# Introductory Dynamic Macroeconomics

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# Contents

# Preface

These notes are written for the course ECON 3410 /4410 *Introductory Dynamic Macroeconomics* at the University of Oslo. They contain a presentation of the key concepts and models required for starting to address positive macrodynamics in a systematic way, for example in connection with the master thesis in economics. The level of mathematics used does not go beyond basic calculus and simple algebra.

Several students and colleagues have already contributed to this project by offering their comments and corrections, in particular Roger Hammersland, Thomas Lystad, Andreas Ringen, and Øyvind Tveter.

# Chapter 1

# Dynamic models in economics

In this chapter we introduce the distinction between static and dynamic models which underlies modern dynamic economics. The concepts are explained with the aid of the standard supply and demand model of a single marked. The main message is that dynamics and statics are distinctively different methods of analysis, but also that they are related: First, a static relationship can be seen as a limiting case of a dynamic relationship. Second, static equations can be used as a way of describing the stationary state of a dynamic system. In practice, dynamic analysis is facilitated by formulating separate models for the short-run and for the long-run, and this methodology is also introduced in this chapter. Another important conceptual distinction, namely between stock and flow variables, is explained towards the end of the chapter, which is rounded off with an empirical example.

## 1.1 Introduction

In many areas of economics time plays an important role: firms and households do not react instantly to changes in for example taxes, wages and business prospects but take their time to make decisions and to adjust behaviour. Moreover, because of information lags, time often goes by before changes in economic circumstances are recognized and some sort of adaptive action is taken by economic agents (households, firms and the government).

There are also institutional arrangements, social and legal agreements and social norms which imply that we will expect gradual adjustment to be typical of economic behaviour. Annual or biannual wage bargaining is one important example of such an institution, responsible for intermittent pay raises at the individual level, and smooth wage development at the aggregate level. The manufacturing of goods is not instantaneous—although economists often formulate models *as if* production is instantaneous—but takes considerable time, even years in the case of projects with huge capital investment. Dynamic behaviour is also induced by the fact that many economic decisions are heavily influenced by what firms, households and the government anticipate about the future. Often expectation formation will attribute a large weight to past developments, since rational anticipations usually build on experience.

Although the above examples all represent dynamics in the form of gradual and 'slow' reactions to changes in economic incentives, there are other examples of dynamic behaviour which involve large and instantaneous changes at the point in time where a shock occurs. A classic example, which we shall study in some detail in the next section, is the case of a single marked characterized by perfect competition. A demand shift in such a market will lead to an immediate change in the (market clearing) price. In macroeconomics, so called portfolio theories of the market for foreign exchange imply that the nominal exchange rate reacts strongly, and with no time lags, to changes in the net supply of currency to the central bank.

The hallmark of dynamic adjustments therefore, is not that the response to a shock is necessarily delayed, but that the adjustment process takes several time periods. The effects of shock literary 'spill over' to the following periods. Using the the terminology of Ragnar Frisch, who formalized macrodynamics as a dicipline, we can speak of *impulses* to the macroeconomic system which are *propagated* by the system's own mechanism into effects that lasts for several periods after the occurrence of the shock.[1]

Because dynamic behaviour and response patterns are regular features of the macroeconomy, all serious policy analysis is based on a dynamic approach. Hence, those responsible for fiscal and monetary policy use dynamic models as an aid in their decision process. In recent years, monetary policy has come to play an important role in activity regulation, and as we will explain later in the book, central banks in many countries have defined the rate of inflation as the target variable of economic policy. The *instrument* of monetary policy nowadays is the central bank sight deposit rate, i.e., the interest rate on banks' deposits in the central bank. However, no central bank seem to base their policy decisions on the assumption of an immediate and strong effect on the rate of inflation after a change in the interest rate. Rather, because of the many dynamic effects triggered by a change in the interest rate, central bank governors prepare themselves to wait a substantial amount of time before the full effect of the interest rate change hits the target variable. The following statement from the web pages of Norges Bank [The Norwegian Central Bank] is typical of many central banks' view:

> Monetary policy influences the economy with long and variable lags. Norges Bank sets the interest rate with a view to stabilizing inflation at the target within a reasonable time horizon, normally 1-3 years.[2]

[1] One of Frisch's many influential publications is called *Propagation Problems and Impulse Problems in Dynamic Economics*, Frisch (1933).

[2] See Norges Bank's web page on monetary policy: http://www.norges-bank.no/english/monetary_policy/in_norway.html.

Similar statements can be found on the web pages of the central banks in e.g., Autralia, New-

One important goal of this book is to learn enough about dynamic modeling to be able to understand the economic meaning of this and similar statements, and to be able to form an opinion about their realism.

The quotation from Norges Bank is demonstrates that policy is guided by the beliefs the decision makers have about the response lag between a policy change and the effect on the target variable—in other words the dynamic nature of macroeconomic relationships. Formalization of such beliefs require that we develop the necessary modelling tools and concepts.

We continue this chapter by giving the classic definition of static and dynamic models, which is due to Ragnar Frisch (1929,1992).[3] In section 1.2, 1.3 and 1.4 we apply Frisch's concepts to a equilibirum model of a single product market. Section 1.5 introduces another conceptual distinction, namely between stock and flow variables, which plays an important role in macrodynamics, and section 1.6 offers an empirical illustration of a dynamic consumption function for Norway. Finally section 1.7 gives a summary of main points and sketches the road ahead.

## 1.2 Statics and dynamics in economic analysis

The above examples already rationalize that dynamic behaviour by economic agents, and dynamic response in economic systems, are typical features which we want to be able to model by using economic theories and data.

As a first step, we need to establish a definition of a dynamic model, as opposed to a static model. At a general level, static models are used to describe, or to predict, relationship between *state* variables, while dynamic models are used to describe, or predict, the relationships between variables which are in *motion*. Hence, in the language of Ragnar Frisch's early contribution to what he coined macrodynamics, we may talk about state laws (or static law) and laws of motion (dynamic laws) as synonyms of static and dynamic relationships[4]

This definition may however be somewhat difficult to grasp, and in any case a more operational definition of dynamic theory is needed, and again Frisch provides the defining insight: A dynamic theory, or model, is made up of relationships between variables that refer to different time periods. Conversely, when all the variables included in the theory refer to the same time period (or, more generally, the model

Zealand, The United Kingdom and Sweden.

The formulation in the main text has been posted since the summer of 2004. Before that the same passage read:

"A substantial share of the effects on inflation of an interest rate change will occur within two years. Two years is therefore a reasonable time horizon for achieving the inflation target of $2\frac{1}{2}$ per cent".

[3] Frisch (1992) is an English translation of (parts of) Frisch (1929). An edited version of Frisch (1929) is available in Norwegian in Frisch (1995, Ch 11).

[4] In his paper on business cycles, Frisch was the first to use the word "microeconomics" to refer to the study of single firms and industries and "macroeconomics" to refer to the study of the aggregate economy.

is conceptualized without time as an entity), the system of relationships is static.[5] Hence, as Frisch also emphasized, a genuinely dynamic model is not obtained by simply adding subscripts for time period to the variables of an essentially timeless relationship. The defining characteristic of dynamic theory is that one and the same equation (often as part of a system of equations) contains entities that refer to different time periods.

Following convention, we will use the subscript $t$ as an index for time period. To provide a simple example of the differences between static and dynamic relationships we first consider a partial equilibrium model. Let $P_t$ denote the price prevailing in a single marked in time period $t$, and let $X_t$ denotes the demand of a good in period $t$. A static demand schedule, using Frisch's definition, is then

$$X_t = aP_t + b + \varepsilon_{d,t}, \tag{1.1}$$

with $a$ and $b$, as parameters. Since $a$ is the derivative with respect to price, we assume that it is a negative number, $a < 0$. The other parameter, the intercept $b$, is assumed to be positive, $b > 0$.

The greek letter $\varepsilon_{d,t}$ denotes a random term, and is used as a reminder that economic laws, even at their best, are not deterministic, but instead laws that hold on average. Hence you should think of $\varepsilon_{d,t}$ as an erratic shock term, or a so called error-term, which is small in magnitude and which which is zero on average. Since $\varepsilon_{d,t}$ takes an arbitrary value, there is in a sense a whole score of demand curves defined by (1.1), each corresponding to different intersection points along the vertical "$P_t$-axis". However, when we set $\varepsilon_{d,t}$ equal to its mean of zero, we can represent the average relationship between price and demand with the usual downward sloping demand curve, as in figure 1.1.

Next, assume that the supply function is given by

$$X_t = cP_t + d + \varepsilon_{s,t}, \tag{1.2}$$

with parameters $c > 0$ and $d < 0$. The disturbance $\varepsilon_{s,t}$ represents random shocks to the supply-side of the market. The average value of $\varepsilon_{s,t}$ is also zero by assumption.

Taken together, the two equations represent a static model with the two endogenous variables, $X_t$ and $P_t$. The two exogenous variables of the system are $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$. As usual, we can analyze the marked equilibrium with the aid of a graphs which show the demand and supply functions as curves. However, because we have included the random shock terms $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$ in the model, care must be taken when we draw the demand and supply curves.

Figure 1.1 shows an example. In line with conventional analysis, we assume that the initial situation is represented by the point A in the figure, i.e., there are no shocks, and the system is "at rest" at the point where *average supply* equals *average demand*. Hence, the demand and supply curves drawn with solid and thick lines apply to the case when both $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$ take their average values, which are zero.

[5] This definition is adopted directly from Frisch (1947, p. 71).

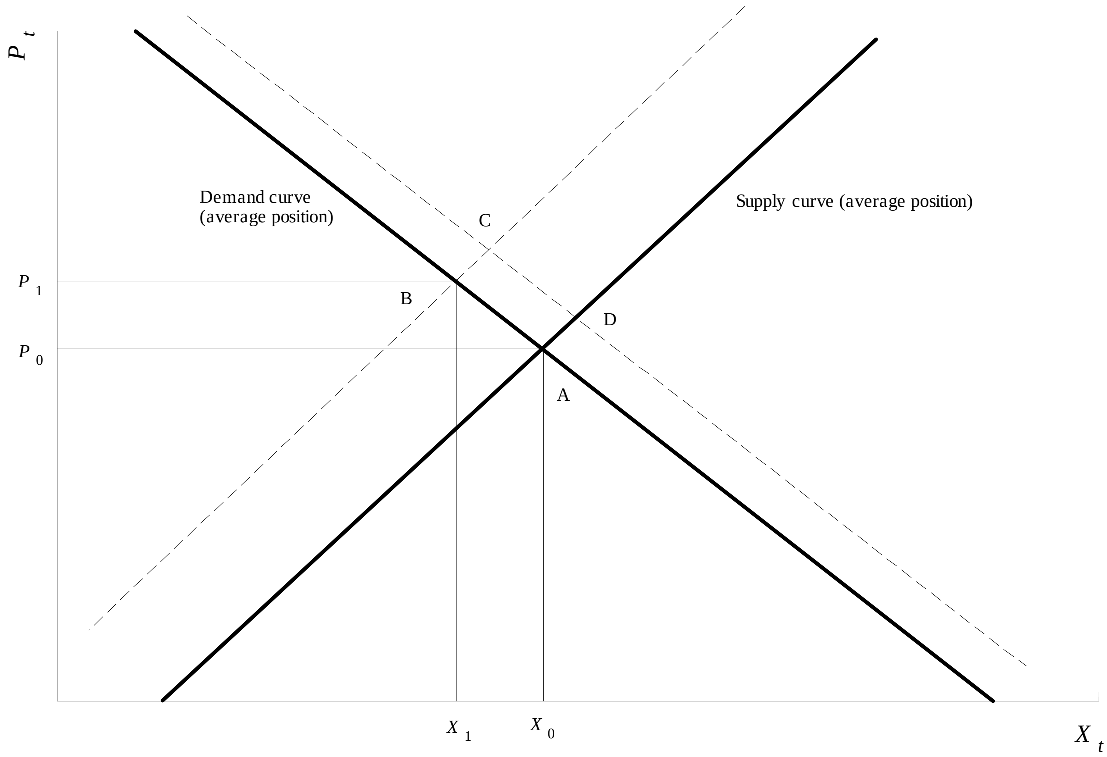


Figure 1.1: A static marked equilibrium model.

In the graph, $\{P_0, X_0\}$ therefore denotes the initial equilibrium values of price and quantity.

We next consider the response of the endogenous variables in the case of a shock to the market. We denote the initial situation, when the equilibrium is given by $\{P_0, X_0\}$ as period $t = 0$. Assume that there is a negative supply shock, in period $t = 1$, so instead of zero we have $\varepsilon_{s,1} < 0$. Graphically, we represent the supply shock by a negative horizontal shift of the supply curve (corresponding to a positive vertical shift), which is shown as the dashed supply curve in the graph.

As any student of economics will know, the new equilibrium predicted by our theory will be at point B, where the dashed supply curve intersects the demand curve (which remains at its average position). Note that the whole effect of the shock is reflected in the new equilibrium values of $P$ and $X$ in period 1. In other words, there are no effects of the period 1 shock that 'spill over' to period 2, 3, or any later periods.[6] There is no *propagation* of impulses from the period of the shock to any of the following periods. Despite the time subscripts of the variables, time does not play an essential role in the model consisting of equation (1.1) and (1.2).

[6]Hence, the partial derivatives of $X_1$ and $P_1$ with respect to $\varepsilon_{s,1}$: $\partial X_1/\partial\varepsilon_{s,1} = a/(a-c)$ and $\partial P_1/\partial\varepsilon_{s,1} = 1/(a-c)$, give the *complete* responses (in absolute values) of supply and demand with respect to the supply shock occurring i period 1.

So far in this example, we have considered a single shock to the supply side of the market. Sometimes joint shocks occur on the two sides of the marked. Whether there is a simple or joint shock makes no difference to *how fast* $P$ and $X$ react, only by how much they change compared to the initial situation. In figure 1.1, point $C$ represents the case of a joint shock in period 1, where a positive demand shock occurs simultaneously with the negative supply shock. In this case, both $P_1$ and $X_1$ will be higher than in the case of single supply shock in period 1. Again, it is important to note that also in this case the adjustments of price and quantity are complete within the same period as the shock occurs.

Let us return to the analysis of the single supply-side shock which occurs in period 1, and ask how the market determined price and quantity will evolve in period $2, 3$ and so forth. The answer, in this static model, depends singularly on the the sequence of random draws of $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$. For example, if $\varepsilon_{d,2} > 0$ and $\varepsilon_{s,2} = 0$, the market equilibrium in period 2 will be at point D. In period 3, if it should happen that $\varepsilon_{d,3} = \varepsilon_{s,3} = 0$, the marked is back at point A, where it started out in period 0. But that would be pure coincidence, and it is in fact much more likely that we will see non-zero values of $\varepsilon_{d,3}$ and $\varepsilon_{s,3}$ and that the period 3 market equilibrium will be at some other point, different from A.

If we specify the values of the four parameters of equation (1.1) and (1.2), and draw random numbers for $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$, it is easy to work out the numerical solution for $X_t$ and $P_t$. Panel a) of figure 1.2 shows an example of a sequence of equilibrium prices $P_t$ over 50 periods, from period 1 to period 50, together with the sequence of random shocks that drives the solution.[7] There are random supply *and* demand shocks in each period, so in the graph we have chosen to show the net demand shocks: $\varepsilon_{d,t} - \varepsilon_{s,t}$.

Note how closely the graph of the equilibrium price follows the graph of random shocks. In fact, the two variables $\varepsilon_{d,t} - \varepsilon_{s,t}$ and $P_t$ are perfectly correlated over the period $t = 1, 2, ...50$. As we have seen, this is because of the static nature of the model, which essentially means that the price in any period, for example period 20, reflects the random excess demand of that period perfectly. No other stochastic shocks, before or after, influence the equilibrium price $P_{20}$.

The underlying model characteristic is therefore that the endogenous variables of the static model, price and quantity in this example, adjust fully to a shock to the system, *within the same period of the shock*. There are no spill-over effects of a shock to the following period, or to subsequent periods. This characteristic trait of static models is also highlighted in panel b) of figure 1.2 which shows the change in the equilibrium price in response to a single random shock in period $t = 1$. As we will see

[7] In this book we often use figures with multiple graphs, such as Figure 1.2. In the text, we refer to the panels as *a)*, *b)* etc, according to the rule

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

for a $2 \times 2$ figure. In the case of a $3 \times 3$ figure for example, *c)* denotes the third panel of the first row, while *e)* is the third panel of the second row.

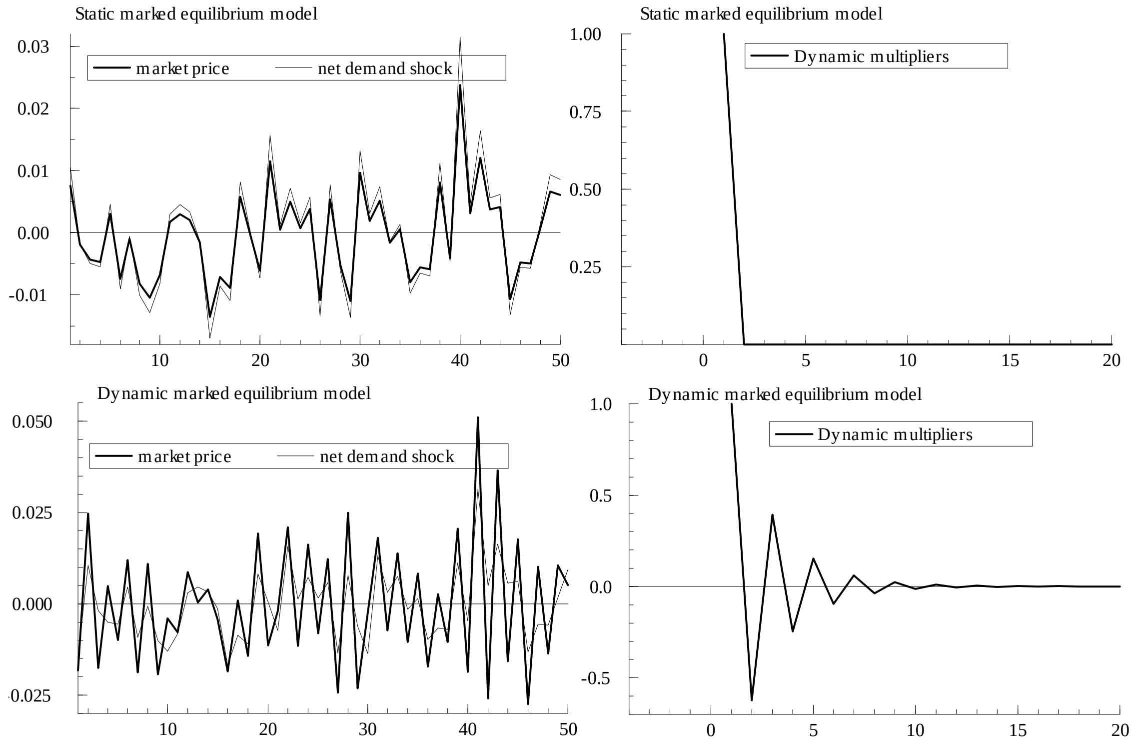


Figure 1.2: Panel a) The marked price (thicker line) and net demand shock (thinner line) of the static model defined by equation (1.1) and (1.2). Panel b) shows the sequence of dynamic multipliers from this model. Panel c) and d) are the corresponding graphs for the dynamic (cobweb) model (1.3) and (1.4).

below, this is actually the plot of the sequence of partial derivatives $\partial P_t/\varepsilon_{d,1}$, over the period $t = 1, 2, ....20$ (only subject to the remark the graph has been scaled so that the first derivative is unity). As you can see, the whole sequence of derivatives is zero with one exception: In period 1, the multiplier is positive because the whole effect of the shock on the market price comes in the period of the shock.

The graph in panel b) 1.2 is in fact an example of a graph of *dynamic multipliers*, and shows how the response of an endogenous variable is distributed between different time periods. In this case, the shape of the graph of the dynamic multiplier reffects that it is dervied from a static model. Models where time plays an *essential* role, are generates dynamic multipliers which have a much more interesting shape. So let us consider genuine dynamics!

In our first example of a dynamic model, we keep the demand equation (1.1):

$$X_t = aP_t + b + \varepsilon_{d,t}, \tag{1.3}$$

but the supply function (1.2) is replaced by

$$X_t = cP_{t-1} + d + \varepsilon_{s,t} \tag{1.4}$$

with parameters $c > 0$ and $d < 0$. In terms of economic interpretation, equation (1.4) may represent the situation when there are important production and delivery lags, so that today's supply depends of the price obtained in the previous period. The classic example is from agricultural economics, where the whole supply, for example of pork, or of wheat, is replenished from one year to the next.[8] An interpretation from modern macroeconoics, which captures that the underlying behaviour of suppliers is governed by expectations, is

$$X_t = cP_t^e + d + \varepsilon_{s,t}$$

where $P_t^e$ denotes the expected period $t$ price, as in Lucas' supply function, see Lucas (1976). To become consistent wity (1.4), we assume that information is either not available or is unreliable in period $t-1$, so that suppliers choose to set $P_t^e = P_{t-1}$, see for example Evans and Honkapoja (2001).

Clearly, using Frisch's definition, equation (1.4) qualifies as a dynamic model of the supplied quantity since "one and the same equation contains entities that refer to different time periods", namely $X_t$ and $P_{t-1}$. What about the model as a whole, i.e., the system of equations (1.3) and (1.4)? Is it a static or dynamic *system*? To answer this question we consider figure 1.3.

In this figure, the line of the demand curve has the same interpretation as in the static model in figure 1.1. However, care must taken when interpreting the supply curve. To understand why, consider the quantity supplied in period $t = 1$, assuming that $t = 0$ represents the initial situation with $\varepsilon_{s,0} = 0$. According to (1.4), supply is given by

$$X_1 = cP_0 + d,$$

saying that supply in period 1 is a function of $P_0$ which is already determined (from history), and not the price in period 1. The supply in period 1, which we can call short-run supply, is therefore completely *inelastic* with respect to the price, $P_1$.

Unlike figure 1.1, where the supply curve represents the short-run response of supply with respect to a price change, the positively sloped supply schedule in figure 1.3 therefore refers to another type of price variation. This variation is counterfactual in nature and applies to hypothetical stationary situations where, $\varepsilon_{s,t} = 0$ for all $t$, and $P_t = P_{t-1} = \bar{P}$, and $X_t = \bar{X}$. It is conventional terminology in economics to use "stationary situation" and "long-run situation" as synonyms, and accordingly the upward sloping curve in in figure 1.3 has been dubbed the long-run supply curve. Mathematically it is defined by:

$$\bar{X} = c\bar{P} + d,$$

and $c$ is called the long-run derivative, which is the rise in supply that would have resulted if the price had been increased by one unit and kept constant at that new level.

[8] Named by Niclas Kaldor, cf. Kaldor (1934).

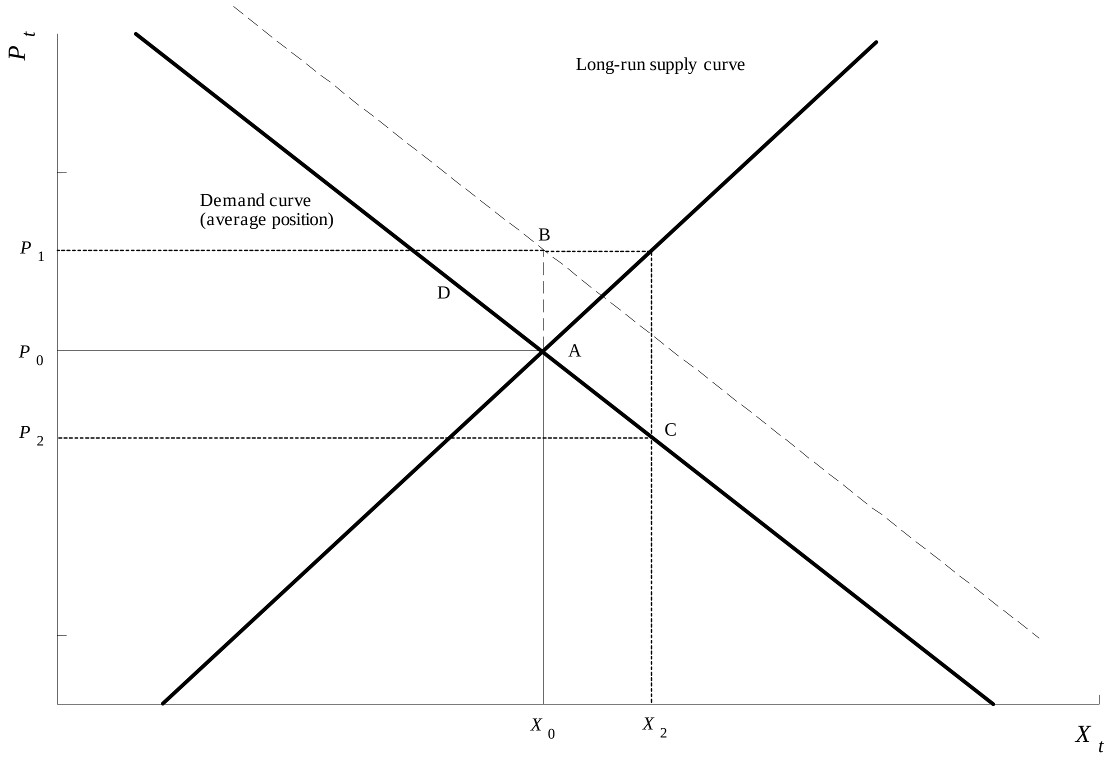


Figure 1.3: A dynamic marked equilibrium model, with inelastic short-run supply—the cobweb model.

We next turn to the system's response to a shock. We consider a positive (horizontal) demand shift, represented by the dashed demand schedule in figure 1.3. In the same way as in the first example, we assume that initially the system is 'at rest' at point $A$, so that $\{P_0, X_0\}$ denotes the initial situation. In period 1 the demand shock ($\varepsilon_{d,1} > 0$) occurs, as indicated. Since short-run supply is completely inelastic, the marked equilibrium in period 1 is at point B, with the market clearing price-quantity combination $\{P_1, X_0\}$. Hence $X_1 = X_0$, since the quantity supplied is already determined from the price that prevailed in the marked in the past period.

In the same way as in the analysis of the static model, we assume that the demand shock disappears in period 2, so the demand schedule shifts back to its average position. However, since the price $P_1$ leads to increased supply in period 2, namely $X_2 - X_1 = c(P_1 - P_0)$, the period 2 equilibrium is given by point C, with price $P_2$ which is lower than the initial price $P_0$. The traded quantity $X_2$ is also different from the initial value $X_0$, refecting that the price was high in period 1. Hence, both price and quantity react dynamically to the demand shock in period 1. The effect of the shock 'spills over' to the following periods, and the adjustment of the endogenous variables is incomplete within the time period of the shock. These are essential differences from the static model in figure 1.1, and they represent a classic case of economic dynamics.

Remember how trivial the change in the model specification seemed at first: we have only changed the time subscript of the right hand side variable in the supply function from $t$ to $t-1$. Yet, this change affects the interpretaion and behaviour of the model fundamentally, reflecting that in the model made up of (1.3) and (1.4) time plays an essential role. Note also that dynamics is a *system property*: both sides of the market are affected, even though the model is made up of one static equation (the demand function) and one dynamic equation (the supply function). This suggest that when we construct larger macroeconomic models which combine static and dynamic equations, all the endogenous variables are affected by the dynamics introduced in the form of one or more dynamic equations. In this way, although there may only be a few dynamic equations in a large system-of-equations, the dynamic relationships represent a dominant feature.

Panel c) and d) of figure 1.2 offer more insight into the dynamics of the model with fixed short-run supply. Panel c) can be compared with panel a) and shows that because of the dynamics, the time series for the market price $P_t$ (the thicker line) is more volatile than the net demand shock. Panel d) can be compared with panel b). It shows the dynamic multipliers of price with respect to a demand shock in period 1. In this case the sequence of multipliers is much more interesting than in the static model. In line with the graphical analysis above, the first multiplier (often called the impact multiplier) is positive, while the second is negative, the third is positive and so on. The graphs shows however, that the absolute values of the multipliers are reduced as we move away from period 1 (when the shock occurs), and after 10 periods they become negligible. In figure 1.1 the dynamic response to the shock can be illustrated by noting that the equilibrium in period 3 will be at point D on the (average) demand curve and noting that by joining up the points a cobweb pattern emerges. The web starts at point B and ends at point A, after a dynamic adjustment period of 10-12 periods. Hence the model defined by equation (1.3) and (1.4) is succinctly named the *cobweb model*.

The cobweb model is only one possible form of dynamics, as we will see in this book, there is a large class of dynamic specifications with relevance for economics. To take one more example of the dynamics of a single market, consider the following two demand and supply equations:

$$X_t = aP_t + b_1 X_{t-1} + b_0 + \varepsilon_{d,t}, \text{ and} \tag{1.5}$$

$$X_t = c_0 P_t + c_1 P_{t-1} + d + \varepsilon_{s,t}. \tag{1.6}$$

In this model, the demand function (1.5) is a alos dynamic equation. The parameter $b_1$ measures by how much an unit increase in $X_{t-1}$ shifts the demand curve. This can be rationalized by habit formation for example, in which case we may set $0 < b_1 < 1$, i.e., high demand today makes for high demand tomorrow as well, *ceteris paribus*. In the model (1.5) and (1.6) we have also relaxed the assumption of completely inelastic supply, the infallible mark of the cob-web model. In (1.6), setting $c_0 = 0$

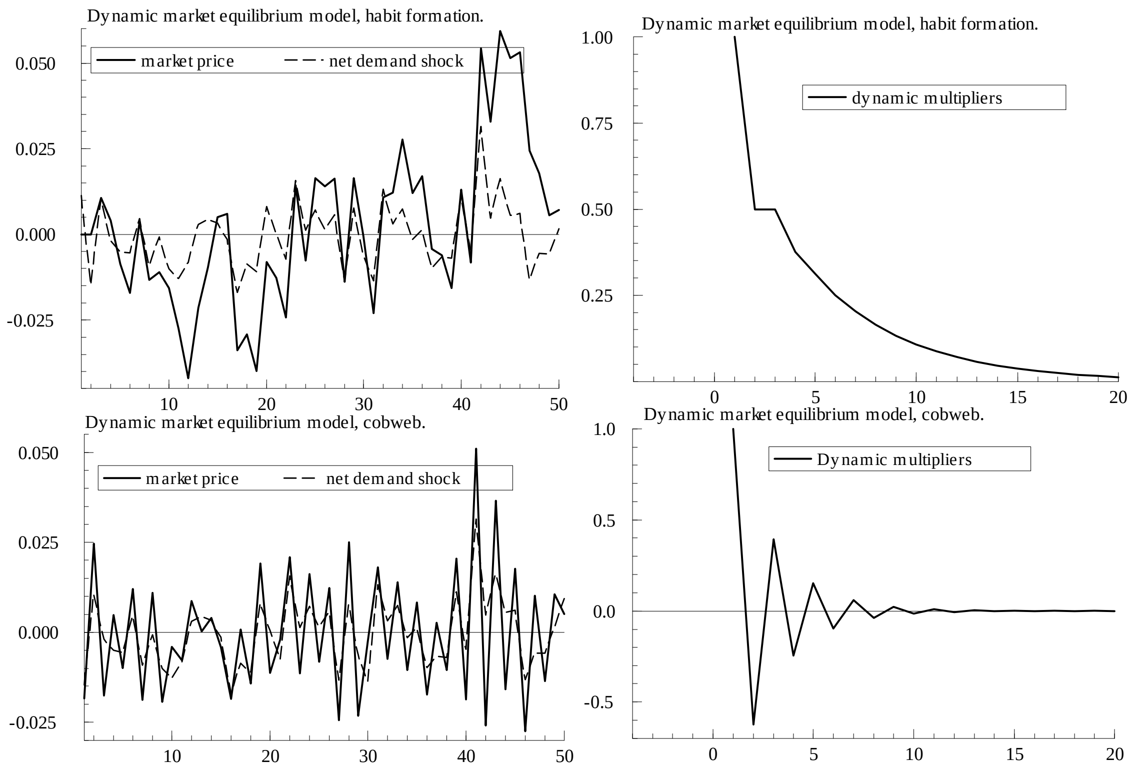


Figure 1.4: Panel a). The marked price (thicker line) and net demand shock (thinner line) of the dynamic model defined by equation (1.5) and (1.6). Panel b) shows the sequence of dynamic multipliers from that model. Panel c) and d) are identical to panel a) and b) of figure 1.2

brings brack inelastich short-run supply. In any given time period $t$, $X_{t-1}$ and $P_{t-1}$ are determined from history and cannot be changed. Hence in this model there are two *pre-determined* variables: $P_{t-1}$ and $X_{t-1}$.

In the same way that we made a distinction between the short and long-run supply equation in the cobweb model, we can now define short and long-run demand schedules. Hence, the coefficient $a$ should now be interpreted as the slope coefficient of the short-run demand curve. Formally, the short-run slope coefficient is found as the partial derivative $X_t$ with respect to $P_t$:

$$\partial X_t / \partial P_t = a$$

The long-run demand schedule is defined for the hypothetical stationary situation where $X_t = X_{t-1} = \bar{X}$, and $P_t = \bar{P}$. The slope coefficient of the long-run demand function is defined by

$$\frac{\partial \bar{X}}{\partial \bar{P}} = a + b_1 \frac{\partial \bar{X}}{\partial \bar{P}},$$

since in the long-run, all values of $X_t$ are affected by a lasting change in the price.

Since we assume that $0 < b_1 < 1$, it is clear that the slope of the long-run demand schedule is less steep than the short-run schedule. In the interpretation adopted here this is due to habit formation: In the (very) short-run, demand is relatively inelastic because households (if we think of a consumption good) have come accustomed to a certain consumption level. However, if the price change persists, consumption habits will adapt and demand is therefore more elastic in the long-run than in the short-run.

Equation (1.6) is an generalization of the supply function which we encountered in the cobweb model. In this case, if both $c_0$ and $c_1$ are positive coefficients, $c_o > 0$ and $c_1 > 0$, the short-run supply schedule is not completely inelastic as in the cobweb model. One interpretation might be that there us some scope for transferring supply from one period to the next (or maybe to bring in supplies form a neighbouring district). But, also in this model, the long-run supply curve is less steep than the short-run schedule.

Figure 1.4, panel a) and b), illustrate the properties of model (1.5) and (1.6). Note that the market price $P_t$, shown as the thicker line on panel a), is more persistent than the net demand shock. The difference from the price behaviour of the cobweb model, which is repeated in panel c) for easy comparison, is striking. The interpretation is that in (1.5) and (1.6), a shock to demand in any given period is propagated into a demand increase (i.e. also at a given price), because of the habit formation effect. For this reason, the market clearing price has a tendency to stay high or low for a longer period of time than in the cobweb model (and the static model). Propagation is in this sense stronger than in the pure cobweb model.

Panel d) shows how the same propagation mechanism leaves its mark on the dynamic multiplier. After a shock to demand there is positive first multiplier, and unlike the cobweb model, the whole sequence of subsequent multipliers are positive for this model. Instead, the price increases after a shock, and is then reduced gradually back to the initial equilibrium level. The dynamic adjustment is therefore smoother than in the previous model, and it is also slower. In panel b) of figure 1.4 there is still some effect left of the initial demand shock after 10 periods, while the adjustment is practically speaking complete in panel d), which is repeated from the cobweb model.

## 1.3 The short-run and the long-run.

In the previous section we applied Frisch's original definitions, namely that dynamic theory is made up of equations which contain variables of different time periods, and studied three simple equilibrium models of a single marked. The first model was a static model since all the variables in that model referred to time period $t$. The two others were examples of dynamic models made up of variables that referred to period $t$ and $t-1$.

The analysis showed that in the static model, the full effect of a shock on the endogenous variables were reached in the same time period as the shock. The telling difference is that in the two dynamic models we analyzed, the endogenous variables' response to shocks were not instantaneous, and the system took several periods to adjust to a shock in period $t$. The exact shapes of the responses, which we dubbed dynamic multipliers above, depend on the values model specification and which values we give to the parameters. In the cobweb model, the multipliers were a oscillating sequence, while in the habit formation model the multipliers (all positive) fell monotonically with time. Later in the book we shall look at several more examples of dynamic multipliers, both from theoretical models and from models that are fitted to real world data, and we shall learn how the properties of the dynamic multipliers can be derived formally.

We may draw from this that static models are best suited when the speed of adjustment of the variables are so fast that we can ignore that 'actually' there is some time delay between the impulse (or shock) and the response. To quote Frisch at some length:

> Hence it is clear that the static model world is best suited to the type of phenomena whose mobility (speed of reaction) is in fact so great that the fact that the transition from one situation to another takes a certain amount of time can be discarded. If mobility is for some reason diminished, making it necessary to take into account the speed of reaction, one has crossed into the realm of dynamic theory.[9]

The choice between a static and a dynamic analysis will therefore depend on what we judge to be realistic or typical of the phenomenon which is the subject of our study: very high speed of reaction to impulses, or more moderate response time. Hence, somewhat paradoxically, phenomena which in an everyday meaning of the word are really dynamic, with lots of volatility, as for example stock market prices, can be analyzed scientifically using a static framework; At least as a first approximation, which brings out that the choice between static and dynamic analysis may also be relative to the level of ambition of our study.

Consider for example the Keynesian income-expenditure model found in any textbook in macroeconomics. This is a static model, but if the assumptions underlying the model (some amount of idle production capacity and involuntary unemployment) are met, there is nothing wrong with using the income-expenditure to analyze the response of GDP to increased government expenditure. We use the model because we think it captures some important aspects of the real-world macroeconomy over a period of 1-2 years for example, and in the recognition that an analysis of the long-run effects would require a dynamic framework. One of the main goals of this book is to extend the analysis of monetary and fiscal policy into the realm of dynamic theory.

Analysis of dynamic models is in increasing demand by central banks, ministries of finance, international organizations and others whose responsibility (or business)

[9] Frisch (1992, p 394), which is an translation of Frisch (1929). A shortened version of Frisch's 1929 paper is accessible in Frisch (1995), the quate is from page 153.

it is to do macroeconomic analysis over a horizon of 1 to 5 years, which economists customarily refer to as the medium run.

Of course long-run analysis is associated with the economics of growth. Incidentally, we note that economics students often have their first encounter with a dynamic model in a course in growth theory, where the Solow model is the standard analytical framework, see Chapter 2.8.3 below. This may have had the side-effect that students come to regard dynamic models as only relevant for growth economics and other really long-term phenomena, covering several decades perhaps This is unfortunate, and may cause misunderstandings, since a dynamic model may be just as relevant for an analysis with a time horizon of 1-5 years. It all comes down to the basic rationale given by Frisch, that "in real life both inertia and friction act as a brake on speed of reaction".[10] Frisch also noted that

> The static theory's assumption regarding an infinitely great speed of reaction contains one of the most important sources of discrepancy between theory and experience.[11]

Therefore Frisch anticipated the increased use of dynamics in economics—it would increase the degree of realism and scope of macroeconomic analysis. Frisch was not alone (for long). It seems that dynamics was "in the air" between the two worlds wars. Frisch published his seminal article called *Propagation Problems and Impulse Problems in Dynamic Economics* in 1933, and the Dutch econometrician Jan Tinbergen developed the first macroeconometric models of the business cycle.[12] Econometricians have continued to play an important rolet in the development of dynamic economics, and the field developed particularly quickly in the 1980s and 1990s. Later in the book we will introduce the concept of error correction (which we will learn to treat synonymously with equilibrium correction) which was coined by the British econometrician Denis Sargan, and which is central to the modern discipline of dynamic econometrics, as documented in Hendry (1995). From another angle, the research programme inititated by Finn Kydland and Edvard Prescott (1982) , and which utilizes relationhips that have been derived from microeconomic theory to model representative macro agents, have become standard in macroeconomic theory. This research started by combining lags in the evolution of physical capital with random technology shocks, and the result was a dynamic macroeconomic model that was dubbed the real business cycle (RBC) model. The RBC model is presented in chapter 2.8.4.

[10] Frisch (1992, p 395).

[11] Frisch (1992, p 395).

[12] Tinbergen's models were commissioned by the League of Nations, and they triggered the first big *metodenstreit* that involved econometrics as a dicipline. Keynes was deeply critical, and Frisch had his own distinct views, see Frisch (1938). Trygve Haavelmo belonged to a group of econometricans who were positive to Tinbergens pathbreaking work, cf. Haavelmo (1943). For those interested in the formative years of dynamic economics and econometrics, the book by Morgan (1990) gives an excellent vantage point.

Table 1.1: Short-run and long-run multipliers of the dynamic model given by equation (1.9) and (1.10).

| | | **Demand shock** | | **Supply shock** | |
|---|---|---|---|---|---|
| | | temp. | perm. | temp. | perm. |
| **short-run** | derivative | | | | |
| Price | $\frac{\partial P_t}{\partial \varepsilon_{i,t}}$ | $\frac{1}{c_0 - a}$ | $\frac{1}{c_0 - a}$ | $\frac{-1}{c_0 - a}$ | $\frac{-1}{c_0 - a}$ |
| Quantity | $\frac{\partial X_t}{\partial \varepsilon_{i,t}}$ | $1 + \frac{a}{c_0 - a}$ | $1 + \frac{a}{c_0 - a}$ | $\frac{-a}{c_0 - a}$ | $\frac{-a}{c_0 - a}$ |
| **long-run** | | | | | |
| Price | $\frac{\partial \bar{P}}{\partial \bar{\varepsilon}_i}$ | 0 | $\frac{\frac{1}{(1-b_1)}}{\kappa}$ | 0 | $\frac{-1}{\kappa}$ |
| Quantity | $\frac{\partial \bar{X}}{\partial \bar{\varepsilon}_i}$ | 0 | $\frac{\frac{c_0+c_1}{(1-b_1)}}{\kappa}$ | 0 | $\frac{\frac{-a}{(1-b_1)}}{\kappa}$ |

Note: $\kappa = (c_0 + c_1) - \frac{a}{1-b_1} > 0$

The RBC model can also be seen as a application of Solow's growth model to a shorter time period that was originally intended. Recently, there has been further development in that direction in the form of micro based dynamic macro model knows as DSGE models (Dynamic Stochastic General Equilibrium models). Later in the book we also present a key element of DSGE models, namely the model of the supply side called the New Keynesian Phillips curve, in chapter 3.6 below.

Returning to the interpretation and role of static models in macroeconomics, it might seem that given that less-than-infinite speed of adjustment is a typical feature of economic systems, static models will only play a limited role in economics: They are only fully realistic when applied to the phenomena which are characterized by very fast response to shocks. That said, static models remain usefull in many cases as a simplified but nevertheless relevant model of short-run macroeconomic response to shocks. As mentioned above, this relevance explains the strong postition that the Keynesian income-expenditure model continue to hold in macroeconomic teaching, as documented in the leading textbooks, see e.g., Blanchard (2008) and Birch Sørensen and Whitta-Jacobsen (2005).

However, there is a second interpretation of static models which will figure prominently in this book: It is that static equations express what the dynamic model would correspond to in a counterfactual situation where no shocks, impulses or changes in incentives occurred. As we become accustomed to dynamic analysis, we will refer to this correspondence by saying that static relationships can represent the stationary state (or steady state) of a dynamic model. We will also, when no misunderstanding are likely to occur, follow custom and use the terms stationary (or steady state) equation interchangeably with the term long-run equation.

It is important to understand that according to this definition, the notion of the "long-run" is completely model dependent, and describes the length of the transition period after a shock, from an initial stationary situation to the next. Hence, in the cobweb model above, the long-run turn out to be approximately 2 years, if the time subscript is taken to denote quarterly observations of the variables $X_t$ and $P_t$. As we explained above, we reach this conclusion since the dynamic multipliers of a shock become negligible after 8 periods.In dynamic models of economic growth, the long-run of course refers to a much longer calender time period, probably decades.

**Box 1.1 (Continuous and discrete time)** *Economic dynamic models are formulated in either continuous time or discrete time. Which one is used is a matter of convenience. In this book we use discrete time to keep the theory as close as possible to applications to time series with real world data which are available for discrete time periods. Consider the continuous time theoretical relationship:*

$$\dot{y} = ay(t) + \varepsilon(t),\ a < 0, \tag{1.7}$$

*where $y(t)$ denotes a variable $y$ which is a continuous function of time, $t$. The exogenous variable $\varepsilon(t)$ is also a continuous function of $t$. The left hand side variable $\dot{y}$ denotes the derivative of $y$ with respect to time. $a$ is a parameter, assumed to be negative (with no loss of generality).*
*Time plays an essential role in model (1.7): If there is an increase in $\varepsilon(t)$, $\dot{y}$ will be larger, and through time there will also be an increase in $y$. Hence, there are propagation effects, just like in the corresponding formulation for discrete time:*

$$y_t = \alpha y_{t-1} + \varepsilon_t,\ \alpha > 0,$$

*or*

$$y_t - y_{t-1} = (\alpha - 1)y_{t-1} + \varepsilon_t,\ \alpha > 0, \tag{1.8}$$

*to make the correspondence even clearer. In (1.8),$y_t - y_{t-1}$ corresponds to $\dot{y}$ in (1.7), and $(\alpha - 1)$ corresponds to $a$.*

## 1.4 Short-run and long-run models

We now recapitulate the definitions and concepts introduced so far in this chapter, and attempt to apply them to the final model of section 1.2, which consisted of the following two equations:

$$X_t = aP_t + b_1X_{t-1} + b_0 + \varepsilon_{d,t}, \text{ (demand)} \quad (1.9)$$
$$X_t = c_0P_t + c_1P_{t-1} + d + \varepsilon_{s,t}, \text{ (supply)}. \quad (1.10)$$

First,by definition, the theory is a dynamic since the relationships consist of variables which refer to different time periods, $t$ and $t-1$. The endogenous variables are $X_t$ and $P_t$. The parameters of this model are; $a$, $b_0, b_1$, $c_0$, $c_1$ and $d$. As explained above, we assume $a < 0$, $0 < b_1 < 1$, $c_0 > 0$, $c_1 > 0$ and $d < 0$ (if the intercept of supply curve negative).

The exogenous variables of the model $X_{t-1}$, $P_{t-1}, \varepsilon_{d,t}$ and $\varepsilon_{s,t}$. $X_{t-1}$ and $P_{t-1}$ are lagged variables of the endogenous variables, and are often referred to as predetermined variables. $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$ are interpreted as random disturbance, or shocks. We have not included economic exogenous variables in the models in this section, but later in the book such variables will usually be included because they have a natural economic interpretation. Hence, if the model was to be applied to a real-world market, it would be reasonable to include for example total consumption expenditure as an exogenous variable in the demand function, and perhaps rainfall or in the supply function.

In figure 1.4, panel a) we have seen an example of the full solution of the dynamic model (1.9)-(1.10), for the endogenous variable $P_t$. Panel b) of figure 1.4 shows the full dynamic response of $P_t$ to a shock that occurs in period 1. Formally the graph shows the sequence of derivatives $\partial P_j/\partial\varepsilon_{d,1}$, for $j = 1, 2, 3, ...$) which, as we have learned, are called the dynamic multipliers.

Although the full mathematical theory of dynamic systems is beyound the scope of this book, it is worth noting that it is in general feasible to solve even quite complicated dynamic models.[13] However, once we move beyond models with two endogenous variables, and one lag, it becomes practical to use computer simulation to find the solution, as explained in chapter 2.7.2. Nevertheless, even when we are unable to derive the full solution of a given dynamic model 'by hand' so to say, an do not have access to a computer, we will still be able to answer the following two important questions:

1. What are the *short-run* effect of a change in an exogenous variable? And

2. What are the *long-run* effects of the shocks.[14]

[13] This is true without qualifications for dynamic systems which are linear (in parameters) and which have a certain property of no charateristic roots outside the unit-circle. The models we consider in this book are of that type, with one exception in chapter 3. For non-linear systems, most analysis is based on a mathematical theorem saying that local stability of non-linear systems can be studied by using a linear approximation of the non-linear system, see Rødseth (2000, Appendix A) for a short and concise introduction. In this book, we show an example of linearization in the review of the Solow growth model in chapter 2.8.3.

[14] In advanced courses in economic dynamics the analysis is often done in terms of so called phase-diagrams, see Rødseth (2000), and Obstfeldt and Rogoff (1998), for example Chapter 2.5.2.2.

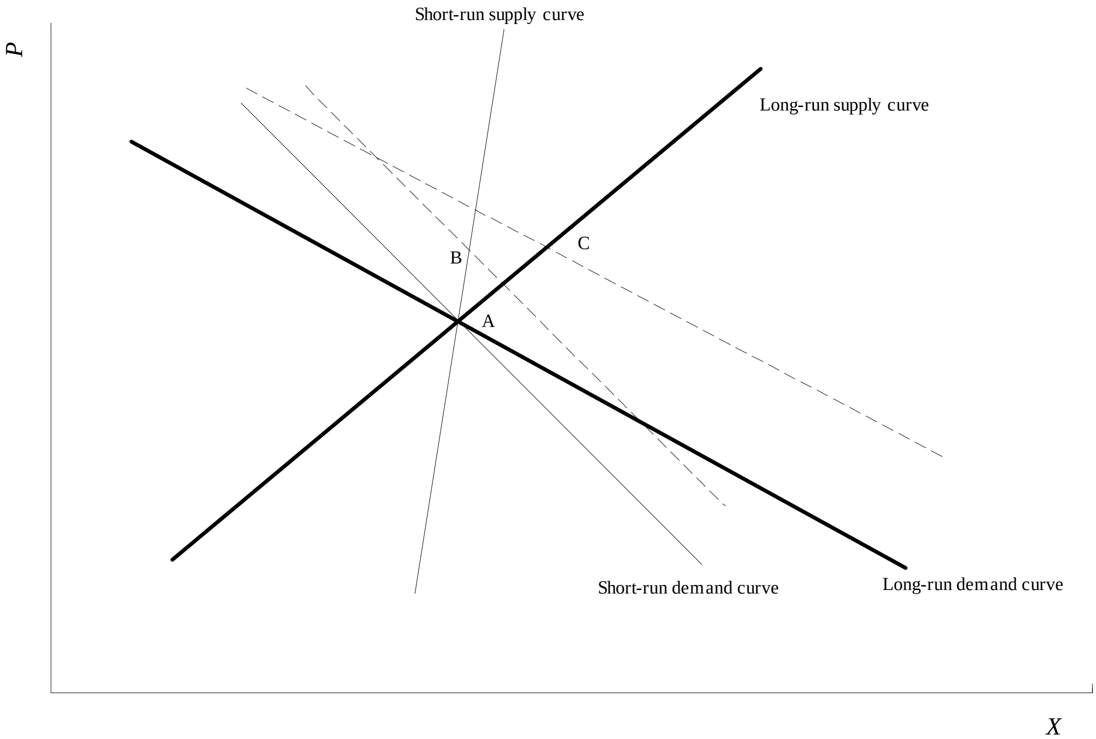


Figure 1.5: Graphical analysis of a demand shock in the model given by equation (1.9) and (1.10)

The technique we will use to answer these questions is based on the distinction between the *short-run* model given by (1.9) and (1.10), and the *long-run model* which is defined for the stationary situation of $X_t = X_{t-1} = \bar{X}, P_t = P_{t-1} = \bar{P}$ and $\varepsilon_{d,t} = \bar{\varepsilon}_d$, $\varepsilon_{s,t} = \bar{\varepsilon}_s$. Hence the (static) long-run model in this example is given by:

$$\bar{X} = \frac{a}{1-b_1}\bar{P} + \frac{b_0}{1-b_1} + \frac{1}{1-b_1}\bar{\varepsilon}_d, \tag{1.11}$$

$$\bar{X} = (c_0 + c_1)\bar{P} + d + \bar{\varepsilon}_s \tag{1.12}$$

where (1.11) is the long-run demand schedule, and (1.12) is the long-run supply schedule. Together they form a model of a hypothetical stationary situation in whihc there are no new shock, and all past shocks have worked their way through (and “out of”) the system.

There are two classes of shocks to consider, temporary shocks and permanent shocks. A temporary shock, as we has defined it above, lasts only one period and then vanishes. A permanent shock last for an infinitely long period of time. Hence the short-run multipliers, also often called the *impact multiplier*, is found as $\partial P_t/\partial \varepsilon_{i,t}$ and $\partial X_t/\partial \varepsilon_{i,t}$ $(i = d, s)$ from the short-run model (1.9)-(1.10). The long-run multipliers $\partial \bar{P}/\partial \bar{\varepsilon}_i$ and $\partial \bar{X}/\partial \bar{\varepsilon}_i$ $(i = d, s)$ are found as the derivatives of the long-run model (1.11)-(1.12).

Table 1.1 summarizes the expressions for the multipliers, and figure 1.5 shows the graphical analysis. As we have explained earlier in the chapter, it is custom to assume that the initial situation is a stationary situation, represented by the intersection point A between the long-run demand curve and the long-run supply curve. Note that the slopes of the long-run curves are steeper than their short-run counterparts, reflecting that both demand and supply are more elastic in the long-run than in the short-run.

The lines in figure 1.5 have been draw in order to illustrate the short- and long-term effects of a shock to demand. As we will see recurrently later in the book, it is often easiest to find the long-run effect first, and then the short-run effect afterwards. For example, if the demand shock is permanent, the new long-run stationary state is C where the dashed "new" long-run demand curve intersects with the long-run demand curve. Compared to the old equilibrium, both $\bar{P}$ and $\bar{X}$ have increased. What about the short-run effect? To answer that question, we simply draw a positive horizontal shift in the short-run demand curve, but note carefully that for a given price, the horizontal shift of the short-run curve is smaller than for the long-run curve demand curve. Again, this is because a part of demand are predetermined by habits in the short-run. Hence the new short-run equilibrium, in the period of the shock, is given by point B in the figure.

So far we have assumed that the demand shock is permanent. In the case of a temporal shock the graphical analysis is very simple: the short-run effect is given by point B, and there are no long-run effects, so the market equilibrium moves back to point A.

## 1.5 Stock and flow variables

In macroeconomic models and analysis there is an important conceptual distinction between *flow* and *stock* variables. Examples of flow variables are GDP and its expenditure components, hours worked and inflation. Flow variables are measured in for example million kroner, or thousand hours worked, *per* year (or quarter, or month). Inflation is measured as a rate, or percentage *per* time period and is another example of a flow variable. Values of stock variables, in contrast, refer to particular points in time. For example, statistical numbers for national debt often refer to the end of the year.

Price indices are also examples of stock variables. They represent the cost of buying a basket of goods with reference to a particular time period. The *annual* consumption price index, CPI for short, is a stock variable which is obtained as the average of the 12 monthly indices (each being a stock variable).

For example $P_t$ may represent the value of the Norwegian CPI in period $t$ (a month, a quarter or a year). As you probably will know from before, the values of $P$ will be *index numbers*. The number 100 (often 1 is used instead) refers to the base period of the index. If $P_t > 100$ the overall price level is higher in period $t$ than in

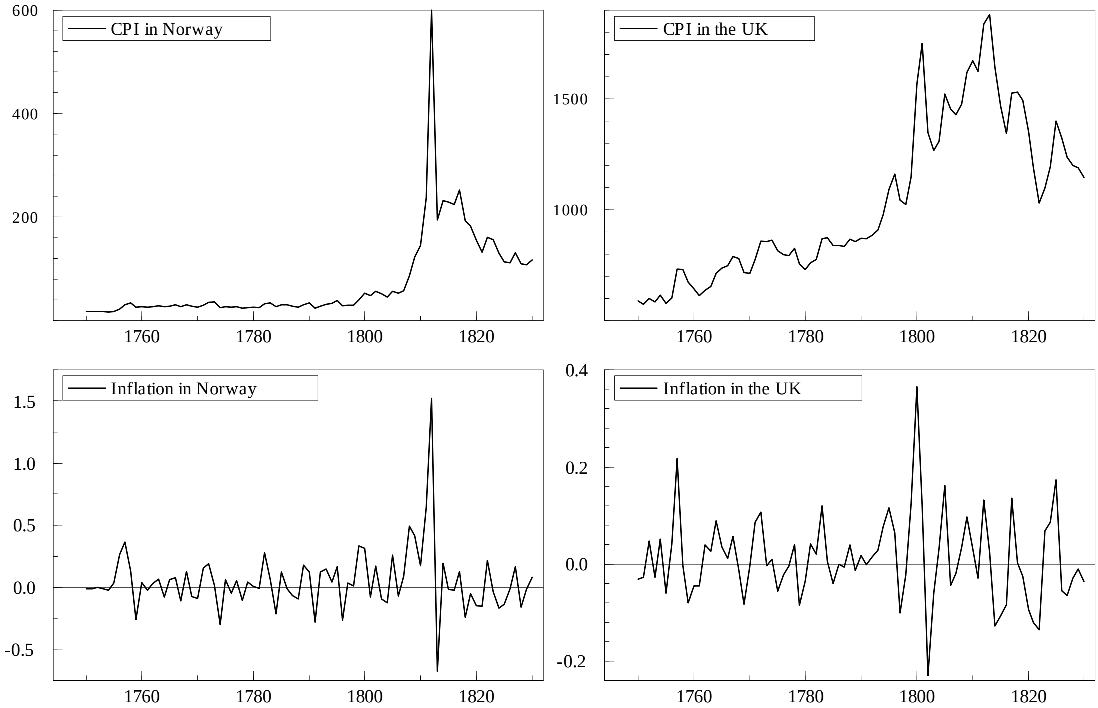


Figure 1.6: Panel a) and b): consumer price indices (stock variables) in Norway and the UK . Panel c) and d): the corresponding rates of change (flow). Annual data 1750-1830.

the base year—there has been a period of inflation.

Starting from a *stock* variable like $P_t$, a flow variable results from obtaining the change, hence

$$\begin{aligned} x_t &= P_t - P_{t-1}, && \text{the (absolute) change} \\ y_t &= \frac{P_t - P_{t-1}}{P_{t-1}}, && \text{the relative change, and} \\ z_t &= \ln P_t - \ln P_{t-1} && \text{the approximate relative change} \end{aligned}$$

are examples of *flow* variables. Note that:

- $y_t \times 100$ is inflation in percentage points. In this book we will often use to the rate formulation , hence we omit the scaling by 100.
- $z_t \approx y_t$ by the properties of the (natural) logarithmic function, see for example the appendix A if you are in doubt.

A typical empirical trait of *stock* variables is that they change gradually, as a summation of often quite small growth rates. Occasionally however, a stock variable *jumps* from one level in period $t$ to another level in period $t+1$. In theory, with continuous time, see Box 1.1 below, the derivative of the variable with respect to time is infinite at the time of the jump. In practice, with discrete time data, the rate of change becomes relatively large in such instances. In the Norwegian "price history" the high rate of change in the consumer price index at the time of the breakdown of the union with Denmark may be regarded an empirical example of jump behaviour in the price level, since the annual rate of inflation increased from approximately 25% to 150%, see figure 1.6, panel a) and c). From panel b) and d) in figure 1.6 we see that the Revolutionary War (which started in 1789 with the Storm on the Bastille) and the Napoleonic wars caused abrupt changes in the CPI in the United Kingdom, compared to the relatively low inflation rates in the late eighteenth century.

In economic theory, when stock variables change gradually, we use explicit dynamic models to account for their evolution. Sometimes though, stock variables can be treated theoretically *as if* they are jump-variables. This is the case of practically infinitively sharp reaction speed which we discussed above, as a condition for giving a dynamic interpretation to a static model. An example of such an approach is the portfolio model for the market for foreign exchange market. In this model, the equilibrium nominal exchange rate is determined by the supply and demand functions for the whole stock of foreign currency, which in a liberalized market is subject to complete and immediate change, over-night, practically speaking. Hence, because the speed of adjustment in the market is so fast, the model of market equilibrium can be given a static specification, at least as a first approximation, see chapter 2.4.3.

Macroeconomic dynamics often arise from the combination of stock and flow variables First consider a nation's net foreign debt $FD_t$, evaluated at the end of period $t$. In principle, the dynamic behaviour of debt (a stock) is linked to the value of the current account (flow) in the following way:

$$FD_t = -CA_t + FD_{t-1} + cor_t.$$

Hence if the *current account*, $CA_t$, is zero over the period, and if there are corrections of the debt value (typically due to financial transactions), this period's net foreign debt will be equal to last period's debt. However, if there is a current account surplus for some time, this will lead to a gradual reduction of debt—or an increase in the nation's net wealth. Conversely, a consistent current account deficit raises a nation's debt.[15]

Figure 1.7 shows the development of the Norwegian current account and of Norwegian foreign deb. At the start of the period, Norway's net debt (a stock variable) was hovering at around 100 billion, despite a current account surplus (albeit small) in the early 1980s. Evidently, there was a substantial debt stemming from the 1970s which the nation's net financial saving (a flow) had not yet been able to wipe out.

[15] Se for example Birch Sørensen and Whitta-Jacobsen (2005, Ch 4.1) for a discussion of wealth accumulation, cf their equation (2) in particular.

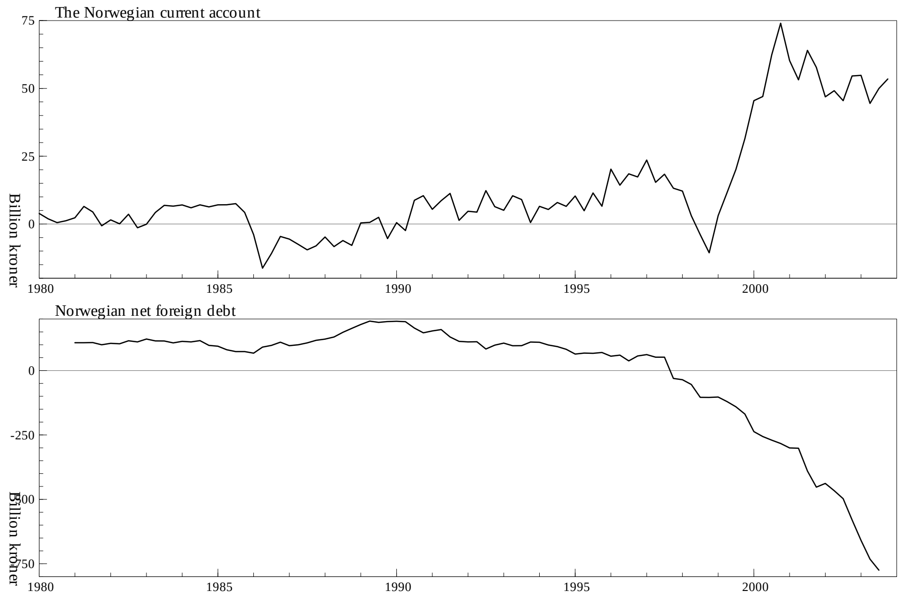


Figure 1.7: The Norwegian current account (upper panel) and net foreign deb (lower panel)t. Quarterly data 1980(1)-2003(4)

In the period 1986-1989 the current account surplus changed to a deficit, and, as one would expect from the "debt equation" above, Noway's debt increased again until it peaked in 1990. Later in the 1990s, the current account surplus returned, and therefore the net debt was gradually reduced. Later, at the end of the millennium, surpluses grew to unprecedented magnitudes, up to 75 billion *per* quarter, resulting in a sharp build up of net financial wealth, accumulating to 750 billion kroner at the end of the period.

Section 2.4.3 discusses the role of the current account in the theory of the market for foreign exchange, in particular why it is fruitful to distinguish between stock and flow variables in the analysis of that market.

The theory of economic growth provides another example of the importance of stock and flow dynamics. For example, the level of production (a flow variable) in the economy depends on the size of the labour stock (literary speaking), and the capital stock. Due to the phenomenon of capital depreciation, some of today's production needs to be saved just in order to keep the capital stock intact in the first period of the future. Moreover, due to population growth (and a declining marginal product of labour), next period's capital stock will have to be larger than it is today if we want to avoid that output per capita declines in the next period. Hence, economic growth in terms of GDP per head requires that the flow of net investment (gross investment minus capital depreciation) is positive. In line with this, the dynamic equation of the capital stock is written as

$$K_{t+1} = K_t + J_t - D_t$$

where $K_t$ denotes the capital stock at the start of period $t$, $J_t$ is the flow of gross investment during period $t$, and $D_t$ is replacement investment (also a flow). A much used assumption is that $D_t$ is proportional to the pre-existing capital stock, i.e., $D_t = \delta K_t$ where the rate of capital depreciation $\delta$ is a positive number which is less than or equal to one. This gives a well known expression for the development of the capital stock

$$K_{t+1} = (1 - \delta)K_t + J_t, \quad 0 < \delta \leq 1, \tag{1.13}$$

which plays an important role in growth theory (see chapter 2.8.3), in real business cycle theory (see chapter 2.8.4), and generally in all macro models with endogenous capital accumulation.

Another case of stock-flow dynamics is the relationship between wages and unemployment, which we will discuss in detail in chapter 3. The rate of unemployment is a stock variable which influences wage growth (a flow variable). On the other hand, the rate of unemployment depends on accumulated wage growth which determines the real wage level (a stock varaible). Similar linkages exist between nominal and real exchange rates, and provide one of the key dynamic mechanisms in the models of the national economy that we encounter in macroeconomics.

## 1.6 An empirical example of a static and dynamic equation

A main use of dynamic models is to describe and predict the behaviour of economic variables over time. A variable $y_t$ is called a time series if we observe it over a sequence of time periods represented by the subscript $t$., for example $\{y_T, y_{T-1}, ..., y_1\}$ represents the case where we have observations from period 1 to $T$. Usually, the simpler notation $y_t$, $t = 1, ...., T$ is used. The interpretation of the time subscript varies from case to case, it can represent a year, a quarter of a year, or a month. In macroeconomics, other time periods are also considered, such as 5-year or 10-year averages of historical data, and daily or even hourly data at the other extreme (e.g., exchange rates, stock prices, money market interest rates). We have already seen examples of theoretically constructed time series data, cf. figure 1.2 and 1.4, and of actual time series data in figure 1.7, showing quarterly observations of the current account and of net debt in Norway. In this section we discuss an example where $y_t$ is (the logarithm) of private consumption, and we consider in detail a dynamic model of consumption where the explanatory variable is private disposable income.

### 1.6.1 A static consumption function

When we consider economic models to be used in an analysis of real world macro data, care must be taken to distinguish between static and dynamic models. The textbook consumption function, i.e., the relationship between real private consumption expenditure ($C$) and real households' disposable income ($INC$) is an example of a *static* equation

$$C_t = f(INC_t),\ f' > 0. \tag{1.14}$$

Consumption in any period $t$ is strictly increasing in income, hence the positive signed first order derivative $f'$ which is called the marginal propensity to consume. To be able to apply the theory to observations of the real economy we also have to specify the function $f(INC_t)$. Two of the most used functional forms in macroeconomics, are the linear and log-linear specifications. For the case of the static consumption function in (1.14), these two specifications are

$$C_t = \beta_0 + \beta_1 INC_t + e_t, \quad \text{(linear), and} \tag{1.15}$$
$$\ln C_t = \beta_0 + \beta_1 \ln INC_t + e_t, \quad \text{(log-linear).} \tag{1.16}$$

For simplicity we use the same symbols for the coefficients in the two equations. However, it is important to note that since the variables are measured on different scales—million kroner at fixed prices in (1.15), the natural logarithm of fixed million kroner in (1.16)—the slope coefficient $\beta_1$ has a different economic interpretation in the two models.[16]

Thus, in equation (1.15), $\beta_1$ is the marginal propensity to consume and is in units of million kroner. Mathematically, $\beta_1$ in (1.15) is the derivative of real private consumption, $C_t$ with respect to real income, $INC_t$:

$$\frac{dC_t}{dINC_t} = \beta_1, \text{ from (1.15).}$$

In the log linear model (1.16), since both real income and real consumption are transformed by applying the natural logarithm to each variable, it is common to say that each variable have been "log-transformed". By taking the differential of the log-linear consumption function we obtain (see appendix A for a short reference on logarithms):

$$\begin{aligned}\frac{dC_t}{C_t} &= \beta_1 \frac{dINC}{INC_t} \text{ or} \\ \frac{dC_t}{dINC_t}\frac{INC_t}{C_t} &= \beta_1.\end{aligned}$$

Hence, in equation (1.16), $\beta_1$ is interpreted as the elasticity of consumption with respect to income: $\beta_1$ represents the (approximate) percentage increase in real consumption due to a 1% increase in income.

[16] In practice, this means that if $LC_t$ is private consumption expenditure in million (or billion) kroner in period $t$, $C_t$ is defined as $LC_t/PC_t$ where $PC_t$ is the price deflator in period $t$. If, for example, $C_t$ is in million 2000 kroner, this means that the base year of the consumer price index $PC_t$ is 2000 (with annual data, $PC_{2000} = 1$, with quarterly data, the annual average is 1 in 2000). $Y_t$ is defined accordingly.

Note that the log-linear specification (1.16) implies that the marginal propensity to consume is itself a function of income. In this sense, the log-linear model is the more flexible of the two functional forms and this is part of the reason for its popularity. To gain familiarity with the log-linear specification, we choose that functional form in the rest of this section, and in the next, but later we will also use the linear functional form, when that choice makes the exposition become easier.

We include a stochastic term $e_t$ in the two specified consumption functions. You can think of $e_t$ as a completely random variable, with mean value of zero, but which cannot be predicted from $C_t$ of $Y_t$ (or from the history of these two variables). In a way, the inclusion of $e_t$ in the models is a concession to reality, since economic theory ($f(INC_t)$ in this case) cannot hope to capture all the vagaries of $C_t$, which instead is represented by the stochastic term $e_t$. Put differently, even if our theory is "correct", it is only true on average. Hence, even if we know $\beta_0$, $\beta_1$ and $INC_t$ with certainty, the predicted consumption expenditure in period $t$ will only be equal to $C_t$ on average, due to the random *disturbance term* $e_t$.[17] You will of course observe that the rationalization of the random term $e_t$ follows exactly the same lines as were used when we motivated the random shock terms in the partial equilibrium models above

There is another reason for including a disturbance term in the consumption function, which has to do with how we confront our theory with the time series evidence. So: let us consider real data corresponding to $C_t$ and $INC_t$, and assume that we have a good way of quantifying the intercept $\beta_0$ and the elasticity of consumption with respect to income, $\beta_1$. You will learn about so called least-squares estimation in courses in econometrics, but intuitively, least-squares estimation is a way of finding the numbers for $\beta_0$ and $\beta_1$ that give the on average best prediction of $C_t$ for a given value of $INC_t$. Using quarterly data for Norway, for the period 1967(1)-2002(4)—the number in brackets denotes the quarter—we obtain by using the least squares method:

$$\ln \hat{C}_t = 0.02 + 0.99 \ln INC_t \tag{1.17}$$

where the "hat" in $\hat{C}_t$ is used to symbolize the fitted value of consumption given the income level $INC_t$. Next, use (1.16) and (1.17) to define the residual $\hat{e}_t$:

$$\hat{e}_t = \ln C_t - \ln \hat{C}_t, \tag{1.18}$$

which is the empirical counterpart of $e_t$.

In Figure 1.8 we show a scatter-plot of the 140 observations of consumption and income (in logarithmic scale), with each observation marked by a +. The upward sloping line represents the linear function in equation (1.17), and for each observation we have also drawn the distance up (or down) to the line. These "projections" are the graphical representation of the residuals $\hat{e}_t$.

[17] Strictly speaking, we should use separate symbols not only for the coefficients, but also for the two disturbances. Logically, the same random variable cannot act as a disturbance in the two different functional forms. However, to economize notation, we have chosen to dodge this formality.

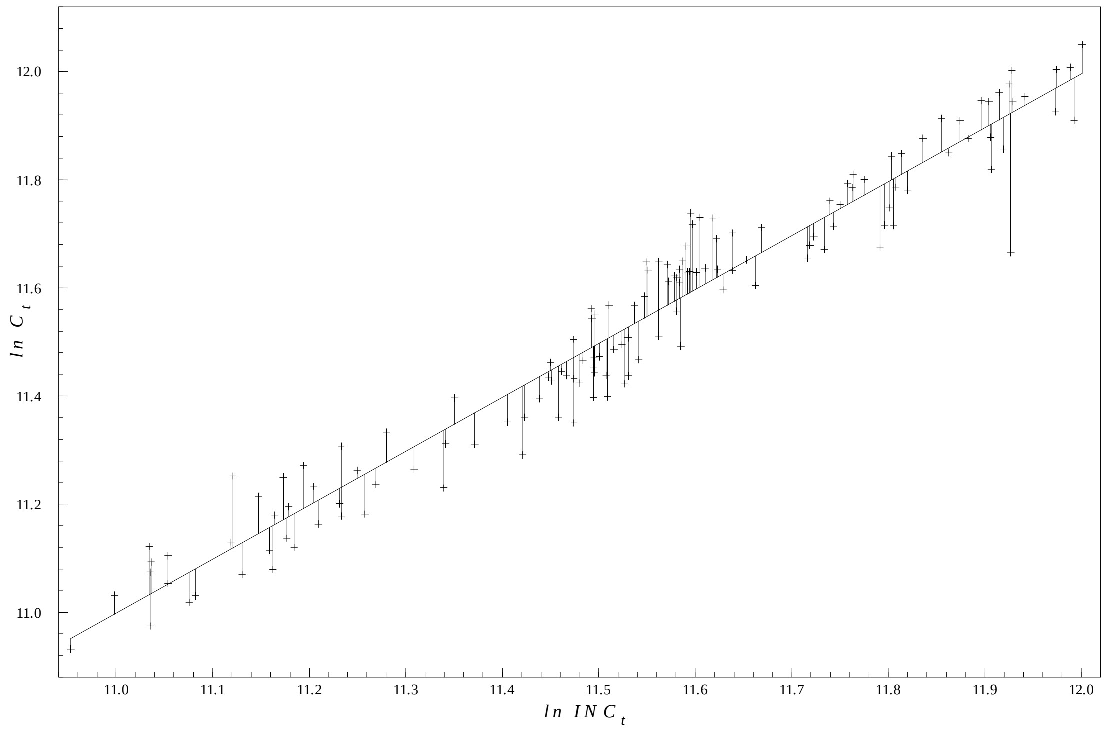


Figure 1.8: The estimated model in (1.17), see text for explanation.

### 1.6.2 A dynamic consumption function

Clearly, if we are right in our arguments about how pervasive dynamic behaviour is in economics, equation (1.16) is a very restrictive formulation. For example, according to (1.16), the whole adjustment to a change in income is completed within a single period, and if income suddenly changes next period, consumer's expenditure changes suddenly too. As noted above, immediate and complete adjustment to changes in economic conditions is seldom seen, and dynamic models account for the typical lags in adjustment. A dynamic model of private consumption allows for the possibility that period $t-1$ income affects consumption, and that for example habit formation induces a positive relationship between period $t-1$ and period $t$ consumption:

$$\ln C_t = \beta_0 + \beta_1 \ln INC_t + \beta_2 \ln INC_{t-1} + \alpha \ln C_{t-1} + \varepsilon_t \tag{1.19}$$

The literature refers to this type of model as an **a**utoregressive **d**istributed **l**ag model, ADL model for short. "Autoregressive" refers to the inclusion of lagged endogenous variable on the right hand side of the equation. "Distributed lag" refers to the presence of both the current and lagged explanatory variable in the model—the effect of a change in the explanatory varaible is distributed between this period and the next. We will study the properties of the ADL model in detail in the next chapter.

How can we evaluate the claim that the ADL model in equation (1.19) gives a better description of the data than the static model? A full answer to that question would take us into the realm of econometrics, but intuitively, one indication would be if the empirical counterpart to the disturbance of (1.19) are smaller and less systematic than the errors of equation (1.16). To test this, we obtain the residual $\hat{e}_t$, again using the method of least squares to find the best fit of $\ln C_t$ according to the dynamic model:

$$\ln \hat{C}_t = 0.04 + 0.13 \ln INC_t + 0.08 \ln INC_{t-1} + 0.79 \ln C_{t-1} \tag{1.20}$$

Figure 1.9 shows the two residual series. It is immediately clear that the dynamic model in (1.20) is a much better description of the behaviour of actual consumption expenditure than the static model (1.17). As already stated, this is a typical finding with macroeconomic data, and it is consistent with Frisch's insight that the static theory's assumptions regarding an infinitely great speed of adjustment is usually the main reason why static models do no explain the data with any degree of accuracy.

Judging from the estimated coefficients in (1.20), one main reason for the improved fit of the dynamic model is the lag of consumption itself, i.e., the autoregressive aspect of the equation. That the lagged value of the endogenous variable is an important explanatory variable is also a typical finding, and just goes to show that dynamic models represent essential tools for macroeconomics. The rather low values of the income elasticities (0.130 and 0.08) may reflect that households find that a single quarterly change in income is "too little to build on" in their expenditure decisions. As we will see in the next chapter the results in (1.20) imply a much higher impact of a permanent change in income than of a temporary rise.

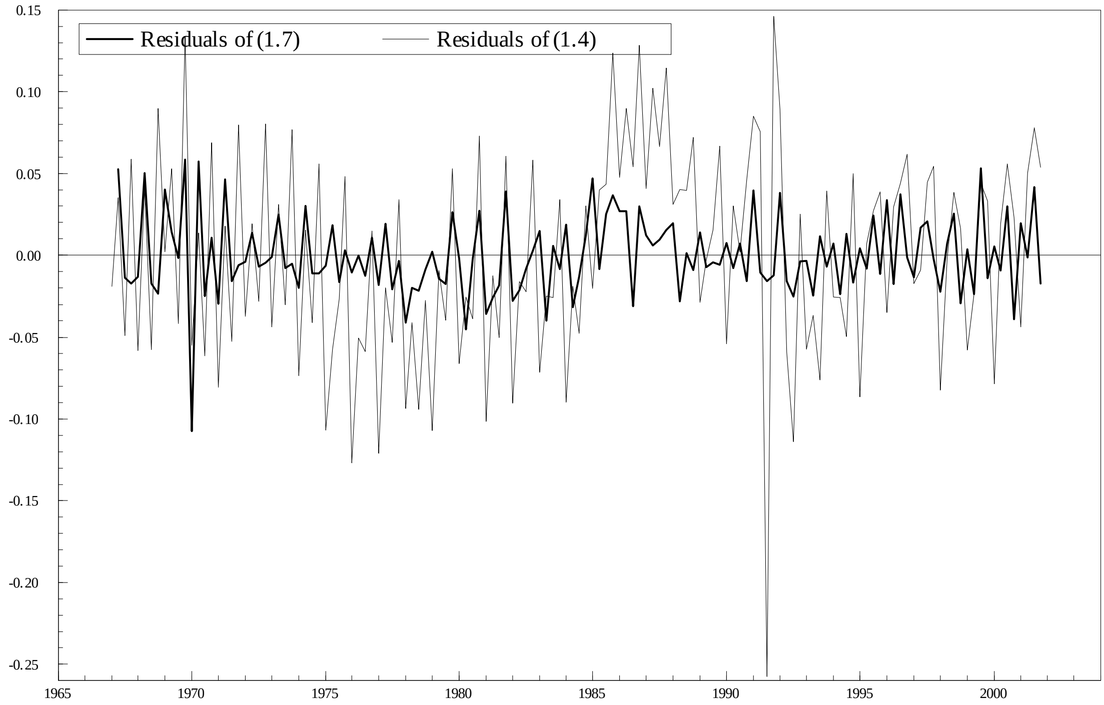


Figure 1.9: Residuals of the two estimated consumptions functions (1.17), and (1.20).

## 1.7 Summary and overview of the rest of the book

The quotation from Norges Bank's web pages at the beginning of the chapter shows that the Central Bank has formulated a view about the dynamic effects of a change in the interest rate on inflation. In the quotation, the Central Bank states that the effect will take place within two years, i.e., 8 quarters in a quarterly model of the relationship between the rate of inflation and the rate of interest. That statement may be taken to mean that the effect is building up gradually over 8 quarters and then dies away quite quickly, but other interpretations are also possible. In order to inform the public more fully about its view on the monetary policy transmission mechanism, the Bank would have to give a more detailed picture of the dynamic effects of a change in the interest rate. Similar issues arise whenever it is of interest to study how fast and how strongly an exogenous perturbation or a policy change affect the economy, for example how private consumption is likely to be affected by a certain amount of tax-cut.

Chapter 2 is an introduction to the modelling tools of dynamics. Building on the motivation of this chapter, we present a general framework for dynamic single equation models, called the autoregressive distributed lag model, or ADL for short. In terms of mathematics, the ADL model corresponds to linear difference equations, but we assume no knowledge of the mathematics of difference equations. Instead, this important model class is motivated by its relevance for economic dynamics, and by the intuitive economic interpretation of the model's parameters.

Having introduced the ADL model, we can use that model that model to establish formally the properties of the dynamic multiplier, which we in fact have come across several times already in the introduction, for example in the analysis of the cobweb model. The observant reader may already have gauged that the dynamic multiplier is a key concept in this course, and once you get a good grip on it, you have a powerful tool which allows you to calculate the dynamic effects of policy changes (and of other exogenous shocks for that matter) in a dynamic model. The ADL model also defines a typology of dynamic equations, as special cases, each with their distinct features and economic meaning. One particular transformation of the ADL model is the error correction model, is particularly useful, since it shows explicitly how dynamic models combines variables that are in terms of changes with respect to time (differenced data), with the past levels of the variables. Finally in chapter 2, the analysis is extended to simple dynamic systems-of-equations. For both the single equations and for systems-of-equations, a number of macroeconomic examples and applications are discussed in details.

In chapter 3 the analytical tools of chapter 2 is applied to wage-and-price setting, which is an essential part of the supply side of modern macroeconomic model. A main difference from existing textbooks is that our approach rationalizes that the careful modelling of bargaining based wage setting leads to a representation of the supply side model which cannot be subsumed in a standard Phillips curve relationship. This changes the premises for stabilization policy as currently perceived. In later versions these models of the supply side will be integrated in a model of the open macroeconomy.

## Exercises

1. Consider the model consisting of equation (1.3) and (1.4). Experiment with different slopes of the demand and (long-run) supply functions, corresponding to different values of the parameters $a$ and $c$, and check whether the cobweb is always "spiraling inwards". Or can other patterns emerge?

2. In equation (1.4), what it the derivative of supply in period $t$, $X_t$, with respect to price in period $t$? What is the derivative of $X_{t+1}$ with respect to $P_t$? Interpret your results in the light of the graph in figure 1.3.

3. In the model defined by equation (1.5) and (1.6), what is the expression for the slope of the long-run demand curve? What happens to the slope if $b = 1$, and can you think of an interpretation of this case?

4. Modify the model in (1.5) and (1.6) by introducing an exogenous variable $Z_t$, representing total consumption expenditure, in the demand equation. Assume that initially $Z_t$ is at a constant level $z_0$ in all time periods. Try to illustrate and explain the effects of a negative expenditure shock that lasts for 2 periods before $Z$ goes back to $z_0$ and stays there.

5. Show that, after setting $e_t = 0$ (for convenience), $MPC \equiv \partial C_t / \partial INC_t = k \cdot \beta_1 INC_t^{\beta_1 - 1}$, where $k = \exp(\beta_0)$.

# Chapter 2

# Linear dynamic models

Section 2.1 introduces an important class of dynamic models called autoregressive distributed lag models, and the concept of the dynamic multiplier, already encountered above, is made precise within that framework. Section 2.2 uses the consumption function as an example of the derivation of dynamic multipliers. A typology of dynamic equations which are often appear in macroeconomic models, is presented in section 2.3, while section 2.4 illustrates how economic models and hypothesis can be formulated in the framework of the ADL. Section 2.5 and 2.6 show that static and dynamic equations can be reconciled, and integrated, by the use of the equilibirum correction model, ECM, which in turn is a *"1-1"* transformation of the ADL. Section 2.7 shows the solution of dynamic models. Finally, section 2.8 sketches how the analysis can be extended to multi equation dynamic models. Two examples of macroecomic system models: the Solow growth model and the real business cycle model, are reviewed in some detail.

## 2.1 The ADL model and dynamic multipliers

In chapter 1 we have motivated several examples of dynamic relationships, e.g., the supply equation of the cobweb model and the dynamic model of private consumption as a function of income. We have also been introduced to the concept of dynamic multiplier, and have learned that while there is only one multiplier for each explanatory variable in a static equation, a dynamic model is characterized by a sequence of multipliers. In this chapter we present the autoregressive distributed lag model as general model of linear dynamic relationships.

In equation (2.1), $y_t$ is the endogenous variable while $x_t$ and $x_{t-1}$ make up the distributed lag part of the model:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t. \tag{2.1}$$

In the same way as before, $\varepsilon_t$ symbolizes the small and random part of $y_t$ which is unexplained by $x_t$, $x_{t-1}$ and $y_{t-1}$.

In many applications, as in the consumption function example, $y$ and $x$ are in logarithmic scale, due to the specification of a log-linear functional form. However, in other applications, different units of measurement are the natural ones to use. Thus, depending of which variables we are modelling, $y$ and $x$ can be measured in million kroner, or in thousand persons, or in percentage points. Mixtures of measurement are also often used in practice: for example in studies of labour demand, $y_t$ may denote the number of hours worked in the economy while $x_t$ denotes real wage costs per hour. The measurement scale does not affect the mathematical derivation of the dynamic multipliers, but care must be taken when interpreting and presenting the results. Specifically, only when both $y$ and $x$ are in logs, are the multipliers directly interpretable as percentage changes in $y$ following a 1% increase in $x$, i.e., they are (dynamic) elasticities.

We now want to show that dynamic multipliers correspond to the derivatives of $y_t$, $y_{t+1}$, $y_{t+2}, ....$, with respect to one ore more of the $x$'s. We first consider a *permanent change* in the explanatory variable. It is then convenient to think of $x_t$, $x_{t+1}$, $x_{t+2}, , ....$ as increasing functions of a continuous variable $h$. When $h$ changes permanently, starting in period $t$,we have $\partial x_{t+j}/\partial h > 0$, for $j = 0, 1, 2, ...$, while there is no change in $x_{t-1}$ and $y_{t-1}$ since those two variables are predetermined. Since $x_t$ is a function of $h$, so is $y_t$, and the effect of $y_t$ of the change in $h$ is found as

$$\frac{\partial y_t}{\partial h} = \beta_1 \frac{\partial x_t}{\partial h}.$$

In the outset, $\partial x_t/\partial h$ can be any number, for example 0.01 may represent a "small change in income". However, because it is convenient, is has become custom to evaluate the multipliers for the case of $\partial x_t/\partial h = 1$ which is referred as the case of a *unit-change*. Following this practice, the first multiplier is

$$\frac{\partial y_t}{\partial h} = \beta_1. \tag{2.2}$$

The second multiplier associated with a permanent change in $x$ is found by considering the equation for period $t + 1$, i.e.,

$$y_{t+1} = \beta_0 + \beta_1 x_{t+1} + \beta_2 x_t + \alpha y_t + \varepsilon_{t+1}. \tag{2.3}$$

and calculating the derivative $\partial y_{t+1}/\partial h$. Since our premise is that the change in $h$ occurs in period $t$, both $x_{t+1}$ and $x_t$ are changed. We need to keep in mind that $y_t$ is a function of $h$, hence:

$$\frac{\partial y_{t+1}}{\partial h} = \beta_1 \frac{\partial x_{t+1}}{\partial h} + \beta_2 \frac{\partial x_t}{\partial h} + \alpha \frac{\partial y_t}{\partial h} \tag{2.4}$$

Again, considering a unit-change, $\partial x_t/\partial h = \partial x_{t+1}/\partial h = 1$, and using (2.2), the second multiplier can be written as

$$\frac{\partial y_{t+1}}{\partial h} = \beta_1 + \beta_2 + \alpha\beta_1 = \beta_1(1 + \alpha) + \beta_2 \tag{2.5}$$

To find the third derivative, or multiplier, consider

$$y_{t+2} = \beta_0 + \beta_1 x_{t+2} + \beta_2 x_{t+1} + \alpha y_{t+1} + \varepsilon_{t+2}. \tag{2.6}$$

Using the same logic as above, we obtain

$$\begin{aligned} \frac{\partial y_{t+2}}{\partial h} &= \beta_1 \frac{\partial x_{t+2}}{\partial h} + \beta_2 \frac{\partial x_{t+1}}{\partial h} + \alpha \frac{\partial y_{t+1}}{\partial h} \\ &= \beta_1 + \beta_2 + \alpha \frac{\partial y_{t+1}}{\partial h} \\ &= \beta_1(1 + \alpha + \alpha^2) + \beta_2(1 + \alpha) \end{aligned} \tag{2.7}$$

where the conventional unit-change, $\partial x_t/\partial h = \partial x_{t+1}/\partial h = 1$, has been used in the second line, and the third line is the result of substituting $\partial y_{t+1}/\partial h$ by the right hand side of (2.5).

Comparison of equation (2.4) with the first line of (2.7) shows that there is a clear pattern: The third and second multipliers are linked by exactly the same form of dynamics that govern $y$ itself. This also holds for higher order multipliers, and means that these multipliers can be computed *recursively*: For example, once we have found the third multiplier, the fourth can be found easily by substituting $\partial y_{t+2}/\partial h$ in

$$\frac{\partial y_{t+3}}{\partial h} = \beta_1 + \beta_2 + \alpha \frac{\partial y_{t+2}}{\partial h}$$

by the final expression in equation (2.7).

In table 2.1, the column named *Permanent unit change in $x_t$* collects the results we have obtained so far. In the table, we use the notation $\delta_j$ ($j = 0, 1, 2, ...$) for the multipliers. For, example $\delta_0$ is identical to $\partial y_t/\partial h$ in (2.2), and $\delta_2$ is identical to the third multiplier, $\partial y_{t+2}/\partial h$ in(2.7). In general, because the multipliers are linked recursively, multiplier $j + 1$ is given as

$$\delta_j = \beta_1 + \beta_2 + \alpha\delta_{j-1}, \text{ for } j = 1, 2, 3, \ldots \tag{2.8}$$

In the consumption function example, we saw that as long as the autoregressive parameter is less than one, the sequence of multipliers is converging towards a long-run multiplier. In this more general case, the condition needed for the existence of a long-run multiplier is that $\alpha$ is less than one in absolute value, formally $-1 < \alpha < 1$. In chapter 2.5 and 2.7, the wider significance of this condition is explained. For our present purpose, we can simply assume that the condition holds, and define the long-run multiplier as $\delta_j = \delta_{j-1} = \delta_{long-run}$. Using (2.8), the expression for $\delta_{long-run}$ is found to be

$$\delta_{long-run} = \frac{\beta_1 + \beta_2}{1 - \alpha}, \text{ if } -1 < \alpha < 1. \tag{2.9}$$

Clearly, if $\alpha = 1$, the expression does not make sense mathematically, since the denominator is zero. Economically, it does not make sense either, since the long-run effect of a permanent unit change in $x$ is an infinitely large increase in $y$ (if

Table 2.1: Dynamic multipliers of the general autoregressive distributed lag model.

| ADL model: | $y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t.$ | |
|---|---|---|
| | Permanent(1) unit change in $x$ | Temporary unit change in $x_t$ |
| 1. multiplier: | $\delta_0 = \beta_1$ | $\frac{\partial y_t}{\partial x_t} = \beta_1$ |
| 2. multiplier: | $\delta_1 = \beta_1 + \beta_2 + \alpha\delta_0$ | $\frac{\partial y_{t+1}}{\partial x_t} = \beta_2 + \alpha\frac{\partial y_t}{\partial x_t}$ |
| 3. multiplier: | $\delta_2 = \beta_1 + \beta_2 + \alpha\delta_1$ | $\frac{\partial y_{t+2}}{\partial x_t} = \alpha\frac{\partial y_{t+1}}{\partial x_t}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| j+1 multiplier | $\delta_j = \beta_1 + \beta_2 + \alpha\delta_{j-1}$ | $\frac{\partial y_{t+j}}{\partial x_t} = \alpha\frac{\partial y_{t+j-1}}{\partial x_t}$ |
| | | |
| long-run(3) | $\delta_{long-run} = \frac{\beta_1+\beta_2}{1-\alpha}$ | 0 |
| | | |
| notes: | (1) As explained in the text, $\partial x_{t+j}/\partial h = 1, \;\; j = 0, 1, 2, ...$ | |

$\beta_1 + \beta_2 > 0$). The case of $\alpha = -1$, may at first sight seem to be acceptable since the denominator is 2, not zero. However, as explained in chapter 2.7, the dynamics is unstable in this case, meaning that the long-run multiplier is not well defined for the case of $\alpha = -1$.

Hence, while we can use the table to calculate dynamic multipliers also for the cases where the absolute value of $\alpha$ is equal to or larger than unity, the long-run multiplier of a permanent change in $x$ does not exist in this case. Correspondingly: the multipliers of a temporary change do not converge to zero in the case where $-1 < \alpha < 1$ does *not* hold.

Notice that, unlike $\alpha = 1$, the case of $\alpha = 0$ is unproblematic. This restriction, which excludes the autoregressive term $y_{t-1}$ from the model, only serves to simplify the dynamic multipliers. All of the dynamic response in of $y$ then due to the distributed lag in the explanatory variable, and it is referred to by economists as the distributed lag model, denoted DL-model.

So far we have made all derivations assuming that the increase in $x$ is permanent. This is qualitatively different from the examples of market price dynamics in chapter 1, where we analyzed how the price reacted to a temporary shock on the demand side and/or the supply side of the market.

In order to formally derive the multipliers resulting from a temporary change in $x$, there is no longer any gains in exposition by invoking the idea that the $x$'s are functions of $h$. This is because, by definition, a temporary change in period $t$ affects only $x_t$, not $x_{t+1}$ or any other $x$'s further into the future. Hence from (2.1) we have directly that the impact multiplier is

$$\frac{\partial y_t}{\partial x_t} = \beta_1.$$

The second multiplier is by definition the partial derivative of (2.3) with respect to $x_t$:

$$\frac{\partial y_{t+1}}{\partial x_t} = \beta_2 + \alpha \frac{\partial y_t}{\partial x_t}$$

which in the literature is referred to as the *interim multiplier* for the first lag. The third multiplier is then the partial derivative of (2.6):

$$\frac{\partial y_{t+2}}{\partial x_t} = \alpha \frac{\partial y_{t+1}}{\partial x_t}$$

which is called the interim multiplier for the second lag. Note that only the impact multiplier is the same as in the case of a permanent change to $x$. Hence, the sequence of multipliers resulting from a permanent change are different in from the interim multipliers arising from a temporary change. For the interim multipliers, as long as $-1 < \alpha < 1$, each multiplier is seen to become smaller in magnitude than the previous, and asymptotically the multipliers are therefore zero. The interim multipliers are collected in the third column of table 2.1.

Heuristically, the effect of a permanent change in the $x$'s can be viewed as the sum of the changes triggered by a temporary change in period $t$. Indeed, there is an algebraic relationship between the interim multipliers and the dynamic multipliers associated with a permanent change. To see this, note that the 2nd multiplier $\delta_1$ in table 2.1 is

$$\delta_1 = \beta_1 + \beta_2 + \alpha\beta_1$$

which is the same at the sum of the impact multiplier and the first interim multiplier:

$$\frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} = \beta_1 + \beta_2 + \alpha\beta_1 = (1+\alpha)\beta_1 + \beta_2 \equiv \delta_1$$

The third dynamic multiplier is the sum of the impact multiplier and the two next interim multipliers:

$$\begin{aligned} \frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} + \frac{\partial y_{t+2}}{\partial x_t} &= \beta_1 + \beta_2 + \alpha\beta_1 + \alpha(\beta_2 + \alpha\beta_1) \\ &= (1 + \alpha + \alpha^2)\beta_1 + (1+\alpha)\beta_2 \equiv \delta_2 \end{aligned}$$

and generally, for the $j$'th dynamic multiplier:

$$\delta_j = \sum\nolimits_{k=0}^{j} \frac{\partial y_{t+k}}{\partial x_t} = (1 + \alpha + ... + \alpha^j)\beta_1 + (1 + \alpha + ... + \alpha^{j-1})\beta_2, \quad j = 1, 2, ....$$

In reflection of these algebraic relationships, the dynamic multipliers due to a permanent change in the $x$'s, are often referred to as the *cumulated interim multipliers.* Similarly, the long-run multiplier $\delta_{long-run}$ can be rationalized as the infinite sum of the interim multipliers, subject to the condition that $-1 < \alpha < 1$.

In a way, the interim multipliers associated with a temporary change are the more fundamental of the two types of dynamic multiplies that we have considered: If we first calculate the effects of a temporary shock, the dynamic effects of a permanent shock can be calculated afterwards by summation of the impact and interim multipliers.

## 2.2 An empirical example: dynamic effects of increased income on consumption.

We now return to the dynamic consumption function in chapter 1.6, to see what the empirical relationship in equation (1.20) implies about the dynamic response in consumption to a change in income. For this purpose there is no point to distinguish between fitted and actual values of consumption, so we drop the ˆ above $C_t$ in equation (1.20).

Assume that income rises by 1% in period $t$, so instead of $INC_t$ we have $INC_t' = INC_t(1 + 0.01)$. Since income increases, consumption also has to rise. Using (1.20) we have

$$\ln(C_t(1 + \delta_0)) = 0.04 + 0.13 \ln(INC_t(1 + 0.01)) + 0.08 \ln INC_{t-1} + 0.79 \ln C_{t-1}$$

where $\delta_0$ denotes the relative increase in consumption in period $t$, the first period of the income increase. Using the approximation $\ln(1 + \delta_0) \approx \delta_0$ when $-1 < \delta_0 < 1$, and noting that

$$\ln C_t - 0.04 - 0.13 \ln INC_t - 0.08 \ln INC_{t-1} - 0.79 \ln C_{t-1} = 0,$$

we obtain $\delta_0 = 0.0013$ as the relative increase in $C_t$.[1] The immediate effect of a one percent increase in $INC$ is a 0.13% rise in consumption. This is a direct derivation of the impact multiplier which we have just seen can be derived more elegantly by taking the derivative of $\ln C_t$ with respect to $\ln INC_t$:

$$\delta_0 = \frac{\partial \ln C_t}{\partial \ln INC_t} = 0.13.$$

As we also have seen, the effect on consumption in the second period depends on whether the rise in income is permanent, or only temporary. It is convenient to first consider the dynamic effects of a *permanent* shock to income. The second (dynamic) multiplier is then

$$\delta_1 = 0.13 + 0.08 + 0.79 \cdot 0.13 = 0.3125$$

and the third becomes:

$$\delta_2 = 0.13 + 0.08 + 0.79 \cdot 0.3125 = 0.4569.$$

both with reference to the results in table 2.1. By using the recursive relationship between the multipliers, we can easily calculate the percentage increase in consumption at all lag lengths. In this example, the sequence of multipliers are increasing,

[1] If you are unfamiliar with the the approximation $\ln(1 + \delta_0) \approx \delta_0$, see appendix A and the references given there.

Note also that the if we had instead used a linear ADL model (with no log transforms of $C_t$ or $INC_t$) in this example, there would have been no need for the approxiation, and the multipliers would have been exact.

Finally, you may note that in practice the issue of the approximation does not arise, since the multiplier will be obtained by computer simulation which will give the exact numbers in an effcient way, see e.g., section 2.7.2 below.

but the increment $(\delta_j - \delta_{j-1})$ becomes smaller and smaller with increasing $j$. From equation (2.8) this is seen to be due to the multiplication of $\delta_{j-1}$ with the coefficient of the autoregressive term, which is less than 1. Eventually, the sequence of multipliers converges to the *long-run multiplier*. Hence, if only $j$ suitaly large, we can set $\delta_{c,j} = \delta_{c,j-1} = \delta_{c,long-run}$ and obtain

$$\delta_{long-run} = \frac{0.13 + 0.08}{1 - 0.79} = 1.0.$$

According to the estimated model in (1.20), a 1% permanent increase in income has a 1% long-run effect on consumption.

If we instead consider a *temporary* 1% rise in income, equation (1.20) implies a different response (appart from the impact multiplier, which is 0.13). The interim multiplier for the first lag is $0.08 + 0.79 \cdot 0.13 = 0.1827$, and the interim multiplier of the second lag is found to be $0.79 \cdot 0.1827 = 0.14433$, so the interim multipliers are rapidly approaching zero, see table 2.2.

Table 2.2: Dynamic multipliers of the estimated consumption function in (1.20), percentage change in consumption after a 1 percent rise in income.

| | Permanent 1% change | Temporary 1% change |
|---|---|---|
| Impact period | 0.13 | 0.13 |
| 1. period after shock | 0.31 | 0.18 |
| 2. period after shock | 0.46 | 0.14 |
| 3. period after shock | 0.57 | 0.11 |
| ... | ... | ... |
| long-run multiplier | 1.00 | 0.00 |

Note that the second multiplier of the permanent change is equal to the sum of the two first multipliers of the transitory shock $(0.13 + 0.18 = 0.31)$. As we saw above, this is due an an underlying algebraic relationship between the two types of multipliers: multiplier $j$ of a permanent shock is the cumulated sum of the $j$ first interim multipliers associated with a temporary shock.

Figure 2.1 shows graphically the two classes of dynamic multipliers for our consumption function example. Panel a) shows the temporary change in income, and below it, in panel c), you find the graph of the interim multipliers. Correspondingly, panel b) and d) show the graphs with permanent shift in income and the corresponding dynamic multipliers (cumulated interim multipliers).[2]

## 2.3 A typology of single equation linear models

The discussion at the end of the last section suggests that if the coefficient $\alpha$ in the ADL model is restricted to for example 1 or to 0, quite different dynamic behaviour

[2] These graphs were constructed using PcGive and GiveWin, but it is of course possible to use Excel or other programs.

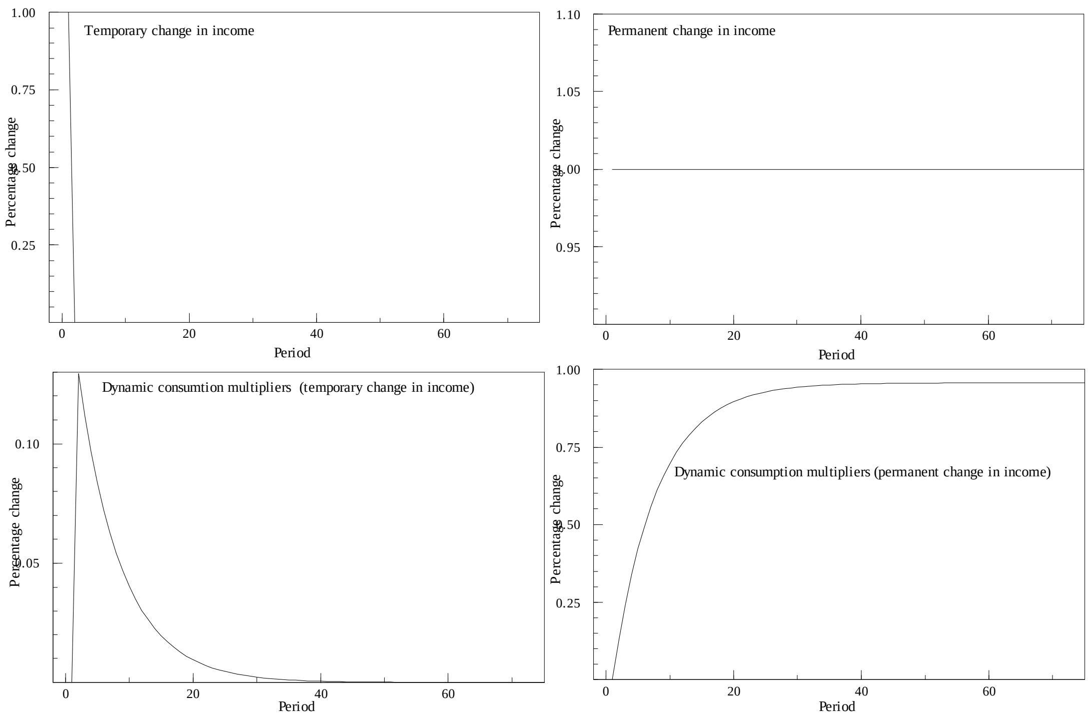


Figure 2.1: Temporary and permanent 1 percent changes in income with associated dynamic multipliers of the consumption function in (1.20).

of $y_t$ is implied. In fact the resulting models are special cases of the unrestricted ADL model. For reference, this section gives a typology of models that are encompassed by the ADL model. Some of these model we have already mentioned, while others will appear later in the book.[3]

Table 2.3 contains three columns, for model Type, defining Equation and Restrictions, where we give the coefficient restrictions that must be true for each models to be a valid simplifications of the ADL. For the ADL model itself no parameter restrictions exists as long as we are interested in all the possible types of solutions for $y_t$, see section 2.7. However, more often than not, economic theory implies that the long-run dynamic multiplier of $y$ with respect to $x_t$ is zero, and in this case we have seen that the restriction $-1 < \alpha < 1$ applies. As section 2.7 will show, this is the same as saying that we restrict our interest to the dynamically stable solutions for $y_t$, which explains why we have entered "$-1 < \alpha < 1$ for *stability*", in the Restrictions column for the ADL.

For the Static model to be a valid simplification of the ADL, both $\beta_2 = 0$ and $\alpha = 0$ must be true. By now, it should be obvious that the danger of using a static model when the restrictions do not hold, is that we get misleading impression of the adjustment lags (the dynamic multipliers). Specifically, the response of $y_t$ to a change in $x_t$ is represented as immediate when it is in fact distributed over several periods.

The autoregressive model (AR in the table), and the random walk model are special cases at the other end of the spectrum, they are so called time series model: In these cases the explanatory variable plays no role for the evolution of $y_t$ through time. The AR model is dynamically stable, while there is no asymptotically stable solution for $y_t$ since the autoregressive coefficient is unity by definition. Nevertheless, the random walk model plays an important role in empirical macroeconomic analysis. In general, it serves as a benchmark against which the performance of other models, with genuine economic content, can be judged. But the random walk model can also be derived from economic theory. Perhaps the most famous example is the random walk model of private consumption.[4] Briefly, the idea is that households relate consumption to their permanent income (or wealth). However, given the information available at the end of time period $t-1$, rational households cannot predict how their income will develop in period $t$, beyond what is already incorporated in the consumption level of period $t-1$, and an average growth rate which is incorporated in $\beta_0$. Hence, the optimal planned consumption level in period $t$ is simply the consumption level of period $t-1$ plus the so called drift term $\beta_0$. Beyond the drift term, changes in consumption from one period to the next is unpredictable on the basis of last periods' information.

For the DL model to be a valid simplification of the ADL, only one coefficient restriction needs to be true, namely $\alpha = 0$. That said, the DL model is also a quite

[3] In fact our typology is partial, and covers only 5 of the 9 models in the full typology of Hendry (1995).

[4] This hypothesis is due to Hall (1978).

Table 2.3: A model typology.

| Type | Equation | Restrictions |
|---|---|---|
| ADL | $y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t.$ | None<br>$-1 < \alpha < 1$ for *stability* |
| Static | $y_t = \beta_0 + \beta_1 x_t + \varepsilon_t.$ | $\beta_2 = \alpha = 0$ |
| AR | $y_t = \beta_0 + \alpha y_{t-1} + \varepsilon_t$ | $\beta_1 = \beta_2 = 0$<br>$-1 < \alpha < 1$ |
| Random walk | $y_t = \beta_0 + y_{t-1} + \varepsilon_t$ | $\beta_1 = \beta_2 = 0$, $\alpha = 1.$ |
| DL | $y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \varepsilon_t$ | $\alpha = 0$ |
| Differenced data[1] | $\Delta y_t = \beta_0 + \beta_1 \Delta x_t + \varepsilon_t$ | $\beta_2 = -\beta_1$, $\alpha = 1$ |
| ECM | $\Delta y_t = \beta_0 + \beta_1 \Delta x_t + (\beta_1 + \beta_2) x_{t-1}$<br>$+(\alpha - 1) y_{t-1} + \varepsilon_t$ | Same as ADL |
| Homogenous ECM | $\Delta y_t = \beta_0 + \beta_1 \Delta x_t$<br>$+(\alpha - 1)(y_{t-1} - x_{t-1}) + \varepsilon_t$ | $\beta_1 + \beta_2 = -(\alpha - 1)$ |

[1] $\Delta$ is the difference operator, defined as $\Delta z_t \equiv z_t - z_{t-1}$.

restrictive model of the dynamic response to a shock, since the whole adjustment of $y_t$ to a change in $x_t$ is completed in the course of only two periods.

The fourth model in Table 2.3, called the Differenced data model is included since it is popular in modern macroeconomics.[5] However, the economic interpretation of the Differenced data model ('growth rate model' if $y_t$ and $x_t$ are in logs) is problematic, since behaviour according to this models can only be rationalized if there are no cost of being out of equilibrium in terms of the levels variables $y_t$ and $x_t$.[6] At present, this implication may not be obvious, but it section 2.5 it will be explained when we review the properties of the two last models in Table 2.3, the error-correction model, ECM,and the Homogeneous ECM. These two models do not have the shortcomings of the Static, DL or Differenced data models. Specifically, the ECM, is consistent with a long-run relationship between $y$ and $x$, and it also describes the behaviour of $y_t$ outside equilibrium. As explained in section 2.5 below, the ECM is a re-parameterization of the ADL, hence none other restrictions applies to the ECM than to the dynamically stable ADL. The Homogenous ECM has the same advantages as the ECM in terms of economic interpretation, but the long-run multiplier is restricted to unity.

As noted, restricting the long-run multiplier to unity is not generic to the ECM, but is implied by some economic theories. One example is so called Purchasing Power Parity (PPP) theory, in which case $y_t$ is the log of the domestic price index (most often the CPI), and $x_t$ is the log of an index of foreign prices, denoted in domestic currency. The hypothesis of purchasing power parity states that the elasticity of the domestic price level with respect to the foreign price level is unity, and the most realistic implementation of the PPP hypothesis is to set $\beta_1 + \beta_2 = (1 - \alpha)$ which implies $\delta_{long-run} = 1$, as we have seen. In section 2.6, a different interpretation of the PPP hypothesis is discussed as well. Other examples of Homogenous ECMs motivated by economic theory include consumption functions that are consistent with a constant long-run saving rate, and theories of wage setting that imply that the wage-share is constant in the long-run, as discussed in Chapter 3.

The ADL model in equation (2.1) is general enough to serve as an introduction to most aspects of dynamic analysis in economics. However, the ADL model, and the dynamic multiplier analysis, can be extended in several directions to provide

[5]For example, Blanchard and Katz (1997) present the standard model of the natural rate of unemployment in U.S.A in the following way (p. 60): *U.S. macroeconometric models...determine the natural rate through two equations, a "price equation" ...and a "wage equation"*. The "wage equation" is a wage Phillips curve, see Chapter 3 below, and the "price equation" is an Differenced data model, with the price growth rate on the left hand side, and the growth rate of wages (or unit labour costs) on the right hand side.

[6]A more technical motivation for the Differenced data model is that, by taking the difference of $y_t$ and $x_t$ prior to estimation, the econometric problem of residual autocorrelation is reduced. However, unless the two restrictions that define the model are empirically acceptable, choosing the Differenced data model rather than the ADL model creates more problems than it solves. For example, the regression coefficient of $\Delta x_t$ will neither be a correct estimate of the impact multiplier $\beta_1$, nor of the long-run multiplier $\delta_{long-run}$. Neverthless, estimation of models with only differenced data, often large systems which are dubbed differenced data vector autoregressive models (dVAR) has become stanadard in applied macroeconomics.

additional flexibility in applications. The most important extensions are:

1. Several explanatory variables
2. Longer lags
3. Systems of dynamics equations

In economics, more than one explanatory variable is usually needed to provide a satisfactory explanation of the behaviour of a variable $y_t$. The ADL model (2.1) can be generalized to any number of explanatory variables. However, nothing is lost in terms of understanding by only considering the case of two exogenous variables, $x_{1,t}$ and $x_{2,t}$. The extension of equation (2.1) to this case is

$$\begin{aligned} y_t = \beta_0 + \beta_{11}x_{1t} + \beta_{21}x_{1t-1} + \\ \beta_{12}x_{2t} + \beta_{22}x_{2t-1} + \alpha y_{t-1} + \varepsilon_t, \end{aligned} \tag{2.10}$$

where $\beta_{ik}$ is the coefficient of the $i'th$ lag of the explanatory variable $k$. The dynamic multipliers of $y_t$ can be with respect to either $x_1$ of $x_2$, the derivation being exactly the same as above. Formally, we can think of each set of multipliers as corresponding to partial derivatives. In applications, the dynamic multipliers of the different explanatory variables are often found to be markedly different. For example, if $y_t$ is the (log of) the hourly wage, while $x_1$ and $x_2$ are the rate of unemployment and productivity respectively, dynamic multipliers with respect to unemployment is usually much smaller in magnitude than the multipliers with respect to productivity for reasons that are explained in chapter 3.

Longer lags in either the $x$'s or in the autoregressive part of the model makes for more flexible dynamics. Even though the dynamics of such models can be quite complicated, the dynamic multipliers always exist, and can be computed by following the logic of section 2.2 above. However, as such calculation quickly become tiring, practitioners use software when they work with such models (good econometric software packages includes options for dynamic multiplier analysis). In this course, we do not consider the formal analysis of higher order autoregressive dynamics.

Often, an ADL equation is joined up with other equations to form dynamic system of equations. For example, assume that $x_{2t}$ is an exogenous explanatory variable in (2.10) but that $x_{1,t}$ is an endogenous variable. This implies that (2.10) is supplemented by a second equation which has $x_{1t}$ on the left hand side and with $y_t$ as one of the variables on the right hand side. Hence, $x_{1t}$ and $y_t$ are determined in a simultaneous two equation system. Another, equally relevant example, is where the equation for $x_{1t}$ contains the lag $y_{t-1}$, but not the contemporaneous $y_t$. Also in this case are $x_{2t}$ and $y_t$ jointly determined, in what is referred to as a *recursive* system of equations. In either case the true multipliers of $y_t$ with respect to the exogenous variable $x_{2t}$, cannot be derived from equation (2.10) alone, because that would make us miss the feed-back that a change in $x_{2,t}$ has on $y_t$ *via* the endogenous variable $x_{1,t}$. To obtain the correct dynamic multipliers of $y$ with respect to $x_1$ we must use the two equation system.

Many more economic examples of dynamic systems, and of how they can be used in analysis, will crop up as we proceed. In particular, section 2.8 takes the dynamic version of the Keynesian multiplier models as one specific example, and the Solow growth model, and a Real Business Cycle model as two other examples. In fact, the reader will have noticed that already in Chapter 1 we analyzed a dynamic system of supply and demand in a single-market, without any formalism though. Other applications of dynamic systems are found in the chapters on wage-price dynamics and on economic policy analysis.

## 2.4 More economic examples of ADL models

The ADL model covers a wide range of economic models, in fact all theories that can be expressed as linear models of causal or future-independent processes. A *causal process* is characterized by its solution (see section 2.7) at time $t$ being independent of future values of the explanatory variable and of the disturbance. Hence future values $\{x_{t+1}, x_{t+2}, ...\}$ and $\{\varepsilon_{t+1}, \varepsilon_{t+2},..\}$ do not affect affect the solution for $y_t$. Although causal models are practical, and therefore widely used in economics and in other disciplines, macroeconomics make use of non-causal models as well. The extension of dynamics to *non-causal processes*, where the dynamically stable solution for $y_t$ depends on $\{x_{t+1}, x_{t+2}, ...\}$ and $\{\varepsilon_{t+1}, \varepsilon_{t+2},..\}$ stems from the forward-looking behaviour which is assumed in many modern theories. In chapter 3.6, we present the New Keynesian Phillips Curve, which is a model of price adjustment that illustrate many of the typical traits of non-causal models. In this section we give examples of standard models that are covered by the causal ADL model.

### 2.4.1 The dynamic consumption function (again)

This has been the main example so far, and in section 2.2 we discussed the dynamics of a log-linear specification in detail. Of course exactly the same analysis and algebra apply to a linear functional form of the consumption function, except that the multipliers will be in units of million kroner (at fixed prices) rather than percentages. In section 2.8 the linear consumption function is combined with the general budget equation to form a dynamic system.

In modern econometric work on the consumption function, more variables are usually included than just income. Hence, there are other multipliers to consider than the ones with respect to $INC_t$. The most commonly found additional explanatory variables are wealth, the real interest rate and indicators of demographic developments, see Erlandsen and Nymoen (2008).

### 2.4.2 The price Phillips curve

In Chapter 3, and several times later in the book, we will consider the so called expectations augmented Phillips curve. An example of such a relationship is

$$\pi_t = \beta_0 + \beta_{11} u_t + \beta_{12} u_{t-1} + \beta_{21} \pi^e_{t+1} + \varepsilon_t. \tag{2.11}$$

$\pi_t$ denotes the rate of inflation, $\pi_t = \ln(P_t/P_{t-1})$, where $P_t$ is an index of the price level of the economy. $u_t$ is the rate of unemployment—or its natural logarithm. The distinction between $u_t$ as the unemployment rate (or percentage), or the log of the rate of unemployment is an example of the care that needs to be taken in choice between different functional forms. If $u_t$ is a rate (as with the unemloyment percentage), the Phillips curve is linear, and the effect of a small change in $u_t$ on $\pi_t$ is independent of the initial unemployment percentage.

If on the other hand $u_t$ is defined as $u_t$ = ln(the rate of unemployment), the Phillips curve is non-linear. In that case, and starting from a low level, a rise in the rate of unemployment leads to a larger reduction in $\pi_t$ than if the initial rate of unemployment is high. The graph of the price Phillips curve, with the rate of unemployment on the horizontal axis, is curved towards the origin (a convex function).[7] In many countries, a convex Phillips curve is known to give a better fit to the data than the linear function. In these economies, a policy that aims at controlling inflation indirectly, via the rate of unemployment, has a better chance of success if the initial level of unemployment is relatively low, compared to the case where the rate of unemployment is relatively high. Thus we see that an apparently technical detail, namely the choice of functional form, has important implications for the effectiveness of policy. This is important to keep this in mind when we later in the this book, for reasons of exposition, often choose to define $u_t$ as the rate of unemployment.

In equation (2.11), the distributed lag in the rate of unemployment captures several interesting economic hypotheses. For example, if the rise in inflation following a fall in unemployment is first weak but then gets stronger, we might have that both $\beta_{11} < 0$ and $\beta_{12} < 0$. On the other hand, some economist have argued the opposite: that the inflationary effects of changes in unemployment are likely to be strongest in the first periods after shock to unemployment, in which case we might expect to find that $\beta_{11} < 0$ while $\beta_{12} > 0$.

Finally, in equation (2.11), $\pi^e_{t+1}$ denotes the expected rate of inflation one period ahead, and in the same manner as earlier in this chapter, $\varepsilon_t$ denotes a random disturbance term. In sum, (2.11) includes two explanatory variables: the rate of unemployment, and the expectation of the future value of the endogenous variable. The rate of unemployment is observable, but expectations are usually not. In order to make progress from (2.11) it is therefore necessary to specify a hypothesis of expectations. The simplest hypothesis is that expectations build on the last observed rate of inflation, hence

$$\pi^e_{t+1} = \tau \pi_{t-1}, \tag{2.12}$$

[7] On Norwegian data, wage Phillips curves, and also the error correction models of wage formation that we encounter below, have alternatively included a reciprocal term, i.e., $1/u_t$, where $u_t$ is the rate of unemployment, or even the double reciprocal $1/u_t^2$. This functional form is "more non-linear" than the log-form. Beyond a certain level of unemployment, there is no more inflation reduction to be hauled from further increases of unemployment, see Johansen (1995).

where the parameter $\tau$ is typically positive, but not larger than 1, i.e., $0 < \tau \leq 1$.[8] Substitution of $\pi^e_{t+1}$ by (2.12), the Phillips curve in (2.11) becomes

$$\pi_t = \beta_0 + \beta_{11} u_t + \beta_{12} u_{t-1} + \alpha \pi_{t-1} + \varepsilon_t, \tag{2.13}$$

which is an ADL equation (with $\alpha = \beta_{21}\tau$).

As explained, the hypothesis in equation (2.12) is just one out of many possible formulations about expectations formation. Alternative specifications give rise to other dynamic models of the rate of inflation. Consider for example a situation where the government have introduced an explicit inflation target, which we denote $\bar{\pi}$. In this case, it is reasonable to believe that firms and households will base their expectations of future inflation on the attainment of the inflation target. In the simplest case we may then set

$$\pi^e_{t+1} = \bar{\pi} \tag{2.14}$$

in place of (2.12). As an exercise, you should convince yourself that equations (2.11) and (2.14) imply an equation for inflation which is an example of a distributed lag model (DL model). More generally, firms and households take into consideration the possibility that future inflation is not exactly on target. Hence they may adopt a more robust forecasting rule, for example

$$\pi^e_{t+1} = (1 - \tau)\bar{\pi} + \tau \pi_{t-1}, \quad 0 < \tau \leq 1. \tag{2.15}$$

In this case, the derived dynamic equation for inflation again takes the form of an ADL model.

### 2.4.3 Exchange rate dynamics

The market for foreign exchange plays a central role in open economy macroeconomics. As always in commodity markets there is a demand and supply side. In the market for foreign exchange, the good in question is foreign currency. Without loss of generality, we can focus on a single country and assume that there is a single foreign country. The price of the good—the nominal exchange rate—is then the price of the foreign currency in terms of the currency of the domestic country. For simplicity we can for example refer to the foreign currency as US dollars, or euro, and to the domestic currency as kroner. The exchange rate is then the price of dollars (or euros) in kroner, for example 7 kroner per dollar.[9]

In the theory of the foreign exchange market a change of emphasis has taken place in recent years. The standard open economy macro model used to treat the net supply of currency to the domestic central bank as a flow variable, primarily determined by the current account. In times of a current account surplus for example,

[8] This fits real word data for the case where we think of $\pi_t$ as the annual rate of inflation. With monthly or quarterly observations of the rate of inflation, seasonal variation calls for more flexible dynamics (more lags) than in (2.11) and/or (2.12).

[9] We refer to Rødseth (2000, Ch 1 and 3.1) for an excellent exposition of the portfolio model of the market for foreign exchange.

there is an excess supply of currency to the central bank. Hence, in the flow model of the market for foreign exchange, a positive net supply of foreign currency, explained by a surplus on the current account, is expected to lead to currency appreciation (in the case of a floating exchange rate regime).

The modern approach to the foreign exchange market does not deny that the needs for currency exchange of exporters and importers make out a part of the net supply of foreign exchange. However, it is a fact that in modern open economies most of the trade in foreign currency is *capital movements.* Domestic and foreign investors buy and sell huge amounts of foreign currency in a drive to change the composition of asset portfolios in a profitable direction. During one day (or week, or month) of trade on the foreign exchange market, the billions of euros traded on the grounds of portfolio decisions totally dominate the amount of trade which can be put down to exports and imports of goods and services. In accordance with the fact that large stocks of wealth can be shifted from one currency to another at any point in time, the modern approach views the net supply of foreign currency as a *stock variable.*

The stock approach to the market for foreign exchange is often called the portfolio model, since the idea is that investors invest their financial wealth in assets that are denominated in different currencies. Investors will change the composition of their financial investments, their portfolios, when expectations about returns change. In the portfolio model, the unit of measurement for quantity is units of foreign currency, for example euros, not euros per units of time as in the flow approach.

In the portfolio model of the foreign exchange market the net supply of foreign currency to the domestic central bank depends in particular on the *risk premium* defined as

$$\textit{risk premium} = i_t - i_t^* - e^e$$

where $i_t$ and $i^*$ are the domestic and foreign nominal interest rates respectively, and $e^e$ denotes the expected rate of depreciation. The risk premium reflects by how much extra the investors get paid over the expected return on euros to take the risk of investing in kroner in the spot foreign exchange market. Heuristically, the net supply of foreign currency to the central bank is a rising function of the risk premium, since investors find kroner a more profitable asset when the risk premium increases. Hence, in a model of a floating exchange rate regime where the net demand of foreign currency is exogenous, in the form of a fixed stock of foreign exchange reserves, the nominal exchange rate (kroner/euro) $E_t$ is increasing in the risk premium. We can represent this basic implication of the portfolio model with the aid of the following relationship:

$$\ln E_t = \beta_0 - \beta_1(i_t - i_t^* - e^e) + \varepsilon_t,\ \beta_1 > 0, \tag{2.16}$$

where $\varepsilon_t$ as usual represents a disturbance, which by the way might be quite large in comparison with the variability of $\ln E_t$, meaning that we cannot expect a particularly close fit between our model and the actual nominal exchange rate. In passing, you may also note that the parameter $-\beta_1$ is called a semi-elasticity. It measures the relative change in $E_t$ due to a unit increase in the risk premium.

When using equation (2.16), it must be understood that $\beta_0$ is only a constant parameter subject to whole range of *ceteris paribus* conditions. For example, a revaluation of investors' wealth (due to a price level shock internationally, for example), will induce a change in $\beta_0$. Moreover, $\beta_0$ can only be seen as invariant to the current account over a limited time period. A current account deficit which lasts for several months, or maybe years, will inevitably affect the stock of net supply of currency, and we can think of this as a gradual increase in $\beta_0$. In this way we can in principle bridge the gap between the stock approach to the foreign exchange rate market, and the older flow approach. Formally, we can write $\beta_0$ as a function of a distributed lag of past current accounts: $\beta_0(CA_t, CA_{t-1}, ...CA_{t-j})$, with negative partial derivatives, and with $j$ normally a quite large number (6-12 months for example). Hence, the link to the current account defines a dynamic model of the exchange rate:

$$\ln E_t = \beta_0(CA_t, CA_{t-1}, ...CA_{t-j}) - \beta_1(i_t - i_t^* - e^e) + \varepsilon_t,\ \beta_1 > 0$$

Another source of dynamics, which is probably quite important in the short time span of say 0-2 years, has to do with expectations. So far we have implicitly assumed that the anticipated rate of depreciation is $e^e$ constant. In reality, the $e^e$ is likely to be highly variable, and to depend of a long list of sentiments and also of macroeconomic variables. However, for modelling purposes, we usually assume that the expected degree of depreciation depend on the level of the exchange rate.[10] If expectations are so called 'regressive', meaning that:

$$e^e = -\tau \ln E_{t-1},\ \tau > 0,$$

the equation for $\ln E_t$ can be written as

$$\ln E_t = \beta_0 - \beta_1(i_t - i_t^*) + \alpha \ln E_{t-1} + \varepsilon_t \tag{2.17}$$

with $\beta_1 > 0$ and $\alpha = -\beta_1\tau < 0$. We recognize equation (2.17) as another example of an ADL model. Note that unlike the earlier examples in this chapter (but in line with the cobweb model of chapter 1), the coefficient of the lagged endogenous variable is negative in the portfolio model with regressive anticipations.

## 2.5 The error correction model

If we compare the long-run multiplier of a permanent shock to the estimated regression coefficient (or elasticity) of a static model, there is often a close correspondence. This is the case in our consumption function example where the multiplier is 1.00 and the estimated coefficient in equation (1.17) is 0.99. This is not a coincidence,

[10] See Rødseth (2000, p 21).

since the dynamic formulation in fact accommodates a so called *steady state* relationship in the form of a static equation. In this sense, a static model is therefore already embedded in a dynamic model.

To look closer at the correspondence between the static model formulation and the steady state properties of the dynamic model, we consider again the ADL model:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t. \tag{2.18}$$

Above, we have emphasized two properties of this model. First it usually explains the behaviour of the dependent variable much better than a static relationship, which imposes on the data that all adjustments of $y$ to changes in $x$ take place without delay. Second, it allows us to calculate the dynamic multipliers. But what does (2.18) imply about the long-run relationship between $y$ and $x$, the sort of relationship that we expect to hold when both $y_t$ and $x_t$ are growing with constant rates, a so called steady state situation?

To answer this question it is useful to re-write equation (2.18), so that the relationship between levels and growth becomes clear. The reason we do this is to establish that changes in $y_t$ are not only caused by changes in $x_t$, but also by last period's deviation between $y$ and the steady state equilibrium value of $y$, which we denote $y^*$. Thus, the period-to-period changes in $y_t$ are correcting past deviations from equilibrium, as well as responding to (new) changes in the explanatory variable. The version of the model which shows this most clearly is known as the error correction model, ECM for short, see Table 2.3 above.

To establish the ECM transformation of the ADL, we need to make two algebraic steps, and to establish a little more notation (related to the concept of the steady state). In terms of algebra, we first subtract $y_{t-1}$ from either side of equation (2.18), and then subtract and add $\beta_1 x_{t-1}$ on the right hand side. This gives

$$\begin{aligned} \Delta y_t &= \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + (\alpha - 1) y_{t-1} + \varepsilon_t \qquad (2.19) \\ &= \beta_0 + \beta_1 \Delta x_t + (\beta_1 + \beta_2) x_{t-1} + (\alpha - 1) y_{t-1} + \varepsilon_t \end{aligned}$$

where $\Delta$ is known as the difference operator, defined as $\Delta z_t = z_t - z_{t-1}$ for a time series variable $z_t$. If $y_t$ and $x_t$ are measured in logarithms (like consumption and income in our consumption function example) $\Delta y_t$ and $\Delta x_t$ are their respective growth rates. Hence, for example, in the consumption function example of section 2.2:

$$\Delta \ln C_t = \ln(C_t/C_{t-1}) = \ln(1 + \frac{C_t - C_{t-1}}{C_{t-1}}) \approx \frac{C_t - C_{t-1}}{C_{t-1}}.$$

because of the approximation explained in appendix A. The model in (2.19) is therefore explaining the growth rate of consumption by, first, the income growth rate and, second, the past levels of income and consumption.

The occurrence of both a variable's growth rate and its level is a defining characteristic of genuinely dynamic models. Frisch, in 1929, offered the following as a definition of dynamics, alongside the definitions due to him that we presented in chapter 1:

> A theoretical law which is such that it involves the notion of rate of change or the notion of speed of reaction (in terms of time) is a dynamic law. All other laws are static.[11]

Since the disturbance term $\varepsilon_t$ is the same in (2.18) as in (2.19) it must be true that everything about the evolution of $y_t$ that is explained by the ADL, is equally well explained by the ECM. For this reason, the transformation from ADL to ECM is often referred to as an "*1-1*" transformation, meaning that the two models represent the same underlying economic behaviour. However, the point of the transformation is that the coefficients of the explanatory variables of the ECM have a particularly useful interpretation. To see this, we first collect the level terms $y_{t-1}$ and $x_{t-1}$ in the second line of (2.19) inside a bracket, as in

$$\Delta y_t = \beta_0 + \beta_1 \Delta x_t - (1-\alpha)\left\{y - \frac{\beta_1+\beta_2}{1-\alpha}x\right\}_{t-1} + \varepsilon_t, \quad -1 < \alpha < 1 \qquad (2.20)$$

which is valid mathematically if $\alpha$ is between $-1$ and 1, as indicated in (2.20). From what we already know about dynamics, it is easy to see why it is important to avoid values of $\alpha$ equal to 1, because that would mean that the coefficient of $x$ is infinite. The reason why it is equally important to avoid $\alpha = -1$ is explained in section 2.7 below, where the nature of unstable solutions of dynamic models is explained.

Let us assume that in the long-run, there is a static relationship between $x$ and the equilibrium value $y$, which we denote $y^*$, hence we postulate that

$$y_t^* = k + \gamma x_t, \qquad (2.21)$$

where $k$ and $\gamma$ are long-run parameters, $\gamma$ in particular being the long-run multiplier of $y$ with respect to a permanent change in $x$. However, we know already from section 2.1, that the long-run multiplier of the ADL model (2.18) is equal to $(\beta_1+\beta_2)/(1-\alpha)$. Hence, we can identify the slope coefficient $\gamma$ in the long term model in the following way

$$\gamma \equiv \frac{\beta_1+\beta_2}{1-\alpha}, \ -1 < \alpha < 1, \qquad (2.22)$$

and the expression inside the brackets in (2.20) can be rewritten as

$$y - \frac{\beta_1+\beta_2}{1-\alpha}x = y - \gamma x = y - y^* + k. \qquad (2.23)$$

Using (2.23) in (2.20) we obtain

$$\Delta y_t = \beta_0 - (1-\alpha)k + \beta_1 \Delta x_t - (1-\alpha)\{y - y^*\}_{t-1} + \varepsilon_t, \quad -1 < \alpha < 1 \qquad (2.24)$$

showing that $\Delta y_t$ is explained by two factors: first the change in the explanatory variable, $\Delta x_t$, and second, the correction of the last period's disequilibrium, the

[11] Frisch (1929) and Frisch (1992, p 394).

deviation between $y_{t-1}$ and last periods equilibrium level $y^*_{t-1}$. Using the representation in (2.24) we see that the autoregressive parameter $\alpha$ is transformed to an adjustment coefficient $-(1-\alpha)$ that measures by how much a past disequilibrium is being corrected by this period's adjustment of $y_t$. Heuristically, the size of $-(1-\alpha)$, which is often referred to as the error correction coefficient, or the equilibrium correction coefficient, can be related to an underlying cost of being out of equilibrium, or, which amounts to the same thing, the cost of adjustment. Indeed, the ECM can be rationalized from economic theory by assuming that agents have a desire for a certain equilibrium combination of the levels variables $y$ and $x$, but that the evolution of $x_t$ cannot be foretold perfectly, and that agents then seeks to minimize quadratic adjustment costs.

With reference to Table 2.3, we note that the Homogenous ECM has exactly the same properties and interpretation as the ordinary ECM, but that the long-run multiplier $\gamma$ is unity according to some relevant economic theory of the long-run relationship between $x_t$ and $y_t$. The other models in the typology in contrast entail restrictions on the speed of adjustment, and on the cost of being out of levels equilibrium.

Consider next a theoretical steady state situation in which growth rates are constants, $\Delta x_t = g_x$, $\Delta y_t = g_y$, and the disturbance term is equal to its average value of zero, $\varepsilon_t = 0$. Imposing this in (2.24), and noting that $\{y - y^*\}_{t-1} = 0$ by definition of a steady state, gives

$$g_y = \beta_0 + \beta_1 g_x - (1-\alpha)k,$$

meaning that the constant term in the long-term relationship (2.21) can be expressed as

$$k = \frac{-g_y + \beta_0 + \beta_1 g_x}{1-\alpha}, \tag{2.25}$$

again subject to a condition $-1 < \alpha < 1$. Often we only consider a static steady states, with no growth, so $g_y = g_x = 0$. In that case $k$ is simply $\beta_0/(1-\alpha)$.

In sum, there is an important correspondence between the dynamic model and a static relationship like (2.21) motivated by economic theory:

1. A theoretical linear relationship $y^* = k + \gamma x$ can be retrieved as the steady state solution of the dynamic model (2.18). This generalizes to theory models with more than one explanatory variable (e.g., $y^* = k + \gamma_1 x_1 + \gamma_2 x_2$) as long as both $x_{1t}$ and $x_{2t}$ (and/or their lags) are included in the dynamic model. In chapter 3 we will discuss some details of this extension in the context of models of wage and price setting (inflation).

2. The theoretical slope coefficient $\gamma$ is identical to the corresponding long-run multiplier (of a permanent increase in the respective explanatory variables).

3. Conversely, if we are only interested in quantifying a long-run multiplier (rather than the whole sequence of dynamic multipliers), it can be found by using the identity in (2.22).

Returning to another insight in this section, we note that the transformation of the ADL model into levels and differences is often referred to as "the error correction transformation". The name reflects that according to the model, $\Delta y_t$ corrects past deviation from the long-run equilibrium relationship. Error correction models became popular in econometrics in the early 1980s. Since $\Delta y_t$ is actually bringing the level of $y$ towards the long-run relationship, a better name may be equilibrium correction model, and some authors now use that term consistently, see Hendry (1995). However the term error correction model seems to have stuck.

The ECM, not only helps clarify the link between dynamics and the theoretical steady state, it also plays an essential role in econometric modelling of non-stationary time series. In 2003, when Clive Granger and Rob Engle were awarded the Noble Price in economics, part of the motivation was their finding that so called cointegration between two or more non-stationary variables implies error correction, and vice versa.

In common usage, the term error correction model is not only used about equation (2.20), where the long-run relationship is explicit, but also about (the second line) of (2.19). One reason is that the long-run multipliers (the coefficients of the long-run relationship) can be easily established by estimating the linear relationship in (2.20), and then calculating the ratio $\gamma$ in (2.22). Direct estimation of $\gamma$ requires a non-linear estimation method.

## 2.6 The two interpretations of static equations

In the first chapter we discussed the definitional differences between static models and dynamic models. One of the conclusions was that there are two different interpretations of static relationships in macroeconomics:

1. As representations of actual behaviour in macroeconomics, and
2. as corresponding to long-run, steady state, relationships.

In terms of the concepts we have developed in this chapter, the first interpretation corresponds to simplifying the ADL model

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t.$$

by setting $\beta_2 = 0$ and $\alpha = 0$ as in the Static model in the typology in table 2.3. In this interpretation, a static model are seen to be a special case of the ADL model. Since there are no reason to believe *a priori* that the two parameters are empirically found to be zero very often, the static special case is also a restrictive model relative to the ADL. In economic terminology, and as we have noted in the first chapter, the validity of the static model hinges on the assumption that the speed of adjustment is so fast that explicit dynamics can be omitted from the model. In more operational terms we may perhaps put it this way: Sometimes, we can assume (either because it is realistic or because we are willing to simplify) that the dynamic adjustment process is so fast that the adjustment to a change in an exogenous variable is completed within the period of analysis. Hence, at least as a first approximation, we do not need to formulate the dynamic adjustment process explicitly.

One example of this rationale for a static model is the simple Keynesian income-expenditure model. As the reader will know, this model is based on an assumption of fixed wages and prices. When we use the income expenditure model, the focus is on the aggregate short-run effects of for example a rise in government expenditure, and for this purpose, the results of the analysis is believed not to be seriously distorted by the abstraction from price-wage dynamics, which in industrialized economies normally is more sluggish than the response in emplyment and output. It is understood that the relevance of this rationalization depends on the comparative and historical context, and on the stage of evolution of the economy which is the subject to the analysis. Broadly speaking, the assumption is only realistic for economies where the short-run aggregate supply curve is relatively horizontal—for technological and/or institutional reasons. The model is not relevant for historical or contemporary agricultural economies for example, or for industrialized economies that have been damaged by wars or by the collapse of institutions, and which therefore have become hyper-inflation economies. The Weimar republic in Germany after the First World War is an important historical example. Zimbabwe between 1998 and 2008 is a contemporary example—and there are many more.

If the time horizon of the analysis is longer than 1-2 years, the assumption of fixed wage and price levels becomes untenable, also in the case of well functioning industrialized economies. The loss of relevance from ignoring for example the effects of inflation on the real interest rate and on the rate of foreign exchange, is also increasing with the time horizon of the analysis. Hence, the explicit modelling of wage and price dynamics is necessary if the model is to be relevant for fiscal and monetary policy analysis over the business cycle, i.e., a 5 to 10 year period. Chapter 3 therefore gives a comprehensive introduction to dynamic models of wage and price formation.

The second interpretation of a static model allows $\beta_2$ and $a$ to take values different from zero, and in fact only hinges on $\alpha$ being "different from one". This second interpretation of static models is exactly the interpretation that we have developed so far in this chapter, and in section 2.5 on the ECM in particular. This interpretation applies to all asymptotically stable dynamic models, where a model's implied steady state relationships represents an equilibrium situation. That equilibrium can be represented mathematically with the aid of static (timeless) equations of the type given above. Frisch's formulation from 1929 already contained this interpretation of static models as long-relationships:

> Therefore static laws basically express what would happen in *the long-run* if the static theory's assumptions prevailed long enough for the phenomena to have time to have time to react in accordance with these assumptions.[12]

[12] Frisch (1992, p 395), Frisch's italics.

Clearly, to avoid confusion, the two interpretations of a static model should not be mixed-up. Nevertheless, the two interpretation are often confounded, and it is not uncommon that static equations are first presented as long-run relationship, but then, at some point in the argument the same relationship is used as a short-run model, often in a rather complicated system-of-equations. In chapter 3, we show that a standard way of rationalizing a modern Phillips curve relationship suffers from lack of precision in this respect.

Another example which shows that care must be taken when we choose an operational interpretation of generally formulated economic hypotheses, is provided by the theory of Purchasing Power Parity, PPP. This hypothesis is about the behaviour the real exchange rate, $\sigma$, which is defined as

$$\sigma = \frac{EP^*}{P}$$

where $P^*$ represents the foreign price level, $E$ is the nominal exchange rate (kroner/euro) and $P$ is the domestic price level. The purchasing power parity condition (PPP) used in many macroeconomic models says that "the real exchange rate is constant".[13] This is trivial if the empirical counterparts of $E$, $P^*$ and $P$, i.e. the observable time series $E_t$, $P_t^*$ and $P_t$, are constant, but even a casual glance at the data shows that they are not. So, to be able to put the PPP hypothesis to use (or to test), we must ask about the time perspective of the PPP condition. If it is taken as a short-run proposition, he real exchange rate is constant from one period to the next, $\sigma_t = \sigma_{t-1} = k$. In the case of exogenous $E_t$, like in a fixed exchange rate regime, and with an exogenous foreign price level, PPP is seen to imply the following model of the domestic price level $P_t$:

$$\ln P_t = -\ln k + \ln E_t + \ln P_t^*, \tag{2.26}$$

a static price equation, which says that the effect of a currency change in period $t$ on the domestic price level is full and immediate in period $t$. Hence, in this interpretation, the short-run elasticities of the domestic price level with respect to its determinants, $E_t$ and $P_t^*$, are identical and equal to the long-run elasticities. All elasticities are unity. In the exchange rate literature, this case is referred to as the case of full and immediate *pass-through*.

If on the other hand, PPP is interpreted as a long-run proposition, we have instead the interpretation that $\sigma_t = \bar{\sigma}$ in a steady state situation, and thus

$$\Delta \ln P_t = g_E + \pi^* \tag{2.27}$$

where $g_E$ and $\pi^*$ denote the long-run constant growth rates of the nominal exchange rate and of foreign prices. In this interpretation there is full long-run pass-through, but the PPP hypothesis has no implications for the short-run *pass-through*, i.e. for the dynamic multipliers. However, (2.27) is often confounded with the relationship

$$\Delta \ln P_t = \Delta \ln E_t + \Delta \ln P_t^*$$

[13] For a concise introduction to the purchasing power parity, see Rødseth (2000, p 261).

which is a Differenced data model in terms of the typology in table 2.3, and so holds no implication for the long-run dynamics.

## 2.7 Solution and simulation of dynamic models

The reader will have noted that the existence of a finite long-run multiplier, and thereby the validity of the correspondence between the ADL model and long-run relationships, depends on the autoregressive parameter $\alpha$ in (2.18) being different from unity. In section 2.7.1 we show that the parameter $\alpha$ is also crucial for the nature and type of solution of equation (2.18). Several of the insights obtained by considering the solution of the ADL equation (2.18) in some detail, carry over to more complicated equations as well as to systems of equations.

### 2.7.1 The solution of ADL equations

In order to discuss the solution of ADL models, it is useful to first consider the simpler case of a deterministic autoregressive model, where we abstract from the distributed lag part ($x_t$ and $x_{t-1}$) of the model, as well as from the disturbance term ($\varepsilon_t$). One way to achieve this simplification of (2.18) is to assume that both $x_t$ and $\varepsilon_t$ are fixed at their respective constant averages:

$\varepsilon_t = 0$ for $t = 0, 1, ...,$ and

$x_t = m_x$ for $t = 0, 1....$

Hence we follow convention and assume that each $\varepsilon_t$, representing a small and random influence on $y$, has a common mean of zero. For the explanatory variable $x_t$ the mean is denoted $m_x$. We write the simplified model as

$$y_t = \beta_0 + Bm_x + \alpha y_{t-1}, \text{ where } B = \beta_1 + \beta_2. \tag{2.28}$$

In the following we proceed *as if* the the coefficients $\beta_0$, $\beta_1$, $\beta_2$ and $\alpha$ are known numbers. This is a simplifying assumption which allows us to abstract from estimation issues, which in any case belong to a course in time series econometrics.

We assume that equation (2.28) holds for $t = 0, 1, 2, ....$ It is usual to refer to $t = 0$ as the *initial period* or the *initial condition*. The assumption we make about the initial period is crucial for the existence and uniqueness of a solution. An important theorem is the following: If $y_0$ is a fixed and known number, then there is a unique sequence of numbers $y_0, y_1, y_2, ...$ which is the solution of (2.28). This can be seen by induction: Consider first $t = 0$: From the assumption of known initial conditions, $y_0$ then follows. Next, set $t = 1$ in (2.28), and $y_1$ is seen to determined uniquely since we already know $y_0$. And so on: having established $y_{t-1}$ we find $y_t$ be solving (2.28) one period forward. This procedure is also known as solving the equation *recursively* from known initial conditions.

You may note that, unlike other "conditions for solution" in other areas of mathematical economics, the requirement of fixed and known initial conditions is almost trivial, since $y_0$ is simply given "from history". If we open up for the possibility that the initial condition is not determined by history, but that it can "jump" at any point in time, there are other solutions to consider. Such solutions play a role in macroeconomics, as the solutions of the non-causal model mentioned above. A full treatment of the solution of non-causal models belongs to more mathematically more advanced courses in macrodynamics. However, we will give specific examples in chapter 3.6 when we discuss the forward-looking New Keynesian price Phillips curve.

The algebraic properties of the solution of (2.28) carry over to other, less stylized, cases so it is worth considering in more detail. Using the recursive procedure as just explained, we obtain the solution for the three first periods:

$$\begin{aligned}
y_1 &= \beta_0 + Bm_x + \alpha y_0 \\
y_2 &= \beta_0 + Bm_x + \alpha y_1 \\
&= \beta_0(1+\alpha) + Bm_x(1+\alpha) + \alpha^2 y_0 \\
&= (\beta_0 + Bm_x)(1+\alpha) + \alpha^2 y_0 \\
y_3 &= \beta_0 + Bm_x + \alpha y_2 \\
&= (\beta_0 + Bm_x)(1+\alpha+\alpha^2) + \alpha^3 y_0.
\end{aligned}$$

As can be seen there is a clear pattern, which is repeated from period to period, and which allows us to express the solution $\{y_1, y_2, ...\}$ of (2.28) compactly as

$$y_t = (\beta_0 + Bm_x)\sum_{s=0}^{t-1} \alpha^s + \alpha^t y_0, \quad t = 1, 2, ... \tag{2.29}$$

for the case of known initial condition $y_0$. Equation (2.29) is a useful reference for discussion of the three types of solution of ADL models, namely the stable, unstable and explosive solutions. We start with the stable solution, and then consider the two others.

#### 2.7.1.1 Stable solution

The condition

$$-1 < \alpha < 1 \tag{2.30}$$

is the necessary and sufficient condition for the existence of a globally asymptotically *stable* solution. The stable solution has the characteristic that asymptotically there is no trace left of the initial condition $y_0$. From (2.29) we see that as the distance in time between $y_t$ and the initial condition increases, $y_0$ has less and less influence on the solution. When $t$ becomes large (approaches infinity), the influence of the initial

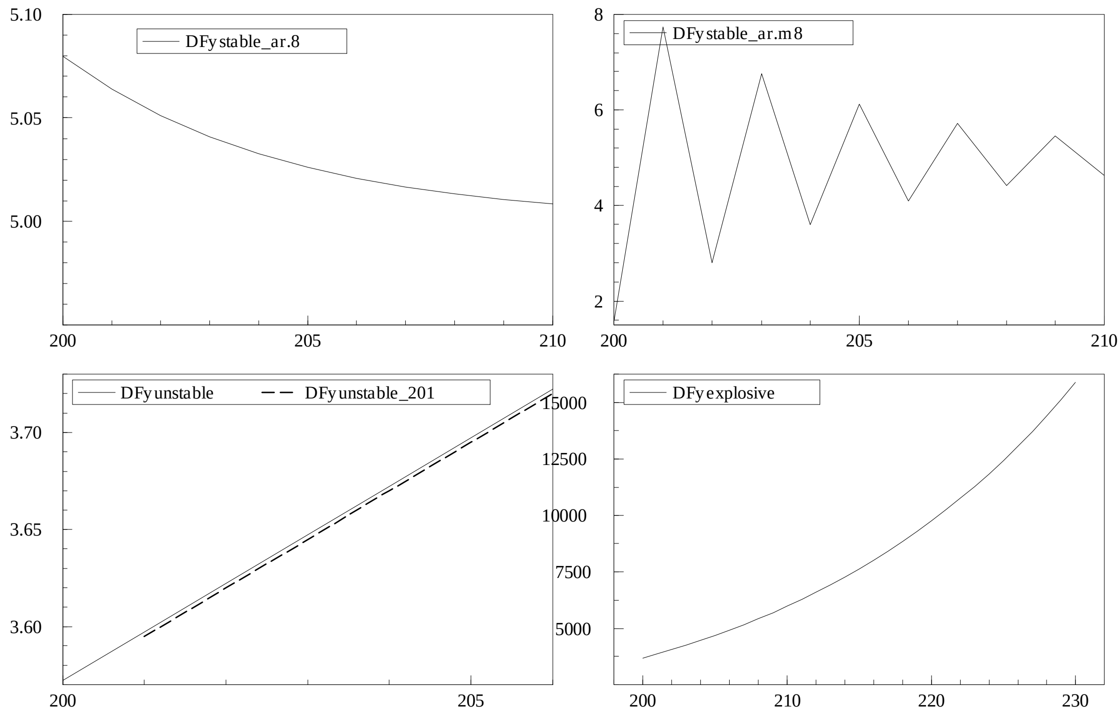


Figure 2.2: Panel a) and b): Two stable solutions of (2.28), (corresponding to positive and negative values of $\alpha$. Panel c): Two unstable solutions (corresponding to different initial conditions). Panel d). An explosive solution, See the text for details about each case.

condition becomes negligible. Since $\sum_{s=0}^{t-1} \alpha^s \to \frac{1}{1-\alpha}$ as $t \to \infty$, we have asymptotically:

$$y^* = \frac{(\beta_0 + Bm_x)}{1-\alpha} \tag{2.31}$$

where $y^*$ denotes the equilibrium of $y_t$. As stated, $y^*$ is independent of $y_0$.

Assume that there is permanent change in $x_t$. Such a change can be implemented as a shift in the mean $m_x$. Keeping in mind that $B = \beta_1 + \beta_2$, the derivative of $y^*$ with respect to a permanent change in $x$ is

$$\partial y / \partial m_x = (\beta_1 + \beta_2)/(1-\alpha),$$

which corresponds to the long-run multiplier of $y_t$ with respect to a permanent change in $x_t$. Finally, note also that (2.25) above, although derived under different assumption about the exogenous variable (namely a constant growth rate), is compatible with (2.31).

The stable solution can be written in an alternative and very instructive way. Note first that by using the formula for the sum of the $t-1$ first terms in a geometric progression, $\sum_{s=0}^{t-1} \alpha^s$ can be written as

$$\sum_{s=0}^{t-1} \alpha^s = \frac{1-\alpha^t}{(1-\alpha)}.$$

Using this result in (2.29), and next adding and subtracting $(\beta_0 + Bm_x)\alpha^t/(1-\alpha)$ on the right hand side of (2.29), we obtain

$$\begin{aligned} y_t &= \frac{(\beta_0 + Bm_x)}{1-\alpha} + \alpha^t(y_0 - \frac{\beta_0 + Bm_x}{1-\alpha}) \\ &= y^* + \alpha^t(y_0 - y^*), \text{ when } -1 < \alpha < 1. \end{aligned} \tag{2.32}$$

In the stable case, the dynamic process is essentially correcting the initial discrepancy (disequilibrium) between the initial level of $y$ and its long-run level.

Panel a) of Figure 2.2 shows a numerical example of a stable solution of equation (2.28). In the example we have set $\alpha = 0.8$ and we have have chosen values of $\beta_0$ and $Bm_x$ so that $y^* = 5$. We use computer generated data and the initial period (corresponding to $t = 0$ in the formulae) is $t = 200$. The initial value is $y_{200} = 5.08$. Since the initial value is higher than the equilibrium value, and $\alpha$ is positive, the solution approaches $y^* = 5$ from above.

Panel b) of the figure illustrates another stable solution, namely the solution for the case of a negative autoregressive coefficient, $\alpha = -0.8$. In this case the initial value is $y_{200} = 1.59$, which is markedly lower than the equilibrium of $y^* = 5$. According to equation (2.32) the solution for period 201 becomes

$$y_{201} = 5 - 0.8 \times (1.59 - 5) = 7.73,$$

and the graph is seen to confirm this. Due to the negative autoregressive coefficient the solution characteristically oscillates towards the equilibrium value.

#### 2.7.1.2 Unstable solution

When $\alpha = 1$, we obtain from equation (2.29):

$$y_t = (\beta_0 + Bm_x)t + y_0, \ \ t = 1, 2, ... \tag{2.33}$$

showing that the solution contains a linear trend and that the initial condition exerts full influence over $y_t$ even over infinitely long distances. There is of course no well defined equilibrium where $y_t$ settles in the long run, and neither is there a finite long-run multiplier. Nevertheless, the solution is perfectly valid mathematically speaking: given an initial condition, there is one and only one sequence of numbers $y_1$, $y_2, ...y_T$ which satisfy the model.

The instability of the solution is however apparent when we consider not a single solution but a *sequence* of solutions. Assume that we first find a solution conditional on $y_0$, and denote the solution $\{y_1^0, y_2^0, ...y_T^0\}$. Note that this is in in fact a *forecast* for the all the periods $t = 1, ..., T$ conditional on the information about the economy which is contained in $y_0$. After one period we will usually want to recalculate the solution because something we did not anticipate occurred in period 1, making $y_1^0 \neq y_1$ and we will want to condition the new forecast on the observed $y_1$. The updated solution is $\{y_2^1, y_3^1, ...y_T^1\}$ since we now condition on $y_1$. From (2.33) we see that as long as $y_1^0 \neq y_1$ (the same as saying that $\varepsilon_1 \neq 0$) we will have $y_2^0 - y_2^1 \neq 0$, $y_3^0 - y_3^1 \neq 0, ..., y_T^0 - y_T^1 \neq 0$. Moreover, when the time arrives to condition on $y_3$, the same phenomenon is going to be observed again. The solution is indeed unstable in the sense that any (small) change in initial conditions have a permanent effect on the solution.

It common that economists like to refer to this phenomenon ($\alpha = 1$) as *hysteresis.* In the literature on European unemployment, the point has been made that failure of wages to respond properly to shocks to unemployment (in fact the long-run multiplier of wages with respect to unemployment is often found to be close to zero ) may lead to hysteresis in the rate of unemployment. The case of $\alpha = -1$ is less common in applications, but it is still useful to check the solution and dynamics implied by (2.29) also in this case.

Panel c) of Figure 2.2 illustrates the unstable case by setting $\beta_0 + Bm_x = 0.025$ and $\alpha = 1$ in (2.28), corresponding to for example 2.5% annual growth if $y_t$ is a variable in logs. Actually there are two solutions. One takes $y_{200} = 3.57$ as the initial period, and the other is conditioned by $y_{201} = 3.59$. Although there is a relatively small difference between the two initial conditions in this example, the lasting influence on the different initial values on the solutions is visible in the graph.

#### 2.7.1.3 Explosive solution

When $\alpha$ is greater than unity in absolute value the solution is called explosive, for reasons that should be obvious when you consult (2.29). In panel d) of Figure 2.2 the explosive solution obtained when setting $\beta_0 + Bm_x = 0.025$ and $\alpha = 1.05$ in (2.28) is shown. We plot the explosive solution over a longer period than the others, in order to allow the explosive nature of the solution to become clearly visible in the graph.

At this point it is worth recalling that everything we have established about the existence and properties of solutions, have been based on making equation (2.18):

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t$$

subject to the two simplifying assumptions: $\varepsilon_t = 0$ for $t = 0, 1, ...,$ and $x_t = m_x$ for $t = 0, 1....$ Luckily, the *qualitative* results (stable, unstable or explosive solution) are quite general and independent of which assumptions we make about the disturbance and the $x$ variable. However, any particular numerical solution that we obtain for (2.18) is conditioned by these assumptions. For example, in the stable case with

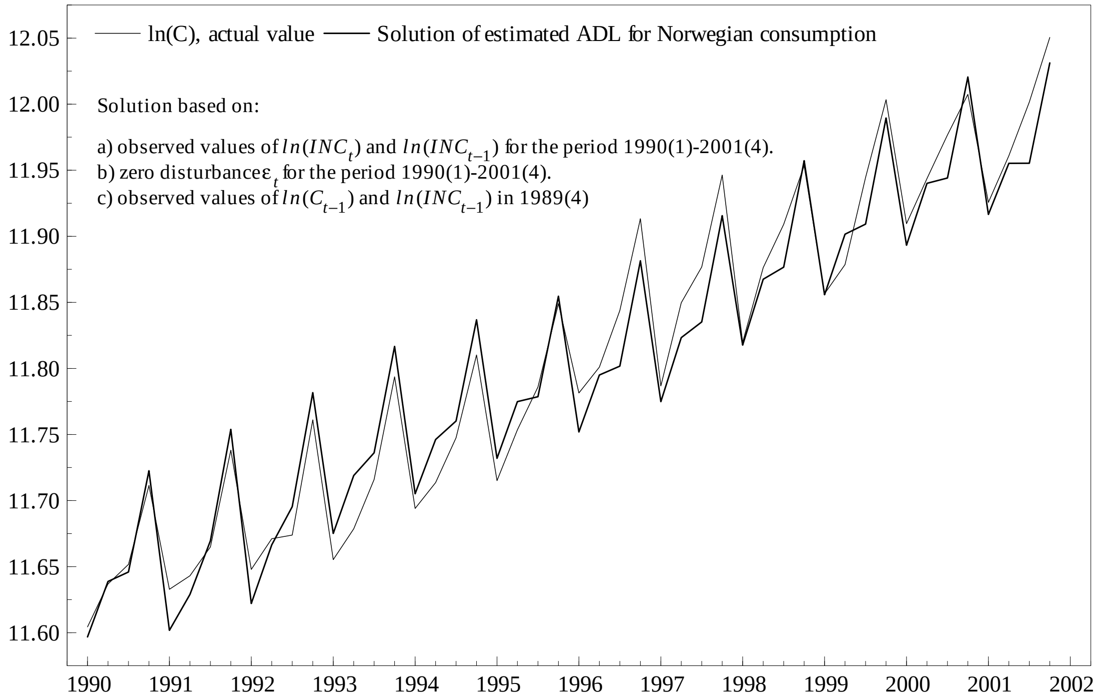


Figure 2.3: Solution of the consumption function in equation (1.20) for the period 1990(1)-2001(4). The actual values of $log(C_t)$ are also shown, for comparison.

$0 < \alpha < 1$, if we instead of $x_t = m_x$ for $t = 0, 1....$ set $x_t = 2m_x$ for $t = 0, 1...$, then both $y^*$ and the solution path from $y_0$ to $y^*$ are affected. Hence the particular solutions that we obtain for ADL equations are conditioned by the values we specify for the disturbances $\varepsilon_t$ $(t = 0, 1, ...)$, and the economic explanatory variables $x_t$ $(t = 0, 1...)$.

Regarding the disturbances, the common practice is to do exactly as we have done above, namely to replace each $\varepsilon_t$ by zero, the mean the disturbances. An important exception is in a forecasting situation when a structural break has been identified Forecasters then use non-zero disturbances in order to avoid unnecessary poor forecasts.

For the explanatory variable, there are several possibilities depending on the purpose of the analysis. For example, if the aim of the analysis is to investigate how well the solution fits the behaviour of $y$ over an historical period, we simply use the observed values of $x$ over the sample period. If the purpose is to produce a forecast of $y$, we have to specify the future values taken by $x$ in the forecast period. These values are of course not observable and have to be forecasted themselves. In the case with little information about future $x$ values, one typically revert to the mean of $x$, in order to at least condition the $y$ forecasts on a representative value of $x$.

Figure 2.3 uses the empirical consumption function in equation (1.20) above to illustrate a particular solution of an ADL model. The point here is to illustrate how well the solution fits the actual data over the sample period, so the solution is conditioned by the actual observations for income. The figure shows the solution for (the logarithm) of consumption over the period 1989(1)-2001(4) together with the observed actual values of consumption over that period. The scale of the vertical axis indicates that real private consumption expenditure has increased by a formidable 50% over the period, and the graph shows that this development is well explained by the solution of the consumption function. That the solution is conditioned by the true values of income, must be taken into account when assessing the close capture of the trend-growth in consumption.

Also apparent from the graph is a marked seasonal pattern: consumption is highest in the 3rd and 4th quarter of each year. Seasonality is a feature of many quarterly or monthly time series. How to represent seasonality in a model is question in its own right and cannot be answered in any detail within the scope of this book. However the interested reader might note that in order to capture seasonality as well as we do in the figure, we have included so called seasonal dummies, but for simplicity, the estimated coefficient have been suppressed in equation (1.20).[14]

### 2.7.2 Simulation of dynamic models

In applied macroeconomics it is not always common to refer to the solution of a dynamic model. The customary term is instead simulation. Specifically, economists use *dynamic simulation* to denote the case where the solution for period 1 is used to calculate the solution for period 2, and the solution for period 2 is in turn used to find the solution for period 3, and so on. In the case of a single ADL equation with one lag, which we have studied extensively above, dynamic simulation clearly amounts to finding the solution of that model. Conveniently, the correspondence between solution and dynamic simulation also holds for equations with several lags and/or several explanatory variables, as well as for the systems of dynamic equations that are used in macroeconomic practice. The dominance of simulation stems from the fact that simulation offers an easy way of finding the solution in the cases where the closed form algebraic solution is impractical.

Reflecting the practical usefulness of simulation in macroeconomics is the fact that modern econometric software packages contain a wide range of built-in possibilities for simulation analysis, as well as rich possibilities for reporting of the simulation results, in numerical or in graphical form. Depending on the purpose of the simulation, it is custom to distinguish between three main types of simulation analyses. These are: track-analysis, forecasting, and multiplier analysis.

When we do track-analysis, simulation is used to find how fell the solution of an estimated macroeconomic model tracks the actual values of the endogenous variables over the sample period) and figure 2.3 provides an example. Dynamic simulation

[14] Seasonal dummies are explained in any introductory textbook in econometrics. To reproduce the detailed results underlying Figure 2.3, consult the file norcons.zip on the web page.

is also the practical way produce forecasts from dynamic models. However, a main complication that distinguishes a historical dynamic simulation from a dynamic forecast is that, since observations of the $x$ variable are only available for the historical sample period ($t = 1, 2, ...T$) they also need to be forecasted before we can simulate $y_t$ for a $H$ periods ahead. $H$ is often referred to as the forecast horizon. Having fixed a set of values $x_{T+1}, x_{T+2}, ..., x_{T+H}$, the simulated values $y_{T+1}, y_{T+1}, ..., y_{T+H}$, correspond to the solution based on the initial values $x_T, x_{T-1.}$ and $y_{T-1}$.

When forecasting, the determination of the future sequence $x$ are is not at all a trivial matter. Sometimes, the forecaster characterizes the sequence of $x$ values as representing a possible future, not necessarily the most likely future realizations of $x$'s. In this case, the forecast is usually referred to as a scenario of the future, and the forecaster often choose to illustrate two or more scenarios for the forecasted variable $y$, corresponding to for example "high" and "low" future realizations of $x$. The other main motivation for a forecasts is to obtain the most likely future development of $y_{T+j}$ and $x_{T+j}$, given the information available in period $T$. In this case, the equation for $y_t$ has to be supplemented with an equation for $x_t$, and the dynamic simulation of that 2-equation model based on $y_T$ and $x_T$ (and knowledge of the parameters) represent the forecast. Such forecasts are called conditional forecasts, because they are conditioned by the history of the system consisting of the equations for $y_t$ and $x_t$ up to period $T$. Another much used term is that the conditional forecast correspond to rational expectations.

Finally, dynamic simulation is used to produce dynamic multipliers. In that case we do two dynamic simulations: first, a *baseline* simulation using a baseline or reference set of values of the $x$ variable (often these are the historical values, or a baseline forecast), and second, an alternative simulation based on an alternative (or "shocked") set of $x$ values. As always, the initial condition has to be specified before the solution can be found. Since the focus is on the effect of a change in $x$, the same initial condition is used for both the baseline and the alternative simulation. If we denote the baseline solution $y_t^b$ and the alternative $y_t^a$, the difference $y_t^a - y_t^b$ are the dynamic multipliers.

Referring back to section 2.2, you will now appreciate that what we did there was nothing but doing dynamic simulation "by hand", in order to obtain the multipliers of consumption with respect to (permanent and temporary changes) in income. Figure 2.1, in contrast is produced "automatically" by the built in multiplier option in the computer programme PcGive, and in exercise 9 you are invited to replicate that graph.

We end this section by noting that care must be taken to distinguish between static simulation, which is also part of the vocabulary used by macroeconomists, and dynamic simulation which we have discussed so far. In a static simulation, the actual (not the simulated) values for $y$ in period $t-1$ is used to calculate $y_t$. Thus the sequence of $y$'s obtained from a static simulation does *not* correspond to the solution of the ADL model. Static and dynamic simulations give an identical result only for the first period of the simulation period. Since, in practice, we use estimated values of the model's coefficient when we simulate a model, the resulting $y-$values from a static simulation over the sample period are identical to the equation's fitted values.[15]

## 2.8 Dynamic systems

In macroeconomics, the effect of a shock or policy change is usually dependent on system properties. As a rule it is not enough to consider only one equation in order to obtain the correct dynamic multipliers. Consider for example the consumption function of section 1.6 where it was assumed that income ($INC$) was an exogenous variable. This exogeneity assumption is only tenable given some further assumptions about the rest of the economy: for example if there is a general equilibrium with flexible prices and the supply of labour is fixed, then output and income may be regarded as independent of $C_t$. However, with sticky prices and idle resources, i.e., the Keynesian case, $INC$ must be treated as logically be considered as an endogenous variable. Generally, because the different sectors and markets of an economy are interlinked, the limitations of the single equation analysis we have considered so far in this chapter is rather obvious.

Nevertheless, the discussion if the solution of a single ADL equation provides essential background for understanding dynamic models, so our efforts so far are not wasted. First, it is often quite easy to bring a system on a form with two reduced form dynamic equations that are of the same form that we have considered above. After this step, we can derive the full solution of each endogenous variable of the system if we so want. Section 2.8.1, 2.8.3 and 2.8.4 show specific examples. Second, in more complicated cases where a full analysis of the system is beyond us (at least without the aid of computer simulation), it is still relatively easy to find the short-run and long-run multipliers of the endogenous variables with respect to changes in the exogenous variables, by drawing on the distinction between the short-run and long-run version of the models that we introduced in chapter 1.4, and which we re-state in section 2.8.2 below. Third, as already hinted, computer simulation represents a practical way of obtaining the solution of systems of dynamic equations with quantified coefficient values and known initial conditions, for i.e., the systems of equations used in practice for forecasting and policy analysis, and the concepts that we have introduced above are essential in the interpretation of the results from such computer based simulation.

### 2.8.1 A dynamic income-expenditure model

We begin by showing how the analysis of the dynamic consumption function with is changed when the assumption of exogenous income is replaced by an assumption of endogeous income. For this purpose it is most practical to specify a consumption

[15] Strictly speaking, this is only true for single equation models. If we consider a system of simultaneous equations, the fitted values of a single endogenous variable are different form the solved out values from a static simulation.

function which is linear in variables, rather than being linear only in parameters as we have used frequently above. Hece, the model is made up of a linear consumption function

$$C_t = \beta_0 + \beta_1 INC_t + \alpha C_{t-1} + \varepsilon_t, \tag{2.34}$$

and a product market equilibrium condition

$$INC_t = C_t + J_t, \tag{2.35}$$

where $J_t$ denotes autonomous expenditure, and where $INC$ is now interpreted as the gross domestic product, GDP. We assume that there are idle resources (unemployment) and that prices are sticky, so that income is determined "from the demand side". The 2-equation dynamic system has two endogenous variables $C_t$ and $INC_t$, while $J_t$ and $\varepsilon_t$ are exogenous.

To find the solution for consumption, substitute $INC$ from (2.35), and obtain

$$C_t = \tilde{\beta}_0 + \tilde{\alpha} C_{t-1} + \tilde{\beta}_2 J_t + \tilde{\varepsilon}_t \tag{2.36}$$

where $\tilde{\beta}_0$ and $\tilde{\alpha}$ are the original coefficients divided by $(1-\beta_1)$, and $\tilde{\beta}_2 = 1/(1-\beta_1)$, and $\tilde{\varepsilon}_t = \varepsilon_t/(1-\beta_1)$.

Equation (2.36) is yet another example of an ADL model, so the theory of the previous sections applies. For a given initial condition $C_0$ and known values for the two exogenous variables (e.g., $\{J_1, J_2, ..., J_T\}$) there is a unique solution. If $-1 < \tilde{\alpha} < 1$ the solution is asymptotically stable. The impact multiplier of consumption with respect to autonomous expenditure is $\tilde{\beta}_2$, while in the stable case, the long-run multiplier is $\tilde{\beta}_2/(1-\tilde{\alpha})$. If the system (2.34) and (2.35) implies a stable solution for $C_t$, there is obviously also a stable solution for $INC_t$, meaning that we do not have to derive the solution for $INC_t$ in order to check the dynamic stability of income.

Equation (2.36) is called the *final equation* for $C_t$. The defining characteristic of a final equation is that (apart from exogenous variables) the right hand side only contains lagged values of the left hand side variable. It is often feasible to derive a final equation for more complex system than the one we have studied here. The conditions for stability is then expressed in terms of the so called characteristic roots of the final equation. The relationship between $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and the characteristic roots goes beyond the scope of this course, but it can be mentioned that a sufficient condition for stability of a final equation with second order dynamics (i.e., not only $y_{t-1}$ but also $y_{t-2}$ is part of the equation for $y_t$) is that both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are less than one. Section 2.8.5 below states these points in a more general framework.

### 2.8.2 The method with a short-run and long-run model

It is worth reminding ourselves about the analysis of dynamic models that was introduced in chapter 1.4. When we are working with models with more than two endogenous variables, or with two or more lags, we are often unable to derive the full solution of the dynamic model, at least without the aid of the mathematical algorithms provided by relevant computer programmes. However, by using the simplified step-wise approach, we can still answer two important questions: i) What are the short-run effect of a change in an exogenous variable, and ii) what are the long-run effects of the shocks. The method, as we saw in chapter 1.4, amounted to using two versions of the model: a short-run version, and a long-run version of the model.

In the income-expenditure model the *short-run model* is given by (2.34) and (2.35), where $C_t$ and $INC_t$ are endogenous variables, $C_{t-1}$ is a predetermined exogenous variable, and $J_t$ and $\varepsilon_t$ are exogenous variables. The long-run model is defined for the situation of $C_t = \overline{C}$, $INC_t = \overline{INC}$, $J_t = \overline{J}$ and $\varepsilon_t = 0$, and is written as

$$\overline{C} = \frac{\beta_0}{1-\alpha} + \frac{\beta_1}{1-\alpha}\overline{INC} \qquad (2.37)$$

$$\overline{INC} = \bar{C} + \overline{J} \qquad (2.38)$$

In exactly the same manner as in chapter 1.4, the impact multiplier of a permanent (or temporary) rise in $J_t$ is obtained from the short-run model (2.34) and (2.35), while subject to dynamic stability, the long-run multipliers can be derived from the static system method (2.37)-(2.38). It is a useful exercise to check that the long-run multipliers from the long-run model are the same as the ones you obtained from the final equations of $C_t$ and $INC_t$.

### 2.8.3 The basic Solow growth model

Growth theory is often the economics student's first encounter with a dynamic model, and here we only briefly review the simplest Solow growth model using the framework developed above.[16] The first equation of the model is the macro production function of the Cobb-Douglas type,

$$Y_t = K_t^{\gamma} N_t^{1-\gamma}\ , \ \ 0 < \gamma < 1 \qquad (2.39)$$

where $Y_t$ is GDP in period $t$, $K_t$ denotes the capital stock, and $N_t$ is employment in period $t$. In the basic version of them model we simply define $N_t$ as the number of employed persons, which in turn is identical to the size the population. Hence full employment is assumed, and we abstract from variations in working time. The size of the population is assumed to grow with a constant rate $n$:

$$\frac{N_t}{N_{t-1}} - 1 = n, \ \ n \geq 0 \qquad (2.40)$$

We have to be precise about the dating of the capital stock, since the third equation of the model links the evolution of the capital stock to the flow of investment. In

[16] A good reference to both the basic version of the Solow model, and to the different extensions of the model with e.g., technological progress is the textbook by Birch Sørensen and Whitta-Jacobsen (2005). The basic Solow model is for example presented in Birch Sørensen and Whitta-Jacobsen (2005, Chapter 3.2).

accordance, with section (1.5) above, we define $K_t$ as the amount of capital available at the start of period $t$. We next assume that all saving, $S_t$, in period $t$ is invested so that the "law of motion" for the capital stock is

$$K_t = (1-\delta)K_{t-1} + S_{t-1},\ 0 < \delta \leq 1 \tag{2.41}$$

where $\delta$ is the rate of depreciation of capital, consistent with (1.13) in chapter 1. In the Solow model, an essential assumption is that saving is proportional to income, hence

$$S_t = sY_t, \qquad 0 \leq s < 1 \tag{2.42}$$

where $s$ is the fraction saved out of income in each period.

Let us first see what the steady state solution of this dynamic model looks like, assuming that a unique and stable steady state solution exists. To formulate the *long-run model*, we first note that if we want to write the model in terms of variables that are independent of time in steady state, we cannot use $Y$, $N$ and $K$ directly since population is growing with rate $n$ in steady state, by assumption. Instead, let us make the inital guess that capital intensity $k_t = K_t/N_t$ is a variable that is a constant $\bar{k}$ in steady state. If this is true, the long-run model, i.e., the model for the steady state takes the form

$$\bar{y} = \left(\bar{k}\right)^{\gamma} \tag{2.43}$$

$$\bar{k} = \frac{1}{\delta} s \cdot \bar{y} \tag{2.44}$$

where GDP per capital is denoted $y = Y/N$, and $\bar{y}$ is the steady state value of GDP per capita. If (2.43) and (2.44) characterizes the steady state, we can combine the two equations to obtained the following equation for $\bar{y}$

$$\bar{y} = \left(\frac{1}{\delta} s \cdot \bar{y}\right)^{\gamma} \tag{2.45}$$

which shows that a permanent increase in the saving rate $s$ has the following long-run effect on GDP per capita:

$$\frac{d \ln \bar{y}}{d \ln s} = \frac{\gamma}{1-\gamma} > 0, \tag{2.46}$$

or, in derivative form:

$$\frac{d\bar{y}}{ds} = \frac{\bar{y}}{s}\frac{\gamma}{1-\gamma} \tag{2.47}$$

which says that a permanent increase in the saving rate has a positive long run effect on GDP per capita, and that the effect is larger the larger $\bar{y}$ is in the initial steady state situation.

Next, we address the more demanding task of checking whether the long-run model made up of (2.43) and (2.44) does indeed represent a stable steady state solution of the Solow model (2.39)-(2.42). For that purpose we need to derive the final equation for the dynamic system. We start with equation (2.41) and divide on both sides by $N_{t-1}$:

$$\frac{K_t}{N_{t-1}} = (1-\delta)\frac{K_{t-1}}{N_{t-1}} + \frac{S_{t-1}}{N_{t-1}}.$$

On the left-and side, multiply by $N_t/N_t$, and use (2.40) and (2.42) to obtain:

$$k_t(1+n) = (1-\delta)k_{t-1} + s\,y_{t-1} \tag{2.48}$$

where $k_t = K_t/N_t$ and $y_t = Y_t/N_t$, consistent with the variables of the long-run model. From the production function (2.39): $y_t = k_t^{\gamma}$ so that the dynamic equation for the capital intensity variable $k_t$ becomes:[17]

$$k_t = \frac{1}{(1+n)}\left\{(1-\delta)k_{t-1} + s\,k_{t-1}^{\gamma}\right\}. \tag{2.49}$$

If it were not for the last term on the right hand side in (2.49), this dynamic equation for $k_t$, which is also the *final equation* for the dynamic Solow system, would have been an autoregressive (AR) model , with autoregressive parameter $\alpha = (1-\delta)/(1+n)$. In terms of economic interpretation this corresponds to the case of $s = 0$, so that all income is consumed in every period. Since $0 < \alpha < 1$, it would then follow that $k_t \to \bar{k} = 0$ in the asymptotically stable solution, and the interpretation would be that from any given initial capital intensity $\bar{k}_0$, the combination of capital depreciation and population growth would drive the capital intensity towards zero. Consequently GDP per head is also zero. Hence, to avoid such a dismal steady state, a strictly positive saving fraction is logically necessary. With $0 < s < 1$, we see that the second term on the right hand side of (2.49) is indeed essential. This term represents the positive contribution from saving to the capital stock, and if it large enough the capital intensity $k_t$ can grow from one period to the next in spite of capital depreciation and population growth. Because of the nature of the production function (2.39), this part of the final equation is a non-linear function of $k_t$ , meaning that (2.49) become a non-linear autoregressive model, unlike the linear models we make use of elsewhere in this book.

Despite the non-linearity, it is straight forward to understand the conditions for stability in (2.49). First subtract $k_{t-1}$ on both sides to obtain

$$(1+n)\Delta k_t = -(\delta+n)k_{t-1} + s\,k_{t-1}^{\gamma} \tag{2.50}$$

and note that

$$\begin{array}{lcl} \Delta k_t > 0 & \iff & s\,k_{t-1}^{\gamma} > (\delta+n)k_{t-1} \\ \Delta k_t < 0 & \iff & s\,k_{t-1}^{\gamma} < (\delta+n)k_{t-1} \\ \Delta k_t = 0 & \iff & s\,k_{t-1}^{\gamma} = (\delta+n)k_{t-1} \end{array} \tag{2.51}$$

The first line in (2.51) states that the capital intensity is growing in all time periods where saving *per capita* is larger than the amount of saving required to compensate

[17] cf. equation (29) in Chapter 3.2 in Birch Sørensen and Whitta-Jacobsen (2005).

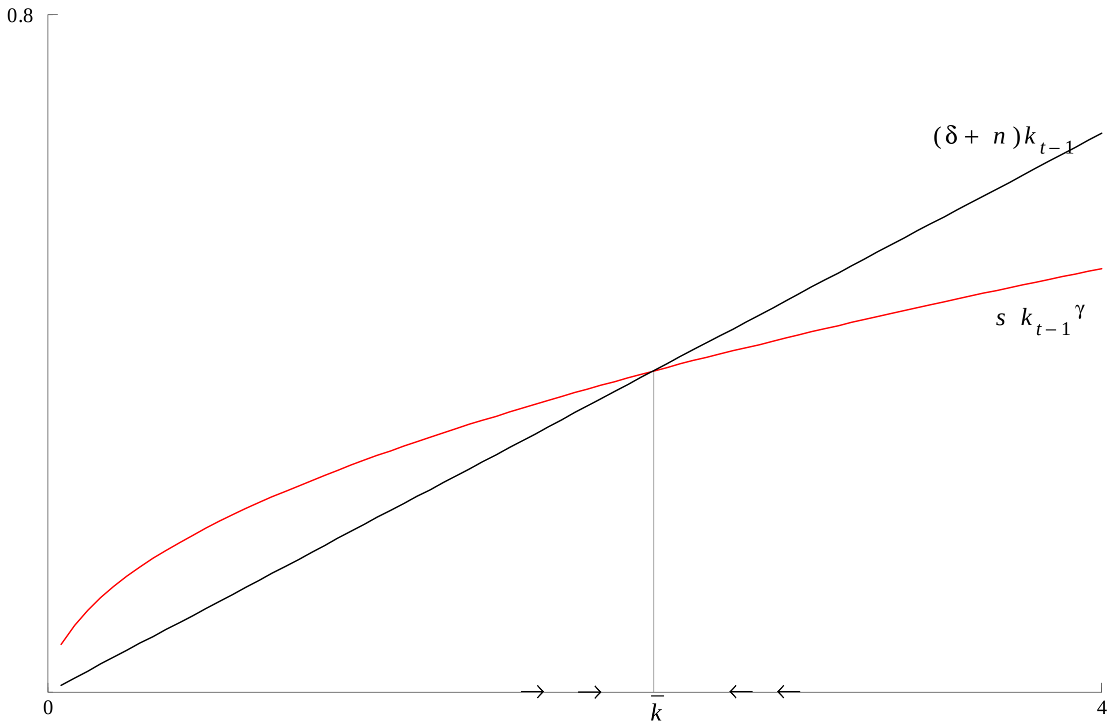


Figure 2.4: Solow growth model: IIlustration of equation (2.50) for the case of $\delta + n = 0.165$) and $s = 0.25$ and $\gamma = 0.5$.

for capital depreciation and population growth. Conversely, line two states that the capital intensity is falling if saving *per capita* is less than that required amount. Finally, $\Delta k_t = 0$ so that $k_t = k_{t-1} = \bar{k}$ when $s\,k_{t-1}^{\gamma} = (\delta + n)k_{t-1}$.

Figure 2.4 illustrates how the dynamics of the capital intensity variable $k_t$ drives the Solow growth model. The two lines represent the first and second terms on the right hand side of the final equation (2.50). The point where the two lines cross defines the (non zero) steady state value $\bar{k}$ of the capital intensity (on the horizontal axis). The figure also illustrates dynamic stability. If the initial capital intensity $k_0$ is below $\bar{k}$, the solution for capital intensity variable will be an increasing sequence towards $\bar{k}$. Conversely, if $k_0 > \bar{k}$, the following periods will be characterized by negative values of $\Delta k_t$, and the capital intensity will then approach $\bar{k}$ from the initially higher value (from the right on the horizontal axis).

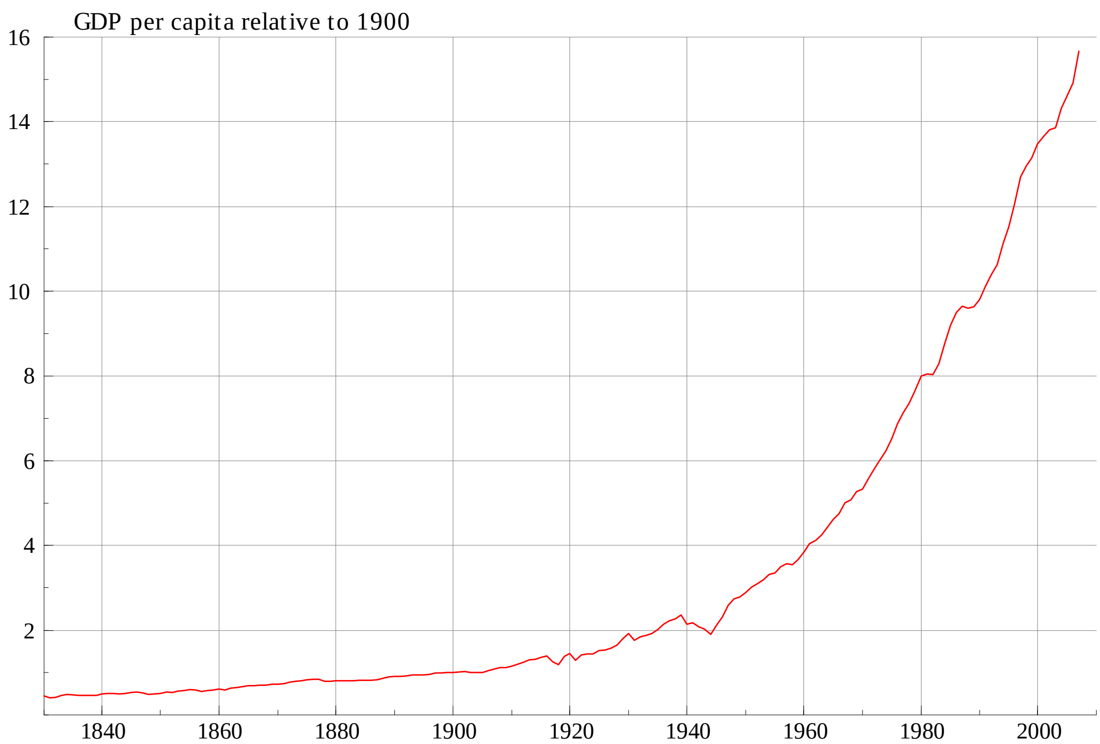


Figure 2.5: GDP per capita relative tol1900. Annual Norwegian data 1830-2007. GDP is in constant 2000 prices.

Figure 2.4 is based on the parameter values that are given in the figure caption, and for values of $k_t$ that vary between 0.05 and 4. It usually helps the understanding to draw a similar graph for some different values of the theoretical parameters. For example: It is easy to see that a higher $s$ than 0.25 will shift the intersection point upwards along the line for capital depreciation and population growth. Likewise, a higher value of $\gamma$ increases the productivity of capital, meaning that $\bar{k}$ and $\bar{y}$ are increased.

Figure 2.4 also shows that the non-linearity is most important when the initial capital intensity is "far from" the steady state value, particularly on to the left of the steady state value $\bar{k}$ . For initial capital intensities that are close to the steady state value, we can therefore approximate the exact dynamics of $k_t$ with the aid of a linear dynamic model. In other words, local dynamic beahviour around the steady state can be modelled using a linear model of the same type that we use elsewhere in the book. To illustrate this point, using the principles of linearization, a so called first order appoximation to (2.49) becomes

$$k_t \approx \left(\frac{1-\delta}{1+n} + \frac{s}{1+n}\gamma\bar{k}^{\gamma-1}\right) k_{t-1} + s\frac{1-\gamma}{1+n}\bar{k}^{\gamma} \tag{2.52}$$

which is an AR equation.[18] Assuming that the autoregressive coefficient is between 0 and 1 (2.52) is consistent with dynamic stability in the neighborhood of the steady state capital intensity $\bar{k}$.

Solow's growth model is the standard model for economic analysis with a time horizon that goes beyond the length of the typical business-cycle of 5-10 years. In order to be able to explain the facts of economic growth the basic model that we have presented here obviously needs to be replaced by a model that incudes more growth explaining factors. As one example, Figure 2.5 shows that GDP per capita in Norway has grown by a factor of 16 since 1900, while the basic Solow model indicates zero growth in steady state. However, we leave it to a course in economic growth theory to show that inclusion of technological progress is one of the modifications that help reconcile the theory's predictions with growth statistics. Instead, we turn to a model that applies the Solow model's framework to a much shorter time horizon than was originally intended.

### 2.8.4 A real business cycle model

The real business cycle (RBC) approach applies the framework of the Solow model the business cycle. So unlike the Solow model, where the time period $t$ typically refers to a 5 or 10 years averages, the time period $t$ in the RBC refers to years or quarters of a year.

The Keynesian income-expenditure model and the RBC model are regarded a contesting explanations of short-run macroeconomic fluctuations. This is because of the differences in assumptions, in particularly regarding the labour market and unemployment. In the Keynesian model, there is involuntary unemployment, the real wage does not correspond to the market clearing real wage of a perfectly competitive labour market, and the business cycle is regarded as disequilibrium phenomenon. In the RBC model, there is no genuine involuntary unemployment. Instead, recorded unemployment is regarded as a misnomer for intertemporal substitution of working time for leisure, and the business cycle is an equilibrium phenomenon.

[18] Appendix B gives the details.

The first equation of the RBC model is the macro production function (2.39), augmented by a technology variable $A_t$

$$Y_t = K_t^{\gamma}(A_t N_t)^{1-\gamma} \ , \ \ 0 < \gamma < 1, \tag{2.53}$$

and where we, because of the change of time horizon and the change in focus to business-cycle fluctuations, change the interpretation of $N_t$ from persons employed to the total number of hours worked, i.e., the number of workers times the average length of the working day. This specification of the production function is referred to as labour augmenting technical progress, since an increase in $A_t$ implies that GDP is increased without any increase in the capital stock (labour becomes more productive).[19] It is important in the following that technical progress is modelled as the sum of two parts: one completely deterministic part and a second part which is random. The deterministic part is a given rate of technological progress $g_A$ multiplied by time, which we write $g_A t$, and the random part is denoted $a_{s,t}$. Hence, the RBC model's theory of technological progress is given by

$$\ln A_t = g_A t + a_{s,t}, \quad 0 < g_A < 1, \tag{2.54}$$

where the random part is given by the autoregressive equation:

$$a_{s,t} = \alpha a_{s,t-1} + \varepsilon_{a,t}, \quad 0 \leq \alpha < 1 \tag{2.55}$$

where $\varepsilon_{a,t}$ is a completely unpredictable technology shock. We can amalgamate the two technology equations into one by lagging (2.54) one period, and then multiplying by $\alpha$ on both sides of the lagged equation:

$$\alpha \ln A_{t-1} = \alpha g_A(t-1) + \alpha a_{s,t-1}. \tag{2.56}$$

Subtraction of equation (2.56) from (2.54) gives

$$\ln A_t = \alpha g_A + g_A(1-\alpha)t + \alpha \ln A_{t-1} + \varepsilon_{a,t}, \tag{2.57}$$

which shows that technological progress $\ln A_t$ is implied to follow an autoregressive model augmented by a deterministic trend $\alpha g_A(t-1)$. We can re-write (2.57) as

$$\ln A_t - \ln A_{t-1} = \alpha g_A + g_A(1-\alpha)t + (\alpha - 1)\ln A_{t-1} + \varepsilon_{a,t}, \tag{2.58}$$

and define a steady state as a situation where all shocks are zero, i.e., $\varepsilon_{a,t} = 0$ for all $t$. Let $\ln \bar{A}_t$ denote steady state productivity. From (2.57) with $\varepsilon_{a,t} = 0$, we note that $\ln \bar{A}_{t-1}$ is given by

$$g_A = \alpha g_A + g_A(1-\alpha)t + (\alpha - 1)\ln \bar{A}_{t-1}$$

or

$$\ln \bar{A}_{t-1} = -g_A + g_A t.$$

[19] We adopt more or less the same assumptions as in the expositition in Birch Sørensen and Whitta-Jacobsen (2005, Chapter 19.4).

since $\ln \bar{A}_t = g_A + \ln \bar{A}_{t-1}$ we have

$$\ln \bar{A}_t = g_A t \tag{2.59}$$

showing that (2.54) can alternatively be written as

$$\ln A_t = \ln \bar{A}_t + a_{s,t}, \tag{2.60}$$

as long as the restriction on $g_A$ is observed.

The RBC model makes uses also of the saving equation (2.42), and the capital evolution equation (2.41), which we however simplify by setting $\delta = 1$, meaning that capital equipment lasts for only one period, so that

$$K_t = S_{t-1}. \tag{2.61}$$

It is the infallible mark of RBC models that the labour marked is assumed to be in equilibrium in each period. Labour supply, $N_t^S$, is a function of the relative wage $w_t/\bar{w}_t$:

$$N_t^S = \bar{N}(\frac{w_t}{\bar{w}_t})^{\epsilon}, \quad \epsilon > 0. \tag{2.62}$$

where $\bar{w}_t$ is the steady state wage and $\epsilon$ is the labour supply elasticity. When the wage is equal to the steady state wage, labour supply is also equal to its steady state value. Though simple in appearance, equation (2.62) is representing optimizing behaviour by households and individuals: They choose to supply labour in excess of the long-run level determined by demography and sociological norms and legislature when the real wage $w_t$ is higher than the steady state real wage $\bar{w}_t$, and to substitute labour for leisure in times when the real wage is low relative its long run level. For simplicity of exposition we set $\bar{N} = 1$ in the following, since in (2.62) this is merely a choice of units.

Labour demand is obtained by assuming optimizing behaviour by a 'macro producer', and the marginal product of labour is therefore set equal to the real wage:

$$w_t = (1-\gamma)\left(\frac{K_t}{A_t N_t}\right)^{\gamma} A_t = (1-\gamma)\left(\frac{Y_t}{N_t}\right). \tag{2.63}$$

The assumption of equilibrium in the labour market means that $N_t^S = N_t$, and we can solve (2.62) and (2.63) for real wage and for employment. First, note that from (2.62):

$$w_t = N_t^{\frac{1}{\epsilon}} \bar{w}_t \tag{2.64}$$

Next, note that the ratio $K_t/A_t N_t$ in (2.63) is a generalization of the capital intensity variable $k_t$ of the Solow model. The generalization is due to the inclusion of the productivity $A_t$ variable in the RBC production function, hence we define $k_{A,t} = K_t/A_t N_t$ as the productivity corrected capital intensity. Moreover, with reference to the Solow model above, we define $\bar{k}_A$ as the steady state value of the productivity corrected capital intensity. The corresponding steady state real wage, from (2.63), is

$$\bar{w}_t = (1-\gamma)\bar{k}_A^{\gamma}\bar{A}_t. \tag{2.65}$$

Remember that both $\bar{w}_t$ and $\bar{k}_A$ refer to the situation with $\varepsilon_{a,t} = 0$ (there are no shocks in the steady state). Using (2.64) and (2.65) we obtain

$$w_t = N_t^{\frac{1}{\epsilon}}(1-\gamma)\bar{k}_A^{\gamma}\bar{A}_t. \tag{2.66}$$

A this point we take the natural logarithms on both sides of (2.63), and (2.66):

$$\begin{aligned} \ln w_t &= \ln(1-\gamma) + \ln Y_t - \ln N_t, \\ \ln w_t &= \frac{1}{\epsilon}\ln N_t + \ln((1-\gamma)\bar{k}_A^{\gamma}\bar{A}_t). \end{aligned}$$

Using these two expressions to solve for $\ln N_t$ gives

$$\ln N_t = \frac{\epsilon}{1+\epsilon}\left\{\ln Y_t - \ln(\bar{k}_A^{\gamma}\bar{A}_t)\right\}. \tag{2.67}$$

Substitution of the this expression together with (2.61) and (2.42) into the log of the production function, (2.53), gives

$$\ln Y_t = \gamma\ln(sY_{t-1}) + (1-\gamma)\left(\frac{\epsilon}{1+\epsilon}\left\{\ln Y_t - \ln(\bar{k}_A^{\gamma}\bar{A}_t)\right\}\right) + (1-\gamma)\ln A_t$$

$$\frac{1+\gamma\epsilon}{1+\epsilon}\ln Y_t = \gamma\ln s + \gamma\ln(Y_{t.-1}) + (1-\gamma)\left(\frac{-\epsilon}{1+\epsilon}\ln(\bar{k}_A^{\gamma}\bar{A}_t)\right) + (1-\gamma)\ln A_t$$

and eventually

$$\ln Y_t = \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon}\ln s + \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon}\ln Y_{t-1} - \frac{(1-\gamma)\epsilon}{1+\gamma\epsilon}\ln(\bar{k}_A^{\gamma}\bar{A}_t) + \frac{(1-\gamma)(1+\epsilon)}{1+\gamma\epsilon}\ln A_t. \tag{2.68}$$

Note that this is the *final equation* for $\ln Y_t$ in the RBC model: it expresses $\ln Y_t$ by its lag and by exogenous variables. Noting that $\ln \bar{A}_t$ and $\ln A_t$ are linked through equation (2.60) above, and that $\ln \bar{A}_t = g_A t$ in (2.59) above, the final equation for $\ln Y_t$ can be written as

$$\ln Y_t = \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon}\ln s - \frac{(1-\gamma)\epsilon}{1+\gamma\epsilon}\ln(\bar{k}_A^{\gamma}) + \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon}\ln Y_{t-1} + \frac{(1-\gamma)}{1+\gamma\epsilon}g_A t + \frac{(1-\gamma)(1+\epsilon)}{1+\gamma\epsilon}a_{s,t} \tag{2.69}$$

where $a_{s,t}$ follows the autoregressive process in equation (2.55) above.

There are several important notes to be made about (2.69). First, dynamic stability hinges on the autoregressive parameter being less than one in magnitude. We see that in (2.69) the stability condition is satisfied, since

$$\frac{\gamma(1+\epsilon)}{1+\gamma\epsilon} = \frac{\gamma(1+\epsilon)+1-1}{1+\gamma\epsilon} = 1 - \frac{1-\gamma}{1+\gamma\epsilon}$$

is a number between 0 and 1, based the assumptions of the model. Second, given stability, there is a deterministic steady state growth path for $Y_t$ with growth rate $g_A$. Third, and heuristically speaking, the only difference between (2.69) and the implied equation for the log of the steady state GDP, $\ln \bar{Y}_t$, is that the equation for $\ln \bar{Y}_t$ does not contain the stochastic technology shock term.[20] Hence, the equation for $\ln \bar{Y}_t$ is:

$$\ln \bar{Y}_t = \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon} \ln s - \frac{(1-\gamma)\epsilon}{1+\gamma\epsilon} \ln(\bar{k}_A^{\gamma}) + \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon} \ln \bar{Y}_{t-1} + \frac{(1-\gamma)}{1+\gamma\epsilon} g_A t, \quad (2.70)$$

meaning that the dynamics of the logarithm of the *output gap*, defined as $(\ln Y_t - \ln \bar{Y}_t)$, is given by the autoregressive equation:

$$(\ln Y_t - \ln \bar{Y}_t) = \frac{\gamma(1+\epsilon)}{1+\gamma\epsilon}(\ln Y_{t-1} - \ln \bar{Y}_{t-1}) + \frac{(1-\gamma)(1+\epsilon)}{1+\gamma\epsilon} a_{s,t}. \quad (2.71)$$

This equation shows that, according to the RBC model, the typical evolution of GDP over time will be characterized by period of economic booms (positive output-gap), and troughs (negative output-gap). This is implied even if the initial situation is characterized by $\ln Y_t = \ln \bar{Y}_t$, and the explanation is that there is a flow of technology shocks, that are propagated into persistent deviations from the steady state by saving and investment dynamics, and by workers willingness to supply more labour in good times, and to substitute work by leisure in economic downturns. Hence, unlike the Keynesian income-expenditure model, periods with below capacity output, and below average recorded employment, is an equilibrium phenomenon in the RBC model—it is a theory of equilibrium business cycles.

[20] This can be made precice by taking the mathematical expectation of (2.69).

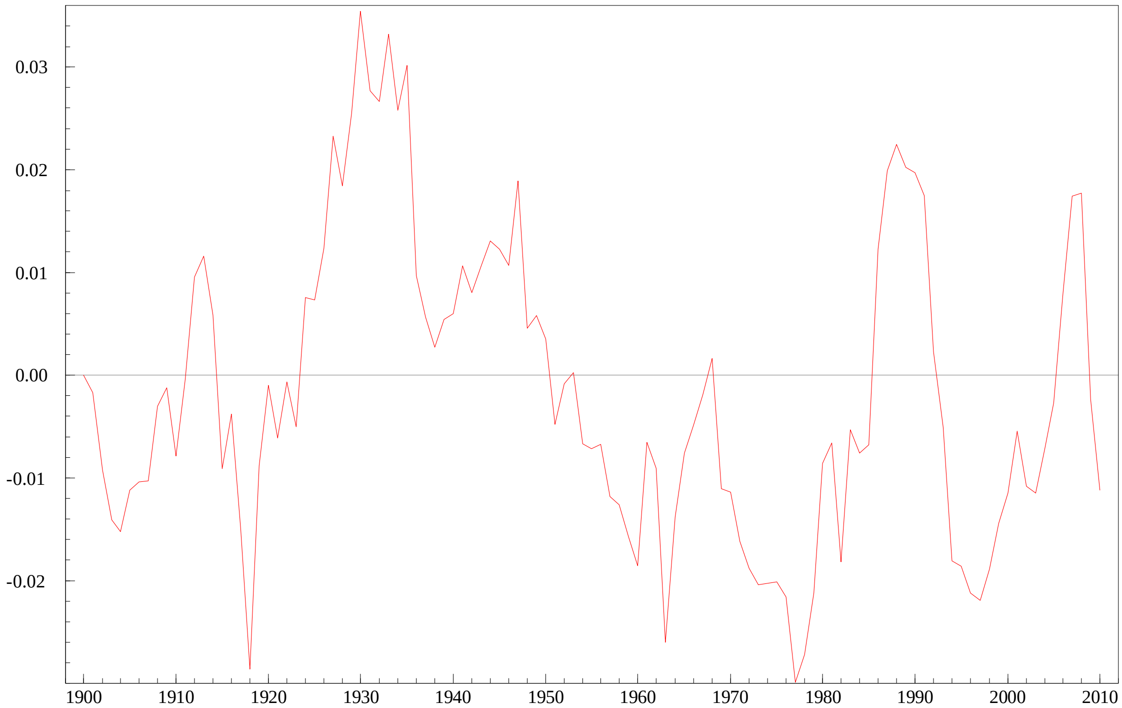


Figure 2.6: A simultated series of the logarthm of the output gap using equation (2.71) and (2.55) with $\omega = 0.1$, $\epsilon = 4$ and $\gamma = 0.5$. The standard deviation of the technology shock variable $\varepsilon_t$ is set to 0.01.

The graph in figure 2.6 shows the solution of (2.71) and (2.55), using the values of the theoretical parameter that are given in the caption to the figure. The solution assumes that the output-gap is zero in the initial year, which we have set to "1900" in the simulation and in the graph. Due to negative productivity shocks in the first years of the solution period, and propagation, the output-gap is negative for the first 10 years, before there is a brief upturn after 12-13 years. The first more lasting boom starts in the mid "1920's" and lasts for more than 25 years. It is followed by a long period of GDP below steady state. At the end of the solution period, the distance between the waves of high income generation is shorter again.

It gives rise to thought that a simple RBC model can generate economic upturns and downturn that are of so long duration as shown in figure 2.6. One lesson may be that one should not jump to conclusions about economic disequilibria or unbalances in the economy on the basis of 10 years of above or below trend economic performance—further analysis is required to determine whether the business cycle is a equilibrium or disequilibrium phenomenon.

### 2.8.5 The solution of the bivariate first order system (VAR)

The income-expenditure model contain first order dynamics because of the lagged term in the consumption function. More generally, if we consider two equations in the two variables $x_t$ and $y_t$, both $x_{t-1}$ and $y_{t-1}$ appear in each of the equations:

$$y_t = \alpha_{11}y_{t-1} + \alpha_{12}x_{t-1} + \varepsilon_{yt}, \quad (2.72)$$

$$x_t = \alpha_{21}y_{t-1} + \alpha_{22}x_{t-1} + \varepsilon_{xt}. \quad (2.73)$$

where $\alpha_{ij}$ are constant coefficients, and $\varepsilon_{yt}, \varepsilon_{xt}$ are two random variables, both with mean zero.

The two equations (2.72) and (2.73) define what is known in the literature as a vector autoregressive model, or VAR. The reason for this name is that if we define the vectors $\boldsymbol{z}_t = (y_t, x_t)'$, and $\boldsymbol{\varepsilon}_t = (\varepsilon_{yt}, \varepsilon_{xt})'$, and a matrix

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

the *vector* $z_t$ is seen to follow a first order autoregressive model:

$$\boldsymbol{z}_t = \boldsymbol{\alpha}\, \boldsymbol{z}_{t-1} + \boldsymbol{\varepsilon}_t. \quad (2.74)$$

Often in economics, the VAR needs to be extended by exogenous explanatory variables, Hence if we let $w_t$ represent an explanatory variable with mean, $m_w \neq 0$, and define the coefficient vector $\boldsymbol{\beta} = (\beta_{10}, \beta_{20})$, the vector autoregressive distributed lag model (often called VAR-X) is given by

$$y_t = \alpha_{11}y_{t-1} + \alpha_{12}x_{t-1} + \beta_{10}w_t + \varepsilon_{yt}, \quad (2.75)$$

$$x_t = \alpha_{21}y_{t-1} + \alpha_{22}x_{t-1} + \beta_{20}w_t + \varepsilon_{xt}. \quad (2.76)$$

or

$$\boldsymbol{z}_t = \boldsymbol{\alpha}\, \boldsymbol{z}_{t-1} + \boldsymbol{\beta} w_t + \boldsymbol{\varepsilon}_t. \quad (2.77)$$

in matrix notation.

The short-run model in this case is clearly given by (2.75) and (2.76)—or by (2.77) if matrices are used.. The impact multipliers are given by the vector $\boldsymbol{\beta}$. If a stable solution exists, the long-run model is given by

$$\overline{y} = \alpha_{11}\overline{y} + \alpha_{12}\overline{x} + \beta_{10}m_w, \quad (2.78)$$

$$\overline{x} = \alpha_{21}\overline{y} + \alpha_{22}\overline{x} + \beta_{20}m_w. \quad (2.79)$$

where $\overline{y}$ and $\overline{x}$ denote the two steady state values of the endogenous variables. By solving the two simultaneous equations (2.78) and (2.79) we find the long-run solution as

$$\overline{y} = \frac{(\alpha_{22} - 1)\beta_{10} - \alpha_{12}\beta_{20}}{\alpha_{11} + \alpha_{22} - \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} - 1} m_w, \quad (2.80)$$

$$\overline{x} = \frac{-\alpha_{21}\beta_{20} + (\alpha_{11} - 1)\beta_{10}}{\alpha_{11} + \alpha_{22} - \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} - 1} m_w. \quad (2.81)$$

The long-run multipliers are the derivatives of (2.80) and (2.81).

A full discussions of the stability conditions for the bivariate first order system goes beyond the scope of this course. But it is possible to gain important insight by deriving the final equation for $y_t$. From the single equation case, we know that stability hinges on the autoregressive part of the model, not on the distributed lag part. The same is true here: what matters is the dynamic interdependence between the two endogenous variables. Hence, to save notation, and we can find the final equation for the case of $\beta_{10} = \beta_{20} = 0$ and $\varepsilon_{yt,} = \varepsilon_{xt} = 0$. First note that (2.72) and (2.73) also holds for period $t-1$.

$$\begin{aligned} y_{t-1} &= \alpha_{11} y_{t-2} + \alpha_{12} x_{t-2} \\ x_{t-1} &= \alpha_{21} y_{t-2} + \beta_{21} x_{t-2}. \end{aligned}$$

From the first equation obtain

$$x_{t-2} = \frac{-1}{\alpha_{12}}(\alpha_{11} y_{t-2} - y_{t-1}),$$

and substitute this for $x_{t-2}$ in the second equation to obtain.

$$x_{t-1} = \alpha_{21} y_{t-2} - \frac{\alpha_{22}\alpha_{11}}{\alpha_{12}} y_{t-2} + \frac{\alpha_{22}}{\alpha_{12}} y_{t-1}.$$

The left hand side of this equation can be substituted for $x_{t-1}$ in equation (2.72) to give the final equation for $y_t$:

$$y_t = (\alpha_{11} + \alpha_{22}) y_{t-1} + (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) y_{t-2}. \tag{2.82}$$

Since this is an autoregressive equation with two lags it is immediately clear that for example $a_{11} < 1$ by itself does not guarantee the existence of a stable solution. Conversely, it is possible that the system is dynamically stable even when for example $\alpha_{11} = 1$, i.e., if the other coefficients "contributes enough to stability". The necessary and sufficient conditions for stability can be shown to be (see Sydsæter and Berck (2006, p 66, equation 10.18)):

$$\begin{aligned} 1 - (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) &> 0 \\ 1 - (\alpha_{11} + \alpha_{22}) + (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) &> 0 \\ 1 + (\alpha_{11} + \alpha_{22}) + (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) &> 0. \end{aligned}$$

Inspection of these conditions show that they are in terms of the coefficients of the final equation (2.82). As hinted above, the conditions for stability are seen to be fulfilled if both those coefficients are less than one.

## Exercises

1. Use the numbers from the estimated consumption function and check that by using the formulae of Table 2.1, you obtain the same numerical results as in section 2.2.

2. Confirm that in the case of a distributed lag model, the two first multipliers of a temporary change are equal the coefficients of $x_t$ and $x_{t-1}$ respectively, while $\delta_j = 0$ for $j = 2, 3, \ldots$. Show also that in the case of a permanent change, $\delta_{long-run}$ is equal to $\beta_1 + \beta_2$.

3. Using the definitions of dynamics, in what sense would you say that the Differenced data model of section 2.3 qualifies as a dynamic model, and in which sense does it (rather) qualify as a static model?

4. In a dynamic system with two endogenous variables, explain why we only need to derive one final equation in order to check the stability of the system.

5. In the model in section 2.8, derive the final equation for $INC_t$. What is the relationship between the demand multipliers that you know from Keynesian models, and the impact and long-run multipliers of INC with respect to a one unit change in autonomous expenditure?

6. Formulate a dynamic model of the real exchange rate which is consistent with the PPP hypothesis holding as a long-run proposition.

7. Let $\sigma_t$ denote the real exchange rate period $t$. Assume that the time period is annual, and that we are given the following ADL model which explains $\sigma_t$:

$$\sigma_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha \sigma_{t-1} \tag{2.83}$$

   $x_t$ represents an exogenous variable. (For simplicity, the random disturbance $\varepsilon_t$ has been omitted).Economists have suggested that after a shock, the real exchange rate can "overshoot" its (new) long-run level.

   (a) With reference to (2.83), formulate a precise definition of overshooting.

   (b) Illustrate the concept by drawing graphs (by hand or computer) of dynamic multipliers which are consistent with both overshooting with no-overshooting.

   (c) Give one example of specific parameter values which gives rise to overshooting, and of another specific set which does not imply overshooting. Formulate a definition of overshooting that you find meaningful in the light of (2.83)

8. Figure 2.3 in the text shows a solution for $\ln(C_t)$ which trends upwards. This may indicate an unstable solution. Assume that you did not have access to the value of $\alpha$ in the consumption function, only to Figure below 2.7. Why does this figure indicate that the solution is in fact stable?

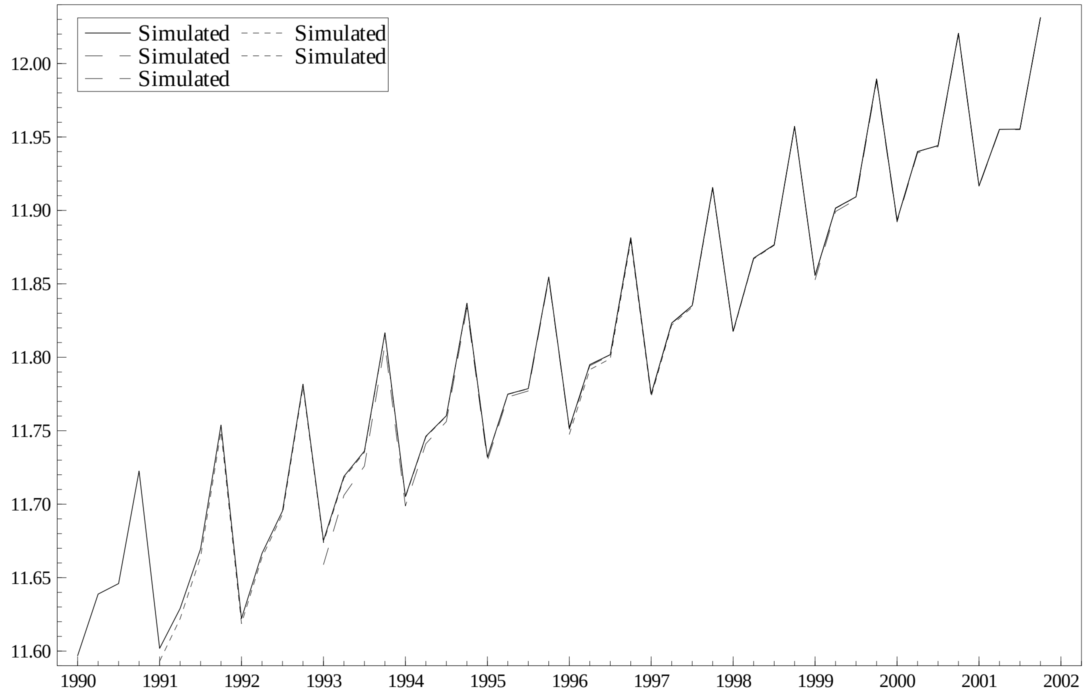


Figure 2.7: 5 solutions of the consumption function in 1.20, corresponding to different initial periods: 1989(4), 1990(4) ,1992(4). 1995(4) and 1998(4).

9. Download the file `norcons.zip` from the web page. Use the instruction provided in the included `.txt` file to replicate the dynamic multipliers in Figure 2.1.

# Chapter 3

# The supply side: Wage-price dynamics

Theories of wage formation and of price setting are important in macroeconomics since they represent the supply-side of macroeconomic models. In this chapter we discuss three alternative models of the supply side: The wage bargain model, the Phillips curve and the New Keynesian Phillips curve. We note that the two first models are consistent with perception that firms and their (organized or unorganized) workers are engaged in a partly cooperative and partly conflictual sharing of the rents generated by the operation of the firm. Despite this common ground, the models have notably different dynamic properties, and have different policy implications. In the standard New Keynesian Phillips curve model there is no role for trade unions. That theory instead places singular emphasis on firms' forward-looking behaviour.

## 3.1 Introduction

In economics the term 'inflation' generally describes the prevailing rate at which the prices of goods and services are increasing. It is commonplace that all prices tend to rise at broadly the same rate, because, when prices of domestic goods are rising, this will generally be true also of wages, and of the price of imported goods. This is because inflation in one sector of the economy permeates rapidly into other sectors. The phrase, a 'high rate of inflation' therefore usually describes a situation in which the money values of all goods in an economy are rising at a fast rate.

History shows many examples of countries which have been hit by extremely high rates of inflation, so called hyperinflation. Any such episode is a result of crisis and disorganization in the economic and political system of a country, and of a breakdown in the public trust in the country's monetary system. The typical case is however that inflation can be moderately high for long periods of time without doing serious damage to the stability of monetary institutions.

Nevertheless, even if we set aside the phenomenon of hyperinflation, it is generally agreed that a high and volatile inflation rate is a cause of concern, and towards the end of last century, the governments of the "Western world" invested heavily in curbing inflation. Institutional changes took place (crushing of labour unions in the UK, revitalization of incomes policies in Norway, a new orientation of monetary policy, the European Union's stability pact, etc.), and governments also allowed unemployment to rise to level not seen since the 2WW.

Was all this necessary, or were there other ways to curb inflation that would have meant a lower toll in terms of unemployment? Finally, given that inflation has now become a prime target of economic policy, what is the inflation outlook and how can inflation be controlled using the policy instruments that are recognized as legitimate in liberalized economies? Answers to any of these important issues are necessarily *model dependent*. By model dependency we mean that, before the answer is given, a view has been formulated, either explicitly or implicitly, about the major determinants of inflation and about which instruments are available for controlling inflation, and so forth. In this chapter we therefore discuss inflation models.

It is a well known fact that prices influence each other: wages follow the cost of living index, which is based on prices on tradable and non-tradable goods, which in their turn (and to varying extent) reflect labour costs that are determined by wage settlements of the past. Hence the 'inflation spiral' has a dynamic structure that results from the interaction of several markets, where the markets for labour and goods are perhaps the two most important. Models of the dynamic structure of wage and price setting are therefore of greatest importance, and they represent an area where the modelling concepts of chapter 2 are indispensable.

In this chapter, we present two important models of wage and price setting behaviour that are relevant to the inflation process of small open economies: the bargaining model, section 3.2.4, and the Phillips curve model, section 3.3. The main premise of the bargaining model is that firms and their (organized or unorganized) workers are engaged in a partly cooperative and partly conflictual sharing of the rents generated by the operation of the firm. In Norway, a system of bargaining based wage setting has been the framework both for analysis and policy decisions for decades.

The Phillips curve is covered by every textbook in macroeconomics, and in section 2.4.2 above we have already introduced a Phillips curve relationship, as an example of how ADL models are applied in macroeconomics. In this chapter, we give a fresh lick to the Phillips curve by comparing it with the bargaining model of inflation. We show that the Phillips curve can be seen as a special case of the richer dynamics of the bargaining model, see section 3.4. The chapter also present some empirical evidence from the Norwegian economy, see section 3.5. Which model receives most support from the data? The Phillips curve or the bargaining model?

In the final section of the chapter we give an appraisal of the new Keynesian Phillips curve, which in a short period of time has become an important model of the supply side in the models used to guide monetary policy, and which also is integrated in the dynamic stochastic general equilibrium models, DSGEs.

## 3.2 Wage bargaining and incomplete competition.

In the course of the 1980s interesting developments took place in macroeconomics. First, the macroeconomic implications of imperfect competition with price-setting firms were developed in several papers and books.[1] Second, the economic theory of labour unions, pioneered by Dunlop (1944), was extended and formalized in a game theoretic framework, see e.g., Nickell and Andrews (1983), Hoel and Nymoen (1988). Models of European unemployment, that incorporated elements from both these developments, appeared in Layard and Nickell (1986), Layard et al. (1991), and Lindbeck (1993).

We refer to the overall framework as the Incomplete Competition Model (ICM) as in Bårdsen and Nymoen (2003) or, interchangeably, as the *wage curve* framework (as opposed to for example the Phillips curve model). *Incomplete competition*, is particularly apt since the model's defining characteristic is the explicit assumption of imperfect competition in both product and labour markets.[2] The ICM was quickly incorporated into the supply side of macroconometric models, see Wallis (1993, 1995), and purged European econometric models of the Phillips curve, at least until the arrival of the New Keynesian Phillips curve late in the 1990s (see below).

Time does not play an essential role in the standard bargaining models. Although the theory is sometimes presented with reference to bargaining stages, real calendar time is not an explicit variable in the model. Hence, the interpretation of the wage equations rationalized by these theories is that they represent hypotheses about the steady state. In line with this interpretation, and following the approach to dynamics presented in the first two chapters, we first present the bargaining model of wage setting as a theory of the steady state in section 3.2.1, and then present the full equilibrium correction model in section 3.2.2.

### 3.2.1 A bargaining theory of the steady state wage

Since the focus of the book is the small open economy, we first establish the conceptual distinction between a tradables sector where firms act as price takers, either because they sell most of their produce on the world market, or because they encounter strong foreign competition on their domestic sales markets, and a non-tradables sector where firms set prices as mark-ups on wage costs. There is monopolistic competition among firms in this sector. The two sectors are dubbed the *exposed* ($e$) and *sheltered* ($s$) sectors of the economy.

We first need to establish some notation: Let $Q_e$ and $Q_s$ denote the producer prices in the two sectors. We assume that the consumer price index $P$ is a weighted average of the two prices:

$$P = Q_s^{\phi} Q_e^{(1-\phi)} \quad 0 < \phi < 1 \tag{3.1}$$

[1] For example., Bruno (1979), and Blanchard and Kiyotaki (1987), Bruno and Sachs (1984) and Blanchard and Fisher (1989, Chapter 8)

[2] The ICM acrynom should not be taken to imply that the Phillips curve contain perfect competition, though.

where $\phi$ is a coefficient that reflects the weight of non-traded goods in private consumption,[3]. More realistically, we could add a third consumption good which is not produced domestically, but with the cost of a more complicated notation.

Due to monopolistic competition, the steady state price in the sheltered sector is proportional to the wage level $W_s$ in that sector:

$$Q_s = \epsilon \frac{W_s}{A_s}. \tag{3.2}$$

$A_s$ is the average labour productivity. It is defined as $A_s = Y_s/L_s$ where $Y_s$ denotes output (measured as value added) in the non-tradable or sheltered sector, and $L_s$ denotes labour input (total number of hours worked). The parameter $\epsilon$ is determined by the elasticity of demand facing each firm $i$ with respect to the firm's own price. It is a common to assume that the elasticity is independent of other firms prices, and that it is identical for all firms. It can be shown that if it is assume that the absolute value of the elasticity of demand is larger than unity, then $\epsilon > 1$ as well, and for that reason, $\epsilon$ is can be referred to as the price *mark-up* coefficient on wages.

In the two equations (3.1) and (3.2), $Q_e$ is by assumption exogenous ($e$-sector firms are price takers), and $A_s$ is also exogenous and represents a technological trend. The wage level in the sheltered sector is determined by the following equation:

$$W_s = W_e \tag{3.3}$$

saying that the sheltered sector wage is proportional to the wage in the exposed sector of the economy. To save notation, the proportionally factor has been set equal to unity.

The hourly wage level $W_e$ is determined by bargaining between labour unions and organizations representing firms. We define real profits, , as

$$\prod = Y_e - \frac{W_e}{Q_e} L_e = (1 - \frac{W_e}{Q_e}\frac{1}{A_e})Y_e.$$

Over the wage bargaining time horizon we assume, for simplicity that output $Y_e$ is equal to the capacity level. In this perspective, is possible that varying profitability results in lower or increased capacity, hence we assume that $Y_e$ is a non-increasing function of real unit labour costs:

$$Y_e = Y_e(\frac{W_e/Q_e}{A_e}), \quad Y_e' \leq 0. \tag{3.4}$$

We assume that the wage $W_e$ is settled in accordance with the principle of maximizing the Nash-product:

$$(\nu - \nu_0)^{\mho} \Pi^{1-\mho} \tag{3.5}$$

[3]For reference, due to the log-form, $\phi = x_s/(1 - x_s)$ where $x_s$ is the share of non-traded goods in consumption.

where $\nu$ denotes union utility and $\nu_0$ denotes the fall-back utility or reference utility. The corresponding break-point utility for the firms has already been set to zero in (3.5), but for unions the utility during a conflict (e.g., strike, or work-to-rule) is non-zero because of compensation from strike funds. Finally $\mho$ is represents the relative bargaining power of unions.

Union utility depends on the consumer real wage of an employed worker and the aggregate rate of unemployment, thus $\nu(\frac{W_e}{P}, U, Z_\nu)$.[4] The partial derivative with respect to wages is positive, and negative with respect to unemployment ($\nu'_W > 0$ and $\nu'_U \leq 0$). $Z_\nu$ represents other factors in union preferences.

The fall-back or reference utility of the union depends on the overall real wage level and the rate of unemployment, hence $\nu_0 = \nu_0(\frac{\bar{W}}{P}, U)$ where $\bar{W}$ is the average level of nominal wages which is one of factors determining the size of strike funds. If the aggregate rate of unemployment is high, strike funds may run low in which case the partial derivative of $\nu_0$ with respect to $U$ is negative ($\nu'_{0U} < 0$). However, there are other factors working in the other direction, for example that the probability of entering a labour market programme, which gives laid-off workers higher utility than open unemployment, is positively related to $U$. Thus, the sign of $\nu'_{0U}$ is difficult to determine from theory alone. However, we assume in following that $\nu'_U - \nu'_{0U} < 0$.

With these specifications of utility and break-points, the Nash-product, denoted $\mathcal{N}$, can be written as

$$\mathcal{N} = \left\{\nu(\frac{W_e}{P}, U, Z_\nu) - \nu_0(\frac{\bar{W}}{P}, U)\right\}^{\mho} \left\{(1 - \frac{W_e}{Q_e}\frac{1}{A_e})Y_e\right\}^{1-\mho}$$

or

$$\mathcal{N} = \left\{\nu(\frac{W_q}{\frac{P}{Q_e}}, U, Z_\nu) - \nu_0(\frac{\bar{W}}{P}, U)\right\}^{\mho} \left\{(1 - W_q\frac{1}{A_e})Y_e\right\}^{1-\mho}$$

where $W_q = W_e/Q_e$ is the producer real wage.

We assume that (3.1), (3.2) and (3.3) are taken into account during the bargaining. Note that $P$ can be written in terms of $W_q$:

$$P = \left(\epsilon\frac{W_e}{Q_e}\right)^{\phi} Q_e \left(\frac{1}{A_s}\right)^{\phi} = W_q^{\phi} Q_e \left(\frac{\epsilon}{A_s}\right)^{\phi}, \tag{3.6}$$

and that the two relative prices $P/Q_e$ and $\bar{W}/P$ are also functions of the real wage $W_q$:

$$\frac{P}{Q_e} = W_q^{\phi}\left(\frac{\epsilon}{A_s}\right)^{\phi} \tag{3.7}$$

[4] This formulation implies that union utility is in terms of the pre-tax real wage, and that changes in the income tax rate has no influence on wage setting. The evdidence is consistent with this formulation, broadly speaking, although there are many caveats, including the many problems of measuring average and marginal tax rates at the aggregate level..

$$\frac{\bar{W}}{P} = \frac{W_e}{\left(\frac{W_e}{Q_e}\right)^{\phi} Q_e \left(\frac{\epsilon}{A_s}\right)^{\phi}} = \frac{\frac{W_e}{Q_e}}{\left(\frac{W_e}{Q_e}\right)^{\phi} \left(\frac{\epsilon}{A_s}\right)^{\phi}} = \frac{{W_q}^{1-\phi}}{\left(\frac{\epsilon}{A_s}\right)^{\phi}} \tag{3.8}$$

Using (3.6)-(3.4), the Nash-product $\mathcal{N}$ can be written as:

$$\mathcal{N} = \left\{ \nu \left( \frac{W_q^{1-\phi} A_s^{\phi}}{\varepsilon^{\phi}}, U, Z_{\nu} \right) - \nu_0 \left( \frac{W_q^{1-\phi} A_s^{\phi}}{\varepsilon^{\phi}} \right) \right\}^{\mho} \left\{ \left(1 - \frac{W_q}{A_e}\right) Y_e \left(\frac{W_q}{A_e}\right) \right\}^{1-\mho} \tag{3.9}$$

The first order condition for a maximum is given by $\mathcal{N}_{W_q} = 0$ which defines the *bargained* real wage $W_q^b$ implicitly as

$$W_q^b = F(A_e, A_s, \mho, U, Z_{\nu}), \tag{3.10}$$

subject to the usual second order conditions for a maximum.

By direct inspection of the Nash-product we see that a one percentage exogenous increase in $A_e$ leads to a one percentage increase in $W_q^b$, meaning that elasticity of $W_q^b$ with respect of $A_e$ is 1. Moreover, since bargaining is about the nominal wage, and $Q_e$ is the exogenous foreign price level, we can write the solution for the bargained *nominal* wage $W_e^b$

$$W_e^b = A_e Q_e \, G(A_s, \mho, U, Z_{\nu}). \tag{3.11}$$

The expected non-negative sign of the partial derivative of the $G$ function with respect to $U$ can be shown to depend on the specification of the utility functions. In a wider interpretation, union bargaining power $\mho$ is negatively related to $U$, meaning that the total derivative of $W_e^b$ with respect to $U$ is negative even though the partial derivative might be positive.

By choosing a log-linear functional form as an approximation to (3.11), we can write

$$w_e^b = m_{e0} + a_e + q_e + \gamma_{e1} u, \; \gamma_{e1} \leq 0 \tag{3.12}$$

where $w_e^b = \ln(W_e^b)$, and $a_e$, $q_e$ and $u$ likewise denote the logs of the corresponding variables in (3.11). For simplicity, we regard $A_s$, and $Z_v$ as constants, and they are therefore subsumed in $m_{e0}$. Equation (3.12) represent our hypothesized long-run or steady state relationship for e-sector wages. For completeness we also express the relationships (3.1), (3.2) and (3.3) in logarithmic form:

$$p = \phi q_s + (1-\phi) q_e, \quad 0 < \phi < 1, \tag{3.13}$$

$$w_s = w_e^b, \tag{3.14}$$

$$q_s = \ln(\epsilon) + w_s - a_s. \tag{3.15}$$

The long-run model consists of (3.12)-(3.15). The four equations determine $w_e^b$,$w_s$, $q_s$ and $p$. The exogenous variables are $a_e, q_e$ and $u$. The next question to ask is whether this steady state is dynamically stable. As we have explained in chapter 2, the answer to this question depends on whether the wage level in period $t$, $w_{et}$, approaches $w_e^b$ in the case where the initial situation is characterized by disequilibrium: $w_{e0} \neq w^b$.

However, before we leave the static wage bargaining model, it is worth making a couple of remarks about the generality of equation (3.12). First, as noted above, the derivation of equation (3.12) is based on the assumption that $A_s$ is a constant. This is a non-trivial simplification since realistically there is a trend like productivity increase in the sheltered sector as well. If we for a minute consider variations in $A_s$, it is easy to see that the impact on $w_e^b$ is negative: A one percent increase in the sheltered sector productivity level reduces the price level $P$ and so "allows" the unions to settle for a lower nominal wage without any reduction on the consumer real-wage. Hence, in our framework, the elasticity of $W_q^b$ with respect to $A_s$ is negative. The assumption that drives this result is that unions acknowledge that the consumer price level is going to be affected by wage increases, i.e., we have substituted equation (3.6) into the Nash-product. In most other derivations of the bargaining wage, $P$ is taken to be an exogenous variable. If we had adopted this alternative assumption, a relationship like (3.12) would again emerge, but with a so called wedge variable $(p - q_q)$ as an additional variable on the right hand side.[5]

A second, less technical remark, concerns how we have modelled output, $Y_e$. Above we made the assumption that $Y_e$ is identical to the capacity of the industry, which is not unreasonable in the time perspective of a steady state, and that capacity is a function of the wage-share. A more conventionally approach, which requires that the time perspective is of the short-run and that the capital stock is fixed, models $\pi$ as a function of the real wage $W_q$, because of short-term profit maximizing behaviour.[6] With this assumption, the resulting equation for the bargained wage does not change qualitatively from what we have in equation (3.12).

Finally, one might consider monopolistic competition also among e-sector firms.[7] In this formulation, e-sector firms face downward sloping, but potentially very flat, demand curves. Since the formulation above, corresponds to a situation with exactly horizontal demand curves, it is perhaps not surprising that the Nash solution with monopolistic e-sector forms also implies an equation like (3.12).

### 3.2.2 Wage bargaining and dynamics

Clearly, it is the ambition of the bargaining model to explain wage dynamics, not just the development of wages in a hypothetical steady state situation. From the discussion in the chapter 2, this extension of the framework can be achieved by integrating (3.12) in an autoregressive distributed lag model, ADL model which is dynamically stable. These ideas can be represented with the aid of the error correction transformation that was also introduced in chapter 2. Thus, we assume

[5] The reason for the name "wedge" is that $p - q_e$ makes out the difference (the wedge) between the (logs) of the consumer real wage and the producer real wage.

[6] See Rødseth (2000, Chapter 5.9).

[7] See Bårdsen et al. (2005a, Ch 5.2).

that $w_{et}$ is determined by the dynamic model

$$w_{et} = \beta_0 + \beta_{11} mc_t + \beta_{12} mc_{t-1} + \beta_{21} u_t + \beta_{22} u_{t-1} + \alpha w_{et-1} + \varepsilon_t. \tag{3.16}$$

Compared to the ADL model in equation (2.18), there are now two explanatory variables ("$x$-es"). The first variable, $mc_t$, is the sum of the logarithms of productivity and product price: $mc_t = a_{et} + q_{et}$ (we will explain the choice of acronym "$mc$" in the next section). The second exogenous explanatory variable in the ADL for e-sector wages is the logarithm of the rate of unemployment, $u_t$. To distinguish the effects of the two variables, we have added a second subscript to the $\beta$ coefficients. Of course, the result from the bargainin model above, namely that $q_e$ and $a_e$ have the same effect on the steady state wage level is not the same as saying that they also have identical effects on the wage in any given period. The use of the combined variable $mc_t$ in (3.16) therefore represents a simplification of the dynamics. This simplification does not represent any loss of generality though.

The endogenous variable in (3.16) is $w_{et}$. As explained above, $mc_t$ is an exogenous variable which displays a dominant trend due to both productivity growth and foreign price growth. The rate of unemployment is also exogenous in the model.

The assumed exogeneity of unemployment is not grounded on realisms, since responses in unemployment to wage changes clearly represent a corrective mechanisms which helps stabilize the wage level around its steady state path. However, it is interesting to study the case of exogenous unemployment first, since we will then see which stabilizing mechanisms in wage formation are at work even at a constant and exogenous rate of unemployment. This is a thought provoking contrast to "natural rate models" of wage dynamics which dominates the macroeconomic policy debate, and which takes it as a given thing that unemployment *has* to adjust in order to bring about constant wage growth and, eventually, stable inflation.[8] We turn to this theory in section 3.3 below.

If we apply the same error-correction transformation as in chapter 2.5, we obtain:

$$\begin{aligned} \Delta w_{et} &= \beta_0 + \beta_{11} \Delta mc_t + \beta_{21} \Delta u_t \\ &\quad + (\beta_{11} + \beta_{12}) mc_{t-1} + (\beta_{21} + \beta_{22}) u_{t-1} + (\alpha - 1) w_{et-1} + \varepsilon_t \end{aligned} \tag{3.17}$$

For the bargaining theory to be a realistic model of long term wage behaviour, it is necessary that (3.16) has a stable solution. Since wages usually show a rather smooth evolution through time we state the stability condition as

$$0 < \alpha < 1, \tag{3.18}$$

thus excluding negative values of the autoregressive coefficient. Hence, subject to the stability condition in (3.18), equation (3.17) can be written as

$$\begin{aligned} \Delta w_{et} = {} & \beta_0 + \beta_{11} \Delta mc_t + \beta_{21} \Delta u_t \\ & - (1 - \alpha) \left\{ w_{et-1} - \frac{\beta_{11} + \beta_{12}}{1 - \alpha} mc_{t-1} - \frac{\beta_{21} + \beta_{22}}{1 - \alpha} u_{t-1} \right\} + \varepsilon_t. \end{aligned} \tag{3.19}$$

[8] See Bårdsen et al. (2005a, Ch 3-6) for an exposition.

To reconcile this with the steady state relationship (3.12), we define

$$\gamma_{e,1} = \frac{\beta_{21} + \beta_{22}}{1 - \alpha},$$

saying that the elasticity of $w^b$ with respect to the rate of unemployment is identical to the long–run multiplier of the actual wage $w_{et}$ with respect to $u_t$, and then impose the following restriction on the coefficient of $mc_{t-1}$:

$$\beta_{11} + \beta_{12} = (1 - \alpha) \tag{3.20}$$

since (3.12) implies that the long-run multiplier with respect to $mc$ is unity. Then (3.19) becomes

$$\begin{aligned} \Delta w_{et} &= \beta_0 + \beta_{11}\Delta mc_t + \beta_{21}\Delta u_t \\ &\quad -(1-\alpha)\{w_{et-1} - mc_{t-1} - \gamma_{e1}u_{t-1}\} + \varepsilon_t \end{aligned} \tag{3.21}$$

which is an example of the homogenous ECM of section 2.3. The short-run multiplier with respect to $mc_t$ is of course $\beta_{11}$, which can be considerably smaller than unity without violating the long-run relationship (3.12).

The formulation in (3.21) is an equilibrium correction model, ECM exactly because the term in brackets captures that wage growth in period $t$ partly corrects last period's deviation from the long-run equilibrium wage level. We can write it as

$$\begin{aligned} \Delta w_{et} &= \beta_0^{'} + \beta_{11}\Delta mc_t + \beta_{21}\Delta u_t \\ &\quad -(1-\alpha)\left\{w_e - w_e^b\right\}_{t-1} + \varepsilon_t \end{aligned} \tag{3.22}$$

where $w_e^b$ is given in (3.12) and $\beta_0^{'}$ is given as $\beta_0' = \beta_0 - (1-\alpha)m_{e0}$.

You should check that the derivations above parallel those given in chapter 2, equation (2.21)-(2.24) in particular. Hence the following equation is equivalent to (3.21) and (3.22)

$$\begin{aligned} \Delta w_{et} &= \beta_0^{'} + \beta_{11}\Delta mc_t + \beta_{21}\Delta u_t \\ &\quad -(1-\alpha)\{w_e - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\} + \varepsilon_t, \end{aligned} \tag{3.23}$$

we only need to keep in mind that $\beta_0' = \beta_0 - (1-\alpha)m_{e0}$ as explained.

Subject to the condition $0 < \alpha < 1$ stated above, wage growth is seen to bring the wage level in the direction of the bargained wage $w_e^b$. For example, assume that the sum of price and productivity growth is constant so that $\Delta mc_t = g_{mc}$, that the rate of unemployment is constant, $\Delta u_t = 0$, and that there is no new shockt, $\varepsilon_t = 0$. If there is disequilibrium in period $t-1$, for example $\left\{w_e - w_e^b\right\}_{t-1} > 0$, wage growth from $t-1$ to $t$ will be reduced, and this leads to $\left\{w_e - w_e^b\right\}_t < \left\{w_e - w_e^b\right\}_{t-1}$ in the next period.

Finally in the steady state equilibrium, with a prevailing constant rate of unemployment, and no random shocks, then $w_{t-1} = w^b_{et-1}$ as well as $w_t = w^b_t$, and steady state wage growth is therefore $\Delta w_{et} = \Delta w^b_{et} = g_{mc}$. At this point teh reader may have noted an apparent inconsistency, since, given these assumptions, the ECM (3.23) gives:

$$\Delta w_{et} = \beta_0' + \beta_{11} g_{mc},$$

and not $\Delta w_{et} = g_{mc}$ as implied by the steady-state path for the wage level. However, this paradox is resolved by noting that the original constant term $\beta_0$ in the ADL can be *defined* to be

$$\beta_0 = (1-\alpha)m_{e0} - (\beta_{11} - 1)g_{mc}. \tag{3.24}$$

If we invoke this definition of $\beta_0$, the dynamic equation (3.23) is seen to give $\Delta w_{et} = (1-\alpha)m_{e0} - (\beta_{11} - 1)g_{mc} - (1-\alpha)m_{e0} + \beta_{11} g_{mc} = g_{mc}$, which is consistent with $\Delta w_{et} = \Delta w^b_{et} = g_{mc}$ in the steady state. At first it may seem a to be a suspect trick to define the constant term of the ECM in the way we do in equation (3.24). Without going into details this is however not the case, and the definition in (3.24) is innocuous. One way to think about this is that if $\beta_0$ measured something different than (3.24) our theory would not account for the systematic part of wage growth, and in that case, a different ECM should have been formulated from the outset. For those with a background in econometrics, (3.24) is seen to correspond to formulating a regression equation by subtracting the means of the variables on both sides of the equation, an operation which does not affect the equation or the estimation results obtained.

### 3.2.3 Wage bargaining and inflation

So far, we have looked at only e-sector wage formation, and not inflation as such. To sketch the theory's implication for inflation, we need to reconsider the three equations (3.13)-(3.15). Equation (3.13) is a definition equation that holds not only in the long run, but also in each time period. Hence, we have

$$p_t = \phi q_{st} + (1-\phi)q_{et}$$

or in terms of growth rates:

$$\Delta p_t = \phi \Delta q_{st} + (1-\phi)\Delta q_{et}. \tag{3.25}$$

The two equations (3.14), for the s-sector wage, and (3.15), for the s-sector price level, have a different interpretation. They are hypotheses about the steady state, and as we learned already in the opening chapter of the book, adding time subscripts to these equations entail that we assume that s-sector price and wage formation are characterized by instantaneous adjustment (infinite speed of adjustment). This is clearly unrealistic, since there is no reason why adjustment lags and friction should not be important for wage and price adjustments in the sheltered sector. Hence, it is *only* in order to simplify the model as much as possible that we write the equations for $\Delta w_{st}$ and $\Delta q_{st}$ as

$$\Delta w_{st} = \Delta w_{et}, \text{ and} \tag{3.26}$$
$$\Delta q_{st} = \Delta w_{et} - \Delta a_{et}. \tag{3.27}$$

Equation (3.23) for e-sector wage growth and (3.25)-(3.27) are 4 equations which determine the endogenous variables $w_{et}$, $w_{st}$, $q_{st}$ and $p_t$ as functions of initial conditions and given values for exogenous variables $mc_t$, $u_t$ and $\varepsilon_t$. The exogenous and pre-determined variables are $w_{et-1}$, $mc_{t-1}$ and $u_{t-1}$. The model is a recursive system of equations: The wage growth rate in the exposed sector is determined first, from (3.23) and then the other growth rates follow recursively

Specifically, the reduced form equation for the rate of inflation, $\Delta p_t$, is found to be:

$$\begin{aligned} \Delta p_t = & \phi\beta_0' \\ & +(\phi\beta_{11} + (1-\phi))\Delta q_{e,t} + \phi\beta_{21}\Delta u_t \\ & +\phi\beta_{11}\Delta a_{et} - \phi\Delta a_{st} \\ & -\phi(1-\alpha)\left\{w_{et-1} - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\right\} \\ & +\phi\varepsilon_t \end{aligned} \tag{3.28}$$

showing that the bargaining model implies the following explanation of inflation in a small-open economy:

1. Autonomous inflation: $\phi\beta_0'$,
2. "Imported inflation": $(\phi\beta_{11} + (1-\phi))\Delta q_{et}$
3. Shock to unemployment: $\phi\beta_{21}\Delta u_t$
4. Productivity growth: $\phi\beta_{11}\Delta a_{et} - \phi\Delta a_{st}$,
5. e-sector equilibrium correction: $-\phi(1-\alpha)\left\{w_{e.t-1} - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\right\}$
6. random shocks: $\phi\varepsilon_t$

In order to get some idea about the size of these factors, we may set the share of non-traded goods in consumption to 0.4. Then $\phi = 0.67$, and setting $\beta_{11} = 0.5$ gives a coefficient of $\Delta q_{et}$ of 0.66. Foreign inflation in the range of $1\% - 5\%$ is of course not uncommon, and our model implies that a 3% inflation abroad implies about 2% "imported inflation". The coefficient of $\Delta a_{et}$ is 0.33, and for $\Delta a_{st}$ we obtain $-0.67$. The net-effect of productivity growth rates at around 2% may therefore be rather small. Note the implication that increased productivity in the exposed sector of the economy *increases* inflation. It occurs because e-sector productivity growth increases the bargained wage in that sector, which inflicts price increases in the sheltered sector via the assumption of relative wage stabilization.

Equilibrium correction in the exposed sector represents a numerically significant factor in this model. With the chosen parameter values, and setting $\alpha = 0.7$ in the wage equation, the coefficient of $\{w_{e.t-1} - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\}$ in equation (3.28) becomes 0.2. The interpretation is that a 1 percentage point deviation from the the steady state wage in period $t-1$ leads to a reduction of the period $t$ inflation rate of 0.2 percentage points, *ceteris paribus*.

As noted, our derivation of the bargaining model inflation equation is very stylized. Practical use of the framework needs to represent $\Delta w_{st}$ and $\Delta q_{st}$ by separate equilibrium correction equations, and not impose instantaneous adjustment as we have done in order to simplify. However, also in derivations which include more realistic models of sheltered sector wage and price adjustment, many of the properties of (3.28) will continue to hold true. For example the strong role of imported inflation, the sign reversal of the two productivity terms, and the role of equilibrium-correction in exposed sector wage setting.

Above, we have repeatedly noted that the inflation model is recursive. In actual wage setting, compensation for cost-of-living increases is always one of the issues. Hence, to increase the degree of realism of the model one would, in most cases, want to include $\Delta p_t$ in the dynamic equation of exposed sector wages setting, with a positive coefficient (less than one, though). Clearly, with this generalization of the model, the inflation model is no longer a recursive system, since instead of (3.23), we have:

$$\begin{aligned}\Delta w_{et} &= \beta_0' + \beta_{11}\Delta mc_t + \beta_{21}\Delta u_t + \beta_{31}\Delta p_t \\ &\quad -(1-\alpha)\{w_{et-1} - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\} + \varepsilon_t.\end{aligned} \tag{3.29}$$

However, since inflation can be expresses as:

$$\Delta p_t = \phi\Delta w_{et} - \phi\Delta a_{st} + (1-\phi)\Delta q_{et}$$

we can derive the following semi-reduced form for $\Delta w_{et}$:

$$\begin{aligned}\Delta w_{et} &= \tilde{\beta}_0' + \tilde{\beta}_{11}\Delta mc_t + \tilde{\beta}_{21}\Delta u_t + \beta_{41}\Delta a_{st} + \beta_{51}\Delta q_{et} \\ &\quad -\widetilde{(1-\alpha)}\{w_{et-1} - mc_{t-1} - \gamma_{e1}u_{t-1} - m_{e0}\} + \tilde{\varepsilon}_t.\end{aligned} \tag{3.30}$$

where the coefficient with an $\tilde{}$ are the original coefficients of (3.29) divided by $(1-\beta_{31}\phi)$, and $\beta_{41} = -\beta_{31}\phi/(1-\beta_{31}\phi)$, $\beta_{51} = \beta_{31}(1-\phi)/(1-\beta_{31}\phi)$. By using (3.30) instead of (3.29) in the system-of-equations, the recursive nature of the model is put back in place.

### 3.2.4 The Norwegian model of inflation

The Norwegian model of inflation was formulated in the 1960s, several decades before the formal bargaining approach was applied to wage setting[9]. It soon became

[9] In fact there were two models, a short-term multisector model, and the long-term two sector model that we re-construct using modern terminology in this chapter. The models were formulated

the framework for both medium term forecasting and normative judgements about "sustainable" centrally negotiated wage growth in Norway.[10] However, it is not the historical importance of the Norwegian model which is our concern here, but instead that the Norwegian model of inflation is consistent with the implications of the bargaining model. Moreover, while the interpretation of the modern bargaining model is often left hanging "in the air", with some researchers even viewing it as a model that is relevant in the short-run, the Norwegian model was originally explicitly formulated as a long-run model. Consistent with this is the Norwegian model's careful analysis of the potentially numerous equilibrating mechanisms, but also the multitude of disequilibrating shocks that realistically characterizes wage dynamics in the short-run. From this point of view, the Norwegian model is a more general model of bargaining dominated wage evolution.

Another name for the Norwegian model is the main-course model for wages, since a joint trend made up of productivity and foreign prices define the scope for wage growth, i.e., the *main-course* followed by the wage level through time. In the following, we use the names *main-course model* and *Norwegian model of inflation* interchangeably.

#### 3.2.4.1 A direct motivation for long-run wage and price relationships

In the same way as above, a central distinction is drawn between a tradables sector where firms act as price takers, and a non-tradables sector where firms set prices as mark-ups on wage costs. In the same manner as above, the two sectors are dubbed the *exposed* ($e$) and *sheltered* ($s$) sectors of the economy.

The model's main propositions are, first, that exposed sector wage growth will follow a long-run tendency defined by the exogenous price and productivity trends which characterize that sector. The joint productivity and price trend is called the *main-course* of e-sector wage development. The relationship corresponds to equation (3.12), for the bargained wage. Second, it is assumed that the relative wage between the two sectors are constant in the long-run, as already introduced in equation (3.3). Third, the development of the price level in the s-sector is a mark-up on the unit-labour costs in that sector, as captured by (3.2). In this section we concentrate on how the first proposition is rationalized in the seminal contribution by Odd Aukrust, from 1977.[11]

in 1966 in two reports by a group of economists who were called upon by the Norwegian government to provide background material for that year's round of negotiations on wages and agricultural prices. The group (Aukrust, Holte and Stolzt) produced two reports. The second (dated October 20 1966, see Aukrust (1977)) contained the long-term model that we refer to as the main-course model. Later, there was similar development in e.g., Sweden, see Edgren et al. (1969) and the Netherlands, see Driehuis and de Wolf (1976).

In later usage the distinction between the short and long-term models seems to have become blurred, in what is often referred to as the Scandinavian model of inflation. We acknowledge Aukrust's clear exposition and distinction in his 1977 paper, and use the terminology *Norwegian main-course model* for the long-term version of his theoretical framework.

[10] On the role of the main-course model in Norwegian economic planning, see Bjerkholt (1998).

[11] Aukrust (1977).

Recall that the wage share of value added output in the e-sector is $\frac{W_e L_e}{Q_e Y_e} = \frac{W_e}{Q_e A_e}$, where average labour productivity is given as $A_e = \frac{Y_e}{L_e}$. Correspondingly, the rate of profit is

$$\frac{Q_e Y_e - W_e L_e}{Q_e Y_e} = 1 - \frac{W_e}{Q_e A_e}.$$

It is reasonable to assume that in order to attract the investments needed to maintain competitiveness and employment in the e-sector, a certain normal rate of profit has to be met, at least in the long term. Since there is a one-to-one link between the profit rate and the wage share, there is also a certain maintainable level of wage share. Denote this long-run wage share $M_e$, and assume that both the product price and productivity are exogenous variables. We can now formulate what we might call the *main-course proposition*:

$$W_e^* = M_e Q_e A_e, \tag{3.31}$$

where $W_e^*$ denotes the long-run equilibrium wage level consistent with the two assumptions of exogenous price and productivity, and the existence of a normal wage share. It is practical to take logs,

$$\text{H1}_{mc}: \quad w_e^* = q_e + a_e + m_e, \tag{3.32}$$

and use the marker $\text{H1}_{mc}$ to indicates that this is the first hypothesis of the theory. As ususal we identify the long-run equilibrium wage level, $W_e^*$ or (in log) $w_e^*$, with the steady state wage level. Equation (3.32) therefore captures that the long-run elasticities of the wage level with respect to productivity and price are both unity.

Equation (3.32) has a very important implication for actual data of wages, prices and productivity in the $e-$sector: Since data for $q_e$ and $a_e$ show trend-like growth over time, actual time series data for the nominal wage, $w_{et}$, should also show a dominant positive growth around a trend defined by price and productivity. If this is not the case, then $w_e^*$, cannot be the equilibrium to which the actual wage converges in a steady state.

The joint trend made up of productivity and foreign prices, traces out a central tendency for wage growth, it represents a long-run sustainable scope for wage growth. Aukrust (1977) aptly refers to this joint trend as the main-course for wage determination in the exposed industries.[12] For reference we therefore define the main-course variable (in logs) as

$$mc = a_e + q_e. \tag{3.33}$$

It is instructive to see, by way of citation, that Aukrust meant exactly what we have asserted, namely that equation (3.32) is a long-run relationship corresponding to a steady state situation. Consider for example the following quotation:

[12] The essence of the statistical interpretation of the theory is captured by the hypothesis of so called cointegration between $w_e$ and $mc$ see Nymoen (1989) and Rødseth and Holden (1990)).

> The relationship between the "profitability of E industries" and the "wage level of E industries" that the model postulates, therefore, is a certainly not a relation that holds on a year-to-year basis. At best it is valid as a long-term tendency and even so only with considerable slack. It is equally obvious, however, that the wage level in the E industries is not completely free to assume any value irrespective of what happens to profits in these industries. Indeed, if the actual profits in the E industries deviate much from normal profits, it must be expected that sooner or later forces will be set in motion that will close the gap. (Aukrust, 1977, p 114-115).

Aukrust coined the term 'wage corridor' to represent the development of wages through time, and he used a graph similar to figure 3.1 to illustrate his ideas. The main-course defined by equation (3.33) is drawn as a straight line since the wage is measured in logarithmic scale. The two dotted lines represent what Aukrust called the "elastic borders of the wage corridor".

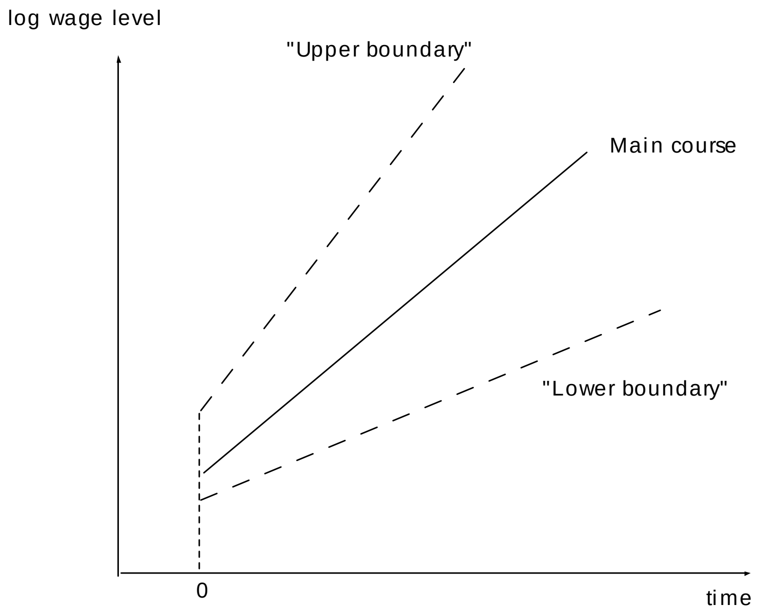


Figure 3.1: The 'wage corridor' in the Norwegian model of inflation.

We understand that Aukrust's model and the bargaining model of wage setting are mutually consistent theories. In particular, the bargaining model's proposition (3.12)

$$w_e^b = m_{e0} + a_e + q_e + \gamma_{e1} u, \; \gamma_{e1} \leq 0$$

is of the same form as $H1_{mc}$, if we set $m_e = m_{e0} + \gamma_{e1} u$. In the wage bargaining model, changes in the rate of unemployment causes the wage mark-up on $mc$ to move up or down. In a similar way, the normal rate of profit in the main-course model is conditioned by economic, social and institutional factors. A long term change in the rate of unemployment is one of the factors that can shift the sustainable profit rate.[13] Therefore, it is fully consistent with the gist of Aukrust's model that we replace $\text{H1}_{mc}$ by the more general hypothesis:

$$\text{H1}_{gmc} \quad w_e^* = m_{e0} + mc + \gamma_{e1} u.$$

As mentioned above, there are two other long-run propositions which are part of Aukrust's theory. The first is an assumption about a constant relative wage between the two sectors, and the second is a normal sustainable wage share also in the sheltered sector of the economy. We dub these two additional propositions $\text{H2}_{mc}$ and $\text{H3}_{mc}$ respectively:

$$\begin{array}{ll} \text{H2}_{mc} & w_s^* - w_e^* = m_{se}, \\ \text{H3}_{mc} & w_s^* - q_s^* - a_s = m_s, \end{array}$$

$m_{se}$ is the log of the long-run ratio between e-sector and s-sector wages. $a_s$ is the exogenous productivity trend in the sheltered sector, and $m_s$ is the (log) of the equilibrium wage rate in the sheltered sector. With reference to section 3.2.1, in that model, $m_{se} = \ln(1)$ and $m_s = -\ln(\epsilon)$.

Note that, if the long-run wage in the exposed sector is determined by the exogenous main course, then $\text{H2}_{mc}$ determines the long-run wage in the sheltered sector, and $\text{H3}_{mc}$ in turn determines the long-run price level of the sheltered sector. Hence, re-arranging $\text{H3}_{mc}$, gives

$$q_s^* = w_s^* - a_s - m_s$$

which is similar to theories of so called normal cost pricing: the price is set as a mark-up on average labour costs, in accordance with (3.15) above.

#### 3.2.4.2 Dynamic adjustment in the Norwegian model

As we have seen, Aukrust was clear about two things. First, the main-course relationship for e-sector wages should be interpreted as a long-run tendency, not as a relationship that governs wage development on a year to year basis. Hence, the theory makes us anticipate that actual observations of e-sector wages will fluctuate around the theoretical main-course. Second, if e-sector wages deviate too much from the long-run tendency, we expect that forces will begin to act on wage setting so that adjustments are made in the direction of the main-course. For example, profitability below the main-course level will tend to lower wage growth, either directly or after a period of higher unemployment. In Aukrust's words:

[13] No doubt, Aukrust's formulation, without unemployment as an explicit variable, was conditioned by the realities of the Norwegian economy of the 1960s, where unemployment had been constant at a low level since the ende of WW2 in line with poltical priorites..

> The profitability of the E industries is a key factor in determining the wage level of the E industries: mechanism are assumed to exist which ensure that the higher the profitability of the E industries, the higher their wage level; there will be a tendency of wages in the E industries to adjust so as to leave actual profits within the E industries close to a "normal" level (for which however, there is no formal definition). (Aukrust, 1977, p 113).

In sum, the basic idea is that after the wage level have been knocked off the main-course, forces are will begin to act on wage setting so that adjustments are made in the direction of the main-course. One equilibrating mechanism, is equilibrium correction of the nominal wage level, and it therefore immediately clear that the analysis and results of section 3.2.2 apply equally to the Norwegian model of inflation.

Figure 3.2 illustrates the dynamics following an exogenous and permanent change in the rate of unemployment: We consider a hypothetical steady state with an initially constant rate of unemployment (see upper panel) and wages growing along the main-course. In period $t_0$ the steady state level of unemployment increases permanently. Wages are now out of equilibrium, since the steady state path is shifted down in period $t_0$, but because of the corrective dynamics, the wage level adjusts gradually towards the new steady state growth path. Two possible paths are indicated by the two thinner line. In each case the wage is affected by $\beta_{21} < 0$ in period $t_0$. Line a corresponds to the case where the short-run multiplier is smaller in absolute value than the long-run multiplier, (i.e.,$-\beta_{21} < -\gamma_{e1}$). A different situation, is shown in adjustment path b, where the short-run effect of an increase in unemployment is larger than the long-run multiplier.

### 3.2.5 A simulation model of wage dynamics

We can use computer simulation to confirm our understanding about the dynamic behaviour wages in the bargaining model. The following three equations make up a representative bargaining model of wage-setting in the exposed sector:

$$w_{et} = 0.1mc_t + 0.3mc_{t-1} - 0.06 \ln U_{t-1} + 0.6w_{et-1} + \varepsilon_{wt}, \quad (3.34)$$

$$mc_t = 0.03 + mc_{t-1} + \varepsilon_{mct} \quad (3.35)$$

$$U_t = 0.005 + 0.005 \cdot S1989_t + 0.8U_{t-1} + \varepsilon_{Ut} \quad (3.36)$$

Equation (3.34) is a simplified version of (3.16), where we omit the constant term and the current value of the rate of unemployment. Note that the equation is written with an explicit log transformation of the rate of unemployment, $U_t$. The disturbance $\varepsilon_{et}$ is has a zero mean and standard error 0.01 (which is representative of the residual standard error found when estimating wage equations on annual data). Equation (3.35) defines the main-course variable as a random walk. The average

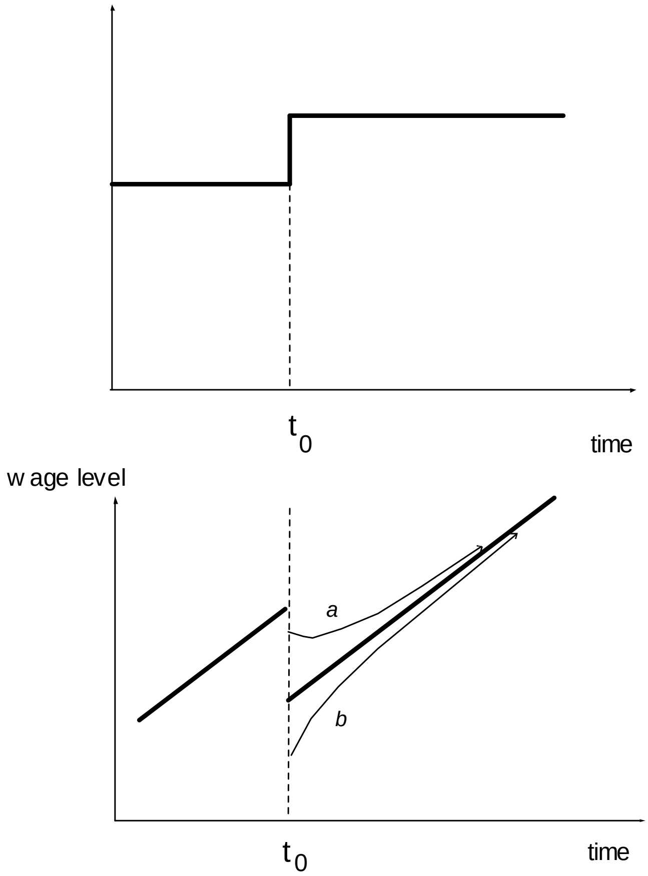


Figure 3.2: The main-course model: A permanent increase in the rate of unemployment, and possible wage responses.

annual growth rate of the main-course is 0.03.[14] The standard error of $\varepsilon_{mct}$ is also set to 0.01. The third equation, (3.36), is the equation for the rate of unemployment. It specifies $U_t$ as an exogenous variable (there is no presence of $w_{et}$ or its lags in the $U_t$ equation), in line with the assumptions underlying our derivation of the wage equation from bargaining theory. The exogeneity of $U_t$ is not meant as a realistic assumption to use in a complete macro model. We make the assumption here to illustrate that the wage rate can be dynamically stable even if the unemployment rate is not influenced by the wage rate. Intuitively, endogenity of $U_t$ of the normal type, where $U_t$ increases with higher $w_{et}$, cannot damage the stability of wage. Instead such a formulation would represent a second equilibrating mechanism, alongside the stabilization inherent in the bargaining model.

Note that the present formulation of the $U_t$ equation has a steady state solution

[14] Due to the disturbance term, the period by period growth rates vary, hence the rate of change, and thus the trend of $mc_t$, is stochastic as the name suggests.

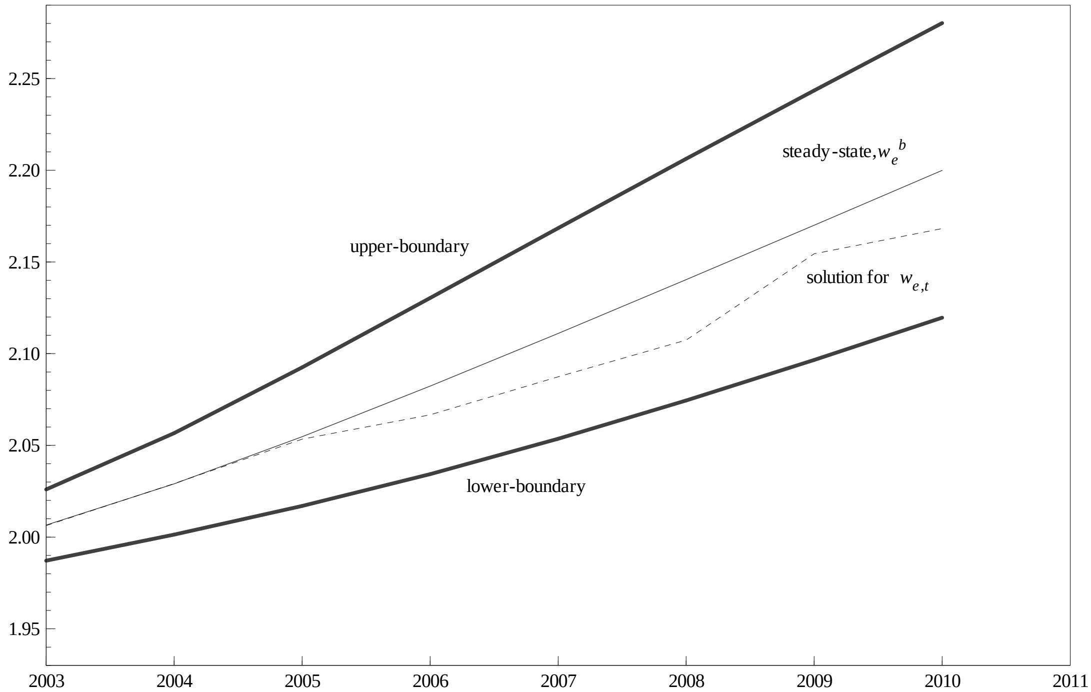


Figure 3.3: The wage evolution resulting from simulation of the calibrated model of equation (3.34) -(3.36).

for $U_t$, since the autoregressive coefficient is set to 0.8. We have added a structural change in the unemployment period, and this explains why the term $0.005 \cdot S1989_t$ is included the model. $S1989_t$ is a so called 'step-dummy', which is zero until 1988 ($t < 1989$) and one for later periods. You can check that this corresponds to a steady state rate of unemployment equal to 2.5% before 1989, and to 5% after. Hence there is a regime shift in the equilibrium rate of unemployment, taking place in 1989.[15]

In Figure 3.3, the dotted line shows the solution for $w_{et}$ over the period 2003-2010, using the initial values $mc_{2002}$ and $U_{2002}$, and the values for the three disturbances, drawn randomly by the computer programme for the solution period $2003 - 2010$. The line closest to the line for the solution is the hypothetical steady state development for the wage level, and it corresponds to the log of the bargained wage ($w_e^b$). Finally, the boundaries of the wage-corridor is given as $\pm 2$ standard errors of the main-course.

Figure 3.3 illustrates the theoretical points of the bargaining and Norwegian model.

1. There is a stable growth in the steady state wage, $w_e^b$.

[15] For reference, the standard error of $\varepsilon_{U,t}$ is set to 0.003.

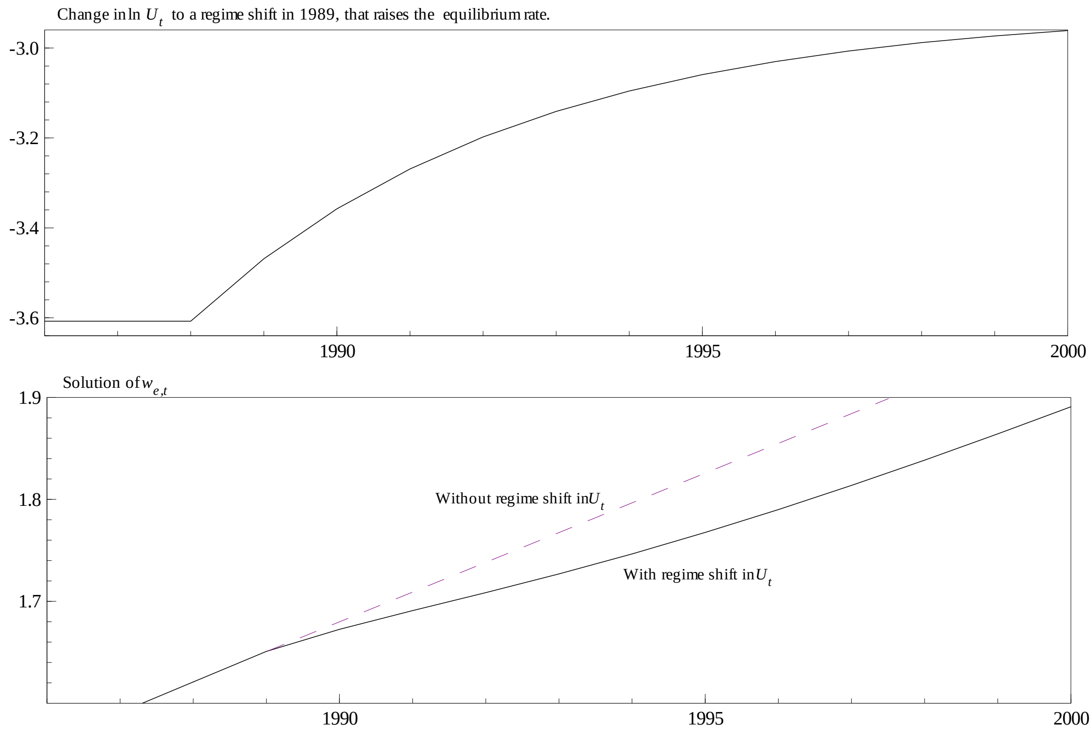


Figure 3.4: Unemployment and wage response to a regime shift in the equilibrium rate of unemployment in 1989. Simulation of the calibrated model (3.34)-(3.36).

2. The solution for the wage, $w_{et}$, shows the same trend as the steady state.

3. Except by coincidence, the graph for the actual wage $w_{et}$ is not identical to the line for the steady state path of the wage rate. This is of course due to random changes in unemployment, wage setting, and in other factors that influence the solution.

Figure 3.4 shows the effects of the regime shift in the rate of unemployment.[16] The important difference from Figure 3.2 is that the adjustment of unemployment to the new steady state is gradual, while it is assumed to be instantaneous in Figure 3.2. The lower panel of the graph shows that the wage level is gradually reduced compared to what we would have observed if the regime shift had not kicked in during 1989 (indicated by the dotted line).

Although this section has used simulation of a calibrated model to illustrate wage and unemployment dynamics, the model remains wholly theoretical. However, in section 3.5.2 we will show an estimated model, using real world data, which has very similar properties.

[16] For simplicity, all disturbances are set to their mean value of zero in the simulations underlying this figure.

### 3.2.6 The main-course model and the Scandinavian model of inflation

So far in this chapter we have discussed two theoretical models of wage dynamics and inflation which, although they originated in different historical period, are mutually consistent. Both models are genuine dynamic models since they are built on an explicit distinction between the short-run and the long-run. The distinction between long-term steady state propositions and dynamics has not always been made clear, though. This is true for the Scandinavian model, which the main-course model is often confounded with.

The Scandinavian model, see Rødseth (2000, Ch 7.6), specifies the same three underlying assumptions as the main-course model: $H1_{mc}$ (we do not need the extended version of the hypothesis for the point we wish to make here), $H2_{mc}$ and $H3_{mc}$. But the distinction between long-run and dynamics is blurred in the Scandinavian model. Hence, for example, the dynamic equation for e-sector wages in the Scandinavian model is usually written as:

$$\Delta w_{e,t} = \beta_0 + \Delta mc_t,$$

(without a disturbance term for simplicity) which is seen to place the rather unrealistic restriction of $\alpha = 1$ on e-sector wage dynamics. Hence, the Scandinavian model specification can only be expected to be reasonable if the time period subscript $t$ refers to a span of several years, so that $\Delta w_{e,t}$ and $\Delta mc_t$ refer to for example 10 years averages of the two growth rates.[17]

### 3.2.7 Wage-price curves and the NAIRU/natural rate of unemployment

Most modern textbooks in macroeconomics contain models of trade unions and firm behaviour. The gist of these models is that firms attempt to mark-up up their prices on unit labour costs, while workers and unions on their part strive to make their real wage reflect the profitability of the firms, thus their real wage claim is a mark-up on productivity. Hence, there is an important and interesting conflict between workers and firms: Both parties are interested in the size of the real wage, but their goals are usually different. Neither of the parties have perfect control over the real wage: Unions have only limited influence on the real wage since their channel of influence is the bargaining process, which is about the size of *nominal* wage increases. Firms also influences the nominal wage as long as they are part of the bargaining, moreover they have a strong channel of influence on the real wage since they decide the size of price adjustment unilaterally through mark-up pricing. Nevertheless, the real wage is neither perfectly controlled nor perfectly foreseeable by the firms.

Although it might already be obvious to the reader, the bargaining model which we have set out above (and therefore also the main-course model) fits into this

[17] For an exposition and appraisal of the Scandinavian model, in terms of contemporary macroeconomics, see Rødseth (2000, Chapter 7.6)

framework. If we use $w_e^b$ in (3.12), or alternatively $w_e^*$ in H1$_{gmc}$, to denote the desired (or bargained) wage, and use $p^* = \phi q_s^* + (1-\phi)q_e$ to define $p^*$, the price level implied by the pricing of firms, we obtain:

$$w_e^* = m_{e0} + q_e + a_e + \gamma_{e1}u, \text{ and} \tag{3.37}$$
$$p^* = \phi(m_{se} - m_s) + \phi(w_e^* - a_s) + (1-\phi)q_e \tag{3.38}$$

where we have utilized H2$_{mc}$ and H3$_{mc}$ to derive (3.38) for the desired price $p^*$ as a mark-up on bargained unit-labour costs $(w_e^* - a_s)$. Due to the openness of the economy, $p^*$ also depends on foreign prices, $q_e$.

However, there is an important difference, since in our model, $w^*$ and $p^*$ corresponds to steady state equilibrium values, while the standard exposition of wage and price setting often sets $w_e = w_e^*$ and $p = p^*$ in each time period. The inevitable implication is that actual wages and prices are determined in each period are by a static model. As we have seen above, this implies that the speed of adjustment of wages and prices are so fast that that there is neglible nominal persistence, which is unrealistic by most standards. It is more reasonable to hold on to the interpretation that (3.37) and (3.38) are relationships that describe a steady state.

The steady state interpretation in its turn raises another important issue about which variables are determined by the two-equation static model. To look into this issue we simplify the notation by setting $a_s = a_e = a$, and $m_{se} = 0$, so that $w_e^* = w_s^* = w^*$ in line with the two sector model used above. In the steady state interpretation of the model can set $w = w^*$ and $p = p^*$, without loss of generality, and (3.37) and (3.38) can then be re-expressed as:

$$w - q_e - a = m_{e0} + \gamma_{e1}u, \tag{3.39}$$
$$w - q_e - a = m_s + \frac{1}{\phi}(p - q_e) \tag{3.40}$$

where (3.40) is simply (3.38) with the (e-sector) wage share on the left hand side of the equation. Equation (3.39) can be represented graphically as downward sloping line in a graph with $w - q_e - a$ along the vertical axis and $u$ along the horizontal axis. This is the *wage-setting* curve which is illustrated in figure 3.5. Equation (3.40) defines a horizontal line in the same graph. This is the *price-setting* curve.

The intersection point between the two curves is seen as the determination of the wage curve "natural rate of unemployment", or "non-accelerating inflation rate of unemployment", NAIRU, which has become a popular acronym. In the literature, the distinction between the natural rate and the NAIRU is often made in terms of whether the steady state unemployment rate depends on the rate of inflation (the NAIRU case), or is independent of the inflation rate. In the following we will use the term natural rate for brevity, and make it clear whenever there is a relationship between the steady state inflation and unemployment rates.

In the figure, $u^w$ denotes the wage curve natural rate of unemployment. There has become custom to identify $u^w$ with the steady state rate of unemployment consistent with wage bargaining and mark-up price setting. That interpretation is succinctly expressed by Layard et al. (1994)

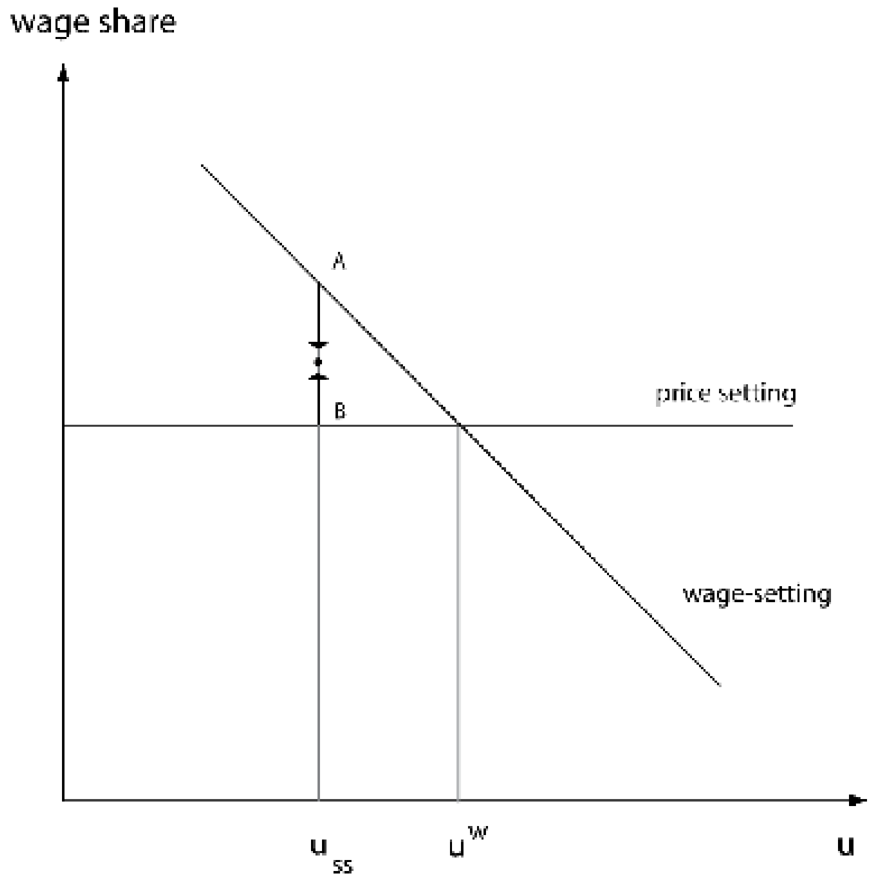


Figure 3.5: Real wage and unemployment determination, NAIRU indicated by $u^w$ .and a steady state rate of unemployment which is determined in part from outside the model of wage and price setting is denoted $u_{ss}$.

> 'Only if the real wage ($W/P$) desired by wage-setters is the same as that desired by price setters will inflation be stable. *And, the variable that brings about this consistency is the level of unemployment'.*[18]

The heuristics of this theory is that actual unemployment below the natural rate, $u_t < u^w$, will create wage increases and expectations of future wage-price adjustments in such a way that the result is $\Delta\pi_t > 0$, using the notation of section 2.4.2 that $\pi_t$ represents the increase in the price level. Conversely, $u_t > u^w$, entails falling rates of inflation $\Delta\pi_t < 0$. $u^w$ is the only (steady state) value of $u_t$ which is consistent with $\Delta\pi_t = 0$.

However, we have seen above that in the Norwegian model of inflation, there is an asymptotically stable equilibrium where both the rate of wage growth and the

[18] Layard et al. (1994, p 18), authors' italics.

inflation rate are constant *for any given constant rate of unemployment*. Since the Norwegian model can be interpreted as a wage-bargaining model, the (emphasized) second sentence in the above quotation has been disproved already: It is not necessary that the steady state rate of unemployment in a bargaining model corresponds to $u^w$ in figure 3.5.[19] The steady state rate of unemployment $u_{ss}$ may be lower than $u^w$, as is the case shown in the graph, or higher. The figure further indicates (by a •) that the steady state wage share in general will reside at a point on the line segment A-B: Heuristically, this is a point where price setters are trying to attain a lower real wage by nominal price increases, at the same time at the wage bargain is delivering nominal wage increases that push real wage upwards. Hence, in the general case the steady state is not characterized by the real wage desired by wage-setters being the same as that desired by price setters. Instead there is a tug-of-war equilibrium where the steady state real wage is a weighted average of the two parties respective targets for the wage share.

Bårdsen et al. (2005a, Ch 6) show which restrictions on the parameters of the dynamic equation for wage and price adjustments that are necessary for $u_t \rightarrow u_{ss} = u^w$ to be an implication, so that the $u^w$ corresponds to the stable steady state. In brief, the model must be restricted in such a way that the nominal wage and price setting adjustment equations become two conflicting dynamic equations for the real wage. In the context of an open economy, in particular, this step is anything but trivial. It is not sufficient to impose a restriction referred to as dynamic homogeneity for example (dynamic homogeneity will be defined below, in the context of the Phillips curve). What is required is n fact to purge the model of all nominal rigidity, which seems to be unrealistic on the basis of both macro and micro evidence.

We have explained that the Layard-Nickell version of the natural-rate/NAIRU concept corresponds to a set of restrictions on the dynamic model of wage and price setting. As will become clear in section 3.3, a similar conclusion is true for the natural rate of unemployment associated with a Phillips curve model for wages and prices.

The conclusion is that, special cases aside, the wage-bargaining model does not imply a unique supply-side determined steady state rate of unemployment. In order to avoid misunderstandings, it is perhaps worth stressing that there is no contradiction between this result, and the realistic belief that the average long-run rate of inflation is strongly influenced by supply-side factors. The implication is only that the steady state rate of unemployment is left undetermined in this particular model of the supply side. A wider iterpretation is that a realistic model of the steady state for unemployment generally requires a richer macro model than only equations for price and wage setting.

We end this section by noting another puzzled that has been more acknowledged than the problem with lack of correspondence between $u^w$ and a steady state rate of

[19] The exposition of wage bargaining dynamics and the Norwgian model was based on exogenous prices in the exposed sector, but this simplification represents no loss of generality regarding this issue about the necessitry (or not) of $u_{ss} = u^w$ withing the framwork of the bargaining model.

unemployment. This problem is understood by noting that the price-curve depends on the relative price $p - q_e$, so an unique intersection point in figure 3.5 does not exist unless we can fix $p - q_e$ at a certain value. Put differently: (3.39) and (3.40) are two equation in three unknown variables, and it is not clear why it is that $u$ should be regarded as endogenous, and not $p - q_e$. The most used argument is that $p - q_e$ can be regarded as determined from outside the system of wage and price setting equations. Since $p - q_e$ is an indicator of the degree of competitiveness in the economy, it makes sense to assume that its long-run value is given by the requirement that the trade balance (or the current account) is balanced in the long term. However, as we have seen in section 3.2.3, the properties of the dynamic wage and price setting system imply that if a steady state exists for wages (and the wage share $w_t - q_{et} - a_t$), a steady state also exists for $p_t - q_{et}$. Hence, from this perspective there is an internal inconsistency in the approach which determines the steady state of the rate of unemployment from static wage and price curves, by imposing a steady-state value of $p - q_e$ from outside the model of wage and price setting.

### 3.2.8 Role of exchange rate regime

Later in the book, a main focus will be the choice of exchange rate regime, and how that choice conditions macrodynamics. So far we have implicitly assumed that the rate of foreign exchange is exogenous, corresponding to a fixed exchange rate regime. In order to make this explicit, we need to expand the notation a little. For example, the sum of the logs of productivity and foreign price

$$mc_t = a_{et} + q_{et}$$

which can be written as

$$mc_t = \Delta a_{et} + \Delta q_{et} + mc_{t-1}, \tag{3.41}$$

or

$$mc_t = \Delta a_{et} + \Delta q^*_{et} + \Delta \ln E_t + mc_{t-1}. \tag{3.42}$$

where $q^*_{et}$ denotes the (log of) foreign prices in foreign currency, and $E_t$ is the nominal exchange rate (kroner/euro).[20] The reason why we have suppressed the nominal exchange rate $E_t$ from the model so far, is that in a fixed exchange rate regime, the $E_t$ has no separate influence on wage growth. It does not matter whether a change in $q_{et}$ "comes from" a change in $q^*_{et}$ or in $E_t$ (a devaluation for example).

However, in the case of a floating exchange rate, we need to specify the model of the market for foreign exchange in more detail. For example, if capital mobility is perfect, the so called risk premium is zero and the market is characterized by

[20] We use the explicit notation $\Delta \ln E_t$ rather than $\Delta e_t$, to avoid conflict of notation with $e$ as a identifier of the exposed sector (for examples in subscripts).

uncovered interest rate parity (see chapter 2.4.3 and Rødseth (2000, Chapter 1)), UIP:

$$i_t - i_t^* = e_t^e$$

where $i_t$ and $i_t^*$ are the domestic and foreign interest rates respectively, and $e_t^e$ denotes the expected rate of depreciation one period ahead. With perfect expectations we thus have

$$\Delta \ln E_t = i_{t-1} - i_{t-1}^* \tag{3.43}$$

where both $i_{t-1}$ and $i_{t-1}^*$are predetermined variables. From equations (3.42) and (3.43) it is clear that in the case when UIP holds, and with perfect depreciation expectations, the main-course variable $mc_t$ depends only on exogenous ($\Delta a_{et}$ and $\Delta q_{et}^*$) variables and predetermined variables ($i_{t-1}$ and $i_{t-1}^*$) and the stability condition $0 < \alpha < 1$ is necessary and sufficient conditions for stability also in the case of a floating exchange rate regime. Hence, it is not valid to *a priori* restrict the validity of the bargaining framework presented above to a fixed exchange rate regime.

However, it is also understood that after a change from a fixed to a floating exchange rate, the continued validity of a wage setting model should be evaluated carefully. The reason is that a change of regime in the market for foreign exchange may lead to structural changes elsewhere in the macroeconomic system, for example in wage setting.

## 3.3 The open economy Phillips curve

The price Phillips curve was introduced in section 2.4.2 as an example of an ADL equation. One often stated difference between the Phillips curve and the wage bargaining model is that the latter views the labour market as the most important source of inflation, while the Phillips curve's focus on product market. However, this difference is more a matter of emphasis than of principle, since both mechanism may be operating together. In this section we show formally how the two approaches can be combined by letting a *wage* Phillips curve take the role of a short-run relationship of nominal wage growth, while the steady state has the same properties as in the wage bargaining model.

An important insight to take away from this section is that also the Phillips curve can be given an equilibrium correction interpretation. The main difference from the Norwegian/bargaining model is the nature of the equilibrating mechanism: The bargaining model represents a way of organizing the economy such that there is enough collective rationality to secure dynamic stability of wages at *any given rate of unemployment*—including the low rates that one speak of "full employment" in practical terms. Wage growth and inflation never gets of control so to speak. The Phillips curve model represents a less sophisticated organization of the economy. According to that theory, unemployment *has* to adjust to a particular level for the rate of inflation to stabilize. This particular equilibrium level of unemployment is logically equivalent to the NAIRU/natural rate $u^w$ of section 3.2.7. However, because we now derive the natural rate of unemployment in a explicit Phillips curve framework, we will denote it $u^{phil}$ as a reminder of the model dependency of the natural rate.

Without loss of generality we concentrate on the wage Phillips curve. We also simplify the long-run model as much as possible, without losing consistency with the wage bargaining model that we presented above. The essential assumptions of the model we formulate is the following:

1. $w_{et}^{*} = m_{e0} + mc_t$, with $mc_t \equiv a_{et} + q_{et}$ exogenous both in the steady state and in each time period. $w_t^{*}$ denotes the asymptotically stable steady state solution for the logarithm of the wage-rate.

2. The unemployment rate $u_t$ is determined in the dynamic system, it has a asymptotically stable solution $u^{phil}$.

The framework we use is an ECM system, with a wage Phillips curve as one of the equations:

$$\begin{aligned} \Delta w_t &= \beta_{w0} + \beta_{w1}\Delta mc_t + \beta_{w2}u_t + \varepsilon_{wt}, \qquad \beta_{w2} < 0, &(3.44)\\ u_t &= \beta_{u0} + \alpha_u u_{t-1} + \beta_{u1}(w - mc)_{t-1} + \varepsilon_{ut} \qquad \beta_{u1} > 0 &(3.45)\\ 0 &< \alpha_u < 1, \end{aligned}$$

where we have simplified the notation somewhat by dropping the "e" sector subscript.[21] Compared to equation (3.17) above, we have also simplified by assuming that only the current unemployment rate affects wage growth. On the other hand, since we now use two equations, we have added a $w$ in the subscript of the coefficients. Note that compared to (3.17), the autoregressive coefficient (which would be $\alpha_w$ in the notation with subscribt for "wage equation") is set to unity in (3.44). This is not a simplification, but instead represents the defining characteristic of the Phillips curve. Equation (3.45) represents the idea that low profitability causes unemployment. Hence if the wage share is high relative to what can be sustained by the exposed sector, unemployment will increase, i.e., $\beta_{u1} > 0$.

(3.17) and (3.44) make up a dynamic system. Building on what we have learnt about stability in chapter 2.7 and 2.8, we know that for a given set of initial values ($w_0$,$u_0$, $mc_0$), the system determines a solution $(w_1, u_1)$, $(w_2, u_2)$, ...$(w_T, u_T)$ for the period $t = 1, 2, ...., T$. We also know that the solution depends on the values of $mc_t, \varepsilon_{w\,t}, \varepsilon_{u,t}$ over that period.

Without further restrictions on the coefficients, it becomes complicated to derive the *final equation* for $w_t$. However, as we also have learned, we can characterize the steady state, assuming that it exists (i.e., assuming that the dynamic the solution is stable). Usually, we can also understand the dynamics in qualitative terms even though the dynamic solution is beyond our capability. We follow this approach in the following.

[21] Alternatively, given $H2_{mc}$, $\Delta w_t$ represents the average wage growth of the two sectors.

As a first step, we derive the steady state solution of the system, assuming stable dynamics, so that a solution exists. We choose the solution based on $\varepsilon_{w\,t} = \varepsilon_{u,t} = 0$ and $\Delta mc_t = g_{mc}$ (i.e., the constant growth rate of the main-course), which gives rise to the following steady state:

$$\begin{aligned} \Delta w_t &= g_{mc}, \\ u_t &= u_{t-1} = u^{phil}, \text{ the equilibrium rate of unemployment.} \end{aligned}$$

since wages cannot reach a steady state unless also unemployment attains its steady state value, which we dub the equilibrium rate of unemployment, $u^{phil}$. Substitution into (3.45) and (3.44) gives the following long-run system:

$$\begin{aligned} g_{mc} &= \beta_{w0} + \beta_{w1} g_{mc} + \beta_{w2} u^{phil} \\ u^{phil} &= \beta_{u0} + \alpha_u u^{phil} + \beta_{u1}(w - mc) \end{aligned}$$

The first equation gives the solution for $u^{phil}$

$$u^{phil} = (\frac{\beta_{w0}}{-\beta_{w2}} + \frac{\beta_{w1} - 1}{-\beta_{w2}} g_{mc}), \tag{3.46}$$

which is the NAIRU/natural rate of unemployment implied by the Phillips curve model. We can also call $u^{phil}$ the "main-course rate of unemployment", since it is the rate of unemployment required to keep wage growth on the main-course.

Next, let us consider the dynamics of the system. Consider the dynamic solution based on $\Delta mc_t = g_{mc}$, and $\varepsilon_{w\,t} = \varepsilon_{u,t} = 0$ in all periods, starting from any historically determined initial condition. In this case, (3.44) becomes:

$$\Delta w_t - g_{mc} = \beta_{w0} + (\beta_{w1} - 1) g_{mc} + \beta_{w2} u_t,$$

or, using (3.46):

$$\Delta w_t - g_{mc} = \beta_{w2}(u_t - u^{phil}). \tag{3.47}$$

Because of the assumption that $\beta_{w2} < 0$, wage growth is higher than the main-course growth as long as unemployment is below the natural rate. Moreover, from the second equation of the system, (3.45):

$$u_t = \beta_{u0} + \alpha_u u_{t-1} + \beta_{u1}(w - mc)_{t-1},\ \beta_{u1} > 0$$

it is seen that a higher value of $w - mc$ contributes to higher unemployment in the next period. This analysis suggests that from any starting point on the Phillips curve, stable dynamics leads to the steady state solution. This indicates that the twin assumption of $\beta_{w2} < 0$ and $\beta_{u1} > 0$ is important for stability. Conversely, for example $\beta_{w2} = 0$ harms stability.

Exactly that the Phillips curve needs to be supplemented with an equilibrating mechanism in the form of the equation for the rate of unemployment is a major point of insight. Without such an equation in place, the system is incomplete, as there is

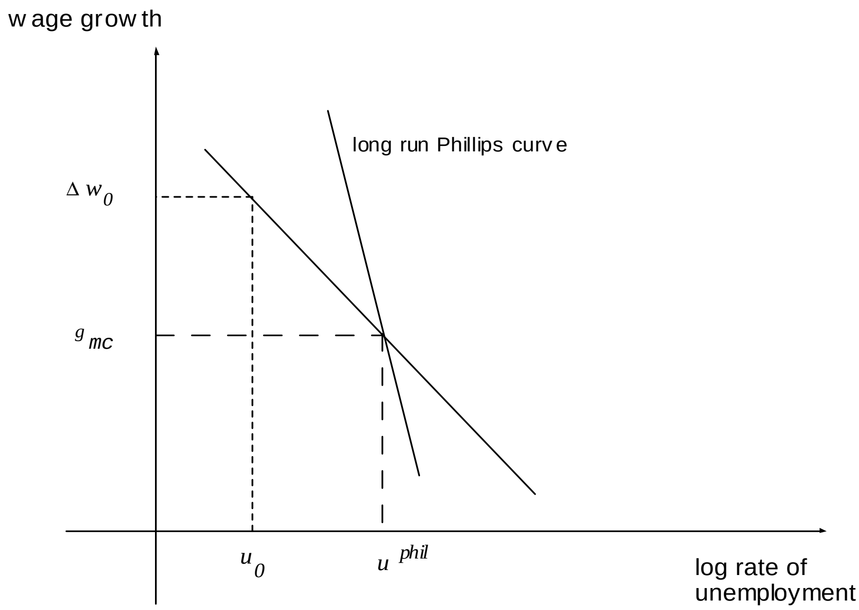


Figure 3.6: Open economy Phillips curve dynamics and equilibrium.

a missing equation. The question about the dynamic stability of the natural rate cannot be addressed in the single equation Phillips curve "system". In the bargaining model, wages equilibrium correct deviations from the main-course directly. In the Phillips curve case, wage equilibrium correction is assumed away. Dynamic stability of the system then becomes dependent on a equilibrating mechanism of a more indirect type which works through the rate of unemployment: In the case of too large wage increases, unemployment is a "disciplining device" which forces wage claims back on to a sustainable path.

The dynamics of the Phillips curve case is illustrated in Figure 3.6. Assume that the economy is initially running at a low level of unemployment, i.e., $u_0$ in the figure. The short-run Phillips curve (3.44) determines the rate of wage inflation $\Delta w_0$. Consistent with equation (3.47), the figure shows that $\Delta w_0$ is higher than the growth in the main-course $g_{mc}$. Given this initial situation, a process starts where the wage share is increasing from period to period. From equation (3.45) is is seen that the consequence must be a gradually increased rate of unemployment, away form $u_0$ and towards the natural rate $u^{phil}$. Hence we can imagine that the dynamic stabilization process takes place "along" the Phillips curve in Figure 3.6. When the rate of unemployment reaches $u^{phil}$ the dynamic process stops, because the impetus of the rising wage share has dried out.

The steep Phillips curve in the figure is defined for the case of $\Delta w_t = \Delta mc_t$. The slope of this curve is given by $-\beta_{w2}/(1-\beta_{w1})$, and it has been dubbed the long-run Phillips curve in the literature. The issue about the slope of the long-run Phillips curve is seen to hinge on the coefficient $\beta_{w1}$, the elasticity of wage growth with respect to the growth in the main-course. In the figure, the long-run curve is downward sloping, corresponding to $\beta_{w1} < 1$ which is referred to as *dynamic inhomogeneity* in wage setting. The converse, referred to as *dynamic homogeneity* in the literature, implies $\beta_{w1} = 1$, and the long-run Phillips curve is then vertical. Subject to dynamic homogeneity, the equilibrium rate $u^{phil}$ is independent of world inflation and productivity growth $g_{mc}$.

**Box 3.1 (A "bargaining based Phillips curve"?)** *Some important textbooks, for example Burda and Wyplosz (2005, Chapter 12.3) give a different message from ours, namely that wage bargaining gives raise to a standard price Phillips curve. However, that argument takes too lightly on the distinction between static relationships and dynamic adjustment. To show how, consider a closed economy version of our static model. Wage bargaining is then for the whole economy so we replace equation (3.12) by:*

$$w^* = m_0 + p^* + a + \gamma_1 u, \quad \gamma_1 < 0 \tag{3.48}$$

*and, the steady state price equation is simply.*

$$p^* = \ln(\epsilon) + w^* - a, \quad \epsilon > 1. \tag{3.49}$$

*Burda and Wyplosz then set $w^* = w$ in both equations. However, in the wage equation $p^*$ is replaced by $p^e$ which denotes the expected price level, while in the price equation $p^* = p$. Finally, time subscripts are added in both equations to give*

$$\begin{aligned} w_t - a_t &= m_{0t} + p_t^e + \gamma_1 u_t, \text{ and} \\ w_t - a_t &= -\ln(\epsilon_t) + p_t. \end{aligned}$$

*Solving for $p_t$ gives*

$$p_t = \ln(\epsilon_t) + m_{0t} + p_t^e + \gamma_1 u_t,$$

*and taking the difference on both sides of the equation gives:*

$$\Delta\pi_t = \Delta(\ln(\epsilon_t) + m_{0t}) + \Delta\pi_t^e + \gamma_1 \Delta u_t. \tag{3.50}$$

*Equation (3.50) is identical to Burda and Wyplosz's equation (12.11), with small qualification that they (implicitly) set $\Delta u_t = 0$. Their "bargaining" Phillips curve also assumes*

$$\Delta(-\ln(\epsilon_t) + m_{0t}) = a(Y_t - \bar{Y}).$$

*where $Y_t$ is GDP and $\bar{Y}$ the full employment output, in other words the output-gap (which is seen as perfectly correlated with the difference between current unemployment and the natural rate).*

The slope of the long-run Phillips curve represented one of the most debated issues in macroeconomics in the 1970 and 1980s. One arguments in favour of a vertical long-run Phillips curve is that workers are able to obtain full compensation for price increases. Therefore, in the context of our model, $\beta_{w1} = 1$ is perhaps the the only reasonable parameter value. The downward sloping long-run Phillips curve has also been denounced on the grounds that it gives a too optimistic picture of the powers of economic policy: namely that the government can permanently reduce the level of unemployment below the natural rate by "fixing" a suitably high level of inflation, see e.g., Romer (1996, Ch 5.5). In the context of an open economy this discussion appears as somewhat exaggerated, since a long-run trade-off between inflation and unemployment in any case does not follow from the premise of a downward-sloping long-run curve. Instead, as shown in figure 3.6, the steady state level of unemployment is determined by the rate of imported inflation and productivity growth as represented by $g_{mc}$. Neither of these are instruments of economic policy.[22]

Both the wage Phillips curve of this section, and the model for wages in section 3.2.4 have been kept deliberately simple. In the real economy, cost-of-living considerations play a significant role in wage setting. Thus, in empirical models one usually includes current and lagged consumer price inflation in the wage equation. Section 3.5 shows an empirical example. The above formal framework above can extended to accommodate this, without changing the conclusion about the steady state solution.

Another important factor which we have omitted so far from the formal analysis, is expectations. For example, instead of (3.44) we might have

$$\Delta w_t = \beta_{w0} + \beta_{w1}\Delta w_t^e + \beta_{w2}u_t + \varepsilon_{wt},$$

or $\Delta w_{t+1}^e$ for that matter. However, as long as expectations are influenced by the main-course variable, we will retrieve the same conclusion as above. For example

$$\Delta w_t^e = \varphi\Delta mc_t + (1-\varphi)\Delta w_{t-1}, \ \ 0 < \varphi \leq 1$$

In a steady state there are no expectations errors, so

$$\Delta w_t^e = \Delta w_{t-1} = g_{mc}$$

as before.

In section 3.2.3 we showed the implications of the bargaining model for domestic inflation. To establish the corresponding result for the case of the Phillips curve model, we repeat the definitional equation for the consumer price index:

$$p_t = \phi q_{st} + (1-\phi)q_{et}, \quad 0 < \phi < 1,$$

or, in differences

$$\Delta p_t = \phi\Delta q_{st} + (1-\phi)\Delta q_{et}. \tag{3.51}$$

[22] To affect $u^{phil}$, policy needs to incur a higher or lower permanent *rate* of currency depreciation.

Using the same simplifying assumptions as in section 3.2.3, for example that sheltered sector price growth is always on the long-run equilibrium path implied by the main-course theory, we obtain the price Phillips curve:

$$\Delta p_t = \phi\beta_{w0} + \{\phi\beta_{w1} + (1-\phi)\}\Delta q_{et} + \phi\beta_{w2}u_t + \phi\beta_{w1}\Delta a_{et} - \phi\Delta a_{st} + \phi\varepsilon_{wt}. \quad (3.52)$$

Hence, the reduced form inflation equation contains basically the same explanatory variables as the bargaining model. The exception is the absence of the wage equilibrium correction term.

Equation (3.52) is the counterpart to the standard price Phillips curve found in all current textbooks, for example Blanchard (2005, Ch 8). Without any logical inconsistencies, the Phillips curve can be augmented with terms that represent expectations, which can originate in wage setting (as hinted immediately above), or in s-sector price setting. As long as these expectation are based on experience, they will (only) imply a more complicated dynamic structure, but they will not affect the essential model properties that we have focused on in this section.

## 3.4 Summing up the open economy model

So far in this chapter we have presented two dynamic models of wage setting. Both are much used in modern macroeconomic thinking and model building. The following 7 points summarize the main results.

1. The first model used bargaining theory to rationalize a long run relationship between the wage level of the exposed sector and its main determinants: Product price ($q_{et}$), productivity ($a_{et}$) and the rate of unemployment ($u_t$). The elasticities of product price and productivity are both unity, and we therefore often subsume these two variables in one variable $mc_t = q_{et} + a_{et}$. With reference to an earlier theoretical development, the variable is called the main-course variable since it defines a long-run trend followed by wages.

2. If the long run wage equation is to correspond to a stable steady state growth path, we showed that there has to be a stable ADL equation for wages (i.e., $0 < \alpha < 1$,in the equation for $w_{et}$), which can be transformed into an ECM for wage growth.

3. The dynamic properties of the wage bargaining model are:

   (a) For a given (exogenously determined) level of $u_t$, wage growth equilibrium-corrects deviations form the main-course.

   (b) Therefore, there is a steady state level for the logarith omf ware share ($w_{et} - mc_t$).

   (c) Stability of wage growth and inflation also follows, even without assumptions about further equilibrium correcting behaviour in s-sector wage setting, or in price formation.

4. The second model postulates a wage Phillips curve, PCM, consistent with setting $\alpha = 1$ in the wage ADL equation. The following results were found to hold for the PCM specification of wage dynamics.

   (a) The PCM (by itself, viewed isolated from the rest of the model economy) gives an unstable solution for the wage share $w_{et} - mc_t$.

   (b) If the PCM is linked up with a second equation, which explains $u_t$ as an increasing function of the wage share, the two equation system implies a steady state level for the ware share $w_{et} - mc_t$.

   (c) The PCM implies a natural rate of unemployment ($u^{phil}$) which corresponds to the steady state level of unemployment implied by the 2-equation system.

5. Hence both models (wage bargaining and the PCM) implies a asymtotically stable wage share, and also stable rates of wage and price inflation.

6. The difference between the models lies in the *mechanism* that secures stability of the wage share

   (a) In the wage brgainig case tere is an amount of collective rationality. For example: Unions adjust their wage claims to the last years profitability. Inflation is stabilized at any given rate of unemployment, also low ones that prevailed in Europe until 1980 and in Scandinavia until the end of last century.

   (b) In the PCM case: There is less collective rationality. Instead unemployment serves as a *disciplining device*: There is *only one* level of unemployment at which the rate of inflation is stable, i.e., the natural rate of unemployment.

7. There seems to be some important implications for policy. For example:

   (a) If PCM is the true model, then self-defeating policy to try to target "full unemployment" below the natural rate.

   (b) If the wage bargaing moodel is the true model then it is not only possible to target full unemployment, it may also be advisable in order to maintain collective rationality (avoid breakdown in the bargaining system/institutions).

## 3.5 Norwegian evidence

In this section we illustrate how well wage equations corresponding to the Phillips curve (PCM), and the wage bargaining model, fit the data for Norwegian manufacturing. Readers with no familiarity with econometrics should read on since we abstract from all technical detail, and explain the economic interpretation of the findings, and how they relate to the two theoretical models presented in this chapter.

### 3.5.1 A Phillips curve model

We use an annual data set for the period 1965-1998. In the choice of explanatory variables and of data transformations, we build on existing studies of the Phillips curve in Norway, cf. Stølen (1990,1993). The variables are in log scale (unless otherwise stated) and are defined as follows:

$wc_t$ = hourly wage cost in manufacturing;

$q_t$ = index of producer prices (value added deflator);

$p_t$ = the official consumer price index (CPI index);

$a_t$ = average labour productivity;

$tu_t$ = rate of total unemployment (i.e., unemployment includes participants in active labour market programmes);

$rpr_t$ = the replacement ratio;

$h$ = the length of the "normal" working day in manufacturing;

$t1$ = the manufacturing industry payroll tax-rate (not log).

Note, with reference to the previous sections, that the main-course variable would be

$$mc_t = a_t + q_t.$$

The estimated Phillips curve, using ordinary least squares, OLS, is shown in equation (3.53). The term of the left hand side, $\widehat{\Delta wc_t}$, is the fitted value of the growth rate of hourly wage costs (hence the OLS residual would be $\Delta wc_t - \widehat{\Delta wc_t}$). On the left hand side, we have the variables that are found to be significant.[23][24] The numbers in brackets below the coefficients are the estimated coefficient standard errors, which are used to judge the statistical significance of the coefficients. A rule-of-thumb says that if the ratio of the coefficient and its standard error is larger than 2 (in absolute value for negatively signed coefficients), the underlying true parameter (the $\beta$) is almost certainly different from zero. Applying this rule, we find that all the coefficients of the Norwegian Phillips curve are significant.

[23] We have followed a strategy of first estimating a quite general ADL, where both lagged wage growth (the AR term) and productivity growth (current and lagged), and the lagged rate of unemployemnt is includedt. A procedure called general-to-specific modelling has been used to derive the final specification in equation (3.53).

[24] Batch file: `aar_NmcPhil_gets.fl`

$$\begin{aligned}\widehat{\Delta wc_t} \;=\; & -\underset{(0.01)}{0.0683} + \underset{(0.11)}{0.26}\,\Delta p_{t-1} + \underset{(0.09)}{0.20}\,\Delta q_t + \underset{(0.06)}{0.29}\,\Delta q_{t-1} \\ & -\underset{(0.004)}{0.0316}\,tu_t - \underset{(0.01)}{0.07}\,IP_t\end{aligned} \tag{3.53}$$

The estimated equation is recognizable as an augmented Phillips curve, although there are some noteworthy departures form the 'pure' wage Phillips curve of section 3.3. First, there is the rate of change in the CPI-index, $\Delta p_{t-1}$, which shows that Aukrust's basic model is too stylized to fit the data. Wage setters in the (exposed) manufacturing sectors obviously care about the evolution of cost-of living, not only about the wage-scope For reference, it might be noted that before the start of each bargaining round in Norway, representatives of unions and of organizations on the firm side, aided by a team of experts, work out a consensus view on the outlook for CPI price increases. It is possible that the significance of the lagged rate of inflation in equation is due these institutionalized forecasts on the actual wage outcome.

A second departure from the theoretical main-course Phillips curve is the absence of productivity growth in equation (3.53) (estimation shows that they are statistically insignificant in this specification). Hence, the variables that represent the main-course are the current an lagged growth rate of the product price index. The rate of unemployment (in log form, remember) is a significant explanatory variable in the model, and is of course what turns this into an empirical Phillips curve. The last left hand side variable, $IP_t$, represents the effects of incomes policies and wage-price freezes, which have been used several times in the estimation period, as part of the wider setting of coordinated and centralized wage bargaining.[25]

As discussed above, a key parameter of interest in the Phillips curve model is the main-course natural rate of unemployment, denoted $u^{phil}$ in equation (3.46). Using the coefficient estimates in (3.53), and setting the growth rate of prices ($\delta_f$) and productivity growth equal to their sample means of 0.06 and 0.027, we obtain an estimated natural rate of 0.0305 (which is as nearly identical to the sample mean of the rate of unemployment (0.0313)).

Figure 3.7 shows the sequence of natural rate estimates over the last part of the sample— together with $\pm 2$ estimated standard errors and with the actual unemployment rate for comparison. The figure shows that the estimated natural rate of unemployment is relatively stable, and that it is appears to be quite well determined. 1990 and 1991 are notable exceptions, when the natural rate apparently increased from to 0.033 and 0.040 from a level 0.028 in 1989. However, compared to confidence interval for 1989, the estimated natural rate has increased significantly in 1991, which represents an internal inconsistency since one of the assumptions of this model is that $u^{phil}$ is a time invariant parameter.

Another point of interest in figure 3.7 is how few times the actual rate of unemployment crosses the line of the estimated natural rate. This suggest very sluggish

[25] $IP_t$ is a so called dummy variable: it is 1 when incomes policy is 'on', and 0 whne it is 'off'.

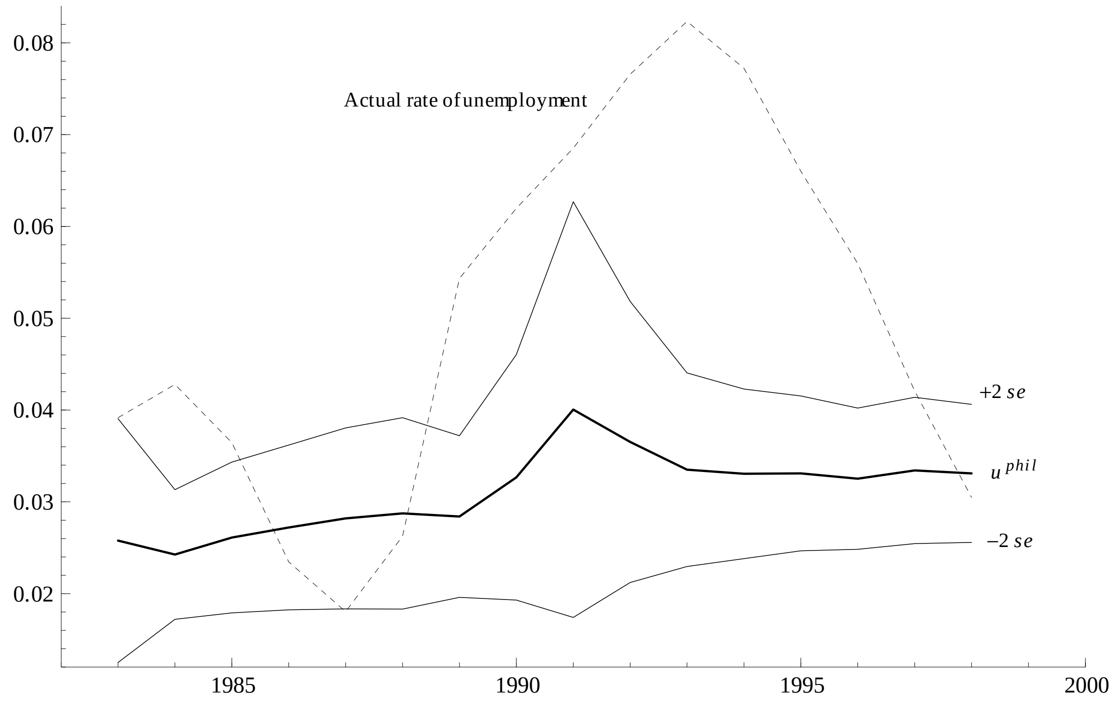


Figure 3.7: Sequence of estimated main-course natural rates, $u^{phil}$ in the figure (with $\pm 2$ estimated standard errors), and the actual rate of unemployment.

adjustment of actual unemployment to the purportedly constant equilibrium rate. In order to investigate the dynamics more formally, we have grafted the Phillips curve equation (3.53) into a system that also contains the rate of unemployment as an endogenous variable, i.e., an empirical counterpart to equation (3.45) in the theory of the main-course Phillips curve. As noted above, the endogeneity of the rate of unemployment is just as much a part of the natural rate framework as the wage Phillips curve itself, since without the "unemployment equation" in place one cannot show that the natural rate of unemployment obtained from the Phillips curve corresponds to a steady state of the system.

We have therefore estimated a version of the dynamic Phillips curve *system* given by equation (3.44)-(3.45) above. We do not give the detailed estimation results here, but Figure 3.8 offers visual inspection of some of the properties of the estimated model. The first four graphs show the actual values of $\Delta p_t$, $tu_t$, $\Delta wc_t$ and the wage share $wc_t - q_t - a_t$ together with the results from dynamic simulation. As could be expected, the fits for the two growth rates are quite acceptable. However, the "near instability" property of the system manifests itself in the graphs for the level of the unemployment rate and for the wage share. In both cases there are several consecutive year of under- or overprediction. The last two displays contain the cumulated dynamic multipliers of $tu$ and and the wage share, resulting from a 0.01 point increase in the unemployment rate. The striking feature is that any evidence of dynamic stability is hard to gauge from the two responses. Instead, it is as if the level of unemployment and the wage share "never" return to their initial values. Thus, in this Phillips curve system, equilibrium correction is found to be extremely weak.

As already mentioned, the belief in the empirical relevance a the Phillips curve natural rate of unemployment was damaged by the remorseless rise in European unemployment in the 1980s, and the ensuing discovery of great instability of the estimated natural rates. In that perspective, the variations in the Norwegian natural rate estimates in figure 3.7 are quite modest, and may pass as relatively acceptable as a first order approximation of the attainable level of unemployment. However, the estimated model showed that equilibrium correction is very weak. After a shock to the system, the rate of unemployment is predicted to drift away from the natural rate for a very long period of time. Hence, the natural rate thesis of asymptotically stability is not validated.

There are several responses to this result. First, one might try to patch-up the estimated unemployment equation, and try to find ways to recover a stronger relationship between the real wage and the unemployment rate, i.e., in the empirical counterpart of equation (3.45). In the following we focus instead on the other end of the problem, namely the Phillips curve itself. In fact, it emerges that when the Phillips curve framework is replaced with a wage model that allows equilibrium correction to any given rate of unemployment rather than to only the "natural rate", all the inconsistencies are resolved.

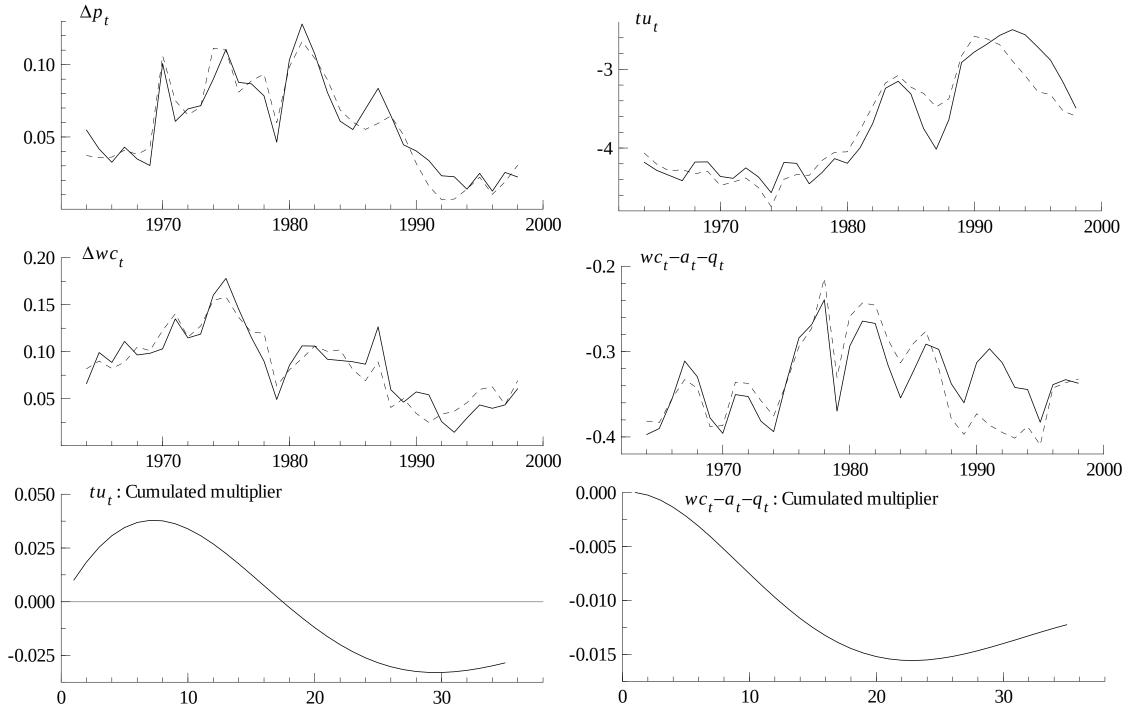


Figure 3.8: Dynamic simulation of the Phillips curve model. Panel a)-d) Actual and simulated values (dotted line). Panel e)-f): multipliers of a one point increase in the rate of unemployment

### 3.5.2 An error correction wage model

In section 3.2.4 we discussed the main-course model and its extensions to modern wage-bargaining theory. Equation (3.54) shows an empirical version of an equilibrium correction model for wages, similar to equation (3.21) above:

$$\begin{aligned} \widehat{\Delta w_t} = & - \underset{(0.01)}{0.197} - \underset{(0.03)}{0.478}\, ecm_{w,t-1} + \underset{(0.05)}{0.413}\, \Delta p_{t-1} + \underset{(0.04)}{0.333}\, \Delta q_t \\ & - \underset{(0.13)}{0.835}\, \Delta h_t - \underset{(0.01)}{0.0582}\, IP_t \end{aligned} \tag{3.54}$$

The first explanatory variable is the error correction term $ecm_{w,t-1}$ which corresponds to $w_{e,t} - w_e^*$ in section 3.2.4.2. The estimated $w_e^*$ is a function of $mc$, with the homogeneity restriction (3.20) imposed.[26] The estimated value of $\gamma_{e,1}$ is $-0.01$, hence we have:

$$ecm_{w,t-1} = wc_{t-1} - mc_{t-1} - 0.01\, tu_{t-1}$$

[26]So-called cointegration techniques have been used to estimate the relationship dubbed $H1_{gmc}$ in section 3.2.4.1. A full analusis is documented in Bårdsen et al. (2005b, chapter 6).

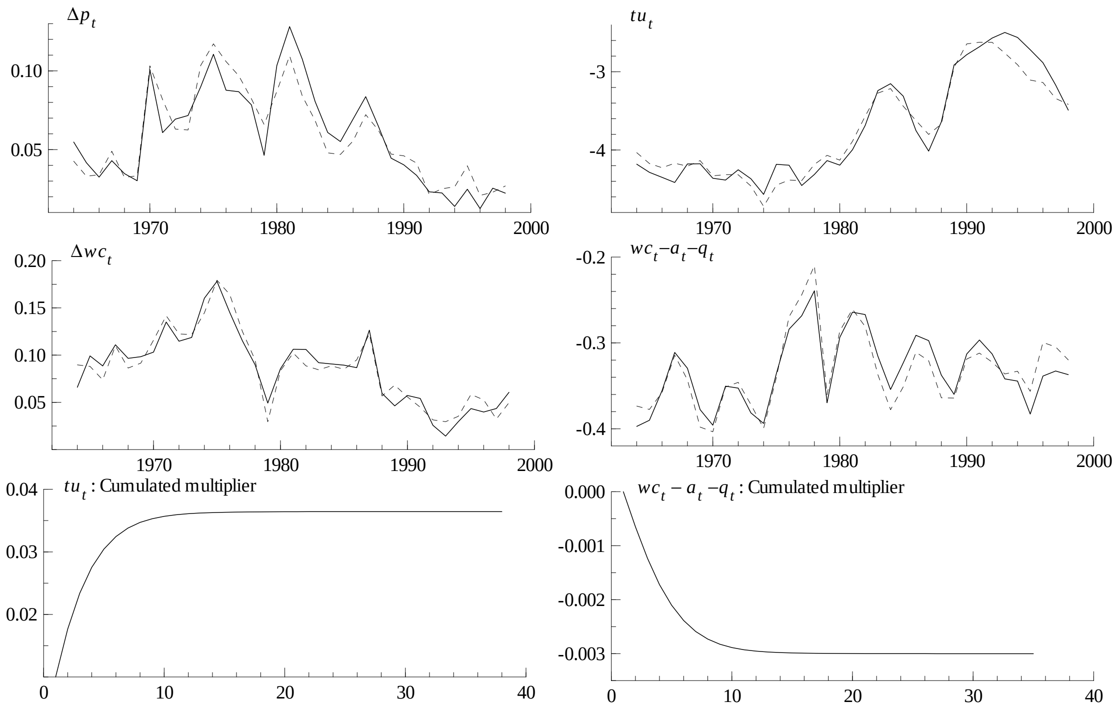


Figure 3.9: Dynamic simulation of the ECM model Panel a)-d) Actual and simulated values (dotted line). Panel e)-f): multipliers of a one point autonomous increase in the rate of unemployment

The variable $\Delta h_t$ represents institutional changes in the length of the working week. The estimated coefficient captures that the pay-losses that would otherwise have followed from the reductions in working hours have been partially compensated in the negotiated wage settlements.

The main motivation here is however to compare this model with the estimated Phillips curve of the previous section. First note that the coefficient of $ecm_{w,t-1}$ is relatively large, a result which is in direct support of Aukrust's view that there are wage-stabilizing forces at work even at a constant rate of unemployment. To make further comparisons with the Phillips curve, we have also grafted (3.54) into dynamic system that also contains an equation for the rate of unemployment (the estimated equation for $tu_t$ is almost identical to its counterpart in the Phillips curve system). Hence we have in fact an estimated version of the calibrated simulation model of section 3.2.5.

Some properties of that system is illustrated in figure 3.9. For each of the four endogenous variables shown in the figure, the model solution (i.e. the lines denoted "simulated values") is closer to the actual value than is the case in Figure 3.8 for the Phillips curve system. The two last panels in the figure show the cumulated dynamic multiplier of an exogenous shock to the rate of unemployment. The difference from figure 3.8, where the steady state was not even "in sight" within the 35 year simulation period, are striking. In figure 3.9, 80% of the long-run effect is reached within 4 years, and the system has reached a new steady state by the end of the first 10 years of the solution period. The conclusion is that this system is more convincingly stable than the Phillips curve version of the main-course model. Note also that the estimated model, which uses real data of the Norwegian economy, has the same dynamic properties as the calibrated theory model of section 3.2.5.

Economists are known to state that it is necessary to assume a vertical Phillips curve to ensure dynamic stability of the macroeconomy. In opposition to this view, the evidence presented here shows not only that the wage price system is stable when the Phillips curve is substituted by a wage equation which incorporates direct adjustment also with respect to profitability (consistent with the Norwegian model and with modern bargaining theory). Quite plainly, the system with the alternative (equilibrium correcting) wage equation has much more convincing stability properties than the Phillips curve system.

## 3.6 The New Keynesian Phillips curve

The New Keynesian Phillips Curve, NPC hereafter, has rapidly gained popularity and has become an integral part of the New Keynesian Model of monetary policy, as the aggregate supply equation in that model.

Compared to the models above which addressed wage and price setting jointly, the theory of the NPC is one-sided it its focus on firms' price setting. Although this may seem to be a drawback, it nevertheless explains some of the success of the NPC. The omission of a theory of trade union behaviour for example makes the NPC fit like hand in glove into dynamic stochastic equilibrium models (DSGE models) which assumes that the labour market is perfectly competitive, and that each individual chooses her own wage in an optimal way. Unemployment is not recognized as an economic pathology, fluctuations on hours worked being instead interpreted as due to intertemporal substitution of labour.[27] In this respect DSGE models are similar to the RBC model in chapter 2.8.4.

### 3.6.1 The 'pure' NPC model

Galí and Gertler (1999) gives a formulation of the NPC which is in line with the model by Calvo (1983) on staggered contracts and rational expectations: They assume that a representative firm takes account of the expected future path of nominal marginal costs when setting its price, given a probability that the price thus set "today" will remain fixed for many time periods ahead. This leads to a an equation of the form

$$\Delta p_t = b_{p_1} \mathsf{E}_t \Delta p_{t+1} + b_{p2} x_t + \varepsilon_{pt}, \;\; b_{p_1} > 0, \, b_{p2} \geq 0, \tag{3.55}$$

[27] The seminal presentation of the DSGE framework is the paper by Smets and Wouters (2003).

where $p_t$ denotes the logarithm of a general price index. Note that we make a slight change in the notation that we have used above, where we distinguished between product prices in the two sectors of the open economy, and $p_t$ denoted the consumer price index of the open economy.

$\mathsf{E}_t\Delta p_{t+1}$ denotes expected inflation one period ahead conditional upon information available at time $t$. As you may have become accustomed to by now, $\varepsilon_{pt}$ in (3.55) denotes a disturbance term. In the NPC in (3.55), $x_t$ denotes the logarithm of the wage share, which Gali and Gertler argue is the best operational measure of the firm's real marginal costs which is the theoretical explanatory variable. With this definition of $x_t$ in mind, note that (3.55) can be written as

$$p_t = \frac{1}{1+b_{p2}}(p_{t-1} + b_{p_1}\mathsf{E}_t\Delta p_{t+1}) + \frac{b_{p2}}{1+b_{p2}}(w_t - a_t) + \varepsilon_{pt}$$

which takes the same general form as the static normal cost pricing model, for example equation (3.15), but the mark-up coefficient is variable in the NPC model, and the elasticity with respect to the unit labour costs is less than one.

Although Gali and Gertler prefer the wage-share formulation, it might be noted that Roberts (1995) has shown that several New Keynesian models have (3.55) as a common representation, but with different operational measures of $x_t$. Hence, in the bigger picture, equations using the output gap, the unemployment rate or the wage share in logs, can all be counted as NPC models of inflation. It is understood that when the unemployment rate is used, $b_{p2} \leq 0$.

The rationale for the forward-looking term in (3.55) is the assumption of staggered price setting. As mentioned, the assumption is that each price setter knows that with a high probability, her next opportunity to adjust the price of her firm's product lies several periods into the future. Hence it is optimal to take the expected price changes of other price setters into account. When everybody behaves in this way at the micro level, the macro outcome is the functional relationship given in equation (3.55), between the expected rate of change in the price level one period ahead, and today's rate of inflation.

The appearance of the forward-looking term $\mathsf{E}_t\Delta p_{t+1}$ has important consequences for the dynamic stability properties of this model of inflation. To understand how, replace the expectations term with the actual forward rate, using $\mathsf{E}_t\Delta p_{t+1} = \Delta p_{t+1} - \eta_{t+1}$, where $\eta_{t+1}$ is the expectations error, to obtain

$$\Delta p_t = b_{p1}\Delta p_{t+1} + b_{p2}x_t + \epsilon_{p,t} \tag{3.56}$$

where $\epsilon_{p,t} = \varepsilon_{pt} - b_{p1}\eta_{t+1}$ is the joint disturbance term made up of the expectations error and the inflation surprise term $\varepsilon_{pt}$. This equation is very much like an ADL model for inflation, for example in the form of the augmented price Phillips curve in equation (2.13) in section 2.4.2 above. There are two trivial differences of notation: as $\pi_t$ in (2.13) is replaced by $\Delta p_t$ in (3.56), and the distributed lag in $u_t$ in (2.13) is replaced by the single explanatory variable $x_t$ in the NPC. The essential difference however is that instead of the lagged inflation term in the augmented Phillips curve, there is a lead term in the NPC. Nevertheless, in direct parallel to the standard backward looking augmented Phillips curve, equation (3.56) implies a unique solution for the rate of inflation, as long as we can pin down an initial condition (remember that this was crucial for the solution also in the ordinary ADL model).

Just as in the case of an ordinary ADL model, the solution is conditional on the assumptions made about the time series $x_t$ and $\epsilon_{p,t}$. In the same way as in section 2.7.1, assume first that $x_t$ is an exogenous variable with known values, and that the disturbance term $\epsilon_{p,t}$ is always zero (i.e., it is replaced by its mean). In the ordinary ADL case, we found the solution by repeated (backward) substitution, and an analogous technique works also in this case, but with *forward* substitution. Thus, by combining equation (3.56) with the NPC equation for period $t+1$, namely

$$\Delta p_{t+1} = b_{p1}\Delta p_{t+2} + b_{p2}x_{t+1}$$

we obtain

$$\begin{aligned}\Delta p_t &= b_{p1}(b_{p1}\Delta p_{t+2} + b_{p2}x_{t+1}) + b_{p2}x_t \\ &= b_{p1}^2\Delta p_{t+2} + b_{p2}x_t + b_{p1}b_{p2}x_{t+1}\end{aligned}$$

and repeated substitution gives

$$\Delta p_t = b_{p1}^j\Delta p_{t+j} + b_{p2}\sum_{i=0}^{j-1} b_{p1}^i x_{t+i}, \quad j = 1, 2, ..... \tag{3.57}$$

Based on the assumptions just made, about known $x$-values in all periods and zero disturbances in all future periods, (3.57) represents a unique solution for $\Delta p_t$ as long as we can fix the value of $\Delta p_{t+j}$, which is often referred to as the terminal condition. Technically peaking, the terminal condition serves the same purpose as the initial condition in the case of the ordinary ADL model of section 2.7.1—namely to secure uniqueness of the solution. However, while knowledge of the initial condition represents a weak requirement, (it is given from history) there is no way of knowing the inflation rate far into the future, which is what the requirement of fixing the value of $\Delta p_{t+j}$ amounts to. In general, there is a continuum of solutions to consider, each solution corresponding to a different value of the terminal condition.

There are two solutions to the non-uniqueness problem. First there are some cases where a particular terminal condition stands out as more natural or relevant. An example might be a central bank which has NPC as its model of price setting, and which runs an inflation targeting monetary policy. For this central bank it is natural to choose $\Delta p_{t+j} = \pi^*$ as the terminal condition, where $\pi^*$ denotes the inflation target. Hence the solution

$$\Delta p_t = b_{p1}^j\pi^* + b_{p2}\sum_{i=0}^{j-1} b_{p1}^i x_{t+i}, \quad j = 1, 2, ..... \tag{3.58}$$

might be said to represent the rate of inflation in the case of a credible inflation targeting regime, given that the NPC model is the correct model of inflation, and given that all future $x-$values are known with certainty.

The other, more general, way around the non-uniqueness problem is to invoke the asymptotically stable solution. In the ordinary ADL model of section 2.7.1, the hallmark of the stable solution was that the influence of the initial condition became negligible as we moved forward in time. In the same manner we note that the role of the terminal condition in the solution (3.57) is reduced as we let $j$ approach infinity, subject to the condition that $b_{p1}$is less than one in absolute value: $-1 < b_{p1} < 1$, the same condition that we saw above applied to the autoregressive coefficient in the ADL model.

Subject to the stability condition, $b_{p1}^{j} \rightarrow 0$ when $j$ gets infinitively large, and we can write the asymptotically stable solution as

$$\Delta p_t = b_{p2} \sum_{i=0}^{\infty} b_{p1}^{i} x_{t+i},$$

which, subject to the further simplifying assumption of constant $x_t$ in all periods ($x_t = m_x$, for all $t$), can be written as

$$\Delta p_t = \frac{b_{p2}}{1 - b_{p1}} m_x \tag{3.59}$$

Note that this solution amounts to using the asymptotic steady state solution for the inflation rate in period $t$, and that the distinction between the dynamic solution and the steady state solution, no longer plays a role in (3.59). This is a consequence of adopting the asymptotically stable solution to get around the non-uniqueness problem. In terms of economic interpretation (3.59) is seen to imply that the short-run effect of a change in $x_t$ is the same as the long-run effect: There is no persistence in inflation adjustment.

### 3.6.2 A NPC system

Above, when we discussed the main-course model and the ordinary Phillips curve model, we learned to regard the dynamic behaviour of inflation as a system property, which depends on the assumptions made about the explanatory variables of the structural price and wage equations. The analysis of the Phillips curve system in section 3.3 provides an example.

In the NPC there is only one explanatory variable, namely $x_t$, and the above solutions for the rate of inflation above are based on exogenous and known $x_t$ (for all $t$). More generally, we can formulate a NPC system, consisting of equation (3.56) and a completing equation for the explanatory variable $x_t$:

$$x_t = b_{x0} + b_{x1} \Delta p_{t-1} + b_{x2} x_{t-1} + \varepsilon_{xt}. \tag{3.60}$$

If $x_t$ is the logarithm of wage share, the sign restriction on the $\Delta p_{t-1}$ coefficient is $b_{x1} \geq 0$, while the case of unemployment is cover by setting $b_{x1} \leq 0$. As for the autoregressive coefficient we may simplify by only considering non-negative values, hence $0 \leq b_{x2} \leq 1$. Having thus specified a NPC system it is easy to see that solution (3.57) corresponds to setting $b_{x1} = 0$, and assuming that we know the values of $b_{x0}$ and $b_{x2}$ as well as all the disturbances $\varepsilon_{xt}$. Solution (3.59) adds further assumptions, namely that $b_{x0} = m_x$, $b_{x2} = 0$ and that all the disturbances are zero.

It is beyond the scope of this book to derive the full solution of the dynamic system made up of equation (3.56) and (3.60), and interested readers are referred to Bårdsen et al. (2005a, Ch 7.3) which discusses several possibilities of dynamic behaviour. Instead, we make use of the method of analysis which we explained in 1.4, namely of (first) considering the stationary solution, assuming that the system is dynamically stable.

Denote the stationary values of inflation and $x$ by $\pi^*$ and $x^*$ respectively. The long-run NPC model is thus

$$\begin{aligned} (1 - b_{p1})\pi^* - b_{p2}x^* &= 0 \\ -b_{x1}\pi^* + (1 - b_{x2})x^* &= b_{x0} \end{aligned}$$

and the solution for $\pi^*$ is

$$\pi^* = \frac{b_{p2}}{(1 - b_{p1})(1 - b_{x2}) - b_{p2}b_{x1}} b_{x0} \tag{3.61}$$

showing that the steady state rate of inflation depends on coefficients from both the $x_t$ equation, $b_{x1}$ and $b_{x2}$, as well as on and the NPC model itself.

The case of exogenous $x_t$ is represented by $b_{x1} = 0$ and in this case (3.61) can be written as

$$\pi^* = \frac{b_{p2}}{(1 - b_{p1})} \frac{b_{x0}}{(1 - b_{x2})}. \tag{3.62}$$

Defining $m_x = b_{xo}/(1 - b_{x2})$ the right hand side can be seen to be identical to equation (3.59), as we would expect.

In the case of exogenous $x_t$ it is always relatively easy to derive the full, dynamic, solution consistent with rational expectations both with regards to inflation and the future values of $x_t$. For example, using the exposition in Bårdsen et al. (2005a, Appendix A.2), we obtain

$$\Delta p_t = \frac{b_{p2}}{(1 - b_{p1}b_{x2})} x_t \tag{3.63}$$

when the disturbance term is ignored (set to zero).[28] The solution in (3.63) is a generalization of equation (3.59) above. When deriving (3.59), the expression $\sum_{i=0}^{\infty} b_{p1}^i x_{t+i}$ was simplified by setting each $x_{t+i}$ equal to the constant mean $m_x$. In (3.63) we have instead set $x_t = x_t$, and $x_{t+i} = b_{x2}^i x_t$ for $i > 0$. This corresponds to the to the full rational expectations solution, i.e. rational expectation both with respects to $\Delta p_{t+1}$ and with respect to the future values of marginal costs.

[28] The derivation is also based on an NPC system where $b_{x0} = 0$.

### 3.6.3 The hybrid NPC model

The ‘pure’ NPC model has been criticized for not being able to explain the observed inflation persistence. To overcome this problem, a so called hybrid Phillips curve which allows a subset of firms to have a backward-looking expectations, have been introduced. The hybrid version of the NPC can be written as

$$\Delta p_t = b^f_{p1} \mathsf{E}_t \Delta p_{t+1} + b^b_{p1} \Delta p_{t-1} + b_{p2} x_t + \varepsilon_{pt}, \tag{3.64}$$

where both $b^f_{p1}$ and $b^b_{p1}$ are assumed to be non-negative coefficients. $b^b_{p1}$ represent the expectations effects that are due to the share of firms who are backward-looking as in the usual expectations augmented Phillips curve. The lagged inflation term of the hybrid NPC has consequences for the solution. Heuristically the solution of the hybrid NPC shows more inflation persistence, and the response to changes in $x_t$ will be less sharp than in the case of the ‘pure’ NPC.

Bårdsen et al. (2005a, Appendix A.2) also give the solution of the rate of inflation in this case, i.e., when the NPC system is given by (3.64) and (3.60), with $b_{x0} = b_{x1} = 0$. Ignoring the disturbances the solution is

$$\Delta p_t = r_1 \Delta p_{t-1} + \frac{b_{p2}}{b^f_{p1}(r_2 - b_{x2})} x_t$$

where $r_1$ is a coefficient which is less than one in magnitude. $r_2$ is a coefficient which is larger than one. If $b^b_{p1} = 0$ it can be shown that $r_1 = 0$ and $r_2 = 1/b^f_{p1}$, so the solution reduces to the simple NPC.

Note the dramatic reduction in the number of explanatory variables compared to the main-course model in section 3.2.3 above, showing that the NPC model is basically a single explanatory variable model of inflation. In particular there are no variables representing open economy aspects in the equations we have presented so far. This has however been changed by subsequent theoretical developments which shows that the relative change in the real import prices is a theoretically valid second explanatory variable in the open economy NPC.

### 3.6.4 The empirical status of the NPC model

As mentioned above the NPC has made a large impact among macroeconomists. The gain in currency of the NPC is based on the apparent combination of theoretical microfoundations with empirical success. For example, using the euro area inflation data set of Galí et al. (2001), and their estimation methodology, we obtain

$$\Delta p_t = \underset{(0.04)}{0.91} \Delta p_{t+1} + \underset{(0.06)}{0.09} x_t + \underset{(0.06)}{0.1} \tag{3.65}$$

for a quarterly data set for the period 1971.3 -1998.1, and using the logarithm of the wage share as $x$ variable. Hence, the estimated value of $b_{p1}$ in the pure NPC model for the euro area is 0.914, and the estimated value of $b_{p2} = 0.088$. Using the rule-of-thumb method for testing significance which we explained in section 3.5.1, we see that the forward-looking term is significantly different form zero, and that the same conclusion applies to the wage share coefficient. We can also assess the significance of $(b_{p1} - 1)$ which is important for the relevance of the forward solution. Since $0.09/0.04 = 2.3$ there is statistical support for the claim that, although positive, $b_{p1}$ is still less than one, so the necessary condition for stability is fulfilled.

For the hybrid NPC, using again the same data and methodology, we obtain

$$\Delta p_t = \underset{(0.07)}{0.68}\ \Delta p_{t+1} + \underset{(0.07)}{0.28}\ \Delta p_{t-1} + \underset{(0.03)}{0.02}\ x_t + \underset{(0.12)}{0.06}\ . \tag{3.66}$$

showing that although the lagged inflation rate is a significant explanatory variable ($b^b_{p1}$ is estimated to be 0.34), the estimated coefficient $b^f_{p1}$ of the forward-term is twice as large. Note however that the $b_{p2}$ coefficient of the wage share is insignificant, so the contribution of the economic explanatory variable is dubious in (3.66). Nevertheless, similar estimation results for the euro area, the US and also for individual countries, are usually seen as a success for the NPC. A second argument recorded in favour of the NPC is the seemingly nice fit between the inflation rate predicted by the model and observed inflation rates. For example, Galí et al. (2001) states that "the NPC fits Euro data very well, possibly better than US data".[29] Even more recently, Galí (2003) writes

> ...while backward looking behaviour is often statistically significant, it appears to have limited quantitative importance. In other words, while the baseline pure forward looking model is rejected on statistical grounds, it is still likely to be a reasonable first approximation to the inflation dynamics of both Europe and the U.S. (Gali (2003, section 3.1).

Nevertheless, an unconditional declaration of success may still prove to be unwarranted, since goodness of fit—always a weak requirement—is saying very little of the quality of the NPC as an approximation to the true inflation process.

Figure 3.10 illustrates the point by graphing actual and fitted values of (3.65) in the first panel—showing a nice fit—and the NPC's fitted values together with the fit of a random walk in the scatter plot in the second panel. The similarity between the two series of fitted values is obvious (a regression line has been added for readability).

How can it be that the NPC with all its theoretical content fits no better than a random walk model of inflation, which (as explained in connection with Table 2.3 above) is completely void of economic interpretation? The answer is not difficult to find if we take a second look at the estimated model in equation (3.65). First, although formally less than one on a statistical test, the coefficient of $\Delta p_{t+1}$ is so large that little is lost in terms of fit by re-writing the estimated equation as

$$\Delta p_t = \Delta p_{t-1} - 0.09 x_{t-1} - 0.1.$$

[29] The *Abstract* of Galí et al. (2001).

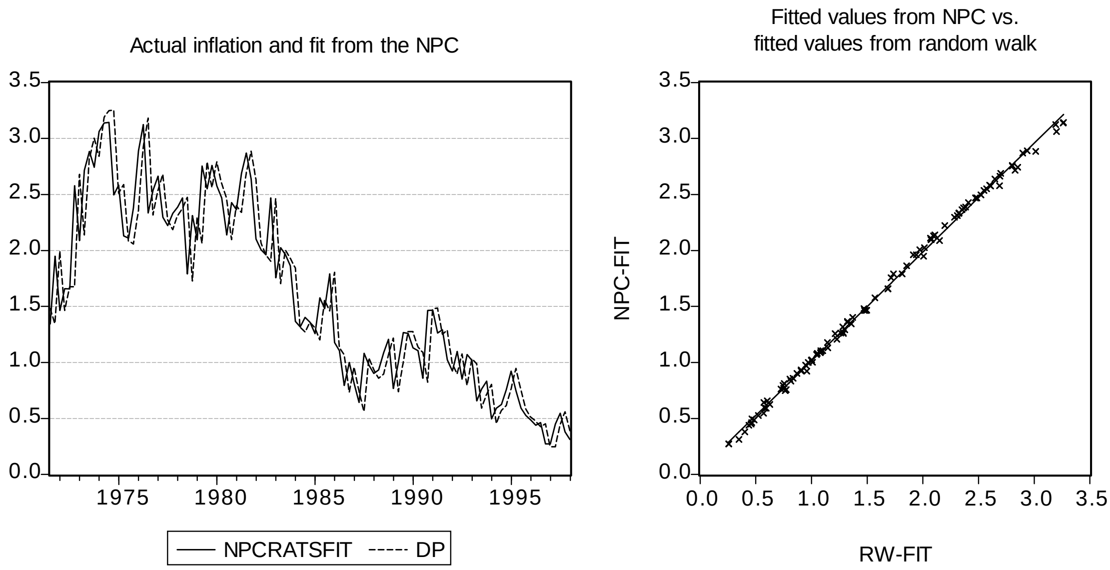


Figure 3.10: Actual euro area infaltion and fitted NPC inflation, together with the fit of a random walk model of euro area inflation.

Second, the contribution of $x_t$ to the NPC fit is actually not very large, because 0.09 is a small number when we take into regard the limited variability of the wage share. In sum therefore, we should not expect a big difference between the NPC fit and the fitted values from the random walk model

$$\Delta p_t = \Delta p_{t-1} + \beta_0 + \varepsilon_t,$$

which is exactly what figure 3.10 shows.

This argument can be extended to the hybrid NPC. Using the estimation results of the euro-area *hybrid* NPC in equation 3.66, the fitted rate of inflation ($\Delta\hat{p}_t$) is given by

$$\Delta\hat{p}_t = 0.09 + 1.06\Delta p_{t-1} + 0.41\Delta^2 p_{t-1} + 0.02x_{t-1}.$$

Note first, that since the variance of $\Delta^2 p_{t-1}$ is of a lower order than $\Delta p_{t-1}$, the sum $1.06\Delta p_{t-1} + 0.41\Delta^2 p_{t-1}$ is dominated by the first factor. Second, since the coefficient of the forcing variable $x_t$ is only 0.02, the variability of $x_{t-1}$ must be huge in order to have a notable numerical influence on $\Delta\hat{p}_t$. But as $x_t$ is the wage share, its variability is normally limited.

It is interesting to note that when the NPC is applied to other data sets, i.e., to such diverse data as US, UK and Norwegian inflation rates, very similar parameter estimates are obtained, see Bårdsen et al. (2002). Is this a sign of support for the

NPC account of inflation? Proponents of the NPC claim so, but another interpretation is that the NPC is almost void of explanatory power, and that it only captures a common feature among different countries data sets, namely autocorrelation.

Hence, the NPC (as an empirical) model fails to corroborate the theoretical message: that rational expectations transmits the movements of the forcing variable strongly onto the observed rate of inflation. Recently, it has been shown by **?** that the typical NPC fails to deliver the expected result that inflation persistence is 'inherited' from the persistence of the forcing variable. Instead, the derived inflation persistence, using estimated NPCs, turns out to be completely dominated by 'intrinsic' persistence (due to the accumulation of disturbances of the NPC equation). Quite contrary to the consensus view, Fuhrer shows that the NPC fails to explain actual inflation persistence by the persistence that inflation inherits from the forcing variable. Fuhrer summarizes that the lagged inflation rate is not a 'second order add on to the underlying optimizing behaviour of price setting firms, it *is* the model'.

More evaluation of the NPC is provided by e.g., Bårdsen et al. (2004) and Bårdsen et al. (2005a, Ch. 7) which also include testing of parameter stability (over sample periods), sensitivity (with respect to estimation methodology), robustness (e.g., inclusion of a variable such as the output-gap in the model) and encompassing (explaining preexisting models.[30] Expect for recursive stability, the results are disheartening for those who believe that the NPC represents a data coherent and theory driven model of price setting. As for recursive stability, it is more apparent than real since the inherent fit of the model is so poor that statistical stability tests have low power , and graphs of the sequence of recursive coefficient estimates of the forcing variable show 'stability around zero'. Even more fundamentally, there is a growing recognition the NPC model fails on an econometric property called identification.[31] Models with weak or no identification are usually met with suspicion from all quarters of the economics discipline, but so far the popularity of the NPC has been unscatched by such criticism.

Hence, after evaluation, the economic interpretation of estimated NPC models 'disappears out of the window' leaving a huge question mark hovering also over the empirical status of New Keynesian DSGE models, of which the DSGE is an integral part. Despite its initial promises, the NPC modelling approach thus seems to be heading towards a failure. One moment's thought suffice to make us recognize that this may not a surprise. The NPC offers a single variable explanation of inflation, which by all other accounts a complex socio-economic phenomenon, and it is more than plausible that a satisfactory inflation model will contain more than one explanatory variable. As we have seen, this insight was present already in the 1960s, and it has been developed further down the decades in theoretical and empirical models that take account the multi-faceted nature of inflation as a socio-economic phenomena.

That no empirically valid "single cause" explanation of inflation can be provided

[30] See also Bårdsen et al. (2002) which includes Norwegian data.

[31] See Mavroeidis (2005).

for any developed economy is a lesson forgotten by the proponents of the NPC model. That said, although the NPC models as they stand, are unsuited for policy related economic analysis, this does not preclude that forward-looking expectations terms could play a role in explaining inflation dynamics within other, statistically well specified, models, for example of the equilibrium correction type.

### Exercises

1. Is $\beta_{22} > 0$ a necessary and/or sufficient condition for path b to occur in figure 3.2?

2. What might be the economic interpretation of having $\beta_{21} < 0$ , but $\beta_{22} > 0$?

3. Assume that $\beta_{21} + \beta_{22} = 0$. Try to sketch the wage dynamics (in other words the dynamic multipliers) following a rise in unemployment in this case!

4. In section 3.3, the expression for the natural rate was found after first establishing the steady state solution for the system. To establish the natural rate of unemployment more directly, rewrite (3.44) as

$$\Delta w_t = \beta_{w1}\Delta mc_t + \beta_{w2}(u_t - \frac{\beta_{w0}}{-\beta_{w2}}) + \varepsilon_{w,t}, \qquad (3.67)$$

   and then the steady state situation: $\Delta w - g_{mc} = 0$, $\varepsilon_{w,t} = 0$. Show that (3.67) defines $u^{phil}$ as

$$0 = \beta_{w2}[u^{phil} - \frac{\beta_{w0}}{-\beta_{w2}}] + (\beta_{w1} - 1)g_{mc}.$$

5. Use the expressions for the wage and price curves in section 3.2.7 to derive the algebraic expression for $u^w$ in figure 3.5.

6. The solution method used for the 'pure' NPC on equation (3.56) was based on forward repeated substitution. Section 2.7.1 showed that the ordinary ADL is solved by backward substitution. Why does not backward substitution work for the NPC?

   Hint: The answer has to do with stability. To see why, note that (3.56) can be re-normalized on $\Delta p_{t+1}$ :

$$\Delta p_{t+1} = (1/b_{p1})\Delta p_t - (b_{p2}/b_{p1})x_t - (1/b_{p1})\epsilon_{p,t}$$

   or

$$\Delta p_t = (1/b_{p1})\Delta p_{t-1} - (b_{p2}/b_{p1})x_{t-1} - (1/b_{p1})\epsilon_{p,t-1} \qquad (3.68)$$

   i.e., an ADL model (albeit with no contemporaneous effect of and the disturbance term is lagged). Show that repeated backward substitution using (3.68) gives an unstable solution if $0 < b_{p1} < 1$, confer section 2.7.1 if necessary.

7. Note that in equation (3.58), giving the solution of the rate of inflation when the inflation target is used as a terminal condition, the number of periods ahead $j$, is a parameter to be determined. With reference to contemporary inflation targeting, and taking the time period to be annual, it is reasonable to set $j = 3$. If we assume that the two parameters are known numbers, for example $b_{p1} = 0.9$ and $b_{p2} = 0.1$, the inflation rate of period $t$ can be determined using $\pi^* = 0\,025$ and $x_t = \ln(0.5)$, $x_{t+1} = \ln(0.55)$, and $x_{t+2} = \ln(0.65)$. Work out $\Delta p_t$, using these numbers in equation (3.58). How is the solution affected if all three wage shares are 10 percentage points lower?

8. Compare the asymptotically stable solution in equation (3.59) with the stable long-run solution (2.31) of the ADL model in section 2.7.1. What seems to be the consequence of forward-looking terms for the stable solution of dynamic relationships in economics

9. Consider the NPC estimated on quarterly Norwegian data:[32]

$$\Delta p_t = \underset{(0.11)}{1.06}\Delta p_{t+1} + \underset{(0.02)}{0.01}x_t + \underset{(0.02)}{0.04}\Delta pb_t + \text{ dummies}$$

Inflation is measured by the quarterly change in the official Norwegian consumer price index. Because of the openness of the economy, the specification has been augmented heuristically with import price growth ($\Delta pb_t$) and dummies for seasonal effects as well as the special events in the economy described in Bårdsen et al. (2002). The estimation period is 1972.4 - 2001.1. $x_t$ is the logarithm of the wage share (total economy minus North-Sea oil and gas production).

Compare this equation with the euro area results.

10. Tveter (2005) estimates a NPC for Norway, explaining the domestic part of the consumer price index. The sample is 1980.1-2003.3 The $x_t$ is defined in the same way as in the equation in exercise 8. For the pure NPC, Tveter reports

$$\Delta p_t = \underset{(0.04)}{0.95}\Delta p_{t+1} - \underset{(0.001)}{0.001}x_t,$$

and for the hybrid model

$$\Delta p_t = \underset{(0.13)}{0.645}\Delta p_{t+1} + \underset{(0.12)}{0.31}\Delta p_{t-1} - \underset{(0.001)}{0.001}x_t.$$

Take notes on the similarities between these results and the euro area results in the main text.

[32], see Appendix B.

# Appendix A

# Variables and relationships in logs

Logarithmic tranformations of economic variables are used several times in the text, first in section 1.6. Logarithms possess some properties that aid the formulation of economic relationships (model building), and the visual inspection of model and data properties in graphs. This appendix reviews some key properties of the logarithmic function, and the characteristic of graphs.

Throughout we make us of logarithms to the base of Euler's number $e \equiv 2.171828....$, and we use the symbol ln for these natural logarithms. In general, the natural logarithm of a number or variable $x$ is the power to which $e$ must be raised to yield $a$, i.e., $e^{\ln a} = a$.

For reference, we put down the main rules for operating on the natural logarithmic function (both $x$ and $y$ are positive):

$$\ln(xy) = \ln x + \ln y \tag{A.1}$$

$$\ln(\frac{x}{y}) = \ln x - \ln y \tag{A.2}$$

$$\ln x^a = a \ln x, \; a \text{ is any number or variable} \tag{A.3}$$

From these basic rules, additional ones can be constructed. For example

$$\ln(x^a y^b) = a \ln x + b \ln y$$

which is often called a linear combination of the log transformed variables, with weights $a$ and $b$. In the main text, a prime example of a linear combination is the stylized definitional equation for the log of the consumer price index, see e.g. equation (**??**). In that equation the weights sum to one, corresponding to $b = 1 - a$. Another example of a linear combination is the weighted geometric average:

$$\ln((xy)^{1/2}) = 0.5(\ln(x) + \ln(y))$$

or, in general, with $n$ different $x$-es:

$$\ln((x_1 x_2 ... x_n)^{1/n}) = \frac{1}{n}\sum_{i=1}^{n} \equiv \frac{1}{n}(\ln x_1 + \ln x_2 + ... + \ln x_n).$$

Assume next that $y$ is a is function $f(x)$. The relationship is linear if $f(x) = a + bx$. A much used non-linear specification of $f(x)$ is

$$y = Ax^b,\ x > 0 \tag{A.4}$$

where $A$ and $b$ are constant coefficients. Note first that if we apply the definition of the elasticity

$$El_x y = f'(x)\frac{x}{y}$$

to (A.4), we obtain

$$El_x y = b \tag{A.5}$$

showing that the coefficient $b$ is the elasticity of $y$ with respect to $x$. Second, if we apply the rules for logarithm to (A.4) the following log-linear relationship is obtained:

$$\ln y = \ln A + b\ln(x),\ x \text{ is positive.} \tag{A.6}$$

The equation is linear in the logs of the variables, a property which is well captured by the name log-linear. Conveniently, the elasticity $b$ is the slope coefficient of the relationship. This is confirmed by taking the differential of (A.6):

$$\frac{d\ln y}{d\ln x} = b.$$

Drawing the graph of (A.4) is not easy, since the slope is different for each value of $x$. Panel a) of Figure A.1 shows a graph for the case of $A = 0.75$ and $b = 0.5$, while panel b) shows the corresponding graph for $\ln y$ and $\ln x$. Clearly, the slope in panel b) is constant at all values of $\ln x$, and is equal to 0.5.

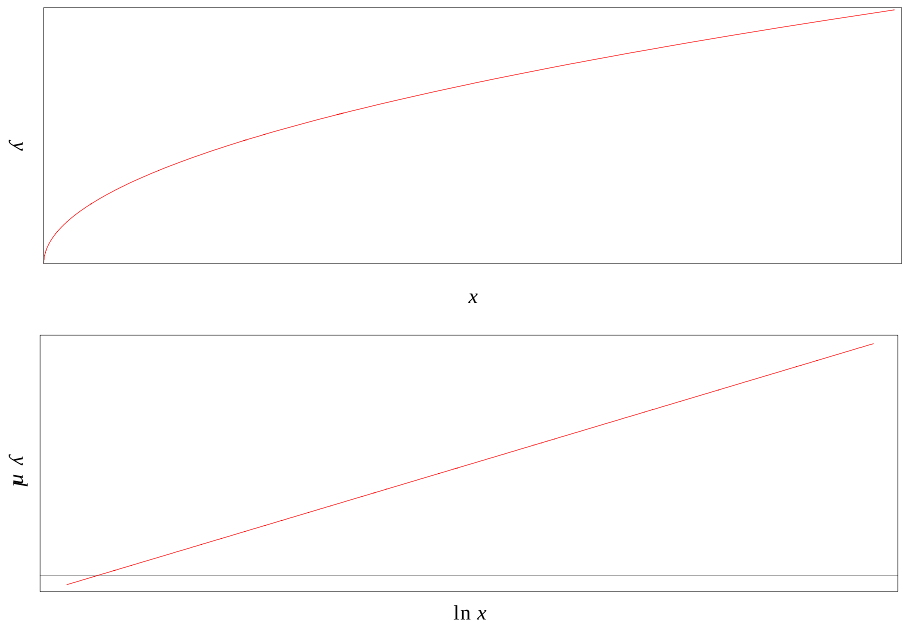


Figure A.1: Upper panel: The graph of the function $y = Ax^b$ when $A = 0.75$, $b = 0.5$ and $x$ is a variable with average growth rate 0.03. Bottom panel: The graph og $\ln y = \ln A + b \ln x$ .

The growth rate of a time series variable $x_t$ over its previous value $x_{t-1}$ is $(x_t - x_{t-1})/x_{t-1}$, or using the difference operator explained in the main text: $\Delta x_t / x_{t-1}$. Of course, percentage growth is a hundred times the growth rate, $100 \Delta x_t / x_{t-1}$. These are exact computations. Taking the log of variables provides a short-cut to growth rates. To understand why, consider first the following expansion of the change in the log of $y_t$ on its previous value:

$$
\begin{aligned}
\Delta \ln x_t &\equiv \ln x_t - \ln x_{t-1} \\
&= \ln(\frac{x_t}{x_{t-1}}) = \ln(\frac{x_t}{x_{t-1}} + 1 - 1) \\
&= \ln(1 + \frac{x_t - x_{t-1}}{x_{t-1}})
\end{aligned}
$$

Hence the change in log $x$ is equal to "log of 1 plus the growth rate of $x$". What is so great about this? At first sight we may seem to be stuck, since we know that for example that $\ln(1 + (x_t - x_{t-1})/x_{t-1}) \neq \ln(1) + \ln((x_t - x_{t-1})/x_{t-1})$. However, there is a rule saying that as (a first order approximation) the natural logarithm of one plus a small number is equal that small number itself. This is exactly what we need, since it allows us to write

$$
\Delta \ln x_t \approx \frac{x_t - x_{t-1}}{x_{t-1}}, \quad \text{when} \quad -1 < \frac{x_t - x_{t-1}}{x_{t-1}} < 1 \tag{A.7}
$$

showing that the difference of the log transformed variable is approximately equal to the growth rate of the original variable. Figure A.2 shows an example of how well the approximation works. The graph in panel a) shows 200 observations of a time series with a growth rate of 3%. In fact this is the same series as we used to represent the $x-$variable in Figure A.1. The graph is not completely smooth, since for realism, we have added a random shock to each observation (a stochastic disturbance term). This means that even using the exact computation, each growth rate is likely to be different from 0.03. This is illustrated in panel b) showing the exact periodic growth rates. Evidently, there is a lot of variation around the mean growth rate of 0.03. Panel c), with the graph of $\ln x_t$ shows a linear though not completely deterministic evolution through time. The straight line represents the underlying average growth of rat of 0.03.

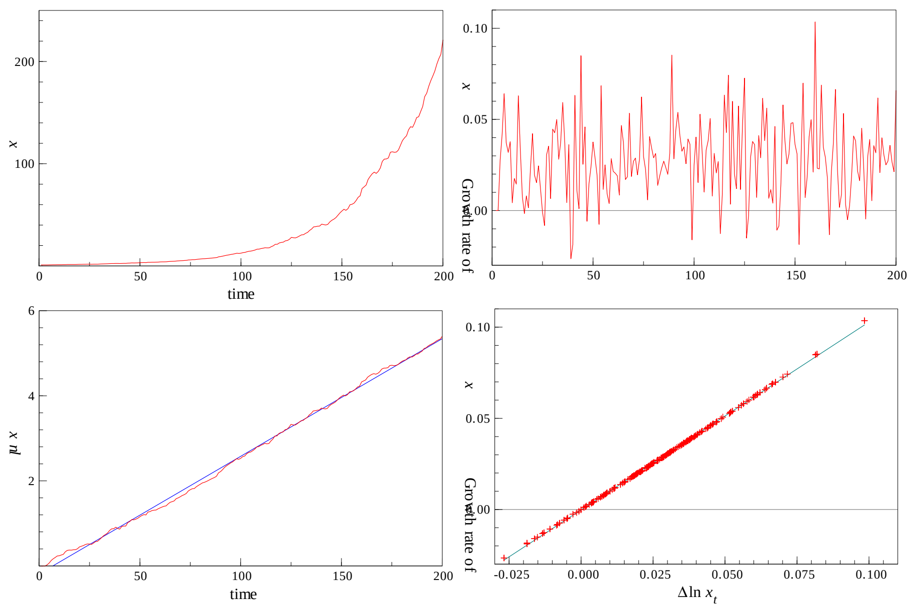


Figure A.2: Panel a): The graph of a time series $x_t$ with an average growth rate of 0.03. Panel b) The time series of the exact growth rates of $x_t$. Panel c): the graph of $lnx_t$. Panel c): The scatter plot of the exact and approximate growth rates. The approximate growth rates is computed as $\Delta lnx_t$

Finally, panel d) in the figure shows the scatter plot of the exact growth rate against the approximated growth rate based the differenced log transformed series in panel c). To the eye at least, the vast majority to observations lies spot on the drawn least squares regression line, which is evidenced that the approximation works really well. However, there is indication that for more sizable growth rates, for example above 10%, the difference between exact and the approximate computation begin to be of practical interest.

The examples considered so far have used computer generated numbers. Figure

A.3 shows the actual and log transformed series of a credit indicator for Norway. Unlike the stylized properties shown in Figure A.1 and A.2, the real world series in this graph shows a more mixed picture. For example, the upper panel suggests two periods of exponential growth in credit: the first ending in 1989 and the second one beginning in 1994. In between lies the period of falling housing prices and the biggest banking crisis since the 1920s. Despite the smoothing function of the log-transform the burst of the bubble is still evident in the bottom panel. Outside the burst period, the linearization works rather well.

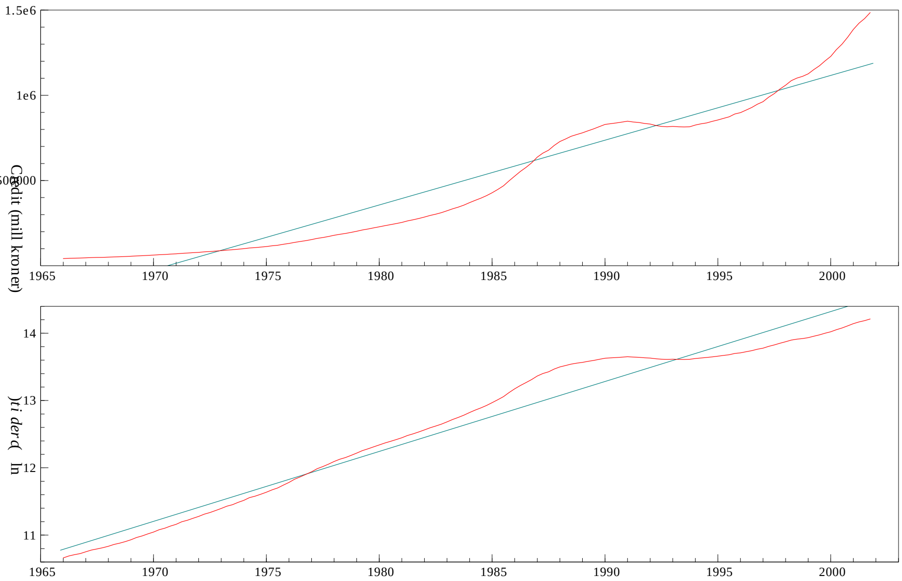


Figure A.3: Upper panel: The time graph of the total credit issued in Norway, together with the estimated linear trend, Bottom panel: The time graph of the logarithmic transform of the credit series. Source: Norges Bank, RIMINI database.

Figure A.4 shows annual data of Norwegian real GDP for the period 1865-1999. Panel a) shows the fixed price series in million kroner, 1990 is the base year. The break in series is due to 2WW, when the country was occupied. Over such a long period of time, the exponential growth pattern is visible and the log transform (panel b) therefore shows a much more linear relationship. In the panel c) and d), the exact and approximate growth rates are shown.

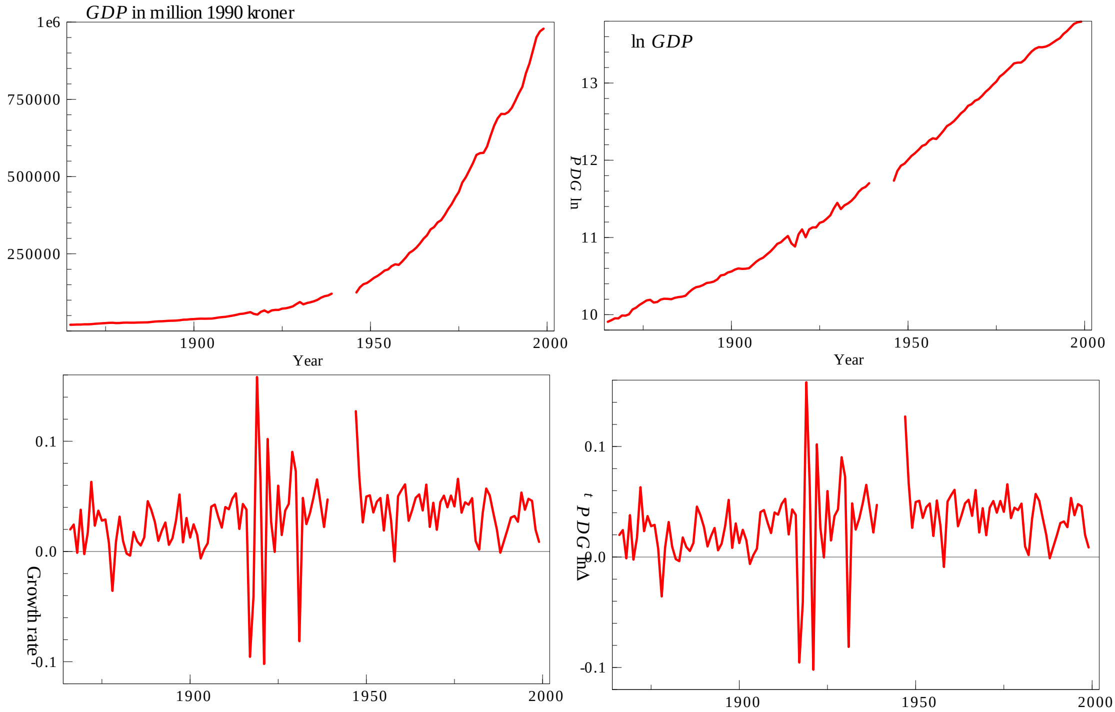


Figure A.4: Panel a: The time graph of Norwegian real GDP, in million1990 kroner. Panel b) The natural logarithm og GDP. Panel c) The exact growth rate of GDP. Panel d) The approximate growth rate using $\Delta lnGDP_t$. Source: Statistics Norway.

**References** All the formulae in this appendix are standard and can be found in any textbook in mathematical analysis. For example, in Sydsæter (2000), the logarithmic function is presented in chapter 3.10 and rules for derivation in chapter 5.11. Elasticities are found in chapter 5.12. The approximation used in equation (A.7) is discussed on page 253 of Sydsæter (2000). A good reference in English is Sydsæter and Hammond (2002).

# Appendix B

# Linearization of the Solow model

In chapter 2.8.3 we derived the well known dynamic equation (2.49) for the capital intensity variable $k_t$:

$$k_t = \frac{1}{(1+n)} \left\{ (1-\delta)k_{t-1} + s\, k_{t-1}^{\gamma} \right\}$$

which is non-linear because of the last term inside the brackets. With the aid of a first order Taylor expansion, see for example Sydsæter and Berck (2006, p. 50), of that term we obtain a linearization of the whole expression which gives the approximate dynamics of $k_t$ "in the neighbourhood" of the steady state $\bar{k}$.

Define the function

$$f(k_{t-1}) = s\, k_{t-1}^{\gamma}.$$

The first order Taylor expansion around the steady-state gives

$$\begin{aligned} f(k_{t-1}) &\approx f(\bar{k}) + f'(\bar{k})(k_{t-1} - \bar{k}) \\ &= s\,\bar{k}^{\gamma} + \gamma s \bar{k}^{\gamma-1}(k_{t-1} - \bar{k}) \end{aligned} \tag{B.1}$$

Replacing $s\, k_{t-1}^{\gamma}$ in (2.49) by the expression in the second line in (B.1), and collecting terms, gives (2.52) in the main text.

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