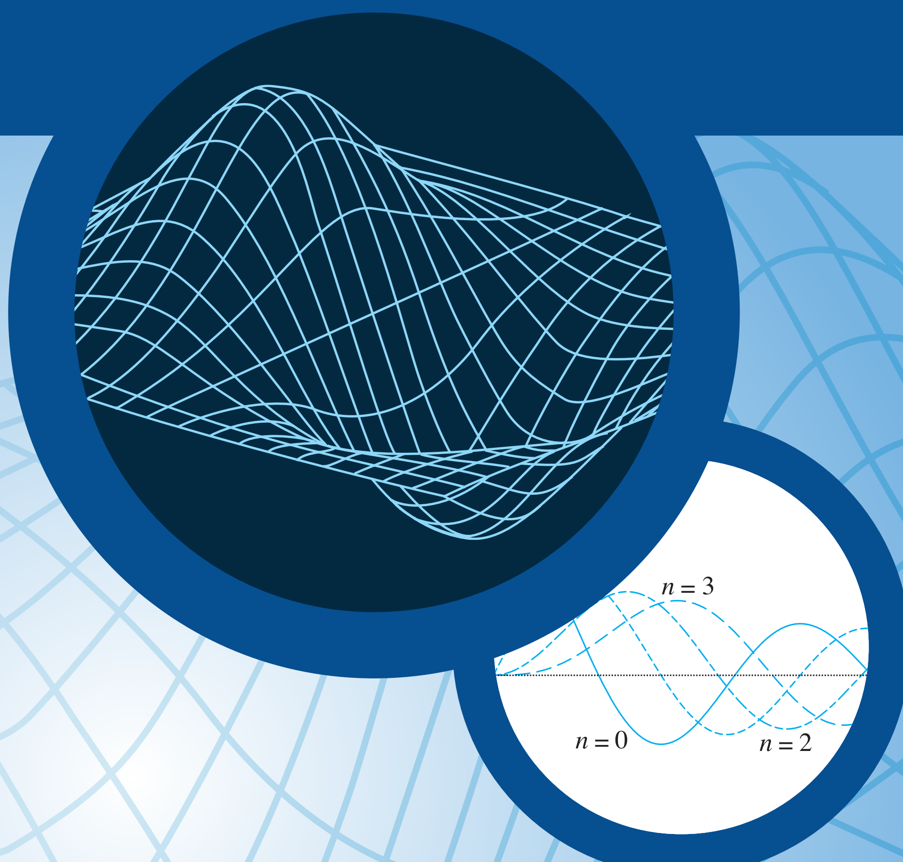
Mathematics for Physicists

B. R. Martin
G. Shaw



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Mathematics for Physicists

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Mathematics for Physicists

B.R. MARTIN

Department of Physics and Astronomy
University College London

G. SHAW

Department of Physics and Astronomy
Manchester University

WILEY

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Editors' preface to the Manchester Physics Series

The Manchester Physics Series is a set of textbooks at first degree level. It grew out of the experience at the University of Manchester, widely shared elsewhere, that many textbooks contain much more material than can be accommodated in a typical undergraduate course; and that this material is only rarely so arranged as to allow the definition of a short self-contained course. The plan for this series was to produce short books so that lecturers would find them attractive for undergraduate courses, and so that students would not be frightened off by their encyclopaedic size or price. To achieve this, we have been very selective in the choice of topics, with the emphasis on the basic physics together with some instructive, stimulating and useful applications.

Although these books were conceived as a series, each of them is self-contained and can be used independently of the others. Several of them are suitable for wider use in other sciences. Each Author's Preface gives details about the level, prerequisites, etc., of that volume.

The Manchester Physics Series has been very successful since its inception over 40 years ago, with total sales of more than a quarter of a million copies. We are extremely grateful to the many students and colleagues, at Manchester and elsewhere, for helpful criticisms and stimulating comments. Our particular thanks go to the authors for all the work they have done, for the many new ideas they have contributed, and for discussing patiently, and often accepting, the suggestions of the editors.

Finally, we would like to thank our publisher, John Wiley & Sons, Ltd., for their enthusiastic and continued commitment to the Manchester Physics Series.

J. R. Forshaw
H. F. Gleeson
F. K. Loebinger
August 2014

Authors' preface

Our aim in writing this book is to produce a relatively short volume that covers all the essential mathematics needed for a typical first degree in physics, from a starting point that is compatible with modern school mathematics syllabuses. Thus, it differs from most books, which include many advanced topics, such as tensor analysis, group theory, etc., that are not required in a typical physics degree course, except as specialised options. These books are frequently well over a thousand pages long and contain much more material than most undergraduate students need. In addition, they are often not well interfaced with school mathematics and start at a level that is no longer appropriate. Mathematics teaching at schools has changed over the years and students now enter university with a wide variety of mathematical backgrounds.

The early chapters of the book deliberately overlap with senior school mathematics, to a degree that will depend on the background of the individual reader, who may quickly skip over those topics with which he or she is already familiar. The rest of the book covers the mathematics that is usually compulsory for all students in their first two years of a typical university physics degree, plus a little more. Although written primarily for the needs of physics students, it would also be appropriate for students in other physical sciences, such as astronomy, chemistry, earth science, etc.

We do not try to cover all the more advanced, optional courses taken by some physics students, since these are already well treated in more advanced texts, which, with some degree of overlap, take up where our book leaves off. The exception is statistics. Although this is required by undergraduate physics students, we have not included it because it is usually taught as a separate topic, using one of the excellent specialised texts already available.

The book has been read in its entirety by one of the editors of the Manchester Physics Series, Jeff Forshaw of Manchester University, and we are grateful to him for many helpful suggestions that have improved the presentation.

B.R. Martin
G. Shaw
April 2015

Notes and website information

'Starred' material

Some sections of the book are marked with a star. These contain more specialised or advanced material that is not required elsewhere in the book and may be omitted at a first reading.

Website

Any misprints or other necessary corrections brought to our attention will be listed on www.wiley.com/go/martin/mathsforphysicists. We would also be grateful for any other comments about the book.

Examples, problems and solutions

Worked examples are given in all chapters. They are an integral part of the text and are designed to illustrate applications of material discussed in the preceding section. There is also a set of problems at the end of each chapter. Some equations which are particularly useful in problem solving are highlighted in the text for ease of access and brief 'one-line' answers to most problems are given at the end of the book, so that readers may quickly check whether their own answer is correct. Readers may access the full solutions to all the odd-numbered problems at www.wiley.com/go/martin/mathsforphysicists. Full solutions to all problems are available to instructors at the same website, which also contains electronic versions of the figures.

1

Real numbers, variables and functions

In this chapter we introduce some simple ideas about real numbers, i.e. the ordinary numbers used in arithmetic and measurements, real variables and algebraic functions of a single variable. This discussion will be extended in Chapter 2 by considering some important examples in more detail: polynomials, trigonometric functions, exponentials, logarithms and hyperbolic functions. Much of the material in these first two chapters will probably already be familiar to many readers and so is covered briefly, but even if this is the case, it is useful revision and sets the scene for later chapters.

1.1 Real numbers

This section starts from the basic rules of arithmetic and introduces a number of essential techniques for manipulating real numerical quantities. We also briefly consider number systems other than the decimal system.

1.1.1 Rules of arithmetic: rational and irrational numbers

The first contact with mathematics is usually via counting, using the positive integers $1, 2, 3, 4, \ldots$ (also called *natural numbers*). Later, fractional numbers such as $\frac{1}{2}$, $\frac{3}{5}$, etc. and negative numbers -1, -3, $-\frac{1}{3}$, $-\frac{7}{9}$, etc. are introduced, together with the rules for combining positive and negative numbers and the basic laws of arithmetic. As we will build on these laws later in this chapter, it is worth reminding oneself of what they are by stating them in a somewhat formal way as follows.

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(i) *Commutativity*: The result of subtracting or dividing two integers is dependent on the order in which the operations are performed, but addition and multiplication are independent of the order. For example,

$$3 + 6 = 6 + 3 \quad \text{and} \quad 3 \times 6 = 6 \times 3, \tag{1.1a}$$

but

$$5 - 3 \neq 3 - 5 \quad \text{and} \quad 5 \div 3 \neq 3 \div 5, \tag{1.1b}$$

where $\neq$ means *not equal to*.

(ii) *Associativity*: The result of subtracting or dividing three or more integers is dependent on the way the integers are associated, but is independent of the association for addition and multiplication. Examples are

$$(2 + 3) + 4 = 2 + (3 + 4) \quad \text{and} \quad 2 \times (3 \times 4) = (2 \times 3) \times 4, \tag{1.2a}$$

but

$$6 - (3 - 2) \neq (6 - 3) - 2 \quad \text{and} \quad 12 \div (6 \div 2) \neq (12 \div 6) \div 2. \tag{1.2b}$$

(iii) *Distributivity*: Multiplication is distributed over addition and subtraction from both left and right, whereas division is only distributed over addition and subtraction from the right. For example, for multiplication:

$$2 \times (4 + 3) = (2 \times 4) + (2 \times 3) \tag{1.3a}$$

and

$$(3 - 2) \times 4 = (3 \times 4) - (2 \times 4), \tag{1.3b}$$

but for division, from the right we have

$$(60 + 15) \div 3 = (60 \div 3) + (15 \div 3), \tag{1.3c}$$

whereas division from the left gives

$$60 \div (12 + 3) = 60 \div 15 \neq (60 \div 12) + (60 \div 3). \tag{1.3d}$$

Positive and negative integers and fractions can all be expressed in the general form n/m, where n and m are integers (with $m \neq 0$ because division by zero is undefined). A number of this form is called a *rational number*. The latter is said to be *proper* if its numerator is less than its denominator, otherwise it is said to be *improper*. The operations of addition, subtraction, multiplication and division, when applied to rational numbers, always result in another rational number. In the case of fractions, multiplication is applied to the numerators and denominators separately; for division, the fraction is inverted and then multiplied. Examples are

$$\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28} \quad \text{and} \quad \frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}. \tag{1.4a}$$

For addition (and subtraction) all the terms must be taken over a common denominator. An example is:

$$\frac{3}{4} - \frac{5}{7} + \frac{1}{3} = \frac{(3 \times 3 \times 7) - (5 \times 3 \times 4) + (1 \times 4 \times 7)}{3 \times 4 \times 7} = \frac{31}{84}. \tag{1.4b}$$

Not all numbers can be written in the form n/m. The exceptions are called *irrational numbers*. Examples are the square root of 2, that is, $\sqrt{2} = 1.414\ldots$, and the ratio of the circumference of a circle to its diameter, that is, $\pi = 3.1415926\ldots$, where the dots indicate a non-recurring sequence of numbers. Irrational numbers, when expressed in decimal form, always lead to such non-recurrence sequences, but even rational numbers when expressed in this form may not always terminate, for example $\frac{2}{11} = 0.1818\ldots$. The proof that a given number is irrational can be very difficult, but is given for one particularly simple case in Section 1.2.2.

In practice, an irrational number may be represented by a rational number to any accuracy one wishes. Thus π is often represented as $\frac{22}{7} = 3.143$ in rough calculations, or as $\frac{355}{113} = 3.141593$ in more accurate work. Rational and irrational numbers together make up the class of so-called *real numbers* that are themselves part of a larger class of numbers called *complex numbers* that we will meet in Chapter 6. It is worth remarking that infinity, denoted by the symbol ∞, is not itself a real number. It is used to indicate that a quantity may become arbitrarily large.

In the examples above of irrational numbers, the sequence of numbers after the decimal point is endless and so, in practice, one has to decide where to terminate the string. This is called *rounding.* There are two methods of doing this: quote either the *number of significant figures* or the *number of decimal places*. Consider the number 1234.567…. To two decimal places this is 1234.57; the last figure has been rounded up to 7 because the next number in the string after 6 is 7, which is greater than 5. Likewise, we would round down if the next number in the string were less than 5. If the next number in the string were 5, then the 5 and the next number following it are rounded up or down to the nearest even multiple of 10, and the zero dropped. For example, 1234.565 to two decimal places would be 1234.56, whereas 1234.575 would be rounded to 1234.58. If we were to quote 1234.567 to five significant figures, it would be 1234.6 and to three significant figures it would be 1230.

1.1.2 Factors, powers and rationalisation

Integer numbers may often be represented as the product of a number of smaller integers. This is an example of a process called *factorisation*, that is, decomposition into a product of smaller terms, or *factors*. For example, 24 is equal to $2 \times 2 \times 2 \times 3$. In this example, the integers in the product cannot themselves be factorised further. Such integers are called *prime numbers*. (By convention, unity is not considered a prime number.) By considering all products of the prime numbers in the factorisation, we arrive at the result that the *factors* of 24 are 1, 2, 3, 4, 6, 8, 12 and 24, that is, these are all the numbers that divide exactly into 24. If we have several numbers, the *highest common factor* (HCF) is the largest factor that can divide exactly into all the numbers. The *lowest common multiple* (LCM) is the smallest number into which all the given numbers will divide exactly. Thus the HCF of 24 and 36 is 12 and the LCM of all three numbers is 72.

In the example of the factorisation of the number 24 above, the factor 2 occurs three times. It is common to encounter situations where a number is multiplied by itself several times. A convenient notation for this is to introduce the idea of a *power* (or *index*) n, such that, for example, $5^n \equiv 5 \times 5 \times 5 \ldots n$ times. To emphasise that this relation *defines* the index n, the usual two-line equality sign has been replaced by a three-line equality sign ($\equiv$). So, using powers, we could also write 24 in the compact prime-number factorised form $24 = 3 \times 2^3$. Any real number p to power zero is *by definition* equal to unity, that is, $p^0 \equiv 1$ for any p.

By writing out in full, it is easy to see that multiplying the *same* integers each raised to a power is equivalent to adding the powers. Thus

$$\begin{aligned} 5^n \times 5^m &= (5 \times 5 \times 5 \ldots n \text{ times}) \times (5 \times 5 \times 5 \ldots m \text{ times}) \\ &= 5 \times 5 \times 5 \ldots (n+m) \text{ times} \\ &= 5^{(n+m)} \end{aligned} \tag{1.5a}$$

and analogously for division,

$$5^n / 5^m = 5^{(n-m)}. \tag{1.5b}$$

A power can also be a fraction or rational number, since, for example, the combination rule (1.5a) implies $5^{1/2} \times 5^{1/2} = 5^1$, so that $5^{1/2} = \sqrt{5}$. Similarly the expression $3^{1/3}3^0 3^{4/3}/27^{1/3}$, for example, can be simplified to give

$$3^{1/3}3^0 3^{4/3}/27^{1/3} = 3^{(1/3+0+4/3-1)} = 3^{2/3}. \tag{1.5c}$$

An example of the use of factors is to express numbers in so-called *scientific notation* (also called *normal form*). In this representation, any real number is written as the product of a number between -10 and $+10$ (excluding the numbers ± 10 themselves), with as many decimal places as required, and a power of 10. The number 1245.678 to four significant figures in scientific notation is therefore 1.246×10^3.

It is conventional to write arithmetical forms in a compact form and to remove as far as possible fractional powers from the denominator of a fraction, a process called *rationalisation*. For example, consider the form

$$\frac{1}{2\sqrt{5}+1} - \frac{1}{2\sqrt{5}+2}. \tag{1.6a}$$

By taking the terms over a common denominator and then multiplying numerator and denominator by $(11 - 3\sqrt{5})$, we have

$$\begin{aligned} \frac{1}{2\sqrt{5}+1} - \frac{1}{2\sqrt{5}+2} &= \frac{(2\sqrt{5}+2)-(2\sqrt{5}+1)}{(2\sqrt{5}+1)(2\sqrt{5}+2)} = \frac{1}{2(11+3\sqrt{5})} \\ &= \frac{(11-3\sqrt{5})}{2(11+3\sqrt{5})(11-3\sqrt{5})} = \frac{11-3\sqrt{5}}{152}, \end{aligned} \tag{1.6b}$$

which is the rationalised form of (1.6a).

Example 1.1

Simplify the following forms:

(a) $(7\frac{1}{9})^{3/2}$, (b) $5^{1/3}25^{-1/2}/5^{-1/6}5^{-2/3}$, (c) $3^{1/2}27^{-2/3}9^{-1/2}$.

Solution

(a) $(7\frac{1}{9})^{3/2} = (64/9)^{3/2} = (8/3)^3 = 512/27$,

(b) $5^{1/3}25^{-1/2}/5^{-1/6}5^{-2/3} = 5^{(1/3-1+1/6+2/3)} = 5^{1/6}$,

(c) $3^{1/2}27^{-2/3}9^{-1/2} = 3^{1/2}3^{-2}3^{-1} = 3^{-5/2} = 3^{2/5}$.

Example 1.2

Rationalise the numerical expressions:

(a) $\dfrac{1}{\sqrt{3}-1} + \dfrac{1}{\sqrt{3}+1}$, (b) $\dfrac{1}{\sqrt{2}+1}$, (c) $\dfrac{(2-\sqrt{5})}{(3+2\sqrt{5})}$.

Solution

(a) Taking both terms over a common denominator gives

$$\frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}+1+\sqrt{3}-1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \sqrt{3}.$$

(b) Multiplying numerator and denominator by $(\sqrt{2}-1)$ gives

$$\frac{1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\sqrt{2}-1.$$

(c) Multiplying numerator and denominator by $(3-2\sqrt{5})$ gives

$$\frac{(2-\sqrt{5})}{(3+2\sqrt{5})}=\frac{(2-\sqrt{5})(3-2\sqrt{5})}{(3+2\sqrt{5})(3-2\sqrt{5})}=\frac{7\sqrt{5}-16}{11}.$$

*1.1.3 Number systems[1]

All the numbers in the previous sections are expressed in the decimal system, where the 'basis' set of integers is $0, 1, 2, \ldots, 9$. Real numbers in this system are said to be to '*base 10*'. In the number 234, for example, the integers 2, 3 and 4 indicate how many powers of 10 are present, reading from the right, i.e.

$$234=(2\times 10^2)+(3\times 10^1)+(4\times 10^0). \tag{1.7}$$

Any other base could equally well be used and in some circumstances other number systems are more appropriate. The most widely used number system other than base 10 is the *binary system*, based on the two integers 0 and 1, that is, *base 2*, so we will only discuss this case. Its importance stems from its use in computers, because the simplest electrical switch has just two states, 'open' and 'closed'. To distinguish numbers in this system we will write them with a subscript 2.

As an example, consider the number 123. In the binary system this is 1111011_2. To check: in the decimal system,

$$1111011_2=2^6+2^5+2^4+2^3+2^1+2^0=123. \tag{1.8}$$

Fractions are accommodated by using negative values for the indices. Thus the number 6.25 in the binary system is 110.01_2. To check: in the decimal system,

$$110.01_2=2^2+2^1+2^{-2}=4+2+0.25=6.25. \tag{1.9}$$

To convert a number in one basis to another is straightforward, if rather tedious. Consider, for example, the conversion of the number 51.78 to the binary system. We start with the integer 51 and find the largest value of an integer n such that 2^n is less than or equal to

[1] The reader is reminded that the results of starred sections are not needed later, except in other starred sections, and therefore they may prefer to omit them on a first reading.

51 and then note the remainder $R = 51 - 2^n$. This is then repeated by again finding the largest number n such that 2^n is less than or equal to R, and continued in this way until the remainder is zero. We thus obtain:

$$51 = 2^5 + 19 = 2^5 + 2^4 + 3 = 2^5 + 2^4 + 2^1 + 1 = 2^5 + 2^4 + 2^1 + 2^0,$$

so that in the binary system

$$51 = 110011_2. \tag{1.10a}$$

Similarly, we can convert the numbers after the decimal point using negative powers. This gives

$$0.78 = 2^{-1} + 0.28 = 2^{-1} + 2^{-2} + 0.03 \approx 2^{-1} + 2^{-2} + 2^{-5},$$

so again in the binary system,

$$0.78 \approx 0.11001_2 \tag{1.11a}$$

and finally,

$$51.78 = 110011.11001_2, \tag{1.11b}$$

in the binary system, which represents the decimal number to an accuracy of two decimal places.

All the normal arithmetic operations of addition, subtraction, multiplication and division can be carried out in any number system. For example, in the binary system, we have the basic result $1_2 + 1_2 = 10_2$. So adding the numbers 101_2 and 1101_2 gives $101_2 + 1101_2 = 10010_2$. To check, we can again use the decimal system. Thus,

$$101_2 = 2^2 + 2^0 = 5, \text{ and } 1101_2 = 2^3 + 2^2 + 2^0 = 13, \tag{1.12a}$$

with

$$10010_2 = 2^4 + 2^1 = 18. \tag{1.12b}$$

As an example of multiplication, consider the numbers 5 and 7. In the binary system these are 101_2 and 111_2, respectively, and multiplying them together gives, using $1_2 + 1_2 = 10_2$,

$$\begin{array}{r} 1\,0\,1 \\ 1\,1\,1 \\ \hline 1\,0\,1\,0\,0 \\ 1\,0\,1\,0 \\ 1\,0\,1 \\ \hline 1\,0\,0\,0\,1\,1 \end{array}$$

Once again, we can check the result using the decimal system:

$$100011_2 = 2^5 + 2^1 + 2^0 = 35. \tag{1.13}$$

As an example of division, consider the numbers 51 and 3. In the binary system these are 110011_2 and 11_2, respectively, and dividing them we have

$$\begin{array}{r} 10001 \\ 11 | \overline{110011} \\ \underline{110000} \\ 11 \\ \underline{11} \\ 00 \end{array}$$

So the quotient is 10001_2, which in the decimal system is $2^4 + 2^0 = 17$, as required.

Example 1.3

Write the decimal number 100 in base 3 and base 4.

Solution

(a) The decimal number 100 written as powers of 3 is $100 = 3^4 + (2 \times 3^2) + 3^0$, so to base 3 it is 10201_3.

(b) The decimal number 100 written as powers of 4 is $100 = 4^3 + (2 \times 4^2) + 4^1$, so to base 4 it is 1210_4.

Example 1.4

Consider the base 3 numbers $p = 201_3$ and $q = 112_3$. Find (a) $p + q$, (b) $p - q$, (c) $p \times q$ and (d) p/q to two decimal places and check your results in the decimal system.

Solution

In base 3,

(a) $p + q = 201_3 + 112_3 = 1020_3$, which as a decimal number is $3^3 + (2 \times 3) = 33$,

(b) $p - q = 201_3 - 112_3 = 12_3$, which as a decimal number is $3^1 + (2 \times 3^0) = 5$,

(c) $p \times q = 201_3 \times 112_3 = 100212_3$, which as a decimal number is $3^5 + (2 \times 3^2) + 3^1 + (2 \times 3^0) = 266$,

(d) $p/q = 201_3/112_3 = 1.1002_3 \ldots$, which as a decimal number is $3^0 + 3^{-1} + (2 \times 3^{-4}) \cdots = 1.357\ldots = 1.36$ to two decimal places.

To check these, we have $p = 201_3$ as a decimal number is $(2 \times 3^2) + 3^0 = 19$ and $q = 112_3$ is $3^2 + 3^1 + (2 \times 3^0) = 14$. Thus in the decimal system, (a) $p + q = 19 + 14 = 33$, (b) $p - q = 19 - 14 = 5$, (c) $p \times q = 19 \times 14 = 266$, and (d) $p/q = 19/14 = 1.36$ to two decimal places, as required.

1.2 Real variables

The work in Section 1.1 can be generalised by representing real numbers as symbols, x, y, etc. Thus we are entering the field of *algebra*. This section starts by generalising the methods of Section 1.1 for real numbers to algebraic quantities and also discusses the general idea of algebraic expressions and the important result known as the binomial theorem.

1.2.1 Rules of elementary algebra

Algebra enables us to consider general expressions like, for example, $(x + y)^2$, where x and y can be any real number. When manipulating real numbers as symbols, the fundamental rules of algebra apply. These are analogous to the basic rules of arithmetic given in Section 1.1 and can be summarised as follows.[2]

(i) *Commutativity:* Addition and multiplication are commutative operations, i.e.

$$x + y = y + x \quad \text{commutative law of addition} \tag{1.14a}$$

and

$$xy = yx \quad \text{commutative law of multiplication.} \tag{1.14b}$$

In contrast, subtraction and division are only commutative operations under special circumstances. Thus,

$$x - y \neq y - x \quad \text{unless } x = y$$

and

$$x \div y \neq y \div x \quad \text{unless } x = y \text{ *and* neither equals zero.}$$

(ii) *Associativity:* Addition and multiplication are associative operations, i.e.,

$$x + (y + z) = (x + y) + z \quad \text{associative law of addition} \tag{1.15a}$$

[2] Here and in what follows the explicit multiplication signs between terms are usually omitted if there is no loss of clarity, so that xy is equivalent to $x \times y$ and so on.

and

$$x(yz) = (xy)z \quad \text{associative law of multiplication.} \tag{1.15b}$$

Subtraction and division are not associative operations except in very special circumstances. Thus,

$$x - (y - z) \neq (x - y) - z \quad \text{unless } z = 0$$

and

$$x \div (y \div z) \neq (x \div y) \div z \quad \text{unless } z = 1 \textit{ and } y \neq 0,$$

as is easily verified by choosing any particular values for x, y and z.

(iii) *Distributivity*: The basic rule is

$$x(y + z) = xy + xz \quad \text{distributive law.} \tag{1.16a}$$

Together with the commutative law of multiplication, this implies

$$(x + y)z = xz + yz, \tag{1.16b}$$

since

$$(x + y)z = z(x + y) = zx + zy = xz + yz.$$

In addition, by noting that $(y - z) = (y + (-z))$ etc., one sees that these results imply that multiplication is distributed over addition *and* subtraction from both the left and the right, i.e.

$$x(y \pm z) = xy \pm xz \text{ and } (x \pm y)z = xz \pm yz. \tag{1.16c}$$

Finally, since $(x + y)/z = (x + y)z^{-1}$, equation (1.16b) implies that division is distributed over addition and subtraction from the right, i.e.

$$(x \pm y) \div z = (x \div z) \pm (y \div z), \tag{1.16d}$$

but not from the left, i.e.

$$x \div (y + z) \neq (x \div y) + (x \div z).$$

(iv) *The law of indices:* This is

$$x^n x^m = x^{(n+m)} \quad \text{law of indices,} \tag{1.17}$$

with $x^n/x^m = x^{n-m}$, and where, by definition, $x^0 \equiv 1$.

The nine laws (1.14)–(1.17) are the fundamental laws of elementary algebra. To illustrate their use, consider the proof of the familiar result

$$(x + y)^2 = x^2 + 2xy + y^2.$$

We have,

$$
\begin{aligned}
(x+y)^2 &= (x+y)(x+y) && \text{by the index law (1.17)} \\
&= (x+y)x + (x+y)y && \text{by the distributive law (1.16a)} \\
&= x(x+y) + y(x+y) && \text{by the commutative law (1.14b)} \\
&= x^2 + xy + yx + y^2 && \text{by the distributive law (1.16b)} \\
&= x^2 + 2xy + y^2. && \text{by the commutative law (1.14b)}
\end{aligned}
$$

It should be emphasised that although the above rules are obeyed by the real variables of elementary algebra, in later chapters we will encounter other mathematical quantities, such as vectors and matrices, that do not necessarily obey all these rules.

*1.2.2 Proof of the irrationality of $\sqrt{2}$

Now we have introduced algebraic symbols and the idea of powers, we can return to the discussion of Section 1.1.1 and prove that $\sqrt{2}$ is an irrational number. The proof uses a general method called *reductio ad absurdum*, or *proof by contradiction*; that is, we assume the opposite, and prove it leads to a contradiction. This is a commonly used method of proof in mathematics. Suppose $\sqrt{2}$ is rational. It then follows that

$$\sqrt{2} = p/q, \tag{1.18}$$

where p and q are integers, and we may, without loss of generality, assume that they are the smallest integers for which this is possible, that is, they have no common factors. Then from (1.18), we have

$$p^2 = 2q^2, \tag{1.19}$$

so that p^2 is even. Furthermore, since the square of an odd number is odd and the square of an even number is even, p itself must be even; and since p and q have no common factors, q must be odd, since otherwise both would be divisible by 2. On the other hand, since p is even, we can write $p = 2r$, where r is an integer. Substituting this in (1.19) now gives $q^2 = 2r^2$, so that q is even, in contradiction to our previous result. Hence the assumption that $\sqrt{2}$ is rational must be false, and $\sqrt{2}$ can only be an irrational number.

1.2.3 Formulas, identities and equations

The use of symbols enables general *algebraic expressions* to be constructed. An example is a *formula*, which is an algebraic expression relating two or more quantities. Thus the volume of a rectangular solid, given by volume = length × breadth × height, may be written $V = lbh$. Given numerical values for l, b and h, we can calculate a value for the volume V. Formulas may be manipulated to more convenient forms providing certain rules are respected. These include (1) taking terms from one side to the other reverses their sign; and (2) division (multiplication) on one side becomes multiplication (division) on the other. For example, if $S = ab + c$, then $S - c = ab, a = (S - c)/b$ etc.

As with numerical forms, it is usual to rationalise algebraic expressions where possible. Thus,

$$\frac{x}{\sqrt{x}+\sqrt{y}} + \frac{y}{\sqrt{x}-\sqrt{y}} = \frac{x(\sqrt{x}-\sqrt{y}) + y(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{\sqrt{x}(x+y) - \sqrt{y}(x-y)}{x-y}. \tag{1.20}$$

Sometimes factorisation may be used to simplify the results. For example,

$$\frac{1}{x^2-3x+2} - \frac{1}{x^2+x-2} = \frac{(x^2+x-2)-(x^2-3x+2)}{(x-1)^2(x-2)(x+2)} = \frac{4}{(x-1)(x^2-4)}, \tag{1.21}$$

where we have used the results

$$x^2 - 3x + 2 = (x-1)(x-2) \tag{1.22a}$$

and

$$x^2 + x - 2 = (x-1)(x+2). \tag{1.22b}$$

Equations (1.21)–(1.22) are examples of *identities*, because they are true for *all* values of x, and the three-line equality symbol (mentioned earlier) is also sometimes used to emphasise this, although in this book we will reserve its use for definitions.

In contrast, the expression on the left-hand side of (1.22a) can also be written $f(x) = x^2 - 3x + 2$ and setting $f(x)$ equal to a specific value gives an *equation* that will only have *solutions* (or *roots*) for specific values of x. In the case of (1.22a), setting $f(x) = 0$ yields the two solutions $x = 1$ and $x = 2$.

Example 1.5

Simplify: (a) $\dfrac{2x-y}{x-y} - \dfrac{2x-y}{x+y}$,

(b) $\dfrac{1}{x^3+2x^2+x+2} - \dfrac{1}{2x^2+x-6}$.