ bo'ladi. Bu nuqtalarda x va y o'zgaruvchilar bo'yicha ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial^2 z(x,y)}{\partial x^2} = 6x, \quad \frac{\partial^2 z(0,0)}{\partial x^2} = 0, \quad \frac{\partial^2 z(-1;1)}{\partial x^2} = -6 < 0,$$

$$\frac{\partial^2 z(x,y)}{\partial y^2} = -6y, \quad \frac{\partial^2 z(0,0)}{\partial y^2} = 0, \quad \frac{\partial^2 z(-1;1)}{\partial y^2} = -6 < 0,$$

$$\frac{\partial^2 z(x,y)}{\partial x \partial y} = -3, \quad \frac{\partial^2 z(0,0)}{\partial x \partial y} = -3, \quad \frac{\partial^2 z(-1,1)}{\partial x \partial y} = -3.$$

Endi ekstremumning yetarli shartidan foydalanib, lokal ekstremum nuqtalarni topamiz. $(0, 0)$ nuqtada

$$\Delta(0;0) = \frac{\partial^2 z(0,0)}{\partial x^2}\frac{\partial^2 z(0,0)}{\partial y^2} - \left(\frac{\partial^2 z(0,0)}{\partial x \partial y}\right)^2 = 0 \cdot 0 - (-3)^2 = -3 < 0,$$

demak, bu nuqtada lokal ekstremum yo'q. $(-1, 1)$ nuqtada $\frac{\partial^2 z(-1;1)}{\partial x^2} = -6 < 0$ va

$$\Delta(-1;1) = \frac{\partial^2 z(-1;1)}{\partial x^2}\frac{\partial^2 z(-1,1)}{\partial y^2} - \left(\frac{\partial^2 z(-1,1)}{\partial x \partial y}\right)^2 = -6(-6) - (-3)^2 = 27 > 0.$$

Demak, ekstremumning yetarli shartiga asosan, $(-1, 1)$ nuqta - funksiyaning lokal maksimum nuqtasi bo'ladi. Bu $(-1, 1)$ nuqtadagi funksiyaning qiymati $z_{max} = z(-1, 1) = 1$.

Mustaqil yechish uchun misollar

Quyidagi ikki o'zgaruvchili funksiyalarni ekstremumga tekshiring:

5.1. $u = -x^2 - y^2$. **5.2.** $u = 2x^2 + xy + y^2$.

5.3. $u = x^3 - 3xy - 3y$. **5.4.** $u = x^4 + y^4 - 4xy$.

5.5. $u = -2x^2 + xy - 2y^2 + 6x + 6y$. **5.6.** $u = x^3 - 9xy + y^3$.

5.7. $u = x^3 + 6xy + 8y^3 - 1$. **5.8.** $u = xy(1 - x - y)$.

5.9. $u = x^2 - 3xy + y^2 - 4x + 5y + 6$. **5.10.** $u = x^2 + y^2 - 8x - 2$.

5.11. $u = x^2 + xy - y^2 - 3x - 6y$. **5.12.** $u = 3x^2 - x^3 + 3y^2 + 4y$.

5.13. $u = 3x^2 - y^2 + 4y + 5$. **5.14.** $u = x^2 + xy + 2y^2 - x + y$.

5.15. $u = -x^2 - xy - y^2 + 3x + 6y$. **5.16.** $u = (x + y^2)e^{x/2}$.

5.17. $u = x^3 - 3axy + y^3$. **5.18.** $u = x^2 + xy + y^2 - 4\ln x - 10\ln y$.

5.19. $u = \sin x + \sin y + \sin(x + y)$, bunda $0 \le x \le \frac{\pi}{2}$, $0 \le y \le \frac{\pi}{2}$.

5.20. $u = xe^{y + x\sin y}$. **5.21.** $u = \frac{8}{x} + \frac{x}{y} + y$.

5.22. $f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$.

5.23. $f(x, y) = 2x^3 + 3xy + 2y^3$. **5.24.** $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$.

Quyidagi uch o'zgaruvchili funksiyalarni ekstremumga tekshiring:

5.25. $u = x^2 + y^2 + z^2 - 4x + 6y - 2z$. **5.26.** $u = x^2 + y^2 + z^2 - xy + x - 2z$.

5.27. $u = x^2 + y^2 + (z + 1)^2 - xy + x$. **5.28.** $u = x^3 + y^2 + z^2 + 6xy - 4z$.

5.29. $u = xyz(16 - x - y - 2z)$. **5.30.** $u = \frac{226}{x} + \frac{x^2}{y} + \frac{y^2}{z} + z^2$.

5.31. $u = \frac{1}{z} + \frac{z}{y} + \frac{y}{x} + x + 1$. **5.32.** $u = x^{2/3} + y^{2/3} + z^{2/3}$.

Quyidagi ikki o'zgaruvchili funksiyalarni shartli ekstremumga tekshiring:

5.33. $u = xy$, $x + y - 2 = 0$. **5.34.** $u = x^2 + y^2$, $x + y - 1 = 0$.

5.35. $u = x^2 + y^2,\ 3x + 4y - 12 = 0.$ **5.36.** $u = xy,\ 2x + 3y - 5 = 0.$

5.37. $u = xy^2,\ x + 2y - 1 = 0.$ **5.38.** $u = x^2 + y^2 - xy + x + y - 4,\ x + y + 3 = 0.$

5.39. $u = \cos^2 x + \cos^2 y,\ x - y - \frac{\pi}{4} = 0.$ **5.40.** $u = 5 - 3x - 4y,\ x^2 + y^2 = 25.$

5.41. $u = 1 - 4x - 8y,\ x^2 - 8y^2 = 8.$ **5.42.** $u = x^2 + xy + y^2,\ x^2 + y^2 = 1.$

Quyidagi uch o'zgaruvchili funksiyalarni shartli ekstremumga tekshiring.

5.43. $u = 2x^2 + 3y^2 4z^2,\ x + y + z - 13 = 0.$

5.44. $u = xy^2z^3,\ x + y + z - 12 = 0,\ x > 0,\ y > 0,\ z > 0.$

5.45. $u = x - 2y + 2z,\ x^2 + y^2 + z^2 - 9 = 0.$

5.46. $u = xy + 2xz + 2yz,\ xyz = 108.$

5.47. $u = x^2 + y^2 + z^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ a > 0, b > 0,\ c > 0.$

5.48. $f(x,y) = x^3 + y^2$ funksiyaning $x^2 + y^2 = 1$ aylanadagi ekstremum qiymatlarini toping.

5.49. $f(x,y) = x^2 + 3y^2 + 2y$ funksiyaning $x^2 + y^2 \le 1$ doiradagi ekstremum qiymatlarini toping.

5.50. $f(x,y,z) = x - y + z$ funksiyaning $x^2 + y^2 + z^2 = 1$ birlik sferadagi ekstremum qiymatlarini toping.

5.51. $f(x,y,z) = x(y + z)$ funksiyaning $x^2 + y^2 = 1$ to'g'ri doiraviy konus va $xz = 1$ giperbolik silindrlarning kesishish chizig'idagi ekstremum qiymatlarini toping.

Quyidagi funksiyalarning ko'rsatilgan D to'plamda eng katta va eng kichik qiymatlarini toping.

5.52. $u = x^3 - 3xy + y^3,\ D = \{(x,y) \in R^2 : 0 \le x \le 2\ \ -1 \le y \le 2\}$

5.53. $u = x - 2y + 5,\ D = \{(x,y) \in R^2 : x \ge 0,\ y \ge 0,\ x + y \le 1\}$

5.54. $u = x^2 - 4x - y^2,\ D = \{(x,y) \in R^2 : x^2 + y^2 \le 9\}$

5.55. $u = xy(4 - x - y),\ D = \{(x,y) \in R^2 : x \ge 0,\ y \ge 0,\ x + y \le 8\}$

5.56. $u = x^3 - 9xy + y^3 + 27,\ D = \{(x,y) \in R^2 : 0 \le x \le 4\ \ 0 \le y \le 4\}$

5.57. $u = \sin x + \sin y + \sin(x + y),\ D = \{(x,y) \in R^2 : 0 \le x \le \pi/2\ \ 0 \le y \le \pi/2\}$

Quyidagi funksiyalarning ko'rsatilgan D to'plamda eng katta va eng kichik qiymatlarini toping:

5.58. $u = xy + x + y,\ D = \{(x,y) \in R^2 : -2 \le x \le 2\ \ -2 \le y \le 2\}$

5.59. $u = x^3 - 6xy + 8y^3 + 1,\ D = \{(x,y) \in R^2 : 0 \le x \le 2\ \ -1 \le y \le 1\}$

5.60. $u = 3 + 2xy,\ D = \{(x,y) \in R^2 : -4 \le x^2 + y^2 \le 9\}$

5.61. $u = x^4 - y^4,\ D = \{(x,y) \in R^2 : x^2 + y^2 \le 9\}$

5.62. $u = x^2 + y^2,\ D = \{(x,y) \in R^2 : (x - \sqrt{2})^2 + (y - \sqrt{2})^2 \le 9\}$

5.63. $u = \cos x \cos y \cos(x + y),\ D = \{(x,y) \in R^2 : 0 \le x \le \pi\ \ 0 \le y \le \pi\}$

Berilgan funksiyalarning berilgan R sohada maksimum va minimum qiymatlarini toping.

5.64. $f(x,y) = x^2 + xy + y^2 - 3x + 3y$, R: birinchi kvadrantda $x + y = 4$ to'g'ri chiziq bilan kesilgan uchburchakli soha.

5.65. $f(x,y) = y^2 - xy - 3y + 2x$, R: $x = \pm 2$ va $y = \pm 2$ to'g'ri chiziqlar bilan chegaralangan kvadratik soha.

5.66. $f(x,y) = x^2 - y^2 - 2x + 4y$, R: pastdan Ox o'q, yuqoridan $y = x + 2$ to'g'ri chiziq va o'ngdan $x = 2$ to'g'ri chiziq bilan chegaralangan uchburchakli soha.

5.67. $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$, R: $x = \pm 1$ va $y = \pm 1$ to'g'ri chiziqlar bilan chegaralangan kvadratik soha.

Mustaqil yechish uchun misollarning javoblari

5.1. $u_{max} = u(0,0) = 0.$ **5.2.** $u_{min} = u(0,0) = 0.$ **5.3.** $(-1;\ 1)$ stasionar nuqtada ekstremum yo'q. **5.4.** $u_{min} = u(1;\ 1) = u(-1;\ -1) = -2$, $O(0;\ 0)$ stasionar nuqtada ekstremum yo'q. **5.5.** $u_{max} = u(2;\ 2) = 12.$ **5.6.** $u_{min} = u(3;3) = -27$, $O(0;\ 0)$ stasionar nuqtada ekstremum yo'q. **5.7.** $u_{max} = u(-1;\ -0{,}5) = 0.$ **5.8.** $u_{max} = u\left(\frac{1}{3};\ \frac{1}{3}\right) = \frac{1}{27}.$

5.9. $\left(\frac{7}{5};\ -\frac{2}{5}\right)$ stasionar nuqtada ekstremum yo'q. **5.10.** $u_{min} = u(4;\ 0) = -18.$

5.11. Ekstremum yo'q. **5.12.** $u_{min} = u\left(0;\ -\frac{2}{3}\right) = -\frac{4}{3}$, $\left(2;\ -\frac{2}{3}\right)$ stasionar nuqtada ekstremum yo'q. **5.13.** Ekstremum yo'q. **5.14.** $u_{min} = u(1;-1) = 0.$ **5.15.** $u_{max} = u(0;\ 3) = 9.$ **5.16.** $u_{min} = u(-2;\ 0) = -\frac{2}{e}.$ **5.17.** $a < 0$ *da* $u_{max} = u(a;a) = -a^3$, .

$a \ge 0$ da $u_{\min} = u(a;a) = -a^3$. **5.18.** $u_{\min} = u(1;\ 2) = 7 - 10\ln 2$. **5.19.**

$u_{\max} = u(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{3}{2}\sqrt{3}$. **5.20.** Ekstremum yo'q. **5.21.** $u = u_{\min}(4;\ 2) = 6$. **5.22.**

$f_{\min} = f(-2,-2) = -8$. **5.23.** $f_{\max} = f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$. **5.24.** $f_{\min} = f(0,2) = -4$,

$f_{\max} = f(-2,0) = 4$.**5.25.** $u_{\min} = u(2;-3;\ 1) = -14$. **5.26.** $u_{\min} = u\left(-\frac{2}{3}; -\frac{1}{3};\ 1\right) = -\frac{4}{3}$.

5.27. $u_{\min} = u\left(-\frac{2}{3}; -\frac{1}{3};\ -1\right) = -\frac{1}{3}$. **5.28.** $u_{\min} = u(6;-18;\ 2) = -112$. **5.29.**

$u_{\max} = u(4;\ 4;\ 2) = 128$. **5.30.** $u_{\min} = u(8;\ 4;\ 2) = 60$. **5.31.**

$u_{\min} = u(1;\ 1;\ 1) = 5$, $u_{\max} = u(-1;\ 1;-1) = -3$. **5.32.** $u_{\min} = u(0;\ 0;\ 0) = 0$. **5.33.** $u_{\max} = u(1;\ 1) = 1$.

5.34. $u_{\min} = u(0{,}5;\ 0{,}5) = 0{,}5$. **5.35.** $u_{\min} = u\left(\frac{36}{25}; \frac{48}{25}\right) = \frac{144}{25}$.

5.36. $u_{\max} = u\left(\frac{5}{4}; \frac{5}{6}\right) = \frac{25}{24}$. **5.37.** $u_{\max} = u(1;\ 0) = 0$ $u_{\min} = u\left(\frac{1}{3}; \frac{1}{3}\right) = \frac{1}{27}$.

5.38. $u_{\min} = u\left(-\frac{3}{2}; -\frac{3}{2}\right) = -\frac{19}{4}$. **5.39.**

$u_{\min} = u\left(\frac{5\pi}{8} + \pi k;\ \frac{3\pi}{8} + \pi k\right) = 1 - \frac{\sqrt{2}}{2}$, $u_{\max} = u\left(\frac{\pi}{8} + \pi k;\ -\frac{\pi}{8} + \pi k\right) = 1 + \frac{\sqrt{2}}{2}$, $k \in Z$.

5.40. $u = u_{\min}(3;4) = -20$, $u = u_{\max}(-3;\ -4) = 30$. **5.41.**

$u = u_{\min}(-4;\ 1) = 9$, $u = u_{\max}(4;\ -1) = -7$. **5.42.** $u = u_{\min}\left(\pm\frac{\sqrt{2}}{2}; \mp\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$,

$u = u_{\max}\left(\pm\frac{\sqrt{2}}{2}; \pm\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$. **5.43.** $u_{\min} = u(6;\ 4;\ 3) = 156$. **5.44.** $u_{\max} = u(2;\ 4;\ 6) = 6912$. **5.45.**

$u_{\max} = u(1;\ -2;2) = 9$, $u_{\min} = u(-1;\ 2;-2) = -9$. **5.46.** $u_{\min} = u(6;\ 6;\ 3) = 108$.

5.47. $u_{\max} = u(\pm a;\ 0;\ 0) = a^2$, $u_{\min} = u(0;\ 0;\ \pm c) = c^2$. **5.48.** $f_{\max} = f(0,\pm 1) = f(1,0) = 1$, $f_{\min} =$ $f(-1,0) = -1$. **5.49.** $f_{\max} = f(0,1) = 5$, $f_{\min} = f(0,-1/3) = -\frac{1}{3}$. **5.50.**

$f_{\max} = f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \sqrt{3}$, $f_{\min} = f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\sqrt{3}$. **5.51.** $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ va $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$- maksimum nuqtalari, ularda funksiya $\frac{3}{2}$ qiymat qabul qilidi; $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ va $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$- minimum nuqtalari, ularda funksiya $\frac{1}{2}$ qiymat qabul qiladi. **5.52.** $u_{e.\,kich.} = u(1;\ 1) = u(0;\ -1) = -1$,

$u_{e.\,kat.} = u(2;-1) = 13$. **5.53.** $u_{e.\,kat.} = u(1;\ 0) = 6$. **5.54.** $u_{e.\,kich.} = u(1;\ -2\sqrt{2}) = u(1;\ 2\sqrt{2}) = -11$,

$u_{e.\,kat} = u(-3;\ 0) = 21$. **5.55.** $u_{e.\,kich.} = u(4;\ 4) = -64$, $u_{e.\,kat.} = u\left(\frac{4}{3}; \frac{4}{3}\right) = \frac{64}{27}$. **5.56.**

$u_{e.\,kich.} = u(3;\ 3) = 0$, $u_{e.\,kat.} = u(4;\ 0) = u(0;\ 4) = 91$. **5.57.** $u_{e.\,kat.} = u(\pi/3;\ \pi/3) = \frac{3}{2}\sqrt{3}$. **5.58.**

$u_{e.\,kich.} = -6$, $u_{e.\,kat.} = 14$. **5.59.** $u_{e.\,kich.} = -7$, $u_{e.\,kat.} = 9 + 4\sqrt{2}$. **5.60.** $u_{e.\,kich.} = -6$, $u_{e.\,kat.} = 12$. **5.61.**

$u_{e.\,kich.} = -81$, $u_{e.\,kat} = 81$. **5.62.** $u_{e.\,kich.} = 0$, $u_{e.\,kat.} = 25$. **5.63.** $u_{e.\,kich.} = -\frac{1}{8}$, $u_{e.\,kat.} = 1$. **5.64.**

$f_{\max} = f(0{,}4) = 28$; $f_{\min} = f\left(\frac{3}{2}, 0\right) = -\frac{9}{4}$. **5.65.** $f_{\max} = f(2,-2)$, $f_{\min} = f\left(-2; \frac{1}{2}\right) = -\frac{17}{4}$.

5.66. $f_{\max} = f(-2{,}0) = 8$, $f_{\min} = f(1{,}0) = -1$.

5.67. $f_{\max} = f(1{,}0) = 4$, $f_{\min} = f(0,-1) = -4$.

6- amaliy mashg'ulot.

SONLI QATORLAR. MUSBAT XADLI QATORLAR

6.1. Yaqinlashuvchi qatorlar va ularning yig'indisi. Ushbu

$$a_1, a_2, \ldots, a_n, \ldots$$

sonlar ketma-ketligi berilgan bo'lsin.

1-ta'rif. Quyidagi

$$a_1 + a_2 + \ldots + a_n + \ldots$$

ifodaga sonli qator yoki cheksiz sonli qator deyiladi. U qisqacha $\sum_{n=1}^{\infty} a_n$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots, \qquad (6.1)$$

bunda $a_1, a_2, \ldots, a_n, \ldots$ lar qatorning hadlari, a_n esa, qatorning umumiy hadi deyiladi. (6.1) sonli qatorning hadlaridan ushbu

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$S_3 = a_1 + a_2 + a_3,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$S_n = a_1 + a_2 + \ldots + a_n,$$
$$\ldots\ldots\ldots\ldots\ldots$$

yig'indilar ketma-ketligini tuzamiz. Bunday tuzilgan $\{S_n\}$ yig'indilar ketma-ketligi (6.1) sonli qatorning qismiy yig'indilar ketma-ketligi deyiladi. Bundan keyin sonli qator deyish o'rniga qator deymiz.

2-ta'rif. Agar $n \to \infty$ da (6.1) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligi chekli limitga ega, ya'ni

$$\lim_{n\to\infty} S_n = S$$

bo'lsa, u holda (6.1) qator yaqinlashuvchi deyiladi. Bu limitning qiymati S son esa, (6.1) qatorning yig'indisi deyiladi va u quyidagicha yoziladi:

$$S = a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n .$$

3-ta'rif. Agar $n \to \infty$ da (6.1) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligining limiti cheksiz bo'lsa yoki mavjud bo'lmasa, (6.1) qator uzoqlashuvchi deyiladi.

1-teorema. Agar (6.1) qator yaqinlashuvchi bo'lsa,

$$\lim_{n\to\infty} a_n = 0 \qquad (A)$$

bo'ladi.

Esdatma.(A) shart qator yaqinlashuvchi bo'lishi uchun zaruriy shart bo'ladi, lekin yetarli shart bo'lmaydi. Agar qatorning umumiy hadi nolga intilmasa, ya'ni $\lim_{n\to\infty} a_n \neq 0$ bo'lsa, (6.1) qator uzoqlashuvchi bo'ladi.

Koshi kriteriysi. (6.1) qator yaqinlashuvchi bo'lishi uchun istalgan musbat $\varepsilon > 0$ son olinganda ham shunday $n_0(\varepsilon) \in \mathrm{N}$ mavjud bo'lib, barcha $n > n_0(\varepsilon)$ va $p \in \mathrm{N}$ lar uchun

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| < \varepsilon \qquad (V)$$

tengsizlikning bajarilishi zarur va yetarli.

Eslatma. (V) shart bajarilmasa, ya'ni $\exists \varepsilon_0 > 0 : \forall k \in N \;\; \exists n \geq k \;\; \exists p \in N$:

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| \geq \varepsilon_0$$

tengsizlik o'rinli bo'lsa, (6.1) qator uzoqlashuvchi bo'ladi.

6.2. Musbat qatorlarning yaqinlashuvchi bo'lishlik sharti. Biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (6.1)$$

qator berilgan bo'lsin.

Agar $a_n \geq 0$, $(n = 1,2,...)$ bo'lsa, (6.1) qator musbat hadli qator yoki qisqacha musbat qator deb ataladi.

1-teorema. (6.1) musbat qator yaqinlashuvchi bo'lishi uchun uning qismiy yiђindilar ketma-ketligining yuqoridan chegaralangan bo'lishi zarur va yetarlidir.

1-natija. Musbat hadli qatorning qismiy yiђindilari ketma-ketligi yuqoridan chegaralanmagan bo'lsa, qator uzoqlashuvchi bo'ladi.

Ikkita

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ..., \qquad (6.1)$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + ... + b_n + ... \qquad (6.2)$$

musbat qatorlar berilgan bo'lsin.

2-teorema. Agar n ning biror $n_0 (n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun $a_n \leq b_n$ tengsizlik o'rinli bo'lsa, (6.2) qatorning yaqinlashuvchi bo'lishidan (6.1) qatorning ham yaqinlashuvchi bo'lishi yoki (6.1) qatorning uzoqlashuvchi bo'lishidan (6.2) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

3-teorema. Agar $n \to \infty$ $\frac{a_n}{b_n} (a_n \geq 0, b_n > 0)$ nisbat ushbu

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k \quad (0 \leq k \leq +\infty)$$

limitga ega bo'lsa, u holda:

a) $k < +\infty$ bo'lganda (6.2) qatorning yaqinlashuvchi bo'lishidan (6.1) qatorning yaqinlashuvchi bo'lishi;

b) $k > 0$ bo'lganda (6.2) qatorning uzoqlashuvchi bo'lishidan (6.1) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

2-natija. Agar ushbu $\lim_{n\to\infty} \frac{a_n}{b_n} = k$ limit o'rinli bo'lib, $0 < k < \infty$ bo'lsa, (6.1) va (6.2) qatorlar bir vaqtda yaqinlashuvchi yoki uzoqlashuvchi bo'ladi.

3-natija. Agar $n \to \infty$ da $a_n \sim b_n$ bo'lsa, (6.1) va (6.2) qatorlar bir vaqtda yaqinlashuvchi, yoki uzoqlashuvchi bo'ladi.

4-teorema. Agar n ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \quad (a_n > b_n, b_n > 0)$$

tengsizlik o'rinli bo'lsa, u holda (6.1) qatorning yaqinlashuvchi bo'lishidan (6.2) qatorning ham yaqinlashuvchi bo'lishi yoki (6.1) qatorning uzoqlashuvchi bo'lishidan (6.2) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

Dalamber alomatining limit ko'rinishi. Agar (6.1) qator uchun

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda \qquad (6.3)$$

mavjud bo'lib, $\lambda < 1$ bo'lsa, (6.1) qator yaqinlashuvchi, $\lambda > 1$ bo'lganda esa, qator uzoqlashuvchi bo'ladi.

Koshi alomatining limit ko'rinishi. Agar (6.1) qator uchun

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lambda \qquad (6.4)$$

limit mavjud bo'lib, $\lambda < 1$ bo'lsa, (6.1) qator yaqinlashuvchi, $\lambda > 1$ bo'lganda esa, uzoqlashuvchi bo'ladi.

5-teorema (Umumlashgan Koshi alomati). Agar $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = q$, $a_n \geq 0$, $(n = 1,2,...)$ bo'lsa, u holda: *a*) $q < 1$ bo'lganda $\sum_{n=1}^{\infty} a_n$ qator yaqinlashadi; *b*) $q > 1$ bo'lganda esa, $\sum_{n=1}^{\infty} a_n$ uzoqlashadi.

Koshining integral alomati. Agar $f(x)$, $[k;+\infty)$ ($k \in N$ - biror son) da aniqlangan, uzluksiz, o'smaydigan va manfiy bo'lmagan funksiya bo'lib, $F(x) = \int_k^x f(t)dt$ funksiya $f(x)$ funksiya uchun boshlanђich funksiya va $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ bo'lsa, $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_k^x f(t)dt$ mavjud va chekli bo'lganda (6.1) qator yaqinlashuvchi, bu limit mavjud bo'lmaganda yoki cheksiz bo'lganda (6.1) qator uzoqlashuvchi bo'ladi.

1-misol. Ushbu $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$ qatorning yig'indisini toping.

Yechilishi. 1) Berilgan qatorning S_n – qismiy yig'indisini tuzamiz va uni hisoblaymiz.

$$S_n=\left(\frac{5}{1}+\frac{1}{1}\right)+\left(\frac{5}{2}+\frac{1}{3}\right)+\left(\frac{5}{4}+\frac{1}{9}\right)+....+\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=$$

$$=\left(5+\frac{5}{2}+\frac{5}{2^2}+...+\frac{5}{2^n}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}+....+\frac{1}{3^n}\right)=$$

$$=5\left(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}\right)+\left(1+\frac{1}{3}+\frac{1}{2^2}+....+\frac{1}{3^n}\right)=$$

$$=5\cdot\frac{1-\frac{1}{2n+1}}{1-\frac{1}{2}}+\frac{1-\frac{1}{3n+1}}{1-\frac{1}{3}}=10+\frac{3}{2}-\frac{1}{2^n}-\frac{1}{2\cdot 3^n}=\frac{23}{2}-\left(\frac{1}{2n+n}+\frac{1}{2\cdot 3^n}\right)$$

Shunday qilib, $S_n=\frac{23}{2}-\frac{1}{2}\left(\frac{1}{2^{n-1}}+\frac{1}{3^n}\right)$.

2) Qator yaqinlashishining ta'rifigi ko'ra, ularning yaqinlashishini isbotlaymiz:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}[\frac{23}{2}-\frac{1}{2}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)]=\frac{23}{2}-\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)=\frac{23}{2}.$$

Demak, $S=\frac{23}{2}$ chekli bo'lgani uchun berilgan qator yaqinlashuvchi.

3) $S=\frac{23}{2}$ berilgan qatorning yig'indisi bo'ladi.

2-misol. $\sum_{n=1}^{\infty}\frac{3+(-1)^n}{2^{n+2}}$. Quyidagi musbat sonli qatorlarni taqqoslash teoremalari yordamida yaqinlashishga tekshiring.

Yechilishi. Ravshanki $2\le 3+(-1)^n\le 4.$ $0<a_n=\frac{3+(-1)^n}{2^{n+2}}\le\frac{1}{2^n}=b_n.$

Ma'lumki, $\sum_{n=1}^{\infty}\frac{1}{2^n}\left(q=\frac{1}{2^n}<1\right)$ geometrik qator yaqinlashuvchi bo'lgani uchun 6.3-teoremaga ko'ra, berilgan qator yaqinlashuvchi.

3-misol. $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5...(2n-1)}{3^n\cdot(n+1)}$ qatorni Dalamber alomatidan foydalanib yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning umumiy hadiga ko'ra,

$$\frac{a_n+1}{a_n}=\frac{1\cdot 3\cdot 5...(2n-1)\cdot(2n+1)}{3^{n+1}\cdot 1\cdot 2\cdot 3...n\cdot(n+1)(n+2)}\cdot\frac{3^n\cdot(n+1)}{1\cdot 3\cdot 5...(2n-1)}$$

iisbatni tuzib, uning $n\to\infty$ dagi limitini topamiz:

$$\lim_{n\to\infty}\frac{a_n+1}{a_n}=\lim_{n\to\infty}\frac{(2n+1)}{3\cdot(n+2)}=\frac{2}{3}<1\cdot$$

Demak, berilgan qator Dalamber alomatiga ko'ra yaqinlashuvchi.

Mustaqil yechish uchun misollar

Quyidagi qatorlarning yaqinlashuvchiligini ko'rsating va yig'indisini toping:

6.1. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}+...$.

6.2. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+...+\frac{1}{(3n-2)\cdot(3n+1)}+...$.

6.3. $\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+...+\frac{1}{n(n+1)(n+2)(n+3)}+...$.

6.4. $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+...+\frac{2n+1}{n^2\cdot(n+1)^2}+...$.

6.5. $\left(\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}\right)+\left(\frac{1}{10^2}+\frac{2}{10^3}+\frac{5}{10^4}\right)+...+\left(\frac{1}{10^n}+\frac{2}{10^{n+1}}+\frac{5}{10^{n+2}}\right)+...$.

6.6. $\frac{1}{1\cdot(1+m)}+\frac{1}{2\cdot(2+m)}+...+\frac{1}{n\cdot(n+m)}+...$ $(m\in N)$.

Quyidagi qatorlar uchun qator yaqinlashuvchiligining zaruriy sharti bajarilmasligini ko'rsating:

6.7. $\sum_{n=1}^{\infty}\left(\frac{3n^3-2}{3n^3+4}\right)^3$. **6.8.** $\sum_{n=1}^{\infty}(n^2+2)\ln\frac{n^2+1}{n^2}$. **6.9.** $\sum_{n=2}^{\infty}\frac{1}{\sqrt[n]{\ln n}}$.

6.10. $\sum_{n=1}^{\infty}\sin n\alpha$, bunda $\alpha\ne\pi m,\ m\in Z$. **6.11.** $\sum_{n=1}^{\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$. **6.12.** $\sum_{n=1}^{\infty}\sqrt[n]{0{,}002}$.

Quyidagi qatorlarning S_n qismiy yig'indilari ketma-ketligini va S yig'indisini toping:

6.13. $\sum_{n=1}^{\infty}\left(\frac{3}{2^{n-1}}+\frac{(-1)^{n-1}}{2\cdot 3^{n-1}}\right)$. **6.14.** $\sum_{n=1}^{\infty}\frac{1}{5}\left(\frac{1}{5n-2}-\frac{1}{5n+3}\right)$.

6.15. $\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$. **6.16.** $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

6.17. $\sum_{n=1}^{\infty}\frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}}$. **6.18.** $\sum_{n=1}^{\infty}\frac{1}{(4n+5)(4n+9)}$.

Koshi kriteriysidan foydalanib, quyidagi qatorlarning yaqinlashuvchiligini ko'rsating:

6.19. $a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}+\ldots(|a_n|<10)$. **6.20.** $\frac{\sin x}{3}+\frac{\sin 2x}{3^2}+\ldots+\frac{\sin nx}{3^n}+\ldots$.

6.21. $\frac{\cos x-\cos 2x}{1}+\frac{\cos 2x-\cos 3x}{2}+\ldots+\frac{\cos nx-\cos(n+1)x}{n}+\ldots$.

6.22. $\frac{\cos x}{1^2}+\frac{\cos x^2}{2^2}+\ldots+\frac{\cos x^n}{n^2}+\ldots$. **6.23.** $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$.

Koshi kriteriysidan foydalanib, quyidagi qatorlarning uzoqlashuvchiligini ko'rsating:

6.24. $1+\frac{1}{2}+\ldots+\frac{1}{n}+\ldots$. 6.25. $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$.

6.26. $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n+1}+\ldots$ . .27. $\frac{2}{5}+\frac{3}{8}+\frac{4}{13}+\ldots+\frac{n+1}{n^2+4}+\ldots$.

Taqqoslash teoremalaridan foydalanib, qatorlarni yaqinlashishga tekshiring:

6.19. $\sum_{n=1}^{\infty}\frac{5+3(-1)^{n+1}}{3^n}$. 6.20. $\sum_{n=1}^{\infty}\frac{\sin^4 3n}{n\sqrt{n}}$. 6.21. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$.

6.22. $\sum_{n=1}^{\infty}\frac{arctgn}{n^2+1}$. 6.23. $\sum_{n=1}^{\infty}\frac{\cos\frac{\pi}{4n}}{\sqrt[5]{2n^5-1}}$. 6.24. $\sum_{n=1}^{\infty}\frac{n^{n-1}}{(2n^2+n+1)^{\frac{n+1}{2}}}$.

6.25. $\sum_{n=1}^{\infty}\frac{n^5}{2^n+3^n}$. 6.26. $\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^{n+1}}$.

6.27. $\sum_{n=1}^{\infty}\frac{1}{n^2-4n+5}$. 6.28. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}$.

Dalamber alomatidan foydalanib, quyidagi qatorlarni yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}a_n$.

6.29. $a_n=\frac{n^{12}}{(n+2)!}$. 6.30. $a_n=\frac{n^4}{4^n}$. 6.31. $a_n=\frac{n!a^n}{n^n}$, $a\neq e$, $a>0$.

6.32. $a_n=\frac{3\cdot 6\ldots(3n)}{(n+1)!}\arcsin\frac{1}{2^n}$. 6.33. $a_n=\frac{(2n)!}{(n!)^2}$. 6.34. $a_n=\frac{n!(2n+1)!}{(3n)!}$.

6.35. $a_n=\frac{2\cdot 5\ldots(3n+2)}{2^n\cdot(n+1)!}$. 6.36. $a_n=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 5\cdot 8\ldots(3n-1)}$.

6.37. $a_n=\frac{n!}{2^n+1}$. 6.38. $a_n=\frac{\ln^{101}n}{4!}$, $n\geq 2$.

Koshi alomatidan foydalanib, quyidagi qatorlarni yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}a_n$.

6.39. $a_n=\frac{1}{(\ln n)^n}$, $n\geq 2$. 6.40. $a_n=\left(\frac{4}{n}\right)^n$.

6.41. $a_n=\left(\frac{n^2+5}{n^2+6}\right)^{n^3}$. 6.42. $a_n=3^{n+1}\left(\frac{n}{n+1}\right)^{n^2}$.

6.43. $a_n=\left(\frac{6n+4}{5n-3}\right)^{\frac{n}{2}}\left(\frac{5}{6}\right)^{\frac{2n}{3}}$. 6.44. $a_n=\frac{n^{\alpha}}{(\ln(n+1))^{n/2}}$.

6.45. $a_n=\left(\frac{2n-1}{2n+1}\right)^{n(n-1)}$. 6.46. $a_n=\frac{3^n}{n(\sqrt{2})^n}$.

6.47. $a_n=\left(\frac{n}{3n-1}\right)^{2n-1}$. 6.48. $a_n=\frac{2^{n-1}}{n^n}$.

Koshining integral alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring:

6.49. $\sum_{n=1}^{\infty}\frac{1}{n(1+\ln n)}$. 6.50. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{6n+5}}$. 6.51. $\sum_{n=1}^{\infty}\frac{5}{4+n^2}$.

$6.52.\ \sum_{n=1}^{\infty}\left(\frac{1+n}{1+n^2}\right)^2.$ $6.53.\ \sum_{n=1}^{\infty}\frac{\ln(n+1)}{(n+1)^2}.$ $6.54.\ \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}.$

$6.55.\ \sum_{n=1}^{\infty} ne^{-n^2}.$ $6.56.\ \sum_{n=1}^{\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}}.$ $6.57.\ \sum_{n=2}^{\infty}\frac{1}{n\ln^{\alpha+1}n},\ (\alpha>0)$

Mustaqil yechish uchun misollarning javoblari

6.1. $S=1$. **6.2.** $S=\frac{1}{3}$. **1.3.** $S=\frac{1}{18}$. **6.4.** $S=1$. **6.5.** $S=\frac{5}{36}$.

6.6. $S=\frac{1}{m}\left(1+\frac{1}{2}+...+\frac{1}{m}\right)$. 6.13. $S_n=\frac{51}{8}-\frac{3}{2^{n+1}}+\frac{(-1)^{n-1}}{8\cdot 3^{n-1}}$, $S=\frac{51}{8}$. 6.14. $S_n=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{5n+3}\right)$, $S=\frac{1}{15}$. 6.15. $S_n=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$, $S=\frac{3}{4}$. 6.16. $S_n=1-\frac{1}{(n+1)^2}$, $S=1$. 6.17. $S_n=\sqrt{\frac{n}{n+1}}$, $S=1$. 2.18. $S_n=\frac{1}{4}\left(\frac{1}{9}-\frac{1}{4n+9}\right)$, $S=\frac{1}{36}$.
6.19. Yaqinlashuvchi. 6.20. Yaqinlashuvchi. 6.21. Yaqinlashuvchi. 6.22. Yaqinlashuvchi. 6.23. Uzoqlashuvchi. 6.24. Yaqinlashuvchi. 6.25. Yaqinlashuvchi. 6.26. Yaqinlashuvchi. 6.27. Yaqinlashuvchi. 6.28. Uzoqlashuvchi. 6.29. Yaqinlashuvchi. 6.30. Yaqinlashuvchi. 6.31. Uzoqlashuvchi. 6.32. Uzoqlashuvchi. 6.33. Uzoqlashuvchi. 6.34. Yaqinlashuvchi. 6.35. Uzoqlashuvchi. 6.36. Yaqinlashuvchi. 6.37. Uzoqlashuvchi. 6.38. Yaqinlashuvchi. 6.39. Yaqinlashuvchi. 6.40. Yaqinlashuvchi. 6.41. Yaqinlashuvchi. 3.42. Uzoqlashuvchi 6.43. Yaqinlashuvchi. 6.44. Ixtiyoriy α lar uchun yaqinlashuvchi. 6.45. Yaqinlashuvchi. 6.46. Uzoqlashuvchi. 6.47. Yaqinlashuvchi. 6.48. Yaqinlashuvchi. 6.49. Uzoqlashuvchi. 6.50. Uzoqlashuvchi. 6.51. Yaqinlashuvchi. 6.52. Yaqinlashuvchi. 6.53. Yaqinlashuvchi. 6.54. Yaqinlashuvchi. 6.55. Yaqinlashuvchi. 5.56. Yaqinlashuvchi. 6.57. Yaqinlashuvchi.

7- amaliy mashg'ulot.

IXTIYORIY ISHORALI QATORLAR VA ULARNING YAQINLASHUVCHILIGI

7.1. Ishorasi almashinuvchi qatorlar.

1-ta'rif. Ushbu

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-...+(-1)^{n+1}\cdot a_n+.... \quad (7.1)$$

(bunda $a_n\geq 0$ yoki $a_n\leq 0$, $\forall n\in N$) qator ishorasi almashinuvchi yoki Leybnis qatori deyiladi.

1-teorema (Leybnis alomati). Agar ishorasi almashinuvchi (7.1) qatorning hadlari absolyut qiymati bo'yicha monoton kamayuvchi, ya'ni

$$a_n\geq a_{n+1}>0 \quad (\forall n\in N) \quad (7.2)$$

va

$$\lim_{n\to\infty}a_n=0 \quad (7.3)$$

bo'lsa, (7.1) qator yaqinlashuvchi bo'ladi.

1-eslatma. Absolyut yaqinlashuvchi qatorlar uchun Leybnis alomatining shartlari bajarilmasa ham ishorasi almashinuvchi qator yaqinlashuvchi bo'lishi mumkin.

2-eslatma. Absolyut yaqinlashuvchi bo'lmagan ishorasi almashinuvchi, hadlari monoton kamayuvchi qatorlarda qator yaqinlashuvchi bo'lishi uchun Leybnis alomatidagi shartlarning bajarilishi zarur va yetarli.

3-eslatma. Leybnis alomatidagi har uchta shart ham, ya'ni qatorning hadlarini ishora almashinuvchiligi, absolyut qiymati bo'yicha monotonligi va ularning nolga intilishi absolyut yaqinlashuvchi bo'lmagan qatorlarning yaqinlashishi uchun muhim shart bo'lib hisoblanadi. Shulardan birortasi buzilsa, u holda qator uzoqlashuvchi bo'ladi.

Bundan keyin, Leybnis alomatining shartlarini qanoatlantiruvchi qatorlarni Leybnis tipidagi qatorlar deb ataymiz.

Natija. Leybnis tipidagi qatorlarda $\forall n\in N$ uchun quyidagi

$$S_{2n}<S\leq S_{2n-1},\quad |S-S_n|\leq a_{n+1},\quad 0<S<a_1$$

tengsizliklar o'rinli bo'ladi.

7.2. Qatorlarning absolyut va shartli yaqinlashuvchiligi. Hadlari ixtiyoriy ishorali

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...a_n+... \quad (7.4)$$

qator berilgan bo'lsin. Bu qator hadlarining absolyut qiymatlaridan ushbu

$$\sum_{n=1}^{\infty}|a_n|=|a_1|+|a_2|+...|a_n|+... \quad (7.5)$$

qatorni tuzamiz.

1-ta'rif. Agar (7.5) qator yaqinlashuvchi bo'lsa, (7.4) qator absolyut yaqinlashuvchi qator deyiladi.

2-ta'rif. Agar (7.4) qator yaqinlashuvchi bo'lib, (7.5) qator uzoqlashuvchi bo'lsa, (7.4) qator shartli yaqinlashuvchi deyiladi.

1-teorema. Agar (7.5) qator yaqinlashuvchi bo'lsa, (7.4) qator ham yaqinlashuvchi bo'ladi.

2-teorema. Agar (7.4) qator absolyut yaqinlashuvchi bo'lib, $\{b_n\}$ ketma-ketlik esa chegaralangan bo'lsa, ya'ni $\exists M>0:\forall n\in N$ uchun $|b_n|\le M$ bo'lsa, $\sum_{n=1}^{\infty}a_nb_n$ qator absolyut yaqinlashuvchi bo'ladi.

3-teorema. Agar ixtiyoriy ishorali $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar absolyut yaqinlashuvchi bo'lsa, barcha $\lambda,\mu\in R$ o'zgarmas sonlar uchun

$$\sum_{n=1}^{\infty}(\lambda a_n+\mu b_n)$$

qator ham absolyut yaqinlashuvchi bo'ladi.

4-teorema. Agar (7.4) qator absolyut yaqinlashuvchi bo'lsa, (7.4) qator hadlarining o'rinlarini almashtirish natijasida tuzilgan

$$\sum_{n=1}^{\infty}\tilde{a}_n$$

qator ham absolyut yaqinlashuvchi bo'ladi va uning yig'indisi (7.4) qatorning yig'idisiga teng bo'ladi.

5-teorema. Agar (7.4) qator absolyut yaqinlashuvchi bo'lsa, u holda

$$\sum_{n=1}^{\infty}Ca_n \quad (C \text{ - o'zgarmas son})$$

qator ham absolyut yaqinlashuvchi bo'ladi.

6-teorema. Agar

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...a_n+... \qquad (A)$$

$$\sum_{n=1}^{\infty}b_n=b_1+b_2+...b_n+.... \qquad (B)$$

qatorlar absolyut yaqinlashuvchi bo'lib, ularning yig'indilari mos ravishda S', S'' ga teng bo'lsa, ular hadlarining istalgan tartibdagi $a_i\cdot b_j$ ko'paytmasidan tuzilgan qator ham absolyut yaqinlashuvchi bo'ladi, va uning yig'indisi $S'\cdot S''$ ga teng bo'ladi.

1-eslatma. (7.5) qatorning uzoqlashuvchi bo'lishidan (7.4) qatorning uzoqlashuvchi bo'lishi har doim ham kelib chiqavermaydi.

2-eslatma. Agar (A) va (B) qatorlarning biri yaqinlashuvchi, ikkinchisi absolyut yaqinlashuvchi bo'lsa, u holda qatorlarni ko'paytirishda Koshi qoidasi o'rinli bo'ladi:

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}a_n\cdot\sum_{n=1}^{\infty}b_n,\quad c_n=a_1b_n+a_2b_{n-1}+...+a_nb_1.$$

3-eslatma. (A) va (B) qatorlar shartli yaqinlashuvchi bo'lganda, ularning ko'paytmasi uzoqlashuvchi bo'lishi ham mumkin. Masalan, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ qatorningLeybnis alomatiga ko'ra shartli yaqinlashuvchi ekanligini ko'rsatish qiyin emas.

1-misol. $\sum_{n=2}^{\infty}(-1)^n\cdot\frac{\ln n}{n}$. qatorni absolyut va shartli yaqinlashuvchilikga tekshiring.

Yechilishi. 1) Berilgan qator hadlarining absolyut qiymatlaridan tuzilgan ushbu $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ qatorni yaqinlashishga tekshiramiz. Bu qatorining umumiy hadi $a_n=\frac{\ln n}{n}=f(n)$ da $n=x$ $f(x)=\frac{\ln x}{x}$ funksiya $[2;+\infty)$ da musbat, uzluksiz $f(x)=\frac{\ln x}{x}$ funksiyani monotonlikka tekshiramiz:

$$f'(x)=\left(\frac{\ln x}{x}\right)'=\frac{1-\ln x}{x^2}$$

Agar $x>e$ bo'lsa, $f'(x)<0$ bo'ladi, ya'ni $f(x)$ funksiya monoton kamayuvchi. Demak, $\frac{\ln x}{x}$ funksiya Makloren Koshi alomatining hamma shartlarini qanoatlantiradi. Shuning uchun, $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ qatorga Makloren -Koshining integral alomatini qo'llaymiz:

$$F(x)=\int\frac{\ln x}{x}dx=\frac{\ln^2 x}{2}\underset{x\to\infty}{\to}+\infty$$

bo'lgani uchun $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ - qator uzoqlashuvchi.

2) Endi $\sum_{n=2}^{\infty}(-1)^n\cdot\frac{\ln n}{n}$ qatorni yaqinlashishiga tekshiramiz. Ravshanki bu qatorning umumiy hadi Leybnis teoremasining hamma shartlarini qanoatlantiradi, ya'ni $C_n=(-1)^n\cdot\frac{\ln n}{n}$ absolyut qiymati bo'yicha monoton kamayuchi va $n\to\infty$ bu qator shartli yaqinlashuvchi

Quyidagi qatorlarning absolyut yaqinlashuvchiligini isbotlang:

7.1. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{ne^{\sqrt{n}}}$. 7.2. $\sum_{n=1}^{\infty}(-1)^n \frac{n^5}{2^n+3^n}$. 7.3. $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\sin^2\frac{\pi}{n}\right)$. 7.7.

$\sum_{n=1}^{\infty}(-1)^n \sqrt[3]{n} arctg\frac{2n+1}{n^3+2}$. 7.5. $\sum_{n=1}^{\infty}\frac{(-n)^n}{(2n)!}$. 7.6. $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{\left(3+\frac{1}{n}\right)^n}$.

7.7. $\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}} \frac{n^{100}}{2^n}$. 7.8. $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}}{n^3+\sin\frac{\pi n}{4}}$. 7.9. $\sum_{n=1}^{\infty}\left(\frac{1}{n\sin\frac{1}{n}}-\cos\frac{1}{n}\right)\cos\pi n$.

Ishorasi almashinuvchi qatorlarning absolyut, shartli yaqinlashishini yoki uzoqlashishini tekshiring:

7.10. $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n^2+1}$. 7.11. $\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}$.

7.12. $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{3n+2}$. 7.13. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+2}}{2^{n^2}}$.

7.14. $\sum_{n=3}^{\infty}(-1)^n \frac{\sqrt{n+1}-\sqrt{n-2}}{n}$. 7.15. $\sum_{n=1}^{\infty}(-1)^n \left[\frac{(2n)!!}{(2n-1)!!}\right]^2$.

Quyidagi qatorlarni Dirixle va Abel alomatlari bo'yicha yaqinlashishga tekshiring:

7.16. $\sum_{n=1}^{\infty}(-1)^n \frac{\sin^2 n}{n}$. 7.17. $\sum_{n=1}^{\infty}\frac{\sin nx}{n^{\alpha}}, \alpha>0$.

7.18. $\sum_{n=2}^{\infty}\frac{\cos\frac{\pi n^2}{n+1}}{\ln^2 n}$. 7.19. $\sum_{n=1}^{\infty}\frac{\sin n\sin n^2}{n}$.

Quyidagi qatorlarni yaqinlashuvchi ekanligini isbotlang va ularning yig'indisini toping

7.20. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\dots$. 7.21. $1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\dots$.

7.22. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$.

7.10. Absolyut yaqinlashuvchi. 7.11. Shartli yaqinlashuvchi. 7.12. Shartli yaqinlashuvchi. 7.13. Absolyut yaqinlashuvchi. 7.14. Shartli yaqinlashuvchi. 7.15. Uzoqlashuvchi. 7.16. Yaqinlashuvchi. 7.17. $\forall x\in R$-lar uchun yaqinlashuvchi. 7.18. Yaqinlashuvchi. 7.19. Yaqinlashuvchi. 7.20. $\frac{2}{9}$. 7.21. $\frac{10}{3}$. 7.22. $\ln 2$.

8- amaliy mashg'ulot.

FUNKSIONAL KETMA-KETLIKLAR VA QATORLAR

8.1. Funksional ketma-ketliklar va ularning yaqinlashuvchiligi. Elementlari biror $X\subset R$ to'plamda aniqlangan

$$f_1(x), f_2(x), \dots, f_n(x), \dots \qquad (8.1)$$

funksiyalar ketma-ketligi berilgan bo'lsin. Bu ketma-ketlik funksional ketma-ketlik deb ataladi va qisqacha $\{f_n(x)\}$ kabi belgilanadi. Umumiy holda $\{f_n(x)\}$ ketma-ketlik turli hadlarining aniqlanish sohasi, umuman aytganda, turlicha bo'lishi ham mumkin. Biz buyerda X sifatida shu soxalarning umumiy qismini olamiz. (8.1) ketma ketlikdagi $f_n(x)$ funksiya shu ketma-ketlikning umumiy hadi deyiladi. X to'plamdan x_0 ($x_0\in X$) nuqtani olib, (8.1) ketma-ketlik har bir hadining shu nuqtadagi qiymatini hisoblab, natijada

$$f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots$$

sonlar ketma-ketligini hosil qilamiz.

1-ta'rif. Agar $\{f_n(x_0)\}$ sonlar ketma-ketligi yaqinlashuvchi (uzoqlashuvchi) bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi.

2-ta'rif. Agar $\{f_n(x)\}$ funksional ketma-ketlik X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, u X to'plamda yaqinlashuvchi (uzoqlashuvchi) deyiladi.

1-eslatma $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish sohasi $\{f_n(x)\}$ funksional ketma-ketlikning aniqlanish sohasiga teng yoki uning bir qismi, yoki bo'sh to'plam ham bo'lishi mumkin.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik $X\subset R$ to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0\in X$ uchun unga mos kelgan,

$$f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots$$

ketma-ketlik chekli limitga ega bo'ladi, ya'ni

$$\lim_{n\to\infty} f_n(x_0) = f(x_0).$$

Agar X to'plamdan olingan har bir x ga, unga mos kelgan $f_1(x), f_2(x),..., f_n(x),...$ ketma-ketlikning limitini mos qo'ysak, ya'ni

$$f : x \to \lim_{n\to\infty} f_n(x),$$

unda X to'plamda aniqlangan biror $f(x)$ funksiya hosil bo'ladi. $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi deb ataladi va uni

$$\lim_{n\to\infty} f_n(x_0) = f(x) \ (x \in X) \qquad (8.2)$$

kabi yozamiz yoki qisqacha

$$f_n(x) \overset{X}{\to} f(x_0)$$

deb belgilaymiz. (8.2) ni "ε" tilida quyidagicha ham yozish mumkin:

$$\forall \varepsilon > 0 \ \ \exists n_0 = n_0(\varepsilon, x) \ \ \forall n \ge n_0, \ \ \forall x \in X \ \Rightarrow \ |f_n(x) - f(x)| < \varepsilon.$$

8.2. Funksional qatorlar va ularning yaqinlashuvchiligi. Biror X ($X \subset R$) to'plamda $u_1(x), u_2(x),..., u_n(x),...$ funksiyalar ketma- ketligi berilgan bo'lsin.

3-ta'rif. Ushbu

$$u_1(x) + u_2(x) + ... + u_n(x) + ...$$

ifodaga funksional qator deyiladi va u $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$u_1(x) + u_2(x) + ... + u_n(x) + ... = \sum_{n=1}^{\infty} u_n(x) \qquad (8.3)$$

Bunda $u_1(x), u_2(x),..., u_n(x),...$ lar qatorning hadlari, $u_n(x)$ esa funksional qatorning umumiy hadi deb ataladi. (8.3) funksional qatorning hadlaridan tuzilgan ushbu

$$S_1(x) = u_1(x)$$
$$S_2(x) = u_1(x) + u_2(x)$$
$$.. \qquad (8.4)$$
$$S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$$
$$..$$

yig'indilar ketma-ketligi (8.3) funksional qatorning qismiy yig'indilari ketma-ketligi deyiladi va u $\{S_n(x)\}$ kabi belgilanadi.

2-eslatma.

$$\sum_{n=1}^{\infty} u_n(x)$$

funksional qator turli hadlarining aniqlanish sohalari (to'plamlari), umuman aytganda, turlicha bo'ladi. Biz bu yerda X to'plam sifatida shu sohalarning umumiy qismini tushunamiz.

X to'plamdan $x_0 (x_0 \in X)$ nuqtani olib, (8.3) funksional qator har bir $u_n(x)$ $(n = 1,2,...)$ hadlarining shu nuqtadagi qiymatini hisoblab, ushbu

$$\sum_{n=1}^{\infty} u_n(x_0) = u_1(x_0) + u_2(x_0) + ... + u_n(x_0) + ... \qquad (8.5)$$

sonli qatorni hosil qilamiz.

4-ta'rif. Agar (8.5) sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (8.3) funksional qator x_0 nuqtada *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

5-ta'rif. Agar (8.3) funksional qator X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (8.3) funksional qator X to'plamda yaqinlashuvchi (uzoqlashuvchi) deyiladi.

Faraz qilaylik, (8.3) funksional qator X to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan (8.5) qator yaqinlashuvchi bo'ladi va uning yig'indisi biror S_0 songa teng bo'ladi. Agar X to'plamdan olingan har bir x ga, unga mos kelgan

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$

qatorning yig'indisini mos qo'ysak, u holda X to'plamda aniqlangan biror $S(x)$ funksiya hosil bo'ladi. Bu $S(x)$ funksiya

$$\sum_{n=1}^{\infty} u_n(x)$$

funksional qatorning yig'indisi deyiladi va u

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

kabi yoziladi.

Sonli qatorlarning yaqinlashish (uzoqlashish) ta'rifiga asosan, funksional qatorning x_0 nuqtadagi yaqinlashish (uzoqlashish) ta'rifini quyidagicha ham berish mumkin.

6-ta'rif. Agar $n \to \infty$ da (8.4) funksional ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (8.3) funksional qator x_0 nuqtada yaqinlashuchi (uzoqlashuvchi) deyiladi.

Agar $n \to \infty$ da $\{S_n(x)\}$ funksional ketma-ketlik X to'plamda $S(x)$ limit funksiyaga ega bo'lsa, ya'ni

$$\lim_{n\to\infty} S_n(x) = S(x)$$

bo'lsa, $S(x)$ funksiya (8.3) qatorning yig'indisi deyiladi.

7-ta'rif Agar

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + ... + |u_n(x)| + ... \qquad (8.6)$$

funksional qator $x = x_0$ nuqtada yaqinlashuvchi bo'lsa, (8.3) funksional qator x_0 nuqtada absolyut yaqinlashuvchi deyiladi.

8-ta'rif. Agar X to'plamning har bir nuqtasida (8.6) qator yaqinlashuvchi bo'lsa, (8.3) funksional qator X to'plamda absolyut yaqinlashuvchi deb ataladi.

Agar $x = x_0$ nuqtada (8.6) qator uzoqlashuvchi bo'lib, (8.3) qator yaqinlashuvchi bo'lsa, (8.3) qator $x = x_0$ nuqtada shartli yaqinlashuvchi deyiladi.

(8.3) va (8.6) qatorlar yaqinlashadigan nuqtalar to'plami mos ravishda (8.3) qatorning yaqinlashish va absolyut yaqinlashish sohasi deyiladi.

3-eslatma. Berilgan (8.3) funksional qatorning yaqinlashish va absolyut yaqinlashish sohasini topishda sonli qatorlar mavzusida ko'rib o'tilgan Dalamber va Koshi alomatlaridan foydalanish mumkin.

8.3. Funksional ketma-ketliklarning tekis yaqinlashuvchiligi. Ushbu

$$f_1(x), f_2(x), ..., f_n(x), ... \qquad (8.7)$$

funksional ketma-ketlik $X\ (X \subseteq R)$ to'plamda yaqinlashuvchi va uning limit funksiyasi $f(x)$ bo'lsin.

9-ta'rif. Agar $\forall \varepsilon > 0$ son olganda ham $\exists m_\varepsilon \in N$ nomer topilib, $\forall n > m$ va $\forall x \in X$ lar uchun bir vaqtda

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashadi deyiladi va u qisqacha

$$f_n \overset{X}{\underset{\to}{\to}} f(x)$$

kabi belgilanadi.

8.1-eslatma. 9-ta'rifdagi m natural son faqat ε ga bog'liq bo'lib, x larga bog'liq bo'lmaydi.

10-ta'rif. $\forall m \in N$ olinganda ham, $\exists \varepsilon_0 > 0$, $\exists n \geq m$ va $x_0 \in X$ mavjud bo'lib,

$$|f_n(x_0) - f(x_0)| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashmaydi deyiladi va u qisqacha

$$f_n(x) \overset{X}{\nRightarrow} f(x)$$

kabi belgilanadi.

11-ta'rif. Agar $f_n(x) \overset{X}{\to} f(x)$ bo'lib, lekin $f_n(x) \overset{X}{\nRightarrow} f(x)$ bo'lsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga tekis yaqinlashmaydi (notekis yaqinlashadi) deyiladi.

Xususiy holda, agar $f_n(x) \overset{X}{\to} f(x)$ va $\exists \varepsilon_0 > 0$, $\forall m \in N$ $\exists n \geq m$ va $\exists x_n \in X$

$$|f_n(x_n) - f(x_n)| \geq \varepsilon_0 \qquad (8.8)$$

shart bajarilsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga tekis yaqinlashmaydi deyiladi.

1-teorema. (8.7) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x \in X} |f_n(x) - f(x)| = 0 \qquad (8.9)$$

shartning bajarilishi zarur va yetarli.

2-teorema. (8.7) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun shunday $\{a_n\}$ sonli ketma-ketlik (bunda $\lim_{n\to\infty} a_n = 0$) va shunday m nomer mavjud bo'lib, barcha $n > m$ va barcha $x \in X$ lar uchun

$$|f_n(x) - f(x)| < a_n$$

tengsizlikning bajarilishi zarur va yetarli.

8.4. Funksional qatorlarning tekis yaqinlashuvchiligi. Ushbu

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \qquad (8.10)$$

funksional qator $X(X \subseteq R)$ to'plamda yaqinlashuvchi va uning yig'indisi $S(x)$ bo'lsin, ya'ni

$$\lim_{n\to\infty} S_n(x) = S(x) = \sum_{n=1}^{\infty} u_n(x).$$

12-ta'rif. Agar (8.10) funksional qatorning $\{S_n(x)\}$ qismiy yig'indilari ketma-ketligi X to'plamda $S(x)$ ga tekis yaqinlashsa, (8.10) funksional qator X to'plamda $S(x)$ ga tekis yaqinlashadi deyiladi va u qisqacha

$$S_n(x) \overset{x}{\underset{\to}{\to}} S(x) \qquad (8.11)$$

kabi belgilanadi.

1-eslatma. Funksional qatorlarning tekis yaqinlashuvchiligi (yaqinlashmovchiligi) tushunchasi ham ularning oddiy yaqinlashuvchiligi singari, funksional ketma-ketliklarning tekis yaqinlashuvchilik (yaqinlashmovchiligi) tushunchasi orqali kiritiladi.

12-ta'rifni qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall \varepsilon > 0 \exists m(\varepsilon) : \forall n > m \forall x \in X \to |S_n(x) - S(x)| < \varepsilon. \qquad (8.12)$$

13-ta'rif. (8.10) qatorning dastlabki n ta hadini tashlab yuborgandan so'ng, hosil bo'lgan ushbu

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots = \sum_{k=n+1}^{\infty} u_k(x)$$

qatorga (8.10) funksional qatorning n ta hadidan keyingi qoldig'i deyiladi. Bunda

$$r_n(x) = S(x) - S_n(x)$$

bo'ladi. U holda (8.11) shartni quyidagi ko'rinishda ifodalash mumkin:

$$r_n(x) \overset{X}{\underset{\to}{\to}} 0. \qquad (8.13)$$

(8.11) va (8.13) shartlar teng kuchli.

14-ta'rif. Agar X to'plamda $S_n(x)$ ketma-ketlikning limit funksiyasi mavjud bo'lsa va (8.10) shart bajarilsa, ya'ni

$$\forall \varepsilon_0 > 0 : \forall k \in N\ \exists n \geq k\ \forall \bar{x} \in X \to |S_n(\bar{x}) - S(\bar{x})| \geq \varepsilon_0$$

bo'lsa, $S_n(x)$ ketma-ketlik X to'plamda $S(x)$ ga notekis yaqinlashadi deyiladi.

3-teorema. (8.10) funkional qatorning X da tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x \in X} |r_n(x)| = 0 \qquad (8.14)$$

shartning bajarilishi zarur va yetarlidir.

4-teorema(zaruriy shart). Agar (8.10) funksional qator X da tekis yaqinlashuvchi bo'lsa, u holda uning umumiy hadi $u_n(x)\ (n = 1,2,...)$ $u_n(x) \overset{X}{\underset{\to}{\to}} 0$ bo'ladi.

5-teorema (Veyershtrass alomati). Agar (8.10) funksional qatorning har bir hadi X da aniqlangan bo'lib, $\forall x \in X$ va $\forall n > n_0$ uchun

$$|u_n(x)| \le c_n$$

tengsizlikni qanoatlantirsa va

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \ldots + c_n + \ldots$$

sonli qator yaqinlashuvchi bo'lsa, u holda (8.10) funksional qator X da absolyut va tekis yaqinlashuvchi bo'ladi.

Natija. Agar

$$\sum_{n=1}^{\infty} a_n$$

sonli qator yaqinlashuvchi bo'lsa, bunda $a_n = \sup_{x \in R} |u_n(x)|$, (8.10) funksional qator tekis yaqinlashuvchi bo'ladi.

1-misol. $f_n(x) = n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right)$, $X = [0; +\infty)$. funksional ketma-ketlikning X to'plamdagi $f(x)$ limit funksiyasini toping.

Yechilishi. Berilgan funksional ketma-ketlikni ushbu $f_n(x) = \frac{1}{\sqrt{x + \frac{1}{n}} + \sqrt{x}}$ ko'rinishga keltirib, so'ngra uning limit funksiyasini topamiz:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\sqrt{x + \frac{1}{n}} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Demak, $f(x) = \frac{1}{2\sqrt{x}}$.

2-misol. Ko'rsatilgan oraliqda funksional qatorning tekis yaqinlashuvchiligini Veyershtrass alomatidan foydalanib ko'rsating:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{(3n+1)\cdot 3^n}, \quad X = [-1;3].$$

Yechilishi. $\forall x \in [-1;3]$ uchun $|U_n(x)| = \frac{|x-1|^n}{(3n+1)\cdot 3^n} \le \frac{2^n}{(3n+1)\cdot 3^n}$ o'rinli.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n}{(3n+1)\cdot 3^n}$ - sonli qatorni Dalamber alomatidan foydalanib, yaqinlashishga tekshiramiz: $a_n = \frac{(\frac{2}{3})^n}{3n+1}$, $a_{n+1} = \frac{(\frac{2}{3})^{n+1}}{3n+4}$. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{3}\frac{(3n+1)}{(3n+4)} = \frac{2}{3} < 1$ bo'lgani uchun

majorant sonli qator yaqinlashuvchi. Demak, Veyershtrass alomatiga ko'ra, berilgan funksional qator $[-1;3]$ da tekis yaqinlashuvchi.

Mustaqil yechish uchun misollar

X to'plamda quyidagi $\{f_n(x)\}$ funksional ketma-ketliklarning limit funksiyasi $f(x)$ topilsin.

8.1. $f_n(x) = \frac{1}{x^n + 2n}$, $X = (-\infty; \infty)$. **8.2.** $f_n(x) = \frac{n^3+1}{x^2+n^3}$, $X = (-\infty; \infty)$.

8.3. $f_n(x) = x^n - 4x^{n+3} + 3x^{n+4}$, $X = [0;1]$. **8.4.** $f_n(x) = x^4 \cos\frac{1}{xn}$, $X = (0;\infty)$.

8.5. $f_n(x) = \sqrt{x^2 + \frac{1}{\sqrt{n}}}$, $X = (-\infty;\infty)$. **8.6.** $f_n(x) = n(x^{\frac{1}{n}} - 1)$, $X = [1;3]$.

8.7. $f_n(x) = n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right)$, $X = (0;\infty)$, **8.8.** $f_n(x) = \frac{arctgn^2 x}{\sqrt{n^3 + x^2}}$, $X = (-\infty;\infty)$.

Quyidagi funksional qatorlarning (absolyut va shartli) yaqinlashish sohalarini toping.

8.9. $\sum_{n=1}^{\infty} \ln^n x$. **8.10.** $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\left(\frac{2x}{1+x^2}\right)^n$. **8.11.** $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$.

8.12. $\sum_{n=1}^{\infty} 2^n \sin\frac{x}{3^n}$. **8.13.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(x-2)^n}$. **8.14.** $\sum_{n=1}^{\infty} \frac{tg^n x}{n}$.

X da $\{f_n(x)\}$ funksional ketma–ketlikning tekis yaqinlashuvchiligini isbotlang:

8.15. $f_n(x) = e^{-nx^2}$, $X = [1;\infty)$. **8.16.** $f_n(x) = \frac{x^n}{1+x^n}$, $X = [0; 1-\varepsilon], 0 < \varepsilon < 1$.

8.17. $f_n(x) = x^{4n}$, $X = [0;\varepsilon]$, $0 < \varepsilon < 1$. **8.18.** $f_n(x) = \ln(1 + \frac{\cos nx}{\sqrt{n+x}})$, $X = [0;+\infty)$.

8.19. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $X = (-\infty;+\infty)$. **8.20.** $f_n(x) = e^{-(x-n)^2}$, $X = [-4;4]$.

X da $\{f_n(x)\}$ funksional ketma–ketlikni tekis hamda notekis yaqinlashuvchilikka tekshiring:

8.21. $f_n(x) = \frac{x^n}{1+x^n}$ $a) X = [1-\varepsilon; 1+\varepsilon]$; $b) X = [1+\varepsilon; +\infty)$, $\varepsilon > 0$.

8.22. $f_n(x) = e^{-(x-n)^2}$, $X = (-\infty;+\infty)$. **8.23.** $f_n(x) = \frac{\ln nx}{nx^2}$, $X = [1;+\infty)$.

8.24. $f_n(x) = x^n + x^{2n} - 2x^{3n}$, $X = [0;1]$.

X_1 va X_2 to'plamlarda $\{f_n(x)\}$ funksional ketma–ketlikni tekis hamda notekis yaqinlashuvchilikka tekshiring:

8.28. $f_n(x) = e^{-(x-n)^2}$, $X_1 = (-4;4)$ $X_2 = (-\infty;+\infty)$.

8.26. $f_n(x) = \frac{nx^2}{n^3+x^3}$, $X_1 = [0;1]$, $X_2 = [0;+\infty)$.

8.27. $f_n(x) = narctg\frac{1}{nx}$, $X_1 = (0;2)$, $X_2 = (2;+\infty)$.

8.28 . $f_n(x)=\sqrt{n}\sin\frac{x}{\sqrt{n}}$, $X_1=[0;\pi]$, $X_2=[\pi;+\infty)$.

8.29 . $f_n(x)=\frac{2nx}{1+n^2x^2}$, $X_1=[0;1]$, $X_2=(1;+\infty)$.

8.30 . $f_n(x)=\frac{1}{x^2}\sqrt{1+\frac{x}{n}}$, $X_1=(0;1)$, $X_2=(1;+\infty)$.

Quyidagi funksional qatorlarning ko'rsatilgan oraliqlarda tekis yaqinlashuvchiligini, Veyershtrass alomatidan foydalanib, isbotlang:

8.31. $\sum_{n=1}^{\infty}\frac{1}{(n+x)^4}$, $X=[0;\infty)$. **8.32.** $\sum_{n=1}^{\infty}\frac{x^4}{2+\sqrt[3]{n^4x^4}}$, $X=(-\infty;+\infty)$.

8.33. $\sum_{n=1}^{\infty}\frac{\cos^2 2nx}{\sqrt[4]{n^5+x^4}}$, $X=(-\infty;\infty)$. **8.34.** $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^4+\ln^2 x}}$, $X=(1;\infty)$.

8.35. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3+e^{-3x}}}$, $X=[-3;3)$. **8.36.** $\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{3^n+\cos x}$, $X=(-\infty;\infty)$.

8.37. $\sum_{n=1}^{\infty}\frac{\cos 3nx}{\sqrt{n^3+x^3}}$, $X=[0;\infty)$. **8.38.** $\sum_{n=1}^{\infty}\frac{x}{4+n^4x^3}$, $X=[0;\infty)$.

X to'plamda quyidagi funksional qatorlarning tekis yoki notekis yaqinlashuvchiligini aniqlang:

8.39. $\sum_{n=1}^{\infty}3^n\sin\frac{1}{4^nx}$, $X=(0;+\infty)$. **8.40.** $\sum_{n=2}^{\infty}\frac{(-1)^n}{n+\sin x}$, $X=[0;2\pi]$.

8.41. $\sum_{n=1}^{\infty}\left(arctg\frac{x}{x^2+n^2}\right)^2$, $X=[0;+\infty)$. **8.42.** $\sum_{n=1}^{\infty}\ln^2\left(1+\frac{x}{1+n^2x^2}\right)$, $X=[0;\infty)$.

8.43. $\sum_{n=1}^{\infty}\frac{\cos\frac{2\pi n}{3}}{\sqrt{n^2+x^2}}$, $X=(-\infty;+\infty)$. **8.44.** $\sum_{n=1}^{\infty}\frac{\sin x\cdot\sin nx}{\sqrt{n+x}}$, $X=[0;\infty)$.

Misollarning javoblari

8.1. $f(x)=0$. **8.2.** $f(x)=1$. **8.3.** $f(x)=0$. **8.4.** $f(x)=x^4$. **8.5.** $f(x)=|x|$. **8.6.** $f(x)=\ln x$. **8.7.** $f(x)=\frac{1}{2\sqrt{x}}$. **8.8.** $f(x)=0$. **8.9.** $(\frac{1}{e},e)$-absolyut yaqinlashuvchi. **8.10.** $x\neq 1$- absolyut yaqinlashuvchi; $x=-1$- shartli yaqinlashuvchi. **8.11.** $(-\infty,+\infty)$- absolyut yaqinlashuvchi. **8.12.** $(-\infty;+\infty)$- absolyut yaqinlashuvchi. **8.13.** $(-\infty;1)\cup(3;+\infty)$- absolyut yaqinlashuvchi. **8.14.** $|x-\pi k|<\frac{\pi}{4}$-absolyut yaqinlashuvchi; $x=-\frac{\pi}{4}+\pi k$-shartli yaqinlashuvchi, $k\in Z$. **8.21.** a) $f(x)=0$ ga tekis yaqinlashadi. b) $f(x)=1$ ga tekis yaqinlashadi. **8.22.** $f(x)=0$ ga notekis yaqinlashadi. **8.23.** $f(x)=0$ ga tekis yaqinlashadi. **8.24.** $f(x)=0$ ga notekis yaqinlashadi. **8.28.** funksiyaga X_1 tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **8.28.** $f(x)=0$ funksiyaga X_1 tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **8.27.** $f(x)=\frac{1}{x}$ funksiyaga X_2 tekis yaqinlashadi, X_1 da esa

notekis yaqinlashadi. **8.28.** $f(x)=x$ funksiyaga X_1 tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi.
8.29. $f(x)=0$ funksiyaga X_2 tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi.
8.30. $f(x)=\frac{1}{x^2}$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi.
8.39. Notekis yaqinlashadi. **8.40.** Tekis yaqinlashadi. **8.41.** Tekis yaqinlashadi.
8.42. Tekis yaqinlashadi. **8.43.** Tekis yaqinlashadi. **8.44.** Tekis yaqinlashadi.

9- amaliy mashg'ulot.

DARAJALI QATOR, UNING YAQINLASHISH RADIUSI VA INTERVALI

9.1. Darajali qator, uning yaqinlashish radiusi va intervali. Ushbu

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n+\cdots \quad (9.1)$$

qatorga darajali qator deyiladi. Bunda $a_0,a_1,a_2,...,a_n,...$ o'zgarmas haqiqiy sonlar darajali qatorning koeffisiyentlari deyiladi, x_0 esa, ixtiyoriy o'zgarmas son. (9.1) darajali qator ushbu

$$\sum_{n=0}^{\infty}u_n(x)$$

funksional qatorning xususiy holi bo'lib hisoblanadi:

$$u_n(x)=a_n(x-x_0)^n,\ n=0,1,2,...$$

$x-x_0=t$ belgilash yordamida (9.1) darajali qatorni

$$\sum_{n=0}^{\infty}a_nt^n=a_0+a_1t+a_2t^2+\cdots+a_nt^n+\cdots \quad (9.2)$$

ko'rinishga keltirish mumkin. Shuning uchun biz bundan keyin ushbu

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$$

ko'rinishdagi qatorni o'rganish bilan kifoyalanamiz.

1-teorema (Abel teoremasi). Agar (9.2) darajali qator x ning $x=x_0$ $(x_0\neq 0)$ qiymatiga yaqinlashuvchi bo'lsa, u holda x ning $|x|<|x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida (9.2) darajali qator absolyut yaqinlashuvchi bo'ladi.

Natija. Agar (9.2) qator x ning $x=x_0$ qiymatida uzoqlashuvchi bo'lsa, u x ning $|x|>|x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida uzoqlashuvchi bo'ladi.

2-teorema. Har qanday darajali (9.2) qator uchun $\exists\rho$ ($\rho\geq 0$ con yoki $+\infty$) son mavjud bo'lib:

a) agar $\rho \neq 0$ va $\rho \neq +\infty$ bo'lsa, u holda (9.2) qator $K = \{x : |x| < \rho\}$ intervalda absolyut yaqinlashuvchi bo'ladi va K intervalning tashqarisida uzoqlashuvchi bo'ladi;

b) agar $\rho = 0$ bo'lsa, (9.2) darajali qator faqat $x = 0$ nuqtada yaqinlashuvchi bo'lib, sonlar o'qining qolgan hamma nuqtalarida uzoqlashuvchi bo'ladi;

c) agar $\rho = +\infty$ bo'lsa, (9.2) darajali qator sonlar o'qining hamma joyida yaqinlashuvchi bo'ladi.

1-ta'rif. 2-teoremadagi ρ soni (9.2) darajali qatorning yaqinlashish radiusi, $K = \{x \in R : |x| < \rho\}$ esa darajali qatorning yaqinlashish intervali deyiladi.

1-eslatma. K intervalning chegarasida, ya'ni $x = \pm\rho$ da (9.2) darajali qator yaqinlashuvchi ham, uzoqlashuvchi bo'lishi ham mumkin. K ga nisbatan kichik istalgan $K_1 = \{x : |x| \le \rho_1 < \rho\}$ intervalda (9.2) qator absolyut va tekis yaqinlashuvchi bo'ladi.

3-teorema (Koshi-Adamar). Agar: 1) chekli yoki cheksiz

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

mavjud bo'lsa, u holda (9.2) qatorning yaqinlashish radiusi ρ uchun

$$\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \qquad (9.3)$$

formula o'rinli:

2) chekli yoki cheksiz

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

mavjud bo'lsa, u holda (9.2) darajali qatorning yaqinlashish radiusi ρ uchun

$$\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad (9.4)$$

formula o'rinli.

2-eslatma. Darajali qatorlarning har bir hadi $(-\infty;+\infty)$ da berilgan funksiya bo'lsa ham, tabiiyki, darajali qatorlar ixtiyoriy nuqtada yaqinlashuvchi bo'ladi, deb ayta olmaymiz.

3-eslatma. $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ darajali qatorning yaqinlashish intervali $(x_0 - \rho; x_0 + \rho)$ bo'ladi. Bunda ρ ushbu $\sum_{n=0}^{\infty} a_n x^n$ qatorning yaqinlashish radiusi.

4-eslatma. (9.3)-(9.4) limitlar mavjud bo'lmasligi ham mumkin. Ammo, (9.2) darajali qatorning yaqinlashish radiusini hisoblash uchun umumiy formulaga egamiz, ya'ni

$$\rho = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}. \qquad (9.5)$$

(9.5) formula Koshi-Adamar formulasi deyiladi.

1-misol. Quyidagi berilgan darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

$$\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{3n+1} x^n.$$

Y e ch i l i sh i. Berilgan darajali qatorda $a_n = \left(\frac{n}{3n-1}\right)^{3n+1}$. Darajali qatorning yaqinlashish radiusini (7.23) formulaga binoan topamiz:

$$\rho = \frac{1}{\lim_{n\to\infty} \sqrt[n]{a_n}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{3n-1}\right)^{3n+1}}} = \lim_{n\to\infty} \frac{1}{\left(\frac{n}{3n-1}\right)^{\frac{3n+1}{n}}} =$$

$$= \lim_{n\to\infty} \frac{3^{\frac{3n+1}{n}}}{\left[(1 + \frac{1}{3n-1})^{3n-1}\right]^{\frac{3n+1}{n(3n-1)}}} = 3^3 = 27.$$

Demak, darajali qatorning yaqinlashish radiusi $\rho = 27$, yaqinlashish intervali esa, (-27; 27) dan iborat. Endi yaqinlashish intervalining chegaralarida darajali qatorni yaqinlashishga tekshiramiz. $x = 27$ bo'lganda $\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{3n+1} (27)^n$ qator hosil bo'ladi. Bu qatorni yaqinlashishga tekshirishda Koshi alomatidan foydalanamiz:

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{3n-1}\right)^{3n+1} (27)^n} = 27 \lim_{n\to\infty} \left(\frac{n}{3n+1}\right)^{\frac{3n+1}{n}} = \frac{27}{27} = 1, \ k = 1.$$

Mustaqil yechish uchun misollar

Quyidagi darajali qatorning yaqinlashish radiusi, yaqinlashish intervali hamda yaqinlashish sohasini toping:

9.1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-3)^{2n}}{n5^n}$. **9.2.** $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n3^n \ln n}$. **9.3.** $\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{3n+1} x^n$.

9.4. $\sum_{n=1}^{\infty} \ln^3\left(1 + \frac{1}{\sqrt{n}}\right) x^n$. **9.5.** $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n + 3^n}$. **9.6.** $\sum_{n=1}^{\infty} tg \frac{\pi}{3^n} x^{2n}$.

9.7. $\sum_{n=1}^{\infty} \frac{5^n + (-3)^n}{n+1} x^n$. **9.9.** $\sum_{n=1}^{\infty} 3^n (n^3 + 2)(x-1)^{2n}$.

Quyidagi qatorlarning yaqinlashish sohalarini toping:

9.9. $\sum_{n=1}^{\infty} x^{2n+1} \sin \frac{\pi}{2^n}$. **9.10.** $\sum_{n=1}^{\infty} \frac{\lg^n x}{n}$. **9.11.** $\sum_{n=1}^{\infty} (\sin(\sqrt{n+1} - \sqrt{n}))(x+1)^n$.

9.12. $\sum_{n=1}^{\infty}\left(arctg\frac{1}{5^n}\right)(x-5)^n$. **9.13.** $\sum_{n=1}^{\infty}\frac{2^n n}{n^n}(x-1)^{2n}$.

Quyidagi darajali qatorlarning yig'indilarini hadma-had differensiallash yordamida toping:

9.14. $x+\frac{x^3}{3}+\frac{x^5}{5}+...$ **9.15.** $x-\frac{x^3}{3}+\frac{x^5}{5}-...$

Quyidagi darajali qatorlarning yig'indilarini hadma-had integrallash yordamida toping:

9.16. $x+2x^2+3x^3+...$ **9.17.** $x+\frac{1}{2}x^2+\frac{1}{3}x^3+...+\frac{1}{n+1}x^{n+1}+...$

Misollarning javoblari

9.1. $\rho=\sqrt{5}$, $(3-\sqrt{5};\ 3+\sqrt{5})$, $[3-\sqrt{5};\ 3-\sqrt{5}]$. **9.2.** $\rho=3$, $(-3;3)$, $[-3;3]$. **9.3.** $\rho=27$, $(-27;27)$. **9.4.** $\rho=1, (-1;1), [-1;1]$. **9.5.** $\rho=3$, $(-2;4)$. **9.6.** $\rho=\sqrt{3}$, $(-\sqrt{3};\sqrt{3})$. **9.7.** $\rho=0$, $x=0$. **9.9.** $\rho=\infty$, $(-\infty;\infty)$. **9.9.** $(-\sqrt{2};\sqrt{2})$. **9.10.** $[-10^{-1};10]$. **9.11.** $(-2;0)$. **9.12.** $(0;6)$. **9.13.** $\left(1-\sqrt{\frac{e}{2}};1+\sqrt{\frac{e}{2}}\right)$. **9.14.** $\frac{1}{2}\ln\frac{1+x}{1-x}$ $(|x|<1)$. **9.15.** $arctgx$ $(|x|\le 1)$. **9.16.** $\frac{x}{(1-x)^2}$ $(|x|<1)$. **9.17.** $-\ln|1-x|$ $(|x|<1)$.

10- amaliy mashg'ulot.

TEYLOR QATORI. FUNKSIYALARNI DARAJALI QATORLARGA YOYISH

10.1. Funksiyalarni Teylor qatoriga yoyish. $f(x)$ funksiya x_0 $(x_0\in R)$ nuqtaning biror

$$U_\delta(x_0)=\{x\in R: x_0-\delta<x<x_0+\delta\}\ \ (\delta>0)$$

atrofida berilgan bo'lib,u shu atrofda istalgan tartibdagi hosilaga ega bo'lsa, ushbu

$$f(x_0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (10.1)$$

darajali qator, u yaqinlashuvchi bo'ladimi, yaqinlashuvchi bo'lib, uning yig'indisi $f(x)$ funksiyaga teng bo'ladimi yoki yo'qmi, bundan qat'iy nazar, $f(x)$ funksiyaning $x=x_0$ nuqtadagi Teylor qatori deyiladi. Bu qator (10.1) darajali qatorga o'xshash bo'lib, bunda

$$f(x_0)=a_0,\ \frac{f^{I}(x_0)}{1!}=a_1,\ \frac{f^{II}(x_0)}{2!}=a_2,\ \frac{f^{III}(x_0)}{3!}=a_3,\ ...,\ \frac{f^{(n)}(x_0)}{n!}=a_n,...$$

lar Teylor koeffisiyentlari deyiladi.

Xususiy holda, ya'ni $x_0=0$ bo'lganda (10.1) Teylor qatori

$$f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

ko'rinishga keladi. Bu qator ko'p hollarda Makloren qatori deb yuritiladi.

1-teorema. $f(x)$ funksiya biror $U_\delta(x_0)$ to'plamda istalgan tartibdagi hosilaga ega bo'lib, (10.1) qator uning $x=x_0$ nuqtadagi Teylor qatori bo'lsin. Bu qator $U_\delta(x_0)$ da $f(x)$ ga yaqinlashishi uchun uning

$$f(x)=f(x_0)+\frac{f^{(1)}(x_0)}{1!}(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x).$$

Teylor formulasi qoldiq hadining $\forall x\in U_\delta(x_0)$ da nolga intilishi, ya'ni $\lim_{n\to\infty}r_n(x)=0$ bo'lishi zarur va yetarli.

Ma'lumki, Teylor formulasi qoldiq hadi:

a) integral ko'rinishda

$$r_n(x)=\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt;$$

b) Lagranj ko'rinishida

$$r_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1},$$

bunda $c=x_0+\theta(x-x_0)$, $0<\theta<1$;

s) Koshi ko'rinishida

$$r_n(x)=\frac{f^{(n+1)}(c)}{n!}(1-\theta)^n(x-x_0)^{n+1},\ \ c=x_0+\theta(x-x_0),\ \ 0,<\theta<1;$$

d) Peano ko'rinishida

$$r_n(x)=0\ ((x-x_0)^n)$$

bo'ladi.

2-teorema. $f(x)$ funksiya $(x_0-\rho,x_0+\rho)$ $(\rho>0)$ oraliqda darajali qatorga yoyilgan bo'lsa:

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+...+a_n(x-x_0)^n+...,$$

bu qator $f(x)$ funksiyaning Teylor qatori bo'ladi, bunda

$$a_0=f(x_0),\ a_1=\frac{f'(x_0)}{1!},\ a_2=\frac{f''(x_0)}{2!},\ a_3\frac{f'''(x_0)}{3!},...,a_n=\frac{f^{(n)}(x_0)}{n!},...$$

3-teorema. Agar $f(x)$ funksiya biror $(x_0-\rho,x_0+\rho)$ intervalda istalgan tartibdagi hosilaga ega bo'lib, shunday o'zgarmas $M>0$ son topilsaki, barcha $x\in(x_0-\rho,x_0+\rho)$, hamda barcha $n\in N$ lar uchun

$$|f^{(n)}(x)|\le M$$

bajarilsa, u holda (x_o-p,x_0+p) intervalda $f(x)$ funsiya Teylor qatoriga yoyiladi, ya'ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (A)$$

bo'ladi.

10.2.Elementar funksiyalarning Teylor qatorlari. (A) formulada $x_0 = 0$ deb, amaliyotda ko'p uchraydigan ba'zi elementar funksiyalarning darajali qatorlari yoyilmalarini keltiramiz:

1.Ko'rsatkichli funksiyalar:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty < x < +\infty, \ \ \rho = \infty). \qquad (10.2)$$

$$a^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ln^n a, \quad a > 0, a \neq 1 \ \ (-\infty < x < +\infty, \ \ \rho = \infty). \qquad (10.3)$$

2.Trigonometrik funksiyalar:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty < x < +\infty, \ \ \rho = \infty). \qquad (10.4)$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \quad (-\infty < x < +\infty, \ \ \rho = \infty). \qquad (10.5)$$

3.Darajali funksiyalar:

$$(x+1)^{\alpha} = 1 + \sum_{n=1}^{\infty} C_{\alpha}^n x^n \qquad (10.6)$$

bunda $C_{\alpha}^n = \dfrac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-(n-1))}{(\alpha-n)!}$.

Agar $\alpha \neq 0, \alpha \neq n, (n \in N)$ bo'lsa, (10.6) qatorning yaqinlashish radiusi 1 ga teng bo'ladi.

(10.6) formulaning muhim hususiy hollari:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ , \ \ (|x| < 1, \ \ \rho = 1); \qquad (10.7)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \ , \ (|x| < 1, \ \ \rho = 1); \qquad (10.8)$$

1-misol. $f(x) = \cos^2 x$ Funksiyalarni $x_0 = \frac{\pi}{4}$ nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping:

Yechilishi. Berilgan funksiyani quyidagi ko'rinishda tasvirlaymiz:

$f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x.$ $x - \frac{\pi}{4} = t$ deb belgilab, $\cos 2x = -\sin 2t$ ekanligini topamiz. Natijada $f(x) = g(t) = \frac{1}{2} - \frac{1}{2}\sin 2t$. Endi $\sin t$ ning makloren qatoriga yoyilmasida foydalanib $g(t) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2\pi+1}}{(2n+1)!} t^{2n+1}$ ni topamiz. Bu yerda

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2\pi}}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1}$$

Hosil bo'lgan qatorning yaqinlashish radiusini tekshirishda $\rho = \lim_{n\to\infty}\left|\frac{an}{an+1}\right|$ formulasidan foydalanamiz:

$$\rho = \lim_{n\to\infty}\left|\frac{2^{2n}}{(2n+1)!} \cdot \frac{(2n+3)!}{2^{2n+2}}\right| = \frac{1}{4}\lim_{n\to\infty}(2n+3) = +\infty .$$

Demak,darajali qatorning yaqinlashish radiusi $\rho = \infty$. Shunday qilib, hosil bo'lgan darajali qator $\forall x \in (-\infty;\infty)$ uchun $f(x) = \cos^2 x$ funksiyaga yaqinlashadi.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni x ning darajalari bo'yicha darajali qatorga yoying:

10.1. $y = \sin^3 x$. **10.2.** $y = \dfrac{x}{6 - x - x^2}$. **10.3.** $y = \ln(1 + x + x^2 + x^3)$.

10.4. $y = \dfrac{1}{(1 - x^3)^2}$. **10.5.** $y = \cos^2 x$. **10.6.** $y = \arcsin x^3$.

10.7. $y = e^{-x^2}$. **10.8.** $y = 4^x$.

Quyidagi funksiyalarni ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va bu qatorlarning yaqinlashish radiusini toping:

10.9 $f(x) = \sin\dfrac{2\pi x}{3}, \ x_0 = 3.$ **10.10.** $f(x) = \dfrac{1}{x^2 + 4x + 7}, \ \ x_0 = -2.$

10.11. $f(x) = e^x, \ \ x_0 = 3$. **10.12.** $f(x) = \sqrt{x}, \ \ x_0 = 1.$

10.13. $f(x) = \dfrac{x}{x^2 - 5x + 6}, \ \ x_0 = 5.$ **10.14.** $f(x) = 2^x, \ \ x_0 = 1.$ Quyida

Keltirilgan sonlarni ko'rsatilgan aniqlikda hisoblang:

10.15. $\cos 18^{\circ}$, 0,0001. **10.16.** $e^{\frac{1}{2}}$, 0,00001. **10.17.** $\ln 0{,}98$, 0,0001.

Integral ostidagi funksiyani darajali qatorga yoyish yordamida, berilgan integralni ko'rsatilgan aniqlikda hisoblang:

10.18. $\int_0^{1/4} e^{-x^2} dx$, 0,001. **10.110** $\int_0^{0,5} \frac{\sin 2x}{x} dx$, 0,001.

Darajali qatorlar yordamida quyida keltirilgan limitlarni hisoblang:

10.20. $\lim_{x\to 0}\dfrac{x - arctgx}{x^3}$. **10.21.** $\lim_{x\to 0}\dfrac{1 - \cos x}{e^x - 1 - x}$.

Misollarning javoblari

10.1. $\frac{3}{4}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1}, (|x|<+\infty)$. **10.2.** $-\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n-\frac{1}{5}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n, (-2<x<2)$.

10.3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+[1+(-1)^n](-1)^{\frac{n}{2}+1}}{n}x^n, \ (-1<x\le 1)$ **10.4.** $\sum_{n=1}^{\infty}(n+1)\cdot x^{3n}, (|x|<1)$

10.5. $1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}}{(2n)!}x^{2n}, (|x|<+\infty)$

10.6. $x^3+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!(2n+1)}x^{6n+3}, (|x|\le 1)$ **10.7.** $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}, (|x|<\infty)$

10.8. $\sum_{n=0}^{\infty}\frac{\ln^n 4}{n!}x^n, (|x|<+\infty)$ **10.9** $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\pi}{3}\right)^{2n+1}(x-3)^{2n+1}, R=\infty.$

10.10. $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x+2)^{2n}, R=\sqrt{3}.$ **10.11.** $e^3\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n, R=\infty.$

10.12. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{n!2^n}(x-1)^n, R=\frac{5}{4}.$ **10.13.**
$x\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-5)^n-x\sum_{n=1}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-5)^n, \ R=2.$ **10.14.** $2\sum_{n=0}^{\infty}\frac{\ln^n 2}{n!}(x-1)^n, R=\infty.$ **10.15.** 0,9551. **10.16.** 1,648719. **10.17.** $\ln 0,98\approx -0,0202$.
10.18. 0,245. **10.110** 0,946. **10.20.** $\frac{1}{3}$. **10.21.** 1.

11- amaliy mashg'ulot.

FUNKSIYALARNI FURYE (TRIGONOMETRIK) QATORIGA YOYISH.

1-ta'rif. Koeffisiyesntlari

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx \ , \tag{11.1}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi}{l}xdx, \tag{11.2}$$

$$b_k=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi}{l}xdx \tag{11.3}$$

formulalar orqali topiladigan ushbu

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x) \tag{11.4}$$

trigonometrik qatorga $f(x)$ funksiyaning *Furye qatori,* a_0, a_k, b_k koeffisiyentlarni esa uning *Furye koeffisiyesntlari* deyiladi.

(11.1),(11.2) va (11.3) integrallarning mavjud bo'lishi uchun $f(x)$ funksiyaning $[-l,l]$ oraliqda integrallanuvchi bo'lishi yetarli. Shuning uchun har bir $[-l,l]$ oraliqda integrrallanuvchi $f(x)$ funksiyaga koeffisiyesntlari (11.1)-(11.3) formulalar bilan aniqlanadigan (11.4) trigonometrik qatorni mos qo'yish mumkin:

$$f(x)\approx\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x) \tag{11.5}$$

Umuman olganda $f(x)$ funksiyadan integrallanuvchanligidan tashqari boshqa shart talab qilinmasa, (11.5) da tenglik ishorasini qo'yib bo'lmaydi.

1-teorema. Agar $f(x)$ funksiya $[-l,l]$ kesmada bo'lakli silliq bo'lsa, u holda bu funksiyaning

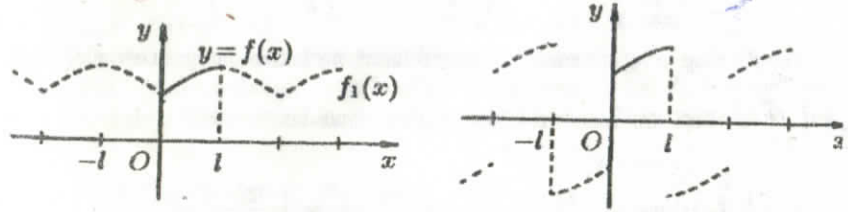
$$S(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x) \tag{11.6}$$

Furye triyegonometrik qatori $[-l,l]$ ga qarashli istalgan x uchun yaqinlashuvchi va uning yig'indisi $S(x)$ uchun quyidagilar o'rinlidir:

1) barcha $x\in(-l,l)$ uchun $S(x)=\frac{f(x-0)+f(x+0)}{2}$ bo'lib, agar x nuqta $f(x)$ ning uzluksizlik nuqtasi bo'lsa, u holda $f(a-0)=f(x+0)=f(x)$, demak, $S(x)=f(x)$ bo'ladi;

2) kesmaning chegaraviy nuqtalarida esa, qator yig'indisi ushbu $S(-l)=S(l)=\frac{f(-l+0)+f(+l-0)}{2}$ tenglik bilan aniqlanadi.

11.2. $[0,l]$ **da berilgan funksiyani faqat kosinuslar yoki sinuslar bo'yicha Furye qatoriga yoyish** $f(x)$ funksiya $[0,l]$ da aniqlangan bo'lib, u shu oraliqda bo'lakli uzluksiz va bo'lakli silliq bo'lsin. Uni $[-l,0]$ oraliqqa har xil davom ettirish mumkin, xususiy holda: 1) juft va 2) toq davom ettirish mumkin.



1) holda $[-l,l]$ da juft funksiya hosil bo'ladi. Shuning uchun

$$a_0=\frac{2}{l}\int_0^l f(\xi)d\xi, \ a_k=\frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi}{l}\xi d\xi, \ b_k=0 \tag{11.7}$$

bo'ladi, $[-l,l]$ dagi Furye qatori

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}\xi \qquad (11.8)$$

ko'rinishda bo'ladi.

2) holda $[-l,l]$ da toq funksiya hosil bo'lib, uning Furye koeffisiyentlari,

$$a_0 = 0,\ a_k = 0,\ b_k = \frac{2}{l}\int_0^l f(\xi)\sin\frac{k\pi}{l}\xi d\xi \qquad (11.9)$$

Furye qatori esa,

$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{l}\xi \qquad (11.10)$$

ko'rinishda bo'ladi. $(0,l)$ oraliqda har ikkala (11.8) va (11.10) qatorlar $f(x)$ ga yaqinlashadi ($f(x)$ ning uzluksizlik nuqtalarida).

1-misol. $f(x)=\begin{cases} -x, & -\pi \le x \le 0, \\ \frac{x^2}{\pi}, & 0 < x \le \pi. \end{cases}$

funksiyani $[-\pi;\pi]$ kesmada Furye qatoriga yoying.

Yechilishi. Berilgan funksiya $[-\pi;\pi]$ kesmada uzluksiz. Uning hosilasi x ning $x_n = \pi n$, $(n = 0, \pm 1, \pm 2, ...)$ nuqtalardan boshqa hamma qiymatlarida uzluksiz va o'zining aniqlanish sohasida chegaralangan.

Demak, berilgan fuknsiyaning Furye qatori x ning hamma qiymatlarida $f(x)$ ga yaqinlashadi. Furye koeffisiyentlarini topamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)dx + \frac{1}{\pi}\int_0^{\pi}\frac{x^2}{\pi}dx = -\frac{x^2}{2\pi}\Big|_{-\pi}^{0} + \frac{x^3}{3\pi^2}\Big|_0^{\pi} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5}{6}\pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\cos nx\,dx + \int_0^{\pi}\frac{x^2}{\pi}\cos nx\,dx\right).$$

Tenglikning o'ng tomonidagi integrallarni bo'laklab integrallash natijasida, $a_n = \frac{3(-1)^n - 1}{\pi n^2}$ bo'lishni topamiz. Endi b_n larni topamiz:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\sin nx\,dx + \int_0^{\pi}\frac{x^2}{\pi}\sin nx\,dx\right) = \frac{2[(-1)^n - 1]}{\pi^2 n^3},$$

bundan

$$b_n = \begin{cases} 0, & n = 2m,\ m \in N, \\ -\frac{4}{\pi^2(2m-1)^3}, & n = 2m-1,\ m \in N. \end{cases}$$

Shunday qilib, berilgan funksiyaning Furye qatori, quyidagi

$$f(x) = \frac{5}{12}\pi + \sum_{m=1}^{\infty}\left[\frac{3(-1)^m - 1}{\pi m^2}\cos mx - \frac{4}{\pi^2(2m-1)^3}\sin(2m-1)x\right]$$

ko'rinishda bo'ladi. Bu yoyilmada $x = \pi$ deyilsa, u holda

$$\pi = \frac{5\pi}{12} + \sum_{m=1}^{\infty}\frac{3(-1)^m - 1}{\pi m^2}(-1)^m.$$

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni $(-\pi;\pi)$ da Furye qatoriga yoying.

11.1. $f(x) = \sin^4 x$. 11.2. $f(x) = \cos^4 x$

11.3. $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases}$ 11.4. $f(x) = \begin{cases} 2, & -\pi < x < 0, \\ -2, & 0 < x < \pi \end{cases}$

11.5. $f(x) = 1 - 2x$ 11.6. $(x) = \frac{1}{2}x - 3$

11.7. $f(x) = \begin{cases} 5, & -\pi < x < 0, \\ -3, & 0 < x < \pi \end{cases}$ 11.8. $f(x) = \begin{cases} -7, & -\pi < x < 0, \\ 2, & 0 < x < \pi \end{cases}$

Quyidagi funksiyalarni $[-\pi;\pi]$ da Furye qatoriga yoying.

11.10. $f(x) = x^2$. 11.11. $f(x) = |x|$

Quyidagi funksiyalarni $(-\pi;\pi)$ oraliqda Furye qatoriga yoying:

11.12. $f(x) = \sin ax$ (a – butun son emas)

11.13. $f(x) = \begin{cases} -x, & -\pi < x < 0, \\ 0, & 0 < x < \pi \end{cases}$. 11.14. $f(x) = \cos ax\ (a \notin z)$.

11.15. $f(x) = \pi^2 - x2$ 11.16. $f(x) = e^{ax}\ x \in (-h,h)$

Quyidagi davriy funksiyalarning Furye qatoriga yoying:

11.17. $f(x) = \mathrm{sgn}(\cos x)$ 11.18.. $f(x) = \arcsin(\sin x)$ 11.19. $f(x) = \arcsin(\cos x)$.

Quyidagi funksiyalarni $(0;\pi)$ oraliqda sinus bo'yicha Furye qatoriga yoying.

11.20. $f(x) = x$. 11.21. $f(x) = \frac{\pi}{4} - \frac{x}{2}$. 11.22. $f(x) = \begin{cases} x, & 0 \le x < \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

Quyidagi funksiyalarni $[0;\pi]$ da kosinun bo'yicha Furye qatoriga yoying.

11.23. $f(x) = x$. 11.24. $f(x) = \frac{1}{2}x - 1$. 11.25 $f(x) = -x^2$

Misollarning javoblari

11.1. $\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$. 11.2. $\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$. 11.3. $f(x)=\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$.

11.4. $f(x)=\frac{-8}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$. 11.5. $f(x)=1+4\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$.

11.6. $f(x)=-3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$. 11.7. $f(x)=1-\frac{16}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$. 11.8. $\frac{-5}{2}+\frac{18}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$.

11.10. $f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$. 11.11. $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$

11.12. $\frac{2\sin\pi a}{\pi}\left(\frac{\sin x}{1a^2}-\frac{\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-.....\right)$ 11.13. $-\frac{2}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}+\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\sin nx$

11.14. $\frac{2\sin\pi a}{\pi}\left[\frac{1}{2a}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a\cos nx}{n^2-a^2}\right]$ 11.15. $\frac{2}{3}\pi^2+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$

11.16. $2Shah\left[\frac{1}{2ah}+\sum_{n=1}^{\infty}(-1)^n\frac{ah\cos\frac{n\pi x}{h}-\pi n\sin\frac{n\pi x}{h}}{(ah)^2+(\pi n)^2}\right]$

11.17. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}\left\{(-1)^\kappa\frac{\cos(2k+1)x}{2\kappa+1}\right\}$. 11.18. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}(-1)^k\frac{\sin(2k+1)x}{(2k+1)^2}$ 11.19. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$.

11.20. $2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin x}{n}$. 11.21. $\sum_{n=1}^{\infty}\frac{\sin 2nx}{2n}$. 11.22. $\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{(-1)^{\kappa+1}\sin(2\kappa-1)x}{(2\kappa-1)x}$.

11.23. $\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$. 11.24. $-1+\frac{\pi}{4}-\frac{2}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$

11.25. $\frac{-\pi^2}{3}-4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$

12- amaliy mashg'ulot.

FURYE INTEGRALI

$f(x)$ funksiya $(-\infty,\infty)$ cheksiz integralda aniqlangan va unda absalyut integrallanuvchi bo'lsin, ya'ni

$$\int_{-\infty}^{\infty}|f(x)|dx=Q$$

integral mavjud bo'lsin.

Ushbu

$$f(x)=\int_0^{\infty}[A(\lambda)\cos\lambda x+B(x)\sin\lambda x]dx \qquad (1)$$

tenglikning o'ng tomoni $f(x)$ funksiya uchun Furye integrali deb ataladi, bunda $A(\lambda)$, $B(\lambda)$ quyidagicha aniqlanadi:

$$A(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos\lambda\, t\, dt,$$

$$B(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(t)\sin\lambda t dt$$

Ushbu $F(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\cos\lambda t dt$ va $\Phi(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\sin\lambda t dt$ funksiyalarga $f(x)$ funksiya uchun furyening mos ravishda kosinus va sinus almashtirishlari deyiladi.

Ushbu

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(t)e^{-i\lambda(t-x)}dt\right]d\lambda$$

formulaning o'ng qismi $f(x)$ funksiya uchun kompleks formadagi Furye integrali deb aytiladi.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni Furye integralini tasvirlang

12.1. $f(x)=\begin{cases}1, & |x|<1,\\ 0, & |x|>1.\end{cases}$ 12.2. $f(x)=\begin{cases}signx, & |x|<1,\\ 0, & |x|>1.\end{cases}$

12.3. $f(x)=sign(x-a)-sign(x-b)$ $(b>a)$. 12.4. $f(x)=\frac{1}{a^2+x^2}$ $(a>0)$.

12.5. $f(x)=\frac{x}{a^2+x^2}$ $(a>0)$. 12.6. $f(x)=e^{-a|x}$ $(a>0)$.

12.7. $f(x)=\begin{cases}\sin x, & |x|\le\pi,\\ 0, & |x|>\pi.\end{cases}$ 12.8. $f(x)=\begin{cases}\cos x, & |x|\le\pi/2,\\ 0, & |x|>\pi/2.\end{cases}$

12.9. $f(x)=e^{-a|x|}\cos\beta x$ $(a>0)$. 12.10. $f(x)=e^{-a|x|}\sin\beta x$ $(a>0)$.

Misollarning javoblari

12.1. $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda}{\lambda}\cos\lambda x\,d\lambda$ 12.2. $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos\lambda}{\lambda}\sin\lambda x\,d\lambda$

12.3. $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda(x-a)-\sin\lambda(x-b)}{\lambda}d\lambda$. 12.4. $\frac{1}{a^2+x^2}=\frac{1}{a}\int_0^{+\infty}e^{-a\lambda}\cos\lambda x\,d\lambda$.

12.5. $\frac{x}{a^2+x^2}=\frac{1}{a}\int_0^{+\infty}e^{-a\lambda}\sin\lambda x\,d\lambda$. 12.6. $f(x)=\frac{2a}{\pi}\int_0^{\infty}\frac{\cos\lambda x}{\lambda^2+a^2}d\lambda$

12.7. $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\pi}{1-\lambda^2}\sin\lambda x d\lambda$ 12.8. $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos 0.5\lambda\pi}{1-\lambda^2}\cos\lambda x d\lambda$.

12.9. $f(x)=\frac{a}{\pi}\int_0^{\infty}[\frac{1}{(\lambda-\beta)^2+a^2}+\frac{1}{(\lambda+\beta)^2+a^2}]\cos\lambda x d\lambda$.

12.10. $f(x)=\frac{a}{\pi}\int_0^{\infty}\frac{\lambda\sin\lambda x}{[(\lambda-\beta)^2+a^2][(\lambda+\beta)^2+a^2]}d\lambda$

13- amaliy mashg'ulot.

O'ZGARUVCHILARI AJRALADIGAN BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR. KOSHI MASALASI

13.1. O'zgaruvchilari ajralgan va ajraladigan differensial tenglamalar. Ushbu ko'rinishdagi tenglamani qaraymiz:

$$\frac{dy}{dx} = f(x)\varphi(y) \quad . \qquad \textbf{(13.1)}$$

Bu yerda $f(x)$ va $\varphi(y)\neq 0$ uzluksiz funksiyalardir. Tenglamaning ikkala qismini dx ga ko'paytiramiz: $dy = f(x)\varphi(y)dx$. Endi ikkala qismini $\varphi(y)\neq 0$ ga bo'lamiz:

$$\frac{dy}{\varphi(y)} = f(x)dx \qquad (13.2)$$

(13.2) tenglamaga *o'zgaruvchilari ajralgan tenglama* deyiladi. Ikkala qismini integrallaymiz:

$$\int\frac{dx}{\varphi(y)} = \int f(x)dx + C$$

Bu ifoda yechim y, argument x va o'zgarmas C ni aniqlovchi munosabatdir, ya'ni (13.1) tenglamaning umumiy integralidir.

(13.1) ko'rinishdagi tenglamaga *o'zgaruvchilari ajraladigan differensial tenglama* deyiladi.

O'zgaruvchilari ajraladigan differensial tenglama quyidagi ko'rinishda ham bo'lishi mumkin:

$$f_1(x)\varphi_1(y)y' + f_2(x)\varphi_2(y) = 0 \qquad (13.3)$$

yoki

$$f_1(x)\varphi_1(y)dy + f_2(x)\varphi_2(y)dx = 0 .$$

Bu yerda $f_1(x)$, $\varphi_1(y)$, $f_2(x)$, $\varphi_2(y)$ funksiyalar uzluksizdir. (13.3) tenglamani o'zgaruvchilari ajralgan tenglamaga keltirish uchun dx qatnashgan hadi o'ng tomonga o'tkazamiz va $f_1(x)\neq 0$, $\varphi_2(y)\neq 0$ shartni hisobga olgan holda ikkala qismini $f_1(x)\,\varphi_2(y)\neq 0$ ga bo'lamiz:

$$\frac{\varphi_1(y)dy}{\varphi_2(y)} = -\frac{f_2(x)dx}{f_1(x)} \qquad (13.4)$$

ko'rinishdagi o'zgaruvchilari ajralgan tenglamani hosil qilamiz.

(13.4) tenglamaning umumiy integrali quyidagicha yoziladi:

$$\int\frac{\varphi_1(y)dy}{\varphi_2(y)} = -\int\frac{f_2(x)dx}{f_1(x)} \qquad (13.5)$$

13.2. Birinchi tartibli bir jinsli differensial tenglamalar. Bir jinsliga keltiriladigan differensial tenglamalar.

1-ta'rif. Agar ixtiyoriy $k>0$ uchun $F(kx,ky)=k^nF(x,y)$ tenglik o'rinli bo'lsa, $F(x,y)$ ga *n-darajali bir jinsli funksiya* deyiladi. Masalan,

$$\frac{x^2-y^2}{x^2+y^2},\quad \frac{x^5+xy^4}{x^4+y^4},\quad x^2+y^2-5xy,\quad x^n+x^{n-\lambda}y^\lambda+y^n$$

funksiyalar mos ravishda 0, 1, 2, n-darajali bir jinslidir.

Agar $f(x,y)$ 0-darajali bir jinsli funksiya bo'lsa, u holda ushbu

$$y' = f(x,y) \qquad (13.6)$$

differensial tenglama *bir jinsli differensial tenglama* deyiladi. Agar $M(x,y)$, $N(x,y)$ lar bir xil darajali bir jinsli funksiyalar bo'lsa, $M(x,y)dx + N(x,y)dy = 0$ tenglamalar *bir jinsli differensial tenglama* deyiladi.

Xususiy holda, $y' = f(y/x)$ differensial tenglama *bir jinsli differensial tenglama* deyiladi.

(13.6) tenglamada $x\neq 0$, $f(x,y) = f\left(1,\frac{y}{x}\right) = \varphi\left(\frac{y}{x}\right)$ funksiya x va y ning barcha qaralayotgan qiymatlarida uzluksizdir. Bu tenglama

$$\frac{y}{x}=u \qquad y=ux \qquad y'=u+xu' \qquad (13.7)$$

o'rniga qo'yish bilan o'zgaruvchilari ajraladigan tenglamaga keltiriladi:

$$u + xu' = \varphi(u) \qquad \text{yoki} \qquad x\frac{du}{dx} = \varphi(u) - u .$$

Bundan quyidagi o'zgaruvchilari ajralgan tenglama hosil bo'ladi:

$$\frac{du}{\varphi(u)-u} = \frac{dx}{x}.$$

1-misol. $(1+x)y + (1-y)xy' = 0$ tenglamani yeching.

Yechilishi. $y' = \frac{dy}{dx}$ munosabatdan foydalanib berilgan tenglamani quyidagicha yozib olamiz: $(1+x)ydx + (1-y)xdy = 0$. O'zgaruvchilarga ajratamiz:

$$\frac{(1-y)dy}{y} = -\frac{(1+x)dx}{x} \qquad \text{yoki} \qquad \left(\frac{1}{y}-1\right)dy = -(\frac{1}{x}+1)dx$$

Bu o'zgaruvchilari ajralgan tenglamadir. Integrallab topamiz: $\ln|y| - y = -(\ln|x| + x) + C$ yoki $\ln|xy| + x - y = C$. Oxirgi munosabat berilgan tenglamaning umumiy integralidir.

2-misol. $y' = \frac{y}{x} + \sin\frac{y}{x}$ tenglama yeching.

Yechilishi. (1.1) o'rniga qo'yishni bajarsak: $u + xu' = u + \sin u$ hosil bo'ladi. Bu yerdan

$$x\frac{du}{dx} = \sin u, \qquad \frac{du}{\sin u} = \frac{dx}{x}$$

Oxirgi munosabatni integrallab

$$\ln\left|tg\frac{u}{2}\right| = \ln|x| + \ln C \qquad \text{yoki} \qquad u = 2arctgCx$$

$u = \frac{y}{x}$ dan foydalanib tenglamaning umumiy yechimini aniqlaymiz:

$$y = 2xarctgCx.$$

Mustaqil yechish uchun misollar

O'zgaruvchilari ajraladigan differensial tenglamalarni yeching.

13.1. $(1+y)dx - (1-x)dy = 0.$ 13.2. $\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0.$

13.3. $xyy' = 1 - x^2.$ 13.4. $e^y(1+y') = 1.$

13.5. $y'(1+y) = xy\sin x.$ 13.6. $y' - xy^2 = 0.$

13.7. $2\sqrt{y}dx - dy = 0,\ y(0) = 1$ 13.8. $y' = 8\sqrt{y},\ y(0) = 4$

13.9. $y'\sin x - y\ln y = 0, \left(\frac{\pi}{2}\right) = 1.$ 13.10. $(1+y^2)dx + (1+x^2)dy = 0,\ y(1) = 2$

13.11. Differensial tenglamani o'zgaruvchini almashtirish yo'li bilan yeching

a) $y' = y\sin x^2;$ b) $(2x-y)dx + (4x-2y+3)dy = 0$

v) $y' = \frac{\cos y - \sin y - 1}{\cos x - \sin x + 1};$ g) $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0,\ y(0) = 1;$

d) $y' = 3x - 2y + 1;$ e) $y' = \cos(y-x)$

Birinchi tartibli bir jinsli differensial tenglamalarni yeching.

13.12. $xy' = y + \sqrt{x^2+y^2}$ 13.13. $\frac{dy}{dx} = \frac{y}{x} - \frac{x}{y}.$

13.14. $y = xy' - xe^{\frac{y}{x}}$ 13.15. $xy' - y(\ln y - \ln x) = 0.$

13.16. $y' = \frac{y + 2\sqrt{xy}}{x}$ 13.17. $ss' - 2s + t = 0.$

13.18. $x^2 + y^2 = 2xyy'$ 13.19. $\sqrt{y}(2\sqrt{x} - \sqrt{y})dx + xdy = 0.$

13. 20. Bir jinsliga keltiriladigan differensial tenglamalarni yeching.

a) $y' = \frac{3x-4y-2}{3x-4y-3};$ b) $y' = \frac{x+y-2}{3x-y-2}$

Mustaqil yechish uchun misollarning javoblari

13.1. $(1-x)(1+y) = C.$ 13.2. $\sqrt{1-y^2} = \arcsin x + C$

13.3. $x^2 + y^2 = \ln Cx^2.$ 13.4. $y + \ln|y| = \sin x - x\cos x + C.$

13.5. $y = \ln(1 + Ce^{-x})$ 13.6. $y = -\frac{2}{C + x^2}.$ 7. $y = (x+1)^2.$

13.8. $y = (4x+2)^2.$ 13.9. $y = 1.$ 13.10. $\frac{x+y}{1-xy} = -3.$

13.11. a) $y = Ce^{\int \sin x^2 dx}.$ б) $5x + 10y + C = 3\ln|10x - 5y + 6|;$

в) $tg\frac{y}{2} = C\left(tg\frac{y}{2} + 1\right)\left(1 - tg\frac{x}{2}\right);$ г) $y = 1, x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1;$

д) $4y - 6x + 1 = Ce^{-2x};$ e) $ctg\frac{y-x}{2} = x + C,\ y - x = 2\pi k,\ k \in Z.$

13.12. $y = \frac{Cx^2}{2} - \frac{1}{2C}.$ 13.13. $y^2 = 2x^2\ln\frac{C}{x}.$

13.14. $e^{-\frac{y}{x}} + \ln Cx = 0.$ 13.15. $y = xe^{Cx+1}.$ 13.16. $y = x\ln^2 Cx.$

13.17. $\frac{t}{s-t} = \ln C(s-t).$ 13.18. $x^2 - y^2 = Cx.$ 13.19. $y = x\ln^2\frac{C}{x}$

13.20. a) $x - y + C = \ln|3x - 4y + 1|;$ b) $(y-x)e^{\frac{2x-2}{y-x}} = C,\ y = x.$

14- amaliy mashg'ulot.
DIFFERENSIAL TENGLAMANING TURLARI VA YECHISH USULLARI

1-ta'rif. Izlanayotgan funksiya va uning hosilasiga nisbatan chiziqli bo'lgan tenglamaga *birinchi tartibli chiziqli differensial tenglama* deyiladi.

Uning umumiy ko'rinishi quyidagicha ifodalanadi:

$$A(x)\frac{dy}{dx} + B(x)y = C(x),$$

bu yerda $A(x) \neq 0$ va $A(x)$, $B(x)$, $C(x)$ lar x ning (a,b) dagi qiymatlari uchun uzluksiz funksiyalardir. $A(x) \neq 0$ bo'lgani uchun birinchi tartibli chiziqli differensial tenglamani

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (14.1)$$

ko'rinishda yozish mumkin, bu yerda $P(x) = \frac{B(x)}{A(x)}$ va $Q(x) = \frac{C(x)}{A(x)}$-berilgan uzluksiz funksiyalardir.

Agar $Q(x) = 0$ bo'lsa,

$$y' + P(x)y = 0$$

tenglamaga *chiziqli bir jinsli* differensial tenglama deyiladi.

(14.1) differensial tenglamaning umumiy yechimi

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx} + C^*\right]. \qquad (14.2)$$

ko'rinishda bo'ladi

14.2. Bernulli tenglamasi. Ushbu

$$y' + P(x)y = Q(x)y^n \qquad (14.3)$$

ko'rinishdagi differensial tenglamaga *Bernulli tenglamasi* deyiladi.

Bu yerda $P(x)$ va $Q(x)$-berilgan uzluksiz funksiyalar, $n = const$, $n = 0$ da bu tenglama chiziqli, $n = 1$ da o'zgaruvchilari ajraladigan tenglamaga aylanadi. Differensial tenglamani yechish uchun $n \neq 0, 1$deb faraz qilamiz va ikkala qismini $y^n \neq 0$ ga bo'lamiz

$$y^{-n}y' = -P(x)y^{1-n} + Q(x) \qquad (14.4)$$

Belgilash kiritamiz: $z = y^{1-n}$, $z' = (1-n)y^{-n}y'$. U holda

$$y^{-n}\cdot y' = \frac{1}{1-n}z'$$

z va z' ning ifodalarini (14.4) ga qo'ysak, z ga nisbatan chiziqli tenglamani hosil qilamiz:

$$\frac{1}{1-n}z' = -P(x)z + Q(x) \Rightarrow z' + (1-n)P(x)z = (1-n)Q(x).$$

Bu tenglamani xuddi chiziqli tenglamani yechgandek yechsak (14.3) Bernulli tenglamasining umumiy yechimi hosil bo'ladi

$$y = \left\{e^{(n-1)\int P(x)dx}\left[(1-n)\int Q(x)e^{(1-n)\int P(x)dx} + C^*\right]\right\}^{\frac{1}{1-n}}.$$

1-misol. $y' - y\,ctg\,x = 2x\sin x$ tenglamaning umumiy yechimini toping.

Yechilishi. Bir jinsli tenglamaning umumiy yechimini topib olaylik

$$\frac{dy}{dx} - y\,ctg\,x = 0, \quad \frac{dy}{y} = \frac{\cos x}{\sin x}dx, \quad \int\frac{dy}{y} = \int\frac{\cos x}{\sin x}dx,$$

$$\ln|y| = \ln|\sin x| + \ln C_1, \quad C_1 > 0, \quad y = C\sin x, \quad C \in R.$$

Endi o'zgarmasni variasiyalaymiz, ya'ni berilgan tenglamaning yechimini $y = C(x)\sin x$ ko'rinishida izlaymiz, bu yerda $C(x)$ hozircha noma'lum funksiya.

$$y = C(x)\sin x \quad, \ y' = C'(x)\sin x + C(x)\cos x$$

differensial tenglamani qo'yib quyidagini olamiz

$$C'(x)\sin x + C(x)\cos x - C(x)\cos x = 2x\sin x,$$

$$C'(x) = 2x, \quad C(x) = x^2 + C_2, \quad y = (x^2 + C_2)\sin x.$$

Demak, berilgan differensial tenglamaning umumiy yechimi $y = x^2\sin x + C_2\sin x$ ko'rinishda ekan.

2-misol. $xy' = y - 3x^2y^2$ differensial tenglama o'zgarmasni variasiyalash (Lagranj usuli) va Bernulli usullarida integrallang.

Yechilishi (Lagranj usuli). Bir jinsli differensial tenglamani qaraymiz $xy' = y$. Bu differensial tenglamaning umumiy yechimi $y = Cx$. Faraz qilaylik $y = C(x)x$, u holda $y' = C(x) + xC'(x)$. Bularni berilgan tenglamaga qo'yib, quyidagi ifodani hosil qilamiz:

$$x[C(x) + xC'(x)] = C(x)x - 3x^2C^2(x)x^2 \text{yoki } C'(x) = -3x^2C^2(x).$$

Bu ifodani integrallab, topamiz

$$\int\frac{dC}{C^2} = -3\int x^2dx - C^*, \quad -\frac{1}{C(x)} = -x^3 - C^*, \quad C(x) = \frac{1}{x^3 + C^*}$$

Demak, berilgan differensial tenglamaning umumiy yechimi:

$$y = x\cdot C(x) = \frac{x}{x^3 + C^*}.$$

Bernulli usuli. $y = u(x)v(x)$, $y' = u'(x)v(x) + u(x)v'(x)$ larni differensial tenglamaga qo'yamiz:

$$x\,x[u'(x)v(x) + u(x)v'(x)] = u(x)v(x) - 3x^2u^2(x)v^2(x)$$

yoki

$$\begin{cases} xu' - u = 0 \\ v' = -3xuv^2 \end{cases}$$

Bu yerda $xu' = u$ differensial tenglamani integrallab $u = x$ xususiy yechimini tanlaymiz, so'ngra ikkinchi differensial tenglamani $v' = -3xuv^2$, $v' = -3x^2v^2$ ni integrallaymiz $\frac{dv}{v^2} = -3x^2dx$. Bu yerdan $v(x) = \frac{1}{x^2 + C^*}$ ni topamiz. U holda berilgan differensial tenglamaning umumiy yechimi: $y = u\cdot v = \dfrac{x}{x^2 + C^*}$.

Mustaqil yechish uchun misollar

Birinchi tartibli chiziqli va Bernulli differensial tenglamalarini yeching

14.1. $y' + 2y = 3e^x$. 14.2. $(1+x^2)y' + 2xy = 3x^2$

14.3. $2(x+y^4)y' - y = 0$. 14.4. $y^2dx + (xy-1)dy = 0$

14.5. $xy' + y = \frac{y^2}{2}\ln x$. 14.6. $y' + 2xy = 2xy^3$

14.7. $y' + \frac{1}{x}y = \frac{y^2}{2}\ln x$. 14.8. $x\frac{dy}{dx} + y = 4x^3$

14.9. $y'e^{x^2} - (xe^{x^2} - y^2)y = 0$. 14.10. $x^3y^2y' + x^2y^3 = 1$.

14.11. $y'x^3\sin y - xy' + 2y = 0$. 14.12. $y' - y = \left(x + \frac{1}{x}\right)e^x$

14.13. $y' + \frac{x}{1-x^2}y = 2$. 14.14. $y' - \frac{y}{\sin x} = tg\frac{x}{2}$

14.15. $xy' - y - x^3 = 0, y(2) = 4$ 14.16. $y'\sin x - y\cos x = 1, y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Mustaqil yechish uchun misollarning javoblari

14.1. $y = Ce^{-2x} + e^x$. 14.2. $y = \frac{x^3+C}{x^2+1}$. 14.3. $x = y^4 + Cy^2, y = 0$

14.4. $x = \frac{\ln y + C}{y}, y = 0$. 14.5. $y = \frac{2}{\ln x + Cx + 1}$. 14.6. $y = \frac{1}{\sqrt{Ce^{2x^2}+1}}, y = 0$

14.7. $y = [xC_1 - 0{,}25x\ln^2 x]^{-1}$. 14.8. $y = x^3 + \frac{C}{x}$. 14.9. $y^2(2x+C) = e^{x^2}, y = 0$.

14.10. $y = \frac{\sqrt[3]{3x+C}}{x}$. 14.11. $x^2 = \frac{y}{C - \cos y}, y = 0$.

14.12. $y = e^x\left(C + \ln x + \frac{x^2}{2}\right)$ 14.13. $y = \sqrt{1-x^2}(2\arcsin x + C)$

14.14. $y = (x+C)tg\frac{x}{2}$. 15. $y = \frac{1}{2}x^3$. 14.16. $y = 2\sin x - \cos x$.

15- amaliy mashg'ulot.
TO'LIQ DIFFERENSIAL TEGLAMA. KLERO VA LAGRANJ TEGLAMALARI

15.1. To'liq differensialli tenglamalar.

1-ta'rif. *Agar*

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

tenglamaning chap tomoni birorta $u(x,y)$-ikki o'zgaruvchili funksiyaning to'liq differensiali bo'lsa, bu tenglamaga to'liq differinsial tenglama deyiladi.

(1) tenglamani $du(x,y) = 0$ ko'rinshda yozish mumkin. Oxirgi ifodani integrallab, umumiy integralni hosil qilamiz:

$$\int du(x,y) = C, \quad u(x,y) = C$$

1-teorema. *Ushbu $M(x,y)dx + N(x,y)dy$ ifoda birorta $u(x,y)$ funksiyaning to'liq differensiali bo'lishi uchun qaralayotgan sohaning barcha nuqtalarida*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (2)$$

shart bajarishi zarur va yetarlidir. Bu yerda $M(x,y)$, $N(x,y)$ funksiyalar Oxy tekislikning D sohasida aniqlangan va uzluksiz bo'lib, uzluksiz xususiy hosilalar $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ ga egadir.

Ushbu

$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy + C. \qquad (3)$$

formula to'liq differinsial tenglamasining umumiy yechimini topish formulasi bo'ladi.

13.2. Integral ko'paytuvchi. Agar (1) tenglama uchun (2) shart bajarilmagan bo'lsa, uning chap qismi biror funksiyanig to'liq differensiali bo'la olmaydi. Bunday holatlarga ba'zan (1) tenglamani $\mu(x,y)$ funksiyaga ko'paytirish bilan uni to'la differensialli tenglamaga keltirish mumkin bo'ladi. Bunday holda $\mu(x,y)$ funksiyaga (1) tenglamaning *integrallovchi ko'paytuvchisi* deyiladi.

Agar $\mu(x)$ ifoda x ga bog'liq bo'lmasdan, faqat y ning funksiyasidan iborat bo'lsa, u holda faqat y ga bog'liq bo'lgan integrallovchi ko'paytuvchi $\mu(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy\right]$ bo'ladi.

Agar $\mu(x)$ ifoda y ga bog'liq bo'lmasdan, faqat x ning funksiyasidan iborat bo'lsa, u holda faqat x ga bog'liq bo'lgan integrallovchi ko'paytuvchi $\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx\right]$ bo'ladi.

27.3. Lagranj tenglamasi. Ushbu $y = x\varphi(y') + \phi(y')$ tenglamaga Lagranj tenglamasi deyiladi. Bu tenglamani x bo'yicha differensiallab, $y' = p$ desak,

$$p = \varphi(p) + x\varphi'(p)\frac{dp}{dx} + \phi'(p)\frac{dp}{dx} \qquad (8)$$

yoki

$$[p - \varphi(p)]\frac{dp}{dx} = x\varphi'(p) + \phi'(p). \qquad (9)$$

Bu chiziqli differensial tenglama va qiyinchiliksiz intgrallanadi (3-§, 1 p. ga qarang). (9) ning integrali $\Phi(x,p,C) = 0$ va $y = x\varphi(p) + \phi(p)$ birgalikda Lagranj tenglamasini beradi.

$$\begin{cases} \Phi(x,p,C) = 0, \\ y = x\varphi(p) + \phi(p). \end{cases} \qquad (10)$$

Faqat biz (8) dan (9) ga o'tayotganda tenglikni dp/dx ga bo'lish chog'ida $p = p_i$ o'zgarmas yechimlarni (agar ular mavjud bo'lsa) yo'qotayapmiz, $dp/dx \equiv 0$. p ni o'zgarmas desak, y (8) ni qanoatlantirishi uchun albatta $p-\varphi(p)=0$ tenglamani qanoatlantirishi kerak, chunki $dp/dx \equiv 0$. Demak, agar $p-\varphi(p)=0$ tenglamaning haqiqiy $p = p_i$ yechimlari mavjud bo'lsa, (10) ga uning to'liq bo'lishi uchun $y = x\varphi(p_i)+\phi(p_i)$ ni qo'shib qo'yish kerak. Shunday qilib, umuman integral chiziqlar

$$\Phi(x,p,C)=0,$$

$$y = x\varphi(p)+\phi(p) \qquad (11)$$

yoki

$$y = x\varphi(p_1)+\phi(p_1)$$

dan iborat bo'ladi.

27.4. Klero tenglamasi. Ushbu

$$y = xy'+\phi(y')$$

tenglamaga Klero tenglamasi deyladi. $y' = p$ deb olsak, $y = xp+\phi(p)$ ni olamiz. Differensiallab,

$$p = p + x\frac{dp}{dx}+\phi'(p)\frac{dp}{dx}$$

yoki

$$\left(x+\phi'(p)\right)\frac{dp}{dx}=0$$

tenglikni olamiz. Bundan $\frac{dp}{dx}=0$ yoki $x+\phi'(p)=0$ kelib chiqadi.

Birinchi holda $p = C$ bo'lib, $y = xp+\phi(p)$ dan

$$y = Cx+\phi(C) \qquad (12)$$

integral chiziqlar oilasini olamiz.

Ikkinchi holda yechim

$$y = xp+\phi(p) \quad \text{va} \quad x+\phi(p)=0 \qquad (13)$$

tenglamalar bilan aniqlanadi.

Qiyinchiliksiz shunga ishonch hosil qilish mumkinki, (13) tengliklar bilan aniqlanadigan integral chiziq (12) integral chiziqlar oilasining o'ramasi bo'ladi.

Haqiqatdan ham, qandaydir $\Phi(x,p,C)=0$ chiziqlar oilasining o'ramasi

$$\Phi(x,p,C)=0, \quad \partial\Phi/\partial C=0 \qquad (14)$$

tenglamalar bilan aniqlanadi. Shuning uchun (12) chiziqlar oilasining o'ramasi

$$y = xC+\phi(C), \quad x+\phi'(C)=0$$

tenglamalar bilan aniqlanadi, bular (12) dan faqat parametri bilan farq qiladi, xolos.

1-misol. $\frac{2x}{y^3}dx+\frac{y^2-3x^2}{y^4}dy=0$ tenglamaning umumiy integralini toping.

Yechilishi. Bu misolda $M(x,y)=\frac{2x}{y^3}$, $N(x,y)=\frac{y^2-3x^2}{y^4}$. Bundan ko'rinadiki

$$\frac{\partial u}{\partial x}=M(x,y)=\frac{2x}{y^3} \quad \frac{\partial u}{\partial y}=N(x,y)=\frac{y^2-3x^2}{y^4},$$

$y \neq 0$ shartda

$$\frac{\partial M}{\partial y}=-\frac{6x}{y^4}, \quad \frac{\partial N}{\partial x}=-\frac{6x}{y^4}.$$

Demak, (2.7) shart bajarildi. Berilgan tenglamaning chap qismi qandaydir $u(x,y)$ funksiyaning to'liq differensialini ifodalar ekan. Shu funksiyani aniqlaymiz.

$\frac{\partial u}{\partial x}=\frac{2x}{y^3}$ ifodadan:

$$u(x,y)=\int\frac{2x\,dx}{y^3}+\varphi(y), \quad u(x,y)=\frac{x^2}{y^3}+\varphi(y).$$

Bu munosabatni y bo'yicha differensiyallaymiz: $\frac{\partial u}{\partial y}=-\frac{3x^2}{y^4}+\varphi'(y)$.

Endi $\frac{\partial u}{\partial y}=\frac{y^2-3x^2}{y^4}$ ni hisobga olganda: $\frac{y^2-3x^2}{y^4}=-\frac{3x^2}{y^4}+\varphi'(y) \Rightarrow$

$$\frac{y^2}{y^4}-\frac{3x^2}{y^4}=-\frac{3x^2}{y^4}+\varphi'(y)$$

dan $\varphi'(y)=\frac{1}{y^2}$ hosil bo'ladi, yoki $\frac{d\varphi}{dy}=\frac{1}{y^2}$, bundan $\varphi'(y)=-\frac{1}{y}+C.$

U holda $u(x,y)=\frac{x^2}{y^3}-\frac{1}{y}+C.$ Tenglamaning umumiy integrali

$$\frac{x^2}{y^3}-\frac{1}{y}=C_1.$$

Mustaqil yechish uchun misollar

To'liq differensial tenglamani yeching

15.1. $(3x-5x^2y^2)dx+\left(3y^2-\frac{10}{3}x^3y\right)dy=0.$ 15.2. $(x\cos 2y-3)dx-x^2\sin 2y\,dy=0.$

15.3. $(2x+ye^{xy})dx+(1+xe^{xy})dy=0, y(0)=1.$

15.4. $\left(\frac{x}{\sqrt{x^2+y^2}}+y\right)dx+\left(x+\frac{y}{\sqrt{x^2+y^2}}\right)dy=0, y(\sqrt{2})=\sqrt{2}.$

15.5. $(x^2+2xy+1)dx+(x^2+y^2-1)dy=0.$

15.6. $\sin(x+y)dx+x\cos(x+y)(dx+dy)=0.$ 15.7. $(3x^2+3x^2\ln y)dx-\left(2y-\frac{x^3}{y}\right)dy=0.$

15.8. $3x^2y+\sin x=(\cos y-x^3)y'.$ 15.9. $(3x^2+y^2+y)dx+(2xy+x+e^y)dy=0, y(0)=0$

15.10. $(x^2+2xy)dx+(x^2-y^2)dy=0, y(1)=-1.$ 15.11. $\frac{(x-y)dx+(x+y)dy}{x^2+y^2}=0.$

Klero va Lagranj teglamalarini yeching .

15.12. $y=\sqrt{1-y'^2}+y'.$ 15.13. $y'=\ln(xy'-y).$ 15.14. $2yy'-x(y'^2+4)=0.$

15.15. $y=y'^2e^{y'}.$ 15.16. $y'+y-xy'^2=0.$

15.17. $y=xy'+y'+\sqrt{y}.$ 15.18. $xy'-y=\ln y'.$

Tenglamani integrallovchi ko'paytuvchi usulidan foydalanib yeching

15.19. $(3x+y^2)dx-2xydy=0.$ 15.20. $4xydx+(y^3+4x^2)dy=0$

15.21. $(xy-4)dx+x^2dy=0.$ 15.22. $\frac{6x+y^3}{x}dx-3y^2dy=0$

15.23. $2ydx+(x+7y^3)dy=0$

Mustaqil yechish uchun misollarning javoblari

15.1. $\frac{3}{2}x^2-\frac{5}{3}x^3y^2+y^3=C.$ 15.2. $\frac{x^2}{2}\cos 2y-3x=C.$ 15.3. $x^2+y+e^{xy}=2$

15.4. $\sqrt{x^2+y^2}+xy=4.$ 15.5. $x^3+y^3+3x^2y+3x-3y=C.$

15.6. $x\sin(x+y)=C.$ 15.7. $x^3+x^3\ln y-y^2=C.$ 15.8. $x^3y-\cos x-\sin y=C.$

15.9. $x^3+xy^2+xy+e^y=1.$ 15.10. $x^3+3x^2y-y^3+1=0.$ 15.11. $\frac{1}{2}\ln(x^2+y^2)-arctg\frac{x}{y}=C$

15.12. $\begin{cases} x=\ln p-\arcsin p+C,\\ y=p+\sqrt{1-p^2}.\end{cases}$ 15.13. $y=Cx-e^C,\ y=x\ln x-x.$

15.14. $y=Cx^2+\frac{1}{C},\ y=\pm 2x.$ 15.15. $\begin{cases} x=(p+1)e^p+C,\\ y=p^2e^p,\end{cases}\quad y=0.$

15.16. $\begin{cases} y=xp^2-p,\\ x=\frac{p-\ln p+C}{(p-1)^2}.\end{cases}$ 15.17. $y=Cx+C+\sqrt{C},\ y=-\frac{1}{4(x+1)}.$

15.18. $y=Cx-\ln C,\ y=\ln x+1.$

15.19. $\mu=14/x^2; y^2=x(3\ln|x|+C).$ 15.20. $\mu=y; 10x^2y^2+y^5=C.$ 15.21. $\mu=1/x; xy-4\ln|x|=C.$ 15.22. $\mu=1/x; 6\ln|x|-y^3/x=C.$ 15.23. $\mu=1/\sqrt{y};\ x\sqrt{y}+y^3\sqrt{y}=C$

16- amaliy mashg'ulot.

YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR. KOSHI MASALASI. TARTIBI PASAYADIGAN DIFFERENSIAL TENGLAMALAR.

16.1. $y^{(n)}=f(x)$ **ko'rinishdagi tenglamalar.** Yeng sodda n-tartibli tenglamani qaraymiz:

$$y^{(n)}=f(x) \quad (1)$$

Bu tenglamaning umumiy integralini topamiz. Ikkala qismini x bo'yicha integrallab va $y^{(n)}=(y^{(n-1)})'$ ekanligini hisobga olib,

$$y^{(n-1)}(x)=\int_{x_0}^{x}f(x)dx+C_1$$

ifodani hosil qilamiz. Yana bir marta integrallasak:

$$y^{(n-2)}=\int_{x_0}^{x}\left(\int_{x_0}^{x}f(x)dx+C_1\right)dx+C_2$$

Integrallashni shu tartibda davom ettirsak, n marta integrallashdan so'ng

$$y(x)=\int_{x_0}^{x}...\int_{x_0}^{x}f(x)dx...dx+\frac{C_1(x-x_0)^{n-1}}{(n-1)!}+\frac{C_2(x-x_0)^{n-2}}{(n-2)!}+...+C_n$$

ifodani hosil qilamiz.

16.2. $F(x,y^{(k)},...y^{(n)})=0$ **ko'rinishdagi tenglamalar.** Tartibini pasaytirish mumkin bo'lgan

$$F(x,y^{(k)},...y^{(n)})=0 \quad (2)$$

ko'rinishdagi n- tartibli differensial tenglamani qaraymiz. Izlanayotgan y funksiya va uning $(k-1)$ tartibgacha hosilalarining ishtirok etmasligidir. (2) tenglamani integrallash bilan shug'ullanamiz. $y^{(k)}=p(x)$ deb belgilasak, (2) tenglama k birlikka pasaytiriladi, ya'ni

$$F(x,p,p^{(1)},...p^{(n-k)})=0. \quad (3)$$

(3) tenglamani integrallab, yangi izlanayotgan funksiyani aniqlaymiz:

$$p=\varphi(x,C_1,C_2,...,C_{n-k}),$$

so'ngra $y^{(k)}=\varphi(x,C_1,C_2,...,C_{n-k})$ tenglamani k marta integrallab, umumiy yechimini topamiz.

Eslatma. (3) tenglamani integrallash usuli quyidagi xususiy hollar uchun ham o'rinlidir:

$$F(y',y'')=0,\quad F(x,y'')=0,\quad F(y'')=0$$

Ikkinchi tartibli differensial tenglamaning tartibini pasaytirish usuli bilan yechishni ikkita birinchi tartibli tenglamalar sistemasiga keltirib yechish usuli bilan almashtirish ham mumkin ya'ni,

$$\begin{cases} y'=p\\ F(x,p,p')=0\end{cases}$$

16.3 $F(y,y',y'',...y^{(n)})=0$ **ko'rinishdagi tenglamalar.** Yozilishda x argumentni oshkora o'z ichiga olmagan

$$F(y,y',y'',...y^{(n)})=0 \qquad (4)$$

tenglamani qaraymiz. $y'=p(y)$ o'rniga qo'yish (4) tenglamaning tartibini bir birlikka pasaytirishga imkon beradi. Bunda erkli o'zgaruvchi sifatida u ni qabul qilamiz. Bu ko'rinishdagi tenglamalarni integrallash uchun belgilash kiritamiz:

$$y'=p(y),\ y''=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=pp',$$

$$y'''=\frac{d}{dx}(pp')=\frac{d}{dy}(pp')\frac{dy}{dx}=\left(\frac{dp'}{dy}p+p'\frac{dp}{dy}\right)p=p''p^2+(p')^2p \text{ va h. k.}$$

$y',y'',...y^{(n)}$ larni (4) tenglamaga qo'yib, $(n-1)$- tartibli tenglamaga ega bo'lamiz.

1-misol. Ushbu $x^4y'''+2x^3y''=1$ tenglamaning $y(1)=0.5,\quad y'(1)=0.5,$ $y''(1)=-1$ boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimi toping.

Yechilishi. $y''=p\quad y'''=p'$ deb belgilash kiritib, $x^4p'+2x^3p=1$ yoki $p'+\frac{2}{x}p=\frac{1}{x^4}$ tenglamani hosil qilamiz. Bu chiziqli tenglamadir. (1.5.6) va (1.5.9) munosabatlarga asosan:

$$p=-\frac{1}{x^3}+\frac{C_1}{x^2}$$

umumiy yechimni topamiz. $y''(1)=p(1)=-1$ boshlang'ich shartdan foydalanib $C_1=0$ topamiz. Demak, $y''=-\frac{1}{x^3}$ bo'ladi. Bu yerdan $y'=\frac{1}{2x^2}+C_2$. $y'(1)=0.5$ boshlang'ich shartdan foydalanib $C_2=0$ topamiz. $y'=\frac{1}{2x^2}$ tenglamani integalaymiz, natijada $y=-\frac{1}{2x}+C_3$ yechimga ega bo'lamiz. Endi $y(1)=0.5$ boshlang'ich shartdan foydalanib $C_3=1$ topamiz.

Shunday qilib, berilgan tenglamaning izlanayotgan $y=1-\frac{1}{2x}$ xususiy yechimini topdik.

2-misol. $2yy''+y'^2=0$ tenglamani yeching.

Yechilishi. $y'=p(y),\quad y''=pp'$ bo'lgani uchun $2ypp'=-p^2$ yoki $2yp'=-pp$.

Bu o'zgaruvchilari ajraladigan tenglamadir $\dfrac{dp}{p}=-\dfrac{dy}{2\cdot y}$

$$\ln|p|=-\frac{1}{2}\ln|y|+\ln C_1 \Rightarrow\ p=\frac{C_1}{\sqrt{y}}.$$

Endi $y'=\frac{C_1}{\sqrt{y}}$ tenglamadan $y=(C_1x+C_2)^{2/3}$ umumiy yechimini topamiz.

Mustaqil yechish uchun misollar

Quyidagi differensial tenglamalarni umumiy yechimini toping.

16.1. $y''=\sin 4x+2x-3.$ 16.2 $y''=e^{5x}+\cos x-2x^3.$

16.3. $y''=xe^{x^2}+3^{-x}.$ 16.4. $y''=4\cos^4 x+2\sin^2\frac{x}{2}+\sqrt{x+2}.$

16.5. $y''=(e^{2x}+\sin 3x)x,\ y(0)=1,\ y'(0)=1.$

$y'=p$ almashtirish yordamida quyidagi differensial tenglamalarni umumiy yechimini toping.

16.6. $y''-\frac{2}{x}y'=2x^3.$ 16.7 $(x+1)y''=y'-1.$

16.8. $x^3y''+x^2y'-1=0.$ 16.9. $y''+y('tgx-\sin 2x)=0.$

16.10. $xy''-y'=x^2e^x.$ 16.11. $xy''\ln x=y'$

16.12. $y''tg\,x-y'-1=0.$ 16.13. $xy''+y'+x=0$

16.14. $(1+x^2)y''+2xy'-x^3=0.$

$y''=p'p$ almashtirish yordamida quyidagi differensial tenglamalarni umumiy yechimini toping.

16.15. $y''y^3=1.$ 16.16. $yy''-(y')^2-1=0$

16.17. $1+(y')^2-2yy''=0.$ 16.18. $yy''-3(y')^2=4y^2$

16.19. $y''=y'(1+(y')^2)$ 16.20. $y''=y'\ln y',\ y(0)=0,\ y'(0)=1.$

Mustaqil yechish uchun misollarning javoblari

16.1. $y=-\frac{1}{16}\sin 4x+\frac{x^3}{3}-\frac{3x^2}{2}+C_1x+C_2.$ 16.2. $y=\frac{1}{25}e^{5x}-\cos x-\frac{1}{10}x^5+C_1x+C_2.$

16.3. $y=\frac{1}{2}\int e^{x^2}dx+\frac{3^{-x}}{\ln^2 3}+C_1x+C_2.$

16.4. $y=\frac{5}{4}x^2-\frac{\cos 2x}{2}-\frac{\cos 4x}{32}+\cos x+\frac{4}{15}\sqrt{(x+2)^5}+C_{1x}+C_2$

16.5. $y=x\left(\frac{1}{4}e^{2x}-\frac{1}{9}\sin 3x\right)-\frac{1}{4}e^{2x}-\frac{2}{27}\cos 3x+\frac{5}{4}x+\frac{143}{108}.$

16.6. $y=\frac{x^5}{5}+C_1x^3+C_2$

16.7. $y=C_1(x+1)^2+x+C_2.$ 16.8. $y=\frac{1}{x}+C_1\ln x+C_2.$

16.9. $y = C_1 \sin x - x - x\frac{1}{2}\sin 2x + C_2$. 16.10. $y = (x-1)e^x + C_1x^2 + C_2$

16.11. $C_1x(\ln x - 1) + C_2$. 16.12. $y = -C_1 \cos x + C_2$.

16.13. $y = -\frac{x^2}{4} + C_1 \ln x + C_2$. 16.14. $y = \frac{c_1}{2}\left(e^{\frac{x+c_2}{c_1}} + e^{-\frac{x+c_2}{c_1}}\right)$

16.15. $C_1y^2 - 1 = (C_1x + C_2)^2$. 16.16. $y = \frac{C_1}{2}\left(e^{\frac{x+C_2}{C_1}} + e^{-\frac{x+C_2}{C_1}}\right)$

16.17. $(x - C_1)^2 = 4C_2(y - C_2)$. 16.18. $y\cos^2(x + C_1) = C_2$

16.19. $y = \pm \arcsin e^{x+C_1} + C_2$ va $y = C$. 16.20. $y = x$

17- amaliy mashg'ulot.

BIR JINSLI BO'LGAN CHIZIQLI DIFFERENSIAL TENGLAMALAR

17.1. O'zgarmas koeffisiyentli chiziqli tenglamalar.

1-ta'rif. *Agar $x \in [a,b]$ kesmada bo'lmaydigan* n *ta $\alpha_1, \alpha_2, ..., \alpha_n$ koeffisientlar mavjud bo'lib, ular bir vaqtda nolga teng bo'lganda*

$$\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n = 0 \text{ yoki } \sum_{i=1}^{n} \alpha_i y_i = 0 \qquad (1)$$

munosabat o'rinli bo'lsa, bu funksiyalar sistemasi chiziqli bog'liq deyiladi.

2-ta'rif. *Agar $x \in [a,b]$ kesmada barcha x lar uchun*

$$\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \neq 0 \text{ yoki } \sum_{i=1}^{n} \alpha_i y_i \neq 0 \qquad (2)$$

bo'lsa, ya'ni $y_1, y_2, ..., y_n$ larning har qanday chiziqli kombinasiyasi aynan nol bo'lmasa, u holda funksiyalarning bunday sistema chiziqli erkli deyiladi.

Shunday qilib (2) shart bajarilsa $y_1, y_2, ..., y_n$ xususiy yechimlar *chiziqli erkli* deyiladi.

Ushbu

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_{n-1}y' + a_n y = 0 \text{ yoki } \qquad L(y) = 0 \qquad (3)$$

ko'rinishdagi tenglamaga n-tartibli o'zgarmas koeffisiyentli chiziqli *bir jinsli differensial tenglama* deyiladi, bu yerda $a_1, a_2, ..., a_n$ -o'zgarmas sonlardir.

Differensial tenglamaning umumiy yechimi (3) quyidagi ko'rinishda bo'ladi:

$$y = C_1y_1 + C_2y_2 + ... + C_ny_n = \sum_{i=1}^{n} C_iy_i, C_i = const$$

Differensial tenglamani yechish uchun Eyler usulidan foydalanamiz, yana (3) differensial tenglamaning xususiy yechimini $y = e^{kx}$ ko'rinishda izlaymiz, $k = const$. U holda (3) dan

$$L(e^{kx}) = e^{kx}p(k) = 0, \; p(k) = k^n + a_1k^{n-1} + ... + a_{n-1}k + a_n = 0,$$

chunki $e^{kx} \neq 0$. Demak,

$$p(k) = 0 \quad \text{yoki} \quad k^n + a_1k^{n-1} + ... + a_{n-1}k + a_n = 0 \quad (4)$$

(4) tenglamaga (3) differensial tenglamaning *xarakteristik tenglamasi* deyiladi.

Yechimlarning fundamental sistemasi xarakteristik tenglamaning ildizlariga bog'liqdir. Uchta hol qaraladi.

1) Xarakteristik tenglamaning barcha ildizlari haqiqiy va har xil.

Algebraik (4) tenglamaning darajasiga asosan uning n ta har xil $k_1, k_2, ..., k_n$ ildizlari bo'ladi. Demak, n ta xususiy yechimini topamiz:

$$y_1 = e^{k_1x}, y_2 = e^{k_2x}, ..., y_n = e^{k_nx}$$

Bu xususiy yechimlar sistemasi fundamental bo'lishini isbotlaymiz. n-tartibli Vronskiy determinantini tuzamiz:

$$W(y_1, y_2, ... y_n) = \begin{vmatrix} e^{k_1x} & e^{k_2x} & .... & e^{k_nx} \\ k_1e^{k_1x} & k_2e^{k_2x} & ... & k_ne^{k_nx} \\ ... & ... & ... & ... \\ k_1^{n-1}e^{k_1x} & k_2^{n-1}e^{k_2x} & ... & k_n^{n-1}e^{k_nx} \end{vmatrix} =$$

$$= e^{(k_1+k_2+...+k_n)x} \begin{vmatrix} 1 & 1 & ... & 1 \\ k_1 & k_2 & ... & k_n \\ ... & ... & ... & ... \\ k_1^{n-1} & k_2^{n-1} & ... & k_n^{n-1} \end{vmatrix}$$

Bu determinantni hisoblash uchun umumiy bir qonuniyat topish maqsadida uchinchi tartibli determinantni qaraymiz:

$$W(y_1, y_2, y_3) = e^{(k_1+k_2+k_3)x} \begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{vmatrix} = e^{(k_1+k_2+k_3)x} \begin{vmatrix} 0 & 0 & 1 \\ k_1 - k_3 & k_2 - k_3 & k_3 \\ k_1^2 - k_3^2 & k_2^2 - k_3^2 & k_3^2 \end{vmatrix} =$$

$$= e^{(k_1+k_2+k_3)x} \begin{vmatrix} k_1 - k_3 & k_2 - k_3 \\ k_1^2 - k_3^2 & k_2^2 - k_3^2 \end{vmatrix} = e^{(k_1+k_2+k_3)x}(k_1 - k_3)(k_2 - k_3)(k_2 - k_1).$$

Umumiy holda shunga o'xshash ushbu formula o'rinli bo'ladi

$$W(y_1, y_2, ..., y_n) = e^{(k_1+k_2+...+k_n)x} \; (-1)^n(k_1 - k_2)(k_1 - k_3) ... (k_1 - k_n)$$
$$(k_2 - k_3) \cdot (k_2 - k_4) .... (k_{n-1} - k_n) \neq 0.$$

Demak, $y_1, y_2, ..., y_n$ funksiyalarning

$$y = C_1e^{k_1x} + C_2e^{k_2x} + ... + C_ne^{k_nx}$$

chiziqli kombinasiyasi differensial tenglamaning umumiy yechimidir.

2) Xarakteristik tenglamaning ildizlari karrali (ya'ni bir xil) bo'lgan hol. Bu holda $k_1 = k_2 = ... = k_n \equiv k$, $p(k) = 0$ tenglamaning haqiqiy va karrali ildizi bo'lsin, u holda unda

$$y_1 = e^{kx}, \quad y_2 = xe^{kx}, ..., y_n = x^{n-1}e^{kx}$$

n ta chiziqli erkli xususiy yechimlar mos keladi va tenglamaning umumiy yechimi ushbu $y = (C_1 + C_2x + ... + C_nx^{n-1})e^{kx}$ ko'rinishda bo'ladi.

3) Xarakteristik tenglamaning ildizlari kompleks sonlar bo'lgan hol. $p(k) = 0$ tenglama ildizlari n karrali $k = \alpha \pm i\beta$ qo'shma kompleks sonlardan iborat bo'lsa, $e^{(\alpha\pm\beta)x}, xe^{(\alpha\pm\beta)x}, x^2e^{(\alpha\pm\beta)x}, ..., x^{n-1}e^{(\alpha\pm\beta)x}$ kompleks yechimlar mos keladi. Bu yechimlarni yozamiz

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2e^{\alpha x}\cos\beta x, ..., x^{n-1}e^{\alpha x}\cos\beta x,$$

$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2e^{\alpha x}\sin\beta x, ..., x^{n-1}e^{\alpha x}\sin\beta x.$$

Shunday qilib, n karrali $k = \alpha \pm i\beta$ qo'shma kompleks ildizlarga $2n$ ta chiziqli erkli yechimlar mos keladi. Berilgan (3) differensial tenglamaning umumiy yechimi:

$$y = e^{\alpha x}[(C_1\cos\beta x + C_2\sin\beta x) + x(C_3\cos\beta x + C_4\sin\beta x) + ... + x^{n-1}(C_{n-1}\cos\beta x + C_n\sin\beta x)]$$

yoki

$$y = e^{kx}[(C_1 + C_3x + C_5x^2 + ... + C_{2n-1}x^{n-1})\cos\beta x + (C_2 + C_4x + C_6x^2 + ... + C_{2n}x^{n-1})\sin\beta x] \qquad (5)$$

ko'rinishda bo'ladi.

1-misol. $y''-5y'+6y = 0$ tenglamaning umumiy yechimini toping.

Yechilishi. Dastavval xarakteristik tenglama tuzamiz:

$$k^2 - 5k + 6 = 0$$

$$k_{1,2} = \frac{5 \pm \sqrt{5^2 - 4\cdot1\cdot6}}{2\cdot1} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5\pm1}{2},$$

$$k_1 = \frac{5-1}{2} = \frac{4}{2} = 2, \qquad k_2 = \frac{5+1}{2} = \frac{6}{2} = 3.$$

Xususiy yechimlar: $y_1 = e^{2x}$ va $y_2 = e^{3x}$ ko'rinishda bo'lib fundamental sistema hosil qiladi. Haqiqatdan ham

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{2x}e^{3x}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = e^{(2+3)x}(3-2) = e^{5x}\cdot 1 = e^{5x} \neq 0.$$

Demak, (5.3) ga asosan, qaralayotgan tenglamaning umumiy yechimi quyidagicha $y = C_1e^{2x} + C_2e^{3x}$ bo'ladi.

Mustaqil yechish uchun misollar

Funksiyalarni chiziqli bog'liq yoki erkli ekanligini tekshiring.

17.1. $\arcsin x$ ва $\arccos x$ 17. 2. $\sin x$., $\sin 2x$

17.3. e^x, e^{x^2}. 17.4. $1, x$. 17.5. $\sin x$, $\sin^2 x$.

17.6. $\sin x\cos x$, $\sin 2x$. 17.7. $1 - \cos 2x, \sin^2 x$. 17.8. $1 + \cos 2x, \cos^2 x$

Quyidagi o'zgarmas koeffisiyentli differensial tenglamalarni umumiy yechimini toping.

17.9. $y''-5y'+6y = 0$. 17.10. $2y''+5y'-7y = 0$.

17.11. $y''+4y'-3y = 0$. 17.12. $3y''+y'-2y = 0$.

17.13. $y''+25y' = 0$. 17.14. $4y''-9y' = 0$

17. 15. $4y''-4y'+y = 0$.

17.16. $y''-6y'+9y = 0$. 17.17. $y''-4y'+4y = 0$

17.18. $4y''-12y'+9y = 0$. 17.19. $9y''+12y'+4y = 0$

17.20. $y''+4y = 0$. 17.21. $4y''+9y = 0$

17.22. $y''+y'+y = 0$. 17.23. $y''-y'+6y = 0$.

17.24. $2y''-3y'+5y = 0$. 17.25. $y''-3y'+2y = 0$

17.26. $y''-4y'+3y = 0$, $y(0) = 6$, $y'(0) = 10$

17.27. $y''+4y' = 0$, $y(0) = 7$, $y'(0) = 8$

17.28. $y''-6y'+9y = 0$, $y(0) = 0$. $y'(0) = 2$

17.29. $4y''+4y'+y = 0$, $y(0) = 2$. $y'(0) = 0$

17.30. $y''-4y'+3y = 0$, $y(0) = 6$, $y'(0) = 10$

17.31. $y''-4y'+3y = 0$. 17.32. $y''+4y'+29y = 0$.

17.33. $9y''+6y' = 0$. 17.34. $4y''+12y'+9y = 0$.

17.35. $5y''+y = 0$. 17.36. $5y''+y' = 0$.

17.37. $y'''-2y''-3y' = 0$. 17.38. $y'''+4y''+13y' = 0$

17.39. $y'''+2y''+y = 0$. 17.40. $y'''+2y''-y'-2y = 0$

17.41. $y^{IV} - 16y = 0$. 17.42. $y^{IV} + y = 0$.

Mustaqil yechish uchun misollarning javoblari

17.1. Chiziqli bog'lanmagan. 2. Chiziqli bog'lanmagan. 3. Chiziqli bog'lanmagan. 4. Chiziqli bog'lanmagan. 5. Chiziqli bog'lanmagan. 6. Chiziqli bog'lanmagan. 7. Chiziqli bog'liq. 8. Chiziqli bog'liq.

$17.9.\ y_{oo} = C_1e^{2x} + C_2e^{3x}$. $17.10.\ y_{oo} = C_1e^{-x} + C_2e^{-\frac{7}{2}x}$. $17.11.\ y_{oo} = C_1e^{(-2+\sqrt{7})x} + C_{2e}{}^{(-2-\sqrt{7})x}$.

$17.12.\ y_{oo} = C_1e^{-x} + C_2e^{-\frac{2}{3}x}$. $17.13.\ y_{oo} = C_1 + C_2e^{-25x}$. $17.14.\ y_{oo} = C_1 + C_2e^{\frac{9}{4}x}$.

$17.16.\ y_{oo} = (C_1 + C_2x)e^{3x}$. $17.17.\ y_{oo} = (C_1 + C_2x)e^{2x}$.

$17.18.\ y_{oo} = (C_1 + C_2x)e^{\frac{3}{2}x}$. $17.19.\ y_{oo} = (C_1 + C_2x)e^{\frac{2}{3}x}$. $17.20.\ y_{oo} = C_1\cos 2x + C_2\sin 2x$.

$17.21.\ y_{oo} = C_1\cos\frac{3}{2}x + C_2\sin\frac{3}{2}x$. $17.22.\ y_{oo} = \left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right)e^{-\frac{x}{2}}$

$17.23.\ y_{oo} = \left(C_1\cos\frac{5}{2}x + C_2\sin\frac{5}{2}x\right)e^{\frac{x}{2}}$. $17.24.\ y_{oo} = \left(C_1\cos\frac{\sqrt{31}}{4}x + C_2\sin\frac{\sqrt{31}}{4}x\right)e^{\frac{3}{4}x}$

$17.25.\ y_{oo} = C_1e^{x} + C_2e^{2x}$. $17.26.\ y_{oo} = 4e^{x} + 2e^{3x}$.

$17.27.\ y = 9 - 2e^{-4x}$ $17.28.\ y = 2xe^{3x}$. $17.29.\ y = (2+x)e^{-\frac{1}{2}x}$ $17.30.\ y = 4e^{x} + 2e^{3x}$

$17.31.\ y_{oo} = C_1e^{x} + C_{2e}{}^{3x}$ $17.32.\ y_{oo} = (C_1\cos 5x + C_2\sin 5x)e^{-2x}$. $17.33.\ y_{oo} = C_1 + C_{2e}{}^{-\frac{2}{3}x}$

$17.34.\ y_{oo} = (C_1 + C_2x)e^{-\frac{3}{2}x}$ $17.35.\ y_{oo} = C_1\cos\frac{x}{\sqrt{5}} + C_2\sin\frac{x}{\sqrt{5}}$. $17.36.\ y_{oo} = C_1 + C_{2e}{}^{-\frac{x}{5}}$

$17.37.\ y_{oo} = C_1 + C_2e^{-x} + C_{3e}{}^{3x}$. $17.38.\ y_{oo} = C_1 + (C_2\cos 3x + C_3\sin 3x)e^{-2x}$

$17.39.\ y_{oo} = C_1 + (C_2 + C_3x)e^{-x}$. $17.40.\ y_{oo} = C_1e^{-2x} + C_2e^{-x} + C_3e^{x}$.

$17.41.\ y_{oo} = C_{1e^{2x}} + C_{2e}{}^{-2x} + C_3\cos 2x + C_4\sin 2x$

$17.42.\ y_{oo} = e^{\frac{\sqrt{2}}{2}x}\left(C_1\cos\frac{\sqrt{2}}{2}x + C_2\sin\frac{\sqrt{2}}{2}x\right) + e^{-\frac{\sqrt{2}}{2}x}\left(C_3\cos\frac{\sqrt{2}}{2}x + C_4\sin\frac{\sqrt{2}}{2}x\right)$

18- amaliy mashg'ulot.
BIR JINSLI BO'LMAGAN CHIZIQLI DIFFERENSIAL TENGLAMALAR

18.1. Bir jinslimas differensial tenglamaning xususiy yechimini izlashning aniqmas koeffisiyentlar usuli. n-tartibli chiziqli differensial tenglama quyidagicha yoziladi:

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = f_0(x) \qquad (1)$$

Bu yerda $a_0, a_1, \ldots a_n$, $f_0(x)$ - x ning ma'lum funksiyalari yoki o'zgarmas sonlardir.

Odatda chiziqli (1) differensial tenglamani «keltirilgan» ko'rinishda yozish qabul qilingan, bunda tenglamaning ikkala qismini $a_0 \neq 0$ ga bo'lish bilan erishiladi. Bunday holda n-tartibli chiziqli differensial tenglama quyidagicha yoziladi:

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_{n-1}(x)y = f(x) \qquad (2)$$

yoki

$$L(y) = f(x)$$

Bu yerda $p_1(x), p_2(x), \ldots, p_n(x)$, $f(x)$ lar qaralayotgan sohada uzluksiz funksiyalardir.

Agar $f(x) \neq 0$ bo'lsa (2) tenglama *bir jinslimas* yoki *o'ng tomonli differensial tenglama* deyiladi.

Bir jinslimas (2) tenglamaning o'ng tomoni

$$f(x) = e^{\alpha x}[P_m(x)\cos\beta x + Q_s(x)\sin\beta x] \qquad (3)$$

ko'rinishda berilgan bo'lsin. Bu yerda α va β haqiqiy sonlar bo'lib, o'zaro $\alpha \pm \beta i$ kabi bog'langan. $P_m(x)$ va $Q_s(x)$ - mos ravishda m va s - darajali ko'phadlardir, ya'ni

$$\left.\begin{array}{l} P_m(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m,\ a_0 \neq 0,\ m \geq 0, \\ Q_s(x) = b_0x^s + b_1x^{s-1} + \ldots + b_{s-1}x + b_s,\ b_0 \neq 0,\quad s \geq 0. \end{array}\right\} \qquad (4)$$

Bu yerda $a_0, a_1, \ldots, a_k$ va $b_0, b_1, \ldots, b_n$ lar $P_m(x)$ va $Q_s(x)$ ko'phadlarning koeffisiyentlari bo'lib, oldindan berilgan haqiqiy sonlardir. Agar $m = s \equiv 0$ bo'lsa, $P_m(x) = P_0(x) = a_0$, $Q_s(x) = Q_0(x) = b_0$ bo'ladi.

Tenglamaning o'ng tomoni (4) ko'rinishda berilsa, va $\alpha + i\beta$ xarakteristik tenglamani m karrali ildizi bo'lsa, uning xususiy yechimi ushbu holatda qaraladi:

$$y^* = x^m e^{\alpha x}[F_l(x)\cos\beta x + E_l(x)\sin\beta x] \qquad (5)$$

Bu yerda α, β - haqiqiy sonlar. $F_l(x)$ va $E_l(x)$ lar ushbu ko'rinishdagi noma'lum koeffisiyentli l ($l = \max(m, s)$, $l \geq 0$)-darajali ko'phadlardir.

$$\left.\begin{array}{l} F_l(x) = A_0x^l + A_1x^{l-1} + \ldots + A_{l-1}x + A_l, \\ E_l(x) = B_0x^l + B_1x^{l-1} + \ldots + B_{l-1}x + B_l \end{array}\right\} \qquad (6)$$

Bu yerda $A_0, A_1, \ldots, A_l$ va $B_0, B_1, \ldots, B_l$ – koeffisiyentlarning sonli qiymatlarini aniqlash talab qilinadi.

Noma'lum koeffisiyentlarni aniqlash uchun (6) dagi y^* funksiyadan hosilalar olib, y^*, $y^{*\prime}$ va $y^{*\prime\prime}$ larning ifodalarini $L(y) = f(x)$ bir jinslimas tenglamaga qo'yamiz. Hosil bo'lgan munosabat ayniyatdir. Uning ikkala qismidagi koeffisiyentlarni tenglashtirish usuli bilan bu masala hal qilinadi.

Bu ayniyatdan y^* ning aniq ifodasini (6) ga asosan aniqlab, tenglamaning umumiy yechimini topamiz: $y = \bar{y} + y^*$.

Agar bir jinslimas tenglama (2) ning o'ng tomoni $f(x) = \sum_{i=1}^{n} f_i(x)$ ko'rinishdagi chekli sondagi funksiyalar yig'indisidan iborat bo'lsa, u holda har bir qo'shiluvchini hisobga olgan holda

$$L(y) = f_1(x),\ L(y) = f_2(x), \ldots, L(y) = f_n(x).$$

Tenglamalarning y_1^*, $y_2^*, \ldots, y_n^*$ xususiy yechimlarini (2) formulaga asosan topib (2) ning ya'ni $L(y) = \sum_{i=1}^{n} f_i(x)$ ning umumiy yechimini

$$y = \tilde{y} + y^* = \tilde{y} + \sum_{i=1}^{n} yi^* \qquad (7)$$

kabi aniqlaymiz.

Aniqmas koeffisiyentlar usuli xususiy yechimning shaklini bilishga asoslangan. Xususiy yechimni berilgan differensial tenglamaning o'ng tomonning shakliga o'xshash shakilda izlash kerak. $P_m(x)$ yoki $Q_s(x)$ ko'phadlardan birining tartibi ikkinchisidan kichik bo'lsa ham, $F_l(x)$ va $E_l(x)$ ko'phadlar l ($l = \max(m,s)$, $l \ge 0$)- darajali qilib izlanadi. Xususiy yechimning izlash jadvalini keltiramiz.

№		$f(x)$ o'ng tomonining turi	Xarakteristik tenglama ildizlari	Xususiy yechimning turi
1	a)	$P_m(x)$	0 soni xarakteristik tenglamaning ildizi emas	$F_m(x)$
	b)		0 soni xarakteristik tenglamaning n karrali ildizi	$x^n F_m(x)$
2	a)	$P_m(x)e^{\alpha x}$ $(x \in R)$	α soni xarakteristik tenglamaning ildizi emas	$F_m(x)e^{\alpha x}$
	b)		α soni xarakteristik tenglamaning n karrali ildizi	$x^n F_m(x)e^{\alpha x}$
3	a)	$P_m(x)\cos\beta x + Q_s(x)\sin\beta x$	$\pm\beta i$ sonlar xarakteristik tenglamaning ildizi emas	$F_l(x)\cos\beta x + E_l(x)\sin\beta x$, $l = \max(m,s)$, $l \ge 0$
	b)		$\pm\beta i$ sonlar xarakteristik tenglamaning n karrali ildizlari	$x^n[F_l(x)\cos\beta x + E_l(x)\sin\beta x]$, $l = \max(m,s)$, $l \ge 0$
4	a)	$P_m(x)e^{\alpha x}\cos\beta x + Q_s(x)e^{\alpha x}\sin\beta x$	$\alpha \pm \beta i$ sonlar xarakteristik tenglamaning ildizi emas	$F_l(x)e^{\alpha x}\cos\beta x + E_l(x)e^{\alpha x}\sin\beta x$, $l = \max(m,s)$, $l \ge 0$
	b)		$\alpha \pm \beta i$ sonlar xarakteristik tenglamaning n karrali ildizlari	$x^n[F_l(x)e^{\alpha x}\cos\beta x + E_l(x)e^{\alpha x}\sin\beta x]$, $l = \max(m,s)$, $l \ge 0$

1-misol. $y'' - 5y' + 6y = 2x + e^x$ tenglamaning ushbu boshlang'ich shartda $y(0) = \frac{16}{9}$, $y'(0) = \frac{5}{6}$ xususiy yechimini toping.

Yechilishi. 1) Berilgan tenglamaga mos bir jinsli tenglamani umumiy yechimini topamiz. $y''-5y'+6y = 0$ tenglamaning xarakteristik tenglamasi: $k^2 - 5k + 6 = 0$. Bu tenglamani yechimi:

$$k_{1,2} = \frac{5 \pm \sqrt{25 - 4\cdot 1\cdot 6}}{2\cdot 1} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2},$$

$$k_1 = \frac{5-1}{2} = \frac{4}{2} = 2, \quad k_2 = \frac{5+1}{2} = \frac{6}{2} = 3.$$

U holda

$$\tilde{y} = C_1 e^{2x} + C_2 e^{3x}.$$

2)Berilgan tenglamaning xususiy yechimini aniqlaymiz. Uning o'ng tomoni

$$f(x) = 2x + e^x$$

bo'lgani uchun $f_1(x) = 2x$, $f_2(x) = e^x$ deb olamiz va ikkita:

$$L(y) = 2x, \qquad L(y) = e^x.$$

bir jinslimas tenglamalarni qaraymiz. Bu yerda $L(y) = y''-5y'+6y$. Birinchi tenglama $L(y) = 2x$ uchun $\alpha = 0, \beta = 0;$ $\alpha + i\beta = 0$ dan $n = 0,$ $m = 1,$ $s = 0,$ $l = \max(m,s) = 1$ kelib chiqadi. U holda

$$\overset{*}{y_1} = x^0 \cdot e^{0x}[F_1(x)\cos 0 + E_1(x)\sin 0] = F_1(x) = Ax + B$$

Bundan hosilalar olib tenglamaga qo'ysak

$$0 - 5A + 6Ax + 6B = 2x,$$

$$\left.\begin{array}{l} 6A = 2 \\ -5A + 6B = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A = 1/3 \\ 6B = 5A \end{array}\right\} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B = \frac{5}{18}. \end{cases}$$

Demak, $\overset{*}{y_1} = \frac{1}{3}\left(x + \frac{5}{6}\right)$.

Ikkinchi tenglama $L(y) = e^x$ uchun $\alpha = 1, \beta = 0;$ $\alpha + i\beta = 1$ dan $n = 0,$ $m = 0,$ $s = 0,$ $l = \max(m,s) = 0$ kelib chiqadi.U holda

$$y_2{*} = x^0 e^x[F_0(x)\cos 0 + E_0(x)\sin 0] = e^x F_0(x) = A_0 e^x.$$

Hosilalari esa: $y_2{*}' = A_0 e^x$, $\quad y_2{*}'' = A_0 e^x$. U holda $L(y) = e^x$ dan:

$$A_0 e^x - 5A_0 e^x + 6A_0 e^x = e^x \Rightarrow 2A_0 = 1, \quad A_0 = \frac{1}{2}.$$

Demak, $L(y) = e^x$ tenglamaning xususiy : $y_2{*} = \frac{1}{2}e^x$.

Shunday qilib, berilgan tenglamaning umumiy yechimi:

$$y = C_1e^{2x} + C_2e^{2x} + \frac{1}{3}(x+\frac{5}{6}) + \frac{1}{2}e^x.$$

Xususiy yechimni $y^* = y_1^* + y_2^* = Ax + B + A_0e^x$ ko'rinishda izlab topsak ham bo'ladi.

3) Endi boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimni aniqlaymiz. Umumiy yechimdan hosila olib

$$\begin{cases} y = C_1e^{2x} + C_2e^{3x} + \frac{1}{3}(x+\frac{5}{6}) + \frac{1}{2}e^x \\ y' = 2C_1e^{2x} + 3C_2e^{3x} + \frac{1}{3} + \frac{1}{2}e^x \end{cases};$$

Boshlang'ich shartdan foydalanamiz:

$$\begin{cases} \frac{16}{9} = C_1e^0 + C_2e^0 + \frac{5}{18} + \frac{1}{2}e^0 \\ \frac{5}{6} = 2C_1e^0 + 3C_2e^0 + \frac{1}{3} + \frac{1}{2}e^0 \end{cases}$$

Bu yerdan

$$\left.\begin{array}{l} \frac{16}{9} = C_1 + C_2 + \frac{14}{18}, \\ \frac{5}{6} = 2C_1 + 3C_2 + \frac{5}{6}, \end{array}\right\} \Rightarrow \begin{cases} C_1 + C_2 = 1, \\ 2C_1 + 3C_2 = 0, \end{cases} \Rightarrow \quad \begin{cases} C_1 + C_2 = 1, \\ 2C_1 = -3C_2, \end{cases} \quad C_1 = -\frac{3}{2}C_2,$$

$$(1-\frac{3}{2})C_2 = 1, \quad -\frac{1}{2}C_2 = 1, \quad C_2 = -2,$$

$$C_1 = 1 - C_2 = 1 - (-2) = 1 + 2 = 3.$$

Natijada, berilgan masalaning izlanayotgan xususiy yechimi quyidagicha yoziladi:

$$y = 3e^{2x} - 2e^{3x} + \frac{1}{3}(x+\frac{5}{6}) + \frac{1}{2}e^x.$$

Mustaqil yechish uchun misollar

O'zgarmas koeffisiyentli chiziqli bir jinslimas differensial tenglamalarni aniqmas koeffisiyentlar usuldan foydalanib umumiy yechimini toping

18.1. $y''-3y'+2y = 10e^{-x}$. 18.2. $y''-6y'+9y = 2x^2 - x + 3$.

18.3. $y''-3y'+2y = 2x^3 - 30$. 18.4. $y''-2y'+2y = 2x$.

18.5. $y''+4y'-5y = 1$. 18.6. $2y''-y'-y = 4xe^{2x}$.

18.7. $y''-7y'+6y = \sin x$. 18.8. $2y''+5y' = 29\cos x$.

18.9. $y''-4y = e^{2x}\sin 2x$. 18.10. $y''-2y'-8y = -8\cos 2x$.

18.11. $y''+4y'+4y = (2x+3)\sin x + \cos x$.

18.12. $y''-2y = 2x(\cos x - \sin x)e^x$.

O'zgarmas koeffisiyentli chiziqli bir jinslimas differensial tenglamalarni boshlang'ich shartni qanoatlantiruvchi xususiy yechimini toping.

18.13. $y''+y = 4x^x, y(0) = -2, y'(0) = 0$.

18.14. $y''+y = 4\sin x, y(0) = 1, y'(0) = 2$.

18.15. $y''-2y'-3y = e^{4x}, y(0) = \frac{26}{5}, \ y'(0) = \frac{39}{5}$.

18.16. $y''+2y'-3y = 48x^2e^x, y(0) = 1, y'(0) = -\frac{3}{2}$.

18.17. $y''+4y'+4y = 32xe^{2x}, y(0) = -1, y'(0) = 1$.

18.18. $y''-\dot{y} = 2e^x - x^2; \ y(0) = 2, y'(0) = 1$.

18.19. $y''+3y'+2y = 2\sin 3x + 6\cos 3x, y(0) = y'(0) = 0$.

18.20. $y''+9y = 6\cos 3x, y(0) = 1, y'(0) = 3$.

18.21. $y''-y' = \frac{1}{1+e^x}, \ y(0) = 1, y'(0) = 2$.

18.22. $4y''+y = ctg\frac{x}{2}, y(\pi) = 3, y'(\pi) = \frac{1}{2}$.

Mustaqil yechish uchun misollarning javoblari

18.1. $y_{o.n} = C_1e^x + C_2e^{2x} + \frac{5}{3}e^{-x}$. 18.2. $y_{o.n} = (C_1 + C_2x)e^{3x} + \frac{2}{9}x^2 + \frac{5}{27}x + \frac{11}{27}$.

18.3. $y_{o.n} = C_1e^x + C_2e^{2x} + x^3 + \frac{9}{2}x^2 + \frac{21}{2}x - \frac{15}{4}$. 18.4. $y_{o.n} = (C_1\cos x + C_2\sin x)e^x + x + 1$.

18.5. $y_{o.n} = C_1e^x + C_1e^{-5x} - \frac{1}{5}$. 18.6. $y_{o.n} = C_1e^x + C_2e^{-\frac{1}{2}x} + \left(\frac{4}{5}x - \frac{28}{25}\right)e^{2x}$.

18.7. $y_{o.n} = C_1e^{6x} + C_2e^x + \frac{5}{74}\sin x + \frac{7}{74}\cos x$. $y_x = A\sin x + B\cos x$

18.8. $y_{o.n} = C_1 + C_2e^{-\frac{5}{2}x} + 5\sin x - 2\cos x$. $y_ч = A\sin x + B\cos x$

18.9. $y_{o.n} = C_1e^{-2x} + C_2e^{2x} - \left(\frac{1}{20}\sin 2x + \frac{1}{10}\cos x\right)e^{2x}$. $y_ч = e^{2x}(A\sin 2x + B\cos 2x)$

18.10. $y_{o.n} = C_1e^{-2x} + C_2e^{4x} + \frac{3}{5}\cos 2x + \frac{1}{5}\sin 2x$. $y_ч = A\cos 2x + B\sin 2x$

18.11. $y_{o.n} = (C_1 + C_2x)e^{-2x} + \left(\frac{6}{25}x\frac{57}{125}\right)\sin x - \left(\frac{8}{25}x + \frac{1}{25}\right)\cos x$.

18.12. $y_{o.n} = C_1e^{-\sqrt{2}x} + C_2e^{\sqrt{2}x} + (x\sin x + \cos x)e^x$. $y_ч = [(Ax+D)\sin x + (Cx+D)\cos x]e^x$.

18.13. $y = (2x-2)e^x$. 18.14. $y = \cos x + 4\sin x - 2x\cos x$.

18.15. $y = 2e^{-x} + 3e^{3x} + \frac{1}{5}e^{4x}$. 18.16. $y = e^{-3x} + \left(4x^3 - 3x^2 + \frac{3}{2}x\right)e^x$.

18.17. $y = xe^{-2x} + (2x-1)e^{2x}$. 18.18. $y = xe^x + x^2 + 2$.

formulasidan foydalanib tuzish mumkin. Bu $e^{\alpha x}\cos\beta x$ va $e^{\alpha x}\sin\beta x$ ko'rinishdagi funksiyalarga ega bo'lgan haqiqiy yechimlar juftini beradi.

Bu ildizlarga ushbu yechimlar mos keladi:

$$\left.\begin{array}{l} x_1^{(1)}(t)=p_1^{(1)}\cdot e^{k_1 t}\equiv p_1^{(1)}\cdot e^{(\alpha+i\beta)t},\\ x_2^{(1)}(t)=p_2^{(1)}\cdot e^{k_1 t}\equiv p_2^{(1)}\cdot e^{(\alpha+i\beta)t},\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ x_n^{(1)}(t)=p_n^{(1)}\cdot e^{k_1 t}\equiv p_n^{(1)}\cdot e^{(\alpha+i\beta)t}.\end{array}\right\}$$

va

$$\left.\begin{array}{l} x_1^{(2)}(t)=p_1^{(2)}\cdot e^{k_2 t}\equiv p_1^{(2)}\cdot e^{(\alpha-i\beta)t},\\ x_2^{(2)}(t)=p_2^{(2)}\cdot e^{k_2 t}\equiv p_2^{(2)}\cdot e^{(\alpha-i\beta)t},\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ x_n^{(2)}(t)=p_n^{(2)}\cdot e^{k_2 t}\equiv p_n^{(2)}\cdot e^{(\alpha-i\beta)t}.\end{array}\right\}$$

yoki qisqa qilib yozilganda:

$$\left.\begin{array}{l} x_j^{(1)}=p_j^{(1)}e^{k_1 t}=p_j^{(1)}\cdot e^{(\alpha+i\beta)t},\\ x_j^{(2)}=p_j^{(2)}e^{k_2 t}=p_j^{(2)}\cdot e^{(\alpha-i\beta)t}.\end{array}\right\}\quad (j=1,\overline{n}) \qquad (6)$$

Bu yerda $P_j^{(1)}$ va $P_j^{(2)}$ – koyeffisiyentlar (6) sistemadan $k_1=\alpha+i\beta$ va $k_2=\alpha-i\beta$ holatlar uchun aniqlanadi.

Ularning ayrimlari komrleks sonlar bo'lishi ham mumkin. $P_1^{(1)}=P_1^{(2)}\equiv 1$ deb olish maqsadga muvofiqdir.

Komrleks yechim (6) larning haqiqiy va mavhum qismlari yana yechim bo'lishini. (5) ga asosan ko'rsatish mumkin, ya'ni:

$$\begin{cases}\overline{x}_1^{(1)}=\dfrac{x_1^{(1)}+x_1^{(2)}}{2}\\ \overline{x}_1^{(2)}=\dfrac{x_1^{(1)}-x_1^{(2)}}{2i}\end{cases}\quad \text{va}\quad \begin{cases}\overline{x}_2^{(1)}=\dfrac{x_2^{(1)}+x_2^{(2)}}{2}\\ \overline{x}_2^{(2)}=\dfrac{x_2^{(1)}-x_2^{(2)}}{2i}\end{cases}$$

va hokazo.

Shunday qilib, biz ikkita xususiy yechimini hosil qilamiz:

$$\left.\begin{array}{l}\overline{x}_j^{(1)}=e^{\alpha t}\left[\lambda_j^{(1)}\cos\beta t+\lambda_j^{(2)}\sin\beta t\right]\\ \overline{x}_j^{(2)}=e^{\alpha t}\left[\lambda_j^{(1)}\cos\beta t+\lambda_j^{(2)}\sin\beta t\right]\end{array}\right\} \qquad (7)$$

Bunda $\lambda_j^{(1)}$, $\lambda_j^{(2)}$, $\overline{\lambda}_j^{(1)}$, $\overline{\lambda}_j^{(2)}$, lar $P_j^{(1)}$ va $P_j^{(2)}$ lar orqali aniqlanadigan haqiqiy sonlardir.

(7) funksiyalarning mos kombinasiyalari (1) sistemaning umumiy yechimiga kiradi.

3) Xarakteristik tenglamaning ildizlari haqiqiy va karrali.

Faraz qilaylik (1) sistemaning (7) xarakteristik tenglamasini n– ta bir xil (karrali) ildizi bo'lsin, ya'ni $k=k_1=k_2=...=k_n$ – ildizlar o'zaro teng bo'lsin va $k_i\neq 0$ $\quad (i=1,\overline{n})$.

U holda bu ildizlarga mos keladigan yechimlar (5) ga asosan quyidagicha bo'ladi.

$$\left.\begin{array}{l} x_1(t)=P_1(t)e^{kt}\\ x_2(t)=P_2(t)e^{kt}\\ \dots\dots\dots\dots\\ x_n(t)=P_n(t)e^{kt}\end{array}\right\} \qquad (8)$$

Bu yerda

$P_1,\ P_2,\dots,P_n$ – $(n-1)$ tartibli ko'rxadlardir, ya'ni

$$\left.\begin{array}{l} P_1(t)=A_0+A_1t+A_2t^2+\dots+A_{n-1}\cdot t^{n-1},\\ P_2(t)=B_0+B_1t+B_2t^2+\dots+B_{n-1}\cdot t^{n-1},\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ P_n(t)=E_0+E_1t+E_2t^2+\dots+E_{n-1}\cdot t^{n-1}.\end{array}\right\}$$

Bu yerda ko'rxadning koeffisiyentlari ixtiyoriy o'zgarmaslar bo'lib, ularni aniqlash kerak bo'ladi.

Shunday qilib masalaning yechimlari (8) ya'ni qisqacha ushbu ko'rinishda yozish mumkin.

$$x_i(t)=P_i(t)\cdot e^{kt} \qquad (i=1,2,\dots,n) \qquad (9)$$

$P_i(t)$ – koeffisiyentlar qanday bo'lishidan qat'iy nazar har bir $x_i(t)$ – funksiyalar o'zining n- tartibli differensial tenglamasini qanoatlantiradi. Bu koeffisiyentlarni aniqlashning usullaridan biri, $x_i(t)$ - ni n-ta noma'lumli n-ta differensial tenglamalar sistemasi (1) ga qo'yamiz va barcha hadlarni $e^{kt}\neq 0$ ga bo'lib, t-ning bir xil darajalari oldidagi koeffisiyentlarini taqqoslaymiz. Bu koeffisiyentlar uchun hosil bo'lgan tenglamalar sistemasini yechamiz va bu yechimni (9) ga qo'yib umumiy yechimni hosil qilamiz.

1-misol. Ushbu

$$\begin{cases}\dfrac{dx}{dt}=x+y;\\ \dfrac{dy}{dt}=x-y\end{cases}$$

differensial tenglamalar sistemasining boshlang'ich sharti $x(0)=2$, $\quad y(0)=0$ qanoatlantiruvchi xususiy yechimini toping.

Yechilishi. Sistema birinchi tenglamasining ikkala qismini t bo'yicha diffensiallaymiz.

$$\frac{d^2x}{dt^2}=\frac{dx}{dt}+\frac{dy}{dt}$$

Bu yerda $\dfrac{dy}{dt}$ ning o'rniga sistemaning ikkinchi tenglamasidan qo'ysak:

$$\frac{d^2x}{dt^2} = x + y + x - y$$

yoki

$$\frac{d^2x}{dt} = 2x$$

Bu tenglama uchun xarakteristik tenglama (4.1) ga ko'ra quyidagicha bo'ladi.

$$k^2 - 2 = 0;\ k^2 = 2,\ k_{1,2} = \pm\sqrt{2} \qquad k_1 = -\sqrt{2};\ k_2 = \sqrt{2};$$

U holda (4.1.4) ga asosan tenglamaning umumiy yechimi:

$$x(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}$$

Sistemaning birinchi tenglamasidan $y = \frac{dx}{dt} - x$ bo'lgani uchun:

$$\frac{dx}{dt} = \sqrt{2}\left(-C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}\right)$$

U holda $y(t) = -C_1 e^{-\sqrt{2}t}\left(\sqrt{2}+1\right) + C_2 e^{\sqrt{2}t}\left(\sqrt{2}-1\right)$

Shunday qilib sistemaning umumiy yechimi

$$\left.\begin{aligned} x(t) &= C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}, \\ y(t) &= -C_1 e^{-\sqrt{2}t}(\sqrt{2}+1) + C_2 e^{\sqrt{2}t}\left(\sqrt{2}-1\right) \end{aligned}\right\}$$

Masalaning boshlang'ich shartlaridan foydalanib, C_1 va C_2 o'zgarmas koeffisiyentlarni aniqlaymiz. Shu maqsadda ushbu

$$\begin{cases} 2 = C_1 e^{-\sqrt{2}\cdot 0} + C_2 e^{\sqrt{2}\cdot 0}, \\ 0 = -C_1 e^{-\sqrt{2}\cdot 0}\cdot\left(\sqrt{2}+1\right) + C_2 e^{\sqrt{2}\cdot 0}\cdot\left(\sqrt{2}-1\right) \end{cases}$$

sistemani hosil qilamiz. Bundan,

$$\begin{cases} C_1 + C_2 = 2, \\ \left(\sqrt{2}+1\right)C_1 - \left(\sqrt{2}-1\right)\cdot C_2 = 0 \end{cases}$$

yoki

$$\begin{cases} c_1 + c_2 = 2; \\ \sqrt{2}(C_1 - C_2) + (C_1 + C_2) = 0 \end{cases}, \qquad \begin{cases} C_1 + C_2 = 2, \\ C_1 - C_2 = -\frac{2}{\sqrt{2}} \end{cases}$$

hadma-had qo'shsak: $2C_1 = 2 - \sqrt{2};\ C_1 = 1 - \frac{\sqrt{2}}{2}$,

hadma-had ayirsak: $2C_2 = 2 + \sqrt{2};\ C_2 = 1 + \frac{\sqrt{2}}{2}$. U holda, masalaning xususiy yechimi quyidagi

$$\left.\begin{aligned} x(t) &= \left(1 - \frac{\sqrt{2}}{2}\right)e^{-\sqrt{2}t} + \left(1 + \frac{\sqrt{2}}{2}\right)e^{\sqrt{2}t}, \\ y(t) &= \frac{\sqrt{2}}{2}\left(-e^{-\sqrt{2}t} + e^{\sqrt{2}t}\right) \end{aligned}\right\}$$

ko'rinishda bo'ladi.

Mustaqil yechish uchun misollar

Tenglamalar sistemasini yeching:

19.1. $\begin{cases} x' = y + z, \\ y'' = 3x + z, \\ z' = 3x + y \end{cases}$ 19.2. $\begin{cases} x' = 3x + -2y \\ y' = 2x - y \end{cases}$ $x(0) = 1,\ y(0) = 2$

19.3. $\begin{cases} x' + 5x + y = e^t \\ y' - x - 3y = e^{2t} \end{cases}$ 19.4. $\begin{cases} x' = -x + y + z, \\ y' = x - y + z, \\ z' = x + y - z \end{cases}$

19.5. $\begin{cases} 4x' - y' = \sin t - 3x, \\ x' = \cos t - y \end{cases}$ 19.6. $\begin{cases} x' = 2x + y, \\ y' = 3x + 4y. \end{cases}$

19.7. $\begin{cases} x' = x - 2y - z, \\ y' = -x + y + z, \\ z' = x - z \end{cases}$ 19.8. $\begin{cases} x' = y \\ y' = x + e^t + e^{-t} \end{cases}$

Mustaqil yechish uchun misollarning javoblari

19.1. $\begin{cases} x = -C_2 e^{2t} + \frac{2}{3}C_3 e^{3t}, \\ y = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{3t}. \\ z = -C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{3t} \end{cases}$ 19.2. $\begin{cases} x = \frac{1}{2}e^t\left(2C_1 + C_2 + 2C_2 e^t\right), \\ y = e^t\left(C_1 + C_2 t\right); \end{cases}$ $\left\{x_4 = \frac{e^t}{2}\left(\frac{10}{3} - \frac{4}{3}e^t\right),\right.$

19.3. $\begin{cases} x = C_1 e^{(-1+\sqrt{15})t} + C_2 e^{(-1-\sqrt{15})t} + \frac{2}{11}e^t + \frac{1}{6}e^t, \\ y = \left(-4+\sqrt{15}\right)C_1 e^{(-1-\sqrt{15})t} - \left(4-\sqrt{15}\right)C_2 e^{(1-1\sqrt{15})t} - \frac{1}{11}e^t - \frac{7}{6}e^{2t} \end{cases}$

19.4. $\begin{cases} x = C_1e^t + C_2e^{-2t}, \\ y = C_1e^t + C_3e^{-2t} \\ z = C_1e^t - (C_2 + C_3)e^{-2t} \end{cases}$ 19.5. $\begin{cases} x_{00} = C_1e^{-t} + C_2e^{-3t} \\ y_{00} = C_1e^t + 3C_2e^{-3t} + \cos t \end{cases}$

19.6. $\begin{cases} x = C_1e^t + C_2e^{5t} \\ y = -C_1e^t + 3C_2e^{5t} \end{cases}$ 19.7. $\begin{cases} x = C_1 + 3C_2e^{2t} \\ y = -2C_2e^{2t} + C_3e^{-t} \\ z = C_1 + C_2e^{2t} - 2C_3e^{-t} \end{cases}$

19.8. $\begin{cases} x = C_1e^t + C_2e^{-t} + \frac{1}{2}t(e^t - e^{-t}), \\ y = C_1e^t - C_2e^{-t} + \frac{1}{2}(e^t - e^{-t}) + \frac{t}{2}(e^t + e^{-t}) \end{cases}$

20- amaliy mashg'ulot.
OPERASION HISOB. ASL VA TASVIR FUNKSIYA. ASL LAR O'RAMASI.

20.1-ta'rif. Haqiqiy o'zgaruvchili $f(t) = u(t) + iv(t)$ kompleks funksiya quyidagi shartlarni qanoatlantirsa:

1) $(-\infty, \infty)$ oraliqda $f(t)$ o'zining $n-$ tartibli hosilasi bilan birlikda bir qiymatli uzluksiz yoki bo'lakli uzluksiz.

2) $t < 0$ uchun $f(t) = 0$;

3) Shunday $M > 0$ va $S > 0$ musbat sonlar topiladiki, barcha $t > 0$ uchun $|f(x)| \le Me^{st}$ tengsizlik o'rinli bo'lsa, u holda $f(t)$ funksiya *original* deyiladi.

Eslatma. S songa $f(t)$ funksiyaning o'sish ko'rsatgichi deyiladi, 1)-3) shartlarga bo'ysinuvchi funksiyalar to'plamni $K(s)$ sinf deb belgilaymiz. Masalan,

$$f(t) = \begin{cases} 0, & t < 0 \\ (5t - 3)e^{2t}, & t > 0 \end{cases}$$

- original bo'lib, $\forall t$ lar uchun $|f(t)| \le Me^{(2+\varepsilon)t}$ bo'ladi. $S = 2 + \varepsilon,\ \varepsilon > 0$ o'sish ko'rsatgichi, $f(t) \in K(2 + \varepsilon)$.

20.2-ta'rif. Ushbu

$$F(p) = \int_0^{+\infty} e^{-pt} f(t)dt$$

Laplas integral bilan aniqlanadigan kompleks o'zgaruvchili $F(p)$ funksiyaga, $f(t)$ original funksiyaning *tasviri* deyiladi, original funksiyadan tasviriga o'tish quyidagicha

$$L(f(t)) = F(p) \text{ yoki } f(t) \stackrel{\cdot}{=} F(p)$$

simvollar bilan, tasvirdan original o'tish esa, $L^{-1}\{F(p)\} = f(t)$ yoki $f(t) \stackrel{\cdot}{=} F(p)$ simvollar bilan belgilanadi.

Ba'zi bir funksiyalarning tasvirini topamiz:

20.3-ta'rif. Ushbu

$$\eta(t) = \begin{cases} 1, & t > 0, \\ 0 & t < 0, \end{cases}$$

original funksiyaga *Xevisaydning birlik funksiyasi* deyiladi (1 - chizma)

Xevisayd funksiyasining $L-$ tasvirini topamiz:

$$L\{\eta(t)\} = F(p) = \int_0^{+\infty} e^{-pt}\eta(t)dt =$$

$$= \lim_{a\to+\infty} \int_0^a e^{-pt}dt = \lim_{a\to+\infty} \frac{e^{-pt}}{-p}\Big|_0^a = \lim_{a\to+\infty}\left(\frac{1}{p} - \frac{1}{p}e^{-pa}\right)$$

Agar $\operatorname{Re} p > 0$ bo'lsa, $\lim_{a\to+\infty} e^{-pa} = 0$.

Demak, $1 \stackrel{\cdot}{=} \frac{1}{p}$, $\operatorname{Re} p > 0$ yoki $\eta(t) \stackrel{\cdot}{=} \frac{1}{p} = F(p)$.

1-misol. $f(t) = e^{\lambda t}$ funksiyaning tasviri. Laplas almashtirish formulasiga asosan, $F(p)$ ni topamiz:

$$F(p) = \int_0^{\infty} e^{-pt}e^{\lambda t}dt = \lim_{a\to+\infty}\int_0^a e^{(\lambda-p)t}dt = \lim_{a\to+\infty}\frac{1}{\lambda - p}e^{(\lambda-p)t}\Big|_0^a = \lim_{a\to+\infty}\frac{1}{\lambda - p}\left(e^{(\lambda-p)a} - 1\right).$$

Agar $\operatorname{Re}(p - \lambda) > 0$ bo'lsa, $\lim_{a\to+\infty} e^{-(p-\lambda)a} = 0$.

Shunday qilib,

$$e^{\lambda t} \stackrel{\cdot}{=} \frac{1}{p - \lambda},\quad \operatorname{Re} p > \operatorname{Re}\lambda.$$

20.2. Laplas almashtirishlarni qoidalari. Endi differensial tenglamalarni yechishda zarur bo'lib qolishi mumkin bo'lgan Laplas almashtirishlari uchun asosiy qoidalar majmuini isbotsiz keltiramiz.

20.1-teorema (chiziqlilik xossasi). $\{f_i(t)\}$ va $\{C_i\}$lar n ta funksiya va n ta son sistemalari bo'lsin. Agar $f_i(t)\doteq F_i(p)$, $(i=1,2,...,n)$ bo'lsa,

$$\sum_{i=1}^{n} C_i f_i(t) \doteq \sum_{i=1}^{n} C_i F_i(p),$$

ya'ni originallarning chiziqli kombinatsiyasiga tasvirlarning chiziqli kombinasiyasi mos keladi va aksincha.

20.2-teorema (o'xshashlik teoremasi (original argumenti macshtabining o'zgarishi)). Agar $a>0$ va $f(t)\doteq F(p)$ bo'lsa, u holda

$$f(at)\doteq \frac{1}{a}F\left(\frac{p}{a}\right)$$

bo'ladi.

20.3-teorema (Tasvirni saqlash teoremasi). $f(t)\doteq F(p)$ bo'lsin. U holda istalgan p_0 uchun $e^{-p_0 t}f(t)\doteq F(p+p_0)$ o'rinli bo'ladi.

20.4-teorema (Originalning kechikish teoremasi). Agar $t_0>0$ bo'lsa, u holda $f(t)\doteq F(p)$ dan $f(t-t_0)\doteq e^{-t_0 p}F(p)$ kelib chiqadi.

20.5-teorema (Originalning o'zib ketish teoremasi). Agar $t_0>0$ bo'lsa, u holda $f(t)\doteq F(p)$ dan $f(t+t_0)\doteq e^{t_0 p}\left[F(p)-\int_0^{t_0} e^{-pt}f(t)dt\right]$ kelib chiqadi.

20.6-teorema (Originalni differensiallash). $f(t)$ funksiya $[0,\infty)$ da uzluksiz differensiallanuvchi va $f'(t)$ hosila tasvir mavjudligi 1^0 - 3^0 xossalarini qanoatlantirsin. U holda:

a) agar $f(t)\doteq F(p)$ bo'lsa, u holda $f'(t)\doteq pF(p)-f(0)$, xususan, agar $f(0)=0$ bo'lsa, $f'(t)\doteq pF(p)$, ya'ni funksiyani differensiallashga tasvirni p ga ko'paytirish (balki uning noldagi qiymatini ayirish) mos keldi.

b) agar $f^{(n)}(t)$ mavjud bo'lsa va 1^0 - 3^0 xossalarga bo'ysunsa, u holda $f(t)\doteq F(p)$ dan $f^{(n)}(t)\doteq p^n F(p)-[p^{n-1}f(0)+p^{n-2}f'(0)+...+f^{(n-1)}(0)]$ kelib chiqadi, xususan, agar $f(t)$ boshlang'ich nol shartlar $f(0)=f'(0)=...=f^{(n-1)}(0)$ ni qanoatlantirsa, u holda $f^{(n)}(t)\doteq p^n F(p)$

20.7-teorema (Originalni integrallash). $f(t)$ funksiya $[0,\infty)$ da uzluksiz, tasvir mavjudligining 1^0 - 3^0 shartlarni qanoatlantirsa va $f(t)\doteq F(p)$ bo'lsin. U holda

$$\int_0^t f(\tau)d\tau \doteq \frac{1}{p}F(p),$$

ya'ni funksiyani integrallash tasvirini p ga bo'lish mos keladi.

20.8-teorema (tasvirni differensiallash). $f(t)\doteq F(p)$ bo'lsin, u holda:

a) $-tf(t)\doteq F'(p)$; b) $(-1)^n t^n f(t)\doteq F^{(n)}(p)$

20.9-teorema (tasvirni integrallash). $f(t)\doteq F(p)$ va $\frac{f(t)}{t}$ kasr tasvir mavjudligining 1^0 - 3^0 shartlarni qanoatlantirsin. U holda

$$\frac{f(t)}{t}\doteq \int_p^{\infty} F(q)dq.$$

20.10-teorema (O'rta haqida teorema (tasvirlarni ko'paytirish teoremasi)). Bu teoremani bayon qilishdan avval o'rash amalini (yoki o'ramaning) ta'rifini keltirishga to'g'ri keladi. U * simvoli Bilan belgilanadi.

Biror $[\alpha,\beta]$ oraliqda aniqlangan $f_1(t)$ va $f_2(t)$ funksiyalar berilgan bo'lsin. Ularning bu kesmadagi o'rama deb

$$f(t)=\int_\alpha^\beta f_1(\tau)f_2(t-\tau)d\tau = f_1(t)*f_2(t)$$

tenglik bilan aniqlanadigan yangi $f(t)$ funksiyaga aytiladi. $[\alpha,\beta]$ kesma uchun $[0,t]$ kesmani olamiz.

Agar $f_1(t)$ va $f_2(t)$ lar 1^0 - 3^0 shartlarni qanoatlantirsa, ular o'ramasining tasviri ko'paytuvchilar tasvirlarining ko'paytmasidan iborat bo'ladi, ya'ni $f_1(t) \doteq F_1(p)$ va $f_2(t) \doteq F_2(p)$ dan $f_1(t) * f_2(t) \doteq F_1(p)F_2(p)$ kelib chiqadi.

20.11-teorema(Dyuamel teoremasi). Bu teorema oldingi teoremaning umumlashtirilgani sifatida qaralishi mumkin, u ikkita funksiya o'ramasi hosilasining tasviri uchun ifoda beradi.

Agar $f_1(t)$, $f_2(t)$ funksiyalar $[0,\infty)$ da uzluksiz hosilalarga ega bo'lib va $f_1(t) \doteq F_1(p), f_2(t) \doteq F_2(p)$ bo'lsa, u holda $\frac{d}{dt}[f_1(t) * f_2(t)] \doteq pF_1(p)F_2(p)$.

Ba'zi elementar funksiyalarning tasvirlari

№	Original	Tasvir
1	$\eta(t)$	$\frac{1}{p}$
2	$e^{-\alpha t}$	$\frac{1}{p+a}$
3	$\sin at$	$\frac{a}{p^2+a^2}$
4	$\cos at$	$\frac{p}{p^2+a^2}$
5	$sh\ at$	$\frac{a}{p^2-a^2}$
6	$ch\ at$	$\frac{p}{p^2-a^2}$
7	t	$\frac{1}{p^2}$
8	t^n	$\frac{n!}{p^{n+1}}$
9	$e^{-\alpha t}\sin \varpi t$	$\frac{\varpi}{(p+a)^2+\varpi^2}$
10	$e^{-\alpha t}\cos \varpi t$	$\frac{p+a}{(p+a)^2+\varpi^2}$
11	$e^{-\alpha t}\ sh\ at$	$\frac{\varpi}{(p+\alpha)^2-\varpi^2}$
12	$e^{-\alpha t} c\ h\ at$	$\frac{p+a}{(p+\alpha)^2+\varpi^2}$
13	$te^{-\alpha t}$	$\frac{1}{(p+\alpha)^2}$
14	$t^n e^{-\alpha t}$	$\frac{n!}{(p+a)^{n+1}}$
15	$t \sin at$	$\frac{2ap}{(p^2+a^2)^2}$
16	$t \cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$

Mustaqil yechish uchun misollar

20.1 – misol. Ushbu

$$f(t) = \begin{cases} e^{2t}\sin 3t, & t>0 \\ 0, & t<0 \end{cases}$$

funksiya original funksiya bo'lishini ko'rsating.

Yechilishi. $f(t)$ funksiya lokal integrallanuvchi, ya'ni har qanday (t_1, t_2) chekli oraliqda integral

$$\int_{t_1}^{t_2} e^{2t}\sin 3t\, dt$$

mavjud.Misol shartidan kelib chiqib 2^0 - shart bajariladi, ya'ni $t<0$ bo'lganda $f(t)=0$. $\exists M$ va $\exists S_0>1$, $S_0=2$ mavjudki barcha haqiqiy t lar uchun

$$\left|e^{2t}\sin 3t\right| \le e^{2t}$$

Demak, berilgan funksiya original ekan.

20.2. Quyidagi funksiyalar original funksiyalar bo'lishini ko'rsating:

1) $f(t)=\frac{1}{t-3}\eta(t)$, $\quad \eta(t)=\begin{cases}1, & t>0, \\ 0 & t<0.\end{cases}$

2) $f(t)=e^{-t}\cos t\,\eta(t)$, $\eta(t)=\begin{cases}1, & t>0, \\ 0 & t<0.\end{cases}$

20.3 – misol. Ta'rifdan foydalanib, Ushbu $f(t)=e^{2t}$ funksiyaning tasvirini toping.

Yechilishi. $f=e^{2t}$ funksiya uchun $S_0=2$ bo'ladi. Shuning uchun $F(p)$ tasvir funksiya $\operatorname{Re} p>2$ yarim tekislikda aniqlangan va analitik. Laplas almashtirishidan foydalanib, tasvir funksiyasini topamiz:

$$F(p)=\int_0^{+\infty} e^{2t}e^{-pt}dt=\int_0^{+\infty} e^{-(p-2)t}dt=\lim_{a\to+\infty}\int_0^a e^{-(p-2)t}dt=\lim_{a\to+\infty}\frac{-1}{p-2}e^{-(p-2)t}\Big|_0^a=$$

$$=\lim_{a\to+\infty}\frac{1}{2-p}\left(e^{-(p-2)a}-e^0\right)=\frac{1}{p-2} \quad (\operatorname{Re} p=s>2).$$

Demak, $F(p)=\frac{1}{p-2}\doteq e^{2t}=f(t)$. Bu funksiya $p=2$ nuqtadan tashqari qolgan barcha $\operatorname{Re} p>2$ tekislikda analitik.

20.4. Ta'rifdan foydalanib, quyidagi funksiyalarning tasvirlarini toping.

1) $f(t)=te^t$. $\qquad$ 2) $\frac{3}{p^2+9}$.

20.5 – misol. O'xshashlik teoremasidan foydalanib, quyidagi $f(t)=\sin 4t$ funksiyaning tasvirini toping.

Echilishi. Bizga ma'lumki, $\sin t$ funksiyaning tasviri

$$\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \frac{1}{2i}\left(\frac{1}{p-i} - \frac{1}{p+i}\right) = -\frac{1}{2i}\frac{2_i}{p^2+1} = \frac{1}{p^2+1} = F(p)$$

O'xshashlik teoremasidan foydalanib, quyidagilarni yozamiz: $f(\alpha t) = \frac{1}{\alpha}F\left(\frac{p}{\alpha}\right)$

$$f(4t) = \sin 4t = \frac{1}{4}F\left(\frac{p}{4}\right) = \frac{1}{4}\frac{1}{\left(\frac{p}{4}\right)^2+1} = \frac{4}{p^2+16} = F_1(p).$$

20.6. O'xshashlik teoremasidan foydalanib, quyidagi funksiyalarning tasvirini toping.

1) $f(t) = \cos at$. 2) $f(t) = sh3t$.

20.7 – misol. Chiziqlilik va o'xshashlik teoremalaridan foydalanib quyidagi $f(t) = \cos^3 t$ funksiyalarni tasvirini toping.

Yechilishi. Bizga ma'lumki, $\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$ ko'rinishda yozish mumkin. O'xshashlik teoremasidan foydalanib $\cos t$ va $\cos 3t$ funksiyalarning tasvirlarini topamiz:

$$\cos t = \frac{p}{p^2+1}, \quad \cos 3t = \frac{1}{3}\frac{\frac{p}{3}}{\left(\frac{p}{3}\right)^2+1} = \frac{p}{p^2+9}.$$

Chiziqlilik teoremasiga asosan,

$$f(t) = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t = \frac{3}{4}\frac{p}{p^2+1} + \frac{p}{p^2+9} = F(p)$$

bo'lamiz.

20.8. Chiziqlilik va o'xshashlik teoremalaridan foydalanib quyidagi funksiyalarni tasvirlarini toping.

1) $f(t) = \sin mt \cos nt$.

2) $f(t) = 2 + t^3 + t\cos 2t$; 3) $f(t) = 3^t$; 4) $f(t) = \cos^2 t$.

20.9 – misol. Original funksiyani differensiallash teoremasidan foydalanib $f(t) = \sin^2 t$ funksiyaning tasvirini toping.

Echilishi. $f(t) = F(p)$ bo'lsin, u holda $f'(t) = pF(p) - f(0)$ bo'ladi, bunda $f(0) = 0$, $f'(t) = 2\sin t\cos t = \sin 2t = \frac{2}{p^2+4}$. Demak, $\frac{2}{p^2+4} = pF(p)$, bu yerdan $F(p)$ni topamiz: $F(p) = \frac{2}{p(p^2+4)} = \sin^2 t$.

20.10. Original funksiyani differensiallash teoremasidan foydalanib quyidagi funksiyalarni tasvirlarini toping.

1) $f(t) = \sin^3 t$ 2) $f(t) = t\cos \omega t$

3). $f(t) = t^2\cos 2t$. 4). $f(t) = t^3 \sin t$. 5). $f(t) = tsh3t$.

6). $f(t) = t\,ch2t$. 7). $f(t) = te^t \sin t$.

20.11 – misol. Tasvirni differensiallash haqidagi teoremaga asosan $f(t) = t^2 e^t$ funksiyaning tasvirini toping.

Yechilishi. Ma'lumki, $e^t = \frac{1}{p-1}$. Tasvirni differensiallash haqidagi teoremaga asosan, $\left(\frac{1}{p-1}\right)' = -te^t$ bo'ladi. Bundan $\left(\frac{1}{p-1}\right)' = -te^t$. Oxirgi tenglikning chap tomonidan yana hosila olamiz:

$$\left(\frac{1}{(p-1)^2}\right)' = -t(te^t) \Rightarrow \frac{-2}{(p-1)^3} = -t^2e^t \Rightarrow \frac{2!}{(p-1)^3} = t^2e^t.$$

20.12. Tasvirni differensiallash haqidagi teoremaga asosan $f(t)$ funksiyaning tasvirini toping

1) $F(p) = \frac{7}{p^3}$; 2) $F(p) = \frac{4}{(p+1)^4} - \frac{3}{(p-1)^2}$; 3) $F(p) = \frac{4}{p^2-6p+13}$.

20.13. Tasvirni differensiallash haqidagi teoremaga asosan, funksiyaning tasvirini toping.

1) $f(t) = t^2\cos t$, 2) $f(t) = (1+t)\sin 2t$.

20.14 – misol. $F(p) = \frac{1}{p(p-1)(p^2+4)}$ funksiya uchun originalni toping.

Yechilishi. $F(p)$ funksiyani soda kasrlarga yoymiz:

$$\frac{1}{p(p-1)(p^2+4)} = \frac{A}{p} + \frac{B}{p-1} + \frac{Cp+D}{p^2+4}$$

A, B, C, D noma'lum koeffisentlarni topamiz: $A = -1$, $B = \frac{1}{5}$, $C = \frac{4}{5}$, $D = -\frac{1}{5}$.

$$F(p) = -\frac{1}{p} + \frac{1}{5}\frac{1}{p-1} + \frac{4}{5}\frac{p}{p^2+4} - \frac{1}{5}\frac{1}{p^2+4} \qquad (*)$$

(*) tenglikning chap tomonidagi har bir oddiy kasr uchun originalni topish oddiy. Chiziqlilik teoremasidan foydalanib, orginalni topamiz:

$$f(t) = -1 + \frac{1}{5}e^t + \frac{4}{5}\cos 2t - \frac{1}{10}\sin 2t$$

Mustaqil yechish uchun misollarning javoblari

20.4. 1) $\frac{1}{(p-1)^2}$. 2) $\frac{3}{p^2+9}$. **20.6.** 1) $F(p) = \frac{p}{p^2+a^2}$. 2) $F(p) = \frac{3}{p^2-9}$.

20.8. 1) $F(p) = \frac{m(p^2+m^2-n^2)}{(p^2+m^2+n^2)^2+4m^2n^2}$. 2) $F(p) = \frac{2}{p} + \frac{6}{p^4} + \frac{p^2-4}{(p^2+4)^2}$.

3) $F(p) = \frac{1}{p-\ln 3}$. 4) $F(p) = \frac{1}{2p} + \frac{p}{2(p^2+4)}$.

20.10. 1) $F(p) = \frac{6}{(p^2+1)(p^2+9)}$. 2) $F(p) = \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$

3). $F(p)=\frac{2p^3-24p}{(p^2+4)^3}$. 4). $F(p)=\frac{24p(p^2-1)}{(p^2+1)^4}$. 5). $F(p)=\frac{6p}{(p^2-9)^2}$.

6). $F(p)=\frac{p^2+4}{(p^2-4)^2}$. 7). $F(p)=\frac{2p-2}{(p^2-2p+2)^2}$.

20.12. 1) $f(t)=\frac{7}{2}t^2$. 2) $f(t)=\frac{2}{3}e^{-t}t^3-3e^t t$. 3) $f(t)=2e^{3t}\sin 2t$.

20.13. 1) $\frac{2p^3-6p}{(p^2+1)^3}$. 2) $\frac{2p^2+4p+8}{(p^2+4)^2}$.

21- AMALIY MASHG'ULOT.

OPERASION USULLARINI DIFFERENSIAL TENGLAMALAR VA ULARNING SISTEMALARINI YECHISHGA TATBIQ ETISH

O'zgarmas koeffisiyentli chiziqli differensial tenglama berilgan bo'lsin:

$$x^{(n)}(t)+a_1x^{(n-1)}(t)+...+a_{n-1}x'(t)+a_nx(t)=f(t), \quad (1)$$

bu tenglamaning

$$x(0)=x_0,\ x'(0)=x'_0,...,x^{(n-1)}(0)=x_0^{(n-1)} \quad (2)$$

boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini topish talab etiladi. Bunda $f(t)$ funksiya kabi izlanayotgan yechim ham Laplas bo'yicha tasvirning mavjudlik shartlariga bo'ysunadi deb faraz qilinadi. (1) tenglamaning ikala qismiga Laplas almashtirishini tatbiq qilamiz.

$x(t)$ va $f(t)$ ning tasvirlarini mos ravishda $X(p)$ va $F(p)$ orqali belgilaymiz:

$$x(t)\doteq X(p),\ f(t)\doteq F(p).$$

Originalni differensiallash qoidasini qo'llanib quyidagini topamiz:

$$x'(t)\doteq pX(p)-x_0,$$

$$x''(t)\doteq p^2X(p)-[px_0+x'_0],$$

....

$$x^{(n-1)}(t)\doteq p^{n-1}X(p)-[p^{n-2}x_0+...+x^{(n-2)}],$$

$$x^{(n)}(t)\doteq p^nX(p)-[p^{n-1}x_0+p^{n-2}x'_0+...+px_0^{(n-2)}+x_0^{(n-1)}].$$

Laplas almashtirishining chiziqliligiga binoan chap tomoning tasvirini topish uchun hosil qilingan ifodalarnii tegishli a_i koeffisentlarga ko'paytirish va qo'shish kifoya, o'ng tomonining tasviri esa $F(p)$ ga teng. Shunday qilib, qo'yidagiga egamiz:

$$\varphi(p)X(p)-\psi(p)=F(p), \quad (3)$$

bu yerda

$$\varphi(p)=p^n+a_1p^{n-1}+...+a_{n-1}p+a_n$$

ifoda chiziqli tenglama uchun xarakteristik ko'phad, $\psi(p)$ esa

$$\psi(p)=[p^{n-1}x_0+p^{n-2}x'_0+...+px_0^{(n-2)}+x_0^{(n-1)}]+$$
$$+a_1[p^{n-2}x_0+...+x_0^{(n-2)}]+...+a_{n-2}[px_0+x'_0]+a_{n-1}x_0.$$

Nol boshlang'ich shartlarda $\psi(p)\equiv 0$ bo'lib, (3) tenglamaning chap tomonini $\varphi(p)X(p)$ ko'rinishni olishini qayd qilib o'tamiz, bunga (1) tenglamada differensiallash operatorini p ko'paytuvchi bilan almashtirib, $X(p)$ni qavs tashqarisiga chiqarish orqali kelish mumkin. (3) tenglama boshlang'ich shartlar sistemasi (2) bo'lgan (1) tenglama uchun yordamchi tenglamadir. Uni yana tasvirlovchi (yoki operator) tenglama deb ham ataladi, (3) tenglamani $X(p)$ ga nisbatan yechib,

$$X(p)=\frac{F(p)+\psi(p)}{\varphi(p)} \quad (4)$$

ko'rinishga ega bo'lgan tasvirlovchi yoki operator yechimini hosil qilamiz.

Original o'tish izlanayotgan $x(t)$ xususiy yechimni topishga imkon beradi. Bunda (4) operator yechimning o'ng tomoni odatda rasional kasr bo'lib chiqadi va tasvirlar jadvalidan foydalanishni yengillatish uchun o'ng tomonini elementar kasrlarga yoyish kerak.

Mustaqil yechish uchun misollar

21.1 – misol. Differensial tenglamani yeching

$x''+x=2\cos t,\ x(0)=0, x'(0)=-1$

Yechilishi. $x(t)\doteq X(p),\ x'(t)\doteq pX(p)-x(0)=pX(p)$

$$x''(t)\doteq p^2X(p)-px(0)-x'(0)=p^2X(p)+1,\ \cos t\doteq\frac{p}{p^2+1},\quad p^2X(p)+1+X(p)=\frac{2p}{p^2+1}$$

bu yerda $X(p)=\frac{2p}{(p^2+1)^2}-\frac{1}{p^2+1}$. $X(p)$ funksiya uchun originalni topamiz. $\frac{1}{p^2+1}$ tasvir funksiya uchun original $\sin t$ bo'ladi, ya'ni $\frac{1}{p^2+1}\doteq\sin t$.

$\frac{2p}{(p^2+1)^2}$ tasvir funksiya uchun original funksiyanin topishda tasvirni differensiallash haqidagi teoremadan foydalanamiz:

$$\frac{2p}{(p^2+1)^2}=-\left(\frac{1}{p^2+1}\right)'_p \doteq t\sin t$$

Demak, $X(p)\doteq t\sin t-\sin t=(t-1)\sin t$.

Shunday qilib, berilgan differensial tenglamaning yechimi $x(t)=(t-1)\sin t$ bo'ladi.

21.2. Quyidagi differensial tenglamalarni yeching:

1) $x'+x=e-t,\ x(0)=1$; 2) $x''+x'=1,\ x(0)=0,\ x'(0)=1$;

3) $x'+2x=\sin t,\ x(0)=0$.

21.3. Differensial tenglamalarni boshlang'ich shartini qanoatlantiruvchi xususiy yechimini toping.

a) $x'-x=1,\ x(0)=-1;$ *b)* $x''-2x'+2x=2t-2,\ x(0)=x'(0)=0;$

v) $x'''-x''=4e^{2t},\ x(0)=1,\ x'(0)=2,\ x''(0)=4.$

21.4-misol. $\int_0^{+\infty}\frac{\sin t}{t}dt$ ni hisoblang.

Yechilishi. Ravshanki, $f(t)=\sin t=\frac{1}{p^2+1}=F(p)$. $\int_0^{+\infty}\frac{f(t)}{t}=\int_0^{\infty}F(p)dp$ formulaga asosan,

$$\int_0^{+\infty}\frac{\sin}{t}dt=\int_0^{\infty}\frac{1}{p^2+1}dp=arctg\,p\Big|_0^{\infty}=\frac{\pi}{2}$$

21.5. Qyidagi integrallarni hisoblang

1) $\int_0^{\infty}\frac{e^{-at}-e^{-bt}}{t}dt,\ (a>0,b>0)$. 2) $\int_0^{\infty}\frac{e^{-\alpha t}\sin at}{t}dt,\ (\alpha>0,\ a>0)$.

Differensial tenglamalar sistemasining yeching.

21.6. $\begin{cases}x'+y=0,\\ y'+x=0,\end{cases}$ $x(0)=1, y(0)=-1.$ 21.7. $\begin{cases}x'-3x-4y=0,\\ y'-4x+3y=0,\end{cases}$ $x(0)=y(0)=1.$

21.8. $\begin{cases}x'+x-y=2,\\ y'+x+y=2t,\end{cases}$ $x(0)=0, y(0)=-1.$

Mustaqil yechish uchun misollarning javoblari

21.2. 1) $x(t)=e(1-e^{-t})+1-t.$ 2) $x(t)=t.$ 3) $x(t)=\frac{1}{5}(e^{-2t}-\cos t+2\sin t).$

21.3. *a)* $x(t)=-1$. b) $x(t)=t-\sin t\cdot e^t$. *v)* $x(t)=e^{2t}$. **21.5.** 1) $\ln\frac{b}{a}$.

Ko'rsatma: $f(t)=e^{at}-e^{bt}=\frac{1}{p+a}\cdot\frac{1}{p-b}=F(p)$. 2) $arctg\ \frac{a}{\alpha}$.

21.6. $x(t)=e^t, y(t)=-e^t$. 21.7. $x(t)=\frac{6}{5}e^{5t}-\frac{1}{5}e^{-5t}$ $y(t)=\frac{3}{5}e^{5t}+\frac{2}{5}e^{-5t}$.

21.8. $x(t)=t,\ y(t)=t-1.$

22- amaliy mashg'ulot.

IKKI KARRALI INTEGRALLARNI HISOBLASH

22.1. Ikki karrali integrallarni hisoblash.

1-teorema. $f(x,y)$ funksiya $D=\{(x,y)\in R^2: a\le x\le b, c\le y\le d\}$ sohada berilgan va integrallanuvchi bo'lsin. Agar $x\in[a,b]$ o'zgaruvchining har bir tayin

qiymatida $I(x)=\int_c^d f(x,y)dy$ integral mavjud bo'lsa, u holda $\int_a^b\left[\int_c^d f(x,y)dy\right]dx$ integral ham mavjud bo'ladi va $\iint_{(D)}f(x,y)dD=\int_a^b\left[\int_c^d f(x,y)dy\right]dx$ formula o'rinli.

2-teorema. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar $y\in[c,d]$ o'zgaruvchining har bir tayin qiymatida $I(y)=\int_a^b f(x,y)dx$ integral mavjud bo'lsa, u holda $\int_c^d\left[\int_a^b f(x,y)dx\right]dy$ integral ham mavjud bo'ladi va

$$\iint_{(D)}f(x,y)dD=\int_c^d\left[\int_a^b f(x,y)dx\right]dy$$ formula o'rinli.

1-natija. Agar $f(x,y)$ funksiya chegaralangan yopiq $(D)((D)\subset R^2)$ sohada berilgan va uzluksiz bo'lsa,

$$\iint_{(D)}f(x,y)dD,\quad \int_a^b\left[\int_c^d f(x,y)dy\right]dx,\quad \int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

integrallarning har biri mavjud va ular o'zaro teng bo'ladi.

3- teorema. $f(x,y)$ funksiya

$(D)=\{(x,y)\in R^2: a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}\ (\varphi_i(x)\in C[a,b],\ i=1,2)$ sohada berilgan va integrallanuvchi bo'lsin. Agar $x\in[a,b]$ o'zgaruvchining har bir tayin qiymatida $I(x)=\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy$ integral mavjud bo'lsa, u holda

$$\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy\right)dx$$

integral ham mavjud bo'ladi va

$$\iint_{(D)}f(x,y)dD=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy\right]dx$$

tenglik o'rinli.

4- teorema. $f(x,y)$ funksiya

$(D)=\{(x,y)\in R^2: c\le x\le d,\ \psi_1(y)\le x\le\psi_2(y)\}\ (\psi_i(x)\in C[c;d],\ i=1,2)$ sohada berilgan va integrallanuvchi bo'lsin. Agar $y\in[c,d]$ o'zgaruvchining har bir tayin qiymatida $I(y)=\int_{\psi_1(y)}^{\psi_2(y)}f(x,y)dx$ integral mavjud bo'lsa, u holda ushbu

$$\int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$$

integral ham mavjud bo'ladi va

$$\iint_{(D)} f(x,y)dD = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right] dy$$

tenglik o'rinli.

1-misol. (D) soha: $x=2y,\ y=2x,\ x+y=6$ to'g'ri chiziqlar bilan chegaralangan uchburchakdan iborat.

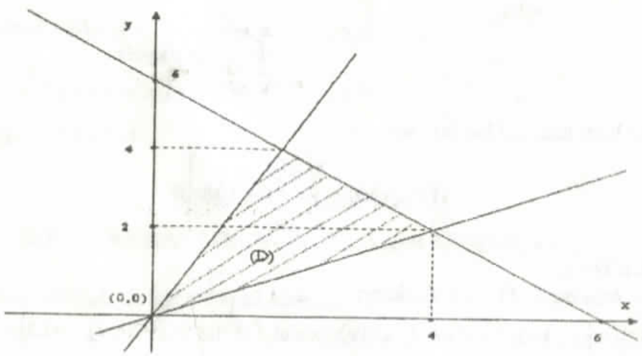
Yechilishi. (D) uchburchak 1- chizmada tasvirlangan. (D) sohani $x=2$ to'g'ri chiziq yordamida (D_1) va (D_2) sohalarga ajratamiz:

$$(D_1)=\left\{(x,y)\in R^2:\ 0\le x\le 2,\ \frac{x}{2}\le y\le 2x\right\},\ (D_2)=\left\{(x,y)\in R^2: 2\le x\le 4,\ \frac{x}{2}\le y\le 6-x\right\}.$$

(*) formulalarga asosan,

$$\iint_{(D)} f(x,y)dxdy = \iint_{(D_1)} f(x,y)dx\,dy + \iint_{(D_2)} f(x,y)dxdy = \int_0^2 dx \int_{\frac{x}{2}}^{2x} f(x,y)dy + \int_2^4 dx \int_{\frac{x}{2}}^{2x} f(x,y)dy$$

yoki $\iint_{(D)} f(x,y)dxdy = \int_0^2 dy \int_{\frac{y}{2}}^{2y} f(x,y)dx + \int_2^4 dy \int_{\frac{y}{2}}^{6-y} f(x,y)dx$ tenglik o'rinli bo'ladi.



1-chizma.

2-misol. $J=\int_0^1 dx \int_{2x}^1 \sqrt[4]{1-y^2}\,dy$ xisoblang.

Yechilishi. Ichki integral y ga nisbatan murakkab bo'lgani uchun takroriy integrallarning chegarasi bo'lgan

$$(D)=\{(x,y)\in R^2: 0\le x\le 1,\ 2x\le y\le 1\}$$

uchburchakni ushbu

$$(D)=\left\{(x,y)\in R^2: 0\le y\le 1,\ 0\le x\le \frac{y}{2}\right\}$$

ko'rinishda ifodalaymiz. Bundan

$$J=\int_0^1 dy \int_0^{\frac{y}{2}} \sqrt[4]{1-y^2}\,dx = -\frac{1}{5}\cdot\left(1-y^2\right)^{\frac{5}{4}}\Big|_0^1 = \frac{1}{5}.$$

Mustaqil yechish uchun misollar

Quyidagi berilgan ikki kaprali integrallarni hisoblang:

22.1 $\iint_{(D)} xydxdy,\ (D)=\{(x,y)\in R^2: 0\le x\le 1,\ 0\le y\le 1\}$.

22.2. $\iint_{(D)} (3x-2y)dxdy,\ (D)=\{(x,y)\in R^2: 1\le x\le 2,\ 1\le y\le 3\}$

22.3. $\iint_{(D)} (x+y)dxdy,\ (D)=\{(x,y)\in R^2: 0\le x\le 1, 0\le y\le 1\}$.

22.4. $\iint_{(D)} (xy-6)dxdy,\ (D)=\{(x,y)\in R^2: 1\le x\le 3,\ 2\le y\le 5\}$.

22.5. $\iint_{(D)} (\sin x+\cos y)dxdy:\ (D)=\{(x,y)\in R^2:\ 0\le x\le \frac{\pi}{2};\ 0\le y\le \frac{\pi}{2}\}$.

(22.6)-(22.9) da ikki karrali integralda integralash tartibini uzgartirib hisoblang.

22.6. $\int_0^\pi dx \int_x^\pi \frac{\sin y}{y} dy.$ **22.7.** $\int_0^1 dy \int_y^1 x^2 e^{xy} dx.$

22.8. $\int_0^{2\sqrt{\ln 3}} dy \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx.$ **22.9.** $\int_0^{1/16} dy \int_{\sqrt[4]{y}}^{1/2} \cos\left(16\pi x^5\right) dx.$

22.10. $\iint\limits_{(D)}(x-y)dxdy$, bu yerda (D)–uchlari $A(1,1)$, $B(4;1)$, $C(4,4)$nuqtalari bo'lgan uchburchak.

22.11. $\iint\limits_{(D)}(x+2y)dxdy$, bu yerda $(D)-y=\frac{x^2}{2}$ parabola va $y=3x$, $x=1$, $x=2$ to'g'ri chiziqlar bilan chegaralangan soha.

22.12. $\iint\limits_{(D)}xdx\,dy$, bu yerda $(D)-y=\frac{1}{x}$ giperbola va $y=2$, $y=4$, $x+y=6$ to'g'ri chiziqlar bilan chegaralangan soha.

22.13. $\iint\limits_{(D)}(x+2y)dxdy$, bu yerda $(D)-y=x^2$ parabola va $x+y-2=0$, $y=0$ to'g'ri chiziqlar bilan chegaralangan soha.

22.14. $\iint\limits_{(D)}x\ln y\,dxdy$, bu yerda $(D)=\{(x,y)\in R^2 : 0\le x\le 4,\ 1\le y\le e\}$ to'g'ri to'rtburchak.

22.15. $\iint\limits_{(D)}(\cos^2 x+\sin^2 y)dxdy$, bu yerda $(D)=\left\{(x,y)\in R^2 : 0\le x\le\frac{\pi}{4},\ 0\le y\le\frac{\pi}{4}\right\}$

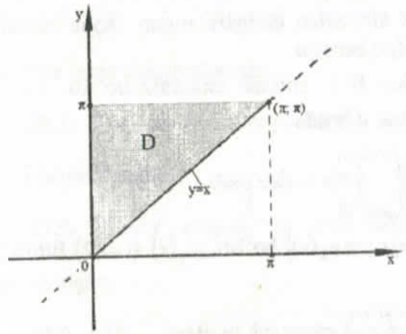
22.16. $\iint\limits_{(D)}e^{x+y}dxdy$, bu yerda $(D)-y=e^x$ egri chiziq va $x=0$, $y=2$ to'g'ri chiziqlar bilan chegaralangan .

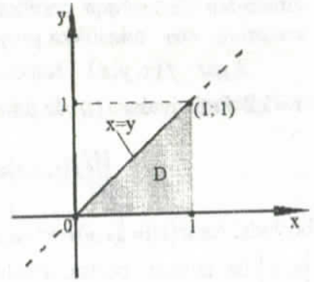
22.17. $\iint\limits_{(D)}xydxdy$, bu yerda $(D)-4x^2+y^2=4$ ellips bilan chegaralangan soha.

22.18. $\iint\limits_{(D)}x,ydxdy$, bu yerda $(D)-y^2=2x$ parabola va $x-y-4=0$ to'g'ri chiziq bilan chegaralangan soha.

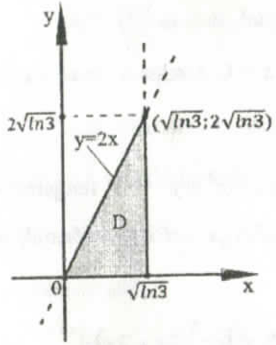
Mustaqil yechish uchun misollarning javoblari

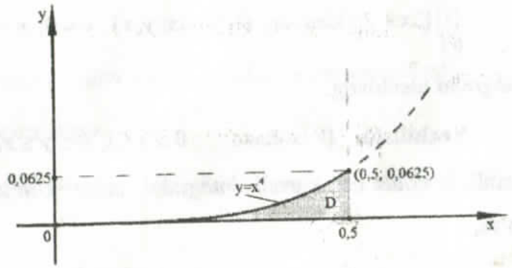
22.1 $\frac{1}{4}$. **22.2.** 1. **22.3.** 1. **22.4.** 18. **22.5.** π.**22.6.** 2. **22.7.** $\frac{e-2}{2}$.





22.8. 2. **22.9.** $1/80\pi$.





22.10. 4,5. **22.11.** $\frac{983}{40}$. **22.12.** $\frac{221}{24}$. **22.13.** 1,45. **22.14.** 8. **22.15** 0,9. **22.16.** e. **22.17.** 0 . **22.18.** 90 .

23- amaliy mashg'ulot.

UCH KARRALI INTEGRALLARNI HISOBLASH

Uch karrali integrallarning mavjudligi, integrallanuvchi funksiyalar sinflari va integralning xossalariga oid teoremalar xuddi ikki karrali integrallardagi kabi bo'ladi.

$f(x,y,z)$ funksiya $(V)=\{(x,y,z)\in R^3 : a\le x\le b,\ c\le y\le d,\ e\le z\le l\}$ sohada berilgan va uzluksiz bo'lsin. U holda

$$\iiint\limits_{(V)}f(x,y,z)dxdydz=\int\limits_a^b\left[\int\limits_c^d\left(\int\limits_e^l f(x,y,z)dz\right)dy\right]dx$$

tenglik o'rinli.

Endi (V) soha — nastdan $z=\psi_1(x,y)$, yuqoridan $z_2=\psi_2(x,y)$ sirtlar bilan, yon tomondan Oz o'qqa parallel silindrik cirt bilan chegaralangan soha bo'lsin. Bu sohaning Oxy tekislikka proyeksiyasi (D) bo'lsin.

Agar $f(x,y,z)$ funksiya shunday (V) sohada uzluksiz bo'lib, $z=\psi_i(x,y)$ $i=1,2$ funksiyalar (D) da uzluksiz bo'lca, u holda

$$\iiint_{(V)} f(x,y,z)dxdydz=\iint_{(D)}\left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right)dxdy$$

bo'ladi. Agar $(D)=\{(x,y)\in R^2: a\le x\le b, \varphi_1(x)\le y\le\varphi_2(x)\}$ bo'lib, $\psi_i(x)$ $(1=1,2)$ funksiyalar $[a;b]$ da uzluksiz bo'lsa, u holda

$$\iiint_{(V)} f(x,y,z)dxdydz=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right)dy\right)dx$$

bo'ladi.

1-misol.

$$\iiint_{(V)}(3x+2y)dxdydz,\ (V)=\{(x,y,z): y=0,\ y=x, x=1,\ z=1, z=1+x^2+y^2\}$$

integralni hisoblang.

Yechilishi. (V)sohada $0\le x\le 1,\ 0\le y\le x,\ 1\le 1+x^2+y^2$ tengsizliklar o'rinli. U holda uch karrali integralni takroriy integrallarga keltirish formulasiga ko'ra,

$$\iiint_{(V)}(3x+2y)dxdydz=\int_0^1 dx\int_0^x dy\int_1^{1+x^2+y^2}(3x+2y)dz=\int_0^1 dx\int_0^x (3x+2y)z\Big|_1^{1+x^2+y^2}dy=$$

$$=\int_0^1 dx\int_0^x(3x+2y)(x^2+y^2)dy=\int_0^1 dx\int_0^x(3x^3+3xy^2+2yx^2+2y^3)dy=$$

$$=\int_0^1\left(3x^3y+xy^3+y^2x^2+\frac{1}{2}y^4\right)\Big|_0^x dx=\int_0^1(3x^4+x^4+x^4+x^4+\frac{1}{2}x^4)dx=\frac{11}{10}.$$

Mustaqil yechish uchun misollar

Quyidagi uch karali integrllar ko'rsatilgan sohada hisoblang:

23.1. $\iiint_{(v)}(x^2+y^2+z^2)dx\,dy\,dz$, bunda (v)– $x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c$ tekisliklar bilanchegaralangan soha.

23.2. $\iiint_{(v)} ydx\,dy\,dz$, bu yerda (v)– $x=0,\ x=2,\ y=0,\ y=1,\ z=0,\ z=1-y$.

23.3. $\iiint_{(v)}(2x+3y-z)dx\,dy\,dz$, bu yerda (v)–soha $x=0, y=0,\ x+y=3, z=0,\ z=4$ tekisliklar bilan chegaralangan.

23.4. $\iiint_{(v)} xyz\ dx\,dy\,dz$, bu yerda (v)- soha $x=0,\ y=0,\ z=0,\ x+y+z=1$ tekisliklar bilan chegaralangan.

23.5. $\iiint_{(v)} xy^2z^3\ dx\,dy\,dz$, bu yerda (v)- soha $x=1,\ y=x\ z=0,\ z=xy$ sirtlar bilan chegaralangan.

23.6. $\iiint_{(v)}(1+x)dx\ dy\,dz$, bu yerda (v)- soha $x=0,\ y=0\ z=0,\ x+y+z=1$ tekisliklar bilan chegaralangan.

23.7. $\iiint_{(v)}\frac{1}{z}dx\,dy\,dz$ bu yerda (v)- soha $z^2=x^2+y^2$ konus va $z=1,\ z=4$ tekislar bilan chegaralangan.

23.8. $\iiint_{(v)} z^2dx\,dy\,dz$, bu yerda (v)- soha $z=x^2+y^2$ elliptik paraboloid va $z=2,\ z=6$ tekisliklar bilan chegaralangan.

23.9. $\iiint_{(v)} z^3dx\,dy\,dz$ bu yerda (v)- soha $z=4-x^2-y^2,\ z=0,\ z=3$ sirtlar bilan chegaralangan.

Mustaqil yechish uchun misollarning javoblari

23.1. $\frac{abc}{3}(a^2+b^2+c^2)$ **23.2.** $\frac{1}{3}$. **23.3.** 54. **23.4.** $\frac{1}{720}$. **23.5.** 2.2 **3.6.** $\frac{5}{24}$. **23.7.** $\frac{15\pi}{2}$. **23.8.** 50π. **23.9.** $162\pi/5$.

24- amaliy mashg'ulot.

KARRALI INTEGRALLARDA O'ZGARUVCHILARNI ALMASHTIRISH. QUTB, SILINDRIK VA SFERIK KOORDINAT SISTEMALARIGA O'TISH USULI

24.1. Ikki karrali integrallarda o'zgaruvchilarni almashtirish. $f(x,y)$ funksiya (D) sohada berilgan va uning chekli

$$\iint_{(D)} f(x,y)dxdy \qquad (1)$$

ikki karrali integrali mavjud va uni hisoblash talab qilingan bo'lsin. Agar $f(x,y)$ funksiya va (D) soha murakkab bo'lsa, (1) integralni hisoblash qiyinlashadi. Bunday hollarda x va y o'zgaruvchilarni ma'lum qoidaga ko'ra, boshqa o''zgaruvchilarga almashtirish natijasida integral ostidagi funksiya ham, integrallash sohasi ham soddalashib, ikki karrali integralni hisoblash osonlashadi.

Oxy hamda *Ouv* koordinatalar sistemasida (D) va (Δ) sohalar berilgan bo'lsin. Bu sohalarning chegaralari, mos ravishda, $\partial(D)$ va $\partial(\Delta)$ lar, sodda, bo'lakli silliq chiziqlardan iborat bo'lsin. (Δ) sohada

$$\begin{cases} x=\varphi(\xi,\eta) \\ y=\psi(\xi,\eta), \end{cases} \quad (\xi,\eta)\in(\Delta)\subset R^2 \qquad (2)$$

uzluksiz funksiyalar sistemasi berilgan bo'lsin. Bu funksiyalar shunday funksiyalar bo'lsinki, ulardan tuzilgan (2) sistema (Δ) dagi (ξ,η) nuqtani (D) sohadagi (x,y) nuqtaga akslantirisin va bu akslantirishni akslaridan iborat $\{(x,y)\}$ to'plam (D)ga qarashli bo'lsin. Demak, (2) sistema (Δ) sohani (D) sohaga akslantiradi.

(2) akslantirish quyidagi shartlarni qanoatlantirsin:

1°. (2) akslantirish o'zaro bir qiymatli bo'lsin, ya'ni (Δ) sohaning turli nuqtalarini (D) sohaning turli nuqtalariga akslantirsin. (D) sohaning har bir nuqtasi uchun (Δ) sohada unga mos keladigan nuqta bittagina bo'lsin. Bu holda (2) sistema ξ va η larga nisbatan bir qiymatli yechiladi:

$$\begin{cases} \xi=\varphi_1(x,y) \\ \eta=\psi_1(x,y), \end{cases} \quad (x,y)\in(D)\subset R^2.$$

2°. $\varphi(\xi,\eta)$, $\psi(\xi,\eta)$ funksiyalar (Δ) sohada, $\varphi_1(x,y),\psi_1(x,y)$ funksiyalar esa, (D) sohada uzluksiz va barcha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar ham uzluksiz bo'lsin.

3°. $\forall(\xi,\eta)\in(\Delta)$ uchun (2) sistemadagi funksiyalarning xususiy hosilalaridan tuzilgan ushbu ikkinchi tartibli determinant

$$\begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} \end{vmatrix} \neq 0 \qquad (\alpha)$$

shartni qanoatlantirsin. Odatda, (α) ikkinchi tartibli determinant - (2) sistemaning *yakobiani* deyiladi va $I(\xi,\eta)$ yoki $\frac{D(x,y)}{D(\xi,\eta)}$ kabi belgilanadi.

$f(x,y)$ funksiya (D)sohada berilgan va uzluksiz bo'lib, (2) akslantirish 1°-, 2°-, 3°- shartlarni qanoatlantirsin. U holda

$$\iint_{(D)} f(x,y)dxdy = \iint_{(\Delta)} f(\varphi(\xi,\eta),\psi(\xi,\eta))|I(\xi,\eta)|d\xi d\eta \qquad (3)$$

formula o'rinli. (3) formula ikki karrali integrallarda *o'zgaruvchilarni almashtirish formulasi* deyiladi.

Dekart koordinatalar sistemasidan

$$x=r\cos\varphi,\ y=r\sin\varphi \qquad (0\le r<+\infty),\ (0\le\varphi<2\pi) \qquad (4)$$

almashtirish yordamida (r,φ) qutb koordinatalar sistemasiga o'tamiz. Natijada (3) formula ushbu

$$\iint_{(D)} f(x,y)dxdy = \iint_{(\Delta)} f(r\cos\varphi, r\sin\varphi)rdrd\varphi \qquad (5)$$

ko'rinishni oladi. Odatda, (5) munosabat, ikki karrali integralda *qutb koordinatalar sistemasiga o'tish formulasi* deyiladi.

24.2. Uch karrali integrallarda o'zgaruvchilarni almashtirish. $f(x,y)$ funksiya (V) sohada berilgan va uzluksiz bo'lib, (V) soha — silliq yoki bo'lakli silliq sirtlar bilan chegaralangan bo'lsin.

$\iiint_{(V)} f(x,y,z)dxdydz$ integralda o'zgaruvchilarni quyidagicha almashtiramiz:

$$\begin{cases} x=\varphi(u,v,\omega), \\ y=\psi(u,v,\omega), \\ z=\chi(u,v,\omega), \end{cases} \quad (u,v,\omega)\in(\Delta)\subset R^3 \ . \qquad (6)$$

(10.8) akslantirish quyidagi shartlarni qanoatlantirsin:

1°. (6) akslantirish o'zaro bir qiymatli bo'lsin, ya'ni (Δ) sohaning turli nuqtalarini (V) sohaning turli nuqtalariga akslantirsin. (V) sohaning har bir nuqtasi uchun (Δ) sohada unga mos keladigan nuqta bittagina bo'lsin. Bu holda (6) sistema u, v va w larga nisbatan bir qiymatli yechiladi:

$$\begin{cases} u=\varphi_1(x,y,z), \\ v=\psi_1(x,y,z), \\ w=\chi_1(x,y,z), \end{cases} \quad (x,y,z)\in(V)\subset R^3. .$$

2°. $\varphi(u,v,w)$, $\psi(u,v,w)$, $\chi(u,v,w)$ funksiyalar (Δ) sohada, $\varphi_1(x,y,z),\psi_1(x,y,z)$, $\chi_1(x,y,z)$ funksiyalar (V) sohada uzluksiz va barcha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar ham uzluksiz bo'lsin.

3°. $\forall(u,v,w)\in(\Delta)$ uchun (6) sistemadagi funksiyalarning xususiy hosilalaridan tuzilgan uchinchi tartibli determinant

$$I(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

shartni qanoatlantirsin. U holda

$$\iiint_{(V)} f(x,y,z)dxdydz = \iiint_{(\Delta)} f(\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w))\,|I(u,v,w)|du\,dv\,dw \qquad (7)$$

bo'ladi. (7) formula uch karrali integrallarda *o'zgaruvchilarni almashtirish* formulasidir.

Ko'pchilik hollarda uch karrali integrallarni hisoblash uchun o'zgaruvchilarni quyidagicha almashtirish maqsadga muvofiq bo'ladi:

a) Quyidagi

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z \qquad (8)$$

almashtirishni qaraylik $(0 \le \rho < +\infty), (0 \le \varphi < 2\pi), (-\infty < z < +\infty)$.Natijada (7) formula ushbu

$$\iiint_{(V)} f(x,y,z)dxdydz = \iiint_{(\Delta)} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho\, d\rho\, d\varphi\, dz$$

ko'rinishni oladi. Odatda (8) almashtirishlar - *silindrik almashtirishlar*, (ρ,φ,z) esa, nuqtaning *silindrik koordinatalari* deyiladi.

Ushbu

$$x = r\cos\varphi\sin\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\theta \qquad (9)$$

almashtirishlarni qaraylik $(0 \le r < +\infty), (0 \le \theta \le \pi), (0 \le \varphi < 2\pi)$. U holda (7) formula quyidagi

$$\iiint_{(V)} f(x,y,z)dxdydz = \iiint_{(\Delta)} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)r^2\sin^2\theta\, dr\, d\theta\, d\varphi.$$

ko'rinishni oladi.

Odatda (9) almashtirishlar - *sferik almashtirishlar,* (r,φ,θ) esa, nuqtaning *sferik koordinatalari* deyiladi(1-chizma).

Mustaqil yechish uchun misollar

24.1. Ushbu $u = x + 2y,\ v = x - y$ sistemadan x va y ni u va v lar orqali ifodalang va $J = \dfrac{D(x,y)}{D(u,v)}$ yakobiani toping.

24.2. (xy) tekislikda $y = 0,\ y = x$ va $x + 2y = 2$ to'g'ri chiziqlar bilan chegaralangan (D) sohada $u = x + 2y,\quad v = x - y$ almashtirishni tasvirini toping. (uv) tekislikda almashtirish chegaralarini aniqlang.

24.3. Ushbu $u = 3x + 2y,\ v = x + 4y$ sistemasidan x va y ni u va v lar orqali ifodalang, so'ngra $J(u,v) = \dfrac{D(x,y)}{D(u,v)}$ yakobiani toping.

24.4. $x = 0, y = 0$ va $x + y = 1$ to'g'ri chiziqlar bilan chegaralangan sohada $u = 3x + 2y,\ v = x + 4y$ almashtirishning tasvirini toping. (uv) tekislikda almashtirish chegaralarini toping.

24.5. Ushbu $u = 2x - 3y,\ v = -x - y$ sistemadan x va y ga nisbatan yechib u va v lar orqali ifodalang, so'ngra $J(u,v) = \dfrac{D(x,y)}{D(u,v)}$ yakobianni toping.

26.6. $x = -3, x = 0,\ y = x$ va $y = x + 1$ to'g'ri chiziqlar bilan chegaralangan sohada $u = 2x - 3y,\ v = -x + y$ almashtirishning tasvirini toping. (uv)tekislikda almashtirish chegaralarini toping.

24.7. $\iint_{(D)} (3x^2 + 14xy + 8y^2)dxdy$, bu yerda (D)- soha birinchi chorakda $y = -\dfrac{3}{2}x + 1,\quad y = -\dfrac{3}{2} + 3,\ y = -\dfrac{1}{4}x$ va $y = -\dfrac{1}{4}x + 1$ to'g'ri chiziqlar bilan chegaralangan.

24.8. $\iint_{(D)} \left(\sqrt{\dfrac{y}{x}} + \sqrt{xy}\right)dxdy$, bu yerda (D)- soha $xy = 1,\ xy = 9$ giperbolalar va $y = x,\ y = 4x$ to'g'ri chiziqlar bilan chegaralangan.

24.9. $\iint_{(D)} (2x - y)dxdy$, bu yerda $(D) - x + y = 1,\ x + y = 2,\ 2x - y = 1, 2x - y = 3$ to'g'ri chiziqlar bilan chegaralangan parallelogramm.

24.10. $\iint_{(D)} \sqrt{xy}dxdy$, bu yerda $(D) - y^2 = ax,\ y^2 = bx,\ xy = p, xy = q\ \ (0 < a < b,\ 0 < p < q)$ egri chiziqlar bilan chegaralangan.

24.11. $\iint_{(D)} xydxdy$, bunda $(D) - y = ax^3,\ y = bx^3,\ y^2 = px,\ y^2 = qx\ (0 < a < b,\ 0 < p < q)$ egri chiziqlar bilan chegaralangan soha.

24.12. $\iint_{(D)} (x + y)^3(x - y)^2 dxdy$, bu yerda $(D) - x + y = 1, x - y = 1, x + y = 3, x - y = -1$ to'g'ri chiziq bilan chegaralan gan kvadrat soha.

Qutb koordinatalar sistemasiga o'tib ikki ikkarali integralni hisoblang.

24.13. $\iint_{(D)} \sqrt{x^2 + y^2}\,dxdy;\ D - I\ chorak,\ \ x^2 + y^2 \le 9$

24.14. $\iint_D y\,dx\,dy;\ D -\ \ x^2 + y^2 \le 1,\ -1 \le x \le 0,\ 0 \le y \le 1$

24.15. $\iint_D \dfrac{dxdy}{\sqrt{x^2 + y^2}};\ D - x^2 + y^2 \ge 1,\ x^2 + y^2 \le 4$

24.16. $\iint_D \dfrac{dxdy}{x^2 + y^2 + 1};\ D - x^2 + y^2 \le 1,\ y \ge 0$

24.17. $\iint_D (x^2 + y^2)dxdy;\ D -\ x^2 + y^2 \le 4x$

Silindrik yoki sferik koordinatalar sistemasiga o'tib, quyidagi sirtlar bilan chegaralangan jismning hajmini hisoblang:

24.18. $\iiint_G (x^2+y^2)dx\,dy\,dz;\ G - z=\sqrt{9-x^2-y^2}$

24.19. $\iiint_G z\sqrt{x^2+y^2}\,dx\,dy\,dz; G - x^2+y^2=2x\ \ z=0, z=3$

24.20. $\iiint_G z\,dx\,dy\,dz; G - z^2=x^2+y^2\quad z=2$

24.21. $\iiint_G (x^2+y^2)dx\,dy\,dz;\ G - 2z=x^2+y^2\ \ z=2$

24.22. $\iiint_G \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}};\ G - x^2+y^2+z^2=4\ x^2+y^2+z^2=16$

24.23. $\iiint_G (x^2+y^2+z^2)dx\,dy\,dz;\ C - x^2+y^2+z^2=9\quad z=\sqrt{x^2+y^2}$.

Mustaqil yechish uchun misollarning javoblari

24.1. $x=\frac{u+2v}{3},\ y=\frac{u-v}{3}\quad J=-\frac{1}{3}$. **24.2.** $v=0,\ u=2\ \ v=u$

24.3. $x=\frac{1}{5}(2u-v),\ y=\frac{1}{10}(3v-u);\ J=\frac{1}{10}$. **24.4.** 3v = u, v = 2u va $3u+v=10$. **24.5.** $x=-u-3v,\ y=\frac{-u-2v}{2},\ J=-\frac{1}{2}$. **26.6.** $u+3v=3,\ u=-3v,\ u=-4v,\ u+4v=2$. **24.7.** $\frac{64}{5}$. **24.8.** $\int_1^2\int_1^3 (u+v)\frac{2u}{v}dudv=8+\frac{52}{3}\ln 2$. **24.9.** $\frac{4}{3}$. **24.10.** $\frac{2}{9}(q^{3/2}-p^{3/2})\ln\frac{b}{a}$. **24.11.** $\frac{5}{48}(a^{-6/5}-b^{-6/5})(q^{8/5}-p^{8/5})$. **24.12.** $\frac{20}{3}$. **24.13.** $9\pi/2$ **24.14.** $1/2$. **24.15.** 2π. **24.16.** $(\pi/2)\ln 2$. **24.17.** 24π. **24.18.** $324\pi/5$. **24.19.** **16.** **24.20.** 4π. **24.21.** $16\pi/3$. **24.22.** 24π. **24.23.** $243(2-\sqrt{2})\pi/5$.

25- amaliy mashg'ulot.

BIRINCHI VA IKKINCHI TUR EGRI CHIZIQLI INTEGRALLAR

25.1.Birinchi tur egri chiziqli integrallarni oddiy integralga keltirish.

1-teorema. Agar $f(x,y)$ funksiya $(K)=(AB)$ egri chiziqda uzluksiz bo'lsa, u holda bu funksiyadan (K) egri chiziq bo'yicha olingan egri chiziqli integral mavjud bo'ladi va u

$$\int_{(K)} f(x,y)ds=\int_0^S f(x(s),y(s))ds \qquad (1)$$

formula bo'yicha hisoblanadi.

Bu teorema egri chiziqli integralning mavjudlik sharti ham deb yuritiladi.

Endi (K) egri chiziq ixtiyoriy

$$x=\varphi(t),\ \ y=\psi(t)\ \ (t_0\le t\le T) \qquad (2)$$

parametrik tenglamasi bilan berilgan bo'lib, bunda $\varphi(t),\ \psi(t)$ funksiyalar uzluksiz va uzluksiz $\varphi'(t),\ \psi'(t)$ hosilalarga ega bo'lsin. Bundan tashqari, (K) chiziq karrali nuqtalarga ega bo'lmasin. Bu holda (K) egri chiziq to'g'rilanuvchi bo'ladi. Ma'lumki, $s'(t)=\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}$. Buni e'tiborga olsak, (1) dan

$$\int_{(K)} f(x,y)ds=\int_{t_0}^{T} f(\varphi(t),\psi(t))\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \qquad (3)$$

kelib chiqadi.

(3) formula, (K) egri chiziq, ixtiyoriy parametrik tenglamasi bilan berilganda, birinchi tur egri chiziqli integralni oddiy Riman integraliga keltirib hisoblash formulasidan iborat.

Agar (K) egri chiziq, $y=y(x)\,(a\le x\le b)$ oshkor shakldagi tenglama bilan berilgan bo'lsa (bunda $y(x)$ $[a;b]$ da uzluksiz va uzluksiz $y'(x)$hosilaga ega), u holda (3) formula,

$$\int_{(K)} (x,y)ds=\int_a^b f(x,y(x))\sqrt{1+[y'(x)]^2}\,dx \qquad (4)$$

shaklga keladi.

(K) egri chiziq, ushbu $\rho=\rho(\theta),\ (\theta_0\le\theta\le\theta_1)$ tenglama bilan qutb koordinatalar sistemasida berilgan bo'lib, $\rho(\theta)$ funksiya $[\theta_0;\theta_1]$ da uzluksiz hosilaga ega bo'lsin. Agar $f(x,y)$ funksiya shu (K) egri chiziqda berilgan va uzluksiz bo'lsa, u holda (3) ning ko'rinishi

$$\int_{(K)} f(x,y)ds=\int_{\theta_0}^{\theta_1} f(\rho\cos\theta,\rho\sin\theta)\sqrt{\rho^2+[\rho']^2}\,d\theta \qquad (5)$$

shaklda bo'ladi.

25.2. Ikinchi tur egri chiziqli integralning mavjudlik sharti va uni hisoblash. (K) egri chiziq o'zining $x=\varphi(t),\ y=\psi(t),\ (\alpha\le t\le\beta)$shakldagi parametrik tenglamasi bilan berilgan bo'lib, $\varphi(t),\psi(t)$ funksiyalar uzluksiz, $\varphi'(t),\ \psi'(t)$ hosilalarga ega, hamda $(\varphi(\alpha),\psi(\alpha))=A,\ B=(\varphi(\beta),\psi(\beta))$ bo'lsin. t parametr α dan β ga qarab o'zgarganda, $(x,y)=(\varphi(t),\psi(t))$ nuqta A dan B ga qarab $(K)=(AB)$ egri chiziqni chizsin.

2-teorema. Agar $f(x,y)$ funksiya (K) egri chiziqda berilgan va uzluksiz bo'lsa, u holda $\int_{(AB)} f(x,y)dx,\quad \int_{(AB)} f(x,y)dy$ egri chiziqli integrallar mavjud bo'ladi va ular

$$\int_{(AB)} f(x,y)dx=\int_\alpha^\beta f(\varphi(t),\psi(t))\varphi'(t)dt,$$

$$\int_{(AB)} f(x,y)dy=\int_\alpha^\beta f(\varphi(t),\psi(t))\psi'(t)dt,$$

formulalar bo'yicha hisoblanadi.

Umumiy holda, yuqoridagi shartlarda

$$\int_{(K)} P(x,y)dx + Q(x,y)dy = \int_{\alpha}^{\beta} [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]dt$$

tenglik o'rinli.

$(K)=(AB)$ egri chiziq tenglamasi $y=y(x), \alpha \le x \le b$, shaklda berilganda, ikkinchi tur egri chiziqli integral

$$\int_{(K)} f(x,y)dx = \int_{a}^{b} f(x,y(x))dx$$

formula bo'yicha hisoblanadi.

Xuddi shunday, agar egri chiziq tenglamasi $x=x(y), c \le y \le d$, ko'rinishda berilgan bo'lsa, u holda ikkinchi tur egri chiziqli integral,

$$\int_{(K)} f(x,y)dy = \int_{c}^{d} f(x(y),y)dy$$

formula bo'yicha hisoblanadi.

Agar $\int_{(AB)} P(x,y)dx$ integral Oy o'qqa parallel bo'lgan (AB) to'g'ri chiziq kesmasi bo'yicha, $\int_{(AB)} Q(x,y)dy$ integral Ox o'qqa parallel bo'lgan (AB) to'g'ri chiziq kesmasi bo'yicha olingan bo'lsa, u holda ularning har biri nolga teng bo'ladi.

1-misol. $\int_{(K)} (3x^2+y)dx + (x-2y^2)dy$, bunda (K): uchlar $O(0;0)$, $A(2;0)$ va $B(0;2)$ nuqtalarda uchburchakning chegarasi - $(OABO)$.

Yechilishi. Berilgan ikkinchi tur egri chiziqli integralning integrallash konturi 2- chizmada tasvirlangan.

$(K)=(OABO)$ chiziqning yo'nalishi 2- chizmada ko'rsatilgan. Berilgan egri chiziqli integralni hisoblash uchun uchburchakning harbir tomoni bo'yicha (ko'rsatilgan yo'nalishda) integralni hisoblab, so'ngra egri chiziqli integralning additivlik xossasiga asosan, uchala tomon bo'yicha hisoblangan integrallarning qiymatlarini qo'shamiz.

1) uchburchak OA tomonining tenglamasi $y=0, dy=0.$ Unda $\int_{(OA)} 3x^2dx = 3\int_0^2 x^2dx = 8.$ 2) uchburchak AB tomonining tenglamasi $: x+y=2,$ bunda $y=2-x,$ va

$$\int_{(AB)} (3x^2+y)dx + (x-2y^2)dy = \int_2^0 [3x^2+(2-x)-x+2(2-x)^2]dx =$$

$$= \int_2^0 (5x^2-10x+10)dx = (\frac{5}{3}x^3-5x^2+10x)\Big|_2^0 = -\frac{40}{3}.$$

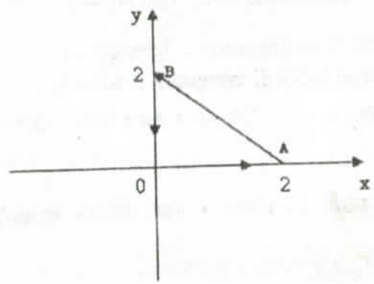
3) uchburchak BO tomonining tenglamasi $: x=0, dx=0,$ va

$$\int_{(BO)} -2y^2dy = -\int_2^0 2y^2dy = \frac{16}{3}.$$

Shunday qilib, 1), 2) va 3) lardan uchala tomon bo'yicha hisoblangan integrallarning qiymatini qo'shsak:

$$\int_{(OABO)} (3x^2+y)dx + (x-2y^2)dy = 8 - \frac{40}{3} + \frac{16}{3} = 0.$$

Bu natijani integralni hisoblamasdan ham olish mumkin, chunki integral ostidagi $Pdx+Qdy = (3x^2+y)dx + (x-2y^2)dy$ ifoda $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1\right)$ shartni qanoatlantiradi.



2-chizma.

Mustaqil yechish uchun misollar

25.1. $\int_{\gamma} (x+y)de$, bunda γ - uchlari $D(0,0)$, $A(1,0)$, $B(0,1)$ nuqtalarda bo'lgan uchburchak konturi.

25.2. $\int_{\gamma} xyde$, bu yerda γ - uchlari $A(-2;2)$, $B(6.1)$, $C(2;5)$ nuqtalarda bo'lgan uchburchak konturi.

25.3. $\int\limits_{\gamma} \frac{de}{\sqrt{x^2+y^2+1}}$ bu yerda γ - tekislikning $0\ (0,0)$ va $A(1,1)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi.

Quyidagi egri chiziqli integrallarni ko'rsatilgan egri chiziq bo'ylab hisoblang

25.4. $\int\limits_{\gamma} y de$, bunda $\gamma = \{(x,y): x = a\cos t,\ y = a\sin t\}$.

25.5. $\int\limits_{\gamma} xy de$, bunda $\gamma: x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$

25.6. $\int\limits_{\gamma} (x+y) dl$, bunda γ-chiziq $r^2 = a^2\cos 2\varphi$ leminskataning o'ng yaprog'i.

25.7. $\int\limits_{\gamma} (2x-3y) dl$, bunda γ-chiziq leminskataning o'ng yoprog'i: $r = a\sqrt{\cos 2\varphi}$.

Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang.

25.8. $\int\limits_{\gamma} y\,dx$, bu yerda $\gamma - y = x^2 (0 \le x \le 1)$ parabola.

25.9. $\int\limits_{\gamma} xy dx$, bu yerda γ - egri chiziq $y = \sin x$ sinusoida chiziqning $(0,0)$ hamda $\left(\frac{\pi}{2};0\right)$ nuqtalar orasidagi qismi.

25.10. $\int\limits_{\gamma} x\,dy$, bu yerda γ - egri chiziq $\frac{x}{3}+\frac{y}{4}=1$ to'g'ri chiziqning $(3,0)$ va $(0,4)$ nuqtalari orasidagi qismi.

25.11. $\int\limits_{\gamma} y dx - (y + x^2) dy$, bu yerda γ - egri chiziq $y = 2x - x^2$ parabola yoyining $A(2,0)(\cdot)$ dan va $B(0,0)$ nuqtagacha bo'lgan qismi.

Quyidagi ikkinchi tur egri chiziqli integralni hisoblang.

25.12. $\int\limits_{\gamma} (2 - y) dx + x\,dy$, bunda $\gamma = \{(x,y): x = t - \sin t,\quad y = 1 - \cos t,\quad 0 \le t \le 2\pi\}$

.

25.13. $\int\limits_{\gamma} y^2 dx + x^2 dy$, bu yerda γ egri chiziq $A(-a,0)$ dan $B(a,0)$ gacha bo'lgan yarim ellips yoyidir: $x = a\cos t,\quad y = b\sin t$.

25.14. $\int\limits_{\gamma} \frac{x^2 dy - y^2 dx}{x^{2/3} + y^{2/3}}$, bu yerda γ-astroidaning $A(R,0)$ nuqtasidan $B(0,R)$ nuqtasigacha bo'lgan yoyi: $x = R\cos^3 t,\quad y = R\sin^3 t$.

Integral ostidagi ifoda to'liq differensial ekanligini tekshirib, berilgan egri chiziqli integralni hisoblang.

25.15. $\int\limits_{(-1,3)}^{(2,3)} x\,dy + y\,dx$.

25.16. $\int\limits_{(0;1)}^{(3;-4)} x\,dx + y\,dy$.

25.17. $\int\limits_{(-1;-2)}^{(1;0)} (2x - y)dx + (3y - x)dy$.

25.18. $\int\limits_{(0,1)}^{(1,0)} (3x^2 - 2xy + y^2)dx - (x^2 - 2xy)dy$.

Mustaqil yechish uchun misollarning javoblari

25.1. $1+\sqrt{2}$. **25.2.** $\frac{31}{16}$. **25.3.** 5. **25.4.** $2a^2$. **25.5.** 0. **25.6.** $a^2\sqrt{2}$. **25.7.** $2\sqrt{2}a^2$. **25.8.** $\frac{1}{3}$. **25.9.** 1. **25.10.** 6. **25.11.** -4. **25.12.** -2π. **25.13.** $\frac{4}{3}ab^2$. **25.14.** $\frac{32}{105}R^{7/3}$. **25.15.** 9. **25.16.** 12. **25.17.** -4. **25.18.** 1.

26- amaliy mashg'ulot.
BIRINCNI VA IKKINCHI TUR SIRT INTEGRALLAR

26.1. Birinchi tur sirt integrallarini ikki karrali integral yordamida hisoblash. R^3 da (S) sirt o'zining $z = z(x,y)$ tenglamasi bilan berilgan bo 'lsin, bunda $z(x,y)$ funksiya chegaralangan yopiq $(D)((D) \subset R^2)$ sohada uzluksiz va uzluksiz $z'_x(x,y),\ z'_y(x,y)$ xususiy hosilalarga ega.

1-teorema. Agar $f(x,y,z)$ - (S) sirtda berilgan va uzluksiz funksiya bo 'lsa, u holda bu funksiyaning (S) sirt bo 'yicha olingan

$$\iint\limits_{(S)} f(x,y,z)dS$$

birinchi tur sirt integrali mavjud va ushbu

$$\iint\limits_{(S)} f(x,y,z)dS = \iint\limits_{(D)} f(x,y,z(x,y))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}\,dxdy \qquad (1)$$

formula bo'yicha hisoblanadi.

1-eslatma. Agar (S) sirt umumiy holda o'zining $x = x(u,v),\ y = y(u,v),\ z = z(u,v),\ ((u,v) \in (\Delta))$ parametrik tenglamasi bilan berilgan bo'lib, unda $f(x,y,z)$ funksiya uzluksiz bo'lsa, u holda birinchi tur sirt integrali mavjud va

$$\iint\limits_{(S)} f(x,y,z)dS = \iint\limits_{(\Delta)} f(x(u,v), y(u,v), z(u,v))\sqrt{EG - F^2}\,dudv =$$

$$= \iint\limits_{(\Delta)} f(x(u,v), y(u,v), z(u,v))\sqrt{A^2 + B^2 + C^2}\,dudv$$

formula o 'rinli, bunda

$$E = (x'_u)^2 + (y'_u)^2 + (z'_u)^2,\ G = (x'_v)^2 + (y'_v)^2 + (z'_v)^2,\ F = x'_u \cdot x'_v + y'_u \cdot y'_v + z'_u \cdot z'_v,$$

$$A = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix},\ B = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix},\ B = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}.$$

2-eslatma. (S) sirt $x = x(y,z), (y = y(x,z))$ tenglama bilan berilgan bo'lib, $x(y,z)(y(x,z))$ funksiya (D) sohada uzluksiz va uzluksiz $x'_y(y,z), x'_z(y,z)$ $(y'_x(x,z), y'_z(x,z))$ xususiy hosilalarga ega bo'lsin.

Agar $f(x,y,z)$ funksiya (S) sirtda berilgan va uzluksiz bo'lsa, u holda bu funksiyadan (S) sirt bo'yicha olingan birinchi tur sirt integrali mavjud bo'ladi va

$$\iint_{(S)} f(x,y,z)dS = \iint_{(D)} f(x(y,z),y,z)\sqrt{1+x'^2_y(y,z)+x'^2_z(y,z)}dydz,$$

$$\left(\iint_{(S)} f(x,y,z)dS = \iint_{(D)} f(x,y(x,z),z)\sqrt{1+y'^2_x(x,z)+y'^2_z(x,z)}dxdz\right)$$

formula o'rinli.

26.2. Ikkinchi tur sirt integralini ikki karrali integral yordamida hisoblash. R^3 fazoda (S) sirt $z = z(x,y)$ tenglama bilan berilgan bo'lib, $z = z(x,y)$ funksiya chegaralangan yopiq (D) sohada uzluksiz $z'_x(x,y)$ va $z'_y(x,y)$ xususiy hosilalarga ega bo'lsin.

2 – teorema. Agar $f(x,y,z)$ funksiya (S) sirtda uzluksiz bo'lsa, u holda bu funksiyadan (S) sirt bo'yicha olingan ikkinchi tur sirt integrali mavjud bo'ladi va u

$$\iint_{(S)} f(x,y,z)dxdy = \iint_{(D)} f(x,y,z(x,y))dxdy \qquad (*)$$

formula orqali hisoblanadi.

Agar integral (S) sirtning yuqori (quyi) tomoni bo'yicha olingan bo'lsa, u holda ikki karrali integral, mos ravishda, musbat (manfiy) ishora bilan olinadi :

$$\iint_{(S)} f(x,y,z)dxdy = \pm \iint_{(D_{xy})} f(x,y,z(x,y))dxdy,$$

$$\iint_{(S)} f(x,y,z)dzdx = \pm \iint_{(D_{zx})} f(x,y(x,z),z)dzdx,$$

$$\iint_{(S)} f(x,y,z)dydz = \pm \iint_{(D_{yz})} f(x(y,z),y,z)dydz,$$

bunda $(D_{xy}), (D_{zx}) (D_{yz})$ lar, mos ravishda, (S) sirtning $Oxy\ (z=0)$, $Oxz\ (y=0)$, $Oyz\ (x=0)$ tekisliklardagi proyeksiyalaridir.

1-misol. $\iint_{(S)} \sqrt{x^2+y^2}ds$, bunda $(S): x^2+y^2 = z^2$ konus sirtning $z=0$ va $z=3$ tekisliklar orasidagi qismi.

Yechilishi. Berilgan sirt tenglamasidan $z = \sqrt{x^2+y^2}$ ekanligini olamiz. Bu sirt qaralayotgan qismining Oxy tekislikdagi proyeksiyasi $(D): x^2+y^2 \le 9$ – doiradan

iborat. Berilgan 1- tur sirt integrali (12.3) formula bilan hisoblanadi: $z'_x = \frac{x}{\sqrt{x^2+y^2}}$, $z'_y = \frac{y}{\sqrt{x^2+y^2}}$ larni e'tiborga olgan holda ,

$$\iint_{(S)} \sqrt{x^2+y^2}ds = \iint_{(D)} \sqrt{x^2+y^2}\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}dx\,dy = \sqrt{2}\iint_{(D)} \sqrt{x^2+y^2}dx\,dy.$$

bo'lishini olamiz. Keyingi ikki karrali integralda $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ almashtirishni bajarib, quyidagiga ega bo'lamiz:

$$\sqrt{2}\iint_{(D)} \sqrt{x^2+y^2}dx\,dy = \sqrt{2}\int_0^{2\pi} d\varphi \int_0^3 \rho^2 d\rho = \sqrt{2}\cdot 2\pi \cdot \frac{27}{3} = 18\sqrt{2}\cdot\pi.$$

Demak, $\iint_{(S)} \sqrt{x^2+y^2}ds = 18\cdot\sqrt{2}\cdot\pi.$

2-misol. $J = \iint_{(S)} xdydz + dxdz + xz^2dxdy$, bunda $(S): x^2+y^2+z^2 = 1$ sfera birinchi oktantdagi qismining yuqori tomoni.

Yechilishi. Berilgan (S) sirtning Oyz, Oxz, Oxy tekisliklardagi proyeksiyalarini, mos ravishda, $(D_x), (D_y)$ va (D_z) kabi belgilab, berilgan J integralni uchta:

$$J_1 = \iint_{(S)} xdydz,\ J_2 = \iint_{(S)} dxdz,\ J_3 = \iint_{(S)} xz^2dxdy$$

integrallar yig'indisi shaklida tasvirlaymiz. J_1 integralda $P = x, Q = R = 0$; J_2 integralda $Q = 1, P = R = 0$; J_3 da esa, $P = Q = 0$, $R = xz^2$. Har bir integral uchun (*) formulani qo'llaymiz:

$$J_1 = \iint_{(D_x)} \sqrt{1-y^2-z^2}dydz,\ J_2 = \iint_{(D_y)} dxdz,\ J_3 = \iint_{(D_z)} x(1-x^2-y^2)dxdy.$$

$(D_x), (D_y)$ va (D_z) sohalar mos koordinatalar tekisliklarida joylashgan radiusi 1 ga teng bo'lgan doiraning to'rtdan bir qismiga teng. Shuning uchun, $J_2 = \frac{\pi}{4}$. J_1 va J_3 integrallarda qutb koordinatalar sistemasiga o'tib, hisoblash bajaramiz:

$$J_1 = \iint_{(D_x)} \sqrt{1-y^2-z^2}dydz = \iint_{(D_x)} \sqrt{1-\rho^2}\cdot\rho\,d\rho\,d\varphi = -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_0^1 (1-\rho^2)^{1/2} d(1-\rho^2) = \frac{\pi}{6}.$$

$$J_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 \rho\cos\varphi(1-\rho^2)\rho d\rho = \sin\varphi\Big|_0^{\frac{\pi}{2}} \cdot \left(\frac{\rho^3}{3} - \frac{\rho^5}{5}\right)\Big|_0^1 = \frac{2}{15}.$$

Demak, $J = J_1 + J_2 + J_3 = \frac{\pi}{6} + \frac{\pi}{4} + \frac{2}{15} = \frac{5\pi}{12} + \frac{2}{15}.$

Mustaqil yechish uchun misollar

Qo'yidagi birinchi tur sirt integrallarini hisoblang.

26.1. $\iint\limits_{(S)}(x+y+z)dS$, bunda (S)- $x\geq 0$, $y\geq 0$, $z\geq 0$ ajratilgan shartda $x+2y+4z=4$ tekislik qismi.

26.2. $\iint\limits_{(S)}(x+y+z)dS$, bunda $(S)-z\geq 0$ ajratilgan shartda $x^2+y^2+z^2=4$ sfera qismi.

26.3. $\iint\limits_{(S)}(x+y+z)dS$, bunda (S) kubning to'liq sirti: $0\leq x\leq a$, $0\leq y\leq a$, $0\leq z\leq a$.

26.4. $\iint\limits_{(S)}(6x+4y+3z)dS$, bunda $(S)-x+2y+3t=6$ tekislikning birinchi oktantdagi qismi.

26.5. $\iint\limits_{(S)}(x^2+y^2+z^2)dS$, bunda $(S)-x^2+y^2+z^2=R^2$ sfera.

26.6. $\iint\limits_{(S)}(x^2+y^2+z^2)dS$, bunda $(S)-|x|\leq a$, $|y|\leq a$, $|z|\leq a$ kub sirti.

26.7. $\iint\limits_{(S)}z^2dxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=a^2$ $(z\geq 0)$ yarim sferaning tashqi tomoni.

26.8. $\iint\limits_{(S)}x^2dydz$, bu erda (S) - ushbu $x\geq 0, y\geq 0, 0\leq z\leq 1$ sohadagi $z=x^2+y^2$ paraboloid sirtning tashqi tomoni.

26.9. $\iint\limits_{(S)}-xdydz+zdzdx+5dxdy$, bu erda (S) - birinchi oktantada joylashgan $2x-3y+z=6$ tekislikning yuqori qismidagi tomoni.

26.10. $\iint\limits_{(S)}xdydz+ydzdx+zdxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

26.11. $\iint\limits_{(S)}x^3dydz+y^3dzdx+z^3dxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

Mustaqil yechish uchun misollarning javoblari

26.1. $\frac{7\sqrt{21}}{3}$. **26.2.** π. **26.3.** $9a^3$. **26.4.** $54\sqrt{14}$. **26.5.** $4\pi R^4$. **26.6.** $40a^4$

26.7. $0{,}5\pi a^4$. **26.8.** $\frac{4}{15}$. **26.9.** -9. **26.10.** $4\pi R^3$. **26.11** $\frac{12}{5}\pi R^5$.

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