A.G'OZIYEV, I.ISRAILOV, M.YAXSHIBOYEV

MATEMATIK ANALIZDAN MISOL VA MASALALAR

O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS
TA'LIM VAZIRLIGI

A. GAZIYEV, I. ISRAILOV, M. YAXSHIBOYEV

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2 - QISM

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan 5460100 – «Matematika» bakalavr yo'nalishi talabalari uchun o'quv qo'llanma sifatida tavsiya etilgan

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Ushbu qo'llanma matematik analizning aniqmas integrallar, aniq integrallar va ularning tadbiqlari hamda ko'p o'zgaruvchili funksiyalar mavzulari bo'yicha talabalarda misol va masalalarni mustaqil yechish ko'nikmasini hosil qilishga mo'ljallangan.

Qo'llanmaning har bir paragrafida, avvalo, mavzuning nazariy qismidan qisqacha axborot berilgan, so'ngra mavzuga mos tipik misol va masalalar batafsil yechib ko'rsatilgan hamda mustaqil ishlash uchun yetarli miqdorda misol va masalalar javoblari bilan berilgan.

O'quv qo'llanma bakalavriatning «matematika», «mexanika», «amaliy matematika va informatika», «fizika» va texnika yo'nalishlarining «Oliy matematika» chuqurlashtirilgan dastur asosida o'qitiladigan talabalari hamda o'qituvchilar uchun mo'ljallangan.

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Taqrizchilar:
A. SOLEYEV – fizika-matematika fanlari doktori, prof.;
A.JALILOV – fizika - matematika fanlari doktori;
U. NARZULLAYEV – fizika - matematika fanlari nomzodi, dots.

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SO‘Z BOSHI

Mazkur qo‘llanma «Matematik analiz» fani bo‘yicha o‘quv-uslubiy majmuaning tarkibiy qismlaridan biri bo‘lib, unda matematik analizning aniqmas integrallar, aniq integrallar va ularning tadbiqlari hamda ko‘p o‘zgaruvchili funksiyalar bo‘limlari bo‘yicha asosiy tushunchalar keltirilgan.

O‘quv qo‘llanma to‘rt bobdan iborat. Birinchi bobda boshlang‘ich funksiya va aniqmas integral qaralgan hamda integrallash usullarining barchasi ochib berilgan. Qo‘llanmaning ikkinchi bobi aniq integral, uni hisoblash usullari bo‘yicha mavzularni o‘z ichiga olgan. Uchinchi bobda aniq integralning fizika, mexanika, iqtisodiyot kabi sohalarda uchraydigan masalalarning yechilishiga tadbiqlari qaralgan. Oxirgi, to‘rtinchi bobda R^n fazo, ko‘p o‘zgaruvchili funksiyalar, ularning limiti, uzluksizligi, xususiy hosilalari, differensiallari, yuqori tartibli xususiy hosilalari, ekstrimumlari qaralgan.

O‘z navbatida, har bir bob tegishli paragraflarga bo‘lingan bo‘lib, har bir paragraf mavzuga taalluqli asosiy ta’riflar, tasdiqlar, teoremalarni o‘z ichiga oladi, shuningdek, ularning har biri an’anaviy misollarni batafsil tahlil yordamida yechish orqali namoyish qilingan. Qo‘llanmada jami 198 ta misol va masalalar yechilgan, 1566 ta mustaqil yechish uchun misol va masalalar tavsiya qilingan hamda ularning javoblari berilgan. Hozirgi vaqtda amaliyotda bir necha yaxshi rivojlangan matematik dasturlar (Mathcad, Maple, Mathematica, Mathlab va h.k.) matematik masalalarni kompyuter imkoniyatlaridan foydalanib yechishda samarali natijalar bermoqda. Shu an’anadan chetda qolmaslik uchun, qo‘llanmada ba’zi bo‘limlar bo‘yicha misol va masalalar yechishda «Maple» tizimining qo‘llanilishi va uning qulayliklari namoyish etilgan.

Ushbu qo‘llanmani yozishga mualliflarni undagan narsa, ularning ko‘p yillar mobaynida Samarqand davlat universitetida matematik analiz kursidan olib borgan ma’ruza va amaliy mashg‘ulotlarida orttirgan tajribasi natijasidir. O‘ylaymizki, qo‘llanma o‘z o‘quvchilarini topadi va boshqa mavjud o‘quv adabiyotlari qatorida matematik analiz kursining aytib o‘tilgan bo‘limlari bo‘yicha ularga bilimlarini oshirishga ko‘mak beradi.

O‘quv qo‘llanma haqidagi fikr-mulohazalar, undagi mavjud kamchiliklar bo‘yicha takliflarni mualliflar mamnuniyat bilan qabul qiladilar.

I bob. ANIQMAS INTEGRALLAR

1-§. Boshlang‘ich funksiya, aniqmas integral tushunchalari. Aniqmas integralning sodda xossalari. Aniqmas integrallar jadvali.

1.1. Boshlang‘ich funksiya tushunchasi. Aniqmas integral. Harakat boshlangandan o‘tgan t vaqt ichida moddiy nuqta $s(t)$ yo‘l o‘tgan bo‘lsin, u holda, $v(t)$ oniy tezlik, $s(t)$ funksiyaning hosilasiga teng, ya’ni $v(t)=s'(t)$. Amaliyotda teskari masala ham uchraydi: moddiy nuqtaning $v(t)$ harakat tezligi berilganda, uning bosib o‘tgan $s(t)$ yo‘lini toping. Amaliyotdagi bunday masala, $f(x)$ funksiyaning *boshlang‘ich funksiyasi* tushunchasiga olib keladi.

$f(x)$ va $F(x)$ funksiyalar biror X (ochiq yoki yopiq: chekli yoki cheksiz) oraliqda aniqlangan bo‘lib, ular

$$F'(x)=f(x) \qquad (1.1)$$

munosabatda bo‘lsin.

1.1-ta’rif. Agar $F(x)$ funksiya biror X oraliqda differensiallanuvchi bo‘lib, $\forall x\in X$ lar uchun (1.1) tenglik o‘rinli bo‘lsa, u holda $F(x)$ funksiyada X oraliqda $f(x)$ *funksiyaning boshlang‘ich funksiyasi deyiladi.*

1.2-ta’rif. Agar $f(x)$ va $F(x)$ funksiyalar $X=[a;b]$ kesmada aniqlangan va $\forall x\in(a,b)$ uchun $F'(x)=f(x)$ yoki $dF(x)=f(x)dx$ bo‘lib, a va b nuqtalarda $F'(a+0)=f(a)$, $F'(b-0)=f(b)$ tengliklar o‘rinli bo‘lsa, u holda, $F(x)$ funksiyaga $X=[a;b]$ kesmada $f(x)$ *funksiyaning boshlang‘ich funksiyasi deyiladi.*

1.1-teorema. Agar $F(x)$ va $\Phi(x)$ funksiyalar X oraliqda differensiallanuvchi bo‘lib, ularning har biri $f(x)$ funksiyaning X oraliqdagi boshlang‘iya funksiysi bo‘lsa, u holda, $F(x)$ va $\Phi(x)$ funksiyalar X da bir-biridan o‘zgarmas songa farq qiladi, ya’ni

$$\Phi(x)=F(x)+C, \qquad x\in X.$$

Agar $f(x)$ funksiyaning X oraliqda biror $F(x)$ boshlang‘ich funksiyasi ma’lum bo‘lsa, uning boshqa istalgan boshlang‘ich funksiya $F(x)+C$ formula orqali topiladi.

1.3-ta'rif. $f(x)$ funksiyaning X oraliqdagi barcha boshlang'ich funksiyalar to'plamiga, $f(x)$ funksiyaning *aniqmas integrali* deyiladi va u

$$\int f(x)\,dx$$

kabi belgilanadi, bunda $\int$ - integral belgisi, $f(x)$ - integral ostidagi funksiya, $f(x)dx$ esa, - *integral ostidagi ifoda deyiladi.*

Agar $F(x)$ funksiya X oraliqda $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsa, u holda $f(x)$ funksiyaning aniqmas integrali

$\int f(x)\,dx = F(x) + C$ (C – ixtiyoriy o'zgarmas son)

yoki

$$\int f(x)\,dx = \{F(x) + C\}$$

kabi yoziladi.

1.1-misol. Berilgan oraliqda berilgan $F(x)$ funksiya berilgan $f(x)$ funksiyaning boshlang'ich funksiyasi ekanligini ko'rsating va aniqmas integralini yozing:

1), $F(x) = \frac{x^4}{4}$, $f(x) = x^3$, $X = (-\infty, \infty) = R$;

2) $F(x) = \frac{\sin ax}{a}$, $f(x) = \cos ax$ $(a = const)$, $X = (-\infty, \infty) = R$;

3) $F(x) = \sqrt{1-x^2}$, $f(x) = -\frac{x}{\sqrt{1-x^2}}$, $X = (-1;1)$;

4) $F(x) = \frac{2}{3}\sqrt{x^3}$, $f(x) = \sqrt{x}$, $X = (0;\infty)$;

5) $F(x) = \sqrt{x} - \cos(x+1)$, $f(x) = \frac{1}{2\sqrt{x}} + \sin(x+1)$, $X \in (0;+\infty)$.

Yechilishi. 1) $f(x) = x^3$ funksiyaning $X = (-\infty, \infty) = R$ oraliqdagi boshlang'ich funksiyasi $F(x) = \frac{x^4}{4}$ bo'ladi, chunki $F'(x) = \left(\frac{x^4}{4}\right)' = x^3$.

Demak, $\int x^3 dx = \frac{x^4}{4} + C$.

2) $f(x) = \cos ax$ $(a = const)$ funksiyaning R dagi boshlang'ich funksiyasi $F(x) = \frac{\sin ax}{a}$ bo'ladi, chunki $F'(x) = \left(\frac{\sin ax}{a}\right)' = \cos ax = f(x)$. Demak, $\int \cos ax dx = \frac{\sin ax}{a} + C$.

3) $f(x) = -\frac{x}{\sqrt{1-x^2}}$ funksiyaning $X = (-1;1)$ oraliqdagi boshlang'ich funksiyasi $F(x) = \sqrt{1-x^2}$ bo'ladi, chunki $F'(x) = (\sqrt{1-x^2})' = -\frac{x}{\sqrt{1-x^2}} = f(x)$.

Demak, $\int \frac{-x}{\sqrt{1-x^2}}dx = \sqrt{1-x^2} + C$.

4) $f(x) = \sqrt{x}$ funksiyaning $X = (0;\infty)$ oraliqdagi boshlang'ich funksiyasi $F(x) = \frac{2}{3}\sqrt{x^3}$ bo'ladi, chunki $F'(x) = \left(\frac{2}{3}\sqrt{x^3}\right)' = \sqrt{x} = f(x)$. Demak, $\int \sqrt{x}dx = \frac{2}{3}\sqrt{x^3} + C$.

5) $f(x) = \frac{1}{2\sqrt{x}} + \sin(x+1)$ funksiyaning $X \in (0;+\infty)$ oraliqdagi boshlang'ich funksiyasi $F(x) = \sqrt{x} - \cos(x+1)$ dan iborat bo'ladi, chunki $F'(x) = \left(\sqrt{x} - \cos(x+1)\right)' = \frac{1}{2\sqrt{x}} + \sin(x+1) = f(x)$.

Demak, $\int\left(\frac{1}{2\sqrt{x}} + \sin(x+1)\right)dx = \sqrt{x} - \cos(x+1) + C$.

1.2-misol. $f(x)$ funksiyaning, grafigi berilgan $A(x_0, y_0)$ nuqta orqali o'tadigan, boshlang'ich funksiyasini toping.

1) $f(x) = x^3$, $A(2;2)$;   2) $f(x) = \sin x + \frac{8}{\pi^2}x$, $A\left(\frac{\pi}{2};3\right)$.

Yechilishi. 1) $f(x) = x^3$ funksiyaning $X = (-\infty, \infty) = R$ oraliqdagi boshlang'ich funksiyasi $F(x) = \frac{x^4}{4} + C$ dan iborat bo'ladi, chunki $F'(x) = \left(\frac{x^4}{4} + C\right)' = x^3 = f(x)$. Endi C o'zgarmas sonni topamiz. $F(x) = \frac{x^4}{4} + C$ funksiyaning grafigi $A(2;2)$ nuqtadan o'tsin. $x = 2, y = 2$ larni $F(x)$ va x larning o'rniga qo'yib, $2 = \frac{2^4}{4} + C$ tenglikni hosil qilamiz, bunda $C = -2$.

Demak, $F(x) = \frac{x^4}{4} - 2$.

2) $f(x) = \sin x + \frac{8}{\pi^2}x$ funksiyaning $X = (-\infty, \infty) = R$ oraliqdagi boshlang'ich funksiyasi $F(x) = -\cos x + \frac{4}{\pi^2}x^2 + C$ dan iborat bo'ladi, chunki $F'(x) = (-\cos x + \frac{4}{\pi^2}x^2 + C)' = \sin x + \frac{8}{\pi^2}x = f(x)$. $F(x) = -\cos x + \frac{4}{\pi^2}x^2 + C$ funksiyaning grafigi $A\left(\frac{\pi}{2};3\right)$ nuqtadan o'tsin. $x = \frac{\pi}{2}, y = 3$ larni $F(x)$ ning ifodasiga qo'yib, $3 = -\cos\frac{\pi}{2} + \frac{4}{\pi^2}\left(\frac{\pi}{2}\right)^2 + C$ ni hosil qilamiz, bunda $C = 2$.

Demak, $F(x) = -\cos x + \frac{4}{\pi^2}x^2 + 2$.

1.2. Aniqmas integralning asosiy xossalari. Elementar funksiyalarning aniqmas integrallari jadvali.

1^0. $f(x)$ funksiya X oraliqda boshlang'ich funksiyaga ega bo'lsa, u holda, $\forall x \in X$ uchun

$$d\left(\int f(x)dx\right) = f(x)dx \qquad (1.2)$$

tenglik o'rinli bo'ladi, ya'ni differensial belgisi d, integral belgisi $\int$ dan oldin kelganda d va $\int$ belgilar o'zaro qisqarib integral ostidagi ifodaga teng bo'ladi;

2^0. Funksiya differensialining aniqmas integrali, shu funksiya bilan o'zgarmas sonning yig'indisiga teng, ya'ni

$$\int dF(x) = F(x) + C \quad \text{yoki} \quad \int F'(x)dx = F(x) + C \qquad (C = const) \qquad (1.3)$$

tenglik o'rinli bo'ladi, ya'ni $\int$ integral belgisi, d differensial belgisidan oldin kelganda, $\int$ va d belgilar o'zaro qisqaradi, lekin bu holda, $F(x)$ funksiyaga ixtiyoriy o'zgarmas son C ni qo'shish kerak.

3^0. Agar $f(x)$ va $g(x)$ funksiyalar X oraliqda boshlang'ich funksiyalarga ega bo'lib, λ ва μ lar haqiqiy o'zgarmas sonlar bo'lsa, u holda $\lambda f(x) + \mu g(x)$ funksiya ham shu oraliqda boshlang'ich funksiyaga ega bo'ladi, hamda

$$\int[\lambda f(x) + \mu g(x)]dx = \lambda\int f(x)dx + \mu\int g(x)dx \qquad (1.4)$$

tenglik o'rinli.

1.2-eslatma. (1.4) formuladagi tenglik, shartli ravishda, ya'ni tenglikning o'ng va chap tomonlari o'zaro ixtiyoriy o'zgarmas son aniqligida teng, deb qaraladi.

Sodda elementar funksiyalarning aniqmas integrallari jadvali qo'yidagichadir:

1. $\int 0 \cdot dx = C.$
2. $\int 1 \cdot dx = x + C.$
3. $\int x^\alpha dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C \ (\alpha \neq -1).$
4. $\int \dfrac{1}{x} dx = \ln|x| + C \ (x \neq 0).$
5. $\int a^x \cdot dx = \dfrac{a^x}{\ln a} + C, \ 0 < a \neq 1, \ \int e^x dx = e^x + C.$

6. $\int \sin x\,dx = -\cos x + C.$

7. $\int \cos x\,dx = \sin x + C.$

8. $\int \frac{1}{\cos^2 x}dx = tgx + C \quad \left(x \neq \frac{\pi}{2} + n\pi,\ n = 0, \pm 1, ...\right).$

9. $\int \frac{1}{\sin^2 x}dx = -ctgx + C \quad (x \neq n\pi,\ n = 0, \pm 1, ...).$

10. $\int shxdx = chx + C.$

11. $\int chxdx = shx + C.$

12. $\int \frac{1}{ch^2 x}dx = thx + C.$

13. $\int \frac{1}{sh^2 x}dx = -cthx + C.$

14. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}arctg\frac{x}{a} + C = -\frac{1}{a}arcctg\frac{x}{a} + C.$

15. $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$

16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C,\ |x| < a.$

17. $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C,\ |x| > |a|.$

Aniqmas integralning ta'rifi va xossalaridan foydalanib, quyida berilgan ba'zi aniqmas integrallarni hisoblaymiz.

1.3- misol. $\int x^2(x+2)(x-3)dx$ integralni hisoblang.

Yechilishi. Aniqmas integralning 3 – xossasi, hamda jadvaldagi 3 – formulaga asosan:

$$\int x^2(x+2)(x-3)dx = \int\left(x^4 - x^3 - 6x^2\right)dx = \frac{x^5}{5} - \frac{x^4}{4} - 2x^3 + C \text{ bo‘ladi.}$$

Tekshirish. Topilgan boshlang‘ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko‘ramiz:

$$\left(\frac{x^5}{5} - \frac{x^4}{4} - 2x^3 + C\right)' = x^4 - x^3 - 6x^2 + 0 = x^2\left(x^2 - x - 6\right) = x^2(x+2)(x-3),\quad x \in R.$$

Demak, berilgan funksiyaning aniqmas integrali to‘g‘ri topilgan ekan.

Misolni Maple tizimidan foydalanib yechish:

> Int((x)^2*(x+2)*(x-3),x)=int((x)^2*(x+2)*(x-3),x);

$$\int x^2(x+2)(x-3)dx = \frac{x^5}{5} - \frac{x^4}{4} - 2x^3.$$

1.4-misol. $\int\left(\left(\frac{4}{x^3} - \frac{2}{x^2} + \frac{3}{x}\right)dx\right)$ integralni hisoblang.

Yechilishi. Integral ostidagi funksiyaning aniqlanish sohasida, aniqmas integrallarning 3-xossasi hamda integrallar jadvalidagi 3 va 4-formulalarga asosan,

$$\int\left(\frac{4}{x^3}-\frac{2}{x^2}+\frac{3}{x}dx\right)=4\int x^{-3}dx-2\int x^{-2}dx+3\int\frac{1}{x}dx=-\frac{2}{x^2}+\frac{2}{x}+3\ln|x|+C\ ,\ \ x\neq 0$$

ekanligini topamiz.

Tekshirish. Topilgan boshlang‘ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko‘ramiz:

$\left(\frac{-2}{x^2}+\frac{2}{x}+3\ln|x|+C\right)'=\frac{4}{x^3}-\frac{2}{x^2}+\frac{3}{x}$. Demak, aniqmas integral to‘g‘ri topilgan.

Misolni Marle tizimidan foydalanib yechish:

> Int(4*(x)^(-3)-2*(x)^(-2)+3/x,x)=int(4*(x)^(-3)-2*(x)^(-2)+3/x,x);

$$\int\frac{4}{x^3}-\frac{2}{x^2}+\frac{3}{x}dx=-\frac{2}{x^2}+\frac{2}{x}+3\ln(x).$$

1.5-misol. $\int\frac{\sqrt{4+x^2}+2\sqrt{4-x^2}}{\sqrt{16-x^4}}dx$ integralni hisoblang.

Yechilishi. Integral ostidagi funksiyaning aniqlanish sohasida, aniqmas integralning 3 – xossasi, hamda integrallar jadvalidagi 16 va 17 formulalaridan foydalanib,

$$\int\frac{\sqrt{4+x^2}+2\sqrt{4-x^2}}{\sqrt{16-x^4}}dx=$$

$$=\int\frac{1}{\sqrt{4-x^2}}dx+2\int\frac{1}{\sqrt{4+x^2}}dx=\arcsin\frac{x}{2}+\ln\left|x+\sqrt{x^2+4}\right|+C,\ x\neq\pm 4$$

ekanligini topamiz.

Tekshirish. Topilgan boshlang‘ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko‘ramiz:

$$\left(\arcsin\frac{x}{a}+2\ln\left|x+\sqrt{x^2+4}\right|+C\right)'=\frac{1}{\sqrt{4-x^2}}+\frac{2\left(\sqrt{4+x^2}+x\right)}{\left(x+\sqrt{4+x^2}\right)\cdot\sqrt{x^2+4}}=\frac{\sqrt{x^2+4}-2\sqrt{4-x^2}}{\sqrt{16-x^4}}.$$

Shunday qilib, integral ostidagi funksiyani hosil qildik, demak aniqmas integral to‘g‘ri topilgan.

1.6-misol. $\int ctg^2x\,dx$ integralni hisoblang

Yechilishi. Integral ostidagi funksiyaning aniqlanish sohasida, aniqmas integralning 3– xossasi va integrallar jadvalining 9 – formulasiga asosan:

$\int ctg^2x\,dx=\int\left(\frac{1}{\sin^2x}-1\right)dx=-ctgx-x+C,\ x\in R$ bo‘lishini topamiz.

Tekshirish. Topilgan boshlang‘ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko‘ramiz:

$$(-ctgx - x + C)' = \frac{1}{\sin^2 x} - 1 = ctg^2 x$$

Demak, aniqmas integral to'g'ri topilgan.

1.7-misol. $\int \frac{2^x + 5^x}{10^x} dx$ integrallarni hisoblang

Yechilishi. Aniqmas integralning 3–xossasi va integrallar jadvalning 5- formulasidan foydalanib,

$$\int \frac{2^x + 5^x}{10^x} dx = \int \frac{1}{5^x} dx + \int \frac{1}{2^x} dx = \int 5^{-x} dx + \int 2^{-x} dx = -\frac{2^{-x}}{\ln 2} - \frac{5^{-x}}{\ln 5} + C, \quad x \in R,$$

bo'lishini topamiz.

Tekshirish. Topilgan boshlang'ich funksiya hosilasining integral ostidagi funksiyaga teng yoki teng emasligini tekshirib ko'ramiz:

$$\left(-\frac{2^{-x}}{\ln 2} - \frac{5^{-x}}{\ln 5} + C\right)' = \frac{1}{2^x} + \frac{1}{5^x} = \frac{2^x + 5^x}{10^x}$$

Demak, topilgan boshlang'ich funksiyaning hosilasi integral ostidagi funksiyaga teng ekan.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning bitta boshlang'ich funksiyasini toping.

1.1. $f(x) = 5x^4$. **1.2.** 2) $f(x) = \cos 2x$. **1.3.** 3) $f(x) = e^{1-3x}$.

1.4. 4) $f(x) = ctg3x$. **1.5.** 5) $f(x) = \frac{1}{\cos^2 3x} + 2^{3x}$. **1.6.** 6) $f(x) = \sqrt{x} + \sqrt[3]{x}$.

Berilgan oraliqda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi ekanligini ko'rsating va aniqmas integralini yozing:

1.7. 1) $f(x) = 2 - \frac{3}{\cos^2 3x}$. **1.8.** 2) $f(x) = ctg^2 5x$.

1.9. 3) $f(x) = 3(3x+5)^4$. **1.10.** 4) $f(x) = \sin 2x \cdot \cos x$.

1.11. 5) $f(x) = \left(\frac{\sin 2x - 2\sin^2 x}{1 - tgx}\right)^2$. **1.12.** 6) $f(x) = \ln^{-1} x - \ln^{-2} x$.

$f(x)$ funksiyaning, grafigi berilgan $A(x_0, y_0)$ nuqta orqali o'tadigan, boshlang'ich funksiyasini toping.

1.13. 1) $f(x) = x^2$, $A(3;2)$.

1.14. 2) $f(x) = \cos x + 2x$, $A\left(\frac{\pi}{2};3\right)$.

1.15. 3) $f(x) = e^x + \sin 2x$, $A\left(0;\frac{7}{2}\right)$.

1.16. 4) $f(x) = \frac{1}{x} + 2x$, $A(e,1)$.

1.17. 5) $f(x)=2\cos^2\frac{x}{2}$, $\quad A(0;3)$.

1.18. 6) $f(x)=e^{3+2x}-4\sin(2x+3)$, $\quad A\left(\frac{-3}{2},0\right)$

Agar *F(x)* funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, berilgan funksiyaning boshlang'ich funksiyalarini toping.

1.19. 1) $5f(5x)$. **1.20.** 2) $3f\left(\frac{x}{2}\right)$. **1.21.** 3) $-4f(-4x+3)$.

1.22. 4) $3f(-3x+2)$. **1.23.** 5) $\frac{4}{5}f\left(\frac{2}{3}x+7\right)$. **1.24.** 6) $cf(ax+b)$.

Quyidagi integrallarni hisoblang:

1.25. $\int 4x^7dx.$ **1.26.** $\int\frac{dx}{x^5}.$

1.27. $\int(5x^3-2x^2+3x-8)dx.$ **1.28.** $\int\frac{3x^4+5x^3-6x\sqrt[4]{x}+4}{x}dx.$

1.29. $\int\frac{2x^4-5x\sqrt[3]{x}+7\sqrt{x}}{x\sqrt{x}}dx.$ **1.30.** $\int\left(x^{3/2}-\frac{2}{x^{3/2}}\right)^2dx.$

Quyidagi integrallarni hisoblang:

1.31. $\int(x-1)(x+2)dx.$ **1.32.** $\int x^2(x+1)(5x-3)dx.$

1.33. $\int\sqrt[3]{x^2}(8\sqrt[3]{x}-1)dx.$ **1.34.** $\int\frac{(\sqrt{x}-1)^3}{x\sqrt{x}}dx.$

1.35. $\int\frac{x-9}{\sqrt{x}+3}dx.$ **1.36.** $\int\frac{125-x}{\sqrt[3]{x}-5}dx.$

Kuyidagi.integrallarni hisoblang:

1.37. $\int 8^x dx.$ **1.38.** $\int 5^{3x}\cdot e^x dx.$

1.39. $\int 5^{x-2}dx.$ **1.40.** $\int\frac{32^x-2^x}{4^x}dx.$

1.41. $\int 6^x(6^x+4)dx.$ **1.42.** $\int(3^x-1)(3^{-x}+1)dx.$

Quyidagi integrallarni hisoblang:

1.43. $\int 8\cos x\,dx.$ **1.44.** $\int\frac{\sin x}{9}dx.$

1.45. $\int\frac{1}{5\sin^2 x}dx.$ **1.46.** $\int\left(\frac{2}{\cos^2 x}-\frac{5}{\sin^2 x}\right)dx.$

1.47. $\int\frac{5-4\cos^3 x}{\cos^2 x}dx.$ **1.48.** $\int\frac{7+2x\sin^2 x}{\sin^2 x}dx.$

Quyidagi integrallarni hisoblang:

1.49. $\int \frac{dx}{\cos^2 x \sin^2 x}$.

1.50. $\int \frac{1-\cos 2x}{6\sin x} dx$.

1.51. $\int \frac{dx}{\sin^2 x + \cos 2x}$.

1.52. $\int \frac{1-4ctg^2 x}{\cos^2 x} dx$.

1.53. $\int (ctgx - tgx)^2 dx$.

1.54. $\int \frac{1+\cos^2 x}{1+\cos 2x} dx$

Quyidagi integrallarni hisoblang:

1.55. $\int \frac{1+\sqrt{4-x^2}}{\sqrt{4-x^2}} dx$.

1.56. $\int \sqrt{\frac{4+x^2}{16-x4}} dx$

1.57. $\int \frac{\sqrt{4+x^2}-\sqrt{4-x^2}}{\sqrt{16-x^4}} dx$

1.58. $\int \sqrt{\frac{3+x^2}{x^4-9}} dx$

1.59. $\int \frac{\sqrt{x^2+9}-6}{x^2+9} dx$

1.60. $\int \frac{4-x^2}{16-x^4} dx$.

Quyidagi integrallarni hisoblang:

1.61. $\int \frac{3+x^2}{1+x^2} dx$

1.62. $\int \frac{1-2x^2}{x^2(1-x^2)} dx$.

1.63. $\int \frac{(2x^2+5)dx}{x^2(x^2+5)}$

1.64. $\int \frac{x^2-7}{9-x^2} dx$.

165. $\int \frac{dx}{16-x^2} dx$

1.66. $\int \frac{dx}{x^4+4x^2}$

Quyidagi integrallarni hisoblang:

1.67. $\int \frac{dx}{ch^2 x\, sh^2 x}$.

1.68. 2) $\int \frac{ch2x}{ch^2 x\, sh^2 x}$.

1.69. $\int sh^2 x dx$.

1.70. $\int ch^2 x dx$.

1.71. $\int th^2 x dx$.

1.72. $\int cth^2 x dx$.

Quyidagi integrallarni hisoblang $(a \neq 0)$:

1.73. $\int e^{ax} dx$.

1.74. $\int \sin(ax+b) dx$.

1.75. $\int \cos(ax+b) dx$.

1.76. $\int b^{ax} dx, \quad b \neq 1, \ b > 0$.

1.77. $\int \frac{dx}{\cos^2 ax}$.

1.78. $\int \frac{dx}{\sin^2 ax}$.

1.79. $\int \frac{dx}{b+ax}$.

1.80. $\int (ax+b)^a dx$.

1.81. $\int \frac{dx}{a^2+b^2x^2}$.

1.82. $\int \frac{dx}{a^2-b^2x^2}$.

1.83. $\int \frac{dx}{\sqrt{a^2+b^2x^2}}$.

1.84. $\int \sqrt{a^2+b^2x^2} dx$.

1.85. $\int \frac{dx}{\sqrt{a^2-b^2x}}$.

1.86. $\int \sqrt{a^2-b^2x^2} dx$.

1.87. $\int \frac{dx}{\sqrt{b^2x^2-a^2}}$.

1.88. $\int shaxdx$.

1.89. $\int sh^2 axdx$. **1.90.** $\int chaxdx$.

1.91. $\int ch^2 axdx$. **1.92.** $\int \frac{dx}{ch^2 ax}$ **1.93.** $\int \frac{dx}{sh^2 ax}$

Quyidagi integrallarni hisoblang:

1.94. $\int e^{3x+4}dx$. **1.95.** $\int (12x-5)^7 dx$. **1.96.** $\int \sqrt{9x+7}\,dx$

1.97. $\int \cos 5x\,dx$. **1.98.** $\int \sin\left(\frac{\pi}{4}-3x\right)dx$. **1.99.** $\int \frac{dx}{(6-5x)^4}$

1.100. $\int \frac{dx}{\sqrt[3]{9x-7}}$. **1.101.** $\int \frac{dx}{6x+5}$. **1.102.** $\int \frac{dx}{7-5x}$

1.103. $\int 6^{3x+2}dx$. **1.104.** $\int \frac{dx}{9+25x^2}$. **1.105.** $\int \frac{dx}{25-16x^2}$.

1.106. $\int \frac{dx}{4x^2-9}$. **1.107.** $\int \frac{dx}{\sqrt{25-9x^2}}$. **1.108.** $\int \frac{dx}{\sqrt{16+25x^2}}$.

Mustaqil yechish uchun misollarning javoblari

1.1. $F(x)=x^5$. **1.2.** $F(x)=\frac{1}{2}\sin 2x$. **1.3.** $F(x)=-\frac{1}{3}e^{1-3x}$ **1.4.** $F(x)=\frac{1}{3}\ln|\sin 3x|$.

1.5. $F(x)=\frac{1}{3}tg3x+\frac{1}{3}\frac{8^x}{\ln 2}$. **1.6.** $F(x)=\frac{2}{3}x\sqrt{x}+\frac{3}{4}x\sqrt[3]{x}$. **1.7.** $2x-tg3x+C$

1.8. $-\frac{1}{5}ctg5x-x+C$. **1.9.** $\frac{1}{5}(3x+5)^5+C$. **1.10.** $-\frac{2}{3}\cos x-\frac{1}{6}\cos 3x+C$.

1.11. $\frac{1}{2}x-\frac{1}{8}\sin 4x+C$. **1.12.** $\frac{x}{\ln x}+C$. **1.13.** $\frac{x^3}{3}-7$. **1.14.** $\sin x+x^2+2-\frac{\pi^2}{4}$.

1.15. $e^x-\frac{1}{2}\cos 2x+3$. **1.16.** $\ln|x|+x^2-e^2$ **1.17.** $x+\sin x+3$.

1.18. $\frac{1}{2}e^{3+2x}+2\sin(2x+3)-2{,}5$. **1.19.** $F(5x)$. **1.20.** $6F\left(\frac{x}{2}\right)$ **1.21.** $F(-4x+3)$.

1.22. $-F(-3x+2)$. **1.23.** $\frac{6}{5}F\left(\frac{2}{3}x+7\right)$ **1.24.** $\frac{c}{a}F(ax+b)$. **1.25.** $\frac{1}{2}x^8+C$.

1.26. $-\frac{1}{4x^4}+C$. **1.27.** $\frac{5}{4}x^3-\frac{2}{3}x^3+\frac{3}{2}x^2-8x+C$.

1.28. $\frac{3}{4}x^4+\frac{5}{3}x^3-\frac{24}{5}x^{\frac{5}{4}}+4\ln|x|+C$. **1.29.** $\frac{4}{7}x^{7/2}-6x^{\frac{5}{6}}+7\ln|x|+C$.

1.30. $\frac{x^4}{4}-4x-\frac{2}{x^2}+C$. **1.31.** $\frac{x^3}{3}+\frac{x^2}{2}-2x+C$. **1.32.** $x^5+\frac{x^4}{2}-x^3+C$.

1.33. $4x^2-\frac{3}{5}x^{5/3}+C$. **1.34.** $x-6\sqrt{x}+3\ln|x|+\frac{2}{\sqrt{x}}+C$. **1.35.** $\frac{2}{3}x^{3/2}-3x+C$.

1.36. $-\frac{3}{5}x^{5/3}-\frac{15}{4}x^{4/3}-25x+C$. **1.37.** $\frac{8^x}{\ln 8}+C$. **1.38.** $\frac{125^x\cdot e^x}{3\ln 5+1}+C$.

1.39. $\frac{5^{x-2}}{\ln 5}+C$. **1.40.** $\frac{16^x+3}{2^x\cdot\ln 2}+C$. **1.41.** $\frac{6^x(6^x+8)}{2\ln 6}+C$. **1.42.** $\frac{3^x+3^{-x}}{\ln 5}+C$.

1.43. $8\sin x+C$. **1.44.** $-\frac{1}{9}\cos x+C$. **1.45.** $-\frac{1}{5}ctgx+C$. **1.46.** $2tgx+5tgx+C$.

1.47. $5tgx-4\sin x+C$. **1.48.** $-7tgx+x^2+C$. **1.49.** $tgx-ctgx+C$.

1.50. $-\frac{1}{3}\cos x+C$. **1.51.** $tgx+C$. **1.52.** $tgx+4ctgx+C$. **1.53.** $tgx-ctgx-4x+C$.

1.54. $\frac{1}{2}(x+tgx)+C$. **1.55.** $\arcsin\frac{x}{2}+x+C$. **1.56.** $\arcsin\frac{x}{4}+C$.

1.57. $\arcsin\frac{x}{2}-\ln\left|x+\sqrt{x^2+4}\right|+C$. **1.58.** $\ln\left|x+\sqrt{x^2-3}\right|+C$.

1.59. $\ln\left|x+\sqrt{x^2+9}\right|-2arctg\frac{x}{3}+C$. **160.** $\frac{1}{2}arctg\frac{x}{2}+C$. **1.61.** $x+2arctgx+C$.

1.62. $\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|-\frac{1}{x}+C$. **1.63.** $\frac{1}{\sqrt{5}}arctg\frac{x}{\sqrt{5}}-\frac{1}{x}+C$. **1.64.** $\frac{1}{3}\ln\left|\frac{3+x}{3-x}\right|-x+C$.

1.65. $\frac{1}{32}\left[2arctg\frac{x}{2}-\ln\left|\frac{x-2}{x+2}\right|\right]+C$. **1.66.** $-\frac{1}{4x}-\frac{1}{8}arctg\frac{x}{2}+C, \quad x\neq 0$.

1.67. $-cthx-thx+C$. **1.68.** $thx-cthx+C$. **1.69.** $\frac{1}{4}sh2x-\frac{1}{2}x+C$.

1.70. $\frac{1}{4}sh2x+\frac{1}{2}x+C$. **1.71.** $x-thx+C$. **1.72.** $x-cthx+C$. **1.73.** $\frac{1}{a}e^{ax}+C$.

1.74. $-\frac{1}{a}\cos(ax+b)+C$. **1.75.** $\frac{1}{a}\sin(ax+b)+C$. **1.76.** $\frac{1}{a}\frac{b^{ax}}{\ln b}+C, \ b\neq 1, \ b>0$.

1.77. $\frac{1}{a}tgax+C$. **1.78.** $-\frac{1}{a}ctgax+C$. **1.79.** $\frac{1}{a}\ln|ax+b|+C$.

1.80. $\frac{(ax+b)^{\alpha+1}}{a(\alpha+1)}+C$. **1.81.** $\frac{1}{ab}arctg\frac{bx}{a}+C=-\frac{1}{ab}arcctg\frac{bx}{a}+C$.

1.82. $\frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|+C$. **1.83.** $\frac{1}{b}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|+C$. **1.84.** $\frac{x\sqrt{a^2+b^2x^2}}{2}+$
$+\frac{a^2}{2b}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|+C$. **1.85.** $\frac{1}{b}\arcsin\frac{bx}{a}+c=-\frac{1}{b}\arccos\frac{bx}{a}+C$.

1.86. $\frac{x\sqrt{a^2-b^2x^2}}{2}+\frac{a^2}{2b}\arcsin\frac{bx}{a}+C$. **1.87.** $\frac{1}{b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|+C$.

1.88. $\frac{1}{a}chax+C$. **1.89.** $\frac{sh2ax}{4a}-\frac{x}{2}+C$. **1.90.** $\frac{1}{a}shax+C$. **1.91.** $\frac{sh2ax}{4a}+\frac{x}{2}+C$.

1.92. $\frac{1}{a}thax+C$. **1.93.** $-\frac{1}{a}cthax+C$. **1.94.** $\frac{1}{3}e^{3x+4}+C$. **1.95.** $\frac{1}{96}(12x-5)^8+C$.

1.96. $\frac{2}{27}\sqrt{(9x+7)^3}+C$. **1.97.** $\frac{1}{5}\sin 5x+C$. **1.98.** $\frac{1}{3}\cos\left(\frac{\pi}{4}-3x\right)+C$.

1.99. $\frac{1}{15}\frac{1}{(6-5x)^3}+C$. **1.100.** $\frac{1}{6}\sqrt[3]{(9x-7)^2}+C$. **1.101.** $\frac{1}{6}\ln|6x+5|+C$.

1.102. $-\frac{1}{5}\ln(7-5x)+C$. **1.103.** $\frac{6^{3x+2}}{3\ln 6}+C$. **1.104.** $\frac{1}{15}arctg\frac{5x}{3}+C$.

1.05. $\frac{1}{40}\ln\left|\frac{5+4x}{5-4x}\right|+C$. **1.106.** $\frac{1}{12}\ln\left|\frac{3-2x}{3+2x}\right|+C$.

1.107. $\frac{1}{3}\arcsin\frac{3x}{5}+C$. **1.108.** $\frac{1}{5}\ln\left|5x+\sqrt{16+25x^2}\right|+C$.

2-§. Integrallash usullari

2.1. O'zgaruvchilarni almashtirish usuli. O'zgaruvchilarni almashtirish usuli aniqmas integralni hisoblashning eng muhim usullaridan biri bo'lib, unda hisoblash talab qilingan integral, hisoblash uchun qulay (oson) bo'lgan integralga almashtiriladi, u esa, quyidagi tasdiqqa asoslanadi:

$t=\varphi(x)$ *funksiya biror* X *(interval yoki segment, yarim o'q yoki son o'qi) oraliqda aniqlangan va differensiallanuvchi bo'lib, uning qiymatlar to'plami* $T=\{t\}$ *bo'lsin. T to'plamda* $g(t)$ *funksiya aniqlangan bo'lib, uning uchun* $G(t)$ *funksiya boshlang'ich funksiya bo'lsin, ya'ni*

$$\int g(t)\,dt=G(t)+C. \qquad (2.1)$$

U holda X *to'plamning hamma nuqtalarida* $G[g(x)]$ *funksiya* $g[\varphi(x)]\varphi'(x)$ *funksiya uchun boshlang'ich funksiya bo'ladi, ya'ni*

$$\int g[\varphi(x)]\varphi'(x)\,dx=G[\varphi(x)]+C \qquad (2.2)$$

Ushbu

$$\int f(x)\,dx \qquad (2.3)$$

integralni hisoblash talab qilingan bo'lsin. Ko'p hollarda, yangi o'zgaruvchi sifatida shunday differensiallanuvchi $t=\varphi(x)$ funksiyani tanlash mumkin bo'ladiki, bunda

$$f(x)\,dx=g[(\varphi(x))]\varphi'(x)\,dx \qquad (2.4)$$

tenglik o'rinli bo'lib, $g(t)$ funksiya, $f(x)$ funksiyaga nisbatan oson integrallanadi, ya'ni

$$\int g(t)\,dt=G(t)+C$$

bundan $t=\varphi(x)$ almashtirish natijasida hisoblash talab qilingan integralni hosil qilamiz:

$$\int f(x)\,dx=G[\varphi(x)]+C \qquad (2.5)$$

(2.3) integralni hisoblashning bu usuli, o'zgaruvchilarni *almashtirish usuli deyiladi.*

Albatta, integrallashning bu usuli hamma integrallarni hisoblash uchun ham qo'llanilavermaydi. Integrallarni hisoblaganda, o'zgaruvchilarni almashtirish usulini qo'llashda, almashtirishni to'g'ri tanlash hisoblovchining mahoratiga bog'liq.

Misol uchun, $\int\frac{\varphi'(x)}{\varphi(x)}dx$ ko'rinishdagi integrallarni hisoblashda, albatta, $t=\varphi(x)$ almashtirish, olish kerak:

$$\int \frac{\varphi'(x)}{\varphi(x)}dx = \int \frac{d\varphi(x)}{\varphi(x)} = \int \frac{dt}{t} = \left(\ln|t| + C\right)\Big|_{t=\varphi(x)} = \ln|\varphi(x)| + C \qquad (2.6)$$

Shu tipdagi integrallar turiga $\int ctgx\,dx$ ham kiradi:

$$\int ctgx\,dx = \int \frac{(\sin x)'}{\sin x}dx = \ln|\sin x| + C .$$

Ba'zi hollarda, integralni hisoblashda, o'zgaruvchilarni almashtirish usulini qo'llash uchun, avvalo, integral ostida funksiyaning shaklini o'zgartirish maqsadga muvofiq bo'ladi. Masalan, $\int \frac{1}{\sin x}dx$ integralni hisoblashda, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ tenglikni e'tiborga olib, integral ostidagi ifodaning shaklini o'zgartirib, (2.6) formulani e'tiborga olsak, natijada

$$\int \frac{1}{\sin x}dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx =$$

$$= \int \frac{1}{2tg\frac{x}{2}\cdot\cos^2\frac{x}{2}}dx = \int \frac{\left(tg\frac{x}{2}\right)'}{tg\frac{x}{2}}dx = \left(\ln|t| + C\right)\Big|_{t=tg\frac{x}{2}} = \ln\left|tg\frac{x}{2}\right| + C \qquad (2.7)$$

bo'lishini topamiz.

2.1 – misol. Quyidagi integrallarni hisoblang:

$1)\int \frac{x^2dx}{3-x^6};\quad 2)\int \frac{\cos xdx}{\sqrt{1+3\sin x}};\quad 3)\int tg^3\varphi\, d\varphi;\quad 3)\int \frac{\sqrt{2+\ln x}}{x},\ x>0;$

$5)\int x\sqrt{a-x}\,dx;\quad 6)\int \frac{dt}{t^2\sqrt{a+t^2}};\quad 7)\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx.$

Yechilishi. 1) $\int \frac{x^2dx}{3-x^6}$ integralni hisoblashda $t = x^3$ almashtirishni olish qulay. Bu almashtirishni bajarib, integrallar jadvalining 15 – formulasiga asosan, quyidagiga ega bo'lamiz:

$$dt = 3x^2dx, \quad x^2dx = \frac{1}{3}dt,$$

$$\int \frac{x^2dx}{3-x^6} = \frac{1}{3}\int \frac{dt}{3-t^2} = -\frac{1}{3}\int \frac{dt}{t^2-(\sqrt{3})^2} = \left(-\frac{1}{6\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C\right)\Big|_{t=x^3} = -\frac{1}{6\sqrt{3}}\ln\left|\frac{x^3-\sqrt{3}}{x^3+\sqrt{3}}\right| + C$$

Misolni Maple tizimidan foydalanib, yechish:

> f:=(x^2)/(3-x^6);

$$f := \frac{x^2}{3-x^6}$$

> Int(f,x);

$$\int \frac{x^2}{3-x^6}\,dx$$

> **int(f,x);**

$$\frac{1}{9}\sqrt{3}\ \operatorname{arctanh}\left(\frac{x^3\sqrt{3}}{3}\right)$$

> **Int(f,x)=int(f,x);**

$$\int \frac{x^2}{3-x^6}\,dx = \frac{1}{9}\sqrt{3}\ \operatorname{arctanh}\left(\frac{x^3\sqrt{3}}{3}\right)$$

2) $\int \frac{\cos x dx}{\sqrt{1+3\sin x}}$ integralni hisoblashda $t = 1+3\sin x$ almashtirishni olish maqsadga muvofiq bo‘ladi. Bundan

$$dt = 3\cos x dx, \qquad \cos x dx = \frac{1}{3}dt,$$

$$\int \frac{\cos x\,dx}{\sqrt{1+3\sin x}} = \frac{1}{3}\int \frac{dt}{\sqrt{t}} = \frac{1}{3}\int t^{-\frac{1}{2}}dt = \frac{1}{3}\left(2t^{\frac{1}{2}}+C\right)\Bigg|_{t=1+3\sin x} = \frac{2}{3}\sqrt{1+3\sin x}+C.$$

Misolni Maple tizimidan foydalanib yechish:

> **f:=cos(x)/sqrt(1+3*sin(x));**

$$f := \frac{\cos(x)}{\sqrt{1+3\sin(x)}}$$

> **Int(f,x);**

$$\int \frac{\cos(x)}{\sqrt{1+3\sin(x)}}\,dx$$

> **int(f,x);**

$$\frac{2}{3}\sqrt{1+3\sin(x)}$$

> **Int(f,x)=int(f,x);**

$$\int \frac{\cos(x)}{\sqrt{1+3\sin(x)}}\,dx = \frac{2}{3}\sqrt{1+3\sin(x)}$$

3) $\int tg^3\varphi d\varphi$ integralni hisoblash uchun $\varphi = arctgt$, $t = tg\varphi$ almashtirish olamiz va

$$d\varphi = \frac{1}{1+t^2}dt,\ \int tg^3\varphi d\varphi = \int \frac{t^3}{1+t^2}dt = \int\left(t-\frac{t}{1+t^2}\right)dt = \frac{t^2}{2}-\frac{1}{2}\int\frac{d(1+t^2)}{1+t^2} =$$

$$= \left(\frac{t^2}{2}-\frac{1}{2}\ln\left|1+t^2\right|+C\right)_{t=tg\varphi} = \frac{tg^2\varphi}{2}-\frac{1}{2}\ln\left|1+tg^2\varphi\right|+C = \frac{tg^2\varphi}{2}+\ln\left|\cos\varphi\right|+C$$

bo‘ladi.

4) $\int\frac{\sqrt{1+\ln x}}{x}dx$ integralni hisoblashda $t=1+\ln x$ almashtirish olamiz. Natijada

$$dt=\frac{dx}{x},\quad \int\frac{\sqrt{1+\ln x}}{x}dx=\int\sqrt{t}\,dt=\int t^{\frac{1}{2}}\,dt=\left(\frac{2}{3}t^{\frac{3}{2}}t+C\right)\Bigg|_{t=1+\ln x}=\frac{2}{3}\sqrt{(1+\ln x)^3}+C$$

bo'ladi.

Misolni Maple tizimidan foydalanib yechish:

> f:=sqrt(2+ln(x))/x;

\>

$$f:=\frac{\sqrt{2+\ln(x)}}{x}$$

> Int(f,x);

$$\int\frac{\sqrt{2+\ln(x)}}{x}\,dx$$

> int(f,x);

$$\frac{2}{3}(2+\ln(x))^{(3/2)}$$

> Int(f,x)=int(f,x);

\>

$$\int\frac{\sqrt{2+\ln(x)}}{x}\,dx=\frac{2}{3}(2+\ln(x))^{(3/2)}$$

5) $\int x\sqrt{a-x}\,dx$ - integralni hisoblashda $t=\sqrt{a-x}$ almashtirishni olish qulay bo'ladi. Bundan, $t^2=a-x$, $dt=-\frac{1}{2\sqrt{a-x}}dx$, $x=a-t^2$, $dx=-2t\,dt$,

$$\int x\sqrt{a-x}dx=-2\int\left(a-t^2\right)t^2dt=-2\int\left(at^2-t^4\right)dt=$$

$$=\left(-2a\frac{t^3}{3}+2\frac{t^5}{5}+C\right)\Bigg|_{t=\sqrt{a-x}}=\frac{2}{15}\left(3x^2-ax-2a^2\right)+C$$

Misolni Maple tizimidan foydalanib yechish:

> Int(x*sqrt(a-x),x)=int(x*sqrt(a-x),x);

$$\int x\sqrt{a-x}\,dx=-\frac{2(2a+3x)(a-x)^{(3/2)}}{15}$$

6) $\int\frac{dt}{t^2\sqrt{a+t^2}}$ integralda $x=\frac{1}{t}$ almashtirish olib, uni quyidagicha hisoblaymiz:

$$\int\frac{dt}{t^2\sqrt{a+t^2}}=-\int\frac{x\,dx}{\sqrt{ax^2+1}}=-\frac{1}{2a}\int\frac{d(ax^2+1)}{\sqrt{ax^2+1}}=-\frac{\sqrt{ax^2+1}}{a}=-\frac{\sqrt{a+t^2}}{at}+C$$

Misolni Maple tizimidan foydalanib yechish:

> Int(1/(t^2*sqrt(a+t^2)),t)=int(1/(t^2*sqrt(a+t^2)),t);

$$\int \frac{1}{t^2\sqrt{a+t^2}}\,dt = -\frac{\sqrt{a+t^2}}{t\,a}$$

7) $\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx$ integralda avvalo, integral ostidagi ifodaning shaklini o'zgartiramiz:

$$\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx = \int \frac{1+\frac{1}{x^2}}{\sqrt{x^2-7+\frac{1}{x^2}}}dx =$$

$$= \int \frac{d\left(x-\frac{1}{x}\right)}{\sqrt{\left(x-\frac{1}{x}\right)^2-5}}dx = \int \frac{dt}{\sqrt{t^2-5}},$$

bunda, $t = x - \frac{1}{x}$. Endi integrallar jadvalining 17-formulasiga asosan, integralni hisoblaymiz. Natijada,

$$\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx = \left(\ln\left|t+\sqrt{t^2-5}\right|+C\right)\Bigg|_{t=x-\frac{1}{x}} = \ln\left|x-\frac{1}{x}+\sqrt{x^2-7+\frac{1}{x^2}}\right|+C.$$

bo'ladi.

Misolni Maple tizimidan foydalanib yechish:

> Int((x^2+1)/sqrt(x^6-7*x^4+x^2),x)=int((x^2+1)/sqrt(x^6-7*x^4+x^2),x);

$$\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx =$$

$$\frac{1}{2}\,\frac{x\sqrt{x^4-7x^2+1}\left(\ln\left(-\frac{7}{2}+x^2+\sqrt{x^4-7x^2+1}\right)+\operatorname{arctanh}\left(\frac{-2+7x^2}{2\sqrt{x^4-7x^2+1}}\right)\right)}{\sqrt{x^6-7x^4+x^2}}$$

2.2. Bo'laklab integrallash usuli. Integrallarni hisoblashda bo'laklab integrallash usuli muhim usullardan biri hisoblanadi. Bu usul quyidagi tasdiqqa asoslangan:

$u(x)$ *va* $v(x)$ *funksiyalar biror* X *oraliqda aniqlangan va differensiallanuvchi bo'lsin. Agar shu oraliqda* $v(x)u'(x)$ *funksiyaning boshlang'ich funksiyasi mavjud bo'lsa, u holda* $u(x)v'(x)$ *funksiyaning ham boshlang'ich funksiyasi mavjud bo'ladi va*

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \qquad (2.8)$$

yoki

$$\int udv = uv - \int vdu \qquad (2.8`)$$

formula o‘rinli. (2.8) yoki (2.8`) formula *bo‘laklab integrallash* formulasi deyiladi. (2.8) formula $\int u(x)v'(x)dx$ integralni hisoblash masalasini, $\int v(x)u'(x)dx$ integralni hisoblash masalasiga olib keladi. Ko‘p hollarda, oxirgi integral, oldingi integralga qaraganda osonroq hisoblanadi.

2.3 - misol. Ushbu

$$\int x^{\alpha} \ln x dx, \quad \alpha \in R$$

integralni hisoblang.

Yechilishi. 1) Avvalo, integralni $\alpha \neq -1$ bo‘lganda xisoblaymiz. $u = \ln x$, $dv = x^{\alpha}dx$ deb belgilab, $du = \frac{dx}{x}$, $v = \frac{x^{\alpha+1}}{\alpha+1}$ formulalarga asosan,

$$J = \int x^{\alpha} \ln x dx = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{1}{\alpha+1}\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C = \frac{x^{\alpha+1}}{\alpha+1}\left(\ln x - \frac{1}{\alpha+1}\right) + C$$

ekanligini topamiz.

2) $\alpha = -1$ bo‘lsin. Bu holda

$$\int x^{-1} \ln x dx = \int \frac{\ln x}{x} dx = \int \ln x\, d(\ln x) = \frac{1}{2}\ln^2 x + C.$$

2. 4 - misol. Ushbu

$$\int x^2 3^x dx$$

integralni hisoblang.

Yechilishi. $u = x^2, dv = 3^x dx$ deb belgilab, (2.8) formulaga asosan, topamiz:

$$du = 2xdx,\ v = \frac{3^x}{\ln 3},$$
$$\int x^2 \cdot 3^x dx = \frac{x^2 \cdot 3^x}{\ln 3} - \frac{2}{\ln 3}\int x \cdot 3^x dx. \qquad (2.9)$$

Oxirgi integralga yana (2.8) formulani qo‘llaymiz: $u = x, dv = 3^x dx$ deb belgilab, (2.8) formulaga asosan,

$$du = dx,\ v = \frac{3^x}{\ln 3}$$
$$\int x \cdot 3^x dx = \frac{x \cdot 3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x dx = \frac{x \cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2}.$$

ekanligini olamiz. Oxirgi natijani (2.9) ga keltirib qo‘ysak, natijada

$$\int x^2 \cdot 3^x = \frac{1}{\ln 3} \cdot x^2 3^x - \frac{2x3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3} = \frac{3^x}{\ln 3}\left(x^2 - \frac{2x}{\ln 3} + \frac{2}{(\ln 3)^2}\right) + C.$$

bo‘lishini topamiz.

Misolni Maple tizimidan foydalanib yechish:

```
> restart:with(student):J=Int(((x)^2)*(3)^x,x);
```

$$J=\int x^2 3^x\,dx$$

```
> J=intparts(Int(((x)^2)*(3)^x,x),x^2);
```

$$J=\frac{x^2\cdot 3^x}{\ln 3}-\frac{2}{\ln 3}\int x\cdot 3^x\,dx$$

```
> intparts(%,x);
```

$$J=\frac{x^2 3^x}{\ln(3)}-\frac{2x3^x}{(\ln 3)^2}+\int\frac{2\,3^x}{(\ln 3)^2}dx$$

```
> value(%);
```

$$J=\frac{x^2 3^x}{\ln(3)}-\frac{2x3^x}{\ln(3)^2}+\frac{2\,3^x}{\ln(3)^3}$$

2.5 - misol. Ushbu

$$\int x^2 arctgx\,dx$$

integralni hisoblang.

Yechilishi. $u=arctgx,\ dv=x^2dx$ deb belgilaylik. U holda, $du=\frac{1}{1+x^2}dx,\ v=\frac{x^3}{3}$. formulalarga asosan, integralning qiymati

$$\int x^2arctgx\,dx=\frac{1}{3}x^3arctgx-\frac{1}{3}\int\frac{x^3}{1+x^2}dx=$$

$$=\frac{1}{3}x^3arctgx-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx=\frac{1}{3}x^3arctgx-\frac{x^2}{6}+\frac{1}{6}\int\frac{d(1+x^2)}{1+x^2}=$$

$$=\frac{1}{3}x^3arctgx-\frac{x^2}{6}+\frac{1}{6}\ln\left(1+x^2\right)+C$$

bo‘ladi.

2.6-misol. Ushbu

$$\int x^2\sin x\,dx$$

integralni hisoblang.

Yechilishi. Berilgan integralni hisoblash uchun $u=x^2,\ dv=\sin x\,dx$ deb olib, (2.8) formulaga asosan, $du=2xdx,\ v=-\cos x,$ bo‘lishini topamiz.

Unda

$$\int x^2\sin x\,dx=-x^2\cos x+2\int x\cos x\,dx \qquad (2.10)$$

bo‘ladi. Oxirgi munosabatning o‘ng tomonidagi integralni hisoblash uchun unga yana bir marta (2.8) formulani qo‘llaymiz:

$$u=x,\ dv=\cos dx,\ du=dx,\ v=\sin x$$

Bundan

$$\int x\cdot\cos x\,dx=x\cdot\sin x-\int\sin x\,dx=x\cdot\sin x+\cos x.$$

ekanligini topamiz. Olingan natijani (2.10) ga keltirib qo‘yamiz.

Natijada, $\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x = \left(2 - x^2\right)\cos x + 2x\sin x + C$.

2.7-misol. $I = \int\sqrt{x^2 + a^2}dx$ integralni hisoblang.

Yechilishi. $u = \sqrt{x^2 + a^2}$ $dv = dx$ deb belgilab, (2.8) formuladan foydalanib, berilgan integralni quyidagi holga keltiramiz:

$$\int\sqrt{x^2 + a^2}dx = x\cdot\sqrt{x^2 + a^2} - \int\frac{x^2dx}{\sqrt{x^2 + a^2}}. \qquad (2.11)$$

(2.11) ning o‘ng tomonidagi integralni hisoblashda, integrallar jadvalining 17 - formulasidan foydalanamiz:

$$\int\frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}}dx = I - a^2\int\frac{dx}{\sqrt{x^2 + a^2}} = I - a^2\ln\left|x + \sqrt{x^2 + a^2}\right|. \qquad (2.12)$$

(2.12) ni (2.11) ga olib borib qo‘yish natijasida,

$$\int\sqrt{x^2 + a^2}dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c \qquad (2.13)$$

ekanligini topamiz. *2-bandda* yechilgan misollarni tahlil qilish natijasida, bo‘laklab integrallash usuli bilan hisoblanadigan integral-larning ko‘proq qismini, shartli ravishda, quyidagi uch guruhga ajratish mumkin.

1. Birinchi guruhga integral ostidagi funksiya tarkibida ko‘pay-tuvchi sifatida

$$\ln x,\ \arcsin x,\ \arccos x,\ \operatorname{arctg}x,\ (\arcsin x)^2,$$
$$(\arccos x)^2,\ (\operatorname{arctg}x)^2,\ \ln\varphi(x),\ldots$$

funksiyalardan biri qatnashgan integrallar kiradi (2.3, 2.5-misollarga qarang).

1-eslatma. Integral ostidagi funksiya tarkibida ko‘paytuvchi sifati-da $(\operatorname{arctg}x)^2, (\arccos x)^2,\ldots$ funksiyalar qatnashsa, integralni hisoblashda (2.8) formula ikki marta qo‘llaniladi.

2. Ikkinchi guruhga $\int p(x)\cos(kx)dx$, $\int p(x)\sin(kx)dx$, $\int p(x)a^{kx}dx$ ko‘ri-nishdagi integrallar kiradi, bunda *p(x)* – o‘zgarmas koeffisientli n – dara-jali ko‘phad, k – o‘zgarmas son.

2-eslatma. Ikkinchi guruhdagi integrallarni hisoblashda *p(x)* ko‘p-hadning darajasi qancha bo‘lsa, shuncha marta (2.8) formula qo‘llani-ladi, bunda *u(x)* sifatida *p(x)* ni olish maqsadga muvofiq bo‘ladi. (2.8) formula har bir qo‘llanganda, *p(x)* ko‘phadning darajasi bittaga kamayadi. (2.4, 2.7-misollarga qarang).

3. Uchinchi guruhga, quyidagi

$$\int e^{ax}\sin bx\,dx,\quad \int e^{ax}\cos bx\,dx,\quad \int\sin(\ln x)dx,\quad \int\cos(\ln x)dx,\ldots$$

integrallar kiradi (2.6-misolga qarang).

Endi, yuqorida keltirilgan uchta guruhning birortasiga ham kirmaydigan, lekin (2.8) bo'laklab integrallash formulasi yordamida hisoblanadigan integrallarning ba'zilarini qaraymiz.

2.8 - misol. Quyidagi

$$J = \int e^{ax}\sin bx\,dx,\ J_1 = \int e^{ax}\cos bx\,dx \quad (a,b = const)$$

integrallarni hisoblang.

Yechilishi. $u = e^{ax}$, $dv = \sin bx dx$ deb belgilab, (2.8) formulaga asosan,

$$du = ae^{ax}dx,\ v = -\frac{\cos bx}{b}$$

$$J = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx dx \qquad (2.14)$$

bo'lishini olamiz. Oxirgi (2.14) munosabatning o'ng tomonidagi integralni hisoblash uchun, unga yana bir marta (2.8) formulani qo'llaymiz: Bunda $u = e^{ax}$, $dv = \cos bx dx$ deb belgilaymiz. U holda,

$$du = ae^{ax}dx,\quad v = \frac{\sin bx}{b}$$

va

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx = \frac{1}{b}\cdot e^{ax}\sin bx - \frac{a}{b}I. \qquad (2.15)$$

(2.15) ni (2.14) ga olib borib qo'ysak, natijada,

$$I = -\frac{e^{ax}\cdot\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I,$$

I ga nisbatan birinchi tartibli chiziqli tenglamani hosil qilamiz. Bu tenglamadan,

$$I = \frac{a\sin bx - b\cos bx}{b^2 + a^2}e^{ax}$$

ekanligi kelib chiqadi.

Shunday qilib,

$$\int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{b^2 + a^2}\cdot e^{ax} + C. \qquad (2.16)$$

Xuddi shunday usul bilan,

$$I_1 = \int e^{ax}\cos bx\,dx = \frac{b\sin bx + a\cos bx}{a^2 + b^2}e^{ax} + C \qquad (2.17)$$

ekanligiga ishonch hosil qilish mumkin.

2.9-misol. Ushbu

$$\int\frac{x}{\sin^2 x}dx$$

integralni hisoblang.

Yechilishi. Bu integral yuqoridagi uch guruh integrallarning birortasiga ham kirmaydi, lekin u, (2.8) bo‘laklab integrallash formulasi yordamida, osongina hisoblanadi. Bunda $u = x,\ dv = \dfrac{dx}{\sin^2 x}$ deb belgilab,

$$\int \frac{x\,dx}{\sin^2 x} = -x\,ctgx + \int ctgx\,dx = -x\,ctgx + \int \frac{\cos x}{\sin x}dx =$$

$$= -x\,ctgx + \int \frac{du}{u} = -x\,ctgx + \ln|u|\Big|_{u=\sin x} + C = -x\,ctgx + \ln|\sin x| + C$$

bo‘lishini topamiz.

2.10-misol. Ushbu

$$I_n = \int \frac{dx}{(x^2 + a^2)^n} \qquad (a = const, \quad n = 1,2,\ldots)$$

integralni hisoblang.

Yechilishi. Bu integral ham yuqorida keltirilgan uch guruh integrallarning birortasiga ham kirmaydi. Bu ko‘rinishdagi integrallarni hisoblash uchun *rekurrent* formula keltirib chiqaramiz: buning uchun $u = \dfrac{1}{(x^2 + a^2)^n},\ dv = dx$ deb belgilasak,

$$du = -2nx(x^2 + a^2)^{-n-1}dx, \qquad v = x.$$

bo‘ladi.

U holda, (2.8) bo‘laklab integrallash formulasiga asosan,

$$I_n = \int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{(x^2 + a^2)^n} + 2n\int \frac{x^2dx}{(x^2 + a^2)^{n+1}} =$$

$$= \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1}.$$

bo‘ladi. Bundan I_{n+1} ni topamiz:

$$I_{n+1} = \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2}I_n. \tag{2.18}$$

Bu rekurrent formula, I_{n+1} integralni hisoblashni I_n integralni hisoblashga, ya’ni indeksi bitta kam bo‘lgan integralni hisoblashga keltiradi. (2.18) rekurrent formula, I_1 integral hisoblanganda, I_2 integralni hisoblashda hech qanday qiyinchilik tug‘dirmaydi. O‘z navbatida, I_2 berilganda, (2.18) formulada $n = 2$ deb, I_3 integral topiladi, va hokazo n ning qolgan boshqa istalgan qiymatiga to‘g‘ri kelgan integralni hisoblash hech qanday qiyinchilik tug‘dirmaydi. I_1 integral esa, integrallar jadvalning 14-formulasiga asosan, hisoblanadi.

2.11 -misol. Ushbu

$$I = \int \sqrt{a^2 - x^2}\,dx, \qquad a \neq 0.$$

integral hisoblansin.

Yechilishi. Berilgan integral yuqoridagi uch guruh integrallarning birortasiga ham kirmaydi, lekin u bo‘laklab integrallash usuli bilan hisoblanadi. Haqiqatan ham, $u=\sqrt{a^2-x^2}$, $dv=dx$ deb olinsa, u holda $du=-\frac{xdx}{\sqrt{a^2-x^2}}$, $v=x$, bo‘ladi va (2.8) formulaga asosan,

$$I=\int\sqrt{a^2-x^2}dx=x\sqrt{a^2-x^2}+\int\frac{x^2dx}{\sqrt{a^2-x^2}}$$

Oxirgi munosabatning o‘ng tomonidagi integralni hisoblash uchun, integral ostidagi funksiyani quyidagicha shakl almashtiramiz:

$$\frac{x^2}{\sqrt{a^2-x^2}}=\frac{x^2+a^2-a^2}{\sqrt{a^2-x^2}}=\frac{a^2}{\sqrt{a^2-x^2}}-\sqrt{a^2-x^2},$$

va hisoblaymiz:

$$\int\frac{x^2}{\sqrt{a^2-x^2}}dx=\int\frac{a^2}{\sqrt{a^2-x^2}}dx-\int\sqrt{a^2-x^2}dx=\frac{a^2}{2}arcsin\frac{x}{|a|}-I.$$

Shunday qilib, berilgan integralni hisoblash uchun,

$$I=x(a^2-x^2)+\frac{a^2}{2}arcsin\frac{x}{|a|}-I$$

munosabatni hosil qilamiz. Bundan,

$$I=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{4}arcsin\frac{x}{|a|}+c. \qquad (2.19)$$

2.12-misol. Ushbu

$$1)\ I_n=\int\frac{dx}{\sin^n x}dx,\ n>2;\quad 2)\ I_n=\int\sin^n x dx,\ n>2$$

ko‘rinishdagi integrallarni hisoblash uchun rekurrent formulalar keltirib chiqaring.

Yechilishi. 1) $\int\frac{dx}{\sin^n x}$ integralni bo‘laklab integrallash usuli yordamida hisoblaymiz:

$$I_n=\int\frac{dx}{\sin^n x}=\int\frac{dx}{\sin^{n-2}x\cdot\sin^2 x}=\left[u=\frac{1}{\sin^{n-2}x},\quad dv=\frac{dx}{\sin^2 x},\right.$$

$$\left.du=(2-n)\sin^{1-n}x\cdot\cos dx,\quad v=-ctgx\right]=-\frac{\cos x}{\sin^{n-1}x}+$$

$$+(2-n)\int\frac{\cos^2 x}{\sin^n x}dx=-\frac{\cos x}{\sin^{n-1}x}+(2-n)\int\frac{dx}{\sin^n x}-$$

$$-(2-n)\int\frac{dx}{\sin^{n-2}x}=-\frac{\cos x}{\sin^{n-1}x}+(2-n)I_n-(2-n)I_{n-2}.$$

Bundan,

$$I_n=-\frac{\cos x}{(n-1)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}. \qquad (2.20)$$

(2.20) rekurrent formulani qo‘llashda, avvalo, $n=3$ deb, $I_1=\int\frac{dx}{\sin x}$ integrallar hisoblanadi. $\int\frac{dx}{\sin x}$ integralni hisoblash yuqorida ko‘rsatildi (2.7 formulaga qarang).

2) $I_n=\int\sin^n x\,dx$ integralni hisoblashning rekurrent formulasini chiqarish uchun, uni quyidagicha bo‘laklab integrallaymiz:

$$I_n=\int\sin^n x\,dx=-\int\sin^{n-1}x\,d\cos x=\left[u=\sin^{n-1}x,\ dv=d\cos x,\right.$$
$$\left.du=(n-1)\sin^{n-2}x\cdot\cos x\,dx,\ \ v=\cos x\right]=-\cos x\cdot\sin^{n-1}x+$$
$$+(n-1)\int\sin^{n-2}x\cos^2 x\,dx=-\cos x\cdot\sin^{n-1}x+(n-1)\int\sin^{n-2}x(1-\sin^2 x)dx=-$$
$$=-\cos x\cdot\sin^{n-1}x+(n-1)I_{n-2}-(n-1)I_n.$$

Bundan

$$I_n=-\frac{\cos x\sin^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}$$

ko‘rinishdagi rekurrent formulani topamiz.

Mustaqil yechish uchun misollar

Quyidagi integrallarni hisoblang:

2.1. $\int x\sqrt{x^2-7}dx.$ **2.2.** $\int\frac{x^2}{\sqrt[5]{3-x^3}}dx.$ **2.3.** $\int\frac{e^{3x}dx}{(e^{3x}-4)^3}.$

2.4. $\int\frac{xdx}{x^2+8}.$ **2.5.** $\int\frac{2x+1}{x^2+x-5}dx.$ **2.6.** $\int\frac{2x-3}{8+3x-x^2}dx.$

2.7. $\int\frac{6x-5}{\sqrt{3x^2-5x+4}}dx.$ **2.8.** $\int\frac{x-1}{\sqrt{2x-x^2}}dx.$ **2.9.** $\int\frac{10x-3x^2}{x^3-5x^2}dx.$

Quyidagi integrallarni hisoblang:

2.10. $\int\frac{e^x dx}{2e^x+7}.$ **2.11.** $\int\frac{e^{\frac{5}{x}}dx}{x^2}dx.$ **2.12.** $\int\frac{1}{x\ln x}dx.$

2.13. $\int\frac{\sqrt[7]{\ln^5 x}}{x}dx.$ **2.14.** $\int x^2e^{1-x^3}dx.$ **2.15.** $\int\frac{5^x dx}{9+5^x}.$

2.16. $\int\frac{e^{5x}}{4-e^{10x}}dx.$ **2.17.** $\int\frac{9^x dx}{\sqrt{81^x+1}}.$ **2.18.** $\int\frac{1}{x\lg x}dx.$

Quyidagi integrallarni hisoblang:

2.19. $\int tgx\,dx.$ **2.20.** $\int ctgx\,dx.$ **2.21.** $\int\sin x\cos^5 x\,dx.$

2.22. $\int\frac{\cos x}{\sqrt[3]{\sin x}}dx.$ **2.23.** $\int\frac{\sin x}{\cos^2 x\sqrt{\cos x}}dx.$ **2.24.** $\int\frac{\sqrt[7]{128tgx}}{\cos^2 x}dx.$

2.25. $\int\frac{dx}{\sqrt[7]{ctg^4x\sin^2 x}}.$ **2.26.** $\int\frac{e^{-1/x^4}}{x^5}dx.$ **2.27.** $\int\frac{\cos x}{\sin^2 x\sqrt{\sin x}}dx.$

Quyidagi integrallarni hisoblang:

2.28. $\int \frac{1-4\arcsin x}{\sqrt{1-x^2}}dx.$ 2.29. $\int \frac{\sqrt[4]{arctgx}}{1+x^2}dx.$ 2.30. $\int \frac{dx}{(x^2+1)arctgx}.$

2.31. $\int e^{3\cos x-1}\sin x dx.$ 2.32. $\int \frac{\sin x dx}{\cos^2 x-5}.$ 2.33. $\int \frac{\cos\frac{2}{x^3}}{x^4}dx.$

2.34. $\int \frac{\cos x dx}{2+\sin^2 x}.$ 2.35. $\int \frac{dx}{x\sqrt{\ln^2 x-5}}.$ 2.36. $\int \frac{x^4 dx}{\sin^2 x^5}.$

Quyidagi integrallarni hisoblang:

2.37. $\int ch^3 x\, shx dx.$ 2.38. $\int \frac{shx}{ch^2 x}dx.$ 2.39. $\int sh^7 x\, chx\, dx.$

2.40. $\int \frac{chx}{sh^2 x}dx.$ 2.41. $\int \frac{cthx}{sh^2 x}dx.$ 2.42. $\int \frac{thx}{ch^2 x}dx.$

Integrallarni hisoblang:

2.43. $\int x\sin x\, dx$ 2.44. $\int (x+3)\cos x dx.$

2.45. $\int (1-4x)\sin x dx.$ 2.46. $\int (2x+4)e^x\, dx.$

2.47. $\int (x+5)6^x\, dx.$ 2.48. $\int x^2\cos x\, dx.$

2.49. $\int (x^2+3)\sin x\, dx.$ 2.50. $\int (2x-x^2)e^{-x}\, dx.$

2.51. $\int \frac{x}{\cos^2 x}dx.$ 2.52. $\int \frac{x}{\sin^2 x}dx.$

Integrallarni hisoblang:

2.53. $\int \ln x dx.$ 2.54. $\int \ln(1+x^2)dx.$

2.55. $\int x^2\ln(x+4)dx.$ 2.56. $\int x\, arctgx\, dx.$

2.57. $\int \frac{\arcsin x}{x^2}dx.$ 2.58. $\int \ln\left(x+\sqrt{1+x^2}\right)dx.$

2.59. $\int \frac{\arccos x}{x^3}dx.$ 2.60. $\int x^3 arctgx dx.$

2.61. $\int x\frac{\arccos x}{\sqrt{1-x^2}}dx.$ 2.62. $\int \frac{\arcsin\frac{x}{2}}{\sqrt{2-x}}dx.$

Quyidagi integlarni hisoblang:

2.63. $\int e^x\cos x dx.$ 2.64. $\int e^x\sin x dx.$

2.65. $\int 2^x\cos x dx.$ 2.66. $\int e^{ax}\sin bx\, dx$, $a^2+b^2\neq 0.$

2.67. $\int e^{ax}\cos bx\, dx$, $a^2+b^2\neq 0.$ 2.68. $\int x^2 e^x\cos x\, dx.$

2.69. $\int xe^x\sin^2 x\, dx.$ 2.70. $\int \sin(\ln x)dx.$

2.71. $\int \cos(\ln x)dx.$ 2.72. $\int x^2\sin(\ln x)dx.$ 2.73. $\int \left(\frac{\cos x}{e^x}\right)^2 dx.$

Quyidagi integrallarni hisoblang:

2.74. $\int x^5\sin x\, dx.$ 2.75. $\int x^6 e^{-x}dx.$

2.76. $\int x^4\cos 3x\,dx$. **2.77.** $\int x^3 e^{-x^2}dx$.

2.78. $\int e^{\sqrt{x}}dx$. **2.79.** $\int \cos^5 x\,dx$. **2.80.** $\int \sin^6 x\,dx$

Quyidagi integrallarni hisoblang:

2.81. $\int \sqrt{x}\sin\sqrt{x}\,dx$. **2.82.** $\int \frac{x\ln x}{\sqrt{1+x}}dx$. **2.83.** $\int \cos^2 \ln x\,dx$.

2.84. $\int \frac{\ln tgx}{\cos^2 x}dx$. **2.85.** $\int x\,arctg\,x^2\,dx$.

2.86. $\int \frac{arctg\sqrt{x+1}}{\sqrt{x+1}}dx$. **2.87.** $\int \arcsin\left(\frac{2\sqrt{x}}{1+x}\right)dx$.

Quyidagi tengliklarni isbotlang $\left(P_n(x)=\sum_{k=0}^{n} a_{n-k}x^k\ ,\ a_0\neq 0\right)$:

2.88. $\int P_n(x)e^{ax}dx=\left(P_n(x)-\frac{P_n'(x)}{a}+\ldots(-1)^n\frac{P_n^{(n)}(x)}{a^n}\right)\frac{e^{ax}}{a}+C$.

2.89. $\int P_n(x)\sin ax\,dx=-\left(P_n(x)-\frac{P_n^{(2)}(x)}{a^2}+\frac{P_n^{(4)}(x)}{a^4}-\ldots\right)\frac{\cos ax}{a}+$

$+\left(\frac{P_n^{(1)}(x)}{a}-\frac{P_n^{(3)}(x)}{a^3}+\frac{P_n^{(5)}(x)}{a^5}-\ldots\right)\frac{\sin ax}{a}+C$.

2.90. $\int P_n(x)\cos ax\,dx=\left(P_n(x)-\frac{P_n^{(2)}(x)}{a^2}+\frac{P_n^{(4)}(x)}{a^4}-\ldots\right)\frac{\sin ax}{a}+$

$+\left(\frac{P_n^{(1)}(x)}{a}-\frac{P_n^{(3)}(x)}{a^3}+\frac{P_n^{(5)}(x)}{a^5}-\ldots\right)\frac{\cos ax}{a}+C$.

Quyidagi integrallarni hisoblang:

2.91. $\int x\ln\left|1+\frac{1}{x}\right|dx$. **2.92.** $\int \sin x\ln tgx\,dx$. **2.93.** $\int\left(\frac{\ln x}{x}\right)^3 dx$.

2.94. $\int \frac{\ln^2 x}{x^2\sqrt{x}}dx$. **2.95.** $\int \cos x\ln\left(1+\sin^2 x\right)dx$.

2.96. $\int x\sin(6x-11)dx$.

2.97. $\int x\cos^2 x\,dx$. **2.98.** $\int x\,tg^2 4x\,dx$. **2.99.** $\int x\ln^3 x\,dx$.

2.100. $\int x\,shx\,dx$. **2.101.** $\int x\,chx\,dx$. **2.102.** $\int x^3 e^{2x}dx$.

2.203. $\int x^2\cos 3x\,dx$. **2.104.** $\int\left(x^2-5x+7\right)e^{4x}dx$. **2.105.** $\int x^3 arctg 2x\,dx$.

2.106. $\int x\sqrt{1-x^2}\,arc\sin x\,dx$. **2.107.** $\int\left(x^2+2\right)^2\cos 3x\,dx$

2.108. $\int xe^x\sin^2 x\,dx$. **2.109.** $\int xe^{3x}\sin\left(2x-\frac{\pi}{4}\right)dx$.

Quyidagi integrallarni hisoblang:

2.110. $\int x^4\cos px\,dx$. **2.111.** $\int x^4\sin px\,dx$.

2.112. $\int x^4 e^{ax}dx$. **2.113.** $\int x^n e^{ax}dx,\ a\neq 0$.

2.114. $\int x^n\cos px\,dx,\ p\neq 0$. **2.115.** $\int x^n\sin px\,dx,\ \ p\neq 0$.

J_n $(n \in N)$ integral uchun rekurrent formula keltirib chiqarilsin:

2.116. $J_n = \int \cos^n x\,dx,\ n > 2.$ **2.117.** $J_n = \int \ln^n x\,dx.$

2.118. $J_n = \int sh^n x\,dx,\ n > 2.$ **2.119.** $J_n = \int ch^n x\,dx,\ n > 2.$

2.120. $J_n = \int \frac{1}{\cos^n x}dx,\ n > 2.$ **2.121.** $J_n = \int \frac{1}{ch^n x}dx,\ n > 2.$

2.122. $J_n = \int x^\alpha \ln^n x\,dx,\ \alpha \neq -1.$ **2.123.** $J_n = \int \frac{x^n}{\sqrt{x^2+a}}dx,\ n > 2.$

2.124. Ushbu $f'(x^2) = \frac{1}{x},\ x > 0$ shartni qanoatlantiruvchi $f(x),\ x \in (0;\infty)$ funksiyani toping.

2.125. Ushbu

$$f'(\ln x) = \begin{cases} 1, & x \in (0;1], \\ x, & x \in (1;+\infty), \end{cases}$$

shartni qanoatlantiruvchi $f(x),\ x \in R$ funksiyani toping.

2.126. Quyidagi

$$x f'(x^2) + g'(x) = \cos x - 3x^2, \qquad f(x^2) + g(x) = \sin x - x^4$$

shartlarni qanoatlantiruvchi $f(x),\ x \in (0;+\infty)$ va $g(x),\ x \in R$ funksiyalarni toping.

2.127. $x > 0$ uchun quyidagi

a) $f(x) + g(x) = x + 1,$
$f'(x) - g'(x) = 0,$
$f'(2x) + g'(-2x) = 1 - 2x^2.$

b) $f(x) + g(x) = \frac{x^4}{6},$
$f'(x) + g'(x) = \sin x,$
$f'(2x) + g'(-2x) = 0.$

shartlarni qanoatlantiruvchi $f(x),\ x \in (0;+\infty)$ va $g(x),\ x \in R$ funksiyalarni toping.

Mustaqil yechish uchun misollarning javoblari

2.1. $\frac{1}{3}\sqrt{(x^2-7)^3} + C.$ **2.2.** $-\frac{5}{12}\sqrt[5]{(3-3^3)^4} + C.$

2.3. $-\frac{1}{6}\frac{1}{(e^{3x}-4)^2} + C.$ **2.4.** $\frac{1}{2}\ln|x^2+8| + C.$ **2.5.** $\ln|x^2+x-5| + C.$

2.6. $-\ln|8+3x-x^2| + C.$ **2.7.** $2\sqrt{3x^2-5x+4} + C.$

2.8. $-\sqrt{2x-x^2} + C.$ **2.9.** $-\ln|x^3-5x^2| + C.$ **2.10.** $\frac{1}{2}\ln|2e^x+7| + C.$

2.11. $-\frac{1}{5}e^{5/x} + C.$ **2.12.** $\ln|\ln x| + C.$ **2.13.** $\frac{7}{12}(\ln x)^{12/7} + C.$ **2.14.** $-\frac{1}{3}e^{1-x^3} + C.$

2.15. $\frac{\ln|9+5^x|}{\ln 5} + C.$ **2.16.** $\frac{1}{20}\ln\left|\frac{e^{5x}-2}{e^{5x}+2}\right| + C.$ **2.17.** $\frac{1}{\ln 9}\ln\left|9^x + \sqrt{81^x+1}\right| + C.$

2.18. $\frac{\ln(\ln x)}{\lg e}+C$. **2.19.** $-\ln|\cos x|+C$. **2.20.** $\ln|\sin x|+C$. **2.21.** $-\frac{\cos^6 x}{6}+C$.

2.22. $\frac{5}{6}\sqrt[5]{\sin^6 x}+C$. **2.23.** $\frac{2}{3\cos x\sqrt{\cos x}}+C$. **2.24.** $\frac{7}{4}tgx\sqrt[7]{tgx}+C$.

2.25. $-\frac{7}{3}\sqrt[7]{ctg^3 x}+C$. **2.26.** $0{,}25e^{-1/x^4}+C$. **2.27.** $-\frac{2}{\sqrt{\sin x}}+C$.

2.28. $\arcsin x-2\arcsin^2 x+C$. **2.29.** $\frac{4}{5}\sqrt[4]{arctg^5 x}+C$. **2.30.** $\ln|arctgx|+C$.

2.31. $-\frac{1}{5}e^{5\cos x-1}+C$. **2.32.** $\frac{1}{2\sqrt5}\ln\left|\frac{\cos x+\sqrt5}{\cos x-\sqrt5}\right|+C$.

2.33. $-\frac{1}{6}\sin\frac{2}{x^3}+C$ **2.34.** $\frac{1}{\sqrt2}arctg\frac{\sin x}{\sqrt2}+C$.

2.35. $\ln\left|\ln x+\sqrt{\ln^2 x-9}\right|+C$. **2.36.** $-\frac{1}{5}ctgx^5+C$. **2.37.** $\frac{1}{4}ch^4x+C$.

2.38. $-\frac{1}{chx}+C$. **2.39.** $\frac{sh^8x}{8}+C$. **2.40.** $-\frac{1}{shx}+C$. **2.41.** $-0{,}5cgh^2x+C$.

2.42. $0{,}5th^2x+C$. **2.43.** $-x\cos x+\sin x+C$. **2.44.** $\sin x-(x+3)\cos x+C$.

2.45. $(4x-1)\cos x-4\sin x+C$. **2.46.** $2e^x(1+x)+C$. **2.47.** $\frac{x\ln 6+5\ln 6-1}{\ln^2 6}6^x+C$.

2.48. $x^2\sin x-2\sin x+2x\cos x+C$. **2.49.** $2x\sin x-(x^2+1)\cos x+C$.

2.50. $e^{-x}x^2+C$. **2.51.** $xtgx+\ln(\cos x)+C$. **2.52.** $\ln(\sin x)-xctgx+C$.

2.53. $x\ln x-x+C$. **2.54.** $x\ln(1+x^2)-2x+2arctgx+C$. **2.55.** $\frac{1}{3}(4+x)^3\ln(x+4)-$
$\frac{352}{9}-\frac{16}{3}x+\frac{2}{3}x^2-\frac{1}{9}x^3-4\ln(x+4)(x+4)^2+16(4+x)+16(4+x)\ln(x+4)+C$

2.56. $\frac{x^2}{2}arctgx-\frac{1}{2}x+\frac{1}{2}arctgx+C$. **2.57.** $-\frac{\arcsin x}{x}-arctg\frac{1}{\sqrt{1-x^2}}+C$.

2.58. $x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$. **2.59.** $-\frac{\arccos x}{2x^2}+\frac{\sqrt{1-x^2}}{2x}+C$.

2.60. $\frac{1}{4}x^4arctgx-\frac{1}{12}x^3+\frac{1}{4}x-\frac{1}{4}arctgx+C$ **2.61.** $-x-\sqrt{1-x^2}\arccos x+C$.

2.62. $-2\sqrt{2-x}\arccos\frac{x}{2}-4\sqrt{2+x}+C$. **2.63.** $\frac{1}{2}e^x\cos x+\frac{1}{2}e^x\sin x+C$.

2.64. $\frac{1}{2}e^x\sin x-\frac{1}{2}e^x\cos x+C$. **2.65.** $\left(2^{1+x}tg\frac{x}{2}-2^x tg^2\frac{x}{2}\ln 2+2^x\ln 2\right)\frac{\cos^2\frac{x}{2}}{1+\ln^2 2}+C$

2.66. $\frac{a\sin bx-b\cos bx}{a^2+b^2}e^{ax}+C$. **2.67.** $\frac{a\cos bx+b\sin bx}{a^2+b^2}e^{ax}+C$.

2.68. $\frac{1}{2}(x^2-1)e^x\cos x-\left(-\frac{1}{2}x^2+x-\frac{1}{2}\right)e^x\sin x+C$.

2.69. $\frac{1}{2}xe^x-\frac{1}{2}e^x-\frac{1}{2}\left(\frac{1}{5}x+\frac{3}{25}\right)e^x\cos 2x+\frac{1}{2}\left(-\frac{2}{5}x+\frac{4}{25}\right)e^x\sin 2x+C$.

2.70. $-\frac{x}{2}\cos(\ln x)+\frac{1}{2}x\sin(\ln x)+C$. **2.71.** $\frac{x}{2}[\cos(\ln x)+\sin(\ln x)]+C$.

2.72. $\dfrac{-\frac{1}{10}x^3+\frac{3}{5}x^3 tg\left(\frac{\ln x}{2}\right)+\frac{x^3}{10}tg^2\left(\frac{\ln x}{2}\right)}{1+tg^2\left(\frac{\ln x}{2}\right)}+C$.

2.73. $\frac{1}{8}[(-2\cos x+2\sin x)\cos x-1]e^{-2x}+C$.

2.74. $-x^5\cos x+5x^4\sin x+20x^3-60x^2\sin x+120\sin x-120x\cos x+C$.

2.75. $-(x^6+6x^5+30x^4+120x^3+360x^2+720x+720)e^{-x}+C$.

2.76. $\frac{1}{3}x^4\sin 3x+\frac{4}{9}x^3\cos 3x-\frac{4}{9}x^2\sin 3x+\frac{8}{81}\sin 3x-\frac{8}{27}x\cos 3x+C$.

2.77. $-\frac{1}{2}(1+x^2)e^{-x^2}+C$. **2.78.** $2(\sqrt{x}-1)e^{\sqrt{x}}+C$.

2.79. $\frac{1}{5}\cos^4 x\sin x+\frac{4}{15}\cos^2 x\sin x+\frac{8}{15}\sin x+C$.

2.80. $-\frac{1}{6}\sin^5 x\cos x-\frac{5}{24}\sin^3 x\cos x-\frac{5}{16}\sin x\cos x+\frac{5}{16}x+C$.

2.81. $-2x\cos\sqrt{x}+4\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+C$. **2.82.** $\left(-\frac{4}{9}\sqrt{1+x}+\frac{2}{3}\ln x\sqrt{1+x}\right)x-$

$-\frac{20}{9}+\frac{20}{9}\sqrt{1+x}+\frac{1}{2}\ln x\left(\frac{8}{3}-\frac{8}{3}\sqrt{1+x}\right)-\frac{8}{3}\ln\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+x}\right)+C$.

2.83. $\frac{3}{5}x tg^4\left(\frac{\ln x}{2}\right)-\frac{4}{5}x tg^3\left(\frac{\ln x}{2}\right)+\frac{2}{5}x tg^2\left(\frac{\ln x}{2}\right)+\frac{4}{5}x tg\left(\frac{\ln x}{2}\right)+\frac{3}{5}x+C$.

2.84. $tgx\ln tgx-tgx+C$. **2.85.** $\frac{1}{2}x^2 arctgx^2-\frac{1}{4}\ln(1+x^4)+C$.

2.86. $2\sqrt{1+x}\,arctg\sqrt{1+x}-\ln(2+x)+C$.

2.87. $-2sign(1-x)\sqrt{x}+(1+x)\arcsin\frac{2\sqrt{x}}{1+x}+C$.

2.91. $\frac{x^2}{2}\ln\left|1+\frac{1}{x}\right|-\frac{1}{2}\ln|x+1|+\frac{x}{2}+C$. **2.92.** $\ln\left|tg\frac{x}{2}\right|-\cos x\cdot\ln tgx+C$.

2.93. $-\frac{1}{2x^2}\left(\ln^3 x+\frac{3}{2}\ln^2 x+\frac{3}{2}\ln x+\frac{3}{4}\right)+C$.

2.94. $-\frac{8}{27}x^{-3/2}\left(\frac{9}{4}\ln^2 x+3\ln x+2\right)+C$.

2.95. $\sin x\ln(1+\sin^2 x)-2\sin x+2arctg\sin x+C$.

2.96. $\frac{1}{36}\sin(6x-11)-\frac{1}{36}[6x+22)\cos(6x-11)+C$.

2.97. $x\left(-\frac{1}{4}\sin 2x+\frac{1}{2}x\right)+\frac{1}{4}\sin^2 x-\frac{1}{4}x^2+C$.

2.98. $\frac{1}{4}x tg 4x-\frac{1}{2}x^2-\frac{1}{32}\ln(1+tg^2 4x)+C$.

2.99. $\frac{1}{2}x^2\ln^3 x-\frac{3}{4}x^2\ln^2 x+\frac{3}{4}x^2\ln x-\frac{3}{2}x^2+C$. **2.100.** $x\,chx-shx+C$.

2.101. $x\,shx-chx+C$. **2.102.** $\frac{1}{8}\left(4x^3-6x^2+6x-3\right)e^{2x}+C$.

2.103. $\frac{1}{27}\left(9x^2-2\right)\sin 3x+\frac{2}{9}x\cos 3x+C$. **2.104.** $\frac{1}{32}\left(8x^2-44x-67\right)e^{4x}+C$.

2.105. $\frac{1}{64}\left(16x^4-1\right)arctg\,2x-\frac{1}{24}x^3+\frac{1}{24}x+C$. **2.106.** $-\frac{1}{3}\sqrt{(1-x^2)^3}\arcsin x+C$.

2.107. $\frac{1}{81}\left(27x^4+72x^2+92\right)\sin 3x+\left(12x^3+16x\right)\cos 3x+C$.

2.108. $\frac{1}{2}(x-1)e^x-\frac{1}{2}\left(\frac{1}{5}x+\frac{3}{25}\right)e^x\cos 2x+\frac{1}{2}\left(-\frac{2}{5}x+\frac{4}{25}\right)\cdot e^x\sin 2x+C$.

2.109. $-\frac{3}{13}e^{3x}\left(\cos 2x+\frac{\pi}{4}\right)-\frac{2}{13}e^{3x}\sin\left(2x+\frac{\pi}{4}\right)+C$.

2.110. $\frac{1}{p^4}\left(4p^2x^3-24x\right)\cos px+\frac{1}{p^5}\left(p^4x^4-12p^2x^2-24\right)\sin px+C$.

2.111. $\frac{1}{p^4}\left(4p^2x^2-24x\right)\sin px-\frac{1}{p^5}\left(p^4x^4-12p^2x^2+24\right)\cos px+C$.

2.112. $\frac{a^4x^4-4a^3x^3+12a^2x^2-24ax+24}{a^5}e^{ax}+C$.

2.113. $e^{ax}\left[\frac{x^n}{a}-\frac{nx^{n-1}}{a^2}+\frac{n(n-1)x^{n-2}}{a^3}-\ldots+(-1)^{n-1}\frac{n!x}{a^n}+(-1)^n\frac{n!}{a^{n+1}}\right]$.

2.114. $\frac{x^n}{p}\sin px+\frac{nx^{n-1}}{p^2}\cos px-\frac{n(n-1)}{p^2}\int x^{n-2}\cos px\,dx$.

2.115. $-\frac{x^n}{p}\cos px+\frac{nx^{n-1}}{p^2}\sin px-\frac{n(n-1)}{p^2}\int x^{n-2}\sin px\,dx$.

2.116. $J_n=\frac{\sin x\cos^{n-1}x}{n}+\frac{n-1}{n}J_{n-2}$. **2.117.** $J_n=x\ln^n x-nJ_{n-1}$.

2.118. $J_n=\frac{chx\,sh^{n-1}x}{n}-\frac{n-1}{n}J_{n-2}$. **2.119.** $J_n=\frac{shx\,ch^{n-1}x}{n}+\frac{n-1}{n}J_{n-2}$.

2.120. $J_n=-\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}J_{n-2}$. **2.121.** $J_n=\frac{shx}{(n-1)ch^{n-1}x}+\frac{n-2}{n-1}J_{n-2}$.

2.122. $J_n=\frac{x^{\alpha+1}\ln^n x}{\alpha+1}-\frac{n}{\alpha+1}J_{n-1}$. **2.123.** $J_n=\frac{x^{n-1}\sqrt{x^2+a}}{n}-\frac{n-1}{n}aJ_{n-2}$.

2.124. $f(x)=2\sqrt{x}+C$. **2.125.** $f(x)=\begin{cases}x+1+C, & x\le 0,\\ e^x+C, & x>0.\end{cases}$

2.126. $f(x)=C-\frac{x^2}{2}$, $g(x)=\sin x-\frac{x^4}{4}+C$.

2.127. a) $f(x)=\frac{x}{2}+x-C$, $g(x)=\begin{cases}\frac{x}{2}+C, & x>0\\ \frac{x}{2}-x^3+C, & x\le 0\end{cases}$

b) $f(x)=\frac{x^4}{12}-\frac{\cos x}{2}+x+C,\quad g(x)=\begin{cases}\frac{x^4}{12}+\frac{\cos x}{2}-C, & x>0,\\ \frac{x^4}{12}-\frac{\cos x}{2}+1-C, & x\le 0.\end{cases}$

3-§. Rasional funksiyalarni integrallash

3.1. Noma'lum koeffisientlar usuli. Ikkita algebraic ko'phadning nisbatiga, ya'ni

$$f(x)=\frac{P_m(x)}{Q_n(x)} \tag{3.1}$$

ifodali *rasional funksiya* yoki *rasional kasr* deyiladi. Bunda, $P_m(x)=b_0+b_1x+...+b_mx^m$ va $Q_n(x)=a_0+a_1+...+a_nx^n$ ($b_m, a_n\neq 0,\ m\geq 0,\ n\geq 1$) haqiqiy koeffisientli ko'phadlar, deb faraz qilinadi.

Agar $m<n$ bo'lsa, u holda $\frac{P_m(x)}{Q_n(x)}$ *to'g'ri kasr rasional funksiya*, $m\geq n$ bo'lganda esa, *noto'g'ri kasr rasional funksiya* deyiladi. Agar (3.1) rasional kasr, noto'g'ri kasr bo'lsa, u, kasrning suratini maxrajiga bo'lish yo'li bilan,

$$f(x)=w(x)+\frac{P_k(x)}{Q_n(x)}\qquad (k<n) \tag{3.2}$$

ko'rinishga keltiriladi, bunda $w(x)$ – biror ko'phad.

Oliy algebra kursidan ma'lumki, har qanday $Q_n(x)$ ko'phadni, ushbu

$$Q_n(x)=a_n(x-\alpha)(x-\beta)...(x-\nu) \tag{3.3}$$

(bunda a_n – $Q_n(x)$ ko'phadda $x$ning yuqori darajasi oldidagi koeffisient, $\alpha,\beta,...,\nu$ lar – $Q_n(x)=0$ tenglamaning ildizlari) ko'rinishda tasvirlash mumkin.

Agar ko'phadning ildizlari ichida o'zaro tenglari bo'lsa, u holda, ko'phad,

$$Q_n(x)=a_n(x-\alpha)^r(x-\beta)^s...(x-\nu)^t \tag{3.4}$$

ko'rinishga keltiriladi, bunda $r,s,...,t$ – butun sonlar, $\alpha,\beta,...,\nu$ sonlar esa, mos ravishda, $Q_n(x)$ ko'phadning $r,s,...,t$ karrali ildizlari deyiladi va $r+s+...+t=n$ bo'ladi.

Ko'phadning (3.3) dagi ildizlari ichida kompleks ildizlar ham bo'lishi mumkin. Algebra kursidan ma'lumki, agar $\alpha=a+ib$ haqiqiy koeffisientli ko'phadning r – karrali ildizi bo'lsa, u holda unga qo'shma $\bar{\alpha}=a-ib$ son ham ko'phadning r – karrali ildizi bo'ladi. Boshqacha aytganda, agar (3.4) ning tarkibida $(x-\alpha)^r (\alpha = a+ib)$ bo'lsa, u holda, (4) ning tarkibidagi $(x-\alpha)^r$ va $(x-\overline{\alpha})^r$ larning ko'paytmasi, quyidagicha bo'ladi:

$$(x-\alpha)^r (x-\overline{\alpha})^r = \{[x-(a+bi)]\cdot[x-(a-bi)]\}^r =$$
$$= \left[x^2 - x(a+bi) - x(a-bi) + a^2 + b^2\right]^r = \left[x^2 - 2ax + a^2 + b^2\right]^r =$$
$$= \left(x^2 + 2px + q\right)^r,$$

bunda $p=-a,\qquad q=a^2+b^2,\ p^2-q<0$, p va q – haqiqiy sonlar.

Xuddi shunday yuqoridagi mulohazalarni boshqa kompleks ildizlar uchun ham yuritsak, u holda, (3.4) quyidagi ko'rinishni oladi:

$$Q_n(x) = A_n(x-\alpha)^r\cdot(x-\beta)^s \ldots \left(x^2+2px+q\right)^t\left(x^2+2ux+v\right)^k, \ldots, \qquad (3.5)$$

bunda, $\alpha, \beta, \ldots, p, q, u, v$ – haqiqiy sonlar, $r, s, \ldots, t, k$ – natural sonlar.

Algebra kursida quyidagi teorema isbot qilinadi.

3.1-teorema. Agar $\frac{P_m(x)}{Q_n(x)}$ to'g'ri rasional kasr tarkibidagi $Q_n(x)$ ko'phad (3.5) shaklda tasvirlangan bo'lsa, u holda, rasional kasr, yagona ravishda,

$$\frac{P_m(x)}{Q_n(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_r}{(x-\alpha)^r} + \ldots +$$
$$+ \ldots + \frac{M_1x+N_1}{x^2+2px+q} + \frac{M_2x+N_2}{\left(x^2+2px+q\right)^2} + \ldots + \frac{M_tx+N_t}{\left(x^2+2px+q\right)^t} + \ldots \qquad (3.6)$$

(bunda, $A_1 A_2, \ldots, A_r, \ldots, M_1N_1, M_2N_2, \ldots, M_tN_t, \ldots$ – noma'lum haqiqiy sonlar) ko'rinishda tasvirlanadi. (3.6) tenglik, x ning $Q_n(x)$ ko'phadning haqiqiy ildizlariga teng bo'lmagan hamma qiymatlarida o'rinli.

(3.6) dagi noma'lum koeffisientlarni topish uchun (3.6) ni umumiy maxrajga keltirib (umumiy maxraj $Q_n(x)$) ikki ko'phadning tengligi haqidagi teoremaga asosan, o'ng tomonidagi suratdagi hosil bo'lgan ko'phad bilan $P_m(x)$ ko'phaddagi x ning bir xil darajalari oldidagi koeffisientlarni tenglashtirish natijasida, noma'lum koeffisientlarga nisbatan chiziqli algebraik tenglamalar sistemasi hosil bo'ladi. Bu sistemadan noma'lum koeffisientlarni topib, topilgan qiymatlarni (3.6) tenglikka keltirib qo'yamiz. Kasrning yoyilmasidagi noma'lum koeffisientlarni topishning bu usuli, *noma'lum koeffisientlar usuli* deyiladi.

Shunday qilib, $f(x)=\frac{P_m(x)}{Q_n(x)}$ rasional kasrning integralini hisoblash, $w(x)=c_0x^k+c_1x^{k-1}+\ldots+a_k$ shakldagi ko'phadni integrallashga va quyidagi

$$I.\ \frac{A}{x-\alpha},\quad II.\ \frac{A}{(x-\alpha)^r},\quad III.\ \frac{Mx+N}{x^2+px+q},\quad IV.\ \frac{Mx+N}{(x^2+px+q)^r} \tag{3.7}$$

$(r>1)$ (bunda A, M, N, α, p, q – haqiqiy sonlar, $q-\frac{p^2}{4}>0$) ko'rinishdagi sodda kasrlarni integrallashga keltiriladi. Bu sodda kasrlarning integrallari quyidagicha hisoblanadi.

Quyidagi, $\frac{A}{(x-a)^m}$, $\frac{Bx+C}{(x^2+px+q)^k}$ $(A, B, C, a, p, q \in R\ ,\ k, m \in N)$ ko'rinishdagi *sodda kasrlar*ni qaraymiz. U holda:

1) $m=1$ bo'lganda, $\int \frac{A}{x-a}dx = A\int \frac{dx}{x-a} = A\ln|x-a|+C$.

2) $m>1$ bo'lganda, $\int \frac{A}{(x-a)^m}dx = A\int (x-a)^{-m}dx = \frac{A}{1-m}\frac{1}{(x-a)^{m-1}}+C$.

3) $k=1$ bo'lganda, $a^2=q-\frac{p^2}{4}$, $x+\frac{p}{2}=t$ almashtirish olib, kvadrat uchhadni, $x^2+px+q=a^2+t^2$ ko'rinishga keltiramiz, va berilgan kasrning aniqmas integralini topamiz:

$$\int \frac{Bx+D}{x^2+px+q}dx = B\int \frac{tdt}{a^2+t^2} + \frac{(2D-Bp)}{2}\int \frac{dt}{a^2+t^2} =$$

$$= \frac{B}{2}\ln(x^2+px+q) + \frac{2D-Bp}{\sqrt{4q-p^2}}\, arctg\frac{2x+p}{\sqrt{4q-p^2}} + C$$

4) $k>1$ bo'lganda, $a^2=q-\frac{p^2}{4}$, $x+\frac{p}{2}=t$ almashtirish olib, kvadrat uchhadni $x^2+px+q=a^2+t^2$ ko'rinishga keltiramiz va berilgan kasrning aniqmas integralini topamiz:

$$\int \frac{Bx+D}{(x^2+px+q)^k}dx = \frac{B}{2}\frac{1}{1-k}\frac{1}{(a^2+t^2)^{k-1}} - \frac{(2D-Bp)}{2}\int \frac{dt}{(a^2+t^2)^k}$$

Oxirgi ifodadagi integral esa, quyidagi

$$\int \frac{dt}{(a^2+t^2)^k} = \frac{1}{2(k-1)a^2}\left(\frac{t}{(a^2+t^2)^{k-1}} - (2k-3)\int \frac{dt}{(a^2+t^2)^{k-1}}\right)$$

rekurrent formula orqali topiladi.

I va II tipdagi kasrlarning integrallari, $t=x-\alpha$ almashtirish yordamida hisoblanadi:

$$\int \frac{A}{x-\alpha}dx = A\int \frac{dt}{t} = A\ln|t|\Big|_{t=x-\alpha} + c = A\ln|x-\alpha|+c.$$

$$\int \frac{A}{(x-\alpha)^r}dx = A\int \frac{dt}{t^r} = \left(\frac{A}{r-1}\cdot\frac{1}{t^{r-1}}+C\right)\Big|_{t=x-\alpha} = -\frac{A}{(r-1)}\cdot\frac{1}{(x-\alpha)^{r-1}}+C.$$

III tipdagi kasrning integralini hisoblash uchun, kvadrat uchhadni quyidagicha shakl almashtiramiz:

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right).$$

$a=\sqrt{q-\frac{p^2}{4}}$, $t=x+\frac{p}{2}$ deb belgilab olib, integralni hisoblaymiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\int\frac{Mt+\left(N-\frac{Mp}{2}\right)}{t^2+a^2}dt=$$

$$=\frac{M}{2}\int\frac{2tdt}{t^2+a^2}+\left(N-\frac{Mp}{2}\right)\int\frac{dt}{t^2+a^2}=$$

$$=\frac{M}{2}\int\frac{d(t^2+a^2)}{t^2+a^2}+\left(N-\frac{Mp}{2}\right)\frac{1}{a}\int\frac{d\left(\frac{t}{a}\right)}{\left(\frac{t}{a}\right)^2+1}=$$

$$\left(\frac{M}{2}\right)\ln\left(t^2+a^2+\frac{2N-Mp}{2a}artng\frac{t}{a}+c\right)=$$

$$=\frac{M}{2}\ln(x^2+px+q)+\frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C. \qquad (3.8)$$

IV tipdagi kasrning integralini hisoblashda yuqoridagi $t=x+\frac{p}{2}$, $a=\sqrt{q-\frac{p^2}{4}}$ belgilashlardan foydalanamiz:

$$\int\frac{Mx+N}{(x^2+px+q)^r}dx=\int\frac{Mt+\left(N-\frac{Mp}{2}\right)}{(t^2+a^2)^r}dt=$$

$$=\frac{M}{2}\int\frac{d(t^2+a^2)}{(t^2+a^2)^r}+\left(N-\frac{Mp}{2}\right)\int\frac{dt}{(t^r+a^2)^r}=\frac{M}{2(r-1)}\cdot\frac{1}{(t^2+a^2)^{r-1}}+$$

$$+\left(N-\frac{Mp}{2}\right)\int\frac{dt}{(t^2+a^2)^r}=\frac{M}{2(r-1)}\cdot\frac{1}{(t^2+a^2)^{r-1}}+$$

$$+\left(N-\frac{Mp}{2}\right)\frac{dt}{(t^2-a^2)^r}=-\frac{M}{2(r-1)}\frac{1}{(x^2+px+q)^{r-1}}+\left(N-\frac{Mp}{2}\right)\int\frac{dt}{(t^2+a^2)^r}$$

keyingi integral (2.18) rekurrent formula orqali hisoblanadi.

Shunday qilib, har qanday haqiqiy koeffisientli haqiqiy o'zgaruvchili rasional (rasional kasr) funksiyaning boshlang'ich funksiyasi – logarifm, arktangens va rasional funksiya orqali ifodalanar ekan.

3.1-misol. Ushbu

$$\int\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}dx$$

integralni hisoblang.

Yechilishi. Integral ostidagi kasr – noto‘g‘ri kasr bo‘lganligi uchun, (3.2) ga asosan, $P_6(x)=x^6-2x^4+3x^3-9x^2+4$ ko‘phadni $Q_5(x)=x^5-5x^3+4x$ ko‘phadga bo‘lib, $w(x)=x$ bo‘linma va $P_4(x)=3x^4+3x^3-13x^3+4$ qoldiqni topamiz, ya’ni

$$\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}=x+\frac{3x^4+3x^3-13x^2+4}{x^5-5x^3+4x}.$$

Ravshanki, $Q_5(x)=x^5-5x^3+4x$ ko‘phad $x=1$ haqiqiy ildizga ega, $Q_5(x)=x^5-5x^3+4x$ ko‘phadni $x-1$ *ga* bo‘lib,

$$Q_5(x)=x(x-1)(x+1)(x-2)(x+2)$$

ko‘rinishga keltiramiz.

3.1-teoremaga asosan, $\frac{P_4(x)}{Q_5(x)}$ kasr (3.6) ko‘rinishdagi sodda kasrlar yig‘indisi sifatida tasvirlanadi, ya’ni

$$\frac{3x^4+3x^3-13x^2+4}{x(x-1)(x+1)(x-2)(x+2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+1}+\frac{D}{x-2}+\frac{E}{x+2}. \qquad (3.9)$$

Oxirgi tenglikni umumiy maxrajga keltirib, ushbu

$$3x^4+3x^3-13x^2+4=A(x^2-1)(x^2-4)+Bx\cdot(x+1)(x^2-4)+$$
$$+C\cdot x(x-1)(x^2-4)+D\cdot x(x^2-1)(x+2)+E\cdot x(x^2-1)(x-2)$$

tenglikni hosil qilamiz. Tenglikning o‘ng tomonidagi qavslarni ochib, ko‘phadlarning o‘zaro tengligi haqidagi xossadan foydalanib, x ning bir xil darajalari oldidagi koeffisientlarni tenglashtiramiz:

$$\left.\begin{array}{l|l} x^4 & 3=A+B+C+D+E \\ x^3 & 3=B+C+2D-2E \\ x^2 & -13=-5A-4B-4C-D-E \\ x & 0=-4B+4C-2D+2E \\ x^0 & 4=4A \end{array}\right\} \qquad (3.10)$$

Natijada, A,B,C,D,E noma’lumlarga nisbatan beshta chiziqli algebraik tenglamalar sistemasi hosil qilindi. Bu sistemani yechib, noma’lum koeffitsiyentlarni topamiz: bunda, $A=1$, (3.10) sistemaning ikkinchisi va to‘rtinchisini birga yechib, $B=C-1$ eqanligini, birinchi va uchinchisini birga yechib, $D=-E$ ekanligini topamiz. Bularni e’tiborga olib, to‘rtinchi tenglamadan, $C=\frac{3}{2}$, $B=\frac{1}{2}$ ekanligini topamiz. Shunday qilib, noma’lum koeffitsiyentlarning hammasi topildi:

$$A=1,\ B=\frac{1}{2},\ C=\frac{3}{2},\ D=1,\ E=-1.$$

Demak,

$$\int\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}dx=\int\left(x+\frac{3x^4+3x^3-13x^3+4}{x(x-1)(x+1)(x-2)(x+2)}\right)dx=$$

$$=\int x\,dx+\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x-1}+\frac{3}{2}\int\frac{dx}{x+1}+\int\frac{dx}{x-2}-\int\frac{dx}{x+2}=\frac{x^2}{2}+\ln|x|+$$

$$+\frac{1}{2}\ln|x-1|+\frac{3}{2}\ln|x+1|+\ln|x-2|-\ln|x+2|+C=\frac{x^2}{2}+\ln\left|\frac{x(x-2)(x+1)\sqrt{|x^2-1|}}{x+2}\right|+C.$$

To‘g‘ri rasional kasrlarni noma’lum koeffisientli (3.6) ko‘rinishdagi sodda kasrlar yig‘indisi shaklida tasvirlaganda, undagi noma’lum koeffitsiyentlarni yuqorida ko‘rsatilgan (3.1-misolga qarang), noma’lum koeffitsiyentlar usulidan foydalanib topishda, chiziqli algebraik tenglamalarni yechishga to‘g‘ri keladi, lekin chiziqli tenglamalar sistemasini yechish, har doim ham engil bo‘lavermaydi. Xususiy hollarda, noma’lum koeffitsiyentlar usuliga qaraganda, qulayroq bo‘lgan, ya’ni noma’lum koeffitsiyentlarni topishda osonroq bo‘lgan, usullar mavjud. Masalan, *Xevisayd usuli, Gorner sxemasi, differensiallashdan foydalanish usullari,* shular jumlasiga kiradi.

3.1.1.Xevisayd usuli. Agar $\dfrac{P_m(x)}{Q_n(x)}$ $(m<n)$ to‘g‘ri kasrning maxraji $Q_n(x)$, ushbu

$$Q_n(x)=a_n(x-\alpha_1)(x-\alpha_2),\ldots,(x-\alpha_n),\; a_n=1, \tag{3.11}$$

ko‘rinishda tasvirlansa, $\alpha_1,\alpha_2,\ldots,\alpha_n$ – haqiqiy sonlar bo‘lib, $Q_n(\alpha_i)=0$, $(i=\overline{1,n})$, uning (3.6) shakldagi sodda kasrlarga yoyilmasidagi koeffitsiyentlarni, Xevisayd usulidan foydalanib, topish maqsadga muvofiq bo‘ladi. Bu usulni, qisqacha, quyidagi tartibda amalga oshiramiz:

1-qadam. $\dfrac{P_m(x)}{Q_n(x)}$ $(m<n)$ to‘g‘ri kasrning maxraji $Q_n(x)$ ning (3.11) ifodadagi ko‘paytuvchilari orqali yozish, ya’ni

$$\frac{P_m(x)}{Q_n(x)}=\frac{P_m(x)}{(x-\alpha_1)(x-\alpha_2)..(x-\alpha_n)}.$$

2-qadam. $Q_n(x)$ ning $(x-\alpha_i)(i=\overline{1,n})$ ko‘paytuvchisini vaqtincha yopib, i ning har bir qiymatida yopilmagan ko‘paytuvchilarda x ni α_i son bilan almashtiramiz. Bu esa, har bir α_i ildiz uchun A_i sonni beradi:

$$A_1=\frac{P_m(\alpha_1)}{(\alpha_1-\alpha_2)...(\alpha_1-\alpha_n)},$$

$$A_2=\frac{P_m(\alpha_2)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)...(\alpha_2-\alpha_n)},$$

....

$$A_n=\frac{P_m(\alpha_n)}{(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)...(\alpha_n-\alpha_{n-1})}.$$

3-qadam. $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ rasional kasrni

$$\frac{P_m(x)}{Q_n(x)}=\frac{A_1}{x-\alpha_1}+\frac{A_2}{x-\alpha_2}+...+\frac{A_n}{x-\alpha_n}$$

ko'rinishda yozish.

Misol uchun, bu usulni 3.1 - misolga tatbiq qilsak, A, B, C, D, E koeffitsiyentlarni osongina topish mumkin:

1 - qadam.

$$\frac{P_4(x)}{Q_5(x)}=\frac{3x^4+3x^3-13x^2+4}{x(x-1)(x+1)(x-2)(x+2)},\ \alpha_1=0,\ \alpha_2=1,\alpha_3=-1,\ \alpha_4=2,\alpha_5=-2.$$

2 - qadam. $A=\frac{4}{(-1)\cdot1\cdot(-2)\cdot2}=1$

$$B=\frac{-3}{1\cdot2\cdot(-1)\cdot3}=\frac{1}{2};$$

$$C=\frac{3-3-13+4}{(-1)\cdot(-2)\cdot(-3)\cdot1}=\frac{3}{2}.$$

$$D=\frac{3\cdot16+3\cdot8-13\cdot4+4}{2\cdot1\cdot3\cdot4}=1,$$

$$E=\frac{3\cdot16-3\cdot8-13\cdot4+4}{(-2)\cdot(-3)\cdot(-1)\cdot(-4)}=-1$$

3 - qadam. Berilgan $\frac{P_4(x)}{Q_5(x)}$ tug'ri kasrni,

$$\frac{P_4(x)}{Q_5(x)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}+\frac{D}{x-2}+\frac{E}{x+2}=$$

$$=\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x-1}+\frac{3}{2}\cdot\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}$$

ko'rinishda yozamiz.

Ko'p hollarda x ning qiymatlarini, masalan, $x=0,\pm1,\pm2,...$ kabilarni, tanlash yordamida ham noma'lum koeffitsiyentlarni topish qulay bo'ladi.

3.2-misol. x ning sonli qiymatlari yordamida

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

ifodadagi A, B, C koeffitsiyentlar topilsin.

Yechilishi. Dastlab tenglamada kasrdan qutilamiz:

$$x^2+1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2). \qquad (3.12)$$

Endi, A, B, C koeffitsiyentlarni topish uchun, mos ravishda, (3.12) tenglikda $x=1, x=2, x=3,$ qiymatlarni ketma-ket qo'yamiz:

$$1+1=A(-1),(-2),2=2A, A=1$$

$$2^2+1=-B, B=-5,$$

$$3^2+1=2C, C=5.$$

Natijada,

$$\frac{x^2+1}{(x-1)(x-2)(x-3)(x-4)}=\frac{1}{x-1}-\frac{5}{x-2}+\frac{5}{x-3}$$

bo'ladi.

3.3 - misol. Ushbu

$$\frac{x}{x^3+6x^2+11x+6}$$

kasrning noma'lum koeffitsiyentli sodda kasrlar yig'indisi shaklidagi ifodasidagi noma'lum koeffitsiyentlarni toping.

Yechilishi. *1-qadam.* Berilgan kasrning maxrajini ko'paytuvchilarga ajratamiz:

$$Q_3(x)=x^3+6x^2+11+6=(x+1)(x^2+5x+6)=(x+1)(x+2)(x+3).$$

Demak, $Q_3(x)$ maxraj, $\alpha_1=-1,\ \alpha_2=-2,\ \alpha_3=-3$ sodda haqiqiy ildizlarga ega. Shunga asosan, berilgan kasrni,

$$\frac{x}{x^3+6x^2+11x+6}=\frac{x}{(x+1)(x+2)(x+3)}$$

ko'rinishda yozib olamiz.

2 - qadam.

$$\alpha=-1: \quad A_1=\frac{-1}{1\cdot 2}=-\frac{1}{2},$$

$$\alpha_2=-2: \quad A_2=\frac{-2}{-1\cdot 1}=2,$$

$$\alpha_3=-3 \quad A_3=\frac{-3}{2\cdot(-1)}=-\frac{3}{2}.$$

3 - qadam. Berilgan kasrni, ushbu

$$\frac{x}{x^3+6x^2+11x+6}=-\frac{1}{2(x+1)}+\frac{2}{x+2}-\frac{3}{2(x+3)}$$

ko'rinishda yozamiz.

3.1.2. Differensiallashdan foydalanish usuli. Bu usulni, $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ to'g'ri kasrning $Q_n(x)$ maxraji, haqiqiy karrali ildizlarga ega bo'lganda qo'llash qulay bo'ladi.

3.4 - misol. Ushbu

$$\frac{x-1}{(x+2)^3}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$$

tenglamada A, B, C noma'lum koeffisientlarni differensiallashdan foydalanish usuli bo'yicha topish jarayonini batafsil qarab chiqamiz.

Yechilishi. *1-qadam.* Dastlab kasrdan qutilamiz:

$$x-1=A(x+2)^2+B(x+2)+C \qquad (3.13)$$

Bu tenglamada, $x=-2$ deb olsak, $C=-3$ bo'lishini topamiz.

2-qadam. (3.13) tenglikning ikkala tomonini x ga nisbatan differensiallab,

$$1=2A(x+2)+B \qquad (3.14)$$

tenglikni hosil qilamiz. Bunda $x=-2$ deb olsak, $B=1$ bo'ladi.

3-qadam. Endi (3.14) tenglikning ikkala tomonini x ga nisbatan differensiallasak, natijada

$$0=2A,\ A=0$$

bo'ladi.

Demak,

$$\frac{x-1}{(x+2)^3}=\frac{1}{(x+2)^2}-\frac{3}{(x+2)^3}$$

3.1.3 Gorner sxemasi. Har qanday n – darajali

$$P_n(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots\ldots+a_2x^2+a_1x+a_0 \qquad (a_n\neq 0)$$

ko'phadni $x-\alpha$ ikkihadning darajalari bo'yicha yoyish mumkin:

$$\tilde{P}_n(x)=A_n(x-\alpha)^n+A_{n-1}(x-\alpha)^{n-1}+\ldots+A_2(x-\alpha)^2+A_1(x-\alpha_1)+A_0\ ,$$

bunda $A_i(i=\overline{0,n})$ noma'lum koeffitsiyentlar. Gorner sxemasini ketma-ket qo'llash yordamida, $A_i(i=\overline{0,n})$ noma'lum koeffitsiyentlarni topish jadvalini keltiramiz:

		x^n	x^{n-1}	x^{n-2}	...	X	x^0
		a_n	a_{n-1}	a_{n-2}	...	a_1	a_0
x^0	α	a_n	$\alpha a_n+a_{n-1}=b_{n-1}$	$\alpha b_{n-1}+a_{n-2}=b_{n-2}$	...	$\alpha b_2+a_1=b_1$	$\alpha b_1+a_0=A_0$
x^1	α	a_n	$\alpha a_n+b_{n-1}=c_{n-1}$	$\alpha c_{n-1}+b_{n-2}=c_{n-2}$	...	$\alpha c_2+b_1=A_1$	
x^2	α	a_n	$\alpha a_n+c_{n-1}=d_{n-1}$	$\alpha d_{n-1}+c_{n-2}=d_{n-2}$	...		
⋮	⋮	⋮	⋮	⋮	⋮		⋮
x^{n-2}	α	a_n	$\alpha a_n+\ell_{n-1}=h_{n-1}$	$\alpha h_{n-1}+\ell_{n-2}=A_{n-2}$			
x^{n-1}	α	a_n	$\alpha a_n+h_{n-1}=A_{n-1}$				

x^n	α	$a_n = A_n$					

$$P_n(x) = A_n(x-\alpha)^n + A_{n-1}(x-\alpha)^{n-1} + ... + A_2(x-\alpha)^2 + A_1(x-\alpha_1) + A_0 ,$$

ko'phadni $(x-\alpha)^{n+1}$ ga bo'lamiz, natijada, $\frac{P_n(x)}{(x-\alpha)^{n+1}}$ to'g'ri rasional kasrni sodda kasrlarga yoygan bo'lamiz:

$$\frac{P_n(x)}{(x-\alpha)^{n+1}} = \frac{A_n}{x-\alpha} + \frac{A_{n-1}}{(x-\alpha)^2} + ... + \frac{A_2}{(x-\alpha)^{n-2}} + \frac{A_1}{(x-\alpha)^{n-1}} + \frac{A_0}{(x-\alpha)^{n+1}}$$

3.5- misol. Ushbu $\frac{x^4-2x^2+3}{(x+1)^5}$ to'g'ri rasional kasrni sodda kasrlarga yoying.

Yechilishi. Gorner sxemasidan foydalanib, $P_4(x) = x^4 - 2x^2 + 3$ ko'phadni $x+1$ ikkihadning darajalari bo'yicha yoyamiz. Dastlab quyidagi jadvalni tuzamiz:

		x^4	x^3	x^2	x	x^0
		1	0	-2	0	3
x^0	-1	1	-1	-1	1	$2=A_0$
x^1	-1	1	-2	1	$0=A_1$	
x^2	-1	1	-3	$4=A_2$		
x^3	-1	1	$-4=A_3$			
x^4	-1	$1=A_4$				

Bu jadvalga asosan, $P_4(x) = (x+1)^4 - 4(x+1)^3 + 4(x+1)^2 + 0(x+1) + 2$ ko'phadni tuzamiz. Endi $P_4(x)$ ko'phadni $(x+1)^5$ ga bo'lib, berilgan to'g'ri rasional kasrni sodda kasrlar orqali yoyilmasiga ega bo'lamiz:

$$\frac{P_4(x)}{(x+1)^5} = \frac{1}{x+1} - \frac{4}{(x+1)^2} + \frac{4}{(x+1)^3} + \frac{2}{(x+1)^5} .$$

3.6 - misol. Ushbu

$$\int \frac{4x^2-8x}{(x+1)^2(x^2+1)^2} dx$$

integralni hisoblang.

Yechilishi. *1 - qadam.* Berilgan integralda $P_2(x) = 4x^2 - 8x$, $Q_6(x) = (x+1)^2(x^2+1)^2$, integral ostidagi funksiya to'g'ri kasrdan iborat bo'lgani uchun, 3.1 - teoremaga asosan, integral ostidagi to'g'ri kasrni quyidagi sodda kasrlar yig'indisi shaklida tasvirlaymiz:

$$\frac{4x^2-8x}{(x+1)^2(x^2+1)^2}=\frac{A}{(x+1)^2}+\frac{B}{(x+1)}+\frac{Dx+E}{(x^2+1)^2}+\frac{Mx+N}{(x^2+1)},$$

bunda A, B, C, D, E, M, N - noma'lum koeffisientlar.

2 - qadam. Oxirgi kasrni umumiy maxrajga keltirish natijasida,

$$4x^2-8x=A(x^2+1)^2+B(x+1)(x^2+1)^2+(Dx+E)(x+1)^2+$$
$$+(Mx+N)(x+1)^2(x^2+1) \qquad (3.15)$$

munosabatga ega bo'lamiz. (3.15) ning ikki tomonidagi ko'phadlarning mos koeffisientlarini tenglashtirish natijasida, noma'lum A, B, C, D, E, M, N koeffitsiyentlarga nisbatan chiziqli tenglamalar sistemasi hosil bo'ladi. Lekin, bu sistemani yechishga nisbatan qulayroq (osonroq) usul mavjud bo'lib, u quyidagicha amalga oshiriladi: (3.15) tenglikda $x=-1$ deb olib, $A=3$ ekanligini topamiz. So'ngra, $x=i$ deb olsak,

$$-4-8i=(Di+E)(i+1)^2=-2D+i2E,$$

bunda haqiqiy va mavhum qiymatlarni tenglashtirib,

$$-4=-2D, \quad -8=2E, \quad D=2, \quad E=-4$$

larni topamiz.

3 - *qadam.* (3.15) tenglikning ikkala tomonini differensiallab, so'ngra $x=-1$ da nolga aylanmaydigan hadlarini yozib olamiz:

$$8x-8=4A(x^2+1)\cdot x+B(x^2+1)^2+\ldots$$

bundan, $x=-1$ deb olsak, $-16=-8A+4B=-24+4B \quad 8=4B; \quad B=2$ bo'ladi.

4 - *qadam.* (3.15) ning ikkala tomonini differensiallab, faqat $x=i$ da nolga aylanmaydigan hadlarni yozib olamiz:

$$8x-8=D(x+1)^2+2(Dx+E)(x+1)+(Mx+N)(x+1)^2\cdot 2x+\ldots$$

Bu tenglikning ikkala tomoniga $x=i$ ni qo'yib, qolgan ikkita koeffitsiyentlarni topamiz:

$$8i-8=2(i+1)^2+2(2i-4)(i+1)+(Mi+N(i+1)^2\cdot 2i$$
$$8i-8=-12-4Mi-4N, \qquad 4-8i=-4N-4Mi,$$
$$4=-4N,\ -8=-4M, \qquad N=-1, \quad M=2.$$

5 - qadam. Shunday qilib,

$$\int\frac{4x^2-8x}{(x+1)^2(x^2+1)^2}=\int\left(\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{Dx+E}{(x^2+1)^2}+\frac{Mx+N}{x^2+1}\right)dx=$$
$$=3\int\frac{dx}{(x+1)^2}+2\int\frac{dx}{x+1}+2\int\frac{x-2}{(x^2+1)^2}+\int\frac{2x-1}{x^2+1}dx=$$
$$=-\frac{3}{x+1}+2\ln|x+1|-\frac{1}{x^2+1}-4\int\frac{dx}{(x^2+1)^2}+\ln|x^2+1|-arctgx.$$

Oxirgi integral (2.18) rekurrent formula orqali topiladi:

$$\int\frac{dx}{(x^2+1)^2}=\frac{x}{2(x^2+1)}+\frac{1}{2}arctgx+C.$$

Demak,

$$\int\frac{4x^2-8x}{(x+1)^2(x^2+1)^2}=-\frac{3}{x+1}-\frac{2x+1}{x^2+1}+\ln(x+1)^2(x^2+1)-3arctgx+C.$$

Rasional kasrlarni sodda kasrlarga yoyib integrallashda ko‘p hollarda, murakkab hisoblashlar bajarishga to‘g‘ri keladi. Ba’zi hollarda, integral ostidagi rasional kasrning shaklini almashtirib, o‘zgaruvchilarni almashtirish, bo‘laklab integrallash usullaridan foydalanish, berilgan integralni hisoblashni yengillashtiradi.

3.7 - misol. Quyidagi:

$$1)\int\frac{dx}{x^2(3+x^2)^2};\quad 2)\int\frac{dx}{x^3(x^2+2)};\quad 3)\int\frac{dx}{x(2-x^3)^2};$$

$$4)\int\frac{x^2+1}{x^4+3x^2+1};\quad 5)\int\frac{x^2dx}{(x^2+3)^3};$$

$$6)\ \frac{1}{x^4(1+x^2)};\qquad 7)\ \frac{2x^9+3x^4}{(x^{10}+1)^2}$$

integrallarni hisoblang.

Yechilishi.

$$1)\int\frac{dx}{x^2(3+x^2)^2}=\frac{1}{3}\int\frac{3-x^2+x^2}{x^2(3+x^2)^2}dx=\frac{1}{3}\int\frac{dx}{x^2(3+x^2)}-\frac{1}{3}\int\frac{dx}{(3+x^2)^2}=$$

$$=\frac{1}{9}\int\left(\frac{1}{x^2}-\frac{1}{3+x^2}\right)dx-\frac{1}{3}\int\frac{dx}{(3+x^2)^2}=$$

$$=-\frac{1}{9x}-\frac{1}{9\sqrt{3}}arctg\frac{x}{\sqrt{3}}-\frac{x}{18(x^2+3)}-\frac{1}{18\sqrt{3}}arctg\frac{x}{\sqrt{3}}=$$

$$=-\frac{1}{9x}-\frac{x}{18(x^2+3)}+\frac{1}{6\sqrt{3}}arctg\frac{x}{\sqrt{3}}+C$$

$$2)\int\frac{dx}{x^3(x^2+2)}=\int\frac{x\,dx}{x^4(x^2+2)}=\frac{1}{2}\int\frac{d(x^2)}{x^4(x^2+2)}=\frac{1}{2}\int\frac{du}{u^2(u+2)}\bigg|_{u=x^2}=$$

$$=\frac{1}{2}\int\frac{u+2-u}{u^2(u+2)}du=\frac{1}{2}\int\frac{du}{u^2}-\frac{1}{2}\int\frac{du}{u(u+2)}=$$

$$=-\frac{1}{2u}-\frac{1}{4}\ln|u|+\frac{1}{4}\ln|u+2|+C=-\frac{1}{2x^2}+\frac{1}{4}\ln\frac{x^2+2}{x^2}+C.$$

$$3)\int\frac{dx}{x(2-x^3)^2}=\int\frac{x^2dx}{x^3(2-x^3)^2}=\frac{1}{3}\int\frac{d(x^3)}{x^3(2-x^3)^2}=\frac{1}{3}\int\frac{du}{u(2-u)^2}\bigg|_{u=x^3}=$$

$$=\frac{1}{6}\int\frac{1}{2-u}\left(-\frac{1}{u}+\frac{1}{2-u}\right)du=\ =\frac{1}{6}\int\frac{du}{(2-u)^2}-\frac{1}{12}\int\frac{du}{2-u}+\frac{1}{12}\int\frac{du}{u}=$$

$$=\frac{1}{6(2-u)}-\frac{1}{12}\ln|2-u|+\frac{1}{12}\ln|u|+C=\frac{1}{6(2-x^3)}+\frac{1}{12}\ln\left|\frac{x^3}{2-x^3}\right|+C$$

$$4)\int\frac{x^2+1}{x^4+3x^2+1}dx=\int\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+3+\frac{1}{x^2}\right)}dx=\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+5}dx=$$

$$=\int\frac{du}{u^2+5}\Bigg|_{u=x-\frac{1}{x}}=\left(\frac{1}{\sqrt5}arctg\frac{u}{\sqrt5}+C\right)\Bigg|_{u=\left(x-\frac{1}{x}\right)}==\frac{1}{\sqrt5}arctg\frac{x^2-1}{x\sqrt5}+C.$$

$$5)\int\frac{x^2dx}{(x^2+3)^3}=\int\frac{x\cdot xdx}{(x^2+3)^3}=\frac{1}{3}\int\frac{xd(x^2+3)}{(x^2+3)^3}=$$

$$\left[u=x,\quad dv=\frac{d(x^2+3)}{(x^2+3)^3},\ du=dx,\ v=-\frac{1}{2(x^2+3)^2}\right]$$

$$=\frac{1}{3}(-\frac{x}{2(x^2+3)}+\frac{1}{2}\int\frac{dx}{(x^2+3)^2})=-\frac{x}{6(x^2+3)}+\frac{1}{36}\frac{x}{x^2+3}+$$

$$+\frac{1}{36\sqrt3}arctg\frac{x}{\sqrt3}+C$$

$$5)\int\frac{x^2dx}{(x^2+3)^3}=\int\frac{x^2+3-3}{(x^2+3)^3}dx=\int\frac{dx}{(x^2+3)^2}-3\int\frac{dx}{(x^2+3)^3}=$$

$$=\frac{x}{6(x^2+3)}+\frac{1}{6\sqrt3}arctg\frac{x}{\sqrt3}-\frac{x}{4(x^2+3)^2}-$$

$$-\frac{1}{8}\frac{x}{(x^2+3)}-\frac{1}{8\sqrt3}arctg\frac{x}{\sqrt3}=\frac{1}{24}\cdot\frac{x}{x^2+3}-\frac{1}{4}\cdot\frac{x}{(x^2+3)^2}+\frac{1}{24\cdot\sqrt3}arctg\frac{x}{\sqrt3}+C$$

$$6)\int\frac{1}{x^4(1+x^2)}dx=\int\frac{1-x^4+x^4}{x^4(1+x^2)}dx=\int\frac{(1-x^2)(1+x^2)+x^4}{x^4(1+x^2)}dx=$$

$$=\int\frac{1-x^2}{x^4}dx+\int\frac{dx}{1+x^2}=-\frac{1}{3x^3}+\frac{1}{x}+arctgx+C$$

$$7)\int\frac{2x^9+3x^4}{(x^{10}+1)^2}dx=\frac{1}{5}\int\frac{d(x^{10}+1)}{(x^{10}+1)^2}+\frac{3}{5}\int\frac{dx^5}{((x^5)^2+1)^2}=\frac{1}{5}\int\frac{du}{u^2}\Bigg|_{u=x^{10}+1}+\frac{3}{5}\int\frac{du}{(u^2+1)^2}\Bigg|_{u=x^5}=$$

$$=-\frac{1}{x(x^{10}+1)}+\frac{x^5}{2(x^5+1)}+\frac{1}{2}arctgx^5+C.$$

3.2. Integralning rasional qismini ajratishda Ostrogradskiy usuli. $\frac{P(x)}{Q(x)}$ to‘g‘ri kasrning maxraji karrali kompleks ildizlarga ega bo‘lganda, uni integrallashda, murakkab hisoblashlar bajarishga to‘g‘ri keladi.

Bunday hollarda, ushbu

$$\int\frac{P(x)}{Q(x)}dx=\frac{P_1(x)}{Q_1(x)}+\int\frac{P_2(x)}{Q_2(x)}dx \qquad (3.16)$$

Ostrogradskiy formulasidan foydalanish qulay bo‘ladi, bunda $Q_2(x)$– ildizlari $Q(x)$ ko‘phadning hamma sodda (bir karrali) ildizlaridan iborat bo‘lgan ko‘phad, $Q(x)=Q_1(x)\cdot Q_2(x)$, $P_1(x)$ va $P_2(x)$ lar no‘malum koeffisientli ko‘phadlar bo‘lib, $\frac{P_1(x)}{Q_1(x)}$ va $\frac{P_2(x)}{Q_2(x)}$ –to‘g‘ri kasrlardan iborat.

$P_1(x)$ va $P_2(x)$ ko‘phadlarni topish uchun, ularni no‘malum koeffitsiyentlar yordamida yozib olib, so‘ngra, (3.16) ning ikkala tomonini differensiallaymiz, natijada, (3.16) tenglikka teng kuchli,

$$\frac{P(x)}{Q(x)}=\left(\frac{P_1(x)}{Q_1(x)}\right)'+\frac{P_2(x)}{Q_2(x)} \tag{3.17}$$

tenglikga ega bo‘lamiz. Bu tenglikdan noma’lum koeffitsiyentlar usulidan foydalanib, $P_1(x)$ va $P_2(x)$ larning tarkibidagi noma’lum koeffitsiyentlarni topamiz. Ostrogradskiy formulasi, integralning rasional qismini (integrallamasdan) ajratishga imkon beradi, $\frac{P(x)}{Q(x)}$ to‘g‘ri kasrni integrallash masalasi, unga nisbatan osonroq integrallanadigan $\frac{P_2(x)}{Q_2(x)}$ to‘g‘ri kasrni integrallashga keltiriladi.

3.8-misol. Ushbu $\int\frac{x-1}{(x^2+x+1)^2}dx$ integralni hisoblang.

Yechilishi. Bu holda, $P(x)=x-1$, $Q(x)=(x^2+x+1)^2$, $Q_1(x)=x^2+x+1$, $Q_2(x)=x^2+x+1$. (1) formulaga asosan,

$$\int\frac{x-1}{(x^2+x+1)^2}dx=\frac{Ax+B}{x^2+x+1}+\int\frac{Cx+D}{x^2+x+1}dx$$

deb yozib olamiz. A,B,C,D noma’lum koeffitsiyentlarni topish uchun, yuqoridagi tenglikning ikkala tomonini differensiallaymiz:

$$\frac{x-1}{(x^2+x+1)^2}=\left(\frac{Ax+B}{x^2+x+1}\right)'+\frac{Cx+D}{x^2+x+1},$$

$$\frac{x-1}{(x^2+x+1)^2}=\frac{A(x^2+x+1)-(Ax+B)(2x+1)}{(x^2+x+1)^2}+\frac{Cx+D}{x^2+x+1},$$

bundan

$$x-1=A(x^2+x+1)-(Ax+B)(2x+1)+(Cx+D)(x^2+x+1).$$

Tenglikning ikkala tomonidagi x ning bir xil darajalari oldidagi koeffitsiyentlarni tenglashtiramiz:

$x^3|\ 0=C,$

$x^2|\ 0=A-2A+C+D,$

$x\ |\ 1=A-A-2B+C+D,$

$x^0|\ -1=A-B+D,$

bundan, $C=0$, $A=D=B=-1$.

Shunday qilib,

$$\int\frac{x-1}{(x^2+x+1)^2}dx=-\frac{x+1}{x^2+x+1}-\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=$$

$$=-\frac{x+1}{x^2+x+1}-\frac{2}{\sqrt{3}}arctg\frac{2x}{\sqrt{3}}+C.$$

Mustaqil yechish uchun misollar

Integrallarni hisoblang:

3.1. $\int\frac{dx}{x(2+x)}$ **3.2.** $\int\frac{dx}{x(5+2x)}$ **3.3.** $\int\frac{dx}{x\left(\frac{3}{2}+\frac{5}{7}x\right)}$

3.4. $\int\frac{dx}{x(a+bx)}$ **3.5.** $\int\frac{dx}{(x+1)(x-3)}$ **3.6.** $\int\frac{dx}{(2x+3)(x+4)}$

3.7. $\int\frac{dx}{(3x+5)(4x+9)}$ **3.8.** $\int\frac{dx}{(a+bx)(c+fx)}$ **3.9.** $\int\frac{x\,dx}{(x+3)(x-4)}$.

3.10. $\int\frac{x\,dx}{(2x+5)(3x-4)}$ **3.11.** $\int\frac{x\,dx}{(a+bx)(c+fx)}$

Integrallarni hisoblang:

3.12. $\int\frac{dx}{x(4+x^2)}$ **3.13.** $\int\frac{dx}{x(3+4x)^2}$ **3.14.** $\int\frac{dx}{x(a+bx)^2}$

3.15. $\int\frac{dx}{(2+x)(x-3)^2}$ **3.16.** $\int\frac{dx}{(3-4x)(5-2x)^2}$ **3.17.** $\int\frac{dx}{(a+bx)(c+fx)^2}$

3.18. $\int\frac{x\,dx}{(x-3)(2x+5)^2}$ **3.19.** $\int\frac{x\,dx}{(a+bx)(c+fx)^2}$ **3.20.** $\int\frac{4x-1}{4x^2-4x+5}dx.$

3.3. Integrallarni hisoblang:

3.21. $\int\frac{dx}{x(a+bx)(c+fx)}$ **3.22.** $\int\frac{dx}{x(x^2+3x+5)}$ **3.23.** $\int\frac{dx}{(x-3)(3x^2+4x+2)}$

3.24. $\int\frac{dx}{(2x+5)(x^2-4x+7)}$ **3.25.** $\int\frac{4x+8}{4x^2+6x-13}dx.$ **3.26.** $\int\frac{x+2}{3x^2-x+5}dx.$

3.27. $\int\frac{\alpha x+\beta}{ax^2+bx+c}dx.$ **3.28.** $\int\frac{(x^2+x+1)dx}{(x-3)(x^2+4)}$

3.29. $\int\frac{3x^3-16x^2-x+51}{3x^2-7x-20}dx.$

3.30. $\int\frac{x^5+x^4-8}{x^3-4x}dx.$

Integrallarni hisoblang:

3.31. $\int\frac{x^6-2x^4-3x^3-9x^2+4}{x^5-5x^3+4x}dx.$ **3.32.** $\int\frac{x^5-2x^2+3}{x^2-4x+4}dx.$

3.33. $\int \frac{2x^4-3x^3+4x^2-5x+6}{x^2-3x+1}dx$ **3.34.** $\int \frac{x^3-x+1}{(x-2)^5}dx.$

3.35. $\int \frac{x^4-2x^2+3}{(x-1)^5}.$

Integrallarni hisoblang:

3.36. $\int \frac{x^2+1}{x(x-1)^3}dx.$ **3.37.** $\int \frac{dx}{(x-2)^2(x+3)^3}.$

3.38. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}.$ **3.39.** $\int \frac{x^2}{(1-x^2)^3}dx.$

3.40. $\int \frac{x^2}{(x-1)(x+2)(x+3)}dx.$ **3.41.** $\int \frac{1+x}{(x-1)(x^2+1)}dx.$

3.42. $\int \frac{x^2}{x^4-16}dx.$ **3.43.** $\int \frac{5x^2+6x-23}{(x-1)^3(x+1)^2(x-2)}dx.$

3.6. Integrallarni hisoblang:

3.44. $\int \frac{x^3+x+1}{x^4-1}dx.$ **3.45.** $\int \frac{dx}{(x^2-4x+5)(x^2-4x+3)}.$

3.46. $\int \frac{x^3-6}{x^4+6x^2+8}dx.$ **3.47.** $\int \frac{(3x^2-2)dx}{9x^4-13x^2+4}.$

3.48. $\int \frac{(3x^2-2)x\,dx}{(x+2)^2(3x^2-2x+4)}.$ **3.49.** $\int \frac{18-11x}{(x^2-9x+20)(x^2+2x+2)}dx.$

Mustaqil yechish uchun misollarning javoblari

3.1. $-\frac{1}{2}\ln\left|\frac{2}{x}+1\right|+C.$ **3.2.** $-\frac{1}{5}\ln\left|\frac{5}{x}+2\right|+C.$ **3.3.** $-\frac{2}{3}\ln\left|\frac{3}{2x}+\frac{5}{7}\right|+C.$

3.4. $-\frac{1}{a}\ln\left|\frac{a}{x}+b\right|+C.$ **3.5.** $-\frac{1}{4}\ln\left|\frac{x+1}{x-3}\right|+C.$ **3.6.** $\frac{1}{5}\ln\left|\frac{2x+3}{x+4}\right|+C.$ **3.7.** $\frac{1}{7}\ln\left|\frac{3x+5}{4x+9}\right|+C.$

3.8. $\frac{1}{bc-af}\ln\left|\frac{a+bx}{c+fx}\right|+C.$ **3.9.** $\frac{1}{7}[3\ln|x+3|+4\ln|x-4|]+C.$

3.10. $\frac{1}{23}[\frac{5}{2}\ln|2x+5|+\frac{4}{3}\ln|3x-4|]+C.$

3.11. $\frac{1}{af-bc}[\frac{a}{b}\ln|a+bx|-\frac{c}{f}\ln|c+fx|]+C.$ **3.12.** $\frac{1}{4}\left(\frac{1}{4+x}-\frac{1}{4}\ln\left|\frac{4}{x}+1\right|\right)+C.$

3.13. $\frac{1}{3}[\frac{1}{3+4x}-\frac{1}{3}\ln\left|\frac{3}{x}+4\right|]+C.$ **3.14.** $\frac{1}{a}[\frac{1}{a+bx}-\frac{1}{a}\ln\left|\frac{a}{x}+b\right|]+C.$

3.15. $\frac{-1}{5(x-3)}+\frac{1}{25}\ln\left|\frac{x+2}{x-3}\right|+C.$ **3.16.** $\frac{1}{14(2x-5)}-\frac{1}{49}\ln\left|\frac{3-4x}{5-2x}\right|+C.$

3.17. $\frac{1}{(bc-af)(c+fx)}+\frac{b}{(af-bc)^2}\ln\left|\frac{a+bx}{c+fx}\right|+C.$ **3.18.** $-\frac{5}{22(2x+5)}+\frac{6}{121}\ln\left|\frac{x-3}{2x+5}\right|+C$

3.19. $\frac{c}{f(af-bc)(c+fx)}-\frac{af}{b(af-bc)^2}\ln\left|\frac{a+bx}{c+fx}\right|+C.$

3.20. $\frac{1}{2}\ln|4x^2-4x+5|+\frac{1}{4}arctg\frac{2x-1}{2}+C.$

3.21. $\frac{1}{ac}\ln|x|+\frac{b}{a(af-bc)}\ln|a+bx|-\frac{c}{c(af-bc)}\ln|c+fx|+C.$

3.22. $\frac{1}{10}\ln\frac{x^2}{x^2+3x+5}-\frac{3}{5\sqrt{11}}arctg\frac{2x+3}{\sqrt{11}}+C.$

3.23. $\frac{1}{82}\ln\left|\frac{(x-3)^2}{3x^2+4x+2}\right|-\frac{11}{41\sqrt{2}}arctg\frac{3x+2}{\sqrt{2}}+C.$

3.24. $\frac{1}{93}\ln\left|\frac{(2x+5)^2}{x^2-4x+7}\right|+\frac{\sqrt{3}}{31}arctg\frac{2x-2}{\sqrt{3}}+C.$

3.25. $\frac{1}{2}\ln|4x^2+6x-13|+\frac{5}{2\sqrt{61}}\ln\left|\frac{4x+3-\sqrt{61}}{4x+3+\sqrt{61}}\right|+C.$

3.26. $\frac{1}{6}\ln|3x^2-x+5|+\frac{13}{3\sqrt{59}}arctg\frac{6x-1}{\sqrt{59}}+C.$

3.27. $$\begin{cases}\frac{\alpha}{2a}\ln|ax^2+bx+c|+\frac{2a\beta-b\alpha}{a\sqrt{4ac-b^2}}arctg\frac{2ax+b}{\sqrt{4ac-b^2}}, & 4ac-b^2>0\\ \frac{\alpha}{2a}\ln|ax^2+bx+c|+\frac{2a\beta-b\alpha}{2a\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b-\sqrt{b^2-4ac}}\right|, & b^2-4ac>0\end{cases}+C.$$

3.28. $\ln|x-3|-\frac{1}{2}arctg\frac{x}{2}+C.$ **3.29.** $\frac{(x-3)^2}{2}+\frac{1}{3}\ln|3x+5|-\ln|x-4|+C.$

3.30. $\frac{x^3}{3}+\frac{x^2}{2}+4x+\ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+C.$ **3.31.** $\frac{x^2}{2}+\ln\left|\frac{x(x-2)(x+1)\sqrt{|x^2-1|}}{x+2}\right|+C.$

3.32. $\frac{x^4}{4}+\frac{4}{3}x^3+6x^2+30x-\frac{27}{x-2}+72\ln|x-2|+C.$

3.33. $\frac{2}{3}x^3+\frac{3}{2}x^2+11x+\frac{5}{2}\ln|x^2-3x+1|+\frac{13}{2\sqrt{5}}\ln\left|\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right|+C.$

3.34. $-\frac{1}{x-2}-\frac{3}{(x-2)^2}-\frac{11}{3(x-2)^3}-\frac{7}{4(x-2)^4}+C.$

3.35. $\ln|x+1|+\frac{4}{x+1}-\frac{2}{(x+1)^2}-\frac{1}{2(x+1)^4}+C.$

3.36. $-\frac{1}{(x-2)^2}+\ln\left|\frac{x-1}{x}\right|+C.$ **3.37.** $\frac{16-21x-6x^2}{250(x-2)(x+3)^2}-\frac{3}{625}\ln\left|\frac{x-2}{x+3}\right|+C.$

3.38. $\frac{9x^2+50x+68}{4(x+2)(x+3)^2}+\frac{1}{8}\ln\left|\frac{(x+1)(x+2)^{16}}{(x+3)^{17}}\right|+C.$ **3.39.** $\frac{x^3+x}{8(1-x^2)^2}-\frac{1}{16}\ln\left|\frac{1+x}{1-x}\right|+C.$

3.40. $\frac{1}{12}\ln|x-1|-\frac{4}{3}\ln|x+2|+\frac{9}{4}\ln|x+3|+C.$ **3.41.** $\frac{1}{2}\ln\frac{(x-1)^2}{x^2+1}+C.$

3.42. $\frac{1}{8}\ln\left|\frac{x-2}{x+2}\right|+\frac{1}{4}arctg\frac{x}{2}+C.$ **3.43.** $\ln\frac{(x-1)(x-2)}{(x+1)^2}+\frac{3}{2}\frac{1}{(x-1)^2}+\frac{4}{x-1}+\frac{1}{x+1}+C.$

3.44. $\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|-\frac{1}{2}arctg\,x+C$. **3.45.**
$\frac{1}{52}\ln|x-3|-\frac{1}{20}\ln|x-1|+\frac{1}{65}\ln(x^2+4x+5)+\frac{7}{130}arctg(x+2)+C$.
3.46. $\ln\frac{x^2+4}{\sqrt{x^2+2}}+\frac{3}{2}arctg\frac{x}{2}-\frac{3}{\sqrt{2}}arctg\frac{x}{\sqrt{2}}+C$. **3.47.** $\frac{1}{10}\ln\frac{3x^2-5x+2}{3x^2+5x+2}+C$.
3.48. $\frac{1}{x+2}+\ln|x+2|-\frac{1}{\sqrt{11}}arctg\frac{3x-1}{\sqrt{11}}+C$. **3.49.** $\ln\left|\frac{x-4}{x-5}\right|+arctg(x+1)+C$.

4-§. Ba'zi irrasional ifodalarni integrallash

Agar integral ostidagi funksiya irrasional funksiya bo'lsa, ba'zi hollarda uning integralini hisoblash masalasi, rasional funksiyaning integralini hisoblashga olib kelinadi. Bu usulga integral ostidagi ifodani *rasionallashtirish usuli* deyiladi. Biz bu paragrafda irrasional ifodalar qatnashgan integrallarning ba'zi turlarini rasionallashtirish usullarini keltiramiz. $x_1, x_2,...,x_n$ o'zgaruvchilarga nisbatan rasional bo'lgan funksiyani $R(x_1,x_2,...,x_n)$ deb belgilaymiz.

4.1. $\int R(x, x^{\alpha}, x^{\beta},..., x^{\gamma})dx$ **ko'rinishdagi ifodalarni integrallash.** Bunda $R(x,x^{\alpha},x^{\beta},...,x^{\gamma})$- o'z argumentlarining rasional funksiyasi. $\alpha=\frac{m_1}{n_1}, \beta=\frac{m_2}{n_2}...\gamma=\frac{m_s}{n_s}$ - rasional sonlar. Bu integralni hisoblash, $x=t^k (k - n_1,n_2,...$ larning eng kichik umumiy bo'linuvchisi) almashtirish yordamida rasional funksiyaning integralini hisoblashga keltiriladi.

4.1-misol. Ushbu $\int\frac{x+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$ integralni hisoblang.

Yechilishi. Integral ostidagi funksiya, $x_1=x,\ x_2=x^{\frac{1}{3}},\ x_3=x^{\frac{1}{6}}$ o'zgaruvchilarga nisbatan rasional funksiya. Berilgan integral 4.1- banddagi integral ko'rinishida bo'lib, $\alpha=\frac{1}{3},\ \beta=\frac{1}{6},\ n_1=3,\ n_2=6,\ k=6$. Uni hisoblash, 4.1- bandga asosan, $x=t^6$ almashtirish yordamida rasional funksiyaning integralini hisoblashga keltiriladi: $dx=6t^5dt$,

$$\int\frac{x+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx=\int\frac{t^6+t}{t^6(1+t^2)}6t^5dt=6\int\frac{t^5+1}{1+t^2}dt=6\int(t^3+1)dt+6\int\frac{t+1}{1+t^2}dt=$$

$$=\frac{3}{2}t^4+3t^3+6arctgt+3\ln(1+t)+C=\frac{3}{2}\sqrt[3]{x^2}+3\sqrt{x}+6arctg\sqrt[6]{x}+3\ln(1+\sqrt[3]{x})+C.$$

4.2. $\int R[x,(ax+b)^{\alpha},(ax+b)^{\beta},...,(ax+b)^{\gamma}]dx,$ (4.1)

$$\int R[x,\left(\frac{ax+b}{cx+d}\right)^{\alpha},\left(\frac{ax+b}{cx+d}\right)^{\beta},...,\left(\frac{ax+b}{cx+d}\right)^{\gamma}]dx\ \ (a,b,c,d\in R,\ ad-cb\neq 0)\ (4.2)$$

ko'rinishdagi ifodalarni integrallash. (4.1) yoki (4.2) ko'rinishidagi integrallar, mos ravishda, $ax+b=t^k$ yoki $\frac{ax+b}{cx+d}=t^k$ almashtirish yordamida rasionallashtiriladi, bunda k - $\alpha, \beta, ..., \nu$ rasional sonlarning eng kichik umumiy bo'linuvchisi.

4.2-misol. Ushbu $\int \frac{1}{x}\sqrt{\frac{x-2}{x}}dx$ integralni hisoblang.

Yechilishi. Berilgan integral (4.2) ko'rinishdagi integral bo'lib, $\frac{x-2}{x}=t^2$ almashtirish yordamida rasionallashtirilib, hisoblanadi:

$$\int \frac{1}{x}\sqrt{\frac{x-2}{x}}dx = \left| x=\frac{2}{1-t^2},\ dx=\frac{4t}{(1-t^2)^2}dt \right| = 2\int \frac{t^2(1-t^2)}{(1-t^2)^2}dt = -2\int dt + 2\int \frac{1}{1-t^2}dt =$$

$$= -2t - \ln\left|\frac{1-t}{1+t}\right| + C = -2\sqrt{\frac{x-2}{x}} - \ln[|x|(1-\sqrt{\frac{x-2}{x}})^2] + C.$$

4.3. $\int R\left(x,\sqrt{a^2-x^2}\right)dx,\ \int R\left(x,\sqrt{a^2+x^2}\right)dx,\ \int R\left(x,\sqrt{x^2-a^2}\right)dx$ **ko'rinishdagi ifodalarni integrallash**.

Quyidagi: $\int R\left(x,\sqrt{a^2-x^2}\right)dx,\ \int R\left(x,\sqrt{a^2+x^2}\right)dx,\ \int R\left(x,\sqrt{x^2-a^2}\right)dx$ (4.3)

ko'rinishdagi integrallar, mos ravishda $x=a\sin t,\ x=atgt,\ x=a\sec t,\ a\in R,\ a\neq 0$, almashtirishlar natijasida rasionallashtirilib, hisoblanadi.

4.3 -misol. Ushbu $\int x^2\sqrt{4-x^2}dx$ integralni hisoblang.

Yechilishi. Berilgan integral (4.3) ko'rinishdagi ifodalarning birinchi integrali ko'rinishida bo'lganligi uchun, $x=2\sin t$ almashtirishni bajarib, hisoblaymiz:

$$\int x^2\sqrt{4-x^2}dx = \left| x=2\sin t,\ dx=2\cos t dt,\ \sqrt{4-4\sin^2 t}=2\cos t \right| =$$

$$= 16\int \sin^2 t\cos^2 t dt = 4\int \sin^2 2t dt = 2\int (1-\cos 4t)dt = 2t - \frac{1}{2}\sin 4t + C.$$

$$\sin 4t = 2\sqrt{1-\sin^2 t}\sin t\left(1-2\sin^2 t\right),\ -\frac{\pi}{2}\le t\le\frac{\pi}{2},\ \sin\left(\arcsin\frac{x}{2}\right)=\frac{x}{2},\ -2\le x\le 2$$

formulalarni e'tiborga olgan holda, eski o'zgaruvchiga qaytib, integralni hisoblaymiz:

$$\int x^2\sqrt{4-x^2}dx = 2t - \frac{1}{2}\sin 4t + C = 2\arcsin\frac{x}{2} + \frac{x}{4}\left(x^2-2\right)\sqrt{4-x^2} + C.$$

4.4. $R\left(x,\sqrt{ax^2+bx+c}\right)dx\ \left(a\neq 0,\ b^2-4ac\neq 0\right)$ **ko'rinishdagi ifodalarni integrallash.**

Quyidagi: $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ (4.3) integralni hisoblash, undagi a,b,c koeffisientlarga bog'liq, uchta almashtirish yordamida rasional funksiyaning integralini hisoblashga keltiriladi:

1-hol. $a>0$ bo'lganda, (4.3) integralda $\sqrt{ax^2+bx+c}=\pm t\pm\sqrt{a}x$ almashtirish bajariladi.

2-hol. $c>0$ bo'lganda, (4.3) integralda $\sqrt{ax^2+bx+c}=\pm xt\pm\sqrt{c}$ almashtirish bajariladi.

3-hol. $a\neq 0,\ b^2-4ac>0$ bo'lganda esa, (4.3) integralda $\sqrt{ax^2+bx+c}=t(x-x_1)$ yoki $\sqrt{ax^2+bx+c}=t(x-x_2)$ almashtirish bajariladi.

Odatda, yuqorida keltirilgan uchta almashtirishlar - *Eyler almashtirishlari* deb aytiladi.

4.4 -misol. Ushbu $\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}$ integralni hisoblang.

Yechilishi. $1+x-x^2$ kvadrat uchhad kompleks ildizga ega va $a<0,\ c>0$ bo'lgani uchun, *2-holga* asosan, $\sqrt{1+x-x^2}=tx-1$ almashtirishni bajaramiz:

$$1+x-x^2=t^2x^2-2tx+1;\ 1-x=t^2x-2t;\ x=\frac{1+2t}{1+t^2},$$

$$dx=\frac{2\left(1-t-t^2\right)}{\left(1+t^2\right)^2}dt,\quad \sqrt{1+x-x^2}=\frac{t^2+t-1}{t^2+1}$$

Shunday qilib

$$\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}=\int\frac{2\left(1-t-t^2\right)}{\left(1+t^2\right)^2\left(1+\frac{1+2t}{t^2+1}\right)\cdot\left(\frac{t^2+t-1}{t^2+1}\right)}dt=$$

$$=-2\int\frac{dt}{(t+1)^2+1}=-2\,arctg(t+1)+C=-2arctg\frac{\sqrt{1+x-x^2}+x+1}{x}+C$$

4.5 -misol. Ushbu $\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx$ integralni hisoblang.

Yechilishi. Integral ostidagi x^2+3x+2 kvadrat uchhad $x_1=-1,\ x_2=-2$ haqiqiy ildizlaga ega bo'lgani uchun, 4.4 - banddagi *3-holga* asosan, $\sqrt{x^2+3x+2}=t(x+1)$ almashtirish bajarib, berilgan integralni hisoblaymiz:

$$(x+1)(x+2)=t^2(x+1)^2;\ x+2=t^2(x+1);\ x=\frac{t^2-2}{1-t^2};$$

$$dx=\frac{2t\left(1-t^2\right)+2t\left(t^2-2\right)}{\left(1-t^2\right)^2}dt=\frac{2t-4t}{\left(1-t^2\right)^2}dx=-\frac{2t}{\left(1-t^2\right)^2}dt;$$

$$\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx=\int\frac{x^2+x^2-3x-2}{\left(x+\sqrt{x^2+3x+2}\right)^2}dt=-\int\frac{3x+2}{\left(x+\sqrt{x^2+3x+2}\right)^2}dx=$$

$$=2\int\frac{\left(\frac{3t^2-6}{1-t^2}+2\right)t\,dt}{(1-t^2)^2\left[\frac{t^2-2}{1-t^2}+t\left(\frac{t^2-2}{1-t^2}+1\right)\right]^2}=2\int\frac{\frac{t(t^2-4)}{1-t^2}}{(1-t^2)^2\left[\frac{t^2-t-2}{1-t^2}\right]^2}dt=$$

$$=2\int\frac{t(t^2-4)dt}{(1-t^2)(t+1)^2(t-2)^2}=2\int\frac{t(t+2)}{(1-t^2)(t+1)^2(t-2)}dt=2\int\frac{t(t+2)}{(1-t)(1+t)^3(t-2)}dt$$

Oxirgi integralda, integral ostidagi ifoda to‘g‘ri kasr bo‘lganligi uchun, 3.1 – teoremaga asosan, uni, sodda kasrlar yig‘indisi shaklida,

$$\frac{2t(t+2)}{(1-t)(1+t)^3(t-2)}=\frac{A}{1-t}+\frac{B}{t-2}+\frac{C}{1+t}+\frac{D}{(1+t)^2}+\frac{E}{(1+t)^3}$$

ko‘rinishda tasvirlab, A, B, C, D, E noma’lum koeffitsiyentlarni topamiz. Yuqoridagi tenglikdan,

$$2t(t+2)=A(t-2)(1+t)^3+B(1-t)(1+t)^3+c(1-t)(2-t)(1+t)^2+$$
$$+D(1-t)(1+t)(t-2)+E(1-t)(t-2)$$

munosabatni hosil qilamiz. Oxirgi tenglikda $t=1, t=-1, t=2$ qiymatlarni ketma-ket qo‘yib, $A=-\frac{3}{4}, E=\frac{1}{3}, B=-\frac{16}{27}$ ekanligini topamiz. D koeffitsiyentni topish uchun,

$$2t(t+2)=A(t-2)(1+t)^3+B(1-t)(1+t)^3+c(1-t)(2-t)(1+t)^2+$$
$$+D(1-t)(1+t)(t-2)+E(1-t)(t-2)$$

tenglikning ikkala tomonini differensiallab, so‘ngra unga $t=-1$ ni keltirib qo‘yamiz. Differensiallanganda tenglikning o‘ng tomonida $t=-1$ da nulga aylanmaydigan hadlarni yozib olamiz:

$$4t+4=D[-2t(t-2)]+E[-(t-2)+(1-t)]$$

Bundan $t=-1$ deb, $0=-6D+5E=-6D+\frac{5}{3}, D=\frac{5}{18}$ ekanligini topamiz.

Xuddi shunday, C koeffitsiyentni topishda ham,

$$2t(t+2)=A(t-2)(1+t)^3+B(1-t)(1+t)^3+C(1-t)(2-t)(1+t)^2+$$
$$+D(1-t)(1+t)(t-2)+E(1-t)(t-2)$$

tenglikning ikkala tomonini differensiallab, tenglikning o‘ng tomonida $t=1$ da nolga aylanmadigan hadlarni yozib olamiz:

$$4t+4=A[(1+t)^3+3(t-2)(1+t)^2]+B[-(1+t)^3]+C[-(t-2)(1+t)^2-]+$$
$$+D[-(1+t)(t-2)]-E(t-2)$$

Bundan, $t=1$ deb olib,

$8=-4A-8B+4C+2D+E=3+\frac{128}{27}+4C+\frac{5}{9}+\frac{1}{3}, C=-\frac{17}{108}$ bo‘lishini topamiz.

Demak,

$$\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx=\frac{3}{4}\ln|t-1|-\frac{16}{27}\ln|t-2|-\frac{17}{108}\ln|1+t|-\frac{5}{18(t+1)}-\frac{1}{6(t+1)^2}+C,$$

bunda, $t=\frac{\sqrt{x^2+3x+2}}{x+1}$.

4.5. Binomial differensiallarni integrallash.

4.1-ta'rif. Ushbu $x^m(a+bx^n)^p dx$-ko'rinishdagi ifodaga, *binomial differensial* deyiladi, bunda a, b - haqiqiy sonlar, m, n, p - lar esa, rasional sonlar.

$$\int x^m(a+bx^n)^p dx \qquad (4.4)$$

ko'rinishdagi integralni hisoblash, quyidagi uchta holda, rasional funksiyani integrallashga keltiriladi:

1-hol. p- butun son. Bu holda m, n kasr sonlar maxrajining eng kichik umumiy bo'linuvchisini λ orqali belgilab, (4.4) integralda $x=t^\lambda$ almashtirish bajarilsa, integral ostidagi ifoda rasional ifodaga aylanib, (4.4) integral rasional funksiyani integrallashga keltiriladi.

2-hol. $\frac{m+1}{n}$- butun son. Bu holda (4.4) integralda $a+bx^n=t^s$, (s son - p kasrning maxraji) almashtirish bajarilsa, integral ostidagi ifoda rasional ifodaga aylanib, (4.4) integralni hisoblash rasional funksiyani integrallashga keltiriladi.

3-hol. $\frac{m+1}{n}+p$-butun son bo'lsin. Bu holda (4.4) integralda, $t^s=ax^{-n}+b$ (s son - p kasrning maxraji) almashtirish bajarilishi natijasida rasional funksiya integralini hisoblashga keltiriladi.

4.6-misol. $\int\sqrt{x}(1+\sqrt[3]{x})^4dx$ integralni hisoblang.

Yechilishi. Integral ostidagi ifoda binomial differensial shaklidagi ifoda bo'lib, bunda $m=\frac{1}{2}, n=\frac{1}{3}, p=4$. p-butun bo'lganligi uchun, 4.5 - banddagi *1*-holga asosan, $t=\sqrt[6]{x},\ x=t^6$ almashtirish olinsa, integral ostidagi ifoda rasional ko'rinishga keladi:

$$\int\sqrt{x}(1+\sqrt[3]{x})^4dx=\int t^3(1+t^2)^4 6t^5dt=6\int t^8(1+4t^2+6t^4+4t^6+t^8)dt=$$

$$=6\left(\frac{t^9}{9}+\frac{4\cdot t^{11}}{11}+6\cdot\frac{t^{13}}{13}+4\cdot\frac{t^{15}}{15}+\frac{t^{17}}{17}\right)+C=$$

$$=\frac{2}{3}t^9+\frac{24}{11}t^{11}+\frac{36}{13}t^{13}+\frac{8}{5}\cdot t^{15}+6\cdot\frac{t^{17}}{17}+C=$$

$$=\frac{2}{3}x\sqrt{x}+\frac{24}{11}x\sqrt[6]{x^5}+\frac{36}{13}x^2\sqrt[6]{x}+\frac{8}{5}x^2\sqrt{x}+\frac{6}{17}x^2\sqrt[6]{x^5}+C.$$

4.7-misol. $\int\frac{\sqrt{1-x^4}}{x^5}dx$ integralni hisoblang.

Yechilishi. Bu holda, $m=-5,\ n=4,\ p=\frac{1}{2}$ bo‘lib, $\frac{m+1}{n}=\frac{-5+1}{4}=-1$ butun bo‘lgani uchun, 4.5 - banddagi 2-holga asosan, $1-x^4=t^2$ almashtirish olinib, integral ostidagi ifoda rasionallashtiriladi:

$$\int\frac{\sqrt{1-x^4}}{x^5}dx=\int x^{-5}\left(1-x^4\right)^{\frac{1}{4}}dx=[x=\left(1+t^2\right)^{\frac{1}{4}},\ dx=-\frac{1}{2}\left(1-t^2\right)^{-\frac{3}{4}}tdt]=$$

$$=-\frac{1}{2}\int\frac{t^2dt}{\left(1-t^2\right)^{5/4}\left(1-t^2\right)^{3/4}}=-\frac{1}{2}\int\frac{t^2}{\left(1-t^2\right)^2}dt=$$

$$=\frac{1}{2}\int\frac{1-t^2-1}{\left(1-t^2\right)^2}dt=\frac{1}{2}\int\frac{dt}{1-t^2}-\frac{1}{2}\int\frac{dt}{\left(1-t^2\right)^2}=$$

$$=-\frac{1}{4}\ln|1-t|+\frac{1}{4}\ln|1+t|+\frac{1}{8}\ln|1-t|-\frac{1}{8(1-t)}-\frac{1}{8}\ln|1+n|+C=$$

$$=\frac{1}{8}\ln\left|\frac{1+t}{1-t}\right|+\frac{1}{8(1+t)}=\frac{1}{8}\ln\left|\frac{1+t}{1-t}\right|-\frac{1}{8}\cdot\frac{t}{1-t^2}+C$$

bunda, $t=\sqrt{1-x^4}$

4.8-misol. $\int\sqrt[3]{x\left(1-x^2\right)}dx$ integralni hisoblang.

Yechilishi. Berilgan integralda $m=\frac{1}{3},n=2,\ p=\frac{1}{3},\ \frac{m+1}{n}+p=1$ butun bo‘lgani uchun, 4.5 -banddagi 3- holga asosan, $\frac{1}{x^2}-1=t^3$ almashtirish bajarib, hisoblaymiz:

$$\int\sqrt[3]{x\left(1-x^2\right)}dx=\int x^{\frac{1}{3}}\left(1-x^2\right)^{\frac{1}{3}}dx=\left|\begin{array}{l}x^2=\frac{1}{1+t^3},\ x=\sqrt{\frac{1}{1+t^3}}\\ dx=-\frac{3}{2}\left(\frac{1}{t^3+1}\right)^{-\frac{3}{2}}t^2dt\end{array}\right|=$$

$$=-\int\left(1+t^3\right)^{-\frac{1}{6}}\cdot\frac{t}{\left(1+t^3\right)^{1/3}}\cdot\frac{3^{t^2}}{2}\left(\frac{1}{t^3+1}\right)^{-\frac{3}{2}}dt=$$

$$=-\frac{3}{2}\int\frac{t^3}{\left(1+t^3\right)^{\frac{1}{6}+\frac{1}{3}+\frac{3}{2}}}dt=-\frac{3}{2}\int\frac{t^3}{\left(1+t^3\right)^2}\cdot dt=$$

$$=-\frac{3}{2}\int\frac{dt}{1+t^3}+\frac{3}{2}\int\frac{dt}{\left(1+t^3\right)^2}=-\frac{3}{2}\int\frac{dt}{(1+t)(t^2-t+1)}+\frac{3}{2}\int\frac{dt}{(1+t)^2\left(t^2-t+1\right)^2}$$

Oxirgi integrallarda, integral ostidagi ifodalar to‘g‘ri kasr bo‘lgani uchun, ularni noma’lum koeffisientli sodda kasrlar yig‘indisi shaklida tasvirlab, so‘ngra noma’lum koeffisientlarni topib, sodda kasrni integrallash usulidan foydalanib, integrallarni hisoblaymiz:

$$\frac{1}{(1+t)(t^2-t+1)}=\frac{A}{1+t}+\frac{Bt+C}{t^2-t+1},$$

bunda, $1=A(t^2-t+1)+(Bt+c)\cdot(1+t)$, $1=A(t^2-t+1)+(Bt+C)\cdot(1+t)$ tenglikda $t=-1$ deb, $A=\frac{1}{3}$ ekanligini topamiz. t ning bir xil darajalari oldidagi koeffisentlarni tenglashtirib, quyidagi

$$\begin{array}{ll} t^2 & 0=A+B \\ t & 0=-A+B+C \\ t^0 & 1=A+C \end{array}$$

sistemani hosil qilamiz. Bundan, $C=\frac{2}{3}$, $B=-\frac{1}{3}$ bo'lishi kelib chiqadi.

Xuddi shunday,

$$\frac{1}{(1+t)^2(t^2-t+1)^2}=\frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{Ct+D}{t^2-t+1}+\frac{Mt-N}{(t^2-t+1)^2}$$

va

$$1=A(1+t)(t^2-t+1)^2+B(t^2-t+1)^2+(ct+D)(1+t)^2(t^2-t+1)+(Mt+N)(1+t)^2$$

tenglikni hosil qilamiz.

Oxirgi tenglikda $t=-1$ deb, $B=\frac{1}{3}$ ekanligini topamiz.

$$1=A(1+t)(t^2-t+1)^2+B(t^2-t+1)^2+(ct+D)(1+t)^2(t^2-t+1)+(Mt+N)(1+t)^2$$

tenglikning ikki tomonini differensiallab, $t=-1$ da nulga aylanmaydigan hadlarni yozib olamiz:

$$0=A(t^2-t+1)^2+B2(2t-1)(t^2-t+1)+\dots$$

Bundan $t=-1$deb olsak, $0=9A-18B$, $A=\frac{2}{3}$ bo'lishi kelib chiqadi.

$$1=A(1+t)(t^2-t+1)^2+B(t^2-t+1)^2+(ct+D)(1+t)^2(t^2-t+1)+(Mt+N)(1+t)^2$$

tenglikning ikkala tomonidagi t ning bir xil darajalari oldidagi koef-fitsiyentlarni tenglashtiramiz va

$$\begin{array}{ll} t^5: & 0=A+C \\ t^4: & 0=-A+B+C+L \\ t^3: & 0=A-2B+D+M \\ t^2: & 0=A+3B+D+2M+N \\ t: & 0=-A-2B+C+D+M+2N \\ t^0: & 1=A+B+D+N \end{array}$$

sistemani hosil qilamiz. Bu sistemadan , $C=-\frac{2}{3}$, $D=1$, $M=-1$, $N=-1$ qiymatlarni topamiz.

Demak,

$$\int\frac{dt}{(1+t)^2(t^2-t+1)^2}=\frac{2}{3}\ln|1+t|-\frac{1}{3(1+t)}-\frac{1}{3}\int\frac{d(t^2-1)}{(t^2-t+1)}+\frac{2}{3}\int\frac{dt}{t^2+t+1}-$$

$$-\frac{1}{2}\int\frac{d(t^2-t+1)}{(t^2-t+1)^2}+\frac{1}{2}\int\frac{dx}{(t^2-t+1)^2}=\frac{2}{3}\ln|1+t|-\frac{1}{3(1+t)}-$$

$$-\frac{1}{3}\ln|t^2-t+1|+\frac{2}{3}\cdot\frac{2}{\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+\frac{1}{2\sqrt{3}}\cdot\frac{\left(t-\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}+$$

$$+\frac{2}{3}\cdot\frac{2}{\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)$$

Shunday qilib, berilgan integral quyidagicha hisoblanadi, ya'ni

$$\int\sqrt[3]{x(1-x^2)}dx=-\frac{1}{2}\ln|1+t|+\frac{1}{4}\ln|t^2-t+1|-\frac{3}{2\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+$$

$$+\ln|1+t|-\frac{1}{2(1+t)}+\frac{1}{2}\ln|t^2-t+1|+\frac{2}{\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+$$

$$+\frac{3}{4\sqrt{3}}\frac{\left(t-\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{2}{\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+C=$$

$$=\frac{1}{2}\ln|1+t|+\frac{3}{4}\ln|t^2-t+1|-\frac{1}{2(1+t)}+\frac{5}{2\sqrt{3}}arctg\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+$$

$$+\frac{3}{4\sqrt{3}}\frac{\left(t-\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}+C,$$

bunda $t=\sqrt[3]{\frac{1}{x^2}-1}$.

4.9- misol. $\int\frac{dx}{\sqrt[4]{1+x^4}}$ integralni hisoblang.

Yechilishi. Bu holda $m=0,\ n=4,\ p=-\frac{1}{4}$ $\quad\frac{m+1}{n}+p=\frac{1}{4}-\frac{1}{4}=0$ bo'lgani uchun, 4.5 - bandning 3- holiga asosan, $t=\sqrt[4]{x^{-4}+1}=\frac{\sqrt[4]{1+x^4}}{x}$ almashtirishni bajarib, berilgan integralni hisoblaymiz:

$$\int\frac{dx}{\sqrt[4]{1+x^4}}=[x=(t^4-1)^{-\frac{1}{4}},\ dx=-t^3(t^4-1)^{-\frac{5}{4}}dx]=$$

$$=-\int\frac{t^2dt}{t^4-1}=\frac{1}{4}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)dt-\frac{1}{2}\int\frac{dt}{t^2+1}=$$

$$=\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right|-\frac{1}{2}arctg\,t+C,$$

bunda $t=\sqrt[4]{x^{-4}+1}$.

Mustaqil yechish uchun misollar

Quyidagi integrallarni hisoblang:

4.1. $\int\frac{dx}{\sqrt{x}+\sqrt[4]{x}}$ **4.2.** $\int\frac{dx}{\sqrt[3]{x}+\sqrt{x}}$

4.3. $\int\frac{dx}{1+\sqrt{x+1}}$ **4.4.** $\int\frac{1+\sqrt{x+1}}{2+\sqrt{x+1}}dx.$

4.5. $\int\frac{dx}{\sqrt[3]{x}+2\sqrt[4]{x}+\sqrt{x}}$ **4.6.** $\int\frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx.$.

4.7. $\int\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx.$ **4.8.** $\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx.$

4.9. $\int\frac{\sqrt{x}-2}{x(\sqrt[3]{x}+1)}dx$ **4.10.** $\int\sqrt{\frac{x-2}{x+2}}\frac{dx}{x}$

4.11. $\int\sqrt{\frac{4x-5}{x+1}}dx.$ **4.12.** $\int(x-2)\sqrt{\frac{1+x}{1-x}}dx.$

4.13. $\int\sqrt{\frac{x}{x+5}}\frac{dx}{x^2}$ **4.14.** $\int\frac{\sqrt[3]{x}+2}{(\sqrt[4]{x}+\sqrt[6]{x})\sqrt{x^5}}dx.$

4.15. $\int\frac{x^2}{(4x-3)\sqrt{4x-3}}dx.$

Quyidagi integrallarni (Eyler almashtirishlaridan foydalanib) hisoblang:

4.16. $\int\frac{dx}{1+\sqrt{x^2+2x+2}}$ **4.17.** $\int\frac{dx}{x\sqrt{4x^2+4x+3}}$ **4.18.** $\int\frac{dx}{(1+x)\sqrt{1+x+x^2}}$

4.19. $\int\frac{dx}{x+\sqrt{x^2-x+1}}$ **4.20.** $\int\frac{dx}{x\sqrt{x^2-x-5}}$ **4.21.** $\int\frac{dx}{x+\sqrt{x^2-x+4}}$

4.22. $\int\frac{dx}{x\sqrt{2+x-x^2}}$ **4.23.** $\int\frac{x\,dx}{\sqrt{(6x-8-x^2)^3}}$

4.24. $\int\frac{dx}{(x-2)\sqrt{(7x-x^2-10)^3}}$ **4.25.** $\int\frac{x^2\,dx}{\sqrt{1-2x-x^2}}$

4.26. $\int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}dx.$ **4.27.** $\int\frac{3x^2-5x}{\sqrt{3-2x-x^2}}dx.$

Quyidagi integrallarni, binomial differensiallarni integrallash usulidan foydalanib,hisoblang:

4.28. $\int\sqrt[3]{x}(1+\sqrt{x})^2dx.$ **4.29.** $\int\frac{1}{\sqrt[6]{x}}(\sqrt[3]{x}-1)^2dx$

4.30. $\int\frac{dx}{\sqrt[4]{x^3}(\sqrt{x}-1)^5}$ **4.31.** $\int\frac{dx}{x(1+\sqrt[3]{x})^2}$

4.32. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx.$

4.33. $\int \frac{\sqrt[3]{1+x^3}}{x^2}dx.$

4.34. $\int \sqrt{x}\sqrt[3]{1+\sqrt[4]{x^3}}\,dx.$

4.35. $\int \frac{dx}{x\sqrt{1+x^3}}.$

4.36. $\int x^3\sqrt[3]{7-3x^2}\,dx.$

4.37. $\int \frac{dx}{\sqrt{x^3}\sqrt[3]{2+\sqrt[4]{x^3}}}.$

4.38. $\int \frac{dx}{x\sqrt[4]{1+x^3}}.$

4.39. $\int \frac{dx}{\sqrt[4]{1+x^4}}.$

4.40. $\int \frac{dx}{x^3\sqrt[5]{1+\frac{1}{x}}}.$

4.41. $\int \frac{dx}{\sqrt[3]{x^2}\left(\sqrt[3]{x}+1\right)^3}.$

4.42 $\int \frac{x^7}{\sqrt{x^2-1}}dx.$

4.43. $\int x^{-11}\left(1+x^4\right)^{-1/2}dx.$

Quyidagi integrallarni, $\int R\left(x,\sqrt{x^2+a^2}\right)dx$, $\int R\left(x,\sqrt{a^2-x^2}\right)dx$ ifodalarni integrallash usulidan foydalanib, hisoblang:

4.44. $\int \sqrt{9-x^2}\,dx.$

4.45. $\int \sqrt{9-16x^2}\,dx.$

4.46. $\int \sqrt{a^2-b^2x^2}\,dx.$

4.47. $\int \frac{dx}{\sqrt{\left(4-x^2\right)^3}}.$

4.48. $\int \frac{dx}{\sqrt{\left(a^2-b^2x^2\right)^3}}.$

4.49. $\int \frac{dx}{x\sqrt{a^2-b^2x^2}}.$

4.50. $\int x\sqrt{\left(a^2-b^2x^2\right)^m}\,dx.$

4.51. $\int x^2\sqrt{a^2-b^2x^2}\,dx.$

4.52. $\int x^2\sqrt{4-x^2}\,dx.$

4.53. $\int x^3\sqrt{\left(a^2-b^2x^2\right)^m}\,dx.$

4.54. $\int \sqrt{4+x^2}\,dx.$

4.55. $\int \sqrt{a^2+b^2x^2}\,dx.$

4.56. $\int \sqrt{\left(9+x^2\right)^3}\,dx.$

4.57. $\int \sqrt{\left(a^2+b^2x^2\right)^3}\,dx.$

4.58. $\int x\sqrt{\left(a^2+b^2x^2\right)^m}\,dx.$

4.59. $\int \frac{\sqrt{a^2+b^2x^2}}{x}dx.$

4.60. $\int \frac{\sqrt{16+x^2}}{x}dx.$

4.61. $\int \frac{\sqrt{25+x^2}}{x^2}dx.$

4.62. $\int \frac{\sqrt{a^2+b^2x^2}}{x^2}dx.$

4.63. $\int \frac{\sqrt{4+x^2}}{x^3}dx.$

Quyidagi integrallarni hisoblang:

4.64. $\int \frac{dx}{\sqrt{\left(9-x^2\right)^3}}.$

4.65. $\int \frac{x^2dx}{\sqrt{25-x^2}}.$

4.66. $\int \frac{x^2dx}{\sqrt{a^2-b^2x^2}}.$

4.67. $\int \frac{x^3}{\sqrt{4-x^2}}dx.$

4.68. $\int \frac{x^3}{\sqrt{a^2 - b^2x^2}}dx.$ **4.69.** $\int \frac{x^4}{\sqrt{a^2 - b^2x^2}}dx.$

4.70. $\int \frac{dx}{x^2\sqrt{a^2 - b^2x^2}}.$ **4.71.** $\int \frac{dx}{x^4\sqrt{a^2 - b^2x^2}}.$

Mustaqil yechish uchun misollarning javoblari

4.1. $2\sqrt{x} - 4\sqrt[4]{x} + 4\ln\left|\sqrt[4]{x} + 1\right| + C.$ **4.2.** $6[\frac{1}{3}\sqrt{x} - \frac{1}{2}\sqrt[3]{x} + \sqrt[6]{x} - \ln\left|1 + \sqrt[6]{x}\right|] + C.$

4.3. $2\sqrt{x+1} - 2\ln\left|\sqrt{x+1} + 1\right| + C.$ **4.4.** $x + 1 - 2\sqrt{x+1} + 2\ln\left|\sqrt{x+1} + 2\right| + C.$

4.5. $2\sqrt{x} - 3\sqrt[3]{x} - 8\sqrt[4]{x} + 6\sqrt[6]{x} + 48\sqrt[12]{x} + + 3\ln\left(1 + \sqrt[12]{x}\right) + \frac{33}{2}\ln\left(\sqrt[6]{x} - \sqrt[12]{x} + 2\right) -$

$-\frac{171}{\sqrt{7}} arctg \frac{2\sqrt[12]{x} - 1}{\sqrt{7}} + C.$

4.6. $6t - 3t^2 + \frac{3}{2}t^4 + \frac{6}{5}t^5 - \frac{6}{7}t^7 + 3\ln\left(1 + t^2\right) - 6arctgt + C, \ t = \sqrt[6]{x+1}.$

4.7. $x + 4\sqrt{x+1} + 4\ln\left|\sqrt{x+1} - 1\right| + C.$ **4.8.** $\frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\ln\left|x + \sqrt{x^2 - 1}\right| + C.$

4.9. $6\,[\sqrt[6]{x} - 2\ln\sqrt[6]{x} + \ln\left(\sqrt[3]{x} + 1\right) - arctg\sqrt[6]{x}] + C$

4.10. $\ln\left|\frac{1+t}{1-t}\right| - 2arctgt + C, \quad t = \sqrt{\frac{x-2}{x+2}}.$

4.11. $\frac{9}{4}\ln\left|\frac{t-2}{t+2}\right| - \frac{9t}{t^2 - 4} + C, \quad t = \sqrt{\frac{4x-5}{x+1}}.$ **4.12.** $\left(1 - \frac{x}{2}\right)\sqrt{1 - x^2} - \arcsin x + C.$

4.13. $-\frac{2}{5}\sqrt{\frac{x+5}{x}} + C.$ **4.14.** $4\sqrt[4]{x} - 6\sqrt[6]{x} + 12\sqrt[12]{x} + 2\ln|x| - 36\ln\left(\sqrt[12]{x} + 1\right) + C.$

4.15. $\frac{-1}{96}\sqrt{(4x-3)^3} + \frac{3}{16}\sqrt{4x-3} - \frac{9}{32\sqrt{4x-3}} + C.$

4.16. $\ln\left(x + 1 + \sqrt{x^2 + 2x + 2}\right) + \frac{2}{x + 2 + \sqrt{x^2 + 2x + 2}} + C.$

4.17. $\frac{1}{\sqrt{3}}\ln\frac{2x + \sqrt{4x^2 + 4x + 3} - \sqrt{3}}{2x + \sqrt{4x^2 + 4x + 3} + \sqrt{3}} + C.$ **4.18.** $-2arctg\frac{1 + x + \sqrt{1 + x - x^2}}{x} + C.$

4.19. $2\ln|t| - \frac{1}{2}\ln|t - 1| + \frac{3}{t+1} - \frac{3}{2}\ln|t + 1| + C,$ bunda $t = \frac{\sqrt{x^2 - x + 1} + 1}{x}.$

4.20. $\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{x^2 - x + 5} + x - \sqrt{5}}{\sqrt{x^2 x + 5} + x + \sqrt{5}}\right| + C.$

4.21. $8\ln|2t + 1| - \frac{1}{2}\ln|t - 1| - \frac{15}{2}\ln|t + 1| + \frac{5}{t+1} + C,$ bunda $t = \frac{\sqrt{x^2 - x + 4} + 2}{x}.$

4.22. $-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2 + x - x^2} + \sqrt{2}}{x} + \frac{1}{2\sqrt{2}}\right| + C.$ **4.23.** $\frac{2(x-2)}{\sqrt{6x - 8 - x^2}} - \frac{\sqrt{6x - 8 - x^2}}{x - 2} + C.$

4.24. $\frac{2}{27}\left(\frac{1}{t} - 2t - \frac{t^3}{3}\right) + C.$ bunda $t = \frac{\sqrt{7x - 10 - x^2}}{x - 2}.$

4.25. $\frac{1}{2}(3-x)\sqrt{1-2x-x^2}+2arc\sin\frac{x+1}{\sqrt{2}}+C.$

4.26. $x\sqrt{x^2-2x+5}-5\ln\left(x-1+\sqrt{x^2-2x+5}\right)+C.$

4.27. $14arc\sin\frac{x+1}{2}-\frac{1}{2}(3x-19)\sqrt{3-2x-x^2}+C.$ **4.28.** $\frac{3}{7}x^{7/3}+\frac{24}{11}x^{11/6}+3x^{4/3}+C.$

4.29. $6\sqrt[6]{x}+\frac{9}{2}\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|-\frac{3\sqrt[6]{x}}{\sqrt[3]{x-1}}+C$ **4.30.** $\frac{-8}{3(\sqrt[8]{x}-1)^3}-\frac{2}{(\sqrt[8]{x}-1)^2}+C.$

4.31. $3\left(\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+\frac{1}{1+\sqrt[3]{x}}\right)+C.$ **4.32.** $2\left(1+\sqrt[3]{x}\right)^{3/2}+C.$

4.33. $\frac{1}{6}\ln\left(\frac{\sqrt[3]{(1+x^3)^2}}{x^2}+\frac{\sqrt[3]{1+x^3}}{x}+1\right)+\frac{1}{\sqrt{3}}arctg\frac{2\sqrt[3]{1+x^3}+x}{\sqrt{3}x}-$

$\frac{1}{3}\ln\left|1-\frac{\sqrt[3]{1+x^3}}{x}\right|-\frac{\sqrt[3]{1+x^3}}{x}+C.$ **4.34.** $\frac{4}{7}\left(1+\sqrt[4]{x^3}\right)^{7/3}-\left(1+\sqrt[4]{x^3}\right)^{4/3}+C.$

4.35. $\frac{1}{3}\ln\left|\frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}\right|+C.$ **4.36.** $\frac{t^7}{42}+\frac{7t^4}{24}+C, \quad t=\sqrt[3]{7-3x^2}.$

4.37. $\sqrt[3]{\left(2x^{-3/4}+1\right)^2}+C.$ **4.38.** $\frac{1}{3}\ln\left|\frac{t-1}{t+1}\right|+\frac{2}{3}arctg\,t+C,$ bunda $t=\sqrt[4]{1+x^3}.$

4.39. $\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-\frac{1}{2}arctg+C,$ bunda $t=\frac{\sqrt[4]{1+x^4}}{x}.$

4.40. $\frac{5}{4}t^4-\frac{5}{9}t^9+C,$ bunda $t=\sqrt[5]{1+\frac{1}{x}}.$ **4.41.** $-\frac{3}{2(\sqrt[3]{x}+1)^2}+C.$

4.42. $\frac{t^7}{7}+\frac{3}{5}t^5+t+C,$ bunda $t=\sqrt{x^2-1}.$

4.43. 16) $-\frac{t^5}{10}+\frac{t^3}{3}-\frac{t}{2}+C,$ bunda $t=\frac{\sqrt{1+x^4}}{x^2}.$ **4.44.** $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}arc\sin\frac{x}{3}+C.$

4.45. $x\frac{\sqrt{9-16x^2}}{2}+\frac{9}{8}arc\sin\frac{3x}{4}+C.$ **4.46.** $\frac{x\sqrt{a^2-b^2x^2}}{2}+\frac{a^2}{2^b}arc\sin\frac{bx}{a}+C.$

4.47. $\frac{x}{4\sqrt{4-x^2}}+C.$ **4.48.** $\frac{x}{a^2\sqrt{a^2-b^2x^2}}+C.$ **4.49.** $-\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-b^2x^2}}{bx}\right|+C.$

4.50. $-\frac{\sqrt{\left(a^2-b^2x^2\right)^{n+2}}}{(m+2)b^2}+C.$ **4.51.** $\frac{2b^2x^3-a^2x}{8b^2}\sqrt{a^2-b^2x^2}+\frac{a^4}{8b^3}\arcsin\frac{bx}{a}+C.$

4.52. $2\arcsin\frac{x}{2}\left(2-x^2\right)\sqrt{4-x^2}+C.$ **4.53.** $\frac{\sqrt{\left(a^2-b^2x^2\right)^{n+4}}}{(m+4)\ b^4}-\frac{a^2\sqrt{\left(a^2-b^2x^2\right)^{m+2}}}{(m+2)\ b^4}+C.$

4.54. $\frac{x\sqrt{4+x^2}}{2}+2\ln\left|x+\sqrt{4+x^2}\right|+C.$ **4.55.** $\frac{x\sqrt{a^2+b^2x^2}}{2}+\frac{a^2}{2b}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|+C.$

4.56. $\frac{45x+2x^3}{8}\sqrt{9+x^2}+\frac{3\cdot 9^4}{8}\ln\left|x+\sqrt{9+x^2}\right|+C.$

4.57. $\frac{5a^2x+2b^2x^3}{8}\sqrt{a^2+b^2x^2}+\frac{3a^4}{8b}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|+C.$

4.58. $\frac{\sqrt{\left(a^2+b^2x^2\right)^{m+2}}}{(m+2)b}+C.$ **4.59.** $\sqrt{a^2+b^2x^2}-a\ln\left|\frac{a+\sqrt{a^2+b^2x^2}}{bx}\right|+C.$

4.60. $\sqrt{16+x^2}-4\ln\left|\frac{4+\sqrt{16+x^2}}{x}\right|+C.$ **4.61.** $-\frac{\sqrt{25+x^2}}{x}+\ln\left|x+\sqrt{25+x^2}\right|+C.$

4.62. $-\frac{\sqrt{a^2+b^2x^2}}{x}+b\ln\left|bx+\sqrt{a^2+b^2x^2}\right|+C.$

4.63. $-\frac{\sqrt{4+x^2}}{2x^2}-\frac{1}{4}\ln\left|\frac{2+\sqrt{4+x^2}}{x}\right|+C.$ **4.64.** $\frac{x}{9\sqrt{9-x^2}}+C.$

4.65. $-\frac{x\sqrt{25-x^2}}{2}+\frac{25}{2}\arcsin\frac{x}{5}+C.$ **4.66.** $-\frac{\sqrt{a^2-b^2x^2}}{2b^2}+\frac{a^2}{2b^3}\arcsin\frac{bx}{a}+C.$

4.67. $-\frac{8+x^2}{3}\sqrt{4-x^2}+C.$ **4.68.** $-\frac{2a^2+b^2x^2}{3b^4}\sqrt{a^2-b^2x^2}+C.$

4.69. $-\frac{2b^2x^3+3a^2x}{8b^4}\sqrt{a^2-b^2x^2}+\frac{3a^4}{8b^5}\arcsin\frac{bx}{a}+C.$

4.70. $-\frac{\sqrt{a^2-b^2x^2}}{a^2x}+C.$ **4.71.** $-\frac{a^2+2b^2x^2}{3a^4x^3}\sqrt{a^2-b^2x^2}+C.$

5-§. Tarkibida trigonometrik funksiyalar qatnashgan ifodalarni integrallash

5. 1. $\int R(\sin x;\cos x)dx$ **ko'rinishdagi integrallarni hisoblash.** Ushbu

$$\int R(\sin x;\cos x)dx \tag{5.1}$$

integralni qaraymiz.

1) $R(\sin x;\cos x)$ - $\sin x$ va $\cos x$ larning rasional funksiyasi bo'lsin. Bu holda (5.1) integralda $t=tg\frac{x}{2}$ $(-\pi<x<\pi)$ *universal* almashtirish olinib, uni hisoblash, t ga nisbatan rasional funksiyaning integralini hisoblashga keltiriladi. Haqiqatan ham, quyidagi

$$\sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t}{1+t^2},\quad dx=\frac{2dt}{1+t^2} \tag{5.2}$$

munosabatlarni e'tiborga olsak, (5.1) integral,

$$\int R(\sin x,\cos x)dx=\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt$$

ko'rinishga keladi.

2) $R(-\sin x;\cos x)=-R(\sin x;\cos x)$ bo'lsa, u holda, $z=\cos x,\ x\in(0;\pi)$ almashtirish bajarilsa, (5.1) integral ostidagi ifoda, z ning rasional funksiyasiga keltiriladi.

3) $R(\sin x; -\cos x) = -R(\sin x; \cos x)$ bo'lsa, u holda, $z = \sin x,\ x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ almashtirish bajarilsa, (5.1) integral ostidagi ifoda, z ning rasional funksiyasiga keltiriladi.

4) $R(-\sin x; -\cos x) = R(\sin x; \cos x)$ bo'lsa, u holda, $z = tgx,\ x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ yoki $z = \cos 2x$ almashtirishlardan biri bajarilsa, (5.1) integral ostidagi ifoda, z ning rasional funksiyasiga keltiriladi.

5.1-misol. Ushbu $\int \frac{dx}{\sin x + 2\cos x + 6}$ integralni hisoblang.

Yechilishi. $t = tg\frac{x}{2}$ universal almashtirish olib va (5.2) formulalarga asosan, uni t ga nisbatan rasional funksiyaning integralini hisoblashga keltiramiz:

$$\int \frac{dx}{\sin x + 2\cos x + 6} = 2\int \frac{dt}{4t^2 + 2t + 8} = \frac{1}{2}\int \frac{dt}{\left(t + \frac{1}{4}\right)^2 + \frac{31}{16}} = \frac{2}{\sqrt{31}} arctg \frac{4}{\sqrt{31}}(t + \frac{1}{4}) + C =$$

$$= \frac{2}{\sqrt{31}} arctg \frac{4}{\sqrt{31}}(tg\frac{x}{2} + \frac{1}{4}) + C.$$

5.2-misol. Ushbu $\int \frac{dx}{\sin^4 x \cos x}$ integralni hisoblang.

Yechilishi. Agar $\frac{1}{\sin^4 x \cos x}$ ifodada $\cos x$ ni $-\cos x$ ga almashtirsak, u holda, uning ishorasi, qarama - qarshi ishoraga o'zgaradi. Shuning uchun, $t = \sin x,\ x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ almashtirishni olish qulay bo'ladi. Bunda,

$$x = \arcsin t,\ dx = \frac{1}{\sqrt{1-t^2}}dt,\ \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1-t^2},$$

va

$$\int \frac{dx}{\sin^4 x \cos x} = \int \frac{1}{t^4\sqrt{1-t^2}} \frac{1}{\sqrt{1-t^2}} dt = \int \frac{dt}{t^4(1-t^2)} =$$

$$= \int \frac{dt}{t^4} + \int \frac{dt}{t^2} + \int \frac{dt}{1-t^2} = -\frac{1}{3t^3} - \frac{1}{t} + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C = -\frac{1}{3\sin x^3} - \frac{1}{\sin x} + \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$

5.3-misol. Ushbu $\int \frac{dx}{\sin^3 x \cos^2 x}$ integralni hisoblang.

Yechilishi. Integral ostidagi ifodada $\sin x$ ni $-\sin x$ ga almashtirgan-da, u o'z ishorasini teskarisiga almashtiradi. Bu holda 2) almashtirishni, ya'ni $t = \cos x,\ x \in (0; \pi)$ deb olish qulay bo'ladi. Bunda,

$$x = \arccos t,\ dx = -\frac{1}{\sqrt{1-t^2}}dt,\ \sin x = \sqrt{1 - \cos^2 x} = \sqrt{1-t^2},\ \sin^3 x = \sqrt{(1-\cos^2 x)^3} = \sqrt{(1-t^2)^3}$$

va

$$\int\frac{dx}{\sin^3 x\cos^2 x}=\int\frac{1}{\sqrt{(1-t^2)^3}}\frac{1}{t^2\sqrt{1-t^2}}dt=\int\frac{1}{t^2(1-t^2)}dt=\int\frac{1}{t^2}dt+\int\frac{1}{1-t^2}dt=$$

$$=-\frac{1}{t}+\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+C=-\frac{1}{\cos x}+\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right|+C.$$

5.4-misol. Ushbu $\int\frac{\sin xdx}{2\sin x+3\cos x}$ integralni hisoblang.

Yechilishi. Integral ostidagi ifoda, $\sin x$ va $\cos x$ larni ,mos ravishda, $-\sin x$ va $-\cos x$ larga almashtirganda, o'z ishorasini o'zgartirmaydi. Bu holda $t=tgx,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ almashtirish bajarilib, berilgan integral ostidagi ifoda t ning rasional funksiyasiga keltiriladi: $x=arctgt,\ dx=\frac{dt}{1+t^2}$ va

$$\int\frac{\sin xdx}{2\sin x+3\cos x}=\int\frac{tgxdx}{2tgx+3}=\int\frac{tdt}{(2t+3)(1+t^2)}.$$

$\frac{t}{(2t+3)(1+t^2)}$ to'g'ri rasional kasrni, noma'lum koeffisientli (3.6) ko'rinishdagi sodda kasrlar yig'indisi shaklida tasvirlaymiz:

$$\frac{t}{(2t+3)(1+t^2)}=\frac{A}{2t+3}+\frac{Bt+C}{1+t^2}.$$

Undagi noma'lum koeffisientlarni topish uchun, yuqorida ko'rsatilgan noma'lum koeffisientlar usulidan foydalanib, chiziqli algebraik tenglamalar sistemasini yechamiz:

$$t=A(1+t^2)+(Bt+C)(2t+3)\Rightarrow\begin{array}{c|c} t^2 & A+2B=0,\\ t & 3B+2C=1,\\ t^0 & A+3C=0,\end{array}\Rightarrow A=-\frac{6}{13},B=\frac{3}{13},C=\frac{2}{13},$$

$$\int\frac{\sin xdx}{2\sin x+3\cos x}=\int\left[\frac{A}{2t+3}+\frac{Bt+C}{1+t^2}\right]dt=-\frac{6}{13}\int\frac{dt}{2t+3}+\frac{1}{13}\int\frac{3t+2}{1+t^2}dt=$$

$$=-\frac{3}{13}\ln|2t+3|+\frac{3}{26}\ln\left|1+t^2\right|+\frac{2}{13}arctgt+C=-\frac{3}{13}\ln|2\sin x+3\cos|+\frac{2}{13}x+C.$$

5.2. $\int\sin\alpha x\cos\beta x\,dx$, $\int\sin\alpha x\sin\beta x\,dx$, $\int\cos\alpha x\cos\beta x\,dx$ ko'rinishidagi integrallarni hisoblash. Bu integrallar ostidagi ifodalarda, quyidagi

$$\sin\alpha x\cos\beta x=\frac{1}{2}[\sin(\alpha+\beta)x+\sin(\alpha-\beta)x], \qquad (*)$$

$$\sin\alpha x\sin\beta x=\frac{1}{2}[\cos(\alpha-\beta)x-\cos(\alpha+\beta)x],$$

$$\cos\alpha x\cos\beta x=\frac{1}{2}[\cos(\alpha+\beta)x+\cos(\alpha-\beta)x]$$

formulalardan foydalanib, ularni integrallash mumkin.

5.5-misol. Ushbu $\int\sin\alpha x\cos\beta xdx$ integralni hisoblang.

Yechilishi. (*) formuladan foydalanib, integralni hisoblaymiz:

$$\int \sin 8x \cos 2x dx = \frac{1}{2}\int (\sin 10x + \sin 6x) dx = -\frac{1}{20}\cos 10x - \frac{1}{12}\cos 6x + C.$$

5.3. $\int \sin^m x \cos^n x dx$ $(n, m \in Z)$ **ko'rinishdagi integrallarni hisoblash.**

I. n, m - lar manfiy bo'lmagan juft sonlar bo'lgan hol. Bu holda, darajani pasaytirish, ya'ni

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \ \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

formulalar qo'llaniladi.

II. n, m - lar natural sonlar bo'lib, hech bo'lmaganda ularning birortasi toq bo'lgan hol. Bu holda, misol uchun, m toq bo'lganda, $\sin x = t$; n toq bo'lganda esa, $\cos x = t$ almashtirish olinadi va $1 - \cos^2 x = \sin^2 x$ yoki $1 - \sin^2 x = \cos^2 x$ formulalarning biridan foydalanishga to'g'ri keladi.

III. n, m - larning ikkalasi ham butun manfiy bo'lib, $|n|$ va $|m|$ sonlar juft yoki toq bo'lgan hol. Bu holda $tgx = t$ yoki $ctgx = t$ almashtirish olinib,

$$1 + tg^2 x = \frac{1}{\cos^2 x}, \quad 1 + ctg^2 x = \frac{1}{\sin^2 x}$$

formulalarning biridan foydalaniladi.

Ushbu

$$\int \frac{dx}{\sin^n x}, n > 0 \ \text{ва} \ \int \frac{dx}{\cos^m}, \ m > 0$$

integrallar ham, *III* holga keltirilib hisoblanadi.

Haqiqatan ham,

$$\int \frac{dx}{\sin^n x} = \frac{1}{2^{n-1}}\int \frac{d\left(\frac{x}{2}\right)}{\sin^n \frac{x}{2} \cdot \cos^n \frac{x}{2}} = \frac{1}{2^{n-1}}\int \frac{du}{\sin^n u \cos^n u}\bigg|_{u=\frac{x}{2}},$$

$$\int \frac{dx}{\cos^m x} = \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin^m\left(x + \frac{\pi}{2}\right)} = \int \frac{du}{\sin^m u}\bigg|_{u=x+\frac{\pi}{2}}.$$

IV. n, m - butun manfiy sonlar bo'lib, $|n|$ va $|m|$ sonlarning biri toq bo'lgan hol. U holda, agar $|m|$ - toq bo'lsa, $\sin x = t$, $|n|$ - toq bo'lganda esa, $\cos x = t$ almashtirish olish maqsadga muvofiq bo'ladi. Ba'zi hollarda $|n|$ va $|m|$ larning darajalari katta bo'lganda integral ostidagi funksiyaning suratida 1 ni $\sin^2 x + \cos^2 x$ yig'indi bilan almashtirish qulay bo'ladi.

V. n - juft son, m - esa, butun manfiy son bo'lgan hol. Bu holda $\sin^2 x = 1 - \cos^2 x$ formuladan foydalanib, integral

$$\int \frac{dx}{\cos^{\alpha} x}, \ \alpha \in N$$

ko'rinishdagi integralga keltiriladi.

m- juft son, n- esa, butun manfiy bo'lganda $\cos^2 x = 1-\sin^2 x$ formuladan foydalanib, hisoblanishi kerak bo'lgan integral,

$$\int \frac{dx}{\sin^{\alpha} x}, \ \alpha \in N$$

ko'rinishdagi integralga keltiriladi.

VI. n - toq, son m - esa, butun manfiy son bo'lsa, bu holda $\cos x = t$ almashtirish olib, $\sin^2 x = 1-\cos^2 x$ formuladan foydalanish kerak.

m - toq, n - esa, butun manfiy son bo'lganda almashtirish olib, $\cos^2 x = 1-\sin^2 x$ formuladan foydalanish kerak.

Ba'zi hollarda, ya'ni $|n|$ va $|m|$ darajalar etarli katta bo'lgan integral ostidagi funksiyaning suratidagi 1 ni $\sin^2 x+\cos^2 x$ ga almashtirish qulay bo'ladi.

5.6 - misol. Quyidagi integrallarni hisoblang:

1) $\int \sin^4 x\cos^2 x dx$. 2) $\int \sin^3 \cos^2 x dx$; 3) $\int \frac{1}{\sin^3 x\cos^5 x}dx$;

4) $\int \frac{\sin^5 x}{\cos^2 x}dx$; 5) $\int \frac{1}{\sin^5 x}dx$; 6) $\int \frac{\cos^3 x}{\sin^2 x}dx$; 7) $\int \frac{1}{\sin^2 x\cos^3 x}dx$.

Yechilishi. 1) Bu erda $n=4$, $m=2$ bo'lgani uchun, 1 - holga asosan, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ formuladan foydalanib, integralni hisoblaymiz:

$$\int \sin^4 x\cos^2 x dx = \frac{1}{4}\int \sin^2 x \cdot \sin^2 2x dx = \frac{1}{4}\int \frac{1-\cos 2x}{2}\cdot\frac{1-\cos 4x}{2}dx =$$

$$= \frac{1}{16}\int (1-\cos 4x-\cos 2x+\cos 2x\cos 4x)dx =$$

$$= \frac{1}{16}x-\frac{1}{64}\sin 4x-\frac{1}{32}\sin 2x+\frac{1}{16}\int(\cos 2x+\cos 6x)dx =$$

$$= \frac{1}{16}x-\frac{1}{32}\sin 2x-\frac{1}{64}\sin 4x+\frac{1}{64}\sin 2x+\frac{1}{162}\sin 6x+C =$$

$$= \frac{1}{16}x-\frac{1}{64}\sin 2x-\frac{1}{64}\sin 4x+\frac{1}{192}\sin 6x+C.$$

2) $\int \sin^3 x \cdot \cos^2 x dx$ integralni hisoblashda II holdagi almashtirishdan foydalanamiz: bunda $n=3$, ya'ni u toq bo'lgani uchun, $\cos x = t$ almashtirishni bajarib, quyidagiga ega bo'lamiz:

$$\int \sin^3 x \cdot \cos^2 x dx = -\int \sin^2 x\cos^2 x d\cos x = -\int \left(1-\cos^2 x\right)\cos^2 x d\cos x =$$

$$= -\int \left(\cos^2 x-\cos^4 x\right)dx = -\frac{\cos^3 x}{3}++\frac{\cos^5 x}{5}+C.$$

3) integralda $n=-3,\ m=-5$ bo‘lib, $|n|$ va $|m|$ larning ikkalasi ham toq bo‘lgani uchun, III holga asosan, $ctgx=t$ almashtirishni bajarib, $1+ctg^2x=\frac{1}{\sin^2 x}$ formuladan foydalangan holda hisoblaymiz:

$$\int\frac{1}{\sin^3 x\cos^5 x}dx=\int\frac{dx}{\sin^2 x\cdot\frac{\cos^5 x}{\sin^5 x}\cdot\sin^6 x}=-\frac{dctgx}{ctg^5x\cdot\frac{1}{(1+ctg^2x)^3}}=$$

$$=-\int\frac{(1+ctg^2x)^3dctgx}{ctg^5x}=-\frac{(1+lu^2)^3du}{u^5}\Bigg|_{u=ctgx}=$$

$$=-\int\left(\frac{1}{u^5}+\frac{3}{u^3}+\frac{3}{u}+u\right)du\Bigg|_{u=ctgx}=\left(+\frac{u^{-4}}{4}+\frac{3}{2}u^{-2}-3\ln|u|-\frac{u^2}{2}+C\right)\Bigg|_{u=ctgx}=$$

$$=\frac{1}{4}\cdot\frac{1}{ctg^4x}+\frac{3}{2}\cdot\frac{1}{ctg^2x}-3\ln|ctgx|+\frac{ctg^2c}{2}+C.$$

4) integralda $n=5,\ m=-2$, ya’ni n - toq, m - butun manfiy bo‘lgani uchun, VI holga asosan, $\cos x=t$ almashtirish olamiz va $\sin^2 x=1-\cos^2 x$ formulani qo‘llab, integralni hisoblaymiz:

$$\int\frac{\sin^5 x}{\cos^2 x}dx=-\int\frac{\sin^4 xd\cos x}{\cos^2 x}=-\int\frac{(1-\cos^2 x)^2d\cos x}{\cos^2 x}=\int\frac{1-2u^2+u^4}{u^2}du\Bigg|_{u=\cos x}=$$

$$=\left(\frac{1}{u}+2u-\frac{u^3}{3}+C\right)\Bigg|_{u=\cos x}=\frac{1}{\cos x}+2\cos x-\frac{\cos^3 x}{3}+C.$$

5) $\int\frac{dx}{\sin^5 x}$ integralni hisoblashda, $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$ formuladan foydalanib, uni quyidagi

$$\int\frac{dx}{\sin^5 x}=\frac{1}{2^4}\int\frac{d\left(\frac{x}{2}\right)}{\sin^5\frac{x}{2}\cos^5\frac{x}{2}}=\frac{1}{24}\int\frac{du}{\sin^5 u\cos^5 u}\Bigg|_{u=\frac{x}{2}}$$

ko‘rinishga keltiramiz. Keyingi integralda $n=-5,\ m=-5$ bo‘lib, n va m larning ikkalasi ham toq bo‘lgani uchun, III - holga asosan, $ctgx=t$ almashtirishni olish qulay bo‘ladi:

$$\int\frac{dx}{\sin^5 x}=\frac{1}{24}\int\frac{dctgu}{ctg^5u\cdot\sin^8 u}=\left[1+ctg^2u=\frac{1}{\sin^2 u}\right]=\frac{1}{16}\int\frac{(1+z^2)^4}{z^5}dz\Bigg|_{z=ctgu}=$$

$$=\frac{1}{16}\int\frac{1+4z^2+6z^4+4z^6+z^8}{z^5}dz=$$

$$=\left(-\frac{1}{64}z^{-4}-\frac{1}{8}z^{-2}+\frac{3}{8}\ln|z|+\frac{1}{8}z^2+\frac{1}{64}z^4+C\right)\Bigg|_{z=ctgu}$$

6) integralda $n=-2,\ m=3$, ya’ni n - butun manfiy, m - toq bo‘lgani uchun, V holga asosan, $\sin x=t$ almashtirish olamiz va $\cos^2 x=1-\sin^2 x$ formuladan foydalanib, integralni hisoblaymiz:

$$\int\frac{\cos^3 x}{\sin^2 x}dx=\int\frac{(1-\sin^2 x)d\sin x}{\sin^2 x}=\int\frac{(1-u^2)du}{u^2}\bigg|_{u=\sin x}=$$

$$=\int u^{-2}du-\int du=\left(-\frac{1}{u}-u+C\right)\bigg|_{u=\sin x}=-\frac{1}{\sin x}-\sin x+C\,.$$

7) integralda $n=-2,\ m=-3$ ya'ni n va m manfiy sonlar bo'lib, $|m|$ toq bo'lgani uchun, IV holga asosan, $\sin x=u$ almashtirishni olib, integralni xisoblaymiz:

$$\int\frac{1}{\sin^2 x\cos^3 x}dx=\int\frac{\cos x}{\sin^2 x\cdot\cos^4 x}dx=\int\frac{d\sin x}{\sin^2 x(1-\sin^2)^2}=\int\frac{du}{u^2\left(1-u^2\right)^2}\bigg|_{u=\sin x}=$$

$$=\int\frac{du}{u^2(1-u)^2(1+u)^2}\,.$$

Mustaqil yechish uchun misollar

Integrallarni hisoblang:

5.1. $\int\cos^2 x dx.$ **5.2.** $\int\cos^3 pxdx.$ **5.3.** $\int\sin^3 pxdx.$

5.4. $\int\cos^4 pxdx.$ **5.5.** $\int\sin^3 xdx.$ **5.6.** $\int tg^3 dx.$

5.7. $\int ct^3 xdx.$ **5.8.** $\int\cos^5 xdx.$ **5.9.** $\int\sin^4 px dx.$

5.10. $\int\cos^7 xdx.$ **5.11.** $\int ctg^4 xdx.$ **5.12.** $\int sh^2 xdx.$

5.13. $\int sh^3 xdx.$ **5.14.** $\int ch^3 x\ shxdx.$ **5.15.** $\int ch^2 xdx.$

5.16. $\int sh^4 xdx.$ **5.17.** $\int ch^3 xdx.$ **5.18.** $\int ch^4 x dx.$

$\int\sin px\cos qx\,dx,$ $\int\sin px\sin qx dx,$ $\int\cos px\ \cos qx\ dx$ ko'rinishdagi integrallarni hisoblang:

5.19. $\int\cos 5x\cos 9x dx.$ **5.20.** $\int\sin 5x\sin 3x dx.$

5.21. $\int\cos 4x\cos x\ dx.$ **5.22.** $\int\sin 7x\cos 3x\ dx.$

5.23. $\int\sin 3x\cos 5x\ dx.$ **5.24.** $\int\cos px\cos qx\ dx.$

5.25. $\int\sin px\sin qx\ dx.$ **5.26.** $\int\sin px\cos qx\ dx.$

5.27. $\int\sin x\sin 2x\sin 3x dx.$ **5.28.** $\int\cos x\ \cos\ \cos 5x\ dx$

5.29. $\int\sin x\sin\frac{x}{2}\sin\frac{x}{3}dx.$ **5.30.** $\int\cos^2 ax\cos^2 bx dx.$

5.31. $\int\sin^3 2x\cos^2 3x dx.$ **5.32.** $\int\cos x\cos 2x\cos 5x dx.$

5.33. $\int sh2x\,sh5x\ dx.$ **5.34.** $\int ch3x\,ch7x\,dx.$

5.35. $\int sh3x\,ch5xdx.$ **5.36.** $\int sh(2x+3)sh(2x+5)dx.$

$\int\frac{dx}{\sin^m x},\ \int\frac{dx}{\cos^m x}$ ko'rinishdagi integrallarni hisoblang:

5.37. $\int \frac{dx}{\sin x}$. 5.38. $\int \frac{dx}{\sin^3 x}$. 5.39. $\int \frac{dx}{\sin^4 x}$.

5.40. $\int \frac{dx}{\sin^5 x}$. 5.41. $\int \frac{dx}{\sin^6 x}$. 5.42. $\int \frac{dx}{\cos x}$.

5.43. $\int \frac{dx}{\cos^3 x}$. 5.44. $\int \frac{dx}{\cos^4 x}$. 5.45. $\int \frac{dx}{\cos^5 x}$.

5.46. $\int \frac{dx}{\cos^6 x}$. 5.47. $\int \frac{dx}{sh^3 x}$. 5.48. $\int \frac{dx}{sh^4 x}$.

5.49. $\int \frac{dx}{ch^3 x}$. 5.50. $\int \frac{dx}{ch^4 x}$. 5.51. $\int \frac{sh^3 x}{chx} dx$.

$\int \frac{\cos^p x}{\sin^q x} dx$, $\int \frac{\sin^p x}{\cos^q x} dx$ ko‘rinishdagi integrallarni hisoblang:

5.52. $\int \frac{\sin^2 x}{\cos x} dx$. 5.53. $\int \frac{\sin^3 x}{\cos x} dx$.

5.54. $\int \frac{\sin^4 x}{\cos x} dx$. 5.55. $\int \frac{\sin x}{\cos^2 x} dx$.

5.56. $\int \frac{\sin^2 x}{\cos^2 x} dx$. 5.57. $\int \frac{\sin^3 x}{\cos^2 x} dx$.

5.58. $\int \frac{\sin^4 x}{\cos^2 x} dx$. 5.59. $\int \frac{\sin^3 x}{\cos^3 x} dx$.

5.60. $\int \frac{\sin^3 x}{\cos^4 x} dx$. 5.61. $\int \frac{\cos^2 x}{\sin x} dx$.

5.62. $\int \frac{\cos^3 x}{\sin x} dx$. 5.63. $\int \frac{\cos^4 x}{\sin x} dx$.

5.64. $\int \frac{\cos^2 x}{\sin^2 x} dx$. 5.65. $\int \frac{\cos^3 x}{\sin^2 x} dx$. 5.66. $\int \frac{\cos^4 x}{\sin^2 x} dx$.

5.67. $\int \frac{\cos^3 x}{\sin^3 x} dx$. 5.68. $\int \frac{\cos^3 x}{\sin^4 x} dx$. 5.69. $\int \frac{\cos^4 x}{\sin^4 x} dx$.

$\int \frac{ch^p x}{sh^q x} dx$, $\int \frac{sh^p x}{ch^q x} dx$ ko‘rinishdagi integrallarni hisoblang:

5.70. $\int \frac{sh^2 x}{chx} dx$. 5.71. $\int \frac{ch^3 x}{chx} dx$. 5.72. $\int \frac{sh^4 x}{chx} dx$.

5.73. $\int \frac{sh^2 x}{ch^2 x} dx$. 5.74. $\int \frac{sh^3 x}{ch^2 x} dx$. 5.75. $\int \frac{sh^2 x}{ch^3 x} dx$.

5.76. $\int \frac{sh^3 x}{ch^4 x} dx$ 5.77. $\int \frac{sh^4 x}{ch^4 x} dx$. 5.78. $\int \frac{sh^2 x}{ch^4 x} dx$.

5.79. $\int \frac{ch^2 x}{shx} dx$. 5.80. $\int \frac{ch^3 x}{shx} dx$. 5.81. $\int \frac{ch^4 x}{shx} dx$.

5.82. $\int \frac{ch^3 x}{shx} dx$. 5.83. $\int \frac{ch^2 x}{sh^2 x} dx$. 5.84. $\int \frac{ch^3 x}{sh^2 x} dx$.

5.85. $\int \frac{ch^4 x}{sh^2 x}dx.$ 5.86. $\int \frac{ch^3 x}{sh^3 x}dx.$ 5.87. $\int \frac{ch^4 x}{sh^3 x}dx.$

5.88. $\int \frac{ch^4 x}{sh^4 x}dx.$ 5.89. $\int \frac{ch^2 x}{sh^4 x}dx.$

$\int \sin^p x \cos^q x dx$ ko'rinishdagi integrallarni hisoblang:

5.90. $\int \sin^2 x \cos^2 dx.$ 5.91. $\int \sin^2 x \cos^3 x dx.$

5.92. $\int \sin^2 x \cos^4 x dx.$ 5.93. $\int \sin^3 x \cos x dx$

5.94. $\int \sin^3 x \cos^2 x dx.$ 5.95. $\int \sin^3 x \cos x dx.$

5.96. $\int \sin^4 x \cos x dx.$ 5.97. $\int \sin^4 x \cos^2 x dx.$

5.98. $\int \sin^4 x \cos^3 x dx.$ 5.99. $\int \sin^4 x \cos^4 x dx.$

$\int sh^p x ch^q x dx$ ko'rinishdagi integrallarni hisoblang:

5.100. $\int shx chx dx.$ 5.101. $\int sh^2 x ch^2 x dx.$

5.102. $\int sh^2 x ch^4 x dx$ 5.103. $\int sh^3 x ch^2 x.$

5.104. $\int sh^3 x ch^3 x dx.$ 5.105. $\int sh^3 x ch^2 x dx.$

5.106. $\int sh^3 x ch^4 x dx.$ 5.107. $\int sh^4 x ch^4 x dx$

$\int \frac{dx}{\sin^p x \cos^q x}$ ko'rinishdagi integrallarni hisoblang:

5.108. $\int \frac{dx}{\sin x \cos x}.$ 5.109. $\int \frac{dx}{\sin \cos^2 x}.$ 5.110. $\int \frac{dx}{\sin x \cos^3 x}.$

5.111. $\int \frac{dx}{\sin \cos^4 x}.$ 5.112. $\int \frac{dx}{\sin^2 x \cos x}.$

5.113. $\int \frac{dx}{\sin^2 x \cos^2 x}.$

5.114. $\int \frac{dx}{\sin^2 \cos^2 x}.$ 5.115. $\int \frac{dx}{\sin^2 x \cos^4 x}.$

5.116. $\int \frac{dx}{\sin^3 x \cos x}.$

5.117. $\int \frac{dx}{\sin^3 \cos^2 x}.$ 5.118. $\int \frac{dx}{\sin^3 x \cos^3 x}.$ 5.119. $\int \frac{dx}{\sin 4x \cos x}.$

5.120. $\int \frac{dx}{\sin^4 x \cos^2 x}.$ 5.121. $\int \frac{dx}{\sin^4 x \cos^4 x}.$ 5.122. $\int \frac{dx}{\sin^4 x \cos^3 x}.$

$\int \frac{dx}{sh^p x ch^q x}$ ko'rinishdagi integrallarni hisoblang:

5.123. $\int \frac{dx}{shx chx}.$ 5.124. $\int \frac{dx}{shx ch^2 x}.$ 5.125. $\int \frac{dx}{shx ch^3 x}.$

5.126. $\int \frac{dx}{shx ch^4 x}.$ 5.127. $\int \frac{dx}{sh^2 x chx}.$ 5.128. $\int \frac{dx}{sh^2 x ch^2 x}.$

5.129. $\int \frac{dx}{sh^3 x\, chx}$. **5.130.** $\int \frac{dx}{sh^3 x\, ch^2 x}$. **5.131.** $\int \frac{dx}{sh^2 x\, ch^3 x}$.

5.132. $\int \frac{dx}{sh^4\, chx}$. **5.133.** $\int \frac{dx}{sh^4 x\, ch^3 x}$. **5.134.** $\int \frac{dx}{sh^4 x\, ch^4 x}$

$\int R(\sin x, \cos x)dx$, (bunda $R-$ rasional funksiya) ko'rinishdagi integrallarni hisoblang:

5.135. $\int \frac{dx}{3+5\cos x}$.

5.136. $\int \frac{dx}{10+8\cos x}$.

5.137. $\int \frac{dx}{5+4\cos x}$

5.138. $\int \frac{dx}{13+5\sin x}$

5.139. $\int \frac{dx}{3+5\sin x}$

5.140. $\int \frac{dx}{5+3\sin x}$.

5.141. $\int \frac{dx}{a+b\cos x}$.

5.142. $\int \frac{dx}{a+b\sin x}$.

5.143. $\int \frac{dx}{4\cos x+3\sin x+5}$.

5.144. $\int \frac{dx}{\sin x+3\cos x+5}$

5.145. $\int \frac{dx}{5-4\sin x+2\cos x}$.

5.146. $\int \frac{3\sin x-2\cos x}{1+\cos x}dx$.

5.147. $\int \frac{dx}{2\sin x+3\cos x+3}$.

5.148. $\int \frac{dx}{3\sin x-4\cos x}$.

5.149. $\int \frac{dx}{a+b\cos x+c\sin x}$

5.150. Ushbu $J_n = \int \frac{dx}{(a\cos x+b\sin x)^n}\left(a^2+b^2 \neq 0, n\in N\right)$ integral uchun

$$J_n = \frac{1}{(n-1)(a^2+b^2)}\left(\frac{a\sin x-b\cos x}{(a\cos x+b\sin x)^{n-1}}+(n-2)J_{n-2}\right),\ n>1$$

rekurrent formulani isbotlang, uning yordamida

$J_3 = \int \frac{dx}{(2\cos x+\sin x)^3}$ integralni hisoblang.

5.151. Ushbu $J_n = \int \frac{dx}{(a\cos x+b)^n}\left(|a| \neq |b|,\ n\in N\right)$ integral uchun

$$J_n = \frac{1}{(n-1)(a^2-b^2)}\left(\frac{a\sin x}{(a\cos x+b)^{n-1}}-(2n-3)bJ_{n-1}+(n-2)J_{n-2}\right),\ n>1$$

rekurrent formulani isbotlang, uning yordamida

a) $\int \frac{dx}{(1+\varepsilon\cos x)^2}, (0<\varepsilon<1)$. b) $\int \frac{dx}{(1+\varepsilon\cos x)^3}, \varepsilon>1$

integrallarni hisoblang.

Mustaqil yechish uchun misollarning javoblari

5.1. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$ **5.2.** $\frac{1}{p}\sin px-\frac{1}{3p}\sin^3 px+C$. **5.3.** $\frac{1}{3p}\cos^3 px-\frac{1}{p}\cos px+C$.

5.4. $\frac{3}{8}x+\frac{1}{4p}\sin 4px+C$. **5.5.** $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$.

5.6. 6) $\frac{1}{2}tg^2x+\ln|\cos x|+C$. **5.7.** $-\frac{1}{2}ctg^2-\ln|\sin x|+C$.

5.8. $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C$. **5.9.** $\frac{3x}{8}-\frac{1}{np}\sin 2px+\frac{1}{32p}\sin 4px+C$.

5.10. $\sin x-\sin^3 x+\frac{3}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C$. **5.11.** $ctgx-\frac{1}{3}ctg^3x+x+C$.

5.12. $\frac{1}{4}sh2x-\frac{x}{2}+C$. **5.13.** $-chx+\frac{1}{3}ch^3x+C$.

5.14. $\frac{1}{4}ch^4x+C$. **5.15.** $\frac{1}{4}sh2x+\frac{x}{2}+C$. **5.16.** $\frac{3}{8}x-\frac{3}{8}shx\,chx+\frac{1}{4}sh^3x\,chx+C$.

5.17. $shx+\frac{1}{3}sh^3x+C$. **5.18.** $\frac{3}{8}x+\frac{3}{8}shx\,chx+\frac{1}{4}shx\,ch^3x+C$.

5.19. $\frac{1}{8}\sin 4x+\frac{1}{28}\sin 4x+C$. **5.20.** $\frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+C$.

5.21. $\frac{1}{6}\sin 3x+\frac{1}{10}\sin 5x+C$. **5.22.** $-\frac{1}{8}\cos 4x-\frac{1}{20}\cos 10x+C$.

5.23. $-\frac{1}{16}\cos 8x+\frac{1}{4}\cos 2x+C$.

5.24. $\frac{\sin(p+q)x}{2(p+q)}+\frac{\sin(p-q)x}{2(p-q)}+C,\ (p^2\neq q^2)$

5.25. $-\frac{\sin(p+q)x}{2(p+q)}+\frac{\sin(q-p)x}{2(q-p)}+C,\ (p^2\neq q^2)$

5.26. $-\frac{\cos(p+q)x}{2(p+q)}-\frac{\cos(p-q)x}{2(p-q)}+C,\ (p^2\neq q^2)$ **5.27.** $\frac{\cos 6x}{24}-\frac{\cos 4x}{16}-\frac{\cos 2x}{8}+C$.

5.28. $\frac{\sin 9x}{36}+\frac{\sin 7x}{28}+\frac{\sin 3x}{12}+\frac{\sin x}{4}+C$.

5.29. $\frac{3}{2}\cos\frac{x}{6}-\frac{3}{10}\cos\frac{5x}{6}-\frac{3}{14}\cos\frac{7x}{6}+\frac{3}{22}\cos\frac{11x}{6}+C$.

5.30. $\frac{x}{4}+\frac{\sin 2ax}{8a}+\frac{\sin 2bx}{8b}+\frac{\sin 2(a-b)x}{16(a-b)}+\frac{\sin 2(a+b)x}{16(a+b)}+C$.

5.31. $-\frac{3}{16}\cos 2x+\frac{3}{64}\cos 4x+\frac{1}{48}\cos 6x-\frac{3}{128}\cos 8x+\frac{1}{192}\cos 2x+C$.

5.32. $\frac{1}{8}\sin 2x+\frac{1}{16}\sin 4x+\frac{1}{24}\sin 6x+\frac{1}{32}\sin 8x+C$. **5.33.** $\frac{1}{14}sh7x-\frac{1}{6}sh3x+C$.

5.34. $\frac{1}{20}sh10x+\frac{1}{8}sh4x+C$. **5.35.** $\frac{1}{16}ch8x-\frac{1}{4}ch2x+C$.

5.36. $-\frac{x}{2}ch2+\frac{1}{8}sh(4x+8)+C$. **5.37.** $\ln\left|tg\frac{x}{2}\right|+C$.

5.38. $-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+C$. **5.39.** $-\frac{\cos x}{3\sin^3 x}-\frac{3}{2}ctgx+C=-\frac{1}{3ctg^3 x}-ctgx+C$.

5.40. $-\frac{\cos x}{4\sin^4 x}-\frac{3}{8}\frac{\cos x}{\sin^2 x}+\frac{3}{8}\ln\left|tg\frac{x}{2}\right|+C$.

5.41. $-\frac{1}{5}ctg^5x-\frac{2}{3}ctg^3x-ctgx+C$. **5.42.** $\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$.

5.43. $\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$. **5.44.** $\frac{\sin x}{3\cos^3 x}+\frac{2}{3}tgx+C=\frac{1}{3}tg^3x+tgx+C$.

5.45. $\frac{\sin x}{4\cos^4 x}+\frac{3}{8}\frac{\sin x}{\cos^2 x}+\frac{3}{8}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$.

5.46. $\frac{\sin x}{5\cos^5 x}+\frac{4}{5}tg^3x+\frac{4}{5}tgx=\frac{1}{5}tg^5x+\frac{2}{3}tg^3x+tgx+C$.

5.47. $-\frac{chx}{2sh^2x}-\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+C$. **5.48.** $-\frac{1}{3}cth^3x+cthx+C$.

5.49. $\frac{shx}{2ch^2x}+\frac{1}{2}arctg(shx)+C$. **5.50.** $-\frac{1}{3}th^3x+thx+C$.

5.51. $\frac{1}{2}ch^2x-\ln chx+C$. **5.52.** $-\sin x+\ln\left|tg\left(\frac{\pi}{2}+\frac{x}{2}\right)\right|+C$

5.53. $\frac{1}{2}\cos^2 x-\ln|\cos x|+C$. **5.54.** $\frac{1}{3}\sin^3 x-\sin x+\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$.

5.55. $\frac{1}{\cos x}+C$. **5.56.** $tgx-x+C$. **5.57.** $\cos x+\frac{1}{\cos x}+C$.

5.58. $tgx+\frac{1}{2}\sin x\cos x-\frac{3}{2}x+C$. **5.59.** $\frac{1}{2\cos^2 x}+\ln|\cos x|+C$.

5.60. $-\frac{1}{\cos x}+\frac{1}{3\cos^3 x}+C$. **5.61.** $\cos x+\ln\left|tg\frac{x}{2}\right|+C$. **5.62.** $\frac{\cos^2 x}{2}+\ln|\sin x|+C$.

5.63. $\frac{1}{3}\cos^3 x+\cos x+\ln\left|tg\frac{x}{2}\right|+C$. **5.64.** $-ctgx+x+C$. **5.65.** $-\sin x-\frac{1}{\sin x}+C$.

5.66. $-ctgx-0{,}5\sin x\cos x-1{,}5x+C$. **5.67.** $-\frac{1}{2\sin^2 x}-\ln|\sin x|+C$.

5.68. $\frac{1}{\sin x}-\frac{1}{3\sin^3 x}+C$. **5.69.** $-\frac{1}{3}ctg^3x+ctgx+x+C$. **5.70.** $shx-arctg(shx)+C$.

5.71. $\frac{1}{2}ch^2x-\ln chx+C$. **5.72.** $\frac{1}{3}sh^3x-shx+arctg(shx)+C$. **5.73.** $x-thx+C$.

5.74. $-\frac{3}{2}x+\frac{1}{4}sh2x+thx+C$. **5.75.** $-\frac{shx}{2ch^2x}+\frac{1}{2}arctg(shx)+C$.

5.76. $-\frac{1}{chx}+\frac{1}{3ch^3x}+C$. **5.77.** $-\frac{1}{3}th^3x-thx+x+C$.

5.78. $\frac{1}{3}th^3x+C$. **5.79.** $chx+\ln\left|th\frac{x}{2}\right|+C$. **5.80.** $\frac{1}{2}ch^2x+\ln|shx|+C$.

5.81. $\frac{1}{3}ch^3x+chx+\ln\left|th\frac{x}{2}\right|+C$. **5.82.** $\frac{1}{2}ch^2x+\ln|shx|+C$. **5.83.** $x-cthx+C$.

5.84. $shx-\frac{1}{shx}+C$. **5.85.** $\frac{3}{2}x+\frac{1}{4}sh2x-cthx+C$. **5.86.** $-\frac{1}{2}cth^2x+\ln|shx|+C$.

5.87. $chx-\frac{chx}{2sh^2x}-\frac{3}{2}\ln\left|th\frac{x}{2}\right|+x+C$. **5.88.** $-\frac{1}{3}cth^3x-cthx+x+C$.

5.89. $-\frac{1}{3}cth^3x+C$. **5.90.** $\frac{1}{8}x-\frac{1}{32}\sin 4x+C$.

5.91. $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$. **5.92.** $\frac{x}{16}+\frac{1}{64}\sin 2x-\frac{1}{64}\sin 4x-\frac{1}{192}\sin 6x+C$.

5.93. $\frac{1}{4}\sin^4 x+C$. **5.94.** $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$.

5.95. $\frac{1}{64}\left(\frac{1}{3}\cos 5x-3\cos 2x\right)+C$. **5.96.** $\frac{\sin^5 x}{5}+C$.

5.97. $\frac{1}{16}x-\frac{1}{64}\sin 2x-\frac{1}{64}\sin 4x+\frac{1}{192}\sin 6x+C$.

5.98. $\frac{1}{7}\sin^3 x\left(\frac{2}{5}+\frac{3}{5}\cos^2 x-\cos^4 x\right)+C$. **5.99.** $\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$.

5.100. $\frac{sh^2x}{2}+C$. **5.101.** $-\frac{x}{8}+\frac{1}{32}sh4x+C$. **5.102.** $\frac{1}{5}sh^3xch^2x+\frac{2}{15}sh^3x+C$.

5.103. $-\frac{1}{16}x-\frac{1}{64}sh2x+\frac{1}{64}sh4x+\frac{1}{192}sh6x+C$.

5.104. $\frac{1}{5}sh^2x\,ch^3x-\frac{2}{15}ch^3x+C$. **5.105.** $\frac{1}{6}sh^6x+\frac{1}{4}sh^4x=\frac{1}{6}ch^6x-\frac{1}{4}ch^4x+C$.

5.106. $\frac{1}{7}sh^2x-\frac{2}{35}sh^2xch^5x+C$. **5.107.** $\frac{3x}{128}-\frac{1}{128}sh4x+\frac{1}{1024}sh8x+C$.

5.108. $\ln|tgx|+C$. **5.109.** $\frac{1}{\cos x}+\ln\left|tg\frac{x}{2}\right|+C$. **5.110.** $\frac{1}{2\cos^2 x}+\ln|tgx|+C$.

5.111. $\frac{1}{\cos x}+\frac{1}{3\cos^3 x}+\ln\left|tg\frac{x}{2}\right|+C$.

5.112. $\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|-\frac{1}{\sin x}+C$. **5.113.** $-2ctg2x+C$.

5.114. $\left(\frac{1}{2\cos^2 x}-\frac{3}{2}\right)\frac{1}{\sin x}+\frac{3}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$. **5.115.** $\frac{1}{3\sin x\cos^3 x}-\frac{8}{3}ctg2x+C$.

5.116. $-\frac{1}{2\sin^2 x}+\ln|tgx|+C$. **5.117.** $-\frac{1}{\cos x}\left(\frac{1}{2\sin^2 x}-\frac{3}{2}\right)+\frac{3}{2}\ln\left|tg\frac{x}{2}\right|+C$.

5.118. $-\frac{2\cos 2x}{\sin^2 2x}+2\ln|tgx|+C$. **5.119.** $-\frac{1}{\sin x}-\frac{1}{3\sin^3 x}+\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$.

5.120. $-\frac{1}{3\cos x\sin^3 x}-\frac{8}{3}ctg 2x+C$. **5.121.** $-8ctg2x-\frac{8}{3}ctg^3 2x+C$.

5.122. $-\frac{2}{\sin x}-\frac{1}{3\sin^3 x}+\frac{\sin x}{2\cos^2 x}+\frac{5}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$. **5.123.** $\ln|thx|+C$.

5.124. $\frac{1}{chx}+\ln\left|tg\frac{x}{2}\right|+C$. **5.125.** $-\frac{1}{2}th^2x+\ln|thx|+C$.

5.126. $\frac{1}{chx}+\frac{1}{3ch^3x}+\ln\left|th\frac{x}{2}\right|+C$. **5.127** $-\frac{1}{shx}-arctg(shx)+C$. **5.128.** $-2cth2x+C$.

5.129. $-\frac{1}{2}cth^2x+\ln|cthx|+C$. **5.130.** $-\frac{1}{chx}-\frac{chx}{2sh^2x}-\frac{3}{2}\ln\left|th\frac{x}{2}\right|+C$.

5.131. $\frac{1}{2}th^2x-\frac{1}{2}cth^2x-2\ln|thx|+C$. **5.132.** $\frac{1}{shx}-\frac{1}{3sh^3x}+arctg(shx)+C$.

5.133. $\frac{2}{shx}-\frac{1}{3sh^3x}+\frac{shx}{2ch^2x}+\frac{5}{2}arctg(shx)+C$.

5.134. $8cth2x-\frac{8}{3}cth^3 2x+C$. **5.135.** $\frac{1}{4}\ln\left|\frac{tg\frac{x}{2}+2}{tg\frac{x}{2}-2}\right|+C$. **5.136.** $\frac{1}{3}arctg\left(\frac{tg\frac{x}{2}}{3}\right)+C$.

5.137. $\frac{2}{3}arctg\left(\frac{tg\frac{x}{2}}{3}\right)+C$. **5.138.** $\frac{1}{6}arctg\left(\frac{13tg\frac{x}{2}+5}{12}\right)+C$.

5.139. $\frac{1}{4}\ln\left|\frac{3tg\frac{x}{2}+1}{3tg\frac{x}{2}+9}\right|+C$. **5.140.** $\frac{1}{2}arctg\left(\frac{5+tg\frac{x}{2}+3}{4}\right)+C$.

5.141.
$$\begin{cases}\frac{2}{\sqrt{a^2-b^2}}arctg\frac{\sqrt{a^2-b^2}\,tg\frac{x}{2}}{a+b}+C,\ a^2>b^2,\\ \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\sqrt{b^2-a^2}\,tg\frac{x}{2}+a+b}{\sqrt{b^2-a^2}\,tg\frac{x}{2}-a-b}\right|+C, a^2<b^2\end{cases}$$

5.142.
$$\begin{cases}\frac{2}{\sqrt{a^2-b^2}}arctg\frac{atg\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C,\ a^2>b^2,\\ \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{atg\frac{x}{2}+b-\sqrt{b^2-a^2}}{atg\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+C, a^2>b^2.\end{cases}$$

5.143. $-\frac{2}{tg\frac{x}{2}+3}+C$.

5.144. $\frac{2}{\sqrt{15}}arctg\left(\frac{1+2tg\frac{x}{2}}{\sqrt{15}}\right)+C$. **5.145.** $\frac{2}{\sqrt{5}}arctg\left(\frac{3t\frac{x}{2}4}{\sqrt{5}}\right)+C$.

5.146. $2tg\frac{x}{2}+3\ln\left|tg^2\frac{x}{2}+1\right|-4arctg\frac{x}{2}+C$. **5.147.** $\frac{1}{2}\ln\left|2tg\frac{x}{2}+3\right|+C$.

5.148. $\frac{1}{5}\ln\left|\frac{tg\frac{x}{2}-\frac{1}{2}}{tg\frac{x}{2}+\frac{1}{2}}\right|+C.$

5.149. $$\begin{cases} \frac{2}{\sqrt{a^2+b^2-c^2}}arctg\frac{(a-b)tg\frac{x}{2}+c}{\sqrt{a^2-b^2-c^2}},\ a^2>b^2+c^2 \\ \frac{1}{\sqrt{b^2+c^2-a^2}\,2}\ln\left|\frac{(a-b)tg\frac{x}{2}+c-\sqrt{b^2+c^2-a^2}}{(a-b)\frac{x}{2}+c+\sqrt{b^2+c^2-a^2}}\right|,a^2<b^2+c^2 \\ \frac{1}{c}\ln\left|a+ctg\frac{x}{a}\right|,\ a=b \\ \frac{-2}{c+(a-b)tg\frac{x}{2}},\ a^2=b^2+c^2 \end{cases}$$

II bob. ANIQ INTEGRAL

6-§. Aniq integralning ta'riflari

6.1. Riman integrali. $f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo'lsin. $[a,b]$ kesmaning $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ shartni qanoatlantiradigan chekli sondagi $\{x_k\}_{k=1}^{n}$ nuqtalar sistemasiga $[a,b]$ kesmaning *bo'linishi* deyiladi va u $P = \{x_k\}_{k=1}^{n}$ kabi belgilanadi. $x_k (k = \overline{1,n})$ nuqta P bo'linishning bo'luvchi nuqtasi $[x_k, x_{k+1}]$ kesma esa, *qism oralig'i* deyiladi. Agar $[a,b]$ kesmaning ixtiyoriy P bo'linishidagi qism oralig'ining uzunliklari bir xil bo'lsa, u holda, bunday bo'linish, $[a,b]$ kesmaning *regular bo'linishi* deyiladi. $d = d(P) = \max\limits_{0 \le k \le n-1} \Delta x_k$ $(\Delta x_k = x_{k+1} - x_k)$, P bo'linishning *diametri,* deb ataladi. Har bir $[x_k, x_{k+1}]$ kesmadan $(k = \overline{0, n-1})$ $\forall \xi_k$ nuqtani olamiz: $x_k \le \xi_k \le x_{k+1}$.

6.1-ta'rif. Ushbu

$$\sigma_P(f) = \sigma_P(f, \{\xi_k\}) = \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k \qquad (6.1)$$

yig'indiga, $f(x)$ funksiyaning, P bo'linishga va ξ_k nuqtani tanlashga mos kelgan, *integral yig'indisi (Riman yig'indisi)* deb ataladi.

6.2-ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ mavjud bo'lib, diametri $d(P) < \delta$ bo'lgan $[a,b]$ kesmaning har qanday P bo'linishida, hamda $\forall \xi_k$ $(x_k \le \xi_k \le x_{k+1})$ nuqtani tanlashga bog'liq bo'lmagan holda,

$$\left| J - \sum_{k=0}^{n-1} \sigma_P(f, \{\xi_k\}) \right| < \varepsilon \qquad (6.2)$$

tengsizlik bajarilsa, u holda, shu J son, *integral yig'indining limiti* deyiladi va u $J = \lim\limits_{d(P) \to 0} \sigma_P(f)$ kabi yoziladi.

6.3- ta'rif. Agar $f(x)$ funksiya uchun, (6.1) integral yig'indining $d(P) \to 0$ da J limiti mavjud bo'lsa, u holda, $f(x)$ funksiya $[a,b]$ kesmada *Riman ma'nosida integrallanuvchi* deyiladi.

Integral yig'indining J limitiga $f(x)$ funksiyadan $[a,b]$ kesma bo'yicha olingan *aniq integral (Riman ma'nosida)* deyiladi va u

$$\int_a^b f(x)dx = J \qquad (6.3)$$

simvol orqali belgilanadi (6.3) da, f - integral ostidagi funksiya, a son-integralning quyi chegarasi, b son esa, integralning yuqori chegarasi, deb ataladi. Integral ostidagi x o'zgaruvchini boshqa o'zgaruvchiga almashtirish ham mumkin, ya'ni

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz$$

va h.k..

Ta'rif bo'yicha, $\int_a^a f(x)dx = 0$, $\int_a^b f(x)dx = -\int_b^a f(x)dx$ ($a < b$ deb olamiz).

6.1-misol. $f(x) = x$ funksiya ixtiyoriy $[a,b]$ kesmada Riman ma'nosida integrallanuvchi ekanligini ko'rsating.

Yechilishi. $[a,b]$ kesmaning $\forall P = \{x_k\}_{k=1}^n$ bo'linishini olamiz. Natijada $[a,b]$ kesma $[x_0, x_1],[x_1, x_2],\ldots,[x_{n-1}, x_n]$ bo'laklarga bo'linadi va $\xi_i = \frac{x_i + x_{i-1}}{2}$ $(\xi_k \in [x_{i-1}, x_i])$, $i = \overline{1,n}$ deb belgilaymiz. P bo'linishga mos kelgan integral yig'indini tuzamiz:

$$\sigma_P(f) = \sigma_P(f, \{\xi_i\}) = \sum_{i=1}^n f(\xi_i)\Delta x_i = \sum_{i=1}^n \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1}) = \sum_{i=1}^n \frac{x_i^2 - x_{i-1}^2}{2} =$$

$$= \frac{1}{2}[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \ldots + (x_n^2 - x_{n-1}^2)] = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(b^2 - a^2).$$

Bundan $\lim\limits_{d(P)\to 0} \sigma_P(f) = \frac{b^2 - a^2}{2} = \int_a^b xdx.$

Demak, $f(x) = x$ funksiya ixtiyoriy $[a,b]$ kesmada Riman ma'nosida integrallanuvchi ekan.

Aniq integralning ta'rifidan, har qanday Riman ma'nosida integrallanuvchi funksiya chegaralangan bo'lishiga ishonch hosil qilish qiyin emas, lekin har qanday chegaralangan funksiya har doim ham integrallanuvchi bo'lavermaydi.

6.2 - misol. Ushbu

$$D(x) = \begin{cases} 1, & x \text{ ratsional son bo'lganda}, \\ 0, & x \text{ irratsional son bo'lganda} \end{cases}$$

Dirixle funksiyasi $[a,b] \subset R$ $(a < b)$ kesmada Riman ma'nosida **integrallanuvchi emasligini ko'rsating.**

Yechilishi. $[a,b]$ kesmaning $\forall P$ bo'linishini olib, quyidagi

$$\sigma_P(D;\{\xi_k\}) = \sum_{k=1}^{n} D(\xi_k)\cdot\Delta x_k,$$

$$\sigma_P(D;\{\eta_k\}) = \sum_{k=1}^{n} D(\eta_k)\Delta x_k$$

yig'indilarni tuzamiz. $\xi_k \in [x_{k-1}, x_k]$ nuqta sifatida $[x_{k-1}, x_k]$ kesmadagi ixtiyoriy rasional nuqtani, η_k sifatida esa ($\eta_k \in [x_{k-1}, x_k]$), shu kesmadagi ixtiyoriy irrasional nuqtani olamiz. U holda, $D(\xi_k)=1,\ D(\eta_k)=0$ bo'ladi. Shuning uchun,

$$\sigma_P(D;\{\xi_k\}) = \sum_{k=1}^{n}\Delta x_k = b-a, \quad \sigma_P(D;\{\eta_k\}) = 0$$

Demak, $b-a \neq 0$ uchun Dirixle funksiyasining integral yig'indisi, 6.2-ta'rifga binoan, limitga ega emas. Shuning uchun, Dirixle funksiyasi $[a,b]$ kesmada integrallanuvchi emas.

6.3-misol. Ushbu

$$f(x) = \begin{cases} x, & x \text{ ratsional son bo'lganda,} \\ -x, & x \text{ irratsional son bo'lganda} \end{cases}$$

funksiyaning $[a,b] \subset R$ $(a<b)$ kesmada Riman ma'nosida integrallanuvchi emasligini ko'rsating.

Yechilishi. $P - J = [a,b]$ kesmaning ixtiyoriy bo'linishi bo'lsin. Unda $J=[a,b]$ kesma $J_1, J_2, ..., J_n$ kesmalarga bo'linadi. $\xi_k \in [x_{k-1}, x_k] = J_k$ nuqta sifatida, $[x_{k-1}, x_k]$ kesmadagi ixtiyoriy rasional nuqtani, η_k ($\eta_k \in [x_{k-1}, x_k]$) sifatida esa, shu kesmadagi ixtiyoriy irrasional nuqtani olamiz. U holda,

$$\sigma_P(f;\{\xi_k\}) = \sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k = \sum_{k=1}^{n}\xi_k\cdot\Delta x_k,$$

$$\sigma_P(f;\{\eta_k\}) = \sum_{k=1}^{n} f(\eta_k)\Delta x_k = \sum_{k=1}^{n}(-\eta_k)\cdot\Delta x_k$$

yig'indilarni tuzamiz. Bunda $\sigma_P(f;\{\xi_k\})$ yig'indi $f_1(x)=x$ funksiya uchun integral yig'indi bo'ladi va u 6.1-misolga asosan,

$$\lim_{d(P)\to 0}\sigma_P(f;\{\xi_k\}) = \frac{b^2-a^2}{2} = \int_a^b x dx.$$

bo'ladi. $\sigma_P(f;\{\eta_k\})$ yig'indi esa, $f_2(x)=-x$ funksiya uchun integral yig'indi bo'lib,

$$\lim_{d(P)\to 0}\sigma_P(f;\{\eta_k\}) = -\frac{b^2-a^2}{2} = -\int_a^b x dx.$$

bo'ladi. Shunday qilib, berilgan integral yig'indi yagona limitga ega emas.

Demak, berilgan funksiya $[a,b]$ kesmada Riman ma'nosida integrallanuvchi emas.

6.4-misol. $[a,b]$ kesmada chegaralanmagan funksiyaning Riman ma'nosida integrallanuvchi emasligini isbotlang.

Yechilishi. $f(x)$ funksiya $[a,b]$ kesmada chegaralanmagan bo'lsin. $[a,b]$ kesmani chekli $P=\{x_k\}_{k=1}^n$ nuqtalar sistemasi yordamida chekli sondagi $[x_{k-1},x_k]$ kesmalarga bo'lamiz, u holda, $f(x)$ funksiya shu kesmalarning, hech bo'lmaganda, birida chegaralanmagan bo'ladi. Umumiylikni buzmaslik uchun, $f(x)$ funksiya $[x_0,x_1]$ kesmada chegaralanmagan bo'lsin, deb faraz qilamiz. Qolgan $[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$ kesmalardan $\forall\xi_2,\xi_3,...,\xi_n$ nuqtalarni olib, ularni belgilaymiz va ushbu

$$\sigma_P(f)=\sigma_P(f,\{\xi_k\})=f(\xi_2)\Delta x_2+...+f(\xi_n)\Delta x_n$$

yig'indini tuzamiz. Endi $f(x)$ funksiyani $[x_0,x_1]$ kesmada qaraymiz. $f(x)$ funksiya $[x_0,x_1]$ kesmada chegaralanmaganligi uchun, ixtiyoriy oldindan berilgan $M>0$ son uchun, bu kesmadan shunday ξ_1 nuqta topiladiki, bu nuqtada $|f(\xi_1)|\geq\frac{(\widetilde{\sigma}_P+M)}{\Delta x_1}$ bo'ladi. Bundan, $|f(\xi_1)|\cdot\Delta x_1\geq|\widetilde{\sigma}_P|+M$ bo'ladi. Shuning uchun, $\forall P=\{x_k\}_{k=1}^n$ bo'linishga mos kelgan integral yig'indi,

$$|\sigma_P(f,\{\xi_k\})|=\left|\sum_{k=1}^n f(\xi_k)\Delta x_k\right|=|f(\xi_1)\Delta x_1+\overline{\sigma}_P(f)|\geq|f(\xi_1)|\Delta x_1-|\overline{\sigma}_P(f)|\geq M$$

tengsizlikni qanoatlantiradi.

$\lim\limits_{n\to\infty}M_n=+\infty$ bo'lgan $\{M_n\}$ sonlar ketma–ketligini, hamda $[a,b]$ kesmaning diametrlari $d_n(p)\to 0$ bo'lgan $P_1,P_2,...,P_n...$ bo'linishlari ketma - ketligini qaraylik. Bu bo'linishlarga mos va $|\sigma_P^n(f)|\geq M_n$ shartni qanoatlantiruvchi $\{\sigma_P^n(f)\}$ integral yig'indilar ketma-ketligini ham qaraymiz. Bu integral yig'indilar ketma-ketligi uzoqlashuvchi bo'ladi, ya'ni $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lmaydi. Shunday qilib, yuqoridagi ta'rif va 6.2 - 6.4- misollardan, quyidagi xulosani chiqarish mumkin.

Agar $f(x)$ funksiya $[a,b]$da integrallanuvchi bo'lsa, u holda, u shu kesmada chegaralangan bo'ladi. Bu tasdiqning teskarisi o'rinli emas (6.2- misolga qarang), ya'ni $f(x)$ funksiyaning integrallanuvchi bo'lishi uchun, uning chegaralanganligi *zaruriy shart* bo'lib, lekin *etarli shart* bo'la olmas ekan.

Aniq integralni, ta'rif bo'yicha, ya'ni integralni integral yig'indining limiti sifatida qarab, hisoblashga doir misollar keltiramiz:

6.5 -misol. Ushbu $\int\limits_0^{\frac{\pi}{2}} \cos dx$ integralni ta'rif bo'yicha hisoblang.

Yechilishi. $[0;\frac{\pi}{2}]$ kesmaning regular P bo'linishini, ya'ni $P=\left\{x_0=0,\ x_1=\frac{\pi}{2n},\dots,x_n=\frac{\pi}{2}\right\}$ bo'linishni qaraymiz. U holda, $\Delta x_k = x_{k+1}-x_k=\frac{\pi}{2n}$ bo'ladi. ξ_k nuqta sifatida, $[x_k, x_{k+1}]$ kesmaning o'ng chetki nuqtasini olamiz, ya'ni $\xi_k = x_{k+1}$. Funksiyaning ξ_k nuqtalardagi qiymatlarini hisoblaymiz:

$$f(\xi_0)=\cos\xi_0=\cos\frac{\pi}{2n},\dots,f(\xi_k)=\cos\xi_k=\frac{(k+1)\pi}{2n},\dots,f(\xi_{n-1})=\cos\xi_{n-1}=0$$

Endi $\cos x$ funksiyaning $[0;\frac{\pi}{2}]$ kesma uchun integral yig'indisini tuzib, uning $n\to\infty$ da limitini hisoblaymiz:

$$\sigma_P(f)=\sigma_P(\cos x)=\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k=\sum_{k=0}^{n-1}\cos\xi_k\cdot\frac{\pi}{2n}=$$

$$=\frac{\pi}{2n}[\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\dots+\cos\frac{(n-1)\pi}{4n}]=\frac{\pi\cdot\sin\frac{\pi}{4}\sin\frac{(n-1)\pi}{4n}}{2n\cdot\sin\frac{\pi}{4n}},$$

$$\lim_{n\to\infty}\sigma_P(f)=\lim_{n\to\infty}\frac{\pi\cdot\sin\frac{\pi}{4}\cdot\sin\frac{(n-1)\pi}{4n}}{2n\cdot\sin\frac{\pi}{4n}}.$$

Bunda, $\lim\limits_{n\to\infty}2n\cdot\sin\frac{\pi}{4n}=\frac{\pi}{2}$, $\sin x$ ning uzluksizligini e'tiborga olsak, u holda,

$$\int\limits_0^{\frac{\pi}{2}}\cos dx=\lim_{n\to\infty}\sigma_P(\cos x)=\frac{\pi\cdot\sin^2\frac{\pi}{4}}{\frac{\pi}{2}}=1.$$

6.6. -misol. Ushbu

$$\int\limits_a^b x^m dx \quad (0<a<b),\ m\neq-1$$

integralni ta'rif bo'yicha hisoblang.

Yechilishi. $[a,b]$ kesmaning regular bo'lmagan bo'linishini, ya'ni $P=\left\{x_0=a<x_1<x_2<\dots<x_n=b \text{ бунда } x_k=aq^k,\ q=\sqrt[n]{\frac{b}{a}}\right\}$ bo'linishni qaraymiz.

ξ_k nuqta sifatida, $[x_k, x_{k+1}]$ kesmaning chap chetki nuqtasini olamiz, ya'ni $\xi_k = x_k$. U holda, integral yig'indining ko'rinishi quyidagicha bo'ladi:

$$\sigma_P(f) = \sigma_P(x^m) = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k = \sum_{k=0}^{n-1} \xi^m (x_{k+1} - x_k) =$$

$$= \sum_{k=0}^{n-1} (aq^k)^m a(q^{k+1} - q^k) = a^{m+1}(q-1)\sum_{k=0}^{n-1} q^{k(m+1)}$$

Bundan, geometrik progressiya hadlari yig'indisi formulasini hisobga olgan holda,

$$\sigma_p(x^m) = \frac{a^{m+1}(q-1)\left[\left(\frac{b}{a}\right)^{m+1} - 1\right]}{q^{m+1} - 1} = \frac{(b^{m+1} - a^{m+1})(q-1)}{q^{m+1} - 1}$$

munosabatni hosil qilamiz, bunda $n \to \infty$ da $q = \sqrt[n]{\frac{b}{a}} \to 1$ va $\lim\limits_{\lambda \to 0} \frac{(1+\lambda)^\mu - 1}{\lambda} = \mu$ formulani e'tiborga olsak,

$$\int_a^b \frac{dx}{x^m} = \lim_{n \to \infty} \sigma_n(x^m) = (b^{m+1} - a^{m+1}) \lim_{q \to 1} \frac{q-1}{q^{m+1} - 1} = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

$m = -1$ bo'lgan holda,

$$\sigma_p(x^{-1}) = n(q-1) = n\left(\sqrt[n]{\frac{b}{a}} - 1\right)$$

bo'ladi. Buyerda $\lim\limits_{\lambda \to 0} \frac{a^\lambda - 1}{\lambda} = \ln a$ formulani e'tiborga olsak, u holda,

$$\int_a^b \frac{dx}{x} = \lim_{n \to \infty} \sigma_p(t) = \lim_{n \to \infty} \sigma_P\left(\frac{1}{x}\right) = \lim_{n \to \infty} n\left(\sqrt[n]{\frac{b}{a}} - 1\right) = \ln b - \ln a$$

ekanligi kelib chiqadi.

6.7-misol. Ushbu $\int_0^1 e^x dx$ aniq integralni ta'rif bo'yicha hisoblang.

Yechilishi. $f(x) = e^x$ funksiya $[0,1]$ kesmada aniqlangan. $[0;1]$ kesmaning

$P = \{x_0 = 0 < x_1 = \frac{1}{n} < .... < x_k = \frac{k}{n} < ... < x_n = \frac{n}{n} = 1\}$ regular bo'linishini qaraymiz.

Ravshanki, $[x_k; x_{k+1}]$ kesmaning uzunligi, $\Delta x_k = \frac{1}{n}$. ξ_k nuqta sifatida, kesmaning chap chetki nuqtasini olamiz, ya'ni $\xi_k = x_k = \frac{k}{n} (k = 0,1,...,n-1)$.

Endi shu P bo'linishga mos kelgan integral yig'indini tuzamiz:

$$\sigma_p(e^x) = \sum_{k=0}^{n-1} e^{\frac{k}{n}} = \left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + ... + e^{\frac{n-1}{n}}\right) \cdot \frac{1}{n}.$$

$1+e^{\frac{1}{n}}+...+e^{\frac{n-1}{n}}$ yig'indi, birinchi hadi $b_0=1$, maxraji $q=e^{\frac{1}{n}}$ bo'lgan geometrik progressiya n ta hadining yig'indisidan iborat bo'lgani uchun, $1+e^{\frac{1}{n}}+...+e^{\frac{n-1}{n}}=\frac{e-1}{e^{\frac{1}{n}}-1}$.

Shunday qilib,

$$\sigma_P(e^x)=\sigma_P\left(e^x,\{\frac{k}{n}\}\right)=\frac{1}{n}\cdot\frac{e-1}{e^{\frac{1}{n}}-1}$$

Bundan,

$$\lim_{n\to\infty}\sigma_P\left(e^x,\{\frac{k}{n}\}\right)=(e-1)\lim_{n\to+\infty}\frac{\frac{1}{n}}{e^{\frac{1}{n}}-1}=e-1$$

Demak,

$$J=\int_0^1 e^x dx=e-1.$$

6.2. Darbu yig'indilari. Biz bundan keyin, hamma vaqt chegaralangan funksiyalarni qaraymiz, chunki chegaralanmagan funksiya Riman ma'nosida integrallanuvchi emas (6.4 – misolga qarang).

$f(x)$ funksiya $[a,b]$ kesmada chegaralangan bo'lsin. $[a,b]$ kesmaning ixtiyoriy $P=\{a=x_0<x_1<x_2<...<x_n=b\}$ bo'linishini qaraymiz. $f(x)$ funksiya $[a,b]$ kesmada chegaralangan bo'lgani uchun, u $\forall[x_{k-1},x_k]$ kesmada ham chegaralangan bo'ladi. $f(x)$ funksiyaning $[x_{k-1},x_k]$ kesmadagi m_k aniq quyi, M_k aniq yuqori chegaralari, ya'ni $m_k=\inf\limits_{x_{k-1}\le x\le x_k} f(x)$, $M_k=\sup\limits_{x_{k-1}\le x\le x_k} f(x)$ mavjud bo'ladi va $\forall\xi_k\in[x_{k-1},x_k]$ uchun $m_k\le f(\xi_k)\le M_k$ tengsizliklar o'rinli.

6.4-ta'rif. Ushbu

$$S_P(f)=M_1\Delta x_1+M_2\Delta x_2+...+M_n\Delta x_n=\sum_{k=1}^{n}M_k\Delta x_k \qquad (6.3)$$

$$s_P(f)=m_1\Delta x_1+m_2\Delta x_2+....+m_n\Delta x_n=\sum_{k=1}^{n}m_k\Delta x_k \qquad (6.4)$$

yig'indilar, mos ravishda, $f(x)$ funksiyaning, berilgan $P=\{x_k\}_{k=1}^n$ bo'linishga mos kelgan, *quyi va yuqori Darbu yig'indilari* deyiladi.

Darbu yig'indilari quyidagi xossalarga ega:

1-xossa. Darbuning har qanday (6.4) quyi yig'indisi, Darbuning (6.3) yuqori yig'indisidan oshmaydi: $s_P(f)\le S_P(f)$.

2-xossa. $\forall\xi_k$ $(x_k \le \xi_k \le x_{k+1})$ nuqtani tanlashga bog'liq bo'lmagan holda, Darbu yig'indilari uchun, $s_P(f) \le \sigma_P(f) \le S_P(f)$ tengsizliklar o'rinli.

3-xossa. Har qanday P bo'linish uchun,

$$S_P(f) = \sup_{\xi_k \in [x_{k-1}, x_k]} \sigma_P(f;\{\xi_k\}), \quad s_P(f) = \inf_{\eta_k \in [x_{k-1}, x_k]} \sigma_P(f;\{\eta_k\})$$

munosabatlar o'rinli.

4-xossa. Darbuning (6.4) quyi yig'indilari to'plami yuqoridan chegaralangan, (6.3) yuqori yig'indilari to'plami esa, quyidan chegaralangan.

5-xossa. Agar $P-[a,b]$ kesmaning ixtiyoriy bo'linishi bo'lib, P' esa, P bo'linishning nuqtalariga chekli sondagi yangi bo'linish nuqtalari qo'shishdan hosil bo'lgan bo'linish bo'lsa, u holda,

$$s_P(f) \le s_{P'}(f), \quad S_{P'}(f) \le S_P(f) \qquad (6.6)$$

munosabatlar o'rinli bo'ladi.

6.4-ta'rif. $\{s_P(f)\}$ to'plamning aniq yuqori chegarasi, $f(x)$ funksiyadan $[a,b]$ kesma bo'yicha olingan Darbuning *quyi integrali* deb ataladi va u,

$$\sup_P\{s_P(f)\} = \underline{J} = \underline{\int_a^b} f(x)dx$$

kabi belgilanadi.

6.4$'$-ta'rif. $\{S_P(f)\}$ to'plamning aniq quyi chegarasi, $f(x)$ funksiyadan $[a,b]$ kesma bo'yicha olingan Darbuning *yuqori integrali* deb ataladi va u,

$$\inf_P\{S_P(f)\} = \overline{J} = \overline{\int_a^b} f(x)dx$$

kabi belgilanadi.

6-xossa. Darbuning quyi integrali, uning yuqori integralidan katta bo'la olmaydi, ya'ni

$$\underline{J} \le \overline{J}.$$

6.5-ta'rif. Agar $f(x)$ funksiyaning $[a,b]$ kesmadagi quyi va yuqori Darbu integrallari bir – biriga teng bo'lsa, u holda $f(x)$ funksiya $[a,b]$ kesmada *Darbu ma'nosida integrallanuvchi* deyiladi va ularning umumiy qiymati

$$\underline{J} = \overline{J} = J,$$

$f(x)$ funksiyaning $[a,b]$ kesmadagi aniq integrali (Darbu ma'nosida) deyiladi va u

$$J = \int_a^b f(x)dx$$

kabi belgilanadi.

Agar $\underline{J} \neq \overline{J}$ bo'lsa, u holda, $f(x)$ funksiya $[a,b]$ kesmada *integrallanuvchi emas* deyiladi.

6.3 -va 6.5 - ta'riflar, o'zaro teng kuchli ta'riflardir.

6.8-misol. $f(x)=x^3$ funksiya uchun, $[-2;3]$ kesmada, ixtiyoriy regular P bo'linishga mos kelgan, quyi va yuqori Darbu yig'indilari tuzilsin.

Yechilishi. $[-2;3]$ kesmaning ushbu $P=\{x_0=-2<x_1=-2+\frac{5}{n}<x_2=-2+\frac{10}{n}<...<x_k=-2+\frac{5k}{n}<...<x_n=3\}$ regulyar bo'linishini qaraymiz, bunda $[x_{k-1},x_k]$ kesmaning uzunligi $\Delta x_k=\frac{5}{n}$ $(k=\overline{1,n})$, $x_k=-2+\frac{5k}{n}$. (6.3) va (6.4) formulalarga asosan, Darbu yig'indilarini tuzamiz.

Berilgan funksiya $[-2;3]$ kesmada o'suvchi bo'lganligi uchun, u o'zining aniq quyi va aniq yuqori chegarasiga, mos ravishda, kesmaning chap va o'ng chetki nuqtalarida erishadi, ya'ni

$$m_k=\left(-2+\frac{5}{n}(k-1)\right)^3,\ M_k=\left(-2+\frac{5}{n}k\right)^3.$$

Shunday qilib,

$$s_P(x^3)=\sum_{k=0}^{n-1}\left(-2+\frac{5}{n}k\right)^3\cdot\frac{5}{n},\qquad S_P(x^3)=\sum_{k=1}^{n}\left(-2+\frac{5k}{n}\right)^3\cdot\frac{5}{n}$$

Endi, Darbu yig'indilarini hisoblaymiz:

$$S_P(x^3)=\frac{5}{n}\sum_{k=1}^{n}\left(-2+\frac{5}{n}k\right)^3=\frac{5}{n}\sum_{k=1}^{n}[\frac{125}{n^3}k^3-3\cdot\frac{25}{n^2}\cdot k^2\cdot 2+$$

$$+3\frac{5}{n}k\cdot 4-8]=\frac{625}{n^4}\sum_{k=1}^{n}k^3-\frac{150\cdot 5}{n^3}\sum_{k=1}^{n}k^2+\frac{300}{n^2}\sum_{k=1}^{n}k-40$$

Ma'lumki,

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2},\quad \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6},\quad \sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

Bu formulalarni e'tiborga olib, oxirgi tenglikdan,

$$S_P(x^3)=16\frac{1}{4}+\frac{175}{2n}+\frac{125}{n^2}.$$

bo'lishini olamiz.

Xuddi shunday,

$$s_P(x^3)=16\frac{1}{4}-\frac{175}{2n}+\frac{125}{n^2}$$

ifoda topiladi.

6.9-misol. $f(x)=2x,\ x\in[0,1],\ P=\left\{0,\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{6}{4},1\right\}$ va $\xi_1=\frac{1}{16},\xi_2=\frac{3}{16},\ \xi_3=\frac{3}{8},$ $\xi_4=\frac{5}{8},\xi_5=\frac{3}{4}$ bo'lsin. U holda:

a) $\Delta x_1,\ \Delta x_2,\ \Delta x_3,\ \Delta x_4,\ \Delta x_5$ qiymatlar;

b) $\|\lambda_P\|=\max\limits_{1\le i\le 5}|\Delta x_i|$;

c) $m_1,\ m_2,\ m_3,\ m_4,\ m_5$ qiymatlar;

d) $f(\xi_1),\ f(\xi_2),\ f(\xi_3),\ f(\xi_4),\ f(\xi_5)$, qiymatlar;

e) $M_1,\ M_2,\ M_3,\ M_4,\ M_5$, qiymatlar

f) $s_P(f)$ quyi Darbu yig'indisi;

g) $S^*(f)$ Riman yig'indisi;

n) $S_P(f)$ yuqori Darbu yig'indisi topilsin va

i) $s_P(f)\le S^*(f)\le S_P(f)$ munosabatning bajarilishi ko'rsatilsin.

Yechilishi. Berilgan $f(x)=2x$ funksiya, $[0;1]$ kesmada uzluksiz, o'suvchi funksiyadan iborat. $[0;1]$ kesma P bo'linish yordamida,5 ta,

$$\left[0,\frac{1}{8}\right],\left[\frac{1}{8},\frac{1}{4}\right],\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},1\right]$$

qism kesmalarga bo'lingan.

a) qism kesmalarning uzunliklarini hisoblaymiz:

$$\Delta x_1=x_1-x_0=\frac{1}{8}-0=\frac{1}{8};\quad \Delta x_2=x_2-x_1=\frac{1}{4}-\frac{1}{8}=\frac{1}{8};$$

$$\Delta x_3=x_3-x_2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4};\ \Delta x_4=x_4-x_3=\frac{3}{4}-\frac{1}{2}=\frac{1}{4};\ \Delta x_5=1-\frac{3}{4}=\frac{1}{4}.$$

Demak,

$$\Delta x_1=\frac{1}{8},\ \Delta x_2=\frac{1}{8};\ \Delta x_3=\Delta x_4=\Delta x_5=\frac{1}{4}\ \text{ ekan.}$$

b) $\|\lambda_P\|=\max\left\{\frac{1}{8},\frac{1}{4}\right\}=\frac{1}{4};$

c) $m_1=f(0)=0;\ m_2=f\left(\frac{1}{8}\right)=\frac{1}{4};\ m_3=f\left(\frac{1}{4}\right)=\frac{1}{2};$

$$m_4=f\left(\frac{1}{2}\right)=1;\quad m_5=f\left(\frac{3}{4}\right)=\frac{3}{2};$$

d) $f(\xi_1)=2\cdot\frac{1}{16}=\frac{1}{8};\ f(\xi_2)=2\cdot\frac{3}{16}=\frac{3}{8};\ f(\xi_3)=2\cdot\frac{3}{8}=\frac{3}{4};$

$$f(\xi_4)=2\cdot\frac{5}{8}=\frac{5}{4};\ f(\xi_5)=2\cdot\frac{3}{4}=\frac{3}{2}.$$

e) $M_1=f\left(\frac{1}{8}\right)=\frac{1}{4};\ M_2=f\left(\frac{1}{4}\right)=\frac{1}{2};\ M_3=f\left(\frac{1}{2}\right)=1;$

$$M_4=f\left(\frac{3}{4}\right)=\frac{3}{2};\ M_5=f(1)=2.$$

$f)\ s_P(f)=m_1\Delta x_1+m_2\Delta x_2+m_3\Delta_3+m_4\Delta x_4+m_5\Delta x_5=$

$=0\frac{1}{8}+\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{4}+\frac{3}{2}\cdot\frac{1}{4}=\frac{1}{32}+\frac{1}{8}+\frac{1}{4}+\frac{3}{8}=$

$=\frac{1}{32}+\frac{1}{4}+\frac{1}{2}=\frac{1+8+16}{32}=\frac{25}{32}.$

$g)\ S^*(f)=f(\xi_1)\Delta x_1+f(\xi_2)\Delta x_2+f(\xi_3)\Delta x_3+f(\xi_4)\Delta x_4+f(\xi_5)\Delta x_5=$

$=\frac{1}{8}\cdot\frac{1}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{3}{4}\cdot\frac{1}{4}+\frac{5}{4}\cdot\frac{1}{4}+\frac{3}{2}\cdot\frac{1}{4}=\frac{1}{64}+\frac{3}{64}+\frac{3}{16}+\frac{5}{16}+\frac{3}{8}=\frac{9}{16}+\frac{3}{8}=\frac{15}{16}$

$h)\ S_P(f)=M_1\Delta x_1+M_2\Delta x_2+M_3\Delta x_3+M_4\Delta x_4+M_5\Delta x_5=$

$=\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{8}+1\cdot\frac{1}{4}+\frac{3}{2}\cdot\frac{1}{4}+2\cdot\frac{1}{4}=\frac{1}{32}+\frac{1}{16}+\frac{1}{4}+\frac{3}{8}+\frac{1}{2}=\frac{1+2+8+12+16}{32}=\frac{39}{32}.$

$i)\ s_P(f)=\frac{25}{32}<S^*(f)=\frac{15}{16}<S_P(f)=\frac{39}{32}$, ya'ni $\frac{25}{32}<\frac{15}{16}<\frac{39}{32}$.

6.10-misol. $f(x)=2x$ funksiya uchun, $[0;1]$ kesmada, $P_1=\left\{0,\frac{1}{4},\frac{1}{2},1\right\}$ va $P_2=\left\{0,\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{2}{3},1\right\}$ bo'linishlarga mos kelgan, Darbu yig'indilarini tuzing va ularni solishtiring.

Yechilishi. 1) P_1 bo'linish, $[0,1]$ kesmani, $[0,\frac{1}{4}],[\frac{1}{4},\frac{1}{2}],[\frac{1}{2},1]$ $(x_0=0;\ x_4=1)$ qism kesmalarga bo'ladi. $f(x)=2x$ funksiya, $[0;1]$ kesmada uzluksiz va o'suvchi bo'lgani uchun, kesmalarning chap chetki nuqtalarida $m_i(i=1,2,3)$ minimal qiymatlariga, o'ng chetki nuqtalarida $M_i(i=1,2,3)$ maksimal qiymatlariga erishadi, ya'ni

$$M_1=f\left(\frac{1}{4}\right)=2\cdot\left(\frac{1}{4}\right)=\frac{1}{2};\ M_2=f\left(\frac{1}{2}\right)=2\cdot\frac{1}{2}=1;\ M_3=f(1)=2\cdot1=2;$$

$$m_1=f(0)=2\cdot0=0;\ m_2=f\left(\frac{1}{4}\right)=2\cdot\frac{1}{4}=\frac{1}{2};\ m_3=f\left(\frac{1}{2}\right)=2\cdot\frac{1}{2}=1.$$

$\Delta x_i(i=1;2;3)$ – qism kesmalarning uzunliklarini topamiz:

$$\Delta x_1=x_1-x_0=\frac{1}{4}-0=\frac{1}{4};\ \Delta x_2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4};\ \Delta x_3=1-\frac{1}{2}=\frac{1}{2}.$$

U holda, (6.3) va (6.4) formulalarga asosan,

$$S_{P_1}(f)=M_1\Delta x_1+M_2\Delta x_2+M_3\Delta x_3=\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{4}+2\cdot\frac{1}{2}=\frac{11}{8},$$

$$s_{P_1}(f)=m_1\Delta x_1+m_2\Delta x_2+m_3\Delta x_3=0\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{2}=\frac{5}{8}.$$

2) $P_2=\left\{0,\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{2}{3},1\right\}$ bo'linish, $[0;1]$ kesmani,

$$[0;\frac{1}{8}],\ [\frac{1}{8};\frac{1}{4}],\ [\frac{1}{4};\frac{1}{2}],\ [\frac{1}{2};\frac{2}{3}],\ [\frac{2}{3};1]$$

qism kesmalarga bo‘ladi. $f(x)=2x$ funksiya, kesmalarning o‘ng chetki nuqtalarida, $M_i(i=\overline{1,5})$ maksimal qiymatlarni, chap chetki nuqtalarida esa, $m_i(i=\overline{1,5})$ minimal qiymatlarni qabul qiladi, ya’ni

$$M_1=f\left(\frac{1}{8}\right)=\frac{1}{4};\quad M_2=f\left(\frac{1}{4}\right)=\frac{1}{2};\quad M_3=f\left(\frac{1}{2}\right)=1;\quad M_4=f\left(\frac{2}{3}\right)=\frac{4}{3};\quad M_5=f(1)=2;$$

$$m_1=f(0)=0;\quad m_2=f\left(\frac{1}{8}\right)=\frac{1}{4};\quad m_3=f\left(\frac{1}{4}\right)=\frac{1}{2};\quad m_4=f\left(\frac{1}{2}\right)=1;\quad m_5=f\left(\frac{2}{3}\right)=\frac{4}{3}$$

$\Delta x_i(i=\overline{1,5})$ qism kesmalarning uzunliklari:

$$\Delta x_1=\frac{1}{8};\quad \Delta x_2=\frac{1}{8};\quad \Delta x_3=\frac{1}{4};\quad \Delta x_4=\frac{1}{6};\quad \Delta x_5=\frac{1}{3}.$$

U holda, (6.3) va (6.4) formulalarga asosan,

$$S_{P_2}(f)=M_1\Delta x_1+M_2\Delta x_2+M_3\Delta x_3+M_4\Delta x_4+M_5\Delta x_5=\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{8}+$$

$$+1\cdot\frac{1}{4}+\frac{4}{3}\cdot\frac{1}{6}+2\cdot\frac{1}{3}=\frac{355}{288};$$

$$s_{P_2}=m_1\Delta x_1+m_2\Delta x_2+m_3\Delta x_3+m_4\Delta x_4+m_5\Delta x_5=0\cdot\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{6}+\frac{4}{3}\cdot\frac{1}{3}=\frac{221}{288}$$

Endi, Darbuning, P_1 va P_2 bo‘linishlarga mos kelgan, quyi va yuqori yig‘indilarini solishtiramiz:

$$S_{P_1}(f)=\frac{11}{8}=1{,}375,\qquad s_{P_1}(f)=\frac{5}{8}=0{,}625,$$

$$S_{P_2}(f)=\frac{355}{288}=1{,}233,\quad s_{P_2}(f)=\frac{221}{288}=0{,}767.$$

Demak, $s_{P_1}(f)<s_{P_2}(f)\left(S_{P_1}(f)>S_{P_2}(f)\right)$, ya’ni P bo‘linishga yangi nuqtalar qo‘shilganda, $s_P(f)$ quyi yig‘indining kamaymasligini, $S_P(f)$ yuqori yig‘indining ortmasligini ko‘ramiz.

6.11-misol. $f(x)=1-x$ funksiya uchun, $[0;2]$ kesmada, $P_1=\{0;\frac{1}{3},\frac{3}{4},1,2\}$ va $P_2=\{0,\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{3}{4},1,\frac{3}{2},2\}$ bo‘linishlarga mos kelgan, Darbu yig‘indilarini tuzing va ularni solishtiring.

Yechilishi. 1) P_1 bo‘linish, $[0;2]$ kesmani, $[0;\frac{1}{3}]$, $[\frac{1}{3},\frac{3}{4}]$, $[\frac{3}{4},1]$, $[1;2]$ qism kesmalarga ajratadi. $f(x)=1-x$ funksiya qaralayotgan kesmalarda kamayuvchi bo‘lganligi uchun, o‘zining $M_i(i=\overline{1,4})$ maksimal qiymatlarini, qism kesmalarning chap chetki nuqtalarida, $m_i(i=\overline{1,4})$ minimal qiymatlarini esa, kesmalarning o‘ng chetki nuqtalarida qabul qiladi.

Demak,

$$M_1=f(x_0)=1-0=1;\quad M_2=f\left(\frac{1}{3}\right)=1-\frac{1}{3}=\frac{2}{3};$$

$$M_3 = f\left(\frac{3}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4};\ M_4 = f(1) = 1 - 1 = 0,$$

$$m_1 = f\left(\frac{1}{3}\right) = \frac{2}{3};\ m_2 = f\left(\frac{3}{4}\right) = \frac{1}{4};\ m_3 = f(1) = 1 - 1 = 0;\ m_4 = f(2) = -1.$$

$\Delta x_i\ (i = \overline{1,4})$ qism kesmalarining uzunliklarini hisoblaymiz:

$$\Delta x_1 = \frac{1}{3} - 0 = \frac{1}{3};\ \Delta x_2 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12};\ \Delta x_3 = 1 - \frac{3}{4} = \frac{1}{4};\ \Delta x_4 = 2 - 1 = 1.$$

Shunday qilib, Darbu yig'indilari:

$$S_{P_1}(f) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + M_4\Delta x_4 = 1\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{5}{15} + \frac{1}{4}\cdot\frac{1}{4} + 0\cdot 1 = \frac{87}{144};$$

$$s_{P_1}(f) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m_4\Delta x_4 = \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{5}{12} + 0\cdot\frac{1}{4} + (-1)\cdot 1 = -\frac{97}{144}.$$

Bunda, $s_{P_1}(f) < S_{P_1}(f)$.

2) P_2 bo'linish, $[0,2]$ kesmani,

$$[0;\frac{1}{6}],\ [\frac{1}{6};\frac{1}{3}],\ [\frac{1}{3};\frac{2}{3}],\ [\frac{2}{3};\frac{3}{4}],\ [\frac{3}{4};1],\ [1;\frac{3}{2}],\ [\frac{3}{2};2]$$

qism kesmalarga ajratadi.

Berilgan funksiya qaralayotgan kesmada uzluksiz, kamayuvchi bo'lganligi uchun, u o'zining $M_i\ (i = \overline{1,7})$ maksimal qiymatlarini, kesmalarning chap chetki nuqtalarida, $m_i\ (i = \overline{1,7})$ minimal qiymatlarini kesmalarning o'ng chetki nuqtalarida qabul qiladi, ya'ni

$$M_1 = f(0) = 1 - 0 = 1,\ M_2 = f\left(\frac{1}{6}\right) = 1 - \frac{1}{6} = \frac{5}{6},\ M_3 = f\left(\frac{1}{3}\right) = \frac{2}{3},\ M_4 = f\left(\frac{2}{3}\right) = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$M_5 = f\left(\frac{3}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4},\ M_6 = f(1) = 1 - 1 = 0,\ M_7 = f\left(\frac{3}{2}\right) = 1 - \frac{3}{2} = -\frac{1}{2}.$$

$$m_1 = f\left(\frac{1}{6}\right) = \frac{5}{6},\ m_2 = f\left(\frac{1}{3}\right) = \frac{2}{3},\ m_3 = f\left(\frac{2}{3}\right) = \frac{1}{3},\ m_4 = f\left(\frac{3}{4}\right) = \frac{1}{4}.$$

$$m_5 = f(1) = 1 - 1 = 0,\ m_6 = f\left(\frac{3}{2}\right) = -\frac{1}{2},\ m_7 = f(2) = -1.$$

$\Delta x_i\ (i = \overline{1,7})$ qism kesmalarning uzunliklarini hisoblaymiz:

$$\Delta x_1 = \frac{1}{6};\ \Delta x_2 = \frac{1}{6};\ \Delta x_3 = \frac{1}{3};\ \Delta x_4\ \frac{1}{12};\ \Delta x_5 = \frac{1}{4};\ \Delta x_6 = \frac{1}{2};\ \Delta x_7 = \frac{1}{2}$$

U holda,

$$s_{P_2}(f) = \sum_{i=1}^{7} m_i\Delta x_i = \frac{5}{6}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{12} + 0\cdot\frac{1}{4} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = -\frac{45}{144};$$

$$S_{P_2}(f) = \sum_{i=1}^{7} M_i\Delta x_i = 1\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{12} + \frac{1}{4}\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{65}{144}.$$

Demak, $s_{P_2}(f) < S_{P_2}(f)$.

Endi, P_1 va P_2 bo'linishlarga mos kelgan, Darbu yig'indilarini taqqoslaymiz:

$$s_{P_1}(f) = -0{,}673; \quad S_{P_1}(f) = \frac{87}{144} \approx 0{,}6042,$$
$$s_{P_2}(f) = -\frac{45}{144} = -0{,}312, \quad S_{P_2}(f) = \frac{65}{144} = 0{,}451;$$
$$s_{P_1}(f) < s_{P_2}(f), \quad S_{P_1}(f) > S_{P_2}(f),$$
ya'ni bu holda ham, P bo'linishga yangi bo'linish nuqtalarini qo'shish bilan, Darbuning quyi yig'indisi kamaymasligi, yuqori yig'indisining ortmasligiga ishonch hosil qilamiz.

6.3. Aniq integral yordamida limitlarni hisoblash.

6.12-misol. Ushbu
$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right)$$
limitni hisoblang.

Yechilishi. $S_n = \frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}$ deb belgilaymiz. Bu yig'indini quyidagi ko'rinishda yozib olamiz:
$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}.$$

Bunda $\frac{1}{1+\left(\frac{1}{n}\right)^2}, \frac{1}{1+\left(\frac{2}{n}\right)^2}, \ldots, \frac{1}{1+\left(\frac{n}{n}\right)^2}$ qo'shiluvchilar, $\frac{1}{1+x^2}$ funksiyaning $x_1 = \frac{1}{n}, \ x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n} = 1$ nuqtalardagi qiymatlarini ifoda qiladi. $[0;1]$ kesmaning regulyar $P = \{x_0 = 0 < x_1 < x_2, \ldots, < x_{n-1} < x_n = 1\}$ bo'linishini olamiz: $[0;1] = [0;x_1] \cup [x_1;x_2] \cup \ldots \cup [x_{n-1};x_n]$, bunda $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$. Ma'lumki, ta'rifga asosan, integral yig'indining limiti, qism kesmalardan olingan ξ_i nuqtalarni tanlashga bog'liq emas. Shuning uchun, $\xi_i = x_i$ deb olsak, u holda, $\sigma_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$

yig'indi, $f(x) = \frac{1}{1+x^2}$ funksiyaning, $[0;1]$ kesmadagi Riman integral yig'indisi bo'lib hisoblanadi.

Demak, ta'rifga asosan,
$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right) = \int_0^1 \frac{1}{1+x^2}dx = \left. arctgx \right|_0^1 = \frac{\pi}{4}.$$

6.13-misol. Ushbu

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$$

limitni hisoblang.

Yechilishi. Ravshanki, $\sigma_n=\frac{1}{n}\sum_{i=1}^{n}\sqrt{1+\frac{i}{n}}$ yig'indi, $f(x)=\sqrt{1+x}$ funksiyaning, $[0;1]$ kesmani teng n ta bo'lakka bo'lgandagi, integral yig'indisini ifodalaydi. Shuning uchun,

$$\lim_{n\to\infty}\sigma_n=\int_0^1\sqrt{1+x}dx=\frac{2}{3}\sqrt{(1+x)^3}\Big|_0^1=\frac{2}{3}\left(2\sqrt{2}-1\right).$$

Mustaqil yechish uchun misollar

6.1. $f(x)=x+1$ funksiya uchun $[-1;2]$ kesmani teng n ta bo'lakka bo'lib, hamda ξ_i $(i=\overline{1,n})$ nuqtani $[x_{i-1};x_i]$ kesmaning o'rtasi deb olib, $\sigma_P(f)$ integral yig'indini tuzing.

Quyidagi berilgan $f(x)$ funksiya uchun ko'rsatilgan kesmani teng n ta bo'lakka bo'lib, Darbuning $s_P(f)$ va $S_P(f)$ yig'indilarini tuzing:

6.2. $f(x)=x^3$, $x\in[-2,3]$. **6.3.** $f(x)=\sqrt{x}$, $x\in[0;1]$.

6.4. $f(x)=2^x$, $x\in[0,10]$.

6.5. $f(x)=x^4$ funksiya uchun, $[1;2]$ kesmani, uzunliklari geometrik progressiya tashkil etadigan n ta bo'laklarga bo'lib , Darbuning quyi yig'indisini tuzing va uning $n\to\infty$ dagi limitini toping.

Quyidagi berilgan $f(x)$ funksiya uchun, berilgan kesmaning berilgan P bo'linishiga mos, Darbuning quyi va yuqori yig'indilarini tuzing:

6.6. $f(x)=2x, x\in[0;1]$, $P=\left\{0,\frac{1}{4},\frac{1}{2},1\right\}$.

6.7. $f(x)=x^2$, $x\in[-1;0]$, $P=\left\{-1,-\frac{1}{2},-\frac{1}{4},0\right\}$.

6.8. $f(x)=x^3+1$, $x\in[0;1]$, $P=\left\{0,\frac{1}{2},1\right\}$.

6.9. $f(x)=x^2$, $x\in[-1;1]$, $P=\left\{-1,-\frac{1}{4},\frac{1}{4},\frac{1}{2},1\right\}$.

6.10. $f(x)=\sin x$, $x\in[0;\pi]$, $P=\left\{0,\frac{\pi}{6},\frac{\pi}{2},\pi\right\}$.

Quyidagi berilgan $f(x)$ funksiya uchun, ko'rsatilgan kesmaning P bo'linishi va qism kesmalarda tanlangan ξ_i $(i=\overline{1,n})$ nuqtalarga mos kelgan $\sigma_P(f)$ Riman integral yig'indisini tuzing va uning qanday figuraning yuzini ifodalashini chizmada ko'rsating:

6.11. $f(x)=x^2,\ x\in[0;1],\ P=\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\},\ \xi_1=\frac{1}{8},\ \xi_2=\frac{3}{8},\ \xi_3=\frac{5}{8},\ \xi_4=\frac{7}{8}.$

6.12. $f(x)=\frac{3}{2}x+1,\ x\in[0;2],\ P=\{0,\frac{1}{4},\frac{3}{4},1,\frac{3}{2},2\},\ \xi_1=\frac{1}{8},\ \xi_2=\frac{1}{2},$
$\xi_3=\frac{7}{8},\ \xi_4=\frac{5}{4},\ \xi_5=\frac{7}{4}.$

6.13. $f(x)=2x,\ x\in[0;1],\ P=\{0,\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{2}{3},1\},\ \xi_1=\frac{1}{16},\ \xi_2=\frac{3}{16},$
$\xi_3=\frac{3}{8},\ \xi_4=\frac{1}{8},\ \xi_4=\frac{7}{8}.$

Quyidagi integrallarni, Riman integral yig'indisining limiti sifatida qarab, hisoblang:

6.14. $\int_0^{\frac{\pi}{2}}\sin x\,dx.$ **6.15.** $\int_1^2\frac{dx}{x}.$

6.16. $\int_0^{10}3^x dx.$ **6.17.** $\int_a^b e^{kx}dx.$

6.18. $\int_0^1 x^2dx.$ **6.19.** $\int_0^1 x(1-x^2)dx.$

6.20. $\int_a^b\frac{1}{x^2}dx\ (0<a<b)$ ($\xi_i=\sqrt{x_ix_{i-1}},\quad (i=\overline{0,n})$ deb oling).

6.21. $\int_1^e\ln x\,dx$ ($[1;e]$ kesmani geometrik progressiyani tashkil qiladigan nuqtalar yordamida qism kesmalarga bo'ling).

6.22. Ushbu $\int_1^4 x^3dx$ integralni quyidagi shartlarda:

1) $[1;4]$ kesmani teng bo'laklarga bo'lib, ξ_i nuqta sifatida qism kesmalarning chap chetki nuqtalarini olib;

2) $[1;4]$ kesmani teng bo'laklarga bo'lib, ξ_i nuqta sifatida qism kesmalarning o'ng chetki nuqtalarini olib;

3) $[1;4]$ kesmani teng bo'laklarga bo'lib, ξ_i nuqta sifatida qism kesmalarning o'rta nuqtalarini olib;

4) $[1;4]$ kesmani $x_0,x_2,x_3,\ldots,x_n$ (bunda $x_k=q^k,q=\sqrt[n]{4}$) nuqtalar yordamida bo'lib , ξ_i nuqta sifatida, qism kesmalarning chap chetki nuqtalarini olib;

5) $[1;4]$ kesmani $x_0, x_2, x_3, \ldots, x_n$ (bunda $x_k = q^k, q = \sqrt[n]{4}$) nuqtalar yordamida bo'lib, ξ_i nuqta sifatida, qism kesmalarning o'ng chetki nuqtalarini olib, hisoblang.

6.23. Ushbu $f(x) = \mathrm{sgn}(\sin\frac{\pi}{x})$, uzilishga ega, funksiyaning $[0;1]$ kesmada integrallanuvchiligini isbotlang.

6.24. Ushbu

$$\varphi(x) = \begin{cases} 0, & x - irratsional\ son\ bo'lganda, \\ \frac{1}{n}, & x = \frac{m}{n}\ bo'lganda \end{cases}$$

(bunda, m va n $(n>1)$ lar o'zaro tub butun sonlar) Riman funksiyasining ixtiyoriy chekli kesmada integrallanuvchiligini isbotlang.

6.25. Aniq integralning tarifiga ko'ra, agar $f(x)$ funksiya biror kesmada integrallanuvchi bo'lsa, u holda uning shu kesmada chegaralangan bo'lishini isbotlang.

6.26. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa, u holda,

$$\lim_{\lambda\to 0} s_P(f) = \lim_{\lambda \to 0} S_P(f) = \int_a^b f(x)dx$$

ekanligini isbotlang $(\lambda = \max|\Delta x_k|)$.

Aniq integral yordamida, quyidagi yig'indilarning limitini hisoblang:

6.27. $\lim\limits_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right)$. **6.28.** $\lim\limits_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{2n-1}{n^2}\right)$.

6.29. $\lim\limits_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + \ldots + e^{\frac{n}{n}}\right)$. **6.30.** $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}[1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}]$.

6.31. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2} + \frac{4}{n^2} + \ldots + \frac{(n-1)^2}{n^2}\right)\frac{1}{n}$. **6.32.** $\lim\limits_{n\to\infty}\left(\frac{1}{n^4} + \frac{8}{n^4} + \ldots + \frac{(4n-1)^3}{n^2}\right)\frac{1}{n}$.

6.33. $\lim\limits_{n\to\infty}\frac{1}{n}[1 + \frac{1}{\cos^2\frac{\pi}{4n}} + \ldots + \frac{1}{\cos^2\frac{\pi n}{4n}}]$. **6.34.** $\lim\limits_{n\to\infty}\frac{1^\alpha + 2^\alpha + \ldots + n^\alpha}{n^{\alpha+1}}$ $(\alpha > 0)$.

6.35. $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$. **6.36.** $\lim\limits_{n\to\infty}\sin\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$.

6.37. $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-2^2}} + \ldots + \frac{1}{\sqrt{4n^2-2^n}}\right)$.

6.38. $\lim\limits_{n\to\infty}\frac{1}{n}[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{(n-1)\pi}{n}]$.

6.39. $\lim\limits_{n\to\infty} n\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(n+n)^2}\right)$.

6.40. $\lim\limits_{n\to\infty}\frac{3}{n}[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\dots+\sqrt{\frac{n}{n+3(n-1)}}]$.

6.41. $\lim\limits_{n\to\infty}\frac{\pi}{2n}[1+\cos\frac{2\pi}{2n}+\cos\frac{3\pi}{2n}+\dots+\cos\frac{(n-1)\pi}{n}]$.

Mustaqil yechish uchun misollarning javoblari

6.1. $\sigma_p(f)=4{,}5$ **6.2.** $s_p(f)=16\frac{1}{4}-\frac{175}{2n}+\frac{125}{4n^2}$, $S_p(f)=16\frac{1}{4}+\frac{175}{2n}+\frac{125}{4n^2}$.

6.3. $s_p(f)=\frac{1}{n}\sum\limits_{i=0}^{n}\sqrt{\frac{i}{n}}$; $S_p(f)=\frac{1}{n}\sum\limits_{i=1}^{n}\sqrt{\frac{i}{n}}$ **6.4.** $s_p(f)=\frac{10230}{n(2^{\frac{10}{n}}-1)}$;

$S_p(f)=\frac{10230\cdot 2^{\frac{10}{n}}}{n(2^{\frac{10}{n}}-1)}$. **6.5.** $s_p(f)=31\cdot\frac{\sqrt[n]{2}-1}{\sqrt[n]{32}-1}$, $\lim\limits_{n\to\infty}s_p(f)=\frac{31}{5}$.

6.6. $s_p(f)=\frac{5}{8}$, $S_p(f)=\frac{11}{8}$. **6.7.** $s_p(f)=\frac{9}{64}$, $S_p(f)=\frac{37}{64}$.

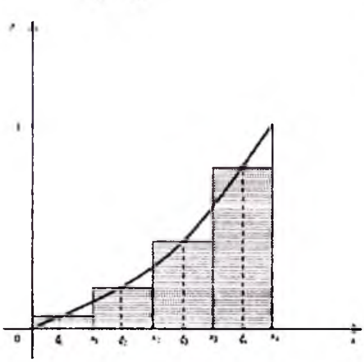
6.8. $s_p(f)=\frac{17}{16}$, $S_p(f)=\frac{25}{16}$. **6.9.** $s_p(f)=\frac{3}{16}$; $S_p(f)=\frac{43}{32}$.

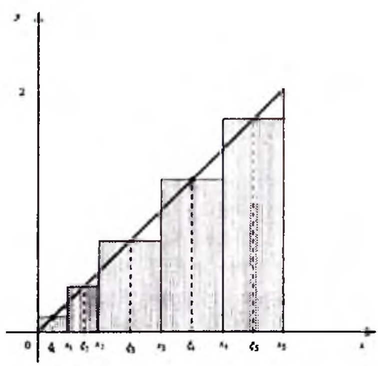
6.10. $s_p(f)=\frac{\pi}{6}$, $S_p(f)=\frac{11\pi}{12}$. **6.11.** $\sigma_p(f)=\frac{21}{64}$, 6.1-chizma.

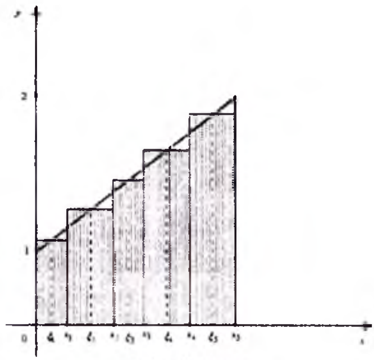
6.12. $\sigma_p(f)=\frac{39}{8}$, 6.2-chizma. **6.13.** $\sigma_p(f)=\frac{27}{32}$, 6.3-chizma.

6.14. 1. **6.15.** $\ln 2$. **6.16.** $\frac{3^{10}-1}{\ln 3}$. **6.17.** $\frac{e^{kb}-e^{ka}}{k}$. **6.18.** $\frac{1}{3}$. **6.19.** $\frac{1}{4}$.

6.20. $\frac{1}{a}-\frac{1}{b}$. **6.21.** 1. **6.22.** $\frac{255}{4}$.


6.1-chizma.


6.2-chizma



6.3-chizma

6.27. $\ln 2$. **6.28.** 4. **6.29.** $e-1$. **6.30.** 2. **6.31.** $\frac{1}{3}$. **6.32.** 16. **6.32.** $\frac{4}{\pi}$. **6.33.** $\frac{1}{\alpha+1}$ **6.34.** $\frac{1}{e}$ **6.35.** $\frac{\pi}{\sqrt{3}}$. **6.36.** $\frac{\pi}{6}$ **6.37.** $\frac{2}{\pi}$ **6.38.** $\frac{1}{2}$ **6.39.** 2. **6.40.** 1.

7-§. Aniq integralning mavjudligi. Integrallanuvchi funksiyalarning sinflari. Aniq integralning xossalari

7.1. Aniq integralning mavjudligi.

7.1-teorema. $[a,b]$ kesmada chegaralangan $f(x)$ funksiya, shu kesmada Darbu ma'nosida integrallanuvchi bo'lishi uchun, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ son topilib, $[a,b]$ kesmaning diametri $d(P) < \delta$ bo'lgan har qanday P bo'linishiga nisbatan, Darbu yig'indilarining,

$$S_P(f) - s_P(f) < \varepsilon \tag{7.1}$$

tengsizlikni qanoatlantirishi zarur va yetarlidir.

Agar $f(x)$ funksiyaning $[x_k, x_{k+1}]$ $(k = \overline{0, n-1})$ kesmadagi tebranishini ω_k deb belgilasak, u holda (7.1) tengsizlik, quyidagi,

$$\sum_{k=0}^{n-1} \omega_k \Delta x_k < \varepsilon$$

tengsizlikka teng kuchli bo'ladi.

7.2-teorema. $f(x)$ funksiyaning $[a,b]$ kesmada Riman ma'nosida integrallanuvchi bo'lishi uchun, uning Darbu ma'nosida integrallanuvchi bo'lishi, zarur va etarlidir.

Bu holda, Riman integrali, Darbu ma'nosidagi integralga teng bo'ladi. Biz bundan keyin, «Darbu ma'nosidagi integral» degan terminni ishlatmasdan, o'sha integralni «Riman ma'nosidagi» integral deb ataymiz.

7.2. Integrallanuvchi funksiyalarning sinflari.

7.3 -teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u holda, $f(x)$ funksiya shu kesmada integrallanuvchi bo'ladi.

7.4 -teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada chegaralangan va monoton bo'lsa, u holda, $f(x)$ funksiya shu kesmada integrallanuvchi bo'ladi.

7.5-teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada chegaralangan va kesmaning chekli sondagi nuqtalarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, u holda, $f(x)$ funksiya shu kesmada integrallanuvchi bo'ladi.

Aniq integralning ta'rifi bo'yicha, ya'ni uni, integral yig'indining limiti sifatida qarab, ba'zi bir sodda funksiyalarning integrallarini hisoblash mumkin. Lekin, ko'pincha, har qanday uzluksiz funksiyaning aniq integralini, integral yig'indining limiti sifatida qarab, hisoblash ancha noqulayliklarga va qiyinchiliklarga olib keladi.

7.1 -misol. Ushbu $\int_0^{\frac{\pi}{2}} \sin x\,dx$ integralni hisoblang.

Yechilishi. $f(x) = \sin x$ funksiya $[0;\frac{\pi}{2}]$ kesmada uzluksiz bo'lgani uchun, 7.3-teoremaga ko'ra, u qaralayotgan kesmada integrallanuvchi bo'ladi.

Demak, berilgan funksiyaning $[0;\frac{\pi}{2}]$ kesma bo'yicha integralini ta'rifga ko'ra hisoblashda, $[0;\frac{\pi}{2}]$ kesmaning P bo'linishini, hamda har bir $[x_k, x_{k+1}]$ qism kesmadan ξ_k nuqtalarni olib, integral yig'indini tuzish va uning limitini hisoblashga qulay qilib olish imkoniyatiga ega bo'lamiz. Shularni e'tiborga olgan holda, $[0;\frac{\pi}{2}]$ kesmani n ta teng bo'lakka bo'lib, $\xi_k\left(k=\overline{0,n-1}\right)$ nuqtalar sifatida $[x_k, x_{k+1}]$ kesmaning chap chetki nuqtalarni olib, $\sigma_P(f)$ integral yig'indini tuzamiz:

$$\sigma_P(f)=\sigma_P(\sin x)=\frac{\pi}{2n}\sum_{k=0}^{n-1}\sin\frac{k\pi}{2n}$$

$\sum_{k=1}^{n}\sin k\alpha=\frac{\sin\frac{(n+1)\alpha}{2}\cdot\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$ formulaga asosan, $\sigma_P(\sin x)$ yig'indini,

$$\sigma_P(f)=\frac{\sqrt{2}\cdot\pi}{4n}\cdot\frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}}$$

ko'rinishda yozib olamiz.

Endi integral yig'indining $n\to\infty$ dagi limitini hisoblaymiz:

$$\lim_{n\to\infty}\sigma_P(f)=\lim_{n\to\infty}\frac{\sqrt{2}\cdot\pi}{4n}\cdot\frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}}=\sqrt{2}\lim_{n\to\infty}\sin\left(\frac{\pi}{4}-\frac{\pi}{4n}\right)\cdot\frac{\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}.$$

Bundan, $f(x)=\sin x$ funksiyaning qaralayotgan kesmada uzluksizligini, hamda $\lim_{n\to\infty}\frac{\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}=1$ ekanligini e'tiborga olsak, u holda,

$$\lim_{n\to\infty}\sigma_P(\sin x)=\int_0^{\pi/2}\sin x\,dx=\sqrt{2}\cdot\sin\frac{\pi}{4}=1$$

bo'ladi.

Demak, $f(x)=\sin x$ funksiya $[0;\frac{\pi}{2}]$ kesmada uzluksiz bo'lgani uchun, u integrallanuvchi bo'lar ekan.

Agar $g(x)$ funksiya $[a,b]$ kesmada aniqlangan va kesmaning chekli sondagi nuqtalarida, $f(x)$ funksiyaning qiymatlaridan farqli (boshqa) qiymatlar qabul qilsa, u holda, $g(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'ladi va

$$\int_a^b g(x)dx=\int_a^b f(x)dx$$

tenglik o'rinli.

7.2-misol. Ushbu

$$f(x)=\begin{cases}\sin\frac{1}{x}, & 0<x\le 1 \ \ bo'lganda,\\ 5, & x=0 \ \ bo'lganda.\end{cases}$$

funksiyani $[0;1]$ kesmada integrallanuvchi ekanligini ko'rsating.

Yechilishi. Berilgan funksiya, $[0;1]$ kesmada chegaralangan va kesmaning $x=0$ nuqtasida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz.

Demak, 7.5- teoremaga asosan, berilgan funksiya, $[0;1]$ kesmada integrallanuvchi bo'ladi.

7.3-misol. $g(x)=\begin{cases}2, x\in[0,3)\cup(3,4]\\ 7, x=3\end{cases}$

funksiyaning $[0.4]$ kesmada integrallanuvchi ekanligini ko'rsating va integralni hisoblang.

Yechilishi. Berilgan funksiya, o'zgarmas $f(x)=2,\quad x\in[0,4]$ funksiyadan, faqat $x=3$ nuqtada farq qiladi.

Ravshanki,

$$\int_0^4 f(x)dx=8.$$

Endi, $[0,4]$ kesmaning ixtiyoriy P (regular yoki regular bo'lmagan) bo'linishida, 8 soni $s_P(g)\le I\le S_P(g)$ tengsizliklarni qanoatlantiradigan yagona I sondan iboratligini ko'rsatsak,

$$\int_0^4 g(x)dx=8$$

bo'lishi ko'rsatilgan (isbotlangan) bo'ladi.

$P-[0,4]$ kesmaning ixtiyoriy bo'linishi bo'lsin. Bu bo'linishdan hosil bo'lgan $[x_{i-1}, x_i]$ qism kesmalarning har birida $m_i=\min\limits_{x\in[x_{i-1},x_i]} g(x)=2,\ i=1,2,...,n,\quad M_i=\max\limits_{x\in[x_{i-1},x_i]} g(x)\ge 2,\ i=1,2,...,n.$ U holda, quyi Darbu yig'indisi,

$$s_P(g)=2\cdot\Delta x_1+2\cdot\Delta x_2+...+2\cdot\Delta x_n=2(\Delta x_1+...+\Delta x_n)=2\cdot\Delta x=2\cdot 4=8,$$

va yuqori Darbu yig'indisi,

$$S_P(g)\ge 2\cdot\Delta x_1+2\cdot\Delta x_2+...+2\cdot\Delta x_n=2(\Delta x_1+...+\Delta x_n)=2\cdot\Delta x=2\cdot 4=8$$

bo'ladi.

Shunday qilib, $[0,4]$ kesmaning barcha P bo'linishlarida, $s_P(g)\le 8\le S_P(g)$ tengsizliklar o'rinli.

Endi yagonalikni isbotlaymiz. $[0,4]$ kesmaning barcha P bo'linishlarida,

$$s_P(g)\le I\le S_P(g) \qquad (7.2)$$

bo'lsin, deb faraz qilamiz. $s_P(g)=8$ munosabat, barcha P bo'linishlar uchun bajarilganligidan, I, hech bo'lmaganda, 8 ga teng bo'ladi. Endi, faraz qilaylikki, $I>8$ bo'lsin va $[0,4]$ kesmaning P bo'linishini shunday o'zgartiramizki, $\lambda_P=\max|\Delta x_i|<\frac{1}{5}(I-8)$ va $0=x_1<x_2<...<x_{i-1}<3<x_i<...<x_n=4$ bo'lsin. U holda,

$$S_P(g)=2\Delta x+\ldots+2\Delta x_{i-1}+7\Delta x_i+2\Delta x_{i+1}+\ldots+2\Delta x_n=$$
$$=2(\Delta x+\Delta x_2+\Delta x_2+\ldots+\Delta x_n)+5\Delta x_i=8+5\Delta x_i<8+\frac{5}{5}(I-8)=I$$

bo'lishi kelib chiqadi, bundan, I sonning (7.2) munosabatni qanoatlantirmasligi kelib chiqadi. Bu qarama–qarshilik, I sonning 8 dan katta bo'lmasligini, ya'ni $I=8$ bo'lishini isbotlaydi.

7.3. Aniq integralning xossalari. Aniq integralning, ta'rifdan bevosita kelib chiqadigan, quyidagi xossalarini keltiramiz:

1^0. $\int\limits_a^b dx=b-a$.

2^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa, u istalgan $[\alpha,\beta]\subset[a,b]$ kesmada ham integrallanuvchi bo'ladi.

3^0. Agar $f(x)$ funksiya $[a,c]$va $[c,b]$ kesmalarda integrallanuvchi bo'lsa, u holda, $f(x)$ funksiya $[a,b]$ da ham integrallanuvchi bo'ladi va ushbu

$$\int\limits_a^b f(x)dx=\int\limits_a^c f(x)dx+\int\limits_c^b f(x)dx,\quad a\le c\le b, \tag{7.3}$$

tenglik o'rinli.

4^0 . Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da integrallanuvchi bo'lsa, u holda, $\forall\lambda,\mu\in R$ lar uchun $\lambda f(x)+\mu g(x)$ funksiya ham $[a,b]$ kesmada integrallanuvchi bo'ladi va

$$\int\limits_a^b[\lambda f(x)+\mu g(x)]dx=\lambda\int\limits_a^b f(x)dx+\mu\int\limits_a^b g(x)dx \tag{7.4}$$

tenglik o'rinli.

7.1 -natija. Agar $f_1(x)$, $f_2(x)$,...,$f_n(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo'lsa, u holda, $C_1f_1(x)+C_2f_2(x)+\ldots+C_nf_n(x)$ $(C_k-const,\ k=\overline{1,n})$ funksiya ham shu kesmada integrallanuvchi bo'ladi va

$$\int\limits_a^b[C_1f_1(x)+\ldots+C_nf_n(x)]dx=C_1\int\limits_a^b f(x)dx+\ldots+C_n\int\limits_a^b f_n(x)dx \tag{7.5}$$

tenglik o'rinli.

5^0. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo'lsa, u holda, ularning $f(x)g(x)$ ko'paytmasi ham shu kesmada integrallanuvchi bo'ladi.

7.2-natija. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa, $\forall n\in N$ uchun, $[f(x)]^n$ funksiya ham shu kesmada integrallanuvchi bo'ladi.

6^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lib, $\forall x \in [a,b]$ uchun, $f(x) \geq 0$ tengsizlik o‘rinli bo‘lsa, u holda,

$$\int_a^b f(x)dx \geq 0 \quad (a > b)$$

tengsizlik o‘rinli.

7.3-natija. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo‘lib, $\forall x \in [a,b]$ uchun, $f(x) \leq g(x)$ tengsizlik o‘rinli bo‘lsa, u holda,

$$\int_a^b f(x) \leq \int_a^b g(x)dx \tag{7.6}$$

tengsizlik o‘rinli.

7^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lsa, $|f(x)|$ funksiya ham shu kesmada integrallanuvchi bo‘ladi va

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \tag{7.7}$$

tengsizlik o‘rinli.

Bu xossaning teskarisi har doim ham o‘rinli emas, ya’ni $|f(x)|$ funksiyaning shu kesmada integrallanuvchi ekanligidan, $f(x)$ funksiyaning integrallanuvchi ekanligi kelib chiqmaydi.

8^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lsa, u holda, shunday μ o‘zgarmas son $(m \leq \mu \leq M)$ mavjud bo‘ladi va

$$\int_a^b f(x)dx = \mu(b-a) \tag{7.8}$$

tenglik o‘rinli, bunda $m = \inf\limits_{a \leq X \leq b} \{f(x)\}$, $M = \sup\limits_{a \leq X \leq b} \{f(x)\}$.

9^0 **(o‘rta qiymat haqidagi teorema)**. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo‘lib, $g(x)$ funksiya shu kesmada o‘z ishorasini saqlasa, u holda, shunday μ o‘zgarmas son $(m \leq \mu \leq M)$ mavjud bo‘ladi va

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \tag{7.9}$$

tenglik o‘rinli.

7.4-natija. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo‘lsa, u holda, $[a,b]$ kesmada shunday ξ $(\xi \in [a,b])$ mavjud bo‘ladi va

$$\int_a^b f(x)g(x)dx = f(\xi)(b-a) \tag{7.10}$$

tenglik o‘rinli.

Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa $(a \le b)$, u holda u aniq integralning 2^0 -xossasiga asosan, $[a,x]$ $(a \le x \le b)$ kesmada ham integrallanuvchi bo'ladi. Aniq integralning yuqori chegarasi b ni x ga, integral ostidagi ifodada x o'zgaruvchini t ga almashtirsak, u holda,

$$\int_a^x f(t)dt$$

x ga bog'liq ifoda hosil bo'ladi, uni $\Phi(x)$ orqali belgilaymiz:

$$\Phi(x) = \int_a^x f(t)dt \qquad (7.11)$$

Bu funksiya quyidagi xossalarga ega:

10^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa, u holda,

$$\Phi(x) = \int_a^x f(t)dt \text{ va } F(x) = \int_x^b f(t)dt$$

funksiyalar ham shu kesmada uzluksiz bo'ladi.

11^0. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lib, biror $x_0 \in [a,b]$ nuqtada uzluksiz bo'lsa, u holda, $\Phi(x)$ va $F(x)$ funksiyalar ham x_0 nuqtada differensiallanuvchi bo'ladi va

$$\Phi'(x_0) = f(x_0),\ F'(x_0) = -f(x_0) \qquad (7.12)$$

tengliklar o'rinli.

7.1-eslatma. Agar x_0 nuqta, $[a,b]$ kesmaning chetki nuqtalari bilan ustma-ust tushsa, u holda, $\Phi(x)$ funksiyaning x_0 nuqtadagi hosilasi, o'ng yoki chap hosila deb tushuniladi.

7.5-natija. $[a,b]$ kesmada uzluksiz bo'lgan har qanday $f(x)$ funksiya, shu kesmada boshlangich funksiyaga ega bo'ladi. Shu boshlang'ich funksiyalardan biri

$$\int f(x)dx = \int_a^x f(t)dt + C \qquad (7.13)$$

bo'ladi. Bu formula aniqmas integral bilan aniq integral orasidagi boshlanishni ifoda qiladi.

7.4-misol. Ushbu

$$f(x) = \begin{cases} x, & 0 \le x < 1/2 \ \text{bo'lganda}, \\ 1, & 1/2 \le x \le 1 \ \text{bo'lganda}. \end{cases}$$

funksiyaning $[0;1]$ kesmada integrallanuvchi ekanligini ko'rsating.

Yechilishi. Berilgan funksiya $[0;1]$ kesmada chegaralangan va kesmaning $x=\frac{1}{2}$ nuqtasida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz.

Demak, 7.5-teorema va 3^0 - xossaga asosan, berilgan funksiya, $[0;1]$ kesmada integrallanuvchi bo'ladi, ya'ni

$$\int_0^1 f(x)dx = \int_0^{1/2} xdx + \int_{1/2}^1 1dx = \frac{1}{8}+\frac{1}{2}=\frac{5}{8}.$$

7.5-misol. $P(x)$ funksiya $[a,b]$ kesmada integrallanuvchi va manfiy emas, $f(x)$ funksiya esa, $[a,b]$ kesmada $m \le f(x) \le M$ munosabatlarni qanoatlantirsin. Agar $f(x)P(x)$ funksiya shu kesmada integrallanuvchi bo'lsa, u holda,

$$m\int_a^b P(x)dx \le \int_a^b P(x)f(x)dx \le M\int_a^b P(x)dx \tag{7.14}$$

tengsizliklar bajarilishini isbotlang.

Yechilishi. Shartga ko'ra, $\forall a \le x \le b$ uchun, $m \le f(x) \le M$ bo'lganligidan, $mP(x) \le f(x)P(x) \le MP(x)$ tengsizliklar o'rinli. Bu tengsizliklarni, a dan b gacha, hadma – had integrallash natijasida, (7.14) tengsizliklarni hosil qilamiz.

7.6-misol. $P(x), f(x), \varphi(x), \psi(x)$ funksiyalar $[a,b]$ kesmada berilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

a) $P(x)$– manfiy emas;

b) $\varphi(x) \le f(x) \le \psi(x)$;

c) $f(x)P(x), \varphi(x)P(x), \psi(x)P(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo'lsin. U holda,

$$\int_a^b \varphi(x)P(x)dx \le \int_a^b f(x)P(x)dx \le \int_a^b \psi(x)P(x)dx \tag{7.15}$$

tengsizlik bajarilishini isbotlang.

Yechilishi. Shartga ko'ra, $P(x) \ge 0$, $\varphi(x) \le f(x) \le \psi(x)$ bo'lganligi uchun, $\varphi(x)P(x) \le f(x)P(x) \le \psi(x)P(x)$ tengsizliklar ham o'rinli bo'ladi. Bundan (7.15) tengsizliklar kelib chiqadi.

7.7-misol. Ushbu

$$0 < \int_0^{\pi} \frac{\sin x}{\sqrt[5]{x^2+2}}dx < \frac{\pi}{\sqrt[5]{2}}$$

tengsizliklarni isbotlang.

Yechilishi. $P(x)=\sin x$, $f(x)=\frac{1}{\sqrt[5]{x^2+2}}$ deb belgilaylik. $[0;\pi]$kesmada $0<\frac{1}{\sqrt[5]{x^2+2}}<\frac{1}{\sqrt[5]{2}}$, $\sin x \geq 0$. 7.5– misoldagi (7.14) tengsizliklarga ko'ra,

$$0<\int_0^\pi \frac{\sin x}{\sqrt[5]{x^2+1}}dx<\frac{1}{\sqrt[5]{2}}\int_0^\pi \sin x dx<\frac{\pi}{\sqrt[5]{2}}$$

ekanligi isbotlanadi.

7.8-misol. Ushbu

$$\frac{1}{10\sqrt{3}} \leq \int_0^1 \frac{x^9}{\sqrt{2+x^2}}dx \leq \frac{1}{\sqrt{210}}$$

tengsizliklarni isbotlang.

Yechilishi. $P(x)=x^9$, $f(x)=\frac{1}{\sqrt{2+x^2}}$ deb belgilasak, u holda, $[0;1]$ kesmada $P(x)\geq 0$ va integrallanuvchi, $\frac{1}{\sqrt{3}}\leq f(x)\leq \frac{1}{\sqrt{2}}$, hamda $f(x)\cdot P(x)$ funksiya integrallanuvchi.

Demak, 7.5 -misoldagi (7.14) tengsizliklarga ko'ra,

$$\frac{1}{\sqrt{3}}\int_0^1 x^9 dx \leq \int_0^1 \frac{x^9}{\sqrt{2+x^2}}dx \leq \frac{1}{\sqrt{2}}\int_0^1 x^9 dx.$$

Bundan, $\frac{1}{10\sqrt{3}} \leq \int_0^1 \frac{x^9}{\sqrt{2+x^2}}dx \leq \frac{1}{\sqrt{2}\cdot 10}$ tengsizliklarning o'rinli ekanligini olamiz.

7.9-misol . $x-\frac{x^3}{6}\leq \sin x \leq x$, $x\geq 0$ tengsizliklardan foydalanib, ushbu

$$\frac{20\sqrt{2}}{21} \leq \int_0^2 \frac{\sin x}{\sqrt{x}}dx \leq \frac{4\sqrt{2}}{3}$$

tengsizliklarni isbotlang.

Yechilishi. $[0;2]$ kesmada $\sqrt{x}\geq 0$ bo'lganligi uchun, 7.6-misoldagi (7.15) tengsizliklarga asosan,

$$\int_0^2 \frac{x-\frac{x^3}{6}}{\sqrt{x}}dx \leq \int_0^2 \frac{\sin x}{\sqrt{x}}dx \leq \int_0^2 \frac{x}{\sqrt{x}}dx.$$

Bundan,

$$\int_0^2 \frac{x-\frac{x^3}{6}}{\sqrt{x}} = \int_0^2 \left(x^{\frac{1}{2}}-\frac{1}{6}x^{\frac{5}{2}}\right)dx = 4\cdot\frac{\sqrt{2}}{3}-\frac{1}{6}\cdot\frac{2^{\frac{7}{2}}}{\frac{7}{2}}=\frac{20\sqrt{2}}{21}.$$

$$\int_0^2 \frac{x}{\sqrt{x}}dx=\int_0^2 x^{\frac{1}{2}}dx=\frac{4\sqrt{2}}{3}$$

bo'lganligidan,

$$\frac{20\sqrt{2}}{21} \le \int_0^{2} \frac{\sin x}{\sqrt{x}}dx \le \frac{4\sqrt{2}}{3}$$

tengsizliklarning o'rinli ekanligi kelib chiqadi.

7.10- misol. Ushbu

$$\frac{4}{25}(e-1) < \int_0^1 \frac{e^x}{(x+2)(3-x)}dx < \frac{1}{6}(e-1)$$

tengsizliklarni isbotlang.

Yechilishi. $f(x)=\frac{1}{(x+2)(3-x)}$ funksiyaning $[0;1]$ kesmadagi $f'(x)=\frac{2x-1}{(x+2)^2(3-x)^2}$ hosilasi, $x=\frac{1}{2}$ nuqtada nolga teng bo'ladi. Funksiyaning hosilasi, $x=\frac{1}{2}$ nuqtadan o'tishda, o'z ishorasini, manfiydan musbatga o'zgartiradi. Shuning uchun, $f(x)$ funksiya bu nuqtada minumumga erishadi, ya'ni $f_{min}\left(\frac{1}{2}\right)=\frac{4}{25}$. Funksiya maksimum qiymatiga $[0;1]$ kesmaning chetki nuqtalarida erishadi, ya'ni $f_{max}(0)=f_{max}(1)=\frac{1}{6}$. Shunday qilib, $\frac{4}{25} \le \frac{1}{(x+2)(3-x)} \le \frac{1}{6}$, $x \ne 0$, $x \ne \frac{1}{2}$ va $x \ne 1$ bo'lganda esa,

$$\frac{4}{25}\int_0^1 e^x dx < \int_0^1 \frac{e^x}{(x+2)(3-x)} < \frac{1}{6}\int_0^1 e^x dx,$$

$$\frac{4}{25}(e-1) < \int_0^1 \frac{e^x}{(x+2)(3-x)}dx < \frac{1}{6}(e-1).$$

Demak, 7.5-misoldagi (7.14) tengsizliklarga asosan,

$$\frac{4}{25}\int_0^1 e^x dx < \int_0^1 \frac{e^x}{(x+2)(3-x)} < \frac{1}{6}\int_0^1 e^x dx.$$

Bundan talab qilingan tengsizlik kelib chiqadi.

7.11 -misol.

Quyidagi: $\int_0^1 \sqrt{1+x^2}dx$ va $\int_0^1 xdx$

integrallarning qaysi biri katta?

Yechilishi. $\sqrt{1+x^2}$ va x funksiyalar $[0;1]$ kesmada uzluksiz (integrallanuvchi). $[0;1]$ kesmada $x<\sqrt{1+x^2}$ tengsizlik o'rinli. U holda (aniq integralning 6^0- xossasiga asosan),

$$\int_0^1 xdx < \int_0^1 \sqrt{1+x^2}dx \text{ t}$$

tengsizlik o'rinli bo'ladi.

7.12– misol. Agar $f(x)$ va $\varphi(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo‘lsa, u holda,

$$\left|\int_a^b f(x)\varphi(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx \cdot \int_a^b \varphi^2(x)dx} \qquad (7.16)$$

tengsizlik o‘rinli bo‘lishini isbotlang. Odatda, bu tengsizlik, Koshi - Bunyakovskiy tengsizligi deb ataladi.

Yechilishi. $[f(x)-\lambda\varphi(x)]^2$ funksiyani qaraymiz. Bunda λ – ixtiyoriy haqiqiy son. Bu funksiya $[a,b]$ kesmada integrallanuvchi. Aniq integralning 6^0 - xossasiga asosan,

$$\int_a^b [f(x)-\lambda\varphi(x)]^2 dx \ge 0$$

yoki

$$\lambda^2\int_a^b \varphi^2(x)dx - 2\lambda\int_a^b f(x)\varphi(x)dx + \int_a^b f^2(x)dx \ge 0.$$

Bu tengsizlikning chap tomoni λ ga nisbatan kvadrat uchhaddan iborat bo‘lib, u λ ning barcha qiymatlarida nomanfiy, demak, kvadrat uchhadning diskriminanti manfiy emas:

$$\int_a^b f^2(x)dx \cdot \int_a^b \varphi^2(x)dx - \left\{\int_a^b f(x)\varphi(x)dx\right\}^2 \ge 0,$$

bundan (7.16) tengsizlikning o‘rinli ekanligi kelib chiqadi.

7.13 -misol. Ushbu

$$\int_0^{\frac{\pi}{2}} x^2 \cdot \sqrt{\cos x}dx \le \frac{\pi^3}{24}$$

tengsizlikni isbotlang.

Yechilishi. $f(x)=x^2, g(x)=\sqrt{\cos x}$ deb olsak, (7.16) formulaga asosan,

$$\int_0^{\frac{\pi}{2}} x^2 \cdot \sqrt{\cos x}dx \le \sqrt{\int_0^{\frac{\pi}{2}} x^2 dx} \cdot \sqrt{\int_0^{\frac{\pi}{2}} \cos dx} = \frac{\pi^3}{24}$$

bo‘lishi kelib chiqadi.

Agar $\varphi(x), \psi(x)$ funksiyalar, $[a,b]$ kesmada differensiallanuvchi bo‘lib, $x\in[a,b]$ uchun, $A\le\varphi(x)\le B$ bo‘lsa, $f(x)$ esa, $[A,B]$ kesmada uzluksiz bo‘lsa, u holda

$$F(x)=\int_{\varphi(x)}^{\psi(x)} f(t)dt; \ a\le x\le b$$

funksiya $[a,b]$ kesmada differensiallanuvchi bo‘ladi va

$$\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(t)dt = f(\psi(x))\cdot\psi'(x) - f(\varphi(x))\cdot\varphi'(x) \qquad (7.17)$$

formula o'rinli.

7.14- misol. Ushbu

$$\frac{d}{dx}\int_{x^2}^{x^3}\frac{dt}{1+t^4}$$

hosilani hisoblang.

Yechilishi. (7.17) formulaga asosan,

$$\frac{d}{dx}\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}} = \frac{1}{\sqrt{(1+x^{12})}}\cdot 3x^2 - \frac{1}{\sqrt{1+x^8}}2x = \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}.$$

7.15-misol. Ushbu

$$g_c(x) = \begin{cases} 0, & x \in [a;b],\ x \neq c, \\ A, & x = c \end{cases}$$

funksiya uchun,

$$\int_a^b g_c(x)dx = 0$$

ekanligini isbotlang.

Isboti. $[a,b]$ kesmaning, $P = \{a = x_0 < x_1 < ... < x_n = b\}$ bo'linishini olamiz. Faraz qilaylik, c nuqta, qism kesmalarning birida yotsin, misol uchun, $c \in [x_m, x_{m+1}]$ bo'lsin. Berilgan $g_c(x)$ funksiyaning P bo'linishga mos kelgan integral yig'indisini tuzamiz: ξ_k deb, qism kesmalarning chap chetki nuqtalarini olamiz:

$$\sigma_P(g_c) = \sum_{k=0}^{n-1} g_c(x)\Delta x_k = g_c(\xi_{m-1})\Delta x_{m-1} + g_c(\xi_m)\Delta x_m,$$

qolgan yig'indilar nolga teng. Shartga ko'ra,

$$|g_c(x)| \leq |A|$$

bo'lgani uchun,

$$|\sigma_P(g_c)| \leq |A|(\Delta x_{m-1} + \Delta x_m).$$

Bundan, $\lim_{d(P)\to 0}\sigma_P(g_c) = 0$, ya'ni $\int_a^b g_c(x)dx = 0$ ekanligi kelib chiqadi.

7.16-misol. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lib, uning qiymatini biror $c \in [a;b]$ nuqtada o'zgartirsak, natijada, yangi hosil bo'lgan $f_1(x)$ funksiya ham, $[a,b]$ kesmada integrallanuvchi bo'lishini va

$$\int_a^b f_1(x)dx = \int_a^b f(x)dx$$

tenglikning o'rinli ekanligini isbotlang.

Yechilishi. Shartga ko'ra,

$$f_1(x)=\begin{cases} f(x), & x\in[a;b],\ x\neq c\ bo'lganda, \\ f_1(x)\neq f(x), & x=c\ bo'lganda. \end{cases}$$

Demak, $f_1(x)=f(x)+g_c(x)$, bunda,

$$g_c(x)=\begin{cases} 0, & x\in[a;b],\ x\neq c\ bo'lganda, \\ A, & x=c\ bo'lganda. \end{cases}$$

Aniq integralning 4^0 - xossasiga, hamda 7.15- misolga asosan,

$$\int_a^b f_1(x)dx=\int_a^b f(x)dx+\int_a^b g_c(x)dx=\int_a^b f(x)dx.$$

7.2 - eslatma. 7.15- va 7.16- misollarga asosan, quyidagi xulosani chiqarish mumkin: $f(x)$ funksiyaning integrallanuvchiligi, uning ba'zi berilgan nuqtalardagi qiymatlariga bog'liq bo'lmaydi.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning berilgan kesmada integrallanuvchi ekanligini ko'rsating:

7.1. $f(x)=\begin{cases} \dfrac{x\ln x}{1-x}, & 0<x<1\ bo'lganda, \\ 0, & x=0\ bo'lganda, \\ -1, & x=1\ bo'lganda. \end{cases}$ **7.2.** $f(x)=\begin{cases} 1, & 0\le x<1\ bo'lganda, \\ 0, & 1\le x<2\ bo'lganda, \\ 3, & 2\le x\le 3\ bo'lganda. \end{cases}$

7.3. $f(x)=\begin{cases} \cos\dfrac{1}{x}, & 0<x\le 1\ bo'lganda, \\ 2, & x=0\ bo'lganda \end{cases}$ **7.4.** $f(x)=\begin{cases} \dfrac{x}{2}+\dfrac{1}{2}, & 1\le x<2\ bo'lganda, \\ x, & 2\le x\le 3\ bo'lganda. \end{cases}$

7.5. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo'lsa, uning shu kesmada chegaralangan ekanligini isbotlang.

7.6. $f(x)$ funksiyaning $[a,b]$ kesmada integrallanuvchi bo'lishi uchun, $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ son topilib, $[a,b]$ kesmaning, diametrlari δ dan kichik bo'lgan, har qanday P_1 va P_2 bo'linishlarida

$$|\sigma_{P_1}(f)-\sigma_{P_2}(f)|<\varepsilon$$

tengsizlikning bajarilishi zarur va etarli ekanligini isbotlang.

7.7. $[a,b]$ kesmada har qanday chegaralangan monoton funksiyaning shu kesmada integrallanuvchi ekanligini isbotlang.

7.8. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada uzluksiz, $P=\{x_k\}_{k=0}^{k_P}$ bu kesmaning bo'linishi, $\xi_i\in[x_{i-1},x_i]$, $\eta_i\in[x_{i-1},x_i]$, $\Delta x_i=x_i-x_{i-1}$, $i=1,2,\dots,k_P$ bo'lsa, u holda,

$$\lim_{\lambda_P \to 0} \sum_{i=1}^{k_P} f(\xi_i) g(\eta_i) \Delta x_i = \int_a^b f(x) g(x) dx$$

tenglik o‘rinli ekanligini isbotlang.

7.9. Agar $f(x)$ funksiya $[0;1]$ kesmada monoton bo‘lsa, $[0;1]$ kesmada

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{k}{n}\right) = O\left(\frac{1}{n}\right), \quad n \to \infty$$

ekanligini isbotlang.

7.10. Agar $f(x)$ funksiya $[a,b]$ kesmada chegaralangan va qavariqligi yuqoriga qaragan (botiq) bo‘lsa, u holda uning shu kesmada integrallanuvchi bo‘lishi va

$$(b-a)\frac{f(a)+f(b)}{2} \le \int_a^b f(x)dx \le (b-a) f\left(\frac{a+b}{2}\right)$$

tengsizlik o‘rinli ekanligini isbotlang.

7.11. Agar $f(x)$ funksiya $[a,b]$ kesmada Riman ma’nosida integrallanuvchi bo‘lsa, u holda,

$$\lim_{\lambda_P \to 0} s_P(f) = \lim_{\lambda_P \to 0} S_P(f) = \int_a^b f(x)dx$$

tenglik o‘rinli ekanligini isbotlang.

7.12. Agar $f(x)$ funksiya $[a,b]$ kesmada integrallanuvchi, $\forall x \in [a,b]$ lar uchun $c \le f(x) \le d$ bo‘lib, $g(x)$ funksiya $[c,d]$ kesmada uzluksiz bo‘lsa, u holda, $g(f(x))$ funksiyaning $[a,b]$ kesmada integrallanuvchi ekanligini isbotlang.

7.13. Kvadrati integrallanuvchi bo‘lib, o‘zi integrallanmaydigan funksiyaga misol tuzing.

7.14. Yig‘indisi integrallanuvchi bo‘lib, qo‘shiluvchilarning har biri integrallanmaydigan funksiyalarga misol tuzing.

7.15. $f(x)g(x)$ funksiyaning $[a,b]$ kesmada integrallanuvchi bo‘lishidan, $f(x)$ va $g(x)$ funksiyalarning $[a,b]$ kesmada har doim integrallanuvchi bo‘lishi kelib chiqadimi?

7.16. Agar $f(x)$ funksiya $[a,b]$ kesmada aniqlangan (a,b) kesmada uzluksiz bo‘lsa, u holda $[a,b]$ kesmada integrallanmaydigan funksiyaga misol tuzing.

Quyidagi integrallarni baholang:

7.17. $J = \int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{1}{2}\sin^2 x}\,dx$.

7.18. $J = \int_0^{2\pi} \frac{dx}{10 + 3\cos x}$.

7.19. $J=\int_0^1 \frac{dx}{\sqrt{2+x-x^2}}$. **7.20.** $J=\int_0^1 e^{-x^2}dx$.

7.21. $J=\int_0^1 \frac{x^6}{\sqrt{1+x}}dx$. **7.22.** $J=\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} x\,arctgx\,dx$.

Quyidagi tengsizliklarni isbotlang:

7.23. $\int_0^{\frac{\pi}{2}} e^{-R\sin x}dx > \frac{\pi}{2R}\left(1-e^{-R}\right)\ (R>0)$. **7.24.** $0{,}5<\int_0^{0{,}5}\frac{dx}{\sqrt{1-x^{2n}}}\le\frac{\pi}{6}\ (n\ge1)$.

7.25. $\frac{\pi}{6}<\int_0^1\frac{dx}{\sqrt{4-x^2-x^3}}<\frac{\pi}{4\sqrt{2}}$. **7.26.** $\frac{2}{\sqrt[4]{e}}<\int_0^2 e^{x^2-x}dx<2e$.

7.27. $\frac{2}{5}<\int_1^2\frac{xdx}{1+x^2}<\frac{1}{2}$. **7.28.** $9<\int_8^{18}\frac{x+1}{x+2}dx<9{,}5$.

7.29. $\frac{1}{k+1}<\int_k^{k+1}\frac{dx}{x}<\frac{1}{k}$ va $\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<\int_1^n\frac{dx}{x}<1+\frac{1}{2}+\ldots+\frac{1}{n-1}$

7.30. $\frac{1}{\sqrt[3]{9}}<\frac{1}{\pi}\int_{-1}^1\frac{\pi+arctgx}{\sqrt[3]{x^2+8}}dx<\frac{3}{2}$. **7.31.** $\frac{\sqrt{2}}{3}<\int_{-1}^1\frac{\cos x}{2+x^2}dx<1$.

Quyidagi integrallarning qaysi biri katta (integralni hisoblamasdan xulosa qiling)?

7.32. $\int_0^1\sqrt{1+x^2}dx$ yoki $\int_0^1 xdx$. **7.33.** $\int_0^{\frac{\pi}{2}}\sin^{10}xdx$ yoki $\int_0^{\frac{\pi}{2}}\sin^2 xdx$.

7.34. $\int_0^1 e^{-x}dx$ yoki $\int_0^{\frac{\pi}{2}} e^{-x^2}dx$. **7.35.** $\int_0^{\pi} e^{-x^2}\cos^2 xdx$ yoki $\int_0^{2\pi} e^{-x^2}\cos^2 xdx$.

7.36. $\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx$ yoki $\int_0^{\pi}\frac{\sin x}{x}dx$. **7.37.** $\int_{1/2}^1\frac{dx}{\sqrt{x}}$ yoki $\int_{1/2}^1\frac{dx}{\sqrt[3]{x}}$.

7.38. $\int_0^1 e^{-x}\sin xdx$ yoki $\int_0^1 e^{-x^2}\sin xdx$. **7.39.** $\int_1^2\frac{dx}{\sqrt{1+x^2}}$ yoki $\int_1^2\frac{dx}{x}$.

7.40. $\int_0^1 x^2\sin^2 xdx$ yoki $\int_0^1 x\sin^2 xdx$ **7.41.** $\int_1^2\ln xdx$ yoki $\int_1^2(\ln x)^2dx$

7.42. $\int_3^4\ln xdx$ yoki $\int_3^4\left(\ln x^2\right)dx$. **7.43.** $\int_0^{\pi} e^{-x^2}\cos^2 xdx$ yoki $\int_{\pi}^{2\pi} e^{-x^2}\cos^2 xdx$

7.44. Agar $f(x)$ funksiya $(p-a,\ p+a)$ kesmada uzluksiz va o‘suvchi bo‘lsa, u holda,

$$2af\left(q-p^2\right)\le\int_{p-a}^{p+a} f\left(x^2-2px+q\right)dx\le 2af\left(a^2+q-p^2\right)$$

tengsizliklarni isbotlang.

Ushbu

$$\int_a^b x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1}, \; m \neq -1, \; \int_0^\pi \sin x dx = 2$$

munosabatlar va 7.6 -misoldagi (7.15) tengsizliklardan foydalanib, quyidagi integrallarni baholang:

7.45. $\int_0^1 \frac{x^7}{\sqrt{1+x^3}} dx.$ **7.46.** $\int_1^4 \frac{\sqrt{1+\sin^2 x}}{x^2} dx.$

7.47. $\int_0^\pi \sqrt{12+x} \cdot \sin x dx.$ **7.48.** $\int_1^e \sqrt[3]{1+\ln^2 x} \; x^2 dx.$

7.6-misoldagi (7.15) formula va $x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6 + \frac{x^5}{120}} (x > 0)$

tengsizlikdan foydalanib quyidagi integralni baholang.

7.49. $J = \int_0^{0.5} \frac{\sin x}{x} dx$ **7.50.** $J = \int_0^{0.64} \sqrt{x} \sin x dx.$ **7.51.** $\int_0^{0.512} \sqrt[3]{x} \sin x dx.$

7.6- misoldagi (7.15) formula va $x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} (x > 0)$

tengsizliklardan foydalanib, quyidagi integrallarni baholang:

7.52. $\int_{0.5}^1 \frac{\ln(1+x)}{x} dx.$ **7.53.** $\int_0^1 \sqrt{x} \ln(1+x) dx.$

Koshi - Bunyakovskiy tengsizligidan foydalanib, quyidagi tengsizliklarni isbotlang:

7.54. $\int_0^1 \sqrt{1+x^2} \sqrt{x^3+1} dx \leq \frac{5}{3}.$ **7.55.** $\int_0^1 \sqrt{1+x^4} dx < \sqrt{1{,}2}.$

7.56. $\int_0^0 x^2 \sqrt{\sin x} dx \leq \frac{2\pi^5}{5}.$ **7.57.** $\int_0^\pi \sqrt{(1+x^2)\sin x} dx \leq 2\pi + \frac{\pi^4}{2}.$

Quyidagi hosilalarni hisoblang:

7.58. $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt.$ **7.59.** $\frac{d}{dx} \int_{\sin x}^{\cos x} \cos \pi t^3 dt.$

7.60. $\frac{d}{dx} \int_{\frac{\pi}{2}}^{\sqrt{x}} \frac{2\sin t}{t} dt.$ **7.61.** $\frac{d}{dx} \int_1^x \ln t dt \; (x > 0).$

7.62. $\frac{d}{dx} \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2) dt \; (x > 0).$ **7.63.** $\frac{d}{da} \int_a^b \sin x^2 dx.$

7.64. Agar $x = \int_1^{t^2} t \ln t dt, \; y = \int_{t^2}^1 t^2 \ln t dt \; (t > 0)$ bo‘lsa, u holda, $\frac{dy}{dx}$ ni toping.

7.65. Agar $x = \int_2^t \frac{\ln z}{z} dz, \; y = \int_5^{\ln t} e^z dz$ bo‘lsa,u holda, $\frac{dy}{dx}$ ni toping.

Quyidagi limitlarni hisoblang:

7.66. $\lim\limits_{x\to 0}\dfrac{\int\limits_0^x \cos x^2 dx}{x}$.

7.67. $\lim\limits_{x\to +0}\dfrac{\int\limits_0^{\sin x}\sqrt{tgx}dx}{\int\limits_0^{tgx}\sqrt{\sin x}dx}$.

Aniq integral yordamida quyidagi funksiyalarning berilgan kesmalardagi o‘rta qiymatini toping.

7.68. $f(x)=\dfrac{1}{x+x^2}$, $[1;15]$.

7.69. $f(x)=\cos^3 x$, $[0;\pi]$.

7.70. $f(x)=\sin^4 x$, $[0;\pi]$.

7.71. $f(x)=\dfrac{1}{e^x+1}$, $[0;2]$.

7.72. $f(x)=\sin^2 x$, $[0;\pi]$.

7.73. $f(x)=20+4\sin x+3\cos x$, $[0;2\pi]$.

7.74. $f(x)=\sqrt{x}$, $[0;100]$.

7.75. $f(x)=\sin x\cdot\sin(x+\alpha)$, $[0;2\pi]$.

7.76. Ellips fokal $r=\dfrac{p}{1-\varepsilon\cos\varphi}\left(0<\varepsilon<1,\ p=\dfrac{b^2}{a}\right)$ radius vektor uzunligining o‘rta qiymatini toping.

7.77. $[0;\frac{T}{4}]$ kesmada $f(t)=\sin\dfrac{2\pi\alpha}{T}t$ funksiyaning o‘rta qiymatini toping, bunda α -parametr.

7.78. φ ning qanday qiymatida $f(t)=\sin\left(\dfrac{2\pi t}{T}+\varphi\right)$ funksiyaning $[0;\frac{T}{4}]$ kesmadagi o‘rta qiymati $\dfrac{2}{\pi}$ ga teng bo‘ladi?

Quyidagi misollarda, ξ ning qanday qiymatida, $\int\limits_a^b f(x)dx=f(\xi)(b-a)$, $a\le\xi\le b$,tenglik o‘rinli bo‘ladi? Bunda

7.79. $f(x)=2x-\frac{3}{4}x^2$, $a=0$, $b=2$.

7.80. $f(x)=tg^2x$, $a=-\frac{\pi}{4}$, $b=0$.

7.81. $f(x)=\ln x$, $a=1$, $b=e^2$

Quyidagi misollarda, ξ ning qanday qiymatida, $\int\limits_a^b f(x)g(x)dx=f(\xi)\int\limits_a^b g(x)dx$ tenglik o‘rinli bo‘ladi? Bunda

7.82. $f(x)=x$, $g(x)=\sqrt{1-x^2}$, $a=0, b=1$

7.83. $f(x)=\sqrt{1-x^2}$, $g(x)=x$, $a=0, b=1$

7.84. $f(x)=x+1$, $g(x)=1/\sqrt{1-x^2}$, $a=0, b=1$.

7.85. $f(x)=1/\sqrt{1-x^2}$, $g(x)=x+1$, $a=0, b=1$.

7.86. $f(x)=x$, $g(x)=\cos x$, $a=0$, $b=\frac{\pi}{2}$.

7.87. $f(x)=\cos x$, $g(x)=x$, $a=0$, $b=\frac{\pi}{2}$

Mustaqil yechish uchun misollarning javoblari

7.10. Ko'rsatma. $\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1+\lambda_2=1,\ \lambda_1 f(x_1)+\lambda_2 f(x_2)\leq f(\lambda_1 x_1+\lambda_2 x_2)$ tengsizlikdan foydalaning. **7.13.** $f(x)=\begin{cases} 1, & x \text{ ratsional son bo'lganda}, \\ -1, & x \text{ irratsional son bo'lganda}. \end{cases}$

7.14. $f(x)=\begin{cases} 1, & x \text{ ratsional son bo'lganda}, \\ 0, & x \text{ irratsional son bo'lganda}. \end{cases}$

$g(x)=\begin{cases} -1, & x \text{ ratsional son bo'lganda}, \\ 0, & x \text{ irratsional son bo'lganda}. \end{cases}$

7.15. $f(x)=g(x)\begin{cases} 1, & x \text{ ratsional son bo'lganda}, \\ -1, & x \text{ irratsional son bo'lganda}. \end{cases}$

7.16. $f(x)=\begin{cases} \dfrac{1}{(x-a)(x-b)}, & x\in(a,b), \\ 0, & x=a,\ x=b. \end{cases}$

7.17. $\frac{\pi}{2}<J<\frac{\pi}{2}\sqrt{\frac{3}{2}}$. **7.18.** $\frac{2}{13}\pi<J<\frac{2}{7}\pi$. **7.19.** $\frac{2}{3}<J<\frac{\sqrt{2}}{2}$.

7.20. $\frac{1}{e}<J<1$. **7.21.** $\frac{\sqrt{2}}{20}<J<\frac{1}{10}$. **7.22.** $\frac{2\pi}{9}<J<\frac{4\pi}{9}$ **7.32.** Birinchisi. **7.33.** Ikkinchisi. **7.34.** Ikkinchisi. **7.35.** Birinchisi. **7.36.**Ikkinchisi. **7.37.** Birinchisi. **7.38.** Birinchisi.**7.39.** Ikkinchisi.**7.40.** Ikkinchisi. **7.41.** Birinchisi. **7.32.** Ikkinchisi. **7.43.** Birinchisi.**7.45.** $\frac{\sqrt{2}}{16}<J<\frac{1}{8}$.

7.46. $\frac{3}{4}<J<\frac{3\sqrt{2}}{4}$. **7.47.** $2\sqrt[3]{12}<J<2\sqrt[3]{12+\pi}$. **7.48.** $\frac{e^3-1}{3}<J<\frac{\sqrt[3]{2}}{3}(e^3-1)$

7.49. $0{,}5-0{,}125/18<J<0{,}5-0{,}125/18+{,}03125/600$.

7.50. $0{,}8[0{,}4(0{,}64)^2-\frac{1}{21}(0{,}64)^4]<J<0{,}8[(0{,}64)^2-\frac{1}{21}(0{,}64)^4+\frac{1}{600}(0{,}64)^6]$.

7.51. $0{,}8[\frac{3}{7}(0{,}512)^2-\frac{1}{26}(0{,}512)^4]<J<0{,}8[\frac{3}{7}(0{,}512)^2-\frac{1}{26}(0{,}512)^4+\frac{1}{760}(0{,}512)^6]$.

7.52. $\frac{5}{16}<J<\frac{3}{8}$. **7.53.** $\frac{9}{35}<J<\frac{313}{945}$. **7.58.** $2x\sqrt{1+x^4}$.

7.59. $-\sin x\cos(\pi\cos^3 x)-\cos x\cos(\pi\sin^3 x)$. **7.60.** $\frac{\sin\sqrt{x}}{x}$. **7.61.** $\ln x$.

7.62. $\frac{\cos x}{2\sqrt{x}}+\frac{1}{x^2}\cos\frac{1}{x^2}$. **7.63.** $-\sin a^2$. **7.64.** $-t^2$. **7.65.** $\frac{t^2}{\ln t}$. **7.66.** 1.

7.67. 1. **7.68.** 0,3648. **7.69.** 0. **7.70.** $\frac{3}{8}$. **7.71.** 0,283. **7.72.** 0,5. **7.73.** 20.

7.74. $6\frac{2}{3}$. **7.75.** $\frac{1}{2}\cos\alpha$.

7.76. $\frac{p}{\sqrt{1-\varepsilon^2}}=b-$ellipsning kichik o'qi uzunligi. **7.77.** $\frac{1-\cos\pi\alpha}{\pi\alpha}$.

7.78. $\varphi = 2k\pi,\ \varphi = \frac{\pi}{2} + 2k\pi,\ k \in Z.$ **7.79.** 0 $\xi_1 = \frac{1}{3},\ \xi_2 = 2.$

7.80. $\xi = -arctg\sqrt{\frac{4-\pi}{\pi}}$. **7.81.** $\ln\xi = \frac{e^2 - 1}{e^2 + 1}$. **7.82.** $\frac{4}{3\pi}$. **7.83.** $\sqrt{\frac{5}{9}}$.

7.84. $\frac{2}{\pi}$. **7.85.** $\sqrt{\frac{\pi-1}{\pi+2}}$. **7.86.** $\frac{\pi-2}{2}$. **7.87.** $\arccos\frac{4\pi-8}{2}$.

8-§. Aniq integralni hisoblash

Har doim, har qanday integrallanuvchi funksiyaning aniq integralini, integral yig'indining limiti sifatida qarab, hisoblash oson bo'lavermaydi, ya'ni integral yig'indini tuzib, uning limitini hisoblashda ancha noqulayliklar va qiyinchiliklarga duch kelinadi. Shuning uchun, aniq integralni yuqoridagi ta'rif bo'yicha hisoblash usulidan boshqa soddaroq usulini topish zaruriyati tug'iladi. Bu usullarni quyida keltirib o'tamiz.

8.1. Nyuton-Leybnis formulasi. Yuqorida ko'rdikki, agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u holda u shu kesmada boshlang'ich funksiyalarga ega bo'ladi. Aniq integralning 11^0 -xossasiga asosan,

$$\Phi(x) = \int_a^x f(t)dt$$

Funksiya, $f(x)$ funksiyaning boshlang'ich funksiyalaridan biridir. $F(x) - f(x)$ funksiyaning $[a,b]$ kesmadagi ixtiyoriy boshlang'ich funksiyasi bo'lsin. Ma'lumki, $\Phi(x)$ va $F(x)$ boshlang'ich funksiyalarning biri, ikkinchisidan o'zgarmas songa farq qiladi, ya'ni

$$\int_a^x f(t)dt = F(x) + C,\ a \le x \le b.$$

Bundan, $x = a$ deb olib,

$$0 = F(a) + C,\ C = -F(a)$$

ekanligini topamiz, ya'ni $\forall x \in [a,b]$ uchun,

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (8.1)$$

Nyuton - Leybnis formulasiga ega bo'lamiz. Odatda, (8.1) formula, *integral hisobning asosiy formulasi*, deb ham yuritiladi.

8.1-**eslatma**. Odatdagidek,

$$F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$$

belgilashni olsak, u holda, (8.1) Nyuton – Leybnis formulasini,

$$\int_a^b f(x)dx = F(x)\Big|_a^b$$

ko'rinishda ham yozish mumkin.

8.1-misol. Ushbu $\int_a^b x^m dx,\ m \neq -1$ integralni Nyuton-Leybnis formulasi orqali hisoblang.

Yechilishi. Ma'lumki, integral ostidagi $f(x) = x^m$ funksiyaning boshlang'ich funksiyasi, $F(x) = \dfrac{x^{m+1}}{m+1}$ dan iborat. Nyuton-Leybnis formulasiga asosan,

$$\int_a^b x^m dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{b^{m+1} - a^{m+1}}{m+1},\ m \neq -1$$

bo'ladi. Xususiy holda, $m = -1$ bo'lganda,

$$\int_a^b \frac{dx}{x} = \ln|x|\Big|_a^b = \ln|b| - \ln|a|\ .$$

Shunday qilib, aniq integralni hisoblash masalasi, integral ostidagi integrallanuvchi funksiyaning boshlang'ich funksiyasini topish masalasiga keltirilar ekan. Lekin har qanday integrallanuvchi funksiyaning ham boshlang'ich funksiyasini topish oson bo'lavermaydi. Shuning uchun, aniq integralni hisoblashda, boshqa usullardan ham foydalanishga to'g'ri keladi.

8.2.Aniq integrallarda o'zgaruvchilarni almashtirish usuli.

8.1-teorema. $x = g(t)$ funksiya $[\alpha, \beta]$ kesmada aniqlangan uzluksiz differensiallanuvchi va uning qiymatlari to'plami $[a,b]$ kesmadan iborat bo'lib, $g(\alpha) = a,\ g(\beta) = b$ bo'lsin. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u holda,

$$\int_a^b f(x)dx = \int_\alpha^\beta f[g(t)]g'(t)dt \qquad (8.2)$$

formula o'rinli.

Bu formulaga, *aniq integralda o'zgaruvchilarni almashtirish* formulasi deyiladi.

8.2 -misol. (8.2) formulani isbotlang.

Yechilishi. $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lgani uchun, 7.5-natijaga asosan, $f(x)$ funksiya, $[a,b]$ kesmada, $\Phi(x)$ boshlang'ich funksiyaga ega bo'ladi. Bu holda, $F(t) = \Phi[g(t)],\ (\alpha \le t \le \beta)$ murakkab funksiya hosilaga ega:

$$F'(t) = \Phi'[g(t)] \cdot g'(t) = f[g(t)] \cdot g'(t).$$

Bundan, Nyuton - Leybnis formulasiga asosan,

$$\int_{\alpha}^{\beta} f[g(t)]g'(t)dt = \int_{\alpha}^{\beta} F'(t)dt = F(\beta)-F(\alpha) = \Phi[g(\beta)]-$$

$$-\Phi[g(\alpha)] = \Phi(b)-\Phi(a) = \int_{a}^{b} f(x)dx.$$

8.3-misol. Ushbu

$$\int_{0}^{\frac{\sqrt{2}}{2}} \sqrt{\frac{1-x}{1+x}}dt$$

integralni hisoblang.

Yechilishi. $x=\cos t$ almashtirishni olamiz. Bu almashtirishning to‘g‘riligini ko‘rsatish uchun, 8.1– teoremaning shartlarini tekshirib ko‘ramiz: $x=g(t)=\cos t$ funksiya R da uzluksiz, yangi t o‘zgaruvchi, $t\in[\frac{\pi}{4};\frac{\pi}{2}]$ kesmada o‘zgarganda, eski $x=g(t)$ o‘zgaruvchi, $\left[0,\frac{\sqrt{2}}{2}\right]$ kesmada o‘zgaradi, ya’ni $g\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, $g\left(\frac{\pi}{2}\right)=0$; $g'(t)=-\sin t$, $[\frac{\pi}{4};\frac{\pi}{2}]$ kesmada uzluksiz. Demak, 8.1– teoremaning hamma shartlari bajariladi.

Shunday qilib, $x=\cos t$, $dx=-\sin t\,dt$, $g\left(\frac{\pi}{2}\right)=0$, ya’ni

$$\beta=\frac{\pi}{2},\ g\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2},\quad \alpha=\frac{\pi}{4},\quad \frac{1-x}{1+x}=\frac{1-\cos t}{1+\cos t}=tg^2\frac{t}{2}$$

Bularni e’tiborga olgan holda,

$$\int_{0}^{\frac{\sqrt{2}}{2}} \sqrt{\frac{1-x}{1+x}}dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} tg\frac{t}{2}\cdot(-\sin t)dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\cos t)dt =$$

$$=[t-\sin t]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2}-1-\frac{\pi}{4}+\frac{\sqrt{2}}{2}=\frac{\pi}{4}-1+\frac{\sqrt{2}}{2}.$$

8.4-misol. Ushbu

$$\int_{0}^{\ln 3}\frac{e^x\sqrt{e^x+1}}{e^x+5}dx$$

integralni hisoblang.

Yechilishi. $\sqrt{e^x+1}=t$ almashtirishni bajaramiz: $x_1=0$ bo‘lganda, $t_1=\sqrt{2}$, $x_2=\ln 3$ bo‘lganda, $t_2=2$. Demak, x o‘zgaruvchi, $[0;\ln 3]$ kesmada o‘zgarganda, yangi t o‘zgaruvchi $[\sqrt{2};2]$ kesmada o‘zgaradi. $x=\ln(t^2-1)$ funksiya, $t=\sqrt{e^x+1}$ funksiyaga teskari, $[\sqrt{2};2]$ kesmada monoton, uzluksiz va $x_t'=\frac{2t}{t^2-1}$.

Shunday qilib, 8.1-teoremaning hamma shartlari o‘rinli, shuning uchun, (8.2) formulaga asosan,

$$\int_0^{\ln} \frac{e^x\sqrt{e^x+1}}{e^x+5}dx = \int_{\sqrt{2}}^{2} \frac{(t^2-1)\cdot t\cdot 2tdt}{(t^2+4)(t^2-1)} = \int_{\sqrt{2}}^{2} \frac{2t^2dt}{t^2+4} =$$

$$= 2\int_{\sqrt{2}}^{2}\left(1-\frac{4}{t^2+4}\right)dt = 2[t-2arctg\frac{t}{2}]_{\sqrt{2}}^{2} = 2[2-2\cdot\frac{\sqrt{\pi}}{4}-\sqrt{2}+2arctg\frac{\sqrt{2}}{2}] =$$

$$= 4-\sqrt{\pi}-2\sqrt{2}+2arctg\frac{\sqrt{2}}{2}.$$

8.3. Bo‘laklab integrallash usuli.

8.2-teorema. Agar $u=u(x)$ va $v=v(x)$ funksiyalar, $[a,b]$ kesmada uzluksiz va uzluksiz hosilalarga ega bo‘lib, ularning xosilalari, shu kesmada integrallanuvchi bo‘lsa, u holda,

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \qquad (8.3)$$

yoki

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx$$

formula o‘rinli.

8.5 -misol. Ushbu

$$\int_0^{\sqrt{3}} x\,arctgx\,dx$$

integralni hisoblang.

Yechilishi. Integral ostidagi funksiyalar, 8.2– teoremaning barcha shartlarini qanoatlantirishiga ishonch hosil qilish qiyin emas. $u = arctgx,\ dv = xdx$ deb olib, (8.3) formulaga asosan,

$$\int_0^{\sqrt{3}} x\,arctg\,x\,dx = [du = \frac{dx}{1+x^2},\ v = \frac{x^2}{2}] =$$

$$= \frac{x^2}{2}arctgx\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2dx}{1+x^2}] = \frac{\pi}{3}\cdot\frac{3}{2} - \frac{1}{2}(x-arctgx)\Big|_0^{\sqrt{3}} =$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{3}-\frac{\pi}{3}\right) = \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Misolni Maple tizimidan foydalanib yechish:

```
> int(x*arctan(x),x=0..sqrt(3));
```

$$\frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

8.6 -misol. Ushbu

$$J_{n,\alpha} = \int_0^1 x^\alpha \ln^n xdx,\ \alpha > 0,\ n \in N, \qquad (8.4)$$

integralni hisoblang.

Yechilishi. $f(x)=x^{\alpha}\ln^{n}x$ funksiya $(0;1]$ kesmada aniqlangan va uzluksiz, hamda $\lim\limits_{x\to+0}f(x)=0$. $f(x)$ funksiyani qayta aniqlaymiz, ya'ni $x=0$ da $f(x)=0$ deb olsak, u holda $f(x)$ funksiya $[0;1]$ kesmada uzluksiz bo'ladi, ya'ni u $[0;1]$ kesmada integrallanuvchi bo'ladi.

Berilgan (8.4) integralni bo'laklab integrallaymiz:

$$J_{n,\alpha}=\int_0^1 x^{\alpha}\ln^{n}x\,dx=[u=\ln^{n}x, dv=x^{\alpha}dx,\ du=n\frac{\ln^{n-1}x}{x}dx,$$

$$v=\frac{x^{\alpha+1}}{\alpha+1}]=\frac{1}{\alpha+1}\cdot x^{\alpha+1}\ln^{n}x\Big|_0^1-\frac{n}{\alpha+1}\int_0^1 x^{\alpha}\ln^{n-1}x\,dx=$$

$$=-\frac{n}{\alpha+1}J_{n-1,\alpha}.$$

Bundan, ketma-ket bo'laklab integrallash natijasida,

$$J_{n,\alpha}=(-1)^{n}\frac{n!}{(\alpha+1)^{n}}J_{0,\alpha}=(-1)^{n}\frac{n!}{(\alpha+1)^{n}}\int_0^1 x^{\alpha}dx=$$

$$=(-1)^{n}\frac{n}{(\alpha+1)^{n+1}}$$

rekurrent formulani hosil qilamiz.

8.7 -misol. Ushbu

$$\int_1^e x^2\ln x\,dx,$$

integralni hisoblang.

Yechilishi.

$$\int_1^e x\ln^2 x\,dx=\left|\begin{matrix}u=\ln^2 x;\ du=(\ln^2 x)'dx=\dfrac{2\ln x}{x}dx;\\ dv=xdx;\ v=\dfrac{x^2}{2}\end{matrix}\right|=\frac{x^2}{2}\ln^2 x\Big|_1^e-\int_1^e\frac{x^2}{2}\frac{2\ln x}{x}dx=$$

$$=\frac{e^2}{2}-\int_1^e x\ln x\,dx\left|\begin{matrix}u=\ln x;\ du=(\ln x)'dx=\dfrac{1}{x}dx;\\ dv=xdx;\ v=\dfrac{x^2}{2}\end{matrix}\right|=\frac{e^2}{2}-\frac{x^2}{2}\ln x\Big|_1^e+\int_1^e\frac{x^2}{2}\frac{1}{x}dx=$$

$$=\frac{e^2}{2}-\frac{e^2}{2}+\frac{1}{2}\int_1^e x\,dx=\frac{e^2}{4}-\frac{1}{4}.$$

Misolni Maple tizimidan foydalanib yechish:

> int(x*(ln(x))^2,x=1..exp(1));

$$\frac{1}{4}e^2-\frac{1}{4}$$

8.8-misol. Ushbu

$$\int_{-1}^{1}Q_m(x)P_n(x)dx=0 \qquad (8.5)$$

tenglikni isbotlang, bunda, $P_n(x)=\frac{1}{2^n n!}\frac{d^n[(x^2-1)]^n}{dx^n}$ – Lejandr ko‘phadi $(n=0,1,2,...)$, $Q_m(x)$ esa, darajasi $m<n$ bo‘lgan ixtiyoriy ko‘phad.

Yechilishi. Ravshanki, $\frac{d^k[(x^2-1)^n]}{dx^k}$ $(k=0,1,2....)$ funksiya $x=-1$ va $x=1$ nuqtalarda nulga aylanadi. Berilgan integralni ketma – ket bo‘laklab integrallash natijasida,

$$\left(\frac{1}{2^n n!}\right)^{-1}\int_{-1}^{1}Q_m(x)P_n(x)dx=\left[u=Q_m(x),\ dv=\frac{d^n[(x^2-1)^n]}{dx^{n-1}}=d[\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right]=$$

$$=Q_m(x)\frac{d^{n-1}[(x^2-1)^n]}{dx^{n-1}}\Big|_{-1}^{1}-\int_{-1}^{1}Q_m'(x)\frac{d^{n-1}[(x^2-2)^n]}{dx^{n-1}}dx=...=(-1)^m Q_m^{(m)}(x)\int_{-1}^{1}\frac{d^{n-1}[(x^2-2)^n]}{dx^{n-1}}=$$

$$=(-1)^m Q_m^{(m)}(x)\frac{d^{n-m-1}[(x^2-1)^n]}{dx^{n-m-1}}\Big|_{-1}^{1}=0$$

ekanligini keltirib chiqaramiz, chunki $Q_m^{(m+1)}(x)=0$. Bundan (8.5) tenglikning o‘rinli ekanligi kelib chiqadi.

Mustaqil yechish uchun misollar

Quyidagi integrallarni, Nyuton-Leybnis formulasiga asosan, hisoblang:

8.1. $\int_0^1(2x-3)dx$. **8.2.** $\int_1^0 5x^4dx.$ **8.3.** $\int_1^4 2\sqrt{x}dx$.

8.4. $\int_1^5 2\sqrt{x-1}dx.$ **8.5.** $\int_{-2}^{0}(x+1)(x-2)dx.$ **8.6.** $\int_1^2\left(3t+\frac{4}{t^2}\right)dt.$

8.7. $\int_0^\pi(1+\cos)dx.$ **8.8.** $\int_0^1(x^2+\sqrt{x})dx.$ **8.9.** $\int_0^1\left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right)dx$

8.10. $\int_0^1(1+x)^{17}dx$ **8.11.** $\int_0^a\left(\sqrt{a}-\sqrt{x}\right)^2dx.$ **8.12.** $\int_1^2\frac{6-t}{t^3}dt.$

Quyidagi integrallarni, Nьyuton – Leybnis formulasiga asosan, hisoblang:

8.13. $\int_{-1}^{1}(3x^2-4x+7)dx.$ **8.14.** $\int_1^2\frac{4}{x^2}dx.$ **8.15.** $\int_1^4\frac{dt}{t\sqrt{t}}.$

8.16. $\int_0^1\frac{36}{(2x+1)^3}dx.$ **8.17.** $\int_{1/8}^{1}x^{-1/3}(1-x^{2/3})^{3/2}dx.$ **8.18.** $\int_0^\pi\sin^2 5xdx.$

8.19. $\int_0^{\pi/3}\sec^2 tdt.$ **8.20.** $\int_\pi^{3\pi}ctg^2\frac{x}{2}dx.$ **8.21.** $\int_{-\frac{\pi}{3}}^{0}\sec xtgxdx.$.

8.22 $\int\limits_{0}^{\pi/2} 5(\sin x)^{3/2}\cos x dx.$ **8.23.** $\int\limits_{0}^{\pi/2} \frac{3\sin x\cos x}{\sqrt{1+3\sin^2 x}}dx.$ **8.24.** $\int\limits_{0}^{\pi/3} \frac{tgt}{\sqrt{2\sec t}}dt.$

8.25. $\int\limits_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x}.$ **8.26.** $\int\limits_{2}^{3} \frac{dx}{\sqrt{5+4x-x^2}}.$

8.27. $\int\limits_{\pi/4}^{3\pi/4} \cos ecx\, ctgx dx.$ **8.28.** $\int\limits_{1}^{\sqrt{2}} \left(\frac{x^2}{2}-\frac{1}{x^5}\right)dx.$

Quyidagi integrallarga, Nyuton-Leybnis formulasini formal ravishda qo‘llaganda, noto‘g‘ri natijaga kelinishini izohlang:

8.29. $\int\limits_{0}^{dx} \frac{dx}{(2+tg^2 x)\cos^2 x}.$ **8.30.** $\int\limits_{-1}^{1} \frac{d}{dx}\left(arctg\frac{1}{x}\right)dx.$ **8.31.** $\int\limits_{-1}^{1} \frac{dx}{x}.$

Quyidagi integrallarni hisoblang:

8.32. $\int\limits_{-2}^{0} (2x+5)dx.$ **8.33.** $\int\limits_{0}^{1} \left(x^2+\sqrt{x}\right)dx.$ **8.34.** $\int\limits_{-1}^{1} \sqrt[3]{x}dx.$

8.35. $\int\limits_{-1}^{2} \sqrt[3]{x}dx.$ **8.36.** $\int\limits_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$ **8.37.** $\int\limits_{2}^{9} \sqrt[3]{x-1}dx.$

8.38. $\int\limits_{0}^{\pi} (1+\cos x)dx.$ **8.39.** $\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos ec\, x dx.$ **8.40.** $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(8y^2+\sin y\right)dy.$

8.41. $\int\limits_{1}^{\sqrt{2}} \left(\frac{u^2}{2}-\frac{1}{4^5}\right)du.$ **8.42.** $\int\limits_{-4}^{4} |x|dx.$ **8.43.** $\int\limits_{-6}^{6} x|x|dx.$

Quyidagi integrallarni hisoblang:

8.44. $\int\limits_{0}^{2} 3^x dx.$ **8.45.** $\int\limits_{0}^{1} \frac{x^2dx}{1+x^6}.$ **8.46.** $\int\limits_{e^2}^{e^3} \frac{dx}{x\ln x}.$

8.47. $\int\limits_{-\pi}^{\pi} \sin^2 x dx.$ **8.48.** $\int\limits_{0}^{1} \frac{dx}{4x^2+4x+5}.$ **8.49.** $\int\limits_{2}^{3} \frac{dx}{x^2-2x-8}.$

8.50. $\int\limits_{1}^{e} \frac{\cos(\ln x)dx}{x}.$ **8.51.** $\int\limits_{1}^{e} \frac{dx}{x(1+\ln^2 x)}$

8.52. $\int\limits_{0}^{1} \sqrt{4-x^2}dx.$ **8.53.** $\int\limits_{0}^{1} \frac{z^3}{z^8+1}dz.$

Quyidagi aniq integrallarni, o‘zgartiruvchilarni almashtirish yordamida hisoblang:

8.54. $\int\limits_{0}^{3} \sqrt{y+1}dy.$ **8.55.** $\int\limits_{-1}^{0} \sqrt{y+1}\,dx.$ **8.56.** $\int\limits_{0}^{\pi} 3\cos^2 x\sin dx.$

8.57. $\int\limits_{2\pi}^{3\pi} \cos^2 x\sin x dx.$ **8.58.** $\int\limits_{-1}^{1} \frac{5x}{\left(4+x^2\right)^2}dx.$ **8.59.** $\int\limits_{0}^{1} \frac{5x}{\left(4+x^2\right)^2}dx.$

8.60. $\int\limits_{0}^{\frac{\pi}{6}} (1-\cos 3t)\sin 3t dt.$ **8.61.** $\int\limits_{\frac{\pi}{6}}^{\pi/3} (1-\cos 3t)\sin 3t dt$**.** **8.62.** $\int\limits_{0}^{2\pi} \frac{\cos t}{\sqrt{4+3\sin t}}dt.$

8.63. $\int\limits_{-\pi}^{\pi} \frac{\cos t}{\sqrt{4+3\sin t}}dt.$ **8.64.** $\int\limits_0^1 \sqrt{x^5+2x(5x^4+2)}dx.$ **8.65.** $\int\limits_0^{\frac{\pi}{6}} \cos^{-3}2x\sin 2x dx.$

8.66. $\int\limits_0^{\frac{\pi}{4}} (1-\sin 2t)^{3/2}\cos 2t dt.$ **8.67.** $\int\limits_0^1 \sin \pi x dx.$ **8.68.** $\int\limits_{\pi/6}^{\pi/3} ctgx dx.$.

8.69. $\int\limits_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos^2 x}dx.$ **8.70.** $\int\limits_1^2 \frac{e^{1/x}}{x^2}dx.$ **8.71.** $\int\limits_0^c \frac{dx}{x^2+c^2}.$

8.72. $\int\limits_1^e \frac{\ln x^3}{x}dx.$ **8.73.** $\int\limits_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}dx.$

8.74. $\int\limits_0^1 (1-2x)^3 dx.$ **8.75.** $\int\limits_0^{\pi} \sin^2\left(1+\frac{t}{2}\right)dt.$

Quyidagi integrallarni, o'zgaruvchilarni almashtirish usulidan foydalanib hisoblang:

8.76. $\int\limits_1^2 \frac{dx}{x\sqrt{1+x^2}}.$ **8.77.** $\int\limits_0^1 \frac{\sqrt{e^x}dx}{\sqrt{e^x+e^{-x}}}.$ **8.78.** $\int\limits_3^6 \frac{\sqrt{x^2-9}}{x^4}dx.$

8.79. $\int\limits_0^a \frac{dx}{x+\sqrt{a^2-x^2}}.$ **8.80.** $\int\limits_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}.$ **8.81.** $\int\limits_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}dx.$

8.82. $\int\limits_1^e \frac{dx}{x\sqrt{1+\ln x}}.$ **8.83.** $\int\limits_{-1}^1 (e^x+e^{-x})tgx dx.$ **8.84.** $\int\limits_{-1}^1 \frac{1+x^2}{1+x^4}dx.$

8.85. Ushbu $\int\limits_1^7 (x^2-6x+13)dx$ integralda, $x^2-6x+13=t$ almashtirishni olish mumkinmi? Javobingizni sharhlang.

8.86. Ushbu $\int\limits_0^1 \sqrt{1-x^2}dx$ integralda, $x=\sin t$ almashtirishni olishmumkinmi? Javobingizni sharhlang.

8.87. Ushbu $\int\limits_0^{\pi} \frac{dx}{1+\sin^2 x}$ integralda, $tgx=t$ almashtirishni olish mumkinmi? Javobingizni sharhlang.

Bo'laklab integrallash formulasi yordamida, aniq integrallarni hisoblang:

8.88. $\int\limits_0^4 xe^{-x}dx.$ **8.89.** $\int\limits_1^2 x\ln x dx.$ **8.90.** $\int\limits_0^{\pi/2} t^2\sin 2t dt.$

8.91. $\int\limits_0^1 x\,arctg(x^2)dx.$ **8.92.** $\int\limits_0^{\pi/3} x\,tg^2 x dx.$ **8.93.** $\int\limits_1^2 x^3\ln x dx.$

8.94. $\int\limits_0^1 x^2e^{-x}dx.$ **8.95.** $\int\limits_1^{e^2} x\ln\sqrt{x}dx.$ **8.96.** $\int\limits_1^{e^2} \ln x dx.$

8.97. $\int\limits_0^{1/2} x\cdot\cos\pi x dx.$. **8.98.** $\int\limits_0^1 \ln(1+x^2)dx.$ **8.99.** $\int\limits_0^{1/4} \arcsin 2x dx.$

Quyidagi integrallarni, bo'laklab integrallash formulasidan foydalanib hisoblang:

8.100. $\int_0^{\frac{\pi}{2}} x\cos x dx$. **8.101.** $\int_1^2 (3x+2)\ln x dx$. **8.102.** $\int_0^{\pi} e^x \sin 2x dx$.

8.103. $\int_0^{\pi} x^3 \sin x dx$. **8.104.** $\int_0^{\sqrt{3}} x\, arctg x\, dx$. **8.105.** $\int_1^2 \frac{\ln x}{x^5} dx$.

8.106. $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^2 x dx$. **8.107.** $\int_0^{\pi} e^x \cos^2 x dx$. **8.108.** $\int_{-a}^{a} \frac{\ln(2a-x)}{\ln(4a^2-x^2)} dx$ $\left(a > \frac{1}{\sqrt{3}}\right)$

8.109. $\int_0^{\pi/2} \sin x\sin 2x\sin 3x dx$. **8.110.** $\int_0^1 x^{15}\sqrt{1+3x^8}dx$. **8.111.** $\int_0^1 arc\cos x dx$.

Quyidagi, uzilishga ega bo'lgan chegaralangan, funksiyalarning aniqmas integrallarini toping:

8.112. $\int sign\, x\, dx$. **8.113.** $\int sign(\sin x)dx$. **8.114.** $\int [x]dx$.

8.115. $\int x[x]dx$. **8.116.** $\int (-1)^{[x]}dx$.

8.117. $\int_0^x f(t)dt$, bunda $f(t)=\begin{cases} 1, & |t|<l \quad bo'lganda, \\ 0, & |t|>l, \quad bo'lganda, \quad l \ge 0. \end{cases}$

Quyidagi, uzilishga ega bo'lgan chegaralangan, funksiyalarning aniq integrallarini hisoblang:

8.118. $\int_0^3 sign(x-x^3)dx$. **8.119.** $\int_0^2 [e^x]dx$. **8.120.** $\int_0^6 [x]\sin\frac{\pi x}{6}dx$.

8.121. $\int_0^{\pi} x\, sign(\cos x)dx$. **8.122.** $\int_0^{n+1} \ln[x]dx, n\in N$. **8.123.** $\int_0^1 sign(\sin\ln x)dx$.

Mustaqil yechish uchun misollarning javoblari

8.1. -2. **8.2.** 1. **8.3.** $28/3$. **8.4.** $32/3$.

8.5. $\frac{2}{3}$. **8.6.** $13/2$. **18.7.** π. **18.8.** 1.

18.9. $-\frac{4}{15}$. **18.10.** $\frac{1}{18}(2^{18}-1)$. **18.11.** $\frac{1}{6}a^2$. **18.12.** $\frac{7}{4}$.

8.13. 16. **8.14.** 2. **8.15.** 2. **8.16.** 8. **8.17.** $\frac{27\sqrt{3}}{160}$.

8.18. $\frac{\pi}{2}$. **8.19.** $\sqrt{3}$. **8.20.** $6\sqrt{3}-2\pi$. **8.21.** -1. **8.22.** 2.

8.23. 1. **8.24.** $\sqrt{2}-1$. **8.25.** $\frac{80\sqrt{3}}{27}$. **8.26.** $\frac{\pi}{2}$. **8.27.** 0.

8.28. $\frac{16\sqrt{2}-19}{48}$. **8.29.** $\frac{1}{\sqrt{2}}arctg\left(\frac{tgx}{\sqrt{2}}\right)$ funksiya, integral ostidagi funksiya uchun boshlang'ich funksiya bo'lib, u, $0\le x\le 2\pi$ da uzilishga ega.

8.30. $arctg\frac{1}{x}$ funksiya, $x=0$ nuqtada uzilishga ega.

8.31. Integral ostidagi $\frac{1}{x}$ funksiya va uning $\ln|x|$ boshlang'ich funksiyasi, $[-1;1]$ kesmada uzilishga ega.

8.32. 6. **8.33.** 1. **8.34.** 0. **8.35** $\frac{3}{4}(2\sqrt{2}-1)$. **8.36.** $\pi/6$. **8.37.** $\frac{45}{7}$.

8.38. π. **8.39.** 0. **8.40.** $\frac{2\pi^3}{3}$. **8.41.** $\frac{16\sqrt{2}-19}{48}$. **8.42** 16. **8.43** 0.

8.44. $\frac{8}{\ln 3}$. **8.45.** $\frac{\pi}{12}$. **8.46.** $\ln 1{,}5$. **8.47.** π. **8.48.** $\frac{1}{4}arctg(4/7)$.

8.49. $\frac{1}{6}\ln(2/5)$. **8.50.** $\sin 1$. **8.51.** $\frac{\pi}{4}$. **8.52.** $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$. **8.53.** $\frac{\pi}{6}$.

8.54. $14/3$. **8.55.** $2/3$. **8.56.** 3) 2. **8.57.** 2. **8.58.** 0.

8.59. $1/8$. **8.60.** $1/6$. **8.61.** $1/2$. **8.62.** 0. **8.63.** 0. **8.64.** $2\sqrt{3}$.

8.65. $\frac{3}{4}$. **8.66.** $\frac{1}{5}$. **8.67.** $2/\pi$. **8.68.** $\frac{1}{2}\ln 3$. **8.69.** 0.

8.70. $e-\sqrt{e}$. **8.71.** $\frac{\pi}{4c}$. **8.72.** $3/2$. **8.73.** $\sqrt{2}$. **8.74.** 0.

8.75. $\frac{\pi}{2}+\sin 2$. **8.76.** $\ln\frac{2(1+\sqrt{2})}{1+\sqrt{5}}$. **8.77.** $\ln\frac{e+\sqrt{1-e^2}}{1+\sqrt{2}}$. **8.78.** $\frac{\sqrt{3}}{72}$.

8.79. $\frac{\pi}{4}$. **8.80.** $\ln 2$. **8.81.** $\frac{\pi}{12}$. **8.82.** $2(\sqrt{2}-1)$. **8.83.** 8) 0. **8.84.** $\frac{\pi}{\sqrt{2}}$.

8.85. Yo'q. **8.86.** Mumkin. **8.87.** Yo'q. **8.88.** $1-5e^{-1}$.

8.89. $\ln 4-\frac{3}{4}$. **8.90.** $\frac{\pi^2-4}{8}$. **8.91.** $\frac{\pi}{8}-\frac{1}{4}\ln 2$. **8.92.** $\frac{\pi\sqrt{3}}{3}-\ln 2-\frac{\pi^2}{18}$.

8.93. $4\ln 2-\frac{15}{16}$. **8.94.** $2-5e^{-1}$. **8.95.** $\frac{3}{8}e^4+\frac{1}{8}$. **8.96.** e^2+1. **8.97.** $\frac{1}{2\pi}-\frac{1}{\pi^2}$.

8.98. $\ln 2+\frac{\pi}{2}-2$. **8.99.** $\frac{\pi}{24}+\frac{\sqrt{3}-2}{4}$. **8.100.** $\frac{\pi}{2}-1$. **8.101.** $10\ln 2-\frac{17}{4}$.

8.102. 0. **8.103.** $\pi^3-6\pi$. **8.104.** $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$. **8.105.** $\frac{15}{256}-\frac{\ln 2}{64}$. **8.106.** $\frac{\pi}{32}$.

8.107. $\frac{3}{5}(e^{\pi}-1)$. **8.108.** a. **8.109.** $\frac{1}{6}$. **8.110.** $\frac{29}{270}$. **8.111.** 1.

8.112. $|x|+C$. **8.113.** $arc\cos(\cos x)+C$. **8.114.** $x[x]-\frac{[x]([x]+1)}{2}+C$

8.115. $\frac{x^2}{2}[x]-\frac{[x]([x]+1)(2[x]+1)}{12}+C$. **8.116.** $\frac{1}{\pi}\arccos x(\cos\pi x)+C$.

8.117. $\frac{1}{2}(|x+l|-|x-l|)$. **8.118.** -1. **8.119.** $14-\ln(7!)$. **8.120.** $\frac{30}{\pi}$.

8.121. $-\frac{\pi^2}{4}$. **8.122.** $\ln(\pi!)$. **8.123.** $-th\frac{\pi}{2}$.

III bob. ANIQ INTEGRALNING TADBIQLARI

9-§. Aniq integral yordamida tekis shaklning yuzini hisoblash

9.1. Dekart koordinatalar sistemasida berilgan tekis shaklning yuzini hisoblash

Tekislikda xOy Dekart koordinatalar sistemasi berilgan bo‘lsin.

9.1-ta’rif. Tekislikning L oddiy (karrali nuqtalarga ega bo‘lmagan) yopiq egri chiziq bilan chegaralangan qismi- *tekis shakl (figura)* deyiladi. Bunda L - tekis shaklning *chegarasi* deyiladi.

O’qlarga nisbatan standart sohalar.

9.2-ta’rif. Koordinatalari. $[a,b]$ kesmada uzluksiz $f_1(x)$ va $f_2(x)$ funksiyalar uchun, $a \le x \le b$, $f_1(x) \le y \le f_2(x)$ munosabatlarni qanoatlantiradigan $M(x,y)$ nuqtalar to‘plami $D - Ox$ o‘qqa nisbatan *standart soha* deyiladi.

Ta’rifning *geometrik ma’nosi* shundan iboratki, D coha chapdan va o‘ngdan, mos ravishda, $x = a$, $y = b$ to‘g‘ri chiziqlar kesmalari bilan (bu kesmalar nuqtalarga aylanishi ham mumkin) chegaralangan; $f_2(x)$ funksiyaning grafigi D cohaning yuqori chegarasidan, $f_1(x)$ funksiyaning grafigi esa, uning quyi chegarasidan iborat (9.1-chizma).

9.3-ta’rif. Koordinatalari, $[c,d]$ kesmada uzluksiz $g_1(y)$ va $g_2(y)$ funksiyalar uchun, $c \le y \le d$, $g_1(y) \le x \le g_2(y)$ munosabatlarni qanoatlantiradigan $M(x,y)$ nuqtalar to‘plami $D_1 - Oy$ o‘qqa nisbatan *standart soha* deyiladi.

Ta’rifning *geometrik ma’nosi* shundan iboratki, D_1 coha yuqoridan va pastdan, mos ravishda, $y = c$, $y = d$ to‘g‘ri chiziqlar kesmalari (bu kesmalar nuqtalarga aylanish ham mumkin) bilan, chapdan va o‘ngdan mos ravishda $g_1(x)$ va $g_2(x)$ funksiyalarning grafiklari bilan chegaralangandir (9.2-chizma).

O‘qlarga nisbatan standart sohalarning yuzalarini hisoblash formulalari:

1. Ox o‘qqa nisbatan *standart* $D = \{(x,y): a \le x \le b, f_1(x) \le y \le f_2(x)\}$ cohaning S_D yuzi

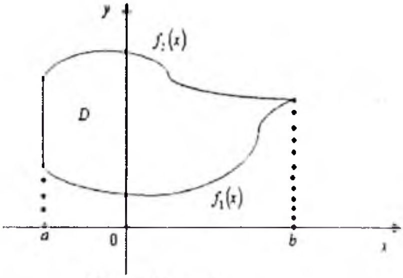
$$S_D = \int_a^b [f_2(x) - f_1(x)]dx$$

formula bo'yicha hisoblanadi.

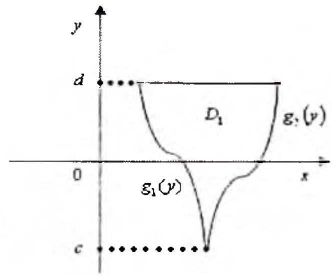
2. Oy o'qqa nisbatan *standart* $D_1 = \{(x,y): c \le y \le d, g_1(y) \le x \le g_2(y)\}$ sohaning S_{D_1} yuzi

$$S_{D_1} = \int_c^d [g_2(y) - g_1(y)]dy$$

formula bo'yicha hisoblanadi.



9.1-chizma.



9.2-chizma.

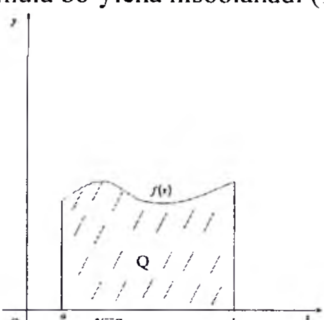
Xususiy holda

3. $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lib, $\forall x \in [a;b]$ da $f(x) \ge 0$ bo'lsin.

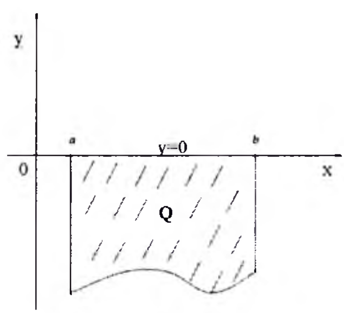
Yuqoridan $f(x)$ funksiyaning grafigi, yon tomonlardan $x = a$ va $x = b$ to'g'ri chiziqlar, pastdan esa, Ox o'q bilan chegaralangan shaklning (odatda bunday shakl, *egri chiziqli trapesiya,* deb yuritiladi) yuzi,

$$S = \int_a^b f(x)dx \qquad (9.1)$$

formula bo'yicha hisoblanadi (9.3-chizma).



9.3-chizma.



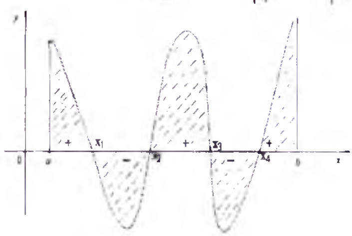
9.4-chizma.

4. Agar $[a;b]$ kesmada aniqlangan, uzluksiz $f(x)$ funksiya manfiy, ya'ni $f(x)<0$ bo'lsa, u holda, asosi $[a;b]$ kesmadan iborat bo'lib, quyidan $y=f(x)$ funksiyaning grafigi bilan chegaralangan (9.4 - chizma) trapesiyaning yuzi manfiy bo'ladi:

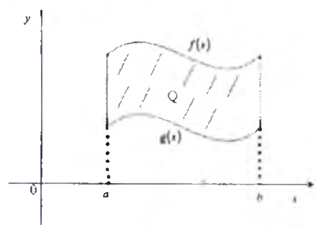
$$S=\left|\int_a^b f(x)dx\right|=-\int_a^b f(x)dx$$

5. Agar $[a;b]$ kesma, chekli sondagi qism oraliqlarga bo'lingan bo'lib, ularning har birida funksiyaning qiymati manfiy emas $(f(x)\geq 0)$ yoki musbat emas $(f(x)\leq 0)$ bo'lsa, u holda, (9.1) integral, chekli sondagi, Ox o'qdan yuqorida va undan pastda joylashgan (yuzi manfiy), egri chiziqli sohalar yuzlarining yig'indisiga teng bo'ladi (9.5 - chizma), ya'ni

$$S=\int_a^{x_1} f(x)dx+\left|\int_{x_1}^{x_2} f(x)dx\right|+\int_{x_2}^{x_3} f(x)dx+\left|\int_{x_3}^{x_4} f(x)dx\right|+\int_{x_4}^{b} f(x)dx.$$



9.5-chizma.



9.6 -chizma.

6. $f(x)$, $g(x)$ funksiyalar $[a;b]$ kesmada aniqlangan uzluksiz, $f(x)\geq 0$, $g(x)\geq 0$ va $\forall x\in[a;b]$ uchun, $f(x)\geq g(x)$ bo'lsin. U holda, $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ chiziqlar bilan chegaralangan sohaning yuzi,

$$S=\int_a^b [f(x)-g(x)]dx \qquad (9.2)$$

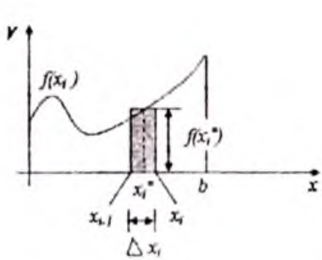
formula orqali topiladi (9.6 - chizma).

Tasavvur qilinadigan (ifodalovchi) to'g'ri to'rtburchaklar.

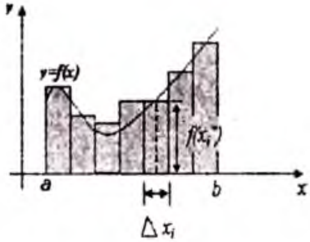
Biz yuqorida ko'rdikki, aniq integral, Riman yig'indisining limiti shaklida, quyidagicha ifodalanadi:

$$\int_a^b f(x)dx=\lim_{|P|\to 0}[f(\xi_1)\Delta x_1+f(\xi_2)\Delta x_2+\ldots+f(\xi_n)\Delta x_n] \qquad (9.3)$$

Bunda ξ_i, $[x_{i-1},x_i]$ oraliqdagi ixtiyoriy tanlangan nuqta, $f(\xi_i)$ esa, $f(x)$ funksiyaning shu oraliqda tasavvur qilinadigan qiymatidir. Agar $f(x)$ funksiya musbat bo'lsa, $f(\xi_i)\Delta x_i$ ko'paytma, 9.7-chizmada ko'rsatilgan *tasavvur qilinadigan to'g'ri to'rtburchakning* yuzini beradi.



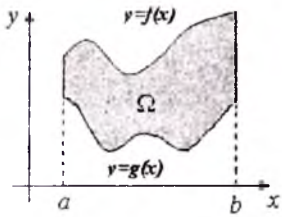
9.7-chizma.



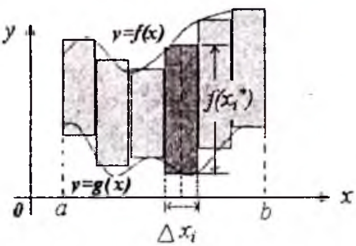
9.8-chizma.

(9.3) formula bizga, berilgan egri chiziqdan pastda joylashgan yuzani, tasavvur qilinadigan to‘g‘ri to‘rtburchaklar yuzalari yig‘indisi sifatida, tasvirlash mumkinligini ko‘rsatadi (9.8-chizma).

Endi Ω soha, yuqoridan $f(x)$ funksiyaning grafigi, pastdan esa, $g(x)$ funksiyaning grafigi bilan chegaralangan bo‘lsin (9.11 - chizma).



9.9-chizma.



9.10-chizma.

Unda Ω sohaning yuzi, $f(x)-g(x)$ funksiyani, $x=a$ dan $x=b$ gacha, x bo‘yicha integrallaash yordamida topiladi (hisoblanadi), ya’ni

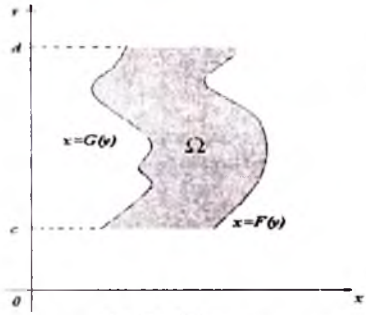
$$S_\Omega = \int_a^b [f(x)-g(x)]dx.$$

Bu holda Riman yig‘indisi,

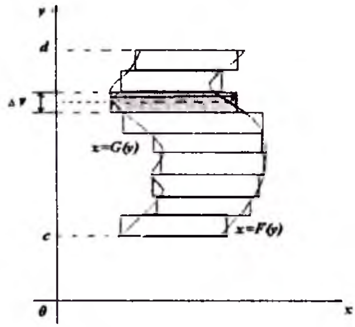
$$[f(\xi_1)-g(\xi_1)]\Delta x_1 + [f(\xi_2)-g(\xi_2)]\Delta x_2 + \ldots + [f(\xi_n)-g(\xi_n)]\Delta x_n$$

shaklida bo‘ladi va tasavvur qilinadigan to‘g‘ri to‘rtburchaklarning o‘lchamlari quyidagicha: $f(\xi_i)-g(\xi_i)$ - «balandligi» va Δx_i - «asosi» (9.11-chizma) bo‘ladi.

Endi y ga nisbatan integrallash yordamida yuzalarni hisoblash formulasini keltirib chiqaramiz. 9.11-chizmada ko‘rsatilgan Ω sohaning chegaralari, x ning funksiyalari bo'lmasdan, ular y ning funksiyalaridan iborat bo'lgan holni qaraymiz.



9.11-chizma.



9.12-chizma.

Bu holda tasavvur qilinadigan to'g'ri to'rtburchaklarni gorizontal ko'rinishda olamiz va yuzani,

$$[F(\eta_1)-G(\eta_1)]\Delta y_1+[F(\eta_2)-G(\eta_2)]\Delta y_2+...+[F(\eta_n)-G(\eta_n)]\Delta y_n$$

Riman yig'indisining limiti sifatida, tasvirlaymiz (9.12-chizma).

Demak, berilgan sohaning yuzi,

$$\int_c^d [F(y)-G(y)]dx$$

integral orqali ifodalanadi. Bu yerda integrallash,

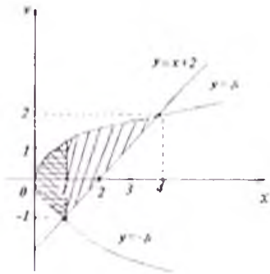
$$F(y)-G(y)$$

«gorizontal bo'linish» ni y ga nisbatan bajaradi.

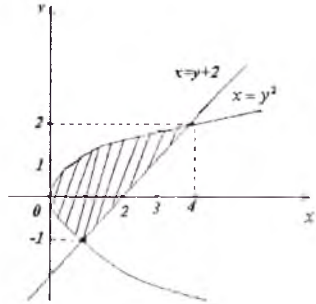
9.1-misol. $x=y^2$ va $x-y=2$ chiziqlar bilan chegaralangan sohaning yuzini: *a*) x ga nisbatan; *b*) y ga nisbatan integrallash yordamida hisoblang.

Yechilishi. Avvalo, berilgan chiziqlarning $(1;-1)$, $(4;2)$ nuqtalarda kesishishiga ishonch hosil qilish mumkin.

a) x bo'yicha integrallash uchun, tasavvur qilinadigan to'g'ri to'rtburchaklarni vertikal joylashtiramiz va tenglamalarni y ga nisbatan echamiz: $x=y^2$ tenglamani y ga nisbatan Yechib. $y=\pm\sqrt{x}$ bo'lishini olamiz, bunda $y=\sqrt{x}$ - parabolaning yuqori yarmidan, $y=-\sqrt{x}$ esa, parabolaning quyi yarmidan iborat. $x-y=2$ to'g'ri chiziq tenglamasini $y=x-2$ shaklida yozamiz (9.13- chizma). Qaralayotgan sohaning yuqori chegarasi, $y=\sqrt{x}$ egri chiziqdan iborat. Uning quyi chegarasi esa, ikkita, har xil tenglamalar orqali ifodalanadi: $x=0$ dan $x=1$ gacha o‘zgarganda, $y=-\sqrt{x}$ egri chiziq, $x=1$ dan $x=4$ gacha o‘zgarganda esa, $y=x-2$ to‘g‘ri chiziq. Shunday qilib, sohaning yuzi,



9.13-chizma.



9.14-chizma.

$$S=\int_0^1[\sqrt{x}-(-\sqrt{x})]dx+\int_1^4[\sqrt{x}-(x-2)]dx=2\int_0^1\sqrt{x}dx+\int_1^4[\sqrt{x}-x+2]dx=$$

$$=[\frac{4}{3}x^{3/2}]\Big|_0^1+[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x]\Big|_1^4=\frac{9}{2}.$$

Y bo‘yicha integrallash uchun, biz tasavvur qilinadigan to‘g‘ri to‘rtburchaklarni gorizontal joylashtiramiz (9.14-chizma). Bunda, o‘ngdan chegaralovchi to‘g‘ri chiziq $x=y+2$ va chapdan chegaralovchi egri chiziq esa, $x=y^2$. Modomiki, y, -1 dan 2 gacha o‘zgarar ekan,

$$S=\int_{-1}^{2}[y+2-y^2]dy=[\frac{1}{2}y^2+2y-\frac{1}{3}y^3]\Big|_{-1}^{2}=\frac{9}{2}.$$

9.2-misol. Ushbu $y=2x$ to‘g‘ri chiziq va $y=3-x^2$ parabola bilan chegaralangan sohaning yuzini toping.

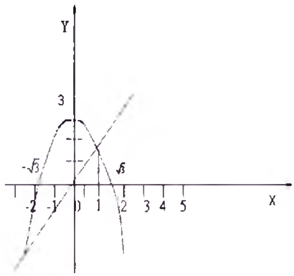
Yechilishi. Berilgan to‘g‘ri chiziq bilan parabolaning kesishish nuqtalarini topamiz:

$$\begin{cases} y=2x \\ y=3-x^2 \end{cases}, \quad 3-x^2=2x, \quad x^2+2x-3=0; \qquad x_1=1,\ x_2=-3 \ .$$

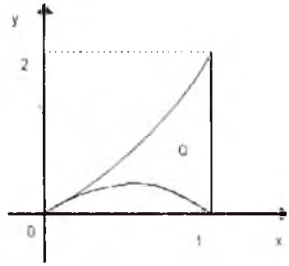
Demak, to‘g‘ri chiziq bilan parabola $(1;2)$ va $(-3;-6)$ nuqtalarda kesishadi.

Shunday qilib, izlanayotgan sohaning yuzi (9.15-chizma),

$$S=\int_{-3}^{1}[3-x^2-2x]dx=\left(3x-\frac{x^3}{3}-x^2\right)\Big|_{-3}^{1}=\frac{5}{3}-(-9+9-9)=\frac{5}{3}+9=\frac{32}{3} \text{ (kv. bir).}$$



9.15 chizma.



9.16- chizma.

9.3-misol. Ushbu $(y-x)^2=x^3$ egri chiziq va $x=1$ to'g'ri chiziq bilan chegaralangan sohaning yuzini toping.

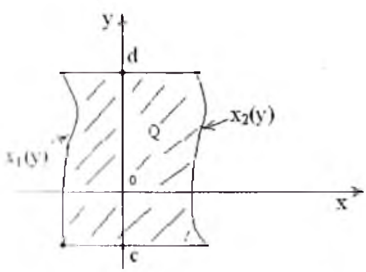
Yechilishi. Ravshanki, y, x ning oshkormas funksiyasi sifatida, $x\geq 0$ uchun aniqlangan. $y-x=\pm x\sqrt{x}$ egri chiziqning $y_1=x+x\sqrt{x}$, $y_2=x-x\sqrt{x}$ shaxobchalaridan birinchisi har doim musbat bo'lib, $x\geq 0$ da, $y_1(x)\geq y_2(x)$ tengsizlikni qanoatlantiradi (9.16 -chizma). (9.2) formulaga asosan,

$$S=\int_0^1[y_1(x)-y_2(x)]dx=\int_0^1\left(x+x\sqrt{x}-x+x\sqrt{x}\right)dx=2\int_0^1 x\sqrt{x}\,dx=\frac{4}{5}\sqrt{x^5}\Big|_0^1=\frac{4}{5}(\text{kv. bir.})$$

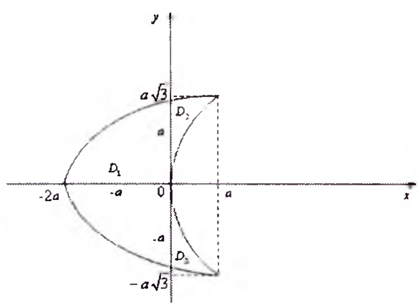
Agar (S)- soha, ushbu $(S)=\{(x,y):c\leq y\leq d,\ x_1(y)\leq x\leq x_2(y)\}$ ko'rinishda bo'lsa (9.17- chizma), u holda uning S yuzi, quyidagi ,

$$S=\int_c^d[x_2(y)-x_1(y)]dy \qquad (9.3)$$

formula orqali topiladi.



9.17- chizma.



9.18-chizma

9.4-misol. Ushbu $x^2+2ax-y^2=0$, $ax-y^2+2a^2=0$, $(a>0)$ egri chiziqlar bilan chegaralangan sohaning yuzini toping.

Yechilishi. Qaralayotgan D soha, Ox o'qqa nisbatan standart bo'lmaganligi uchun, uni, Ox o'qqa nisbatan standart bo'lgan uchta sohalarga ajratamiz (9.18-chizma) :

$$D_1=\{(x,y):\ -2a\le x<0;-\sqrt{2a^2+ax}\le y\le\sqrt{2a^2+ax}\},$$
$$D_2=\{(x,y):0\le x\le a,\ \sqrt{x^2+2ax}\le y\le\sqrt{2a^2+ax}\},$$
$$D_3=\{(x,y):0\le x\le a,-\sqrt{2a^2+ax}\le y\le-\sqrt{x^2+2ax}\}.$$

D- soha Ox o'qqa nisbatan simmetrik bo'lganligi uchun, uning S yuzi Ox o'qqa nisbatan standart bo'lgan

$\overline{D_1}=\{(x,y):\ -2a\le x<0;\ 0\le y\le\sqrt{2a^2+ax}\}$ va D_2 sohalar yuzlarining ikkilanganiga teng:

$$S=2[\int_{-2a}^{0}\sqrt{2a^2+ax}\,dx+\int_{0}^{a}\left(\sqrt{2a^2+ax}-\sqrt{x^2+2ax}\right)dx]=$$
$$=2[\int_{-2a}^{a}\sqrt{2a^2+ax}\,dx-\int_{0}^{a}\sqrt{x^2+ax}\,dx]=2[\int_{-2a}^{a}\sqrt{2a^2+ax}\,dx-\int_{0}^{a}\sqrt{(x+a)^2-a^2}\,dx]=$$
$$=2[\frac{2}{3a}\sqrt{\left(ax+2a^2\right)^3}\Bigg|_{-2a}^{a}-\frac{(x+a)}{2}\sqrt{x^2+2ax}\Bigg|_{0}^{a}+\frac{a^2}{2}\ln\left(x+a+\sqrt{x^2+2ax}\right)\Bigg|_{0}^{a}]=$$
$$=2a^2\sqrt{3}+a^2\ln\left(2+\sqrt{3}\right)$$

Berilgan soha Oy o'qqa nisbatan standart sohadan iborat:

$$D=\{(x,y):-a\sqrt{3}\le y\le a\sqrt{3},\ \frac{y^2-2a^2}{a}\le x\le-a+\sqrt{y^2+a^2}\}$$

Bu Dsohaning simmetrikligidan yana foydalansak, quyidagiga ega bo'lamiz:

$$S=2\int_{0}^{a\sqrt{3}}\left(\sqrt{y^2+a^2}-a-\frac{y^2}{a}+2a\right)dy=$$
$$=2a^2\sqrt{3}+\left(y\sqrt{y^2+a^2}+a^2\ln\left|y+\sqrt{y^2+a^2}\right|-\frac{2y^3}{3a}\right)\Bigg|_{0}^{a\sqrt{3}}=$$
$$=2a^2\sqrt{3}+a^2\ln\left(2+\sqrt{3}\right)$$

9.2. Tenglamasi parametrik shaklda berilgan egri chiziq bilan chegaralangan sohaning yuzini hisoblash. Faraz qilaylik, tekislikdagi shaklni o'rab turuvchi chiziq,

$$\left.\begin{array}{l}x=x(t)\\ y=y(t)\end{array}\right\}\quad(\alpha\le t\le\beta)$$

parametrik tenglamasi bilan berilgan bo'lsin.

1) Agar $x=x(t)$, $y=y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ da $x(t)\ge0$, $y(t)\ge0$ va $x(t)$ funksiya uzluksiz manfiy bo'lmagan $x'(t)$ hosilaga ega bo'lsa, u holda izlanayotgan sohaning yuzi

$$S=\int_{\alpha}^{\beta} y(t)\cdot x'(t)dt \qquad (9.4)$$

formula bo‘yicha topiladi.

2) $x=x(t)$, $y=y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ da $x(t)\geq 0$, $y(t)\geq 0$ va $y(t)$ funksiya, uzluksiz, manfiy bo‘lmagan $y'(t)$ hosilaga ega bo‘lsa, u holda, izlanayotgan sohaning yuzi,

$$S=\int_{\alpha}^{\beta} x(t)\cdot y'(t)dt \qquad (9.5)$$

formula orqali topiladi.

9.5-misol. Ushbu $\left.\begin{array}{l} x=a(t-\sin t),\\ y=a(1-\cos t)\end{array}\right\}$ $(0\leq t\leq 2\pi)$ egri chiziq va Ox o‘q bilan chegaralangan sohaning yuzini toping.

Yechilishi. Sikloidaning bir arkasi, t ning, 0 dan 2π gacha o‘zgarishi natijasida chiziladi, chunki $y(0)=y(2\pi)=0$, t ning qolgan qiymatlarida, $y>0$, $x(0)=0$ va $x(2\pi)=2\pi a$.

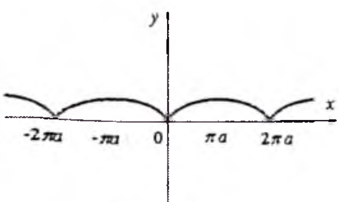
Shunday qilib, izlanayotgan sohaning yuzi,

$$S=\int_{0}^{2\pi a} y\,dx$$

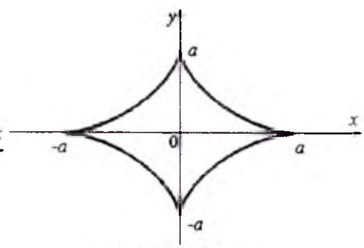
formula orqali topiladi. Bunda, $x=a(t-\sin t)$, $y=a(1-\cos t)$, $dx=a(1-\cos t)dt$ ekanligini e’tiborga olsak, u holda (x, $[0;2\pi a]$ oraliqda o‘zgarganda, t, $[0;2\pi]$ oraliqda o‘zgaradi),

$$S=a\int_{0}^{2\pi}(1-\cos t)\cdot a(1-\cos t)dt=a^2\int_{0}^{2\pi}(1-\cos t)^2dt=3\pi a^2 \text{(kv. bir.)}.$$

Demak, izlanayotgan yuza, radiusi a ga teng bo‘lgan doira yuzining uchlanganiga teng ekan (9.19 - chizma).



9.19-chizma.



9.20-chizma.

9.6-misol. Ushbu $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ egri chiziq bilan chegaralangan sohaning yuzini toping.

Yechilishi. Ravshanki, egri chiziq Ox va Oy o'qlarga nisbatan simmetrik (9.20- chizma) bo'ladi. Shuning uchun, egri chiziqli OAB uchburchakning yuzini topib, uni 4 ga ko'paytirsak, izlanayotgan sohaning yuzi topilgan bo'ladi.

Egri chiziqning tenglamasini, parametrik, $x = a\sin^3 t,\ y = b\cos^3 t\ (0 \le t \le 2\pi)$ shaklda tasvirlaymiz. Bunda, $x(0) = 0,\ x\left(\frac{\pi}{2}\right) = a.\ [0;\frac{\pi}{2}]$ oraliqqa egri chiziqning $\breve{AB}$ yoyi to'g'ri keladi. Egri chiziqli S_{OAB} uchburchakning yuzini, (9.4) formula orqali, hisoblaymiz:

$$S_{OAB} = \int_0^{\frac{\pi}{2}} y(t)x'(t)dt\ . \qquad (*)$$

Bu yuzani hisoblashni soddalashtirish maqsadida, oxirgi integralni bo'laklab integrallaymiz:

$$S_{OAB} = \int_0^{\frac{\pi}{2}} y(t)x'(t)dt = y(t)x(t)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x(t)y'(t)dt = -\int_0^{\frac{\pi}{2}} x(t)y'(t)dt \qquad (**)$$

(*) bilan (**) ni qo'shamiz:

$$S_{OAB} = \frac{1}{2}\int_0^{\frac{\pi}{2}} [y(t)x'(t) - x(t)y'(t)]dt \qquad (***)$$

Unda, (***) formulaga asosan,

$$S_{OAB} = \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos^4 t\sin^2 t + \sin^4 t\cdot\cos^2 t]dt = \frac{3ab}{2}\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt =$$

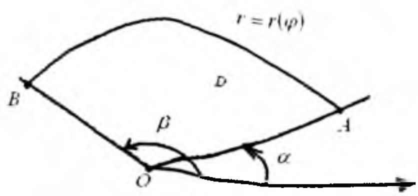
$$= \frac{3ab}{8}\int_0^{\frac{\pi}{2}} \sin^2 2t dt = \frac{3ab}{8}\cdot\int_0^{\frac{\pi}{2}} (1-\cos 4t)dt = \frac{3\pi ab}{32}$$

Shunday qilib, izlanayotgan sohaning yuzi,

$$S = 4\cdot S_{OAB} = 4\cdot\frac{3\pi ab}{32} = \frac{3\pi ab}{8}.$$

9.3. Tenglamasi qutb koordinatalar sistemasida berilgan egri chiziq bilan chegaralangan sohaning yuzini hisoblash.

Faraz qilaylik, $r = r(\varphi)$ funksiya $[\alpha;\beta]$ oraliqda aniqlangan $(0 < \beta - \alpha \le 2\pi)$, uzluksiz va $\forall\varphi \in [\alpha;\beta]$ da $r(\varphi) \ge 0$ bo'lsin. U holda, $r = r(\varphi)$ funksiyaning grafigi hamda $\vec{OA}$ va $\vec{OB}$ radius



9.21-chizma.

vektorlar bilan chegaralangan soha – egri chiziqli sektorni qaraymiz (9.21- chizma).

Qaralayotgan sektorning yuzi,

$$S=\frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)d\varphi$$

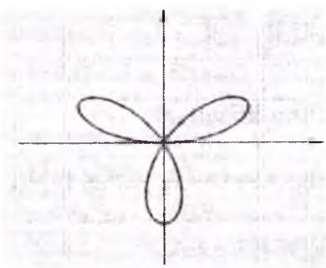
formula bo'yicha hisoblanadi.

9.7-misol. Ushbu $r=a\sin 3\varphi,\ a>0$, egri chiziq bilan chegaralangan sohaning yuzini toping.

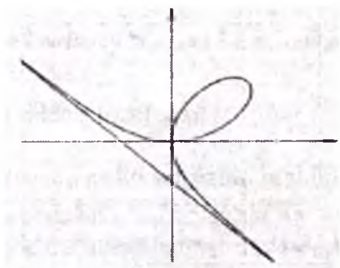
Yechilishi. Berilgan egri chiziq, har biri egri chiziqli sektordan iborat bo'lgan sohani (9.22-chizma) chegaralaydi. Ulardan birinchisini, ya'ni $(S_1)=\left\{(r;\varphi): 0\le\varphi\le\frac{\pi}{3}, 0\le r\le a\sin 3\varphi\right\}$ sektorni qaraymiz, uning yuzi, (9.6) formula yordamida hisoblanadi:

$$S_1=\frac{1}{2}\int_0^{\frac{\pi}{3}} a^2\sin^2 3\varphi\, d\varphi=\frac{\pi a^2}{12}.$$

Shunday qilib, izlanayotgan sohaning yuzi, $S=\frac{3}{12}\cdot\pi a^2=\frac{\pi a^2}{4}$ (kv. bir.)



9.22-chizma.



9.23-chizma.

9.8-misol. Ushbu $x^3+y^3=3axy$ egri chiziq bilan chegaralangan sohaning yuzini hisoblang.

Yechilishi. Berilgan egri chiziqning qutb koordinatalar sistemasidagi tenglamasi quyidagicha bo'ladi: $\rho(\sin^3\varphi+\cos^3\varphi)=3a\cos\varphi\sin\varphi$. Bundan,

$$\rho=\frac{3a\cos\varphi\sin\varphi}{\sin^3\varphi+\cos^3\varphi}.$$

$[0;\frac{\pi}{2}]$ oraliqning ichida, tenglikning o'ng tomoni musbat, oraliqning chetki nuqtalarida, ya'ni $\varphi=0$ va $\varphi=\frac{\pi}{2}$da $\rho=0$ bo'ladi. U holda, $0\le\varphi\le\frac{\pi}{2}$

segment halqaga akslantiriladi (9.23- chizma). Halqaning yuzini, ushbu

$$S=\frac{1}{2}\int_0^{\frac{\pi}{2}}\rho^2 d\varphi$$

formula bo'yicha hisoblaymiz:

$$S=\frac{1}{2}\int_0^{\frac{\pi}{2}}\rho^2 d\varphi=\frac{9a^2}{2}\int_0^{\frac{\pi}{2}}\frac{\cos^2\varphi\sin^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}d\varphi .$$

$\int\frac{\cos^2\varphi\sin^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}d\varphi$ aniqmas integralda, $z=tg\varphi$ almashtirishni olib, uning boshlang'ich funksiyasini topamiz:

$$\int\frac{\cos^2\varphi\sin^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}d\varphi=\int\frac{tg^2\varphi}{(1+tg^3\varphi)^2}\cdot\frac{d\varphi}{\cos^2\varphi}=$$

$$=\int\frac{z^2dz}{(1+z^3)^2}=-\frac{1}{3(1+z^3)}+C=\frac{-\cos^3\varphi}{3(\sin^3\varphi+\cos^3\varphi)}+C.$$

Bundan,

$$S=\frac{9a^2}{2}\left[-\frac{\cos^3\varphi}{3(\sin^3\varphi+\cos^3\varphi)}\right]\Bigg|_0^{\frac{\pi}{2}}=\frac{3a^2}{2}.$$

Mustaqil yechish uchun misollar

Quyidagi chiziqlar bilan chegaralangan sohaning yuzini toping:

9.1. $y=x,\ y=2-x^2.$ **9.2.** $(y-x)^2=x^3,\ x=1.$ **9.3.** $y^2=ax,\ x^2=ay.$

9.4. $y=6x-x^2-7,\ y=x-3.$ **9.5.** $y=\sin x,\ y=0,\ 0\le x\le\pi.$

9.6. $y=x^2+1,\ x+y=3.$ **9.7.** $y^2=2x+1,\ x-y-1=0.$ **9.8.** $y=2-x^2,\ y^2=x^3.$

9.9. $y=\frac{1}{x},\ y=0,\ x=a,\ x=b,\ a>b>0.$ **9.10.** $y=e^{-x},\ x=0,\ y=0,\ x=a.$

Quyidagi chiziqlar bilan chegaralangan sohaning yuzini toping:

9.11. $y=x-\frac{\pi}{2}, y=\cos x,\ x=0.$ **9.12.** $y=\sin x,\ y=\cos x,\ 0\le x\le\frac{\pi}{4}$.

9.13. $y=2^x,\ y=2,\ x=0.$

9.14. $y=(x+1)^2,\ x=\sin\pi y, y=0\ (0\le y\le 1).$

9.15. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ **9.16.** $y=a^x,\ y=a,\ x=0,\ a>1$.

9.17. $y=|\log_a x|,\ y=0,\ x=1/a,\ x=a,\ a>1.$

9.18. $y=tgx,\ y=\frac{2}{3}\cos x,\ x=0.$ **9.19.** $y=-x^2,\ y=x^2-2x-4$

9.20-9.26 misollardagi chiziqlar orqali berilgan sohani chizing. So'ngra sohaning yuzini: *a*) x bo'yicha ; *b*) y bo'yicha integral sifatida tasvirlang:

9.20 $y = x^2$, $y = x + 2$ $\left(x = \sqrt{y},\ x = y - 2\right)$.

9.21. $y = x^3$, $y = 2x^2$ $\left(x = \sqrt[3]{y}, x = \frac{1}{\sqrt{2}}\sqrt{y}\right)$.

9.22. $y = -\sqrt{x}$; $y = x - 6$ $\left(x = y^2,\ x = y + 6\right)$

9.23. $y = |x|$, $3y - x = 8$ $(x = y, y \geq 0,\ x = 3y - 8)$.

9.24. $x + 4 = y^2$, $x = 5$

9.25. $y = 2x$, $x + y = 9$, $y = x - 1$ $\left(x = \frac{y}{2},\ x = 9 - y,\ x = y + 1\right)$

9.26. $y = x^{1/3}$, $y = x^2 + x - 17$.

9.27- 9.32 misollarda berilgan chiziqlar bilan chegaralangan sohani chizing va uning yuzini hisoblang.

9.27. $4x = 4y - y^2$, $4x - y = 0$. **9.28.** $x = y^2$, $x = 3 - 2y^2$

9.29. $x + y - y^3 = 0$, $x - y + y^2 = 0$. **9.30.** $y = \cos x$, $y = \sec^2 x$, $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

9.31. $y = \cos x$, $y = \sin 2x$, $x \in [-\pi, \pi]$. **9.32.** $y = \sin^4 x \cos x$, $x \in [0, \pi/2]$.

9.33. Berilgan $(0, 0)$, $(1, 3)$, $(3, 1)$ nuqtalardan o'tuvchi uchburchakning yuzini, integrallash amalini bajarib, hisoblang.

9.34. Uchlari, $(-2, 2)$, $(1, 1)$, $(5, 1)$, $(7, 2)$ nuqtalarda yotgan trapesiyaning yuzini, integrallash amalini bajarib, hisoblang.

9.35. $y = 6 - x^2$, $y = x(x < 0)$ va $y = -x\ (x > 0)$ chiziqlar bilan chegaralangan sohani chizing va uning yuzini hisoblang.

9.36. $y = x^2$ va $y = 4$ chiziqlar bilan chegaralangan sohani chizing. Bu soha, $y = c$ chiziq orqali, ikkita, teng yuzali, qism sohalarga bo'lingan bo'lsa, c ni toping.

Quyidagi misollarda, qiymati berilgan sohaning yuziga teng bo'lgan, integralni yozing:

9.37 Soha, Ox o'q, $y = \sqrt{3}x$ to'g'ri chiziq va $x^2 + y^2 = 4$ aylana bilan chegaralangan bo'lib, birinchi chorakda yotadi.

9.38. Soha, $x^2 + y^2 = 4$ va $(x-2)^2 + (y-2)^2 = 4$ aylanalarning kesishishi natijasida hosil qilingan.

9.39. To'g'ri to'rtburchakning bir uchi, koordinatalar boshida, qarama-qarshi uchi esa, $y = bx^n$ chiziqning $x = a$ bo'lgan $(a > 0, b > 0, n > 0)$ nuqtasida, uning tomonlari esa, koordinatalar o'qlariga parallel. U holda, to'g'ri to'rtburchakning $y = bx^n$ chiziqdan pastda joylashgan qismi yuzining, to‘g‘ri to‘rtburchakning to‘la yuziga nisbati, faqat n ga bog‘liq bo‘lishini, ya'ni nisbat a va b larga bog‘liq bo‘lmasligini ko‘rsating.

9.40-9.49 misollarda berilgan chiziqlar bilan chegaralangan sohaning yuzini hisoblang:

9.40. $y = 6x^2 - 5x + 1$, $y = \cos \pi x$, $0 \le x \le 0{,}5$. **9.41.** $y = \sin 2x$, $y = \sin x$.

9.42. $y = arctg\sqrt{x}$, $y + x^2 = 0$, $x = 1$. **9.43.** $y = 2x^2 e^x$, $y = -x^3 e^x$.

9.44. $2y = x^2$, $x^2 + y^2 = 4y$, $2y \ge x^2$. **9.45.** $y^2 + x = 4$, $y^2 - 3x = 12$.

9.46. $y = \sqrt{x}$, $y = x - 2$, $x = 0$. **9.47.** $y = \arcsin x$, $y = \arccos x$, $y = 0$.

9.48. $y = x^2$, $y = x^2 + x - 1$, $y = \dfrac{5x}{2}$, $y \le x^2$. **9.49.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, $y = \dfrac{9}{32}x^2$, $y \le \dfrac{9}{32}x^2$.

9.50. $y^2 = 2x$ parabola $x^2 + y^2 = 8$ aylana bilan chegaralangan doiraning yuzini qanday nisbatta bo‘ladi?

9.51. $x^2 + y^2 = a^2$ aylana, $x^2 - 2y^2 = \dfrac{a^2}{4}$ giperbola yordamida uchta qismga bo‘lingan. Shu qismlarning yuzlarini toping.

9.52. $y = x^2 - 2x + 2$ parabola, unga $M(3;5)$ nuqtada o‘tkazilgan urinma va ordinatalar o‘qi bilan chegaralangan sohaning yuzini toping.

9.53 $y = -x^2 + 4x - 3$ parabola va unga $M_1(0;-3)$, $M_2(3;0)$ nuqtalarda o‘tkazilgan urinmalar bilan chegaralangan sohaning yuzini toping.

9.54. $y = x^2 + 4x + 9$ parabola va unga abssissalari $x_1 = -3$, $x_2 = 0$ bo‘lgan nuqtalarda o‘tkazilgan urinmalar bilan chegaralangan sohaning yuzini toping.

Quyidagi, tenglamasi parametrik shaklda berilgan, chiziqlar bilan chegaralangan sohaning yuzini toping:

9.55. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \le t \le 2\pi)$, $y = 0$.

9.56. $x = a\cos^3 t$, $y = a\sin^3 t$.

9.57. $x = at - t^2$, $x = at^2 - t^3$, $a > 0$. **9.58.** $x = 1 + t - t^3$, $y = 1 - 15t^2$.

9.59. $x = at - b\sin t$, $y = a - b\cos t$ $(0 < b < a)$ troxoidaning bir tarmog‘i.

9.60. $x = a\cos t(1 - \cos t)$, $y = a\sin t(1 - \cos t)$ kardoidaning ichida yotgan sohaning yuzini toping:

9.61. $x = a\sin 2t$, $y = a\sin t$, $a > 0$.

9.62. $x = a\left(\dfrac{2}{\pi}t - \sin t\right)$, $y = a(1 - \cos t)$, $a > 0$.

9.63. $x = a(1 + 2\cos t)$, $y = a(tgt + 2\sin t)$, $a > 0$.

Quyidagi tenglamalari qutb koordinatalar sistemasida berilgan, chiziqlar bilan chegaralangan sohaning yuzini toping:

9.64. $r^2 = a^2 \cos 4\varphi$ (lemniskata) . **9.65.** $r = a(1 + \cos\varphi)$ (kardoida) .

9.66. $r\varphi = a$ (giperbolik spiral) $\varphi_1 \le \varphi \le \varphi_2$, $\varphi_2 - \varphi_1 \le 2\pi$.

9.67. $r = \frac{a}{2\pi}\varphi$ (Arximed spirali) $\varphi_1 \le \varphi \le \varphi_2$, $\varphi_2 - \varphi_1 \le 2\pi$.

9.68. $r = Re^{k\varphi}$, $k > 0$ (logarifmik spiral) $\varphi_1 \le \varphi \le \varphi_2$, $\varphi_2 - \varphi_1 \le 2\pi$.

9.69. $r = 2\sqrt{3}a\cos\varphi$, $\rho = 2a\sin\varphi$. **9.70.** $r = 2 - \cos\varphi$, $\rho = \cos\varphi$.

9.71. $r = \frac{p}{1-\cos\varphi}$ $\left(\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}\right)$ (parabola).

9.72. $r = \frac{p}{1+\varepsilon\cos\varphi}$ $\left(\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}\right)$.

9.73. $r = a\frac{\cos\varphi\sin\varphi}{\cos^3\varphi + \sin^3\varphi}$ $\left(0 \le \varphi \le \frac{\pi}{2}\right)$. **9.74.** $r = a(1-\cos\varphi)$, $r = a$.

9.75. $r^2 = a^2(1 - 2\cos\varphi)$, $r = a$, $r \le a$.

9.76. $r = a\sqrt{\cos 2\varphi}$ chiziq bilan chegaralangan va $r = \frac{a}{\sqrt{2}}$ doiraning ichida yotgan sohaning yuzini toping.

9.77. $r = a(1+\cos\varphi)$ chiziq bilan chegaralangan va $r = 3a\cos\varphi$ chiziqning tashqarisida joylashgan sohaning yuzini toping.

Mustaqil yechish uchun misollarning javoblari

9.1. $\frac{9}{2}$ **9.2.** $\frac{4}{5}$ **9.3.** $\frac{a^2}{3}$ **9.4.** $\frac{9}{2}$ **9.5.** 2 **9.6.** $4\frac{1}{2}$ **9.7.** $5\frac{1}{3}$ **9.8.** $2\frac{2}{15}$

9.9. $\ln\left(\frac{a}{b}\right)$ **9.10.** $1-e^{a}$ **9.11.** $1+\frac{\pi^2}{8}$ **9.12.** $\sqrt{2}-1$ **9.13.** $2-\frac{1}{\ln 2} \approx 0{,}56$.

9.14. $\frac{1}{3}+\frac{2}{\pi} \approx 0{,}97$ **9.15.** πab **9.16.** $a+\frac{(1-a)}{\ln a}$ **9.17.** $\frac{a^2-1}{a} - \frac{(a-1)^2}{a\ln a}$

9.18. $\frac{1}{3}+\ln\left(\frac{\sqrt{3}}{2}\right)$ **9.19.** 9.

9.20 $a)\int_{-1}^{2}[(x+2)-x^2]dx$, $b)\int_{0}^{1}[\sqrt{y}-(\sqrt{y})]dy+\int_{1}^{4}[\sqrt{y}-(y-2)]dx$.

9.21. $a)\int_{0}^{2}[2x^2-x^3]dx$, $b)\int_{0}^{8}[y^{\frac{1}{3}}-\left(\frac{1}{2}y\right)^{1/2}]dy$.

9.22. $a)\int_{0}^{4}[0-(-\sqrt{x})]dx+\int_{4}^{6}[0-(x-6)]dx$, $b)\int_{-2}^{0}[(y+6)-y^2]dx$.

9.23. $a)\int_{-2}^{0}[\frac{8+x}{3}-(-x)]dx+\int_{0}^{4}[\frac{8+x}{3}-x]dx$, $b)\int_{0}^{2}[y-(-y)]dy+\int_{2}^{4}[y-(3y-8)]dy$.

9.24. $a)\int_{-4}^{5}[\sqrt{4+x}-(-\sqrt{4+x})]dx$, $b)\int_{-3}^{3}[5-(y^2-4)]dy$

9.25. $a)\int_{-1}^{3}[2x-(x-1)]dx+\int_{3}^{5}[(9-x)-(x-1)]dx$, $b)\int_{-2}^{4}[(y+1)-\frac{1}{2}y]dy+\int_{4}^{6}[(9-y)-\frac{1}{2}y]dy$.

9.26. $a)\int_{-1}^{1}[x^{1/3}-(x^2+x-1)]dx$,

$b)\int_{-5/4}^{-1}[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}\right)-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{4y+5}\right)]dy+\int_{-1}^{1}[-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}-y^3]dy$.

9.27. $9/8$. **9.28.** 4. **9.29.** $\frac{5}{12}$. **9.30.** $2-\sqrt{2}$. **9.31.** 8. **9.32.** $\frac{1}{5}$. **9.33.** 4.

9.34. $\frac{39}{2}$. **9.35.** 27. **9.36.** $4^{\frac{2}{3}}$. **9.37.** $\int_{0}^{\sqrt{3}}[\sqrt{y-y^2}-\frac{1}{\sqrt{3}}y]dy$

9.38. $A=\int_{0}^{2}[\sqrt{4-x^2}-(2-\sqrt{4x-x^2})]dx$. **9.39.** $\frac{1}{n+1}$. **9.40.** $\frac{1}{\pi}-\frac{1}{8}$.

9.41. $\frac{9}{4}$. **9.42.** $\frac{\pi}{2}-\frac{2}{3}$. **9.43.** $18e^{-2}-2$. **9.44.** $\frac{16}{3}+2\pi$. **9.45.** $\frac{32\sqrt{6}}{3}$.

9.46. $\frac{16}{3}$. **9.47.** $\sqrt{2}-1$. **9.48.** $\frac{37}{48}$. **9.49.** $6\pi-6\arcsin\frac{\sqrt{8}}{3}-\frac{4\sqrt{2}}{9}$.

9.50. $\frac{3\pi+2}{9\pi-2}$.

9.51. $a^2[\frac{\pi}{6}-\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})]$, $a^2[\frac{\pi}{6}-\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})]$, $a^2[\frac{2\pi}{3}+\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})]$.

9.52. 9. **9.53.** $2\frac{1}{4}$. **9.54.** $\frac{9}{4}$. **9.55.** $3\pi a^2$. **9.56.** $\frac{3}{8}\pi a^2$. **9.57.** $\frac{a^5}{60}$. **9.58.** 8.

9.59. $\pi(b^2+2ab)$. **9.60.** $\frac{3\pi}{2}a^2$. **9.61.** $\frac{4a^2}{3}$. **9.62.** $a^2\left(\frac{\pi}{2}-\frac{4}{\pi}\right)$.

9.63. $a^2\left(\sqrt{3}+\frac{4\pi}{3}-4\ln(2+\sqrt{3})\right)$. **9.64.** a^2. **9.65.** $\frac{3}{2}\pi a^2$. **9.66.** $\frac{a^2}{2}\left(\frac{1}{\varphi_1}-\frac{1}{\varphi_2}\right)$.

9.67. $\frac{a^2}{24\pi^2}(\varphi_2^3-\varphi_1^3)$. **9.68.** $\frac{R^2}{4k}(e^{2k\varphi_2}-e^{2k\varphi_1})$. **9.69.** $a^2(\frac{5}{6}\pi-\sqrt{3})$. **9.70.** $\frac{17\pi}{4}$.

9.71. $\frac{p^2}{6}(3+4\sqrt{2})$. **9.72.** $\frac{\pi p^2}{\sqrt{(1-\varepsilon^2)^3}}$. **9.73.** $\frac{1}{6}a^2$. **9.74.** $2a^2\left(\frac{5\pi}{8}-1\right)$.

9.75. $a^2(2\pi+3\sqrt{3}-6)/3$. **9.76.** $\frac{a^2}{6}(\pi+6-3\sqrt{3})$. **9.77.** $\frac{a^2\pi}{4}$.

10-§. Aniq integral yordamida chiziqning yoyi uzunligini hisoblash

10.1. Dekart koordinatalar sistemasida berilgan chiziqning yoyi uzunligini hisoblash. Faraz qilaylik, $\breve{AB}$ yoy, fazoda $x=x(t)$, $y=y(t)$, $z=z(t)$, $t\in[\alpha,\beta]$, tenglamalar sistemasi orqali aniqlangan bo‘lsin. Bunda, $x(t),y(t),z(t)$ funksiyalar, $[\alpha;\beta]$ kesmada uzluksiz differensiallanuvchi funksiyalardir. Bu holda, $\breve{AB}$ yoyning uzunligi ,

$$l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt \qquad (10.1)$$

formula bo'yicha hisoblanadi.

$\breve{AB}$ yoy tekislikda berilganda, uning uzunligi,

$$l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^2+(y'(t))^2}\,dt \qquad (10.2)$$

formula orqali hisoblanadi.

Agar egri chiziqning ($\breve{AB}$-yoyning) tenglamasi oshkor, ya'ni $y=f(x),\ x\in[a,b]$ ko'rinishda berilsa, ($f(x)$ – uzluksiz differensiallanuvchi), uning yoy uzunligi,

$$l=\int_{a}^{b}\sqrt{1+[f'(x)]^2}\,dx \qquad (10.3)$$

formula orqali topiladi.

10.1-misol. Ushbu $y=\frac{4}{5}x^{5/4},\ 0\le x\le 9$ egri chiziqning yoy uzunligini toping.

Yechilishi. Berilgan egri chiziqning yoy uzunligini, (10.3) formuladan foydalanib, topamiz:

$$y'_x=\frac{4}{5}\cdot\frac{5}{4}\cdot x^{\frac{1}{4}}=x^{\frac{1}{4}};\ S=\int_0^9\sqrt{1+(y'_x)^2}\,dx=\int_0^9\sqrt{1+x^{\frac{1}{2}}}\,dx$$

Oxirgi integralda, $\sqrt{1+\sqrt{x}}=t$ almashtirishni bajarib,

$$l=4\int_1^2 t^2(t^2-1)\,dt=\frac{232}{15}$$

ekanligini topamiz.

10.2-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$, egri chiziqning yoy uzunligini toping.

Yechilishi. Berilgan egri chiziqning yoy uzunligini, (10.2) formula orqali topamiz:

$$x'_t=a(1-\cos t),\ y'_t=a\sin t,\ \sqrt{(x'_t)^2+(y'_t)^2}=2a\sin\frac{t}{2}dt,$$

$$l=\int_0^{2\pi}2a\sin\frac{t}{2}dt=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=8a.$$

10.3- misol. Ushbu $x=a(1-\sin t),\ y=a(t-\cos t),\ z=4a\sin\frac{t}{2},\ 0\le t\le t_0$, egri chiziqning yoy uzunligini toping.

Yechilishi. Berilgan egri chiziqning yoy uzunligini, (10.1) formula orqali topamiz:

$$x'_t=-a\sin t,\ y'_t=a(1-\cos t),\ z'_t=2a\cos\frac{t}{2},\quad (x'_t)^2+(y'_t)^2+(z'_t)^2=$$

$$= (-a\sin t)^2 + (a(1-\cos t))^2 + (2a\cos\frac{t}{2})^2 = 2a^2(1-\cos t + 2\cos^2\frac{t}{2}) = 4a^2.$$

Demak, $l = \int_0^{t_0} 2a dt = 2at_0$.

10.4-misol. Ushbu

$$x = \int_1^t \frac{\sin}{\varphi} d\varphi;\ y = \int_1^t \frac{\cos\varphi}{\varphi} d\varphi,\ 1 \le t \le t_0$$

tenglama bilan berilgan egri chiziqning uzunligini toping.

Yechilishi. Ravshanki, $t = 1$ bo'lganda, egri chiziq koordinatalar boshidan o'tadi.

$$x_t' = [\int_1^t \frac{\sin\varphi}{\varphi} d\varphi]_t' = \frac{\sin t}{t},\quad y_t' = [\int_1^t \frac{\cos\varphi}{\varphi} d\varphi]_t' = \frac{\cos t}{t},$$

$$\sqrt{(x_t')^2 + (y_t')^2} = \sqrt{\frac{\sin^2 t}{t^2} + \frac{\cos^2 t}{t^2}} = \frac{1}{t}.$$

Shunday qilib, izlanayotgan chiziqning uzunligi, (10.2) formulaga asosan,

$$S = \int_1^{t_0} \sqrt{(x_t')^2 + (y_t')^2} dt = \int_1^{t_0} \frac{dt}{t} = \ln t_0 .$$

10.2. Qutb koordinatalar sistemasida berilgan chiziqning yoyi uzunligini hisoblash. Agar $\breve{AB}$ egri chiziq yoyi qutb koordinatalar sistemasida $r = r(\varphi)$ $(\varphi_1 \le \varphi \le \varphi_2)$ tenglama bilan berilgan bo'lib, $r(\varphi)$ funksiya $[\varphi_1, \varphi_2]$ da uzluksiz va uzluksiz hosilaga ega bo'lsa, u holda, egri chiziqning *yoy uzunligi*,

$$l = \breve{AB} = \int_{\varphi_1}^{\varphi_2} \sqrt{[r(\varphi)]^2 + [r'(\varphi)]^2} d\varphi \qquad (10.4)$$

formula bo'yicha hisoblanadi.

10.5-misol. Ushbu $r = a(1-\sin\varphi),\ -\frac{\pi}{2} \le \varphi \le -\frac{\pi}{6}$ tenglama bilan berilgan chiziq yoyining uzunligini toping.

Yechilishi. Berilgan chiziqning, $r = a(1-\sin\varphi)$ tenglamasidan,

$$r_\varphi' = -a\cos\varphi,\ \sqrt{r^2 + (r_\varphi')^2} = \sqrt{a^2(1-\sin\varphi)^2 + a^2\cos^2\varphi} =$$

$$= a\sqrt{1 - 2\sin\varphi + \sin^2\varphi + \cos^2\varphi} = a\sqrt{2}\sqrt{1-\sin\varphi}$$

ekanligini topamiz. Endi, izlanayotgan chiziq yoyining uzunligi, (10.4) formula yordamida topiladi:

$$l = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} a\sqrt{2}\cdot\sqrt{1-\sin\varphi} d\varphi = a\sqrt{2}\cdot \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \sqrt{1-\sin\varphi} d\varphi = 2a .$$

Mustaqil yechish uchun misollar

Quyidagi berilgan chiziqlarning ko'rsatilgan kesmalardagi yoy uzunliklarini toping:

10.1. $y = \ln x,\ \frac{3}{4} \le x \le \frac{12}{5}$.

10.2. $y = 1 - \ln\cos x,\ 0 \le x \le \frac{\pi}{3}$.

10.3. $y = \sqrt{1 - x^2} + \arcsin x,\ 0 \le x \le \frac{7}{9}$.

10.4. $y = e^x\ (0 \le x \le x_0)$.

10.5. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y\ (1 \le y \le e)$.

10.6. $y^2 = \frac{x^3}{2a - x}\ \left(0 \le x \le \frac{5}{3}a\right)$.

10.7. $y = \sqrt{\frac{x}{3}}(1 - x),\ (0 \le x_0 \le x \le 1)$.

10.8. $x = \frac{2}{3}\sqrt{(y-1)^3}\ \left(0 \le x \le 2\sqrt{3}\right)$.

10.9. $y = \ln\left(x^2 - 1\right),\ 2 \le x \le 5$.

10.10. $y = a\,ch\frac{x}{a}$ chiziqning, $A(0;a)$ nuqtadan, $B(b;h)$ nuqtagacha bo'lgan qismi.

10.11. $y = \ln\sin x$ chiziqning, abssissalari, $\frac{\pi}{2}$ va $\frac{\pi}{3}$ bo'lgan nuqtalar orasidagi qismi.

10.12. $y^2 = x^3$ chiziqning $x = \frac{4}{3}$ to'g'ri chiziq bilan kesilgan qismi.

10.13. $y = \frac{x^2}{2} - 1$ chiziqning Ox o'q bilan kesilgan qismi.

10.14. $y^2 = (x+1)^3$ chiziqning $x = 4$ to'g'ri chiziq bilan kesilgan qismi. **10.15.** $y = \arcsin e^{-x}$ chiziqning, $x = 0$ dan $x = 1$ gacha bo'lgan qismi.

Quyidagi berilgan chiziqlarning ko'rsatilgan kesmalardagi yoy uzunliklarini toping:

10.16. $x = a(t - \sin t),\ y = a(1 - \cos t)\ (0 \le t \le 2\pi)$.

10.17. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),\ \ (0 \le t \le 2\pi)$.

10.18. $x = a\cos^3 t,\ y = a\sin^3 t$.

10.19. $x = e^t(\cos t + \sin t),\ y = e^t(\cos t - \sin t),\ 0 \le t \le 1$.

10.20. $x = a\cos^5 t,\ y = a\sin^5 t\ \ (0 \le t \le 2\pi)$

10.21. $x = a(sht - t),\ y = a(cht - 1)\ (0 \le t \le T)$.

10.22. $x = \sin^4 t,\ y = \cos^2 t,\ \ \left(0 \le t \le \frac{\pi}{2}\right)$

10.23. $x = \int_1^t \frac{\sin\varphi}{\varphi}d\varphi,\ \ y = \int_1^t \frac{\cos\varphi}{\varphi}d\varphi,\ 1 \le t \le t_0$.

10.24. $x = 2a\cos t,\ y = 2a\sin t,\ z = at,\ (0 \le t \le 2\pi)$.

10.25. $x = \frac{2}{3}t^3 + \frac{1}{2}t^2,\ y = -\frac{4}{3}t^3 + \frac{1}{2}t^2,\ z = \frac{1}{3}t^3 + t^2,\ 0 \le t \le 1$.

10.26. $x = e^t(\cos t + \sin t),\ y = e^t(\cos t - \sin t),\ z = ht,\ 0 \le t \le 2\pi.$

10.27. $x = a(\cos t + \ln tg\frac{t}{2}),\ y = a\sin t,\ 0 < t_0 \le t \le \frac{\pi}{2}.$

Quyida berilgan chiziqlarning ko‘rsatilgan kesmalardagi yoy uzunliklarini toping:

10.28. $r = \cos^3\frac{\varphi}{3},\ 0 \le \varphi \le \frac{\pi}{2}.$ **10.29.** $r = \varphi^2,\ 0 \le \varphi \le \pi.$

10.30. $r = a\varphi$ Arximed spiralining radiusi $2a\pi$ bo‘lgan doiraning ichidagi qismi.

10.31. $r = ae^{m\varphi}\ (m > 0,\ 0 < r < a).$ **10.32.** $r = \frac{p}{1+\cos\varphi},\ |\varphi| \le \frac{\pi}{2}.$

10.33. $r = ath\frac{\varphi}{2},\ 0 \le \varphi \le 2\pi.$ **10.34.** $r = a(1-\cos\varphi)$ kardioida.

10.35. $r = a\varphi^3,\ 0 \le \varphi \le 4.$ **10.36.** $r = a\varphi^4,\ 0 \le \varphi \le 3.$

10.37. $r = a(1-\sin\varphi),\ -\frac{\pi}{2} \le \varphi \le -\frac{\pi}{6}.$

10.38. $r = a(1+\cos\varphi)$ kardoidaning uzunligi.

10.39. $\varphi = \frac{1}{2}\left(r + \frac{1}{r}\right),\ 1 \le r \le 3.$

Mustaqil yechish uchun misollarning javoblari

10.1 $\frac{27}{20} + \ln 2.$ **10.2.** $\ln(2+\sqrt{3})$ **10.3.** $\frac{2\sqrt{2}}{3}.$

10.4. $x_0 - \sqrt{2} + \sqrt{1+e^{2x_0}} - -\ln\frac{1+\sqrt{1+e^{2x_0}}}{1+\sqrt{2}}.$ **10.5.** $\frac{e^2+1}{4}.$

10.6. $4a\left(1+\sqrt{3}\ln\frac{1+\sqrt{3}}{\sqrt{2}}\right).$ **10.7.** $10{,}8.$ **10.8.** $14/3.$ **10.9.** $3+\ln 2.$

10.10. $\sqrt{h^2-a^2}.$ **10.11.** $\frac{1}{2}\ln 3.$ **10.12.** $\frac{112}{27}.$ **10.13.** $\sqrt{6}\ln(\sqrt{2}+\sqrt{3})$

10.14. $\frac{670}{27}.$ **10.15.** $\ln\left(e+\sqrt{e^2-1}\right).$**10.16.** $8a.$ **10.17.** $2\pi a^2.$ **10.18.** $6a.$

10.19. $2(e-1).$ **10.20.** $2a\left(5+\frac{5}{2\sqrt{3}}\ln(2+\sqrt{3})\right).$

10.21. $2\left(ch\frac{T}{2}\sqrt{chT}-1\right) - \sqrt{2}\ln\frac{\sqrt{2}ch\frac{T}{2}+\sqrt{chT}}{1+\sqrt{2}}.$

10.22. $0{,}5\sqrt{5}+0{,}25\ln(2+\sqrt{5})$ **10.23.** $\ln t_0.$ **10.24.** $2a\sqrt{5}\pi.$

10.25. $\frac{1}{21}(27\sqrt{3}-2\sqrt{6})$ **10.26.** $\sqrt{h^2+4e^{4\pi}} - \sqrt{h^2+4} + h\ln\left(\frac{\sqrt{4e^{4\pi}+1{,}2}-h}{\sqrt{h^2+4}-h}\right) - 2\pi.$

10.27. $-a\ln\sin t_0.$ **10.28** $\frac{9}{8}(2\pi+3\sqrt{3})$ **10.29.** $\frac{1}{3}[\sqrt{(\pi^2+4)^3}-8].$

10.30. $a\left(\pi\sqrt{4\pi^2+1}+\frac{1}{2}\ln\left(2\pi+\sqrt{4\pi^2+1}\right)\right)$. **10.31.** $\frac{\sqrt{1+m^2}}{m}a$.

10.32. $p[\sqrt{2}+\ln(1+\sqrt{2})]$. **10.33** $a(2\pi-th\pi)$. **10.34** $8a$.

10.35 $\frac{a}{2}\left(205-\frac{81}{4}\ln 3\right)$. **10.36** $\frac{1423a}{15}$. **10.37** $2a$. **10.38.** $16a$.

10.39. $2+\frac{1}{2}\ln 3$.

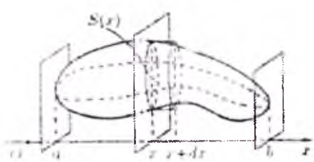
11-§. Aniq integral yordamida aylanma jismning hajmini hisoblash

11.1. Dekart koordinatalar sistemasida berilgan aylanma jismning hajmini hisoblash. Faraz qilaylik, bizga biror T jism berilgan bo'lib, uning Oy o'qqa parallel bo'lgan kesimlarining yuzasi ma'lum bo'lsin. Bu yuza, x o'zgaruvchining funksiyasi bo'ladi, uni $S=S(x)$ orqali belgilaymiz (11.1-chizma).

Agar $S=S(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, T jismning V hajmi, ushbu

$$V=\int_a^b S(x)dx \qquad (11.1)$$

formula bo'yicha hisoblanadi.



11.1-chizma.

11.1-misol. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoid sirt bilan chegaralangan jismning hajmini toping.

Yechilishi. Dastlab berilgan tenglama bo'yicha ellipsoidni yasaymiz.

Ellipsoidni Oxz tekislikka parallel bo'lgan, $y\in[-b,b]$ kesmada o'zgaruvchi, $y=p$ tekisliklar bilan kesamiz. Kesimda ellips hosil bo'ladi:

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1-\frac{p^2}{b^2},\ 1-\frac{p^2}{b^2}>0,\ \ \frac{x^2}{a_1^2}+\frac{z^2}{c_1^2}=1.$$

Bunda ellipsning yarim o'qlari,

$a_1=\frac{a}{b}\sqrt{b^2-p^2}$, $c_1=\frac{c}{b}\sqrt{b^2-p^2}$. Bu kesimlarning yuzlari, p ga bog'liq bo'lgan, ellips bilan chegaralangan yuzaga teng bo'ladi:

$$S(p)=\pi\, a_1 b_1=\frac{\pi\, ac}{b^2}\left(b^2-p^2\right).$$

$S(p)$ kesimlarning yuzasini (11.1) formulaga keltirib quyib, V jismning hajmini topamiz:

$$V=\int_{-b}^{b}\frac{\pi\, ac}{b^2}\left(b^2-p^2\right)dp=2\frac{\pi\, ac}{b^2}\int_0^b\left(b^2-p^2\right)dp=2\frac{\pi\, ac}{b^2}\left(b^2p-\frac{p^3}{3}\right)\Bigg|_0^b=\frac{4}{3}\pi\, abc.$$

Faraz qilaylik, $y=f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lib, $\forall x\in[a,b]$ uchun $f(x)\geq 0$ bo'lsin. Yuqoridan $y=f(x)$ funksiya grafigi, yon tomonlardan $x=a,\ x=b$ vertikal to'g'ri chiziqlar, quyidan Ox o'qdagi $[a,b]$ kesma bilan chegaralangan shaklni Ox o'q atrofida aylanishidan hosil bo'lgan aylanma T jismning hajmi (11.2-chizma),

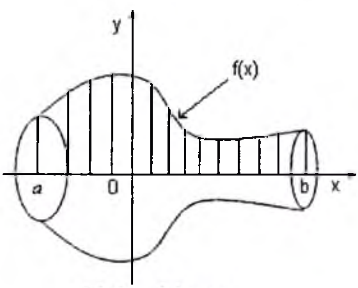
$$V_x=\pi\int_a^b f^2(x)dx \qquad (11.2)$$

formula bo'yicha topiladi.

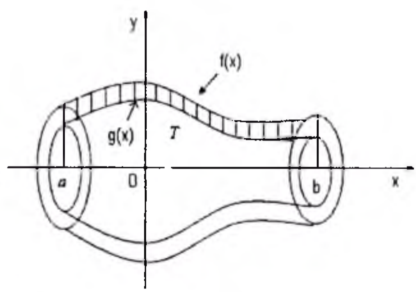
Agar D egri chiziqli trapesiya, yuqoridan $f(x)$, pastdan $g(x)$ uzluksiz egri chiziqlar bilan, yon tomonlardan esa, $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, uning Ox o'q atrofida aylanishidan hosil bo'lgan aylanma T jismning hajmi (11.3-chizma),

$$V_x=\pi\int_a^b\left[f^2(x)-g^2(x)\right]dx \qquad (11.3)$$

formula bo'yicha topiladi.



11.2- chizma.



11.3- chizma.

$y=y(x)$ funksiya, $[\alpha,\beta]$ kesmada $x=x(t)$, $y=y(t)$ parametrik tenglamalari bilan berilgan bo'lsin. Bu funksiyalar $[\alpha,\beta]$ da uzluksiz, $\forall t\in[\alpha,\beta]$ kesmada $y(t)\geq 0$ va $x(t)$ funksiya, uzluksiz, manfiy bo'lmagan $x'(t)$ hosilaga ega, hamda $a=x(\alpha)$, $b=y(\beta)$ bo'lsa, u holda, T aylanma jismning hajmi,

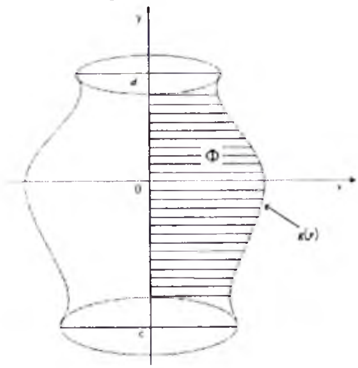
$$V=\pi\int_{\alpha}^{\beta}y^2(t)\cdot x'(t)dt \qquad (11.4)$$

formula bo'yicha topiladi.

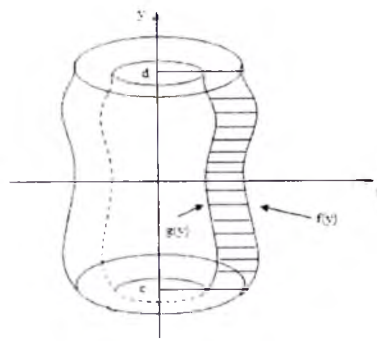
Agar $x(t)$ funksiya $[\alpha,\beta]$ kesmada kamayuvchi va $a=x(\alpha)$, $b=y(\beta)$ bo'lsa, u holda, yuqoridagi shartlar bajarilganda, T aylanma jismning hajmi,

$$V=-\pi\int_{\alpha}^{\beta}y^2(t)\,x'(t)dt$$

formula bo'yicha topiladi.



11.4- chizma.



11.5- chizma.

Oy $(x=0)$o'q atrofida aylantirishdan hosil bo'lgan Φaylanma jismning hajmi (11.4-, 11.5- chizma), yuqoridagi (11.2), (11.3), (11.4) formulalarga o'xshash quyidagi

$$V_y=\pi\int_c^d g^2(y)dy,$$

$$V_y=\pi\int_c^d\left[f^2(y)-g^2(y)\right]dy,$$

$$V=\pi\int_{\alpha}^{\beta}x^2(t)\,y'(t)dt$$

formulalar bo'yicha topiladi.

11.2-misol. Quyidagi, $y^2=2px,\ y=0,\ x=a$ chiziqlar bilan chegaralangan shaklni, Ox $(y=0)$o'q atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmini toping.

Yechilishi. D soha, yuqoridan $y^2=2px$ uzluksiz funksiya bilan, yon tomonlardan, $x=a$ va $x=0$ to'g'ri chiziqlar, pastdan esa, Ox $(y=0)$ o'q bilan chegaralangan. Endi, D egri chiziqli sohani Ox $(y=0)$o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmini, (11.2) formula bo‘yicha hisoblaymiz:

$$V_x = \pi\int_0^a y^2dx = \pi\int_0^a 2pxdx = 2p\pi\left.\frac{x^2}{2}\right|_0^a = \pi pa^2.$$

11.3-misol. Quyidagi, $x = a\,(t-\sin t),\ y = a\,(1-\cos t),\ (1\le t\le 2\pi),\ y=0$ chiziqlar bilan chegaralangan shaklni Ox $(y=0)$ o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmini toping.

Yechilishi. Aylanma jismning hajmini (11.4) formula bo‘yicha topamiz:

$$V_x = \pi\int_0^{2\pi} a^2(1-\cos t)^2 a(1-\cos t)dt = \pi a^3\int_0^{2\pi}(1-\cos t)^3dt = 8\pi a^3\int_0^{2\pi}\left(\sin^2\frac{t}{2}\right)^3 dt = 5\pi a^3.$$

11.2. Qutb koordinatalar sistemasida berilgan aylanma jismning hajmini hisoblash. Agar $\breve{AB}$ egri chiziqning tenglamasi qutb koordinatalar sistemasida $r=r(\varphi)$, $0\le\alpha\le\varphi\le\beta\le 2\pi$, ko‘rinishda berilgan bo‘lib, $[\alpha,\beta]$ kesmada $r(\varphi)$ - uzluksiz bo‘lsa, u holda, qutb nuri atrofida $T=\{(r,\varphi):\alpha\le\varphi\le\beta,\ 0\le r\le r(\varphi)\}$ aylanma sektorning hajmi,

$$V = \frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi d\varphi \qquad (11.5)$$

formula bo‘yicha topiladi.

Agar $\breve{AB}$ egri chiziq tenglamasi qutb koordinatalar sistemasida $r=r(\varphi)$, $-\frac{\pi}{2}\le\alpha\le\varphi\le\beta\le\frac{\pi}{2}$, ko‘rinishda berilgan bo‘lib, $[\alpha,\beta]$ kesmada $r(\varphi)$ uzluksiz bo‘lsa, u holda, $\varphi=\frac{\pi}{2}$ qutb nuri atrofida $T=\{(r,\varphi):\alpha\le\varphi\le\beta,\ 0\le r\le r(\varphi)\}$ aylanma sektorning hajmi,

$$V = \frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\cos\varphi d\varphi$$

formula bo‘yicha topiladi.

11.4-misol. Ushbu $r=a(1+\cos\varphi)$ kardiodaning qutb o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmini toping.

Yechilishi. $r=a(1+\cos\varphi)$ kardiodaning qutb o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmini, (11.5) formulaga ko‘ra, topamiz:

$$V = \frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi d\varphi = \frac{2\pi}{3}\int_0^\pi a^3(1-cos\varphi)^3\sin\varphi d\varphi =$$

$$= \frac{\pi}{6}a^3(1-cos\varphi)^4 = \frac{8\pi}{3}a^3.$$

Mustaqil yechish uchun misollar

Quyidagi chiziqlar bilan chegaralangan figuralarni ko‘rsatilgan o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismlarning hajmlarini toping:

11.1. $y=\sin x,\ 0\le x\le \pi$: *a)* Ox o‘q atrofida. *b)* Oy o‘q atrofida.

11.2. $y=2x-x^2,\ y=0$: *a)* Ox o‘q atrofida. *b)* Oy o‘q atrofida.

11.3. $y=\sin 2x,\ 0\le x\le \frac{\pi}{2},\ y=0$: Ox o‘q atrofida.

11.4. $y=b\sqrt[3]{\left(\frac{x}{a}\right)^2},\ 0\le x\le a$ Ox o‘q atrofida.

11.5. $xy=4,\ x=1,\ x=4,\ y=0$ Ox o‘q atrofida.

11.6. $y=\sqrt{x}e^{-x},\ y=0,\ \ x=a$ Ox o‘q atrofida.

11.7. $y=a\,ch\frac{x}{a}$ Ox o‘q va $x=\pm a$ to‘g‘ri chiziq atrofida.

11.8. $y=\arcsin x,\ y=0,\ x=1$: *a)* Ox o‘q atrofida. *b)* Oy o‘q atrofida.

11.9. $y^2=4x,\ \ y=x$ Ox o‘q atrofida.

Quyidagi chiziqlar bilan chegaralangan figuralarni ko‘rsatilgan o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismlarning hajmlarini toping:

11.10. $x^2+(y-b)^2=a^2\ \ (0<a\le b)$, Ox o‘q atrofida.

11.11. $xy=k^2,\ y=0,\ x=a,\ x=b,\ 0<a<b$, Oy o‘q atrofida.

11.12. $2py=a^2-(x-b)^2,\ y=0,\ 0<a<b$, Oy o‘q atrofida.

11.13. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ a<x<m$, Ox o‘q atrofida.

11.14. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$, Ox o‘q atrofida.

11.15. $y^2=2px,\ y=0,\ x=a$, Ox o‘q atrofida.

11.16. $y^2(x-4a)=ax(x-3a),\ 0\le x\le 3a$, Ox o‘q atrofida.

11.17. $y^2=(x+4)^3,\ x=0$, Oy o‘q atrofida.

11.18. $2py=(x-a)^2,\ 2py=a^2$: *a)* Ox o‘q atrofida. *b)* Oy o‘q atrofida.

Quyidagi chiziqlar bilan chegaralangan figuralarni ko‘rsatilgan o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismlarning hajmlarini toping.

11.19. $x=a(t-\sin t),\ y=a\,(1-\cos t)\ \ (0\le t\le 2\pi)$: *a)* Oy o‘q atrofida; *b)* $y=2a$ to‘g‘ri chiziq atrofida.

11.20 $x=a\sin^3 t,\ \ y=b\cos^3 t\ \ (0\le t\le 2\pi)$: *a)* Ox o‘q atrofida; *b)* Oy o‘q atrofida.

11.21. $x=2t-t^2,\ y=4t-t^3$: *a*) *Ox* o‘q atrofida; *b*) *Oy* o‘q atrofida.

Quyidagi chiziqlar bilan chegaralangan figuralarni ko‘rsatilgan o‘q atrofida aylantirishdan hosil bo‘lgan aylanma jismlarning hajmlarini toping:

11.22. $r=a\sin^2\varphi$, qutb o‘qi atrofida.

11.23. $r=a\sin\varphi$, qutb o‘qi atrofida.

11.24. $r=a\varphi\ (a>0,\ 0\le\varphi\le 2\pi)$ (Arximed spirali), qutb o‘qi atrofida.

11.26. $a\le r\le a\sqrt{2\sin 2\varphi}$, qutb o‘qi atrofida.

11.27. $0\le r\le a\varphi,\ 0\le\varphi\le\pi$, qutb o‘qi atrofida.

11.28. $0\le r\le a\cos^2\varphi$, qutb o‘qi atrofida.

11.29. $0\le r\le a\sqrt[3]{\cos 3\varphi},\ |\varphi|\le\frac{\pi}{6}$, qutb o‘qi atrofida.

11.30. $0\le r\le 2a\frac{\sin^2\varphi}{\cos\varphi},\ 0\le\varphi\le\frac{\pi}{3}$, qutb o‘qi atrofida.

11.31. $a\le r\le a\sqrt{\sin\varphi},\ \varphi=\frac{\pi}{2}$ nur atrofida.

11.32. $a\le r\le a\sqrt[3]{\cos 3\varphi},\ |\varphi|\le\frac{\pi}{6},\ \varphi=\frac{\pi}{2}$ nur atrofida.

11.33. $0\le r\le a\frac{\cos 2\varphi}{\cos\varphi},\ |\varphi|\le\frac{\pi}{3},\ \varphi=\frac{\pi}{3}$ nur atrofida.

Quyidagi sirtlar bilan chegaralangan jismning hajmini toping:

11.34. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ |z|=H,\ H>0.$

11.35. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=\frac{c}{a}x,\ z=0,\ z=0.$

11.36. $\frac{x^2}{y^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ z=\pm c.$

11.37. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ z=c+H.$

11.38. $x+y+z^2=1,\ x=0,\ y=0,\ z=0.$

11.39. $\frac{y^2}{a^2}+\frac{x+z}{a}=1,\ x=0,\ y=0,\ z=0.$

11.40. $x^2+y^2+z^2+xy+yz+zx=a^2.$

11.41. $x^2+z^2=a^2,\ \ y^2+z^2=a^2.$

11.42. $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\ \frac{y}{b}-\frac{z}{c}=0,\ \frac{y}{b}+\frac{z}{c}=0,\ (z\ge 0).$

Quyidagi egri chiziqlar bilan chegaralangan figurani *Ox* o‘q atrofida aylantirishidan hosil bo‘lgan jismning hajmini toping:

11.43. $x^4+y^4=a^2x^2.$

11.44. $x^4+y^4=2axy^2$

11.45. $(x^2+y^2)^3=a^2x^4.$

11.46. $(x^2+y^2)^2=a^2(x^2-y^2)$

Quyidagi egri chiziqlar bilan chegaralangan figurani Oy o‘q atrofida aylantirishidan hosil bo‘lgan jismning hajmini toping:

11.47. $x^4+y^4=ay^3$. **11.48.** $x^4+y^4=2axy^2$.

11.49. $(x^2+y^2)^3=a^2(x^2-y^2)^2$. **11.50.** $(x^2+y^2)^3=a^2(x^2-y^2)^2$.

Mustaqil yechish uchun misollarning javoblari

11.1. $a)\ 0{,}5\pi^2$. $b)\ 2\pi^2$. **11.2.** $a)\ \frac{16}{15}\pi$. $b)\ \frac{8}{3}\pi$. **11.3.** $0{,}25\pi^2$. **11.4.** $\frac{3}{7}\pi ab^2$.

11.5. 12π. **11.7.** $0{,}25\pi[1-e^{-2a}(1+2a)]$. **11.8.** $0{,}25\pi a^3(e^2+4-e^{-2})$

11.9. $a)\ 0{,}25\pi(\pi^2-8)$. $b)\ 0{,}25\ \pi^2$. **11.10.** $\frac{128}{15}\pi$. **11.11.** $2\pi^2a^2b$.

11.12. $2\pi k^2(b-a)$. **11.13.** $\frac{4}{3p}\pi a^3b$. **11.14.** $\frac{1}{3a^2}\pi b^2(m-a)^2(m+2a)$.

11.15. $\frac{4}{3}\pi ab^2$. **11.16.** πpa^2. **11.17.** $0{,}5\pi a^3(15-16\ln 2)$. **11.18.** $58{,}5\pi$.

11.19. $a)\ \frac{32}{15}\pi p^3$. $b)\ \frac{4}{3}\pi p^3$. **11.20.** $a)\ 6\pi^2a^3$. $b)\ 7\pi^2a^3$.

11.21. $a)\ \frac{32}{105}\pi ab^2$. $b)\ \frac{32}{105}\pi a^2b$. **11.22.** $a)\ \frac{64}{35}\pi$. $b)\ \frac{64}{105}\pi$.

11.23. $\frac{64}{105}\pi a^3$. **11.24.** $\frac{\pi^2a^3}{4}$. **11.25.** $\frac{2}{3}(\pi^4-6\pi^2)a^3$. **11.26.** $\frac{2}{3}\pi$

11.27. $2(\pi^4-6\pi^2)a^3/3$. **11.28.** $4\pi a^3/21$. **11.29.** $\pi a^3/24$.

11.30. $\pi a^3(51-64\ln 2)/4$. **11.31.** $\frac{4\pi a^3}{15}$. **11.32.** $\frac{\sqrt{3}\pi a^3}{4}$.

11.33. $\pi a^3(3\pi-8)/3$. **11.34.** $2\pi abH$. **11.35.** $\frac{2}{3}abc$. **11.36.** $\frac{8\pi abc}{3}$.

11.37. $\frac{\pi ab}{3c^2}H^2(H+3c)$. **11.38.** $\frac{4}{15}$. **11.39.** $\frac{4a^3}{15}$. **11.40.** $\frac{4\sqrt{2}\pi a^3}{3}$.

11.41. $\frac{16a^3}{3}$. **11.42.** $\frac{4abc}{3}$. **11.43.** $\frac{2\pi a^3}{3}$. **11.44.** $\frac{2\pi a^3}{3}$. **11.45.** $\frac{4\pi a^3}{21}$.

11.46. $\frac{\pi a^3}{2\sqrt{2}}\left(\ln(\sqrt{2}+1)-\frac{\sqrt{2}}{3}\right)$. **11.47.** $\frac{\pi^2a^3}{16}$. **11.48.** $\frac{4\pi a^3}{3}$. **11.49.** $\frac{8\pi a^3}{15}$.

11.50. $4\pi a^3(16\sqrt{2}-9)/105$.

12-§. Aniq integral yordamida aylanma jism sirtining yuzini hisoblash

12.1. Dekart koordinatalar sistemasida berilgan aylanma sirt yuzini hisoblash. $y=f(x)$ funksiya $[a;b]$ kesmada aniqlangan uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo‘lib, $\forall x\in[a;b]$ uchun, $f(x)\geq 0$ bo‘lsin. Bu funksiya grafigining $A(a;f(a))$ va $B(b;f(b))$ nuqtalar orasidagi $\breve{AB}$ yoyini Ox o'q atrofida aylantirish natijasida hosil bo'lgan sirtning yuzi,

$$Q_x = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx \qquad (12.1)$$

formula orqali hisoblanadi.

Agar $\breve{AB}$ egri chiziq, $x=\varphi(t),\ y=\psi(t),\ \alpha\le t\le\beta$, tenglama bilan berilgan bo'lib, $\psi(t)\ge 0$, $\varphi(t)$ funksiya, t α dan β gacha o'zgarganda, a dan b gacha o'zga-rib, $\varphi(\alpha)=a,\ \varphi(\beta)=b$ bo'lsa, u holda, (12.1) formulada, $x=\varphi(t)$ almashtirish natijasida,

$$Q_x = 2\pi\int_\alpha^\beta \psi(t)\sqrt{(\varphi'(t))^2+(\psi'(t))^2}\,dt \qquad (12.2)$$

formulani hosil qilamiz, ya'ni $\breve{AB}$ egri chiziqning tenglamasi parametrik shaklda berilganda, uning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzini topish formulasini hosil qilamiz.

12.1-misol. Ushbu $y^2=2px$ $(0\le x\le x_0)$ chiziqning: $a)$ Ox o'q ; $b)$ Oy o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzini toping.

Yechilishi. $a)$ Berilgan chiziqning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzi, (12.1) formulaga asosan topiladi, bunda, $f(x)=\sqrt{2px},\ f'(x)=\frac{p}{\sqrt{2px}},\ \sqrt{1+(f'(x))^2}=\sqrt{1+\frac{p}{2x}}.$

Shunday qilib, berilgan chiziqning Ox o'q atrofida aylanishi natijasida hosil bo'lgan sirtning yuzi:

$$Q_x = 2\pi\int_0^{x_0} f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\sqrt{p}\int_0^{x_0}\sqrt{p+2x}\,dx =$$

$$=\frac{2\pi\sqrt{p}}{3}[\sqrt{(p+2x_0)^3}-\sqrt{p^3}]=\frac{2\pi}{3}[(p+2x_0)\sqrt{p^2+2px_0}-p^2].$$

$b)$ Berilgan chiziqning Ox o'qqa nisbatan simmetrikligini e'tiborga olib, izlanayotgan sirtning yuzini,

$$Q_y = 4\pi\int_0^{\sqrt{2px_0}} x(y)\sqrt{1+[x'(y)]^2}\,dy$$

formula orqali topamiz:

$$Q_x=\frac{2\pi}{p}\int_0^{\sqrt{2px_0}} y^2\sqrt{1+\frac{y^2}{p^2}}dy=\frac{2\pi}{p^2}\int_0^{\sqrt{2px_0}} y^2\sqrt{p^2+y^2}dy=$$

$$=[u=y,\ dv=y\sqrt{p^2+y^2}dy;\ du=dy;\ v=\frac{1}{3}(p^2+y^2)^{\frac{3}{2}}]=$$

$$=\frac{2\pi}{p^2}\left\{\frac{y(p^2+y^2)^{3/2}}{4}-\frac{p^2}{8}[y\sqrt{p^2+y^2}+p^2\ln\left(y+\sqrt{p^2+y^2}\right)]\right\}\Bigg|_0^{\sqrt{2px_0}}=$$

$$=\frac{\pi}{4}[(p+4x_0)\sqrt{2x_0(p+2x_0)}-p^2\ln\frac{\sqrt{2x_0}+\sqrt{p+2x_0}}{\sqrt{p}}]$$

12.2-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)\ (0\le t\le 2\pi)$, chiziqning Ox o‘q atrofida aylanishi natijasida hosil bo‘lgan sirtning yuzini toping.

Yechilishi. Izlanayotgan sirtning yuzi, (12.2) formula orqali hisoblanadi:

$$x'_t=a(1-\cos t),\ y'_t=a\sin t,\ ds=\sqrt{(x'_t)^2+(y'_t)^2}dt=$$

$$=\sqrt{a^2(1-\cos t)^2+a^2\sin t dt}=a\sqrt{2}\sqrt{1-\cos t}\cdot dt=2a\sin\frac{t}{2}dt,$$

$$Q=2\pi\int_0^{2\pi}a(1-\cos t)\cdot 2a\cdot\sin\frac{t}{2}dt=2\pi\int_0^{2\pi}4a^2\sin^3\frac{t}{2}dt=$$

$$=16\pi a^2\int_0^{\pi}\sin^3 vdv=\frac{64}{3}\pi a^2$$

12.2. Qutb koordinatalar sistemasida berilgan aylanma sirtning yuzini hisoblash. Agar $\breve{AB}$ egri chiziqning tenglamasi, qutb koordinatalar sistemasida, $r=r(\varphi)$, $\alpha\le\varphi\le\beta$, ko‘rinishda berilgan bo‘lib, $r(\varphi)$ - $[\alpha,\beta]$ kesmada uzluksiz, uzluksiz hosilaga ega bo‘lsa, u holda, (12.2) formula,

$$Q=2\pi\int_\alpha^\beta r(\varphi)\sin\varphi\sqrt{(r(\varphi))^2+(r'(\varphi))^2}d\varphi \tag{12.3}$$

ko‘rinishda bo‘ladi.

12.3 -misol. Ushbu $r^2=a^2\cos 2\varphi$ chiziqning: $a)$ qutb o‘qi; $b)$ $\varphi=\frac{\pi}{2}$ o‘q atrofida aylanishi natijasida hosil bo‘lgan sirtlarning yuzlarini toping.

Yechilishi. $a)$ Ma’lumki, $r=a\sqrt{\cos 2\varphi}$ - Bernulli lemniskatasi bo‘lib, koordinatalar o‘qlariga nisbatan simmetrik joylashgan, koordinatalar boshi - qutb bilan, musbat yarim o‘q - qutb o‘qi bilan mos tushadi. Shuning uchun, izlanayotgan sirtning yuzi, (12.3) formulaga asosan topiladi:

$$Q = 4\pi\int_0^{\frac{\pi}{4}} r(\varphi)\sin\varphi\sqrt{r^2(\varphi)+(r'(\varphi))^2}\,d\varphi,\quad r\sin\varphi = a\sqrt{\cos 2\varphi}\cdot\sin\varphi,$$

$$r'_\varphi = -\frac{a\sin 2\varphi}{\sqrt{\cos 2\varphi}},\quad \sqrt{r^2+(r'_\varphi)^2} = \frac{a}{\sqrt{\cos 2\varphi}},\quad Q_p = 4\pi^2\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi = 2\pi a^2\left(2-\sqrt{2}\right)$$

b) Xuddi yuqoridagi *a)* banddagidek,

$$Q_{\varphi=\frac{\pi}{2}} = 4\pi a^2\int_0^{\frac{\pi}{4}}\cos\varphi\,d\varphi = 2\sqrt{2}\pi a^2.$$

Bu holda, $\varphi=\frac{\pi}{4}$ nurni, r_1,θ sistemadagi qutb o'qi. deb qabul qilamiz, u holda, $r_1(\theta)=r(\varphi)$, $\theta=\varphi-\frac{\pi}{4}$. Shunday qilib, izlanayotgan sirtning yuzi: $Q=Q_1+Q_2$ bo'lib, Q_1 va Q_2 ,mos ravishda,

$$Q_1 = 4\pi\int_{\theta=-\frac{\pi}{2}}^{\theta=-\frac{\pi}{4}} r_1(\theta)|\sin\theta|\cdot\sqrt{r_1^2(\theta)+(r_1'(\theta))^2}\,d\theta, \qquad (*)$$

$$Q_2 = 4\pi\int_{\theta=-\frac{\pi}{4}}^{\theta=0} r_1(\theta)|\sin\theta|\cdot\sqrt{r_1^2(\theta)+(r_1'(\theta))^2}\,d\theta$$

formulalar orqali topiladi(r_1,θ sistemasida, θ , $-\frac{\pi}{2}$ dan 0 gacha o'zgarganda, $\sin\theta$ funksiya manfiy qiymatlar qabul qiladi). (*) formulalarda r_1,θ dan, r va φ larga o'tsak, u holda,

$$Q_1 = 4\pi a^2\int_{-\frac{\pi}{4}}^{0}\sqrt{\cos 2\varphi}\left|\sin(\varphi-\frac{\pi}{4})\right|\frac{d\varphi}{\sqrt{\cos 2\varphi}} = 4\pi a^2\int_{-\frac{\pi}{4}}^{0}\left|\sin(\varphi-\frac{\pi}{4})\right|d\varphi =$$

$$= 4\pi a^2\int_0^{-\frac{\pi}{4}}\sin(\varphi-\frac{\pi}{4})\,d\varphi = -4\pi a^2\cos(\varphi-\frac{\pi}{4})\Big|_0^{-\frac{\pi}{4}} = 2\pi a^2\sqrt{2},$$

$$Q_2 = 4\pi a^2\int_0^{\frac{\pi}{4}}\left|\sin(\varphi-\frac{\pi}{4})\right|d\varphi = -4\pi a^2\int_0^{\frac{\pi}{4}}\sin(\varphi-\frac{\pi}{4})\,d\varphi =$$

$$= 4\pi a^2\cos(\varphi-\frac{\pi}{4})\Big|_0^{\frac{\pi}{4}} = 2\pi a^2(2-\sqrt{2}).$$

Mustaqil yechish uchun misollar

Quyidagi chiziqlarni ko'rsatilgan o'q atrofida aylantirishda hosil bo'lgan aylanma sirtlarining yuzlarini hisoblang:

12.1. $y=\frac{x^2}{2}$ parabolaning $y=1{,}5$ to‘g‘ri chiziq bilan kesilgan qismini, $x=2$ o‘q atrofida.

12.2. $y^2=4+x$ parabolaning $x=2$ to‘g‘ri chiziq bilan kesilgan qismini Ox o‘q atrofida.

12.3. $y=\cos\frac{\pi x}{2a}$ chiziqning, $x_1=-a$ dan $x_2=a$ gacha bo‘lgan qismining Ox o‘q atrofida.

12.4. $3x^2+4y^2=12$ ellipsning Oy o‘q atrofida.

12.5. $y=tgx$ $\left(0\le x\le\frac{\pi}{4}\right)$, Ox o‘q atrofida.

12.6. $y=e^{-x}$ $(0\le x\le a)$, Ox o‘q atrofida.

12.7. $2ay=a^2+x^2$ $(0\le x\le a)$, Ox o‘q atrofida.

12.8. $x=\frac{1}{4}y^2-\frac{1}{2}\ln y(1\le y\le e)$, Oy o‘q atrofida.

12.9. $x=\ln\left(y-\sqrt{y^2-}\right)\left(\frac{5}{4}\le y\le\frac{5}{3}\right)$, Oy o‘q atrofida.

12.10. $y^2+4x=2\ln y$ $(1\le y\le 2)$, Oy o‘q atrofida.

12.11. $y=a\,ch\frac{x}{a}$, $(|x|\le b)$, Oy o‘q atrofida..

12.12. $y=\frac{1}{x}$ $(1\le x\le a)$, Ox o‘q atrofida.

Quyidagi chiziqlarni ko‘rsatilgan o‘q atrofida aylantirishda hosil bo‘lgan sirtlarning yuzlarini hisoblang:

12.13. $x=a(3\cos t-\cos 3t)$, $y=a(3\sin t-\sin 3t)$, $0\le t\le\frac{\pi}{2}$, a) Ox o‘q atrofida; b) Oy o‘q atrofida.

12.14. $x=e^t\sin t$, $y=e^t\cos t$, $0\le t\le\frac{\pi}{2}$, Ox o‘q atrofida .

12.15. $x=a(t-\sin t)$, $y=a(1-\cos t)$ $(0\le t\le 2\pi)$, a) Ox o‘q atrofida; b) Oy o‘q atrofida; c) $y=2a$ to‘g‘ri chiziq atrofida.

12.16. $x=a\cos^3 t$, $y=a\sin^3 t$, $y=x$ to‘g‘ri chiziq atrofida.

12.17. $x=\frac{t^3}{3}$, $y=4-\frac{t^2}{2}$, $|t|\le 2\sqrt{2}$, Ox o‘q atrofida.

12.18. $x=2\sqrt{3}\cos t$, $y=\sin 2t$ Ox o‘q atrofida.

12.19. $x=a(1+t^2)$, $y=\frac{at}{3}(3-t^2)$ Ox o‘q atrofida.

12.20. $x=\frac{a}{t^2+1}$, $y=\frac{a(t^3-t)}{t^2+1}$ Ox o‘q atrofida.

12.21. $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$, $(0\le t\le\pi)$ Ox o‘q atrofida.

Quyidagi chiziqlarni ko'rsatilgan o'q atrofida aylantirishda hosil bo'lgan sirtlarning yuzlarini hisoblang:

12.22. $r = 2a\sin\varphi,\ 0 \le \varphi \le \frac{\pi}{4}$, qutb o'qi atrofida.

12.23. $r = \sqrt{\cos 2\varphi},\ 0 \le \varphi \le \frac{\pi}{4}$, qutb o'qi atrofida.

12.24. $r^2 = a^2 \sin 2\varphi,\ 0 \le \varphi \le \frac{2\pi}{3}$, qutb o'qi atrofida.

12.25. $r = a\sec^2\frac{\varphi}{2},\ 0 \le \varphi \le \frac{\pi}{2}$, qutb o'qi atrofida.

12.26. $r^2 = 2a^2\cos 2\varphi$, $a)$ qutb o'qi atrofida; $b)$ $\varphi = \frac{\pi}{2}$ nur atrofida; $c)$ $\varphi = \frac{\pi}{4}$ nur atrofida.

12.27. $r = a\sqrt{\cos 2\varphi}$ (lemniskata), $y = x$ to'g'ri chiziq atrofida.

Quyidagi silindrik chiziqlarni ko'rsatilgan o'q atrofida aylantirishda hosil bo'lgan sirtlarning yuzlarini hisoblang:

12.28 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$, Ox o'q atrofida.

12.29 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$, Oy o'q atrofida.

12.30. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ -h < y < h$, Oy o'q atrofida.

12.31. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a < x < m = \lambda a$, Ox o'q atrofida.

12.32. $y^2 = 2px,\ 0 < x < a$, Ox o'q atrofida.

12.33. $y^3 = px^2,\ 0 < y < a$, Ox o'q atrofida.

Quyidagi silindrik sirtlarning belgilangan qismidagi sirtlari yuzlarini hisoblang:

12.34. $x^2 + y^2 = a^2,\ 0 \le z \le \frac{h}{a}x,\ x \ge 0.$

12.35 $y = b - \frac{b}{a^2}x^2,\ 0 \le z \le \frac{h}{a}x,\ x \ge 0,\ y \ge 0.$

12.36. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ 0 \le z \le \frac{h}{a}x,\ x \ge 0,\ y \ge 0,\ a > b.$

Mustaqil yechish uchun misollarning javoblari

12.1. $\frac{14\pi}{3}$. **12.2.** $\frac{62\pi}{3}$. **12.3.** $2\sqrt{\pi^2 + 4a^2} + \frac{8a^2}{\pi}\ln\frac{\pi + \sqrt{\pi^2 + 4a^2}}{2a}$.

12.4. $2\pi(4+3\ln 2)$. **12.5.** $\pi[(\sqrt{5}-\sqrt{2})+\ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}]$

12.6. $\pi\left(\sqrt{2}-e^{-a}\sqrt{1+e^{2a}}-\ln\frac{e^{-a}+\sqrt{1+e^{2a}}}{1+\sqrt{2}}\right)$. **12.7.** $\frac{\pi a^2}{8}\cdot(3\ln(\sqrt{2}+1)+7\sqrt{2})$.

12.8. $\frac{\pi}{3}(e-1)(e^2+e+4)$. **12.9.** $\pi(-5-9\ln 2+16\ln 3)/6$. **12.10.** $\frac{10}{3}\pi$.

12.11. $98\pi/3$.

12.12. $\pi\left(\sqrt{2}-\frac{\sqrt{a^4+1}}{a^2}+\ln\frac{\sqrt{a^4+1}+a^2}{\sqrt{2}+1}\right)$. **12.13.** *a)* $9\pi^2a^2$. *b)* $24\pi a^2$.

12.14. $\frac{2\pi\sqrt{2}}{5}(e^{\pi}-2)$. **12.15.** *a)* $\frac{64}{3}\pi a^2$. *b)* $16\pi^2a^2$. *c)* $\frac{32}{3}\pi a^2$.

12.16. $\frac{3\pi}{5}a^2(4\sqrt{2}-1)$ **12.17.** $59{,}2\pi$. **12.18.** $\frac{15\pi}{8}(4+\ln 5)$.

12.19. 7) $3\pi a^2$. **12.20.** $\pi a^3(6\ln 2-4)/3$. **12.21.** $6\pi^2a^2$. **12.22.** $4\pi^2a^2$.

12.23. $2\pi(2-\sqrt{2})$ **12.24.** $4\pi a^2$. **12.25.** $\frac{8}{3}\pi a^2(2\sqrt{2}-1)$

12.26. *a)* $4\pi a^2(2-\sqrt{2})$. *b)* $4\sqrt{2}\pi a^2$. *c)* $8\pi a^2$. **12.27.** $4\pi a^2$.

12.28. $2\pi\,[b^2+ab\frac{\arcsin\varepsilon}{\varepsilon}]$, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.

12.29. $2\pi\,[a^2+\frac{b^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}]$, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.

12.30. $2\pi ah[\sqrt{1+u^2}+\frac{\ln(u+\sqrt{1+u^2})}{u}]$, $u=\frac{h\sqrt{a^2+b^2}}{b^2}$.

12.31. $\pi ab[\lambda\sqrt{\lambda^2u^2-1}-\frac{b}{a}-\frac{1}{e}\ln\frac{e+\sqrt{\lambda^2e^2-1}}{e+\sqrt{e^2-1}}]$, $e=\frac{\sqrt{a^2+b^2}}{b^2}$.

12.32. $\frac{2}{3}\pi\,[(2a+p)\sqrt{2ap+p^2}-p^2]$.

12.33. $\frac{\pi}{15}\left(\frac{8}{9}p\right)^2\left[\sqrt{\left(1+\frac{9a}{4p}\right)^3}\left(\frac{27a}{4p}-2\right)+2\right]$ **12.34.** $2ah$.

12.35. $\frac{h}{2b^2}[\sqrt{(a^2+4b^2)^3}-a^3]$.

12.36. $\frac{ah}{2}\left(1+\frac{1-\varepsilon^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\right)$, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.

13-§. Aniq integralning mexanika masalalariga tadbig'i

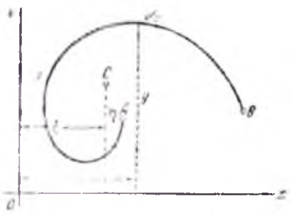
13.1 Silliq egri chiziqning statik momenti va og'irlik markazi. Biror m - massaga ega bo'lgan moddiy nuqtaning biror p o'qqa nisbatan statik momenti deb, uning m massasi bilan undan p o'qqacha bo'lgan d

masofaning ko'paytmasiga, ya'ni $M_p = md$ songa aytiladi. Tekislikda yotgan massalari mos ravishda, $m_1, m_2, ..., m_n$, p o'qqa nisbatan masofalari $d_1, d_2, ..., d_n$ bo'lgan n ta moddiy nuqtalar sistemasining statik momenti,

$$M_p = \sum_{i=1}^{n} m_i d_i$$

bo'ladi, bunda o'qning bir tomonida yotgan nuqtalarning masofalari musbat ishora bilan, ikkinchi tomonida yotganlarining masofalari esa, manfiy ishora bilan olinadi.

Agar massa, ayrim olingan nuqtalarda tarqalmasdan, biror silliq chiziq yoki tekis figura bo'ylab tekis tarqalgan bo'lsa, u holda, statik momentdagi yig'indi o'rniga integral olinadi.



13.1-chizma.

Faraz qilaylik, xOy tekislikda m massali biror (AB) silliq egri chiziq berilgan bo'lib, uning uzunligi l bo'lsin.

(AB) silliq egri chiziqning har bir nuqtasidagi chiziqli zichlik $\rho = \rho(s)$ - s o'zgaruvchining uzluksiz funksiyasi bo'lsin (13.1- chizma). U holda, (AB) silliq egri chiziqning massasi quyidagi formula bo'yicha topiladi:

$$m = \int_0^l \rho(s) ds,$$

bunda, $dl = \sqrt{(dx)^2 + (dy)^2}$ - yoy differensiali, l esa, berilgan chiziqning uzunligi.

Eslatma. Agar (AB) silliq egri chiziqni *birjinsli* deb faraz qilsak, u holda, uning ρ chiziqli zichligi (bir birlik uzunlikdagi massa, uning uzunligi bilan o'lchanadi), o'zgarmas, deb qaraymiz. Soddalik uchun, $\rho = 1$ deb olamiz.

(AB) chiziqning elementar ds bo'lagini ajratamiz. Yuqoridagi farazimiz bo'yicha, ds elementar bo'lakdagi massa, m son orqali ifodalanadi. Chiziqning elementar ds yoyini, taqribiy moddiy nuqta, deb qabul qilsak, u holda, uning Ox o'qqa nisbatan elementar statik momenti,

$$dM_x = y ds$$

bo'ladi. Erkli o'zgaruvchi sifatida, A nuqtadan boshlab hisoblanadigan, ds yoyni olib, elementar statik momentlarni yig'sak, natijada

$$M_x = \int_0^l y(s) \rho(s) ds$$

bo'ladi. Xuddi shunday, Oy o'qqa nisbatan statik moment,

$$M_y = \int_0^l x(s)\rho(s)ds$$

bo'ladi.

m massali, chiziqli zichligi $\rho = \rho(s)$ bo'lgan (AB) egri chiziqning $M(x_M; y_M)$ og'irlik markazi koordinatalari esa, ushbu

$$x_M = \frac{M_y}{m} = \frac{\int_0^l x(s)\rho(s)ds}{\int_0^l \rho(s)ds}, \quad y_M = \frac{M_x}{m} = \frac{\int_0^l y(s)\rho(s)ds}{\int_0^l \rho(s)ds}$$

formulalar yordamida xisoblanadi.

m massali, chiziqli zichligi $\rho = \rho(s)$ bo'lgan (AB) egri chiziqning Ox va Oy o'qlarga nisbatan inersiya momentlari,

$$I_x = \int_0^l y^2(s)\rho(s)ds, \; I_y = \int_0^l x^2(s)\rho(s)ds$$

formulalar yordamida hisoblanadi.

Xususiy holda, birjinsli ($\rho(s) = 1$) bo'lgan (AB) egri chiziqning m massasi, Ox va Oy o'qlarga nisbatan, M_x, M_y statik va I_x, I_y inersiya momentlari, hamda $M(x_M; y_M)$ og'irlik markazining koordinatalari, mos ravishda,

$$m = \int_0^l ds = l, \quad M_x = \int_0^l y(s)ds, \; M_y = \int_0^l x(s)ds$$

$$I_x = \int_0^l y^2(s)ds, \; I_y = \int_0^l x^2(s)ds \qquad (13.1)$$

$x_M = \frac{M_y}{l}, \; y_M = \frac{M_x}{l}$ formulalar yordamida hisoblanadi.

Teorema (Guldinning birinchi teoremasi). (AB) egri chiziqni, uni kesib o'tmaydigan, o'q atrofida aylantirish natijasida hosil bo'lgan aylanish sirtining yuzi, uning og'irlik markazi chizgan aylana uzunligining shu egri chiziqning l yoy uzunligiga ko'paytmasiga teng:

$$2\pi y_M l = 2\pi \int_0^l y dl \quad . \qquad (13.2)$$

13.1-misol. Ushbu $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellips yuqori qismining Ox o'qqa nisbatan statik momentini toping.

Yechilishi. Ellips yuqori qismining Ox o'qqa nisbatan statik momenti

$$M_x=\int_{-a}^{a} y\,dl$$

formula bo'yicha topiladi, bunda dl – elementar yoy uzunligi. Shartga ko'ra, $y\geq 0$, u holda,

$$ydl=y\sqrt{1+y_x'^2}dx=\sqrt{y^2+(yy_x')^2}dx,\ y^2=b^2-\frac{b^2}{a^2}x^2,\ yy'=-\frac{b^2}{a^2}x,$$

$$ydl=\sqrt{b^2-\frac{b^2}{a^2}x^2+\frac{b^4}{a^4}x^2}dx=\frac{b}{a}\sqrt{a^2-\varepsilon^2x^2}dx,\ \varepsilon^2=\frac{a^2-b^2}{a^2},$$

ε – ellipsning ekssentrisiteti.

Shunday qilib,

$$M_x=\int_{-a}^{a}\sqrt{a^2-\varepsilon^2x^2}dx=b\left(b-\frac{a}{\varepsilon}\arcsin\varepsilon\right).$$

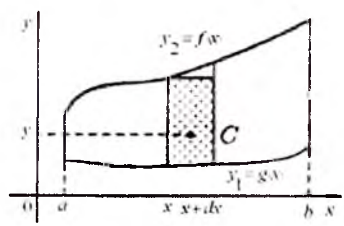
13.2-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)\,(0\leq t\leq 2\pi)$, egri chiziqning og'irlik markazini toping.

Yechilishi. Berilgan birjinsli egri chiziq, $x=\pi a$ to'g'ri chiziqqa nisbatan simmetrik bo'lgani uchun, uning og'irlik markazi shu to'g'ri chiziqda yotadi, ya'ni $x_M=\pi a$. 10.2 - va 12.2 - misollarni e'tiborga olgan holda, Guldinning birinchi teoremasiga asosan, ya'ni (13.2) formuladan,

$$2\pi y_M\cdot 8a=\frac{64}{3}\cdot\pi a^2,\ \eta=\frac{4}{3}a$$

ekanligini olamiz. Demak, sikloidaning og'irlik markazi, $M\left(\pi a;\frac{4}{3}a\right)$.

13.2. Tekis figuraning statik momentlari va og'irlik markazi. Tekis figura, yuqoridan $y_2=f(x)$, quyidan $y_1=g(x)$ funksiyalar, yon tomonlaridan esa, $x=a$ va $x=b$ vertikal chiziqlar bilan chegaralangan bo'lsin. m massali tekis figuraning har bir nuqtasida chiziqli zichligi $\rho=\rho(x)$ x o'zgaruvchining uzluksiz funksiyasi bo'lsin (13.2-chizma).



13.2-chizma.

Tekis figuraning m massasi, Ox va Oy o'qlarga nisbatan, M_x, M_y statik va I_x, I_y inersiya momentlari, hamda $M(x_M;y_M)$ og'irlik markazining koordinatalari, mos ravishda, quyidagi,

$$m=\int_a^b(f(x)-g(x))\rho(x)dx,$$

$$M_x = \frac{1}{2}\int_a^b \left(f^2(x) - g^2(x)\right)\rho(x)dx,\ M_y = \int_a^b x[f(x) - g(x)]\rho(x)dx\ , \qquad (13.3)$$

$$I_x = \frac{1}{3}\int_0^l [f^3(x) - g^3(x)]\rho(x)dx,\ I_y = \int_a^b x^2[f(x) - g(x)]\rho(x)dx,$$

$$x_M = \frac{M_y}{m},\ y_M = \frac{M_x}{m}$$

formulalar yordamida hisoblanadi.

13.1-eslatma. Tekis figura bo'ylab massa tekis tarqalgan, ya'ni uning sirt zichligi ρ o'zgarmas bo'lsin. Umumiylikni buzmasdan, $\rho = 1$ deb olamiz. U holda, tekis figuraning m massasi, Ox va Oy o'qlarga nisbatan M_x, M_y statik va I_x, I_y inersiya momentlari, hamda $M(x_M; y_M)$ og'irlik markazining koordinatalari, mos ravishda,

$$m = \int_a^b (f(x) - g(x))dx = S,$$

$$M_x = \frac{1}{2}\int_a^b \left(f^2(x) - g^2(x)\right)dx,\ M_y = \int_a^b x[f(x) - g(x)]dx,$$

$$I_x = \frac{1}{3}\int_0^l [f^3(x) - g^3(x)]dx,\ I_y = \int_a^b x^2[f(x) - g(x)]dx,$$

$$x_M = \frac{M_y}{S},\ y_M = \frac{M_x}{S}$$

formulalar yordamida hisoblanadi, bu yerda, $S = \int_a^b y(x)dx$ – tekis figuraning yuzi.

Og'irlik markazining ordinatasi formulasidan, $2\pi y_M \cdot S = \pi\int_a^b y^2 dx$ munosabatni hosil qilamiz. Bu formulaning o'ng tomoni, $ABCD$ figurani Ox o'q atrofida aylantirish natijasida hosil bo'lgan jismning hajmini ifodalaydi, chap tomoni esa, og'irlik markazi va u chizgan aylana uzunligini, figuraning yuziga ko'paytmasiga teng bo'lar ekan. Shunday qilib, quyidagi Guldinning ikkinchi teoremasiga kelamiz.

Teorema (Guldinning ikkinchi teoremasi). Tekis figurani, uni kesmaydigan o'q atrofida aylantirish natijasida hosil bo'lgan jismning hajmi, uning og'irlik markazi chizgan aylana uzunligini tekis figuraning yuziga ko'paytmasiga teng, ya'ni $V = 2\pi\eta \cdot S$.

Qutb koordinatalar sistemasida sektor, $0 \le r \le r(\varphi)$, $\alpha \le \varphi \le \beta$, tengsizliklar orqali berilgan bo'lsin, bunda $0 < \beta - \alpha \le 2\pi$, $r(\varphi)$ funksiya $[\alpha, \beta]$ kesmada uzluksiz. Massa sektorda tekis tarqalgan bo'lsin, ya'ni uning ρ zichligi o'zgarmas bo'lsin. Bu erda ham, umumiylikni buzmaslik uchun, $\rho = 1$ deb olamiz. U holda, Ox va Oy o'qlarga nisbatan statik momentlar va og'irlik markazining koordinatalari, quyidagi

$$M_x = \frac{1}{3}\int_{\alpha}^{\beta} r^3(\varphi)\sin\varphi\, d\varphi, \quad M_y = \frac{1}{3}\int_{\alpha}^{\beta} r^3(\varphi)\cos\varphi\, d\varphi,$$

$$x_M = \frac{M_y}{S}, \quad y_M = \frac{M_x}{S}.$$

formulalar orqali topiladi, bunda S - sektorning yuzi, ya'ni

$$S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)\, d\varphi.$$

13.3 - misol. Ushbu $\frac{x}{a} + \frac{y}{b} = 1,\ x = 0,\ y = 0,\ a > 0, b > 0$ chiziqlar bilan chegaralangan tekis figuraning Ox va Oy o'qlarga nisbatan statik momentlari va og'irlik markazining koordinatalarini toping.

Yechilishi. (13.3) formulalarga asosan,

$$M_x = \frac{1}{2}\int_{b}^{a} \frac{b^2}{a^2}(a-x)^2 dx = -\frac{b^2}{2a^2}\frac{(a-x)^3}{3} = \frac{ab^2}{6},$$

$$M_y = \int_0^a x\cdot\frac{b}{a}(a-x)dx = \frac{b}{a}\int_0^a (ax - x^2)dx = \frac{b}{a}\left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{a^2 b}{6},$$

$$S = \int_0^a \frac{b}{a}(a-x)dx = \frac{b}{a}\int_0^a (a-x)dx = \frac{b}{a}\left(ax - \frac{x^2}{2}\right)\Bigg|_0^a = \frac{b}{a}\cdot a^2\left(1 - \frac{1}{2}\right) = \frac{ab}{2},$$

$$x_M = \frac{M_y}{S} = \frac{\frac{a^2 b}{6}}{\frac{ab}{2}} = \frac{a}{3}; \quad y_M = \frac{M_x}{S} = \frac{\frac{ab^2}{6}}{ab} = \frac{b}{3}.$$

Demak, $M(x_M; y_M) = M\left(\frac{a}{3}; \frac{b}{3}\right)$.

13.4-misol. Ushbu $x = a\sin t,\ y = b\cos t,\ |t| \le \frac{\pi}{2},\ y = 0$ chiziqlar bilan chegaralangan sohaning Ox va Oy o'qlarga nisbatan statik momentlari va og'irlik markazining koordinatalarini toping.

Yechilishi. (13.3) formulalarga asosan,

$$M_x = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b^2\cos^2 t\cdot a\cos t\, dt = \frac{ab^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sin^2 t\right) d(\sin t) = \frac{ab^2}{2}\left(\sin t - \frac{\sin^3 t}{3}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2ab^2}{3},$$

$$M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\sin t\cdot b\cos t\, a\cdot\cos t\, dt = -a^2 b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\, d\cos t = -\frac{a^2 b}{3}\cos^3 t\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

$$S=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b\cos t\cdot a\cos t dt = ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t dt = ab\left(\frac{1}{2}t+\frac{\sin 2t}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = ab\cdot\left(\frac{2\cdot\pi}{4}\right)=\frac{ab\pi}{2}.$$

$$x_M=\frac{M_y}{S}=\frac{0}{S}=0;\ y_M=\frac{M_x}{S}=\frac{\frac{2ab^2}{3}}{\frac{ab\cdot\pi}{2}}=\frac{4}{3}\frac{b}{\pi}.$$

13.5-misol. Guldinning ikkinchi teoremasidan foydalanib, $x^2+(y-b)^2=a^2$, $a<b$, doiraning Ox o‘q atrofida aylanishi natijasida hosil bo‘lgan torning hajmini toping.

Yechilishi. Ma’lumki, berilgan doiraning yuzi $S=\pi a^2$, doiraning og‘irlik markazi, uning markazida bo‘ladi. Demak, doiraning Ox o‘q atrofida aylanishi natijasida, uning og‘irlik markazi chizgan aylananing uzunligi, $2\pi b$ ga teng. U holda, Guldinning ikkinchi teoremasiga asosan, izlanayotgan hajm,

$$V=\pi a^2\cdot 2\pi b=2\pi^2a^2b.$$

13.6-misol. Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)$ $(0\le t\le 2\pi)$ sikloida tarmog‘i bilan chegaralangan soha og‘irlik markazining koordinatalarini toping.

Yechilishi. Yuqoridagi 9.5 - va 11.3 - misollar, hamda Guldinning ikkinchi teoremasiga asosan,

$$2\pi y_M\cdot S=5a^3\pi^2;\quad 2\pi y_M\cdot 3\pi a^2=5a^3\pi^2,\quad y_M=\frac{5}{6}a,$$

sohaning simmetrikligidan, $x_M=\pi a$.

Shunday qilib,

$$M_x=\frac{a^3\pi}{3}\left(\pi^2-6\right),\quad M_y=\frac{1}{3}\int_0^{\pi}a^3\varphi\cos d\varphi=a^3\left(4-\pi^2\right),$$

$$S=\frac{1}{2}\int_0^{\pi}a^2\varphi^2 dv=\frac{a^2}{2}\cdot\frac{\pi^3}{3}=\frac{a^2\pi^3}{6}$$

bo‘ladi va

$$x_M=\frac{M_y}{S}=\frac{6a\left(4-\pi^2\right)}{\pi^3}=6\left(4-\pi^2\right)a/\pi^3,\ y_M=\frac{M_x}{S}=2\left(\pi^2-6\right)a/\pi^2.$$

Mustaqil yechish uchun misollar

Quyida berilgan chiziqlarning M_x va M_y statik momentlarini toping:

13.1. $\frac{x}{a}+\frac{y}{b}=1,\ x\ge 0,\ y\ge 0.$

13.2. $x^2+y^2=4,\ y\ge 0.$

13.3. $y=chx,\ \ 0\le x\le 1.$

13.4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, y\ge 0,\ a>b.$

13.5. Birinchi chorakda joylashgan $x^{\frac{2}{3}}+y^{\frac{3}{2}}=a^{\frac{2}{3}}$ astroida yoyining koordinatalar o‘qlariga nisbatan statik momentlarini toping.

13.6. $y=\cos x$, $\left(-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right)$, kosinusoidaning Ox o‘qqa nisbatan statik momentini toping.

13.7. $y=\sin x$, $0\le x\le\pi$, sinusoidaning Ox o‘qqa nisbatan statik momentini toping.

13.8. Ushbu $y^2=2px\left(0\le x\le\frac{p}{2}\right)$ parabola yoyining $x=\frac{p}{2}$ to‘g‘ri chiziqqa nisbatan statik momentini toping.

Tenglamalari parametrik shaklda berilgan quyidagi chiziqning M_x va M_y statik momentlarini toping:

13.9. $x=a\sin t,\ y=b\cos t,\ 0\le t\le\frac{\pi}{2},\ a>b\ \ (\rho=1)$.

13.10. $x=a\sin^3 t,\ y=a\cos^3 t,\ \ 0\le t\le\frac{\pi}{2}\ \ (\rho=1)$.

Qutb koordinatalar sistemasida berilgan chiziqning M_x va M_y statik momentlarini toping:

13.11. $r=2a\cos\varphi,\ 0\le\varphi\le\frac{\pi}{2}\ \ (\rho=1)$. **13.12.** $\eta=a(1+\cos\varphi),\ -\pi\le\varphi\le\pi\ \ (\rho=1)$.

13.13. $r=ae^{\varphi},\ 0\le\varphi\le 2\pi\ \ \ (\rho=1)$.

Quyidagi berilgan chiziqlar og‘irlik markazining x_M va y_M koordinatalarini toping:

13.14. $x^2+y^2=4,\ y\ge 0$.

13.15. $x^{2/3}+y^{2/3}=a^{2/3},\ x\ge 0,\ y\ge 0$.

13.16. $y=ach(x/a),\ |x|\le b$.

13.17. $x=\frac{1}{4}y^2-\frac{1}{2}\ln y,\ 1\le y\le 2$.

13.18. $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$.

13.19. $r=a(1+\cos\varphi),\ 0\le\varphi\le\pi$.

13.20. $r=ae^{\varphi},\ \frac{\pi}{2}\le\varphi\le\pi$.

13.21. Radiusi a ga teng bo‘lgan yarim aylana og‘irlik markazining koordinatalarini Guldin teoremasi yordamida toping.

Quyidagi berilgan chiziqlarning, Ox o‘qqa nisbatan, I_x inersiya momentini toping:

13.22 $y=\sqrt{r^2-x^2},\ -r\le x\le r$.

13.23. $y=e^x,\ 0\le x\le 0{,}5$.

13.24. $y=0{,}5a\left(e^{x/a}+e-e^{-x/a}\right),\ 0\le x\le a$.

13.25. $x=R\cos\varphi,\ y=R\sin\varphi,\ 0\le\varphi\le\alpha\le 2\pi$.

Quyidagi berilgan chiziqning, koordinatalar o‘qlariga nisbatan, I_x va I_y inersiya momentlarini toping:

13.26. $y = chx,\ 0 \le x \le 1.$

13.27. $x = a\cos^3 t,\ y = a\sin^3 t,\ 0 \le t \le \frac{\pi}{2}.$

13.28. $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi.$

Quyida berilgan chiziqlar bilan chegaralangan tekis shakllarning M_x va M_y statik momentlarini toping:

13.29. $x + y = 1, x = 0,\ y = 0.$

13.30. $y = \cos x,\ |x| \le \frac{\pi}{2},\ y = 0.$

13.31. $y = x^2,\ y = \sqrt{x}.$

13.32. $y = \frac{2}{1 + x^2},\ y = x^2,\ x = 0,\ x \ge 0$

13.33. $x = a\cos t,\ y = b\sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2},\ y = 0.$

13.34. $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi,\ y = 0.$

Quyidagi misollarda berilgan egri chiziqlar bilan chegaralangan sohaning grafigini chizing va og'irlik markazini toping:

13.35. $y = 6x - x^2,\ y = x.$

13.36. $x^2 = 4y,\ x - 2y + 4 = 0.$

13.37. $y^3 = x^2,\ 2y = x.$

13.38. $y = x^2 - 2x,\quad y = 6x - x^2.$

13.39 $x + 1 = 0,\ x + y^2 = 0.$

13.40. $y = 2x^2$ va $y = 3 - x^2$ parabolalar bilan chegaralangan sohani qoplovchi yupqa tekis plastinkaning og'irlik markazini toping.

13.41. Birinchi chorakda, Oy o'q, $y = \frac{x^2}{4}$ parabola va $y = 4$ to'g'ri chiziq bilan chegaralangan «uchburchakli» sohani qoplovchi yupqa tekis plastinkaning og'irlik markazini toping.

13.42. $x = a(t - \sin t),\ y = a(1 - \cos t)$, sikloidaning bir arki va uning asosi bilan chegaralangan plastinkaning Ox o'qqa nisbatan statik momentini toping.

Quyidagi misollarda berilgan egri chiziqlar bilan chegaralangan sohaning og'irlik markazi koordinatalarini toping:

13.43. $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\ \ (0 \le x \le a,\ 0 \le y \le b).$

13.44. $x^2 + 4y - 16 = 0$ va Ox o'q bilan chegaralangan.

13.45. $y = \frac{2}{\pi}x$ to'g'ri chiziq va $y = \sin x\ (x \ge 0)$.

13.46. $x^2 + 4y^2 = 4$ va $x^2 + y^2 = 4$(birinchi chorakdagi qismi) .

13.47. $y = h\left(1 - \frac{x^2}{a^2}\right),\ y = 0,\ h > 0, a > 0.$

13.48. $y = \frac{2}{\pi}x,\ y = \sin x,\ \ y = 0.$

13.49. $y^2 = 2px,\ \ x^2 = 2py.$

13.50. $\sqrt{x}+\sqrt{y}=\sqrt{a},\ x=0,\ y=0.$

13.51. $y^2=2x,\ x+y=4.$

13.52. $y=x^3,\ x+y=2,\ x=0.$

13.53. $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi,\ y=0.$

Quyidagi qutb koordinatalar sistemasida berilgan egri chiziqlar bilan chegaralangan sohaning og'irlik markazi koordinatalarini toping:

13.54. $r=a\sin 2\varphi,\ 0\le\varphi\le\frac{\pi}{2}.$

13.55. $r=a\varphi\ (0\le\varphi\le\pi)$ Arximed spirali va $\varphi=0,\ \varphi=\pi$ nurlar bilan chegaralangan.

13.56. $r=a\,(1+\cos\varphi)$ kardoida.

13.57. $r^2=a^2\cos 2\varphi$ – Bernulli lemniskatasining o'ng qismi.

13.58. Uchburchakning asosiga nisbatan inersiya momentini toping.

13.59. Tomonlari a ga teng bo'lgan kvadratning, uning diagonaliga nisbatan, inersiya momentini toping.

13.60. Asosi b, balandligi h bo'lgan $(\rho=1)$ uchburchakning, asosiga nisbatan, statik va inersiya momentlarini toping.

13.61. Yarim o'qlari a va b bo'lgan, birjinsli ellipssimon plastinkaning, uning bosh o'qlariga nisbatan inersiya momentlarini toping $(\rho=1)$.

13.62. Gulden teoremasidan foydalanib, R radiusli yarim aylananing og'irlik markazi koordinatalarini aniqlang.

13.63. Gulden teoremasidan foydalanib, shar sirtning yuzini hisoblang.

13.64. Gulden teoremasidan foydalanib, doiraviy konusning hajmi va yon sirtining yuzini hisoblang.

Mustaqil yechish uchun misollarning javoblari

13.1. $M_x=\frac{b\sqrt{a^2+b^2}}{2},\ M_y=\frac{a\sqrt{a^2+b^2}}{2}.$ **13.2.** $M_x=4,\ M_y=0.$

13.3. $M_x=0{,}25\,(2+sh2),\ M_y=sh1-ch1+1.$ **13.4.** $M_x=b\left(b+\frac{a}{\varepsilon}\arcsin\varepsilon\right),$

$M_y=0,\ \varepsilon=\frac{\sqrt{a^2-b^2}}{a}$ **13.5.** $M_x=M_y=\frac{3}{5}a^2.$ **13.6.** $M_x=\sqrt{2}+\ln(1+\sqrt{2})$

13.7. $\sqrt{2}+\ln(1+\sqrt{2}).$ **13.8.** $M_x=\frac{p^2}{8}\left(\sqrt{2}+5\ln(1+\sqrt{2})\right).$

13.9. $M_x=\frac{ab}{2\varepsilon}\left(\varepsilon\sqrt{1-\varepsilon^2}+\arcsin\varepsilon\right)$, $M_y=\frac{a^2}{2}\left(1+\frac{1-\varepsilon^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\right)$, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.

13.10. $M_x=M_y=\frac{3}{5}a^2$. **13.11.** $M_x=2a^2$, $M_y=\pi a^2$.

13.12. $M_x=0$, $M_y=\frac{32a^2}{5}$. **13.13.** $M_x=\frac{\sqrt{2}}{5}\left(1-e^{4\pi}\right)a^2$, $M_y=\frac{2\sqrt{2}}{6}\left(e^{4\pi}-1\right)a^2$.

13.14. $x_M=0$, $y_M=\frac{4}{\pi}$.

13.15. $x_M=y_M=2a/5$. **13.16.** $x_M=0$; $y_M=\frac{ash(2b/a)+2b}{4sh(b/a)}$.

13.17. $x_M=\frac{27-16\ln 2-4\ln^2 2}{8(3+\ln 4)}$, $y_M=\frac{20}{3(3+\ln 4)}$. **13.18.** $x_M=\pi a$; $y_M=4a/3$.

13.19. $x_M=y_M=4a/5$. **13.20.** $x_M=-\frac{a}{5}\frac{2e^{2\pi}+e^{\pi}}{e^{\pi}-e^{\pi/2}}$; $y_M=\frac{a}{5}\cdot\frac{e^{2\pi}-2e^{\pi}}{e^{\pi}-e^{\pi/2}}$.

13.21. $x_M=0$, $y_M=\frac{4}{3}\cdot\frac{a}{\pi}$. **13.22.** $\frac{\pi r^3}{2}$. **13.23.** $\frac{1}{3}[\sqrt{(1+e)^3}-2\sqrt{2}]$.

13.24. $a^3\left(e-e^{-1}\right)\left(e^2+e^{-2}+10\right)/24$. **13.25.** $\frac{1}{4}(2\alpha-\sin 2\alpha)R^3$.

13.26. $I_x=sh1+\frac{1}{3}sh^3 1$, $I_y=3sh1-2ch1$. **13.27.** $I_x=I_y=\frac{3}{8}a^3$.

13.28. $I_x=\frac{256}{15}a^3$, $I_y=16\left(\pi^2-\frac{128}{45}\right)a^3$. **13.29.** $M_x=\frac{1}{6}$, $M_y=\frac{1}{6}$.

13.30. $M_x=\frac{\pi}{4}$, $M_y=0$. **13.31.** $M_x=M_y=\frac{3}{20}$.

13.32. $M_x=\frac{2}{5}+\frac{\pi}{4}$, $M_y=\ln 2-0{,}25$. **13.33.** $M_x=\frac{2}{3}ab^2$, $M_y=0$.

13.34. $M_x=2{,}5\pi a^3$, $M_y=3\pi^2a^3$. **13.35.** $x_M=\frac{5}{2}$; $y_M=5$.

13.36. $x_M=1$, $y_M=\frac{8}{5}$. **13.37.** $x_M=\frac{10}{3}$, $y_M=\frac{40}{21}$. **13.38.** $x_M=2$, $y_M=4$.

13.39. $x_M=-\frac{3}{5}$, $y_M=0$. **13.40.** $x_M=0$, $y_M=\frac{8}{5}$. **13.41.** $x_M=\frac{3}{2}$, $y_M=\frac{12}{5}$.

13.42. $M_x=\frac{5}{2}\pi a^3$. **13.43.** $x_M=\frac{4a}{3\pi}$, $y_M=\frac{4b}{3\pi}$. **13.44.** $x_M=0$, $y_M=\frac{8}{5}$.

13.45. $x_M=\frac{\pi}{6(4-\pi)}$, $y_M=\frac{12-\pi^2}{12-3\pi}$. **13.46.** $x_M=\frac{8}{3\pi}$, $y_M=\frac{4}{\pi}$.

13.47. $x_M=\frac{59}{7}$; $y_M=\frac{5a}{16}$. **13.48.** $x_M=\frac{\pi^2+12\pi-12}{3(\pi+4)}$, $y_M=\frac{5\pi}{6}(\pi+4)$.

13.49. $x_M=y_M=\frac{9p}{10}$. **13.50.** $x_M=y_M=\frac{a}{5}$. **13.51.** $x_M=\frac{16}{5}$; $y_M=-1$.

13.52. $x_M=\frac{28}{75}$, $y_M=\frac{92}{105}$. **13.53.** $x_M=\pi a$; $y_M=\frac{5a}{6}$.

13.54. $\xi_M=\eta_M=\frac{128a}{105\pi}$. **13.55.** $\xi_M=\frac{6\left(4-\pi^2\right)a}{\pi^3}$; $\eta_M=\frac{2\left(\pi^2-6\right)a}{\pi^2}$.

13.56. $\xi_M = \frac{5a}{6}$; $\eta_M = 0$. **13.57.** $\xi_M = \frac{\pi a\sqrt{2}}{8}$; $\eta_M = 0$. **13.58.** $I_x = \frac{1}{12}\pi h^3$.

13.59. $I_x = \frac{a^4}{12}$. **13.60.** $M_x = \frac{bh^2}{6}$; $I_x = \frac{bh^3}{12}$. **13.61.** $I_a = \frac{\pi ab^3}{4}$, $I_b = \frac{\pi a^3 b}{4}$.

13.62. $x_M = 0$, $y_M = \frac{4R}{3\pi}$. **13.63.** $S = 4\pi R^2$. **13.64.** $V = \frac{1}{3}\pi R^2 H$, $S = \pi RL$.

14-§. Aniq integrallarni taqribiy hisoblash

Ushbu

$$\int_a^b f(x)dx \qquad (14.1)$$

integralni hisoblash talab qilingan bo'lsin, bunda $f(x)$ funksiya $[a,b]$ kesmada uzluksiz, deb faraz qilinadi. Biz, yuqorida, aniq integralni hisoblash usullarini ko'rib o'tdik, lekin ba'zi fizik, mexanik masalalarni yechishda, integral ostidagi funksiyaning boshlang'ich funksiyasini elementar funksiyalar orqali ifodalab bo'lmaydigan integrallar uchraydi. Bunday integrallarni taqribiy hisoblashga to'g'ri keladi. Integrallarni taqribiy hisoblash uchun bir nechta formulalar mavjud bo'lib, biz quyida ularning ba'zilari bilan tanishamiz.

14.1. To'g'ri to'rtburchaklar usuli. $[a,b]$ kesmaning, ixtiyoriy $P = \{a = x_0 < x_2 < ... < x_{2n} = b\}$ regulyar bo'linishini qaraymiz. $[x_{2k-2}, x_{2k}]$ kesmaning o'rtasidagi nuqtani x_{2k-1} orqali belgilaymiz(14.1-chizma). To'g'ri to'rtburchak usuli, (14.1) integralni, mos ravishda, balandliklari $f(x_{2k-1})$, asoslari esa, $x_{2k} - x_{2k-2} = \frac{b-a}{n}$ ga teng bo'lgan, to'g'ri to'rtburchaklar yuzlarining,

$$\frac{b-a}{n}[f(x_1) + f(x_3) + ... + f(x_{2n-1})]$$

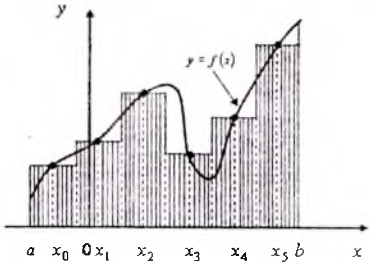
yig'indisiga taqribiy almashtirishdan iborat, ya'ni

$$\int_a^b f(x)dx \approx \frac{b-a}{n}[f(x_1) + f(x_3) + ... + f(x_{2n-1})]\ . \qquad (14.2)$$

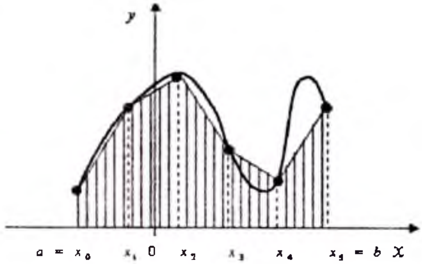
(14.2) formulaga, *to'g'ri to'rtburchak formulasi* deyiladi. Bu formulani, «qo'shimcha» had yordamida, ushbu

$$\int_a^b f(x)dx = \frac{b-a}{n}[f(x_1) + f(x_3) + ... + f(x_{2n-1})] + R \qquad (14.3)$$

ko'rinishda yozish mumkin, bunda $R -$ qoldiq had.



14.1-chizma.



14.2-chizma.

Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz ikkinchi tartibli hosilaga ega bo‘lsa, u holda shu kesmada shunday η nuqta topiladiki, (14.3) formula-dagi qoldiq had uchun,

$$R = \frac{(b-a)^3}{24n^2} f''(\eta)$$

tenlik o‘rinli bo‘ladi.

14.2. Trapesiyalar usuli. (14.1) integralni hisoblash talab qilingan bo‘lsin. $[a,b]$ kesmaning ixtiyoriy $P = \{a = x_0 < x_2 < ... < x_n = b\}$ regular bo‘linishini olamiz va $y = f(x)$ funksiyaning $y_0 = f(x_0),\ y_1 = f(x_1),\ ....,\ y_n = f(x_n)$ qiymatlarini hisoblaymiz. Trapesiyalar usuli, (14.1) integralni,

$$\frac{b-a}{2n}\{[f(x_0)+f(x_1)]+[f(x_1)+f(x_2)]+...+[f(x_{n-1})+f(x_n)]\} =$$

$$= \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right\}$$

yig‘indiga, yoki asoslari, mos ravishda, $f(x_{k-1})$ va $f(x_k)$ larga, balandliklari $x_k - x_{k-1} = \frac{b-a}{n}$ ga teng bo‘lgan, trapesiyalar yuzlarining yig‘indisiga (14.2-chizma) taqribiy almashtirishdan iborat, ya’ni

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n} f(x_k)\right\} \quad . \qquad (14.4)$$

(14.4) formulani, qoldiq had yordamida,

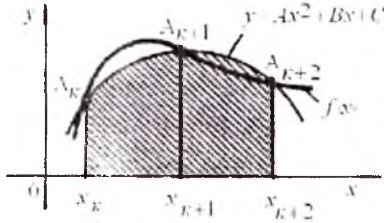
$$\int_a^b f(x)dx = \frac{b-a}{2n}\left\{f(a)+f(b)+2\sum_{k=1}^{n} f(x_k)\right\} + R_n$$

ko‘rinishda yozish mumkin, bunda R_n - qoldiq had. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz ikkinchi tartibli hosilaga ega bo‘lsa, u holda, $[a,b]$ kesmada shunday η nuqta topiladiki, R_{tr} uchun,

$$R_{tr} = -\frac{(b-a)^3}{12n^2} f''(\eta),\ a \le \eta \le b,$$

tenglik o‘rinli bo‘ladi.

14.3. Parabolalar usuli. (14.1) integralni hisoblash uchun, $[a,b]$ kesmaning ixtiyoriy $P=\{a=x_0<x_2<...<x_{2n}=b\}$ regular bo'linishini olamiz. $[x_{2k-2},x_{2k}]$ kesmaning o'rtasidagi nuqtani x_{2k-1} orqali belgilaymiz: $x_{2k-1}=\frac{x_{2k-2}+x_{2k}}{2}$, $k=\overline{1,n}$. Parabolalar usulida berilgan (14.1) integral, $f(x)$ funksiya grafigining, absissalari x_{2k-2}, x_{2k-1} va x_{2k} bo'lgan nuqtalardan o'tuvchi parabolalar ostida joylashgan trapesiyalar yuzlarining



14.3-chizma.

$$\frac{b-a}{6n}\{[f(x_0)+4f(x_1)+f(x_2)]+[f(x_2)+4f(x_3)+f(x_4)]+...+$$

$$+[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})]\}=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})+4\sum_{k=0}^{n-1}f(x_{2k+1})\}$$

yig'indisiga taqribiy almashtiriladi, ya'ni:

$$\int_a^b f(x)dx\approx\frac{b-a}{6n}\{[f(x_0)+4f(x_1)+f(x_2)]+[f(x_2)+4f(x_3)+f(x_4)]+$$

$$+...+[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})]=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})+4\sum_{k=0}^{n-1}f(x_{2k+1})\}$$

yoki

$$\int_a^b f(x)dx=\frac{b-a}{6n}\{f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_{2k})\}+4\sum_{k=0}^{n-1}f(x_{2k+1})+R_{pa},\qquad(14.5)$$

bunda, R_{pa} – qoldiq had.

Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz to'rtinchi tartibli hosilaga ega bo'lsa, u holda $[a,b]$ kesmada shunday nuqta topiladiki, (14.5) formuladagi had R_{pa}- uchun,

$$R_{pa}=-\frac{(b-a)^5}{2880n^4}f^{(4)}(\eta),\quad a\le\eta\le b,$$

tenglik o'rinli bo'ladi. (14.4) formulaga, *parabolalar (Simpson) formulasi* deyiladi (14.3-chizma).

14.1- misol. Ushbu

$$\int_1^9\sqrt{5x+4}dx$$

integralni, integrallash oralig'ini, 8 ta o'zaro teng bo'lakka bo'lib: 1) Nyuton - Leybnis; 2) to'g'ri to'rtburchaklar; 3) trapesiyalar va

4) parabolalar formulalari yordamida hisoblang, so'ngra taqribiy hisoblash formulasida qo'yilgan absolyut va nisbiy xatolarni foizlarda toping.

Yechilishi. 1) Berilgan integralni, Nyuton – Leybnis formulasiga asosan, hisoblaymiz:

$$J=\int_1^9\sqrt{5x+4}dx=\frac{1}{5}\int_1^9(5x+4)^{1/2}d(5x+4)=\frac{2}{15}(5x+4)^{\frac{3}{2}}\Big|_1^9=\frac{2}{15}(343-27)=\frac{632}{15}=42\frac{2}{15}.$$

2) To'g'ri to'rtburchaklar formulasi bo'yicha hisoblaymiz: $[1;9]$ kesmaning, $P=\{1=x_0<x_2<...<x_8=9\}$ regular bo'linishini qaraymiz. $y=f(x)=\sqrt{5x+4}$ funksiyaning, bu kesmalarning o'rtalaridagi qiymatlarini hisoblaymiz: $x_{2k-1}=\dfrac{x_{2k-2}+x_{2k}}{2}$ $(k=\overline{1,8})$;

$$x_1=\frac{3}{2},\qquad f(x_1)=f(\frac{3}{2})=\sqrt{\frac{23}{2}}=3.3911,$$

$$x_3={}^5\!/_2,\qquad f(x_3)=f({}^5\!/_2)=\sqrt{\frac{33}{2}}=4.0620,$$

$$x_5={}^7\!/_2,\qquad f(x_5)=f({}^7\!/_2)=\sqrt{\frac{43}{2}}=4,6368,$$

$$x_7={}^9\!/_2,\qquad f(x_7)=f({}^9\!/_2)=\sqrt{\frac{53}{2}}=5,1478,$$

$$x_9={}^{11}\!/_2,\qquad f(x_9)=f({}^{11}\!/_2)=\sqrt{\frac{63}{2}}=5,6124$$

$$x_{11}={}^{13}\!/_2,\qquad f(x_{11})=f({}^{13}\!/_2)=\sqrt{\frac{73}{2}}=6,0415,$$

$$x_{13}={}^{15}\!/_2,\qquad f(x_{13})=f({}^{15}\!/_2)=\sqrt{\frac{83}{2}}=6,4420,$$

$$x_{15}={}^{17}\!/_2,\qquad f(x_{15})=f({}^{17}\!/_2)=\sqrt{\frac{93}{2}}=6,8190.$$

Har bir kesmaning uzunligi $h=\dfrac{9-1}{8}=1.$ Funksiyaning topilgan qiymatlarini (14.2) to'g'ri turtburchaklar formulasiga keltirib qo'yib, hisoblaymiz:

$$J_1=\int_1^9\sqrt{5x+4}dx=\frac{632}{15}\approx\sqrt{\frac{23}{2}}+\sqrt{\frac{33}{2}}+\sqrt{\frac{43}{2}}+\sqrt{\frac{53}{2}}+\sqrt{\frac{63}{2}}+\sqrt{\frac{73}{2}}+\sqrt{\frac{83}{2}}+\sqrt{\frac{93}{2}}=42{,}1526.$$

Taqribiy hisoblashlardagi absolyut xato:

$$\Delta=|J-J_1|=\left|\frac{632}{15}-42{,}1526\right|=|42{,}1333-42{,}1526|=0{,}0193.$$

Nisbiy xato (foizlarda): $\delta=\dfrac{\Delta}{J}100\%=\dfrac{15\cdot0{,}0193\cdot100}{632}\approx0{,}0458\%.$

3) Trapesiyalar formulasi bo'yicha hisoblaymiz:

$a = x_0 = 1, \quad b = x_8 = 9.$

$$
\begin{array}{ll}
a = x_0 = 1 & f(a) = f(1) = 3, \\
x_1 = 2 & f(x_1) = f(2) = \sqrt{14} = 3{,}7416 \\
x_2 = 3 & f(x_2) = f(3) = \sqrt{19} = 4{,}3588 \\
x_3 = 4 & f(x_3) = f(4) = \sqrt{24} = 4{,}8989 \\
x_4 = 5 & f(x_4) = f(5) = \sqrt{29} = 5{,}3851 \quad 37{,}0934 \\
x_5 = 6 & f(x_5) = f(6) = \sqrt{34} = 5{,}8309 \\
x_6 = 7 & f(x_6) = f(7) = \sqrt{39} = 6{,}2449 \\
x_7 = 8 & f(x_7) = f(8) = \sqrt{44} = 6{,}6332 \\
b = x_8 = 9 & f(x_8) = f(9) = 7
\end{array}
$$

Funksiyaning topilgan qiymatlarini (14.4) formulaga keltirib qo'yib, hisoblaymiz:

$$J_2 = \frac{632}{15} \approx \frac{1}{2}[3 + 7 + 2(\sqrt{14} + \sqrt{19} + \sqrt{24} + \sqrt{29} + \sqrt{34} + \sqrt{39} + \sqrt{44})] = 5 + 37{.}0924 = 42{,}0934.$$

Absolyut xato: $\Delta = |J - J_2| = \left|\frac{632}{15} - 47{,}0934\right| = 42{,}1333 - 42{,}0934 = 0{,}0399.$

Nisbiy xato (foizlarda): $\delta = \frac{\Delta}{J} 100\% = \frac{15 * 0{,}399 * 100\%}{632} \approx 0{,}0946\%.$

4) Parabolalar formulasi bo'yicha hisoblaymiz:

$$
\begin{array}{ll}
x_2 = 2, & f(x_2) = f(2) = \sqrt{14}, \\
x_4 = 3, & f(x_4) = f(3) = \sqrt{19}, \\
x_6 = 4, & f(x_6) = f(4) = \sqrt{24}, \\
x_8 = 5, & f(x_8) = f(5) = \sqrt{29}, \\
x_{10} = 6, & f(x_{10}) = f(6) = \sqrt{34}, \\
x_{12} = 7, & f(x_{12}) = f(7) = \sqrt{39}, \\
x_{14} = 8, & f(x_{14}) = f(8) = \sqrt{44},
\end{array}
$$

Topilgan qiymatlarni (14.5) formulaga keltirib qo'yib, hisoblaymiz:

$$J_3 = \frac{632}{15} \approx \frac{1}{2}[\, 3 + 7 + 2(\sqrt{14} + \sqrt{19} + \sqrt{24} + \sqrt{29} + \sqrt{34} + \sqrt{39} + \sqrt{44}) +$$

$$+ 4\left(\sqrt{\frac{23}{2}} + \sqrt{\frac{33}{2}} + \sqrt{\frac{43}{2}} + \sqrt{\frac{53}{2}} + \sqrt{\frac{63}{2}} + \sqrt{\frac{73}{2}} + \sqrt{\frac{83}{2}} + \sqrt{\frac{93}{2}}\right)] = \frac{1}{6}[10 + 2 \cdot 37{,}0934 + 4 \cdot 42{,}1526] =$$

$$= \frac{10 + 74{,}1868 + 168{,}6104}{6} = 42{,}1328.$$

Absolyut xato: $\Delta = |J - J_3| = 42{,}1333 - 42{,}1328 = 0{,}0005.$

Nisbiy xato (foizlarda): $\delta = \frac{\Delta}{J} 100\% = \frac{15 \cdot 0{,}0005 \cdot 100}{632} \approx 0{,}0111\%.$

14.2-misol. Ushbu $\int_{-2}^{0}(x^2-1)dx$ integralning qiymatini: 1) trapesiyalar usuli $(n=4)$; 2) Simpson usuli $(n=4)$, bo'yicha yaqinlashishlar bajarib, hisoblang.

Yechilishi. 1). Trapesiyalar usulidan foydalanish:

a) $f(x)=x^2-1,[a,b]=[-2,0],\ n=4$ bo'lgani uchun, $[-2,0]$ kesmaning $P=\left\{-2,-\frac{3}{2},-1,\frac{1}{2},0\right\}$ regular bo'linishini qaraymiz, hamda bo'linish nuqtalarida funksiyaning qiymatlarini hisoblaymiz:

$$f(-2)=3;\ f\left(-\frac{3}{2}\right)=\frac{5}{4};\ f(-1)=0;\ f\left(-\frac{1}{2}\right)=-\frac{3}{4},\ f(0)=-1$$

Unda

$$\int_{-2}^{0}(x^2-1)dx\cong\frac{2}{2\cdot4}\left\{f(-2)+f(0)+2[f\left(-\frac{3}{2}\right)+f(-1)+f\left(-\frac{1}{2}\right)]\right\}=$$

$$=\frac{1}{4}\left\{3-1+2\cdot[\frac{5}{4}+0-\frac{3}{4}]\right\}=\frac{1}{4}\{2+1\}=\frac{3}{4}=0{,}75.$$

Endi $f''(\eta)=2,\ h=\frac{1}{2}$ ekanligidan foydalansak,

$$|R_{tr}|\le\frac{2}{12}\cdot\frac{1}{4}\cdot2=\frac{1}{12}\approx0{,}08$$

bo'ladi.

b) Integralni bevosita hisoblaymiz:

$$\int_{-2}^{0}(x^2-1)dx=\int_{-2}^{0}x^2dx-\int_{-2}^{0}dx=\frac{x^3}{3}\Big|_{-2}^{0}-x\Big|_{-2}^{0}=0-\left(\frac{-8}{3}\right)-(0-(-2))=\frac{8}{3}-2=\frac{2}{3}\cong0{,}67.$$

Demak, $|R_{tr}|=|0{,}67-0{,}75|=0{,}08$.

c) Taqribiy hisoblashdagi absolyut xato va nisbiy xatolarni (foizlarda) topamiz:

$$\Delta=\left|\frac{2}{3}-\frac{3}{4}\right|=\left|\frac{8-9}{12}\right|=\left|-\frac{1}{12}\right|=\frac{1}{12},\quad \delta=\frac{3\cdot\frac{1}{12}}{2}100\%=\frac{1}{8}100\%=0{,}125\%.$$

Demak, *I*. *a*) 0,75; 0,08, *b*) $\frac{2}{3}$; 0,08, *c*) 0,125%.

2).Simpson usulidan foydalanish: *a*) $\int_{-2}^{0}(x^2-1)dx$ integralning taqribiy qiymatini, Simpson usuli bo'yicha hisoblash uchun, bizga, funksiyaning, bo'linishda hosil bo'lgan nuqtalardagi qiymatlaridan tashqari, $[x_{i-1},x_i],\ i=1,2,3,4$, qism kesmalar,

$$[-2,-\frac{3}{2}],\ [-\frac{3}{2},-1],\ [1,-\frac{1}{2}],\ [-\frac{1}{2},0]$$

ning o‘rta nuqtalari bo‘lgan, $\xi_1=-\frac{7}{4};\ \xi_2=-\frac{5}{4};\ \xi_3=-\frac{3}{4};\ \ \xi_4=-\frac{1}{4}$ nuqtalarda funksiyaning $f(\xi_1)=\frac{33}{16};\ f(\xi_2)=\frac{9}{16};\ f(\xi_3)=-\frac{7}{16};\ f(\xi_4)=-\frac{15}{16}$ qiymatlari kerak bo‘ladi. Unda

$$\int_{-2}^{0}(x^2-1)dx\approx\frac{2}{6\cdot 4}\{f(-2)+f(0)+2[f\left(\frac{3}{2}\right)+f(-1)+f\left(-\frac{1}{2}\right)]+$$
$$+4[f\left(-\frac{7}{4}\right)+f\left(-\frac{5}{4}\right)+f\left(-\frac{3}{4}\right)+f\left(-\frac{1}{4}\right)]\}=$$
$$=\frac{1}{12}\{3-1+2[\frac{5}{4}+0-\frac{3}{4}]+4[\frac{33}{16}+\frac{9}{16}-\frac{7}{16}-\frac{15}{16}]\}=\frac{1}{12}\{2+2\cdot\frac{1}{2}+4\cdot\frac{33+9-7-15}{16}\}=\frac{2}{3}$$
$$|R_c|\le\frac{2}{180}\cdot\frac{1}{2^4}\cdot f^{(IV)}(\eta)=0\Rightarrow R_c=0.$$

b) Integralni bevosita hisoblaymiz: $\int_{-2}^{0}(x^2-1)dx=\frac{2}{3}$.

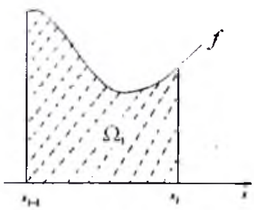
Demak, $|R_c|=\left|\frac{2}{3}-\frac{2}{3}\right|=0$.

c) Taqribiy hisoblashdagi absolyut xato va nisbiy xatolarni (foizlarda) topamiz: 0%. Demak, 2) *a*) $\frac{2}{3}$, 0, *b*) $\frac{2}{3}$, 0, *c*) 0%.

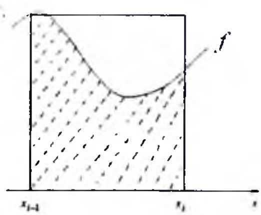
Yana biz ushbu $\int_a^b f(x)dx$ integralni qaraymiz, bunda f funksiyani $[a,b]$ da uzluksiz va chizmada qulay bo‘lishi uchun, uni musbat funksiya, deb faraz qilamiz. $[a,b]$ kesmada regular $P=\{x_0,x_1,x_2,...,x_{n-1},x_n\}$ bo‘linish olamiz, u berilgan kesmani bir xil $\frac{b-a}{n}$ uzunlikdagi n ta qism kesmalarga bo‘ladi:

$$[a,b]=[x_0,x_1]\cup...\cup[x_{i-1},x_i]\cup...\cup[x_{n-1},x_n],\ \Delta x_i=x_i-x_{i-1}=\frac{b-a}{n}.$$

14.3-chizmadagi Ω_i soha bir necha usullar bilan yaqinlashtirilishi mumkin:



14.3-chizma



14.4-chizma.

1. $[x_{i-1}, x_i]$ kesmaning chap chetki nuqtasida funksiyaning qiymati bo'yicha yasalgan to'g'ri turtburchak orqali, unda bu to'g'ri to'rtburchakning yuzi (14.4- chizma),

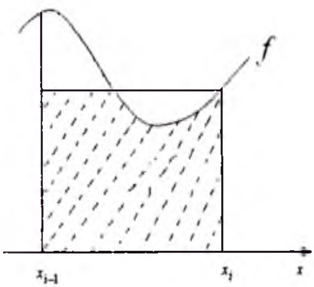
$$\text{yuza} = f(x_{i-1})\Delta x_i = f(x_{i-1})\frac{b-a}{n}$$

bo'ladi.

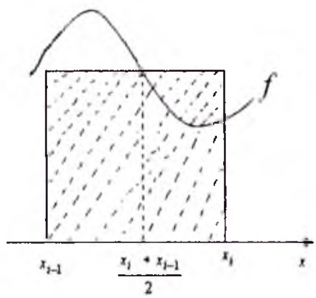
2. $[x_{i-1}, x_i]$ kesmaning o'ng chetki nuqtasida funksiyaning qiymati bo'yicha yasalgan to'g'ri to'rtburchak orqali (14.6-chizma), unda bu to'g'ri to'rtburchakning yuzi

$$\text{yuza} = f(x_i)\Delta x_i = f(x_i)\frac{b-a}{n}$$

bo'ladi.



14.6-chizma.



14.7-chizma

3. $[x_{i-1}, x_i]$ kesmaning o'rta nuqtasida funksiyaning qiymati bo'yicha yasalgan to'g'ri to'rtburchak orqali (14.7- chizma), bu to'g'ri to'rtburchakning yuzi

$$\text{yuza} = f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x_i = f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x_i$$

bo'ladi.

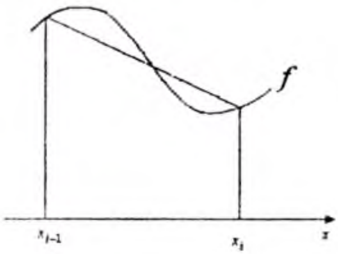
4. Asoslarini $[x_{i-1}, x_i]$ kesmaning chap va o'ng chetlarida funksiyaning qiymatlari tashkil etuvchi trapesiya orqali, uning yuzi (14.8-chizma)

$$\text{yuza} = \frac{1}{2}[f(x_{i-1}) + f(x_i)]\Delta x_i = \frac{1}{2}[f(x_{i-1}) + f(x_i)]\frac{b-a}{n}$$

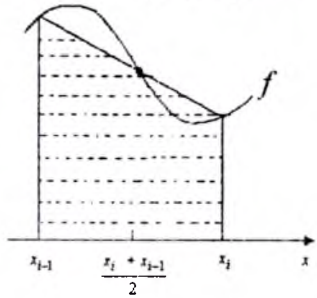
bo'ladi.

5. Parabolik soha orqali (14.9-chizma), bunda yuqorida tilga olingan uchta nuqtalarga mos kelgan nuqtalardan o'tuvchi $y = Ax^2 + Bx + C$ parabola qaraladi, uning yuzi

$$\text{yuza}=\frac{1}{6}[f(x_{i-1})+4f\left(\frac{x_{i-1}+x_i}{2}\right)+f(x_i)]\Delta x_i=[f(x_{i-1})+4f\left(\frac{x_{i-1}+x_i}{2}\right)+f(x_i)]\frac{b-a}{6n}.$$



14.8-chizma.



14.9-chizma

Ravshanki, agar bu uchta nuqtalar bir to‘g‘ri chiziqda yotsa, parabola yoyi to‘g‘ri chiziqqa aylanadi va parabolik soha trapesiyali soha (4-hol) ga keltiriladi.

Shunday qilib, Ω_i sohani biz yuqorida ko‘rib o‘tgan yaqinlashtirishlar $\int_a^b f(x)dx$ integral qiymatining quyidagi yaqinlashishlariga olib keladi.

1. Chap chetki nuqta orqali yaqinlashish:

$$Z_n=\frac{b-a}{n}[f(x_0)+f(x_1)+\cdots+f(x_{n-1})]. \qquad (14.6)$$

2. O’ng chetki nuqta orqali yaqinlashish:

$$R_n=\frac{b-a}{n}[f(x_1)+f(x_2)+\cdots+f(x_n)]. \qquad (14.7)$$

3. O’rta nuqta bo‘yicha yaqinlashish:

$$M_n=\frac{b-a}{n}[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-}+x_n}{2}\right)]. \qquad (14.8)$$

4. Trapesiya bo‘yicha yaqinlashish (trapesiyalar qoidasi):

$$T_n=\frac{b-a}{n}[\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\cdots+\frac{f(x_{n-1})+f(x_n)}{2}]=$$
$$=\frac{b-a}{2n}[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]. \qquad (14.9)$$

5. Parabola bo‘yicha yaqinlashish (Simpson qoidasi):

$$S_n=\frac{b-a}{6n}\{[f(x_0)+f(x_n)+2[f(x_1)+f(x_2)+\cdots+f(x_{n-1})]+$$
$$+4[f\left(\frac{x_0+x_1}{2}\right)+\ldots+f\left(\frac{x_{n-1}+x_n}{2}\right)]\}. \qquad (14.10)$$

Bu yaqinlashishlarning dastlabki uchtasi, ya'ni Z_n, R_n va M_n lar Riman yig'indilari bo'lib, T_n va S_n lar esa, Riman yig'indilari sifatida yozilishi mumkin.

14.3-misol. Ushbu $\int_0^3 \sqrt{4+x^3}dx$ integralning taqribiy qiymatlarini, $n=6$ deb olib: *a*) chap chetki nuqta orqali yaqinlashish; *b*) o'ng chetki nuqta bo'yicha; *c*) o'rta nuqta bo'yicha; *d*) trapesiya bo'yicha (trapesiya qoidasi); *e*) parabola (Simpson qoidasi) bo'yicha yaqinlashishlarini toping.

Yechilishi. $n=6$ bo'lganda har bir $[x_{i-1},x_i]$ qism kesmaning uzunligi, $\frac{b-a}{n}=\frac{3-0}{6}=\frac{1}{2}$ ga teng bo'ladi. Bo'linish nuqtalari

$$x_0=0;\ x_1=\frac{1}{2},\ x_2=1,\ x_3=\frac{3}{2},x_4=2,x_5=\frac{5}{2},\ x_6=3$$

$f(x)=\sqrt{4+x^3}$ funksiyaning bu bo'linish nuqtalaridagi qiymatlarini hisoblaymiz: $f(0)=2;$ $f\left(\frac{1}{2}\right)\cong 2{,}031;$ $f(1)\cong 2{,}236;$ $f\left(\frac{3}{2}\right)\cong 2{,}716;$ $f(2)\cong 3{,}464;$ $f\left(\frac{5}{2}\right)\cong 4{,}430,$ $f(3)\cong 5{,}568$ (bu erda hisoblashlar uchta o'nliklar xonasigacha yaxlitlangan). Unda

$$a)\ Z_6=\frac{1}{2}[f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)]\cong$$

$$\cong\frac{1}{2}[2{,}000+2{,}031+2{,}236+2{,}716+3{,}464+4{,}430]=\frac{1}{2}\cdot 16{,}877=8{,}4385.$$

$$b)\ R_6=\frac{1}{2}[f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)+f(3)]\cong$$

$$\cong\frac{1}{2}[2{,}031+2{,}236+2{,}716+3{,}464+4{,}430+5{,}568]=\frac{1}{2}\cdot 20{,}445=10{,}2225.$$

c) $[x_{i-1},x_i]$ qism kesmalarning $(i=1,2,3,4,5,6)$ o'rta nuqtalarini topamiz:

$$\frac{x_0+x_1}{2}=\frac{0+0{,}5}{2}=\frac{1}{4},\ \frac{x_1+x_2}{2}=\frac{0{,}5+1}{2}=\frac{3}{4},\ \frac{x_2+x_3}{2}=\frac{1+1{,}5}{2}=\frac{5}{4},$$

$$\frac{x_3+x_4}{2}=\frac{1{,}5+2}{2}=\frac{7}{4},\ \frac{x_4+x_5}{2}=\frac{2+2{,}5}{2}=\frac{9}{4},\ \frac{x_5+x_6}{2}=\frac{2{,}5+3}{2}=\frac{11}{4}.$$

Topilgan nuqtalarda $f(x)=\sqrt{4+x^3}$ funksiyaning qiymatlarini hisoblaymiz (bu erda ham hisoblashlarni uchta o'nliklar xonasigacha yaxlitlaymiz):

$$f\left(\frac{1}{4}\right)=2{,}004;\ f\left(\frac{3}{4}\right)=2{,}103;\ f\left(\frac{5}{4}\right)=2{,}439;\ f\left(\frac{7}{4}\right)=3{,}059;\ f\left(\frac{9}{4}\right)=3{,}923;\ f\left(\frac{11}{4}\right)=4{,}980$$

U holda, (14.8) formuladan

$$M_6 \cong \frac{1}{2}[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)]=$$

$$=\frac{1}{2}[2{,}004+2{,}103+2{,}439+3{,}059+3{,}923+4{,}980]=\frac{1}{2}\cdot 18{,}508=9{,}254$$

(14.9) formuladan, $n=6$ bo'lganda quyidagini hosil qilamiz.

$$T_6=\frac{1}{2\cdot 2}[f(0)+2\cdot f\left(\frac{1}{2}\right)+2\cdot f(1)+2\cdot f\left(\frac{3}{2}\right)+2\cdot f(2)+2\cdot f\left(\frac{5}{2}\right)+$$

$$+f(3)]\cong\frac{1}{4}[2{,}000+2\cdot 2{,}031+2\cdot 2{,}236+2\cdot 2{,}716+2\cdot 3{,}464+2\cdot 4{,}430+5{,}568]=$$

$$=\frac{1}{4}[2{,}000+4{,}062+4{,}472+5{,}472+5{,}432+6{,}928+8{,}860+{,}568]==\frac{1}{4}\cdot 37{,}322=9{,}3305$$

(14.10) formulada $n=6$ bo'lganda

$$S_6=\frac{1}{6\cdot 2}\{f(0)+f(3)+2[f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)]+$$

$$+4[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)]\}\cong$$

$$\cong\frac{1}{12}\{2{,}000+5{,}568+2[2{,}031+2{,}236+2{,}716+3{,}464+$$

$$+4{,}430]+4[2{,}004+2{,}103+2{,}439+3{,}059+3{,}923+4{,}980]\}=$$

$$=\frac{1}{12}(7{,}568+2\cdot 14{,}877+4\cdot 18{,}508)=$$

$$\frac{1}{12}(7{,}568+29{,}754+74{,}032)=\frac{1}{12}\cdot 111{,}354=9{,}2795$$

14.5- misol. $\int_4^1\sqrt{x}\,dx$ integralning qiymatini: 1) trapesiyalar qoidasi bo'yicha; 2) Simpson qoidasi bo'yicha yaqinlashtirishlar bajarib hisoblaganda, n ning, absolyut xato berilgan $\varepsilon=0{,}001$ dan kichik bo'lishini kafolatlaydigan qiymatini toping.

Yechilishi. I. Trapesiyalar qoidasi bo'yicha yaqinlashish bajarilganda η ning qoldiq $|R_{tp}|<\varepsilon$ bo'lishini ta'minlaydigan qiymatini topamiz. Buning uchun, formulada $h=\frac{b-a}{n}=\frac{4-1}{n}=\frac{3}{n}$, $\eta=2$ bo'lsin. Unda $f''(\eta)=-0{,}0884$.

Demak,

$$|R_{Tp}|\le\frac{b-a}{12}\cdot\left(\frac{b-a}{n}\right)^2\cdot|f''(\eta)|=\frac{1}{4}\cdot\frac{3^2}{n^2}0{,}0884<\varepsilon$$

yoki

$$\frac{9}{4}\cdot\frac{1}{n^2}\cdot 0{,}0884<0{,}001$$

bo'lishi talab qilinadi. Bu yerdan,

$$\frac{0,1989}{n^2} < 0,001$$

tengsizlik yoki

$$n^2 > \frac{0,199}{0,001} = 199$$

bo‘lishi, ya’ni $n > 14$; $n \geq 15$ bo‘lishini olamiz.

Demak, trapesiyalar qoidasi bo‘yicha yaqinlashish bajarilganda, absolyut xato $\varepsilon = 0,001$ dan kichik bo‘lishini kafolatlash uchun, $[1, 4]$ kesmani, kamida 15 ta qism bo‘laklarga bo‘lish lozim bo‘ladi.

2) Simpson qoidasi bo‘yicha yaqinlashish bajarilganda, n ning, qoldiq

$$|R_c| < \varepsilon = 0,001$$

bo‘lishini ta’minlaydigan qiymatini topamiz. $h = \frac{3}{4}$; $\eta = 2$ bo‘lsin. U holda,

$$f^{(IV)}(\eta) = f^{(IV)}(2) = -0,0828.$$

Demak,

$$|R_c| \leq \frac{3}{12} \cdot \left(\frac{3}{n}\right)^4 \cdot 0,0828 = \frac{1}{4}\left(\frac{3}{n}\right)^4 \cdot 0,0838 < \varepsilon$$

yoki

$$\frac{81}{4} \cdot \frac{1}{n^4} 0,0838 < 0,001$$

bo‘lishi talab qilinadi. Bu yerdan,

$$\frac{1,68}{n^4} < 0,001, \quad n^4 > \frac{1,681}{0,001} = 1681 \Rightarrow n > 6$$

bo‘lishini olamiz. Demak, $n \geq 7$.

Mustaqil yechish uchun misollar

Quyidagi integralni berilgan qadamda to‘g‘ri to‘rtburchak formulasi yordamida taqribiy hisoblang:

14.1. $\int_0^2 x^4 dx, \ h = 0,2$. **14.2.** $\int_1^5 \frac{dx}{1+x^4}, \ h = 0,5$.

14.3. $\int_0^2 e^{x^2} dx, \ h = 0,2$. **14.4.** $\int_0^1 \frac{\ln(1+x)}{x} dx, \ h = 0,2$.

Quyidagi integralni berilgan qadamda trapesiyalar formulasi yordamida hisoblang:

14.5. $\int_2^7 \frac{dx}{\sqrt{x+2}}, \ h = 1$. **14.6.** $\int_0^1 x\sqrt{1-x^2}\, dx, \ h = 0,1$.

14.7. $\int_0^2 \sqrt{1+x^4}\, dx, \ h = \frac{1}{8}$. **14.8.** $\int_0^1 \frac{dx}{1+x^2}, \ h = 0,1$.

Quyidagi integralni berilgan qadamda Simpson formulasi yordamida taqribiy hisoblang:

14.9. $\int_1^3 \sqrt{\frac{x+1}{x}}dx,\ h=0{,}5.$ **14.10.** $\int_0^2 e^{x^2}dx,\ h=0{,}2.$

14.11. $\int_0^2 e^{-x^2}dx,\ h=0{,}5.$ **14.12.** $\int_0^{\pi} \sqrt{3+\cos x}dx,\ h=\pi/6.$

Quyidagi integral uchun berilgan qadamda to‘g‘ri to‘rtburchak formulasi yordamida absolyut xatoni hisoblang:

14.13 $\int_0^2 x^4 dx,\ h=0{,}2.$ **14.14.** $\int_0^{\pi} \sin dx,\ h=\pi/6.$

14.15. $\int_1^5 \frac{dx}{x},\ h=0{,}4.$ **14.16.** $\int_1^3 \frac{\sqrt{x+1}}{x}dx.\ h=0{,}25.$

Quyidagi integral uchun berilgan qadamda trapesiya formulasi yordamida absolyut xatoni hisoblang:

14.17. $\int_2^9 \frac{dx}{x-1},\ h=1.$ **14.18.** $\int_0^1 \frac{dx}{1+x^3},\ h=\frac{1}{3}.$

14.19. $\int_2^7 \frac{dx}{\sqrt{x+2}},\ h=1.$ **14.20.** $\int_2^5 \ln^2 x dx,\ h=0{,}5.$

Quyidagi integral uchun berilgan qadamda Simpson formulasi yordamida absolyut xatoni hisoblang:

14.21. $\int_0^{\pi/2} \cos\frac{x}{2}dx,\ h=\frac{\pi}{8}.$ **14.22.** $\int_1^3 \ln 2x dx,\ h=\frac{1}{3}.$

14.23. $\int_0^6 x\ln(4+x)dx,\ h=\frac{1}{4}.$ **14.24.** $\int_2^5 \frac{dx}{\ln x},\ h=0{,}5.$

Quyidagi integralni to‘g‘ri to‘rtburchak formulasi yordamida hisoblaganda, yo‘l qo‘yilgan xatoning berilgan ε dan oshmasligi uchun h qadam qanday bo‘lishi kerak?

14.25. $\int_0^2 x^4 dx,\ \varepsilon=10^{-2}.$ **14.26.** $\int_0^{\pi/3} \cos 2x dx,\ \varepsilon=10^{-3}.$

14.27. $\int_0^1 \ln(1+x^2)dx,\ \varepsilon=10^{-4}.$ **14.28.** $\int_0^1 \sqrt{1+x^3}dx,\ \varepsilon=10^{-3}.$

Quyidagi integralni trapesiya formulasi yordamida hisoblaganda yo‘l qo‘yilgan xatoning berilgan ε dan oshmasligi uchun h qadam qanday bo‘lishi kerak?

14.29. $\int_0^3 \frac{dx}{x+2},\ \varepsilon=10^{-1}.$ **14.30.** $\int_0^{1,2} e^x dx,\ \varepsilon=10^{-3}.$

14.31. $\int_1^3 \sin x dx,\ \varepsilon=10^{-3}.$ **14.32.** $\int_0^{0,5} \arcsin x dx,\ \varepsilon=10^{-4}.$

Quyidagi integralni, Simpson formulasi yordamida hisoblaganda, yo‘l qo‘yilgan xatoning berilgan ε dan oshmasligi uchun h qadam qanday bo‘lishi kerak?

14.33. $\int\limits_0^{\frac{\pi}{2}} \cos\frac{x}{2}dx,\ \varepsilon = 10^{-3}$. **14.34.** $\int\limits_1^3 \ln 2x dx,\ \varepsilon = 10^{-4}$.

14.35. $\int\limits_1^2 \frac{dx}{x^2},\ \ \varepsilon = 10^{-4}$. **14.36.** $\int\limits_0^{\frac{\pi}{2}} arctg\, x dx,\ \varepsilon = 10^{-4}$.

14.37-14.41 misollardagi talablar ikki qismdan iborat bo‘lib, ulardan biri trapesiya qoidasi, ikkinchisi esa, Simpson qoidasini qo‘llab, integralni taqribiy hisoblash so‘raladi.

I. Trapesiya qoidasi (usuli)dan foydalanish: $a)$ integralni $n = 4$ qadam uchun yaqinlashtiring va

$$|R_{tp}| \le \frac{b-a}{12}h^2 M,\ h = \frac{b-a}{n},\left(|R_{tp}| \le \frac{b-a}{12}|f''(\eta)| \cdot h^2\right),\ \eta \in [a,b]$$

munosabatdan foydalanib, $|R_{tp}|$ uchun yuqori chegarani toping; $b)$ integralni bevosita hisoblang va $|R_{tp}|$ ni toping; $c)$ taqribiy hisoblashdagi nisbiy xatoni (% larda) toping.

II. Simpson (parabola) qoidasi (usuli) dan foydalanish. $a)$ integralni $n = 4$ qadam uchun yaqinlashtiring va

$$|R_{tp}| \le \frac{b-a}{12}h^4 M,\ h = \frac{b-a}{n},\ \left(|R_{tp}| \le \frac{b-a}{12}|h^4 \cdot f^{(4)}(\eta)|\right),\ \eta \in [a,b]$$

munosabatdan foydalanib, $|R_c|$ uchun yuqori chegarani toping; $b)$ integralni bevosita hisoblang va $|R_c|$ ni toping; $c)$ taqribiy hisoblashdagi nisbiy xatoni (%'larda) toping.

14.37. $\int\limits_1^2 x dx$. **14.38.** $\int\limits_{-1}^1 (x^2+1)dx$.

14.39. $\int\limits_0^2 (x^3+x)dx$. **14.40.** $\int\limits_1^2 \frac{1}{x^2}dx$.

14.41. $\int\limits_0^{\pi} \sin x\, dx$.

Quyidagi misollarda javoblarni to‘rtta o‘nliklar xonasigacha yaxlitlab yozing:

14.42. Ushbu $\int\limits_0^{12} x^2 dx$ integralning qiymati: $a)$ chap chetki nuqta bo‘yicha ($n = 12$ deb olib); $b)$ o‘ng chetki nuqta bo‘yicha ($n = 12$ deb olib); $c)$ o‘rta nuqta bo‘yicha ($n = 6$ deb olib); $d)$ trapesiyalar qoidasi bo‘yicha ($n = 12$ deb olib); $e)$ Simpson qoidasi bo‘yicha ($n = 6$ deb olib) yaqinlashishlar bajarib, hisoblang. Natijalarni, integrallashni bajarib, tekshiring (solishtiring).

14.43. Ushbu $\int_0^1 \sin^2 \pi x dx$ integralning qiymatini: $a)$ chap chekti nuqta bo'yicha ($n=3$ deb olib); $b)$ o'ng chetki nuqta bo'yicha ($n=3$ deb olib); $c)$ trapesiyalar qoidasi bo'yicha ($n=6$ deb olib); $d)$ Simpson qoidasi bo'yicha ($n=3$ deb olib) yaqinlashishlar bajarib, hisoblang. Natijalarni integrallashni bajarib, tekshiring (solishtiring).

14.44. Ushbu $\int_0^1 \frac{dx}{1+x^2}$ integralning qiymatini: $a)$ trapesiyalar qoidasi ($n=4$ deb olib); $b)$ Simpson qoidasi ($n=4$ deb olib) bo'yicha yaqinlashishlar bajarib, hisoblang:

14.45. Ushbu $\int_{-1}^{1} \cos x^2 dx$ integralning qiymatini: $a)$ o'rta nuqta bo'yicha ($n=4$ deb olib); $b)$ trapesiyalar qoidasi bo'yicha ($n=8$ deb olib); $c)$ Simpson qoidasi bo'yicha ($n=4$ deb olib) yaqinlashishlar bajarib, hisoblang.

14.46. Ushbu $\int_0^2 e^{-x^2} dx$ integralning qiymatini: $a)$ trapesiyalar qoidasi ($n=10$ deb olib); $b)$ Simpson qoidasi ($n=5$ deb olib) bo'yicha yaqinlashishlar bajarib, hisoblang.

Quyidagi misollarda, integralning qiymatini: $a)$ trapesiyalar qoidasi bo'yicha; $b)$ Simpson qoidasi bo'yicha yaqinlashtirishlar bajarib hisoblaganda n ning absolyut xato berilgan ε dan kichik bo'lishini kafolatlaydigan qiymatini toping:

14.47. $\int_1^4 \sqrt{x} dx, \quad \varepsilon = 0{,}01.$ **14.48.** $\int_1^4 \sqrt{x} dx \quad \varepsilon = 0{,}00001.$

14.49. $\int_0^\pi \sin x dx; \ \varepsilon = 0{,}001.$ **14.50.** $\int_1^3 e^x dx, \ \varepsilon = 0{,}01.$.

14.51. 5) $\int_0^2 e^{-x^2} dx, \quad \varepsilon = 0{,}0001.$

Mustaqil yechish uchun misollarning javoblari

14.1. 6,37. **14.2.** 0,23. **14.3.** 16,1. **14.4.** 0,822. **14.5.** 2,002. **14.6.** 0,3296. **14.7.** 3,482. **14.8.** 0,8350. **14.9.** 2,4859. **14.10.** 16,5. **14.11.** 0,8821. **14.12.** 5,4024. **14.13.** $0 < R < 0{,}16$.

14.14. $-3{,}59\cdot 10^{-2} < R < 0$. **14.15.** $0 < R < 0{,}053$.
14.16. $9{,}12\cdot 10^{-5} < R < 8{,}06\cdot 10^{-1}$. **14.17.** $-1{,}67 < R < -2{,}28\cdot 10^{-3}$.
14.18. $-6{,}95 < R < 1{,}61\cdot 10^{-5}$. **14.19.** $-9{,}77\cdot 10^{-3} < R < -1{,}29\cdot 10^{-3}$.
14.20. $-9{,}59\cdot 10^{-3} < R < 3{,}12\cdot 10^{-3}$. **14.21.** $-1{,}3\cdot 10^{-5} < R < -0{,}91\cdot 10^{-5}$.
14.22. $1{,}01\cdot 10^{-5} < R < 8{,}23\cdot 10^{-4}$ **14.23.** $-1{,}042 < R < -1{,}08\cdot 10^{-6}$.
14.24. $1{,}02\cdot 10^{-2} < R < -3{,}26\cdot 10^{-5}$. **14.25.** $1/20$.
14.26. $\frac{\pi}{42}$. **14.27.** $1/50$. **14.28.** 4) $1/8$. **14.29.** 1. **14.30.** $\frac{3}{55}$.
14.31. $1/13$. **14.32.** $1/18$. **14.33.** $\frac{\pi}{4}$ **14.34.** $1/6$.
14.35. 0,1 **14.36.** $1/5$.
14.37. *I*. *a*) 1,5; 0. *b*) 1,5; 0 . *c*)0% *II*. *a*) 1,5; 0.*b*)1,5;0. *c*)0%.
4.38. *I* *a*) 275; 0,08. *b*) 2,67; 0,08. *c*)$0{,}0312 \approx 3\%$. *II*. *a*) 2,67; 0. *b*) 2,67, 0. *c*)0%
14.39. *I*. *a*) 6,25; 0,5. *b*) 6; 0.25. *c*) $0{,}0417 \approx 4\%$. *II*. *a*)6; 0. *b*) 6; 0. *c*)0%.
14.40. *I*. *a*) 0,509, 0,03125.*b*)0,5, 0,009 *c*) $0{,}018 \approx 2\%$.
II. *a*) 0,5004; 0,002604. *b*)0,5; 0,0004. *c*) 0.08%.
14.41 *I*. *a*)1,8961, 0,161. *b*)2; 0,1039. *c*)$0{,}052 \approx 5\%$. *II*. *a*) 2,00456 0,0066 *b*)2, 0,00457. *c*)0,23%.
14.42. $Z_{12} = 506$; $R_{12} = 650$; $M_6 = 572$, $T_{12} = 578$; $S_6 = 576$.
14.43. $Z_6 \cong 1{,}394$; $R_6 \cong 0{,}9122$; $M_3 \cong 1{,}1852$; $T_6 \cong 1{,}1533$; $S_3 \cong 1{,}1614$.
14.44. $T_4 \cong 0{,}7828$; $S_4 \cong 0{,}7854$.
14.45. $M_4 \cong 1{,}8440$; $T_8 \cong 1{,}7915$; $S_4 \cong 1{,}8090$.
14.46. $T_{10} \cong 0{,}8818$; $S_5 \cong 0{,}8821$. **14.47.** *a*) $n \geq 8$, *b*) $n \geq 2$.
14.48. *a*) $n \geq 238$; *b*) $n \geq 10$ **14.49.** *a*) $n \geq 51$; *b*) $n \geq 4$.
14.50. *a*) $n \geq 37$; *b*) $n \geq 3$. **14.51.** *a*) 78; *b*) 7.

15-§. Aniq integralning fizika masalalariga tadbiqlari

15.1.Jismning bosib o‘tgan yo‘li. Jism to‘g‘ri chiziqli harakat qilganda, uning t vaqt davomida o‘zgarmas v tezlikda bosib o‘tgan yo‘li s, ushbu

$$s = vt \qquad (15.1)$$

formula bo‘yicha aniqlanadi.

Agar jism tekis harakat qilmasa, uning v tezligi t vaqtga bog‘liq ravishda o‘zgaradi, ya’ni

$$v = f(t).$$

Bu holda jismning $t = t_1$ dan $t = t_2$ gacha bo‘lgan vaqt davomida bosib o‘tgan yo‘lini topish uchun, $t_2 - t_1$ vaqt oralig‘ini n ta teng va juda kichik Δt bo‘laklarga bo‘lamiz. Faraz qilaylik, Δt vaqt oraliqlarining har birida jismning tezligi har bir qism vaqt oralig'ining oxirida, sakrashga ega bo'lgan holda, o'zgarmas bo'lib, qolsin, $t_2 - t_1$ vaqt oralig'i $\Delta t = 1$ sek oraliqlarga bo'lingan bo'lsin.

Farazimizga ko'ra, vaqt oralig'ining birinchi sekundida jism tekis harakat qilib, uning oxirida o'z tezligini o'zgartiradi, ikkinchi sekundda esa, olingan tezlik bo'yicha tekis harakat qiladi va ikkinchi sekundning oxirida yangi tezlikka ega bo'ladi va uchinchi sekund davomida tekis harakat qiladi va h.k.

Shuning uchun, jismning Δt vaqtda bosib o'tgan yo'li (15.1) formula orqali topiladi va taqriban $f(t)\Delta t$ ga teng bo'ladi, qaralayotgan $t_2 - t_1$ vaqt oralig'ida bosib o'tilgan yo'l esa,

$$S \approx \sum_{t=t_1}^{t_2} f(t)\Delta t$$

ifodaga teng bo'ladi.

Endi n bo'linishlar sonini ko'paytiramiz, u holda Δt hamda har bir Δt oraliqning oxirida tezlikning o'zgarishidagi sakrashlar, borgan sari kichrayib boradi.

Agar $n \to \infty$ bo'lsa, $\Delta t \to 0$, demak, $f(t)\Delta t \to 0$. Bu shartda jismning tezligi, sakrashlarsiz, ya'ni uzluksiz o'zgaradi va uning bosib o'tgan yo'li,

$$s = \lim_{\Delta t \to 0} \sum_{t=t_1}^{t_2} f(t)\Delta t$$

ifodaga teng bo'ladi, bundan, aniq integralning ta'rifiga ko'ra,

$$s = \lim_{\Delta t \to 0} \sum_{t=t_1}^{t_2} f(t)\Delta t = \int_{t_1}^{t_2} f(t)dt, s = \int_{t_1}^{t_2} f(t)dt. \qquad (15.2)$$

15.1-misol. Jismning harakat tezligi

$$v = (2t^2 + t) \ см/сек$$

tenglama orqali berilgan. Jismning harakat boshlangandan so'ng 6 sekund vaqt davomida o'tgan yo'lini aniqlang.

Yechilishi. (15.2) formulaga asosan,

$$s = \int_0^6 (2t^2 + t)dt = \left(\frac{2}{3}t^3 + \frac{1}{2}t^2\right)\Big|_0^6 = \frac{2}{3}6^3 + \frac{1}{2}\cdot 6^2 = 162 см.$$

15.2. Kuch bajargan ish. Faraz qilaylik, jism o'zgarmas F kuch ta'siri ostida to'g'ri chiziq bo'ylab harakat qilsin. U holda bosib o'tilgan x yo'lda bu F kuch tomonidan bajarilgan A ish,

$$A = Fx \qquad (15.3)$$

formula bo'yicha topiladi, bunda x - metrlarda, F - kilogrammlarda, A esa, kilogrammometrlarda ifodalanadi.

Agar jism o‘zgaruvchan kuch ta’siri ostida harakat qilsa, uning bajargan ishi ancha qiyin aniqlanadi. Bu hol uchun formulani keltirib chiqaramiz.

Faraz qilaylik, O nuqtada tinch holatda turgan jism, o‘tilgan yo‘l x ga bog‘liq ravishda o‘zgaradigan F kuch ta’siri ostida harakat qilayotgan bo‘lsin, ya’ni

$$F = f(x).$$

Shuningdek, vaqtning ba’zi momentlarida jism A va B nuqtalarda bo‘lsin (15.1- chizma), bunda $OA = a$ va $OB = b$ bo‘lsin.

15.1-chizma.

Endi yo‘lning $AB = b - a$ bo‘lagida berilgan kuch bajargan ishni qanday aniqlash mumkinligini ko‘rsatamiz. Buning uchun, yo‘lning ko‘rsatilgan AB bo‘lagini n ta, o‘zaro teng va juda kichik Δx kesmalarga bo‘lamiz. Bu erda ham yuqoridagi yo‘lni topish masalasidagi kabi, har Δx kesmada kuch, o‘zgarmas qolib, uning oxirida, sakrashga ega bo‘lib, o‘zgaradi, deb faraz qilinadi. U holda, (15.1) formulaga asosan, yo‘lning Δx bo‘lagida kuchning bajargan ishi, taqriban

$$f(x)\Delta x$$

ifodaga teng bo‘ladi, kuchning butun $AB = b - a$ yo‘l davomida bajargan ishi esa, taqriban,

$$A \approx \sum_{x=a}^{b} f(x)\Delta x$$

bo‘ladi.

Endi bo‘linishlar soni n ni cheksiz ortirrsak, Δx miqdor, va demak, $f(x)\Delta x$ miqdor ham cheksiz kichik miqdorlar bo‘ladi. Bunda kuch, sakrashlarga ega bo‘lmasdan, uzluksiz o‘zgaradi va izlanayotgan ish,

$$A = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x$$

ifodaga, bundan aniq integralning ta’rifiga ko‘ra,

$$A = \int_a^b f(x)dx \qquad (15.4)$$

bo‘ladi.

15.2-misol. 1 kГ miqdordagi kuch prujinani 3 sm ga cho‘zganda, u qanday ish bajarishini aniqlang.

Yechilishi. Guk qonuniga asosan, kuch prujinaning cho'zilishi yoki qisilishiga to'g'ri proporsional, ya'ni

$$F = kx,$$

bunda x - prujinaning cho'zilish yoki qisilish miqdori, k - proporsionallik koeffitsiyenti. Qaralayotgan masalada k ning qiymatini topish uchun, berilganlarni, Guk qonunini ifodalovchi, $F = kx$ tenglamaga keltirib qo'yamiz: $1 = k \cdot 0{,}03$, bu erdan $k = \frac{1}{0{,}03}$ ekanligi kelib chiqadi.

Demak, prujinani cho'zuvchi kuch

$$F = \frac{1}{0{,}03}x$$

ko'rinishda ifodalanadi.

Kuch tinch holatda bo'lgan prujinaga ta'sir qilgani uchun, (15.4) formuladagi integralning quyi chegarasi $a = 0$ bo'ladi, yuqori chegarasi esa, $b = 0{,}03$ bo'ladi.

Demak, izlanayotgan ish:

$$A = \int_0^{0{,}03} \frac{1}{0{,}03} x dx = \frac{1}{0{,}03} \frac{x^2}{2}\bigg|_0^{0{,}03} = \frac{1}{0{,}03} \cdot \frac{(0{,}03)^2}{2} = 0{,}015 \ \text{кГм}.$$

15.3-misol. Massasi m bo'lgan jismning Yer sirtidan vertikal h masofaga ko'tarishda bajariladigan ishni toping.

Yechilishi. Yerning tortish kuchini F, Yerning massasini m_e orqali belgilasak, u holda, Nyutonning qonuniga asosan, $F = G\frac{m_e m}{x^2}$ bo'ladi, bunda, x - jismdan Yerning markazigacha bo'lgan masofa. U holda $m_e m \cdot G = K$ deb belgilasak, $F = \frac{K}{x^2}$, $R \le x \le h + R$, R - Yerning radiusi. $x = R$ bo'lganda, $F(R)$ - kuch, jismning og'irligi $P = mg$ ga teng bo'ladi, ya'ni

$$\frac{K}{R^2} = P, \ K = P \cdot R^2, \ \ F(x) = \frac{PR^2}{x^2}.$$

Shunday qilib, (13.30) formulasiga asosan, topamiz:

$$A = \int_R^{R+h} \frac{PR^2}{x^2} dx = PR^2 \int_R^{R+h} \frac{dx}{x^2} = -PR^2 \cdot \frac{1}{x}\bigg|_R^{R+h} = \frac{PRh}{R+h}.$$

15.4-misol. Og'irligi $P = 1{,}5m$ bo'lgan raketani Yer sirtidan $H = 2000$ km balandlikka ko'tarish uchun bajarilishi zarur bo'lgan ishni toping.

Yechilishi. Yerning jismni tortish kuchi F yoki jismning og'irligi, uning Yerning markazidan qanday x uzoqlikda joylashganligiga bog'liq bo'ladi:

$F(x) = \frac{\lambda}{x^2}$, bu yerda, λ - o'zgarmas son.

Agar P– jismning, u Yer sirtida, ya'ni Yerning markazidan R radiusga teng masofada joylashgandagi og'irligi bo'lsa, u holda $P=\frac{\lambda}{R^2}$, $\lambda=PR^2$ va ko'tarilayotgan dvigatelning, raketa Yerning markazidan x masofada bo'lgan lahzadagi yengib o'tish F kuchi, x ning ma'lum funksiyasidan iborat bo'ladi: $F(x)=\frac{PR^2}{x^2}$. Endi raketani x balandlikka ko'tarishda uning dvigateli ishini qandaydir $q(x)$ funksiya deb, hamda raketani yana dx kichik balandlikka ko'tarishda, F funksiya o'zgarmaydi deb, bajarilgan ish orrtirmasi uchun,

$$\Delta q \approx F(x)dx = \frac{PR^2}{x^2}dx = dq$$

taqribiy qiymatni topamiz.

Raketani Yer sirtidan H balandlikka ko'tarishda x o'zgaruvchi R dan $R+H$ gacha o'zgaradi. Shu sababli, izlanayotgan A ish,

$$A=\int_R^{R+H}F(x)dx = PR^2\int_R^{R+H}\frac{dx}{x^2}=PR^2\left(-\frac{1}{x}\right)\Big|_R^{R+H}=\frac{PRH}{R+H}$$

integral orqali ifodalanadi.

$P=1{,}5T$, $H=2000\,km$, $R=6400\,km$ bo'lganligidan, $A=2\,285714000\,k\Gamma m\approx$ $\approx 2\,2422854340\,j$ bo'ladi. Endi dvigatelning, raketani Yerning tortilishidan to'la ozod qilish uchun bajarish kerak bo'lgan ishini, H cheksiz ortib borganda, $A(H)$ ishning limiti sifatida aniqlash mumkin:

$$\lim_{H\to\infty}A(H)=\lim_{H\to\infty}\frac{PRH}{R+H}=PR$$

P va H ning yuqorida berilgan qiymatlarida bu ish 9600 000 000 kGm$\approx$94176000 000 j bo'ladi.

15.5-misol. Balandligi $H=1{,}5m$ va radiusi $R=0{,}4m$ bo'lgan silindr (10330 kg/m^2) atmosfera bosimi ostida gaz bilan to'ldirilgan bo'lib, porshen bilan yopilgan. Porshenni silindr ichida $h=1{,}2m$ masofaga ko'chirish uchun gazni izotermik kesishda sarflanadigan ish (miqdori) topilsin.

Yechilishi. Gaz holatining izotermik, ya'ni temperatura o'zgarmagan holda, o'zgarishida v hajm va gazning p bosimi orasidagi bog'lanish, Boyl-Mariott qonuniga asosan, $pv=c=const$ formula orqali ifodalanadi.

Shuning uchun, agar porshen silindrning ichida x masofaga itarilsa, porshenning yuza birligiga gazning $p(x)$ bosimi,

$$p(x)=\frac{c}{v(x)}=\frac{c}{S(H-x)},$$

porshenning to‘la S yuzasiga bo‘lgan bosim esa, $P(x)=Sp(x)=\frac{c}{H-x}$ bo‘ladi.

Endi, porshenni x м ga itarishda sarflanadigan ishni qandaydir $q(x)$ funksiya deb olib, va porshenni kichik dx masofaga itarishda unga ta’sir qiladigan $P(x)$ bosimni o‘zgarmas, deb hisoblab, $q(x)$ funksiya ortirmasi differensialining taqribiy miqdorini topamiz:

$$\Delta q \approx P(x)dx = \frac{c}{H-x}dx = dq$$

Izlanayotgan umumiy A ishga x ning o dan h gacha o‘zgarishi mos keladi, shuning uchun,

$$A = c\int_0^h \frac{dx}{H-x} = -c\ln(H-x)\Big|_0^h = c\ln\frac{H}{H-h}.$$

$H=1{,}5m$, $R=0{,}4m$, $h=1{,}2m$, $p_0=10330k\Gamma/m^2$ bo‘lganda,

$$v_0=\pi R^2H=0{,}24\pi m^3;\ c=p_0v_0=2479{,}2\pi;\ Q\approx 12533{,}3k\Gamma m\approx 122951{,}7\,j.$$

15.3. Iqtisodiy hisob-kitoblarga tadbiqlar bo‘yicha misollar.

15.6-misol. Agar korxonadagi ishchining mehnat mahsuldorligi

$$f(t)=\frac{3}{3t+1}+4$$

funksiya orqali ifodalansa, ishchi ish kunining uchinchi soati davomida qancha hajmdagi mahsulot ishlab chiqarishini aniqlang.

Yechilishi. Agar ishchining t vaqt davomidagi mehnat mahsuldorligi uzluksiz $f(t)$ funksiya orqali ifodalansa, ishchi t_1 dan t_2 gacha o‘tgan vaqt oralig‘ida, hajmi

$$V=\int_{t_1}^{t_2} f(t)dt \qquad (15.5)$$

formula bo‘yicha aniqlanadigan, mahsulot ishlab chiqaradi. Qaralayotgan misolda,

$$f(t)=\frac{3}{3t+1}+4;\ t_1=2,\ t_2=3$$

bo‘lganligidan, izlanayotgan hajm,

$$V=\int_2^3\left(\frac{3}{3t+1}+4\right)dt=\left(\ln(3t+1)+4t\right)\Big|_2^3=$$

$$=\ln 10+12-\ln 7-8=\ln\frac{10}{7}+4$$

bo‘ladi.

15.7-misol. Agar magazinga yangi tovar olib kelinishi $f(t)=2t+5$ funksiya orqali ifodalansa, magazinda uch kun mobaynida qancha hajmda tovar to'planishini aniqlang.

Yechilishi. Qaralayotgan misolda

$$f(t)=2t+5,\ t_1=0,\ t_2=3$$

bo'lganligidan, izlanayotgan hajm uchun, (15.5) formuladan foydalangan holda,

$$V=\int_0^3(2t+5)dt=\left(\frac{2t^2}{2}+5t\right)\Bigg|_0^3=9+15=24$$

bo'lishini olamiz.

15.4. Elektr energiyasining sarflanishini bashorat qilish (oldindan aytish). Ma'lumki, elektr energiyasining har bir chiroq yoki fonar tomonidan sarflanishi, quyosh botgandan to u chiqquncha davom etadi. Kecha qancha qisqa bo'lsa, shuncha kam elektr energiyasi sarflanadi. Yilda eng qisqa tun 22 iyunga to'g'ri kelganligini, eng uzun tun esa, 22 dekabrga to'g'ri kelganligini hisobga olsak, ulardan birinchisida, ikkinchisidan kam energiya sarflanishini olamiz.

Shunday qilib, energiyaning sarflanishi $\omega-$ tebranishli jarayondan iborat ekan. Bu jarayon,

$$w=b+c\cos(2\pi(t+0{,}025)) \qquad (15.6)$$

funksiya orqali ifodalanishi mumkin. Bunda 0,025 qo'shiluvchi, maksimumning, $t=-0{,}025$ qiymatga to'g'ri kelishini, ya'ni har bir yil boshlanishidan $0{,}025\cdot365=9$ kun oldinga, 22 dekabrga to'g'ri kelishini ko'rsatadi; 2π ko'paytuvchi esa, uzunligi 1 ga (yilga) teng bo'lgan davrni aniqlaydi.

15.8-misol. Tarmoq tomonidan, $x=0$ dan $x=1$ gacha, bir yilda energiya sarflanishi, (15.6) funksiya orqali ifodalansin, bu erda b va c qandaydir sonlar. U holda, tarmoq $t=0$ dan $t=1$ gacha o'tgan bir yilda qancha energiya sarflashini aniqlang.

Yechilishi. Yuqoridagi bandlarda bajarilgan ishlarni hisobga olgan holda, dt vaqt davomida sarflangan energiya wdt, yil davomida sarflangan energiya esa,

$$\int_0^1 wdt$$

integralga teng bo'lishini ko'ramiz. U holda,

$$\int_0^1 w\,dt = \int_0^1 [b + c\cos(2\pi(t+0{,}025))]dt =$$

$$= b + c\int_0^1 \cos(2\pi(t+0{,}025))dt.$$

Oxirgi integralni hisoblash uchun, $2\pi(t+0{,}025) = z$ almashtirish olamiz. Natijada,

$$\int_0^1 \cos(2\pi(t+0{,}025))dt = \left| \begin{array}{l} t = \frac{z}{2\pi} - 0{,}025,\ dt = \frac{dz}{2\pi}; \\ t = 0 \Rightarrow z = 0{,}05\pi; \\ t = 1 \Rightarrow z = 2{,}05\pi \end{array} \right| =$$

$$= \frac{1}{2\pi}\int_{0{,}05\pi}^{2{,}05\pi} \cos z\,dz = \frac{1}{2\pi}(\sin z)\Big|_{0{,}05\pi}^{2{,}05\pi} = 0$$

bo‘ladi.

Bu yerdan, bir yilda energiyaning sarflanishi, b birlik quvvatni tashkil etishini olamiz.

15.9-misol. Har bir lampa va fonarning $x=0$ dan $x=1$ gacha energiya sarflashi (15.6) funksiya orqali ifodalangan bo‘lsin,bunda b va $c-$ qandaydir sonlar. Tumanning yoritish tarmog‘i $u = u_0 + at$ chiziqli qonun bo‘yicha o‘ssin (ortsin), bu yerda $t-$ yillar bilan o‘lchanadi. U holda, tarmoqning $t=0$ dan $t=1$ gacha o‘tgan 1 yilda sarflagan energiyasini hisoblang.

Yechilishi. Bu misolda dt vaqt birligi davomida sarflangan energiya $uwdt$ miqdorga, bir yil davomida sarflangan energiya esa,

$$\int_0^1 uwdt$$

integralga teng bo‘lishini ko‘ramiz.

Demak,

$$\int_0^1 uwdt \overset{(1)}{=} \int_0^1 (u_0 + at)[b + c\cos(2\pi(t+0{,}025))]dt =$$

$$= \left| \begin{array}{l} \text{bo'laklab integrallash usuli,} \\ \text{o'zgaruvchilarni almashtirish usuli} \end{array} \right| \approx$$

$$\approx bu_0 + 0{,}5ab + 0{,}025ac.$$

Bu yerdan, tarmoq tomonidan 1 yilda sarflangan energiya miqdori $bu_0 + 0{,}5ab + 0{,}025ac$ birlik quvvatdan iborat bo‘lishi kelib chiqadi.

Mustaqil yechish uchun misollar

15.1. Bo'shliqda pastga tushayotgan jismning tezligi $v = 9.8t$ *м/сек* formula bo'yicha aniqlanadi. Jism tushish boshlangandan 10 sekund vaqt o'tganda qancha yo'l bosib o'tishini aniqlang.

15.2. Jismning harakat tezligi $v = (3t^2 - 2t)$ *см/сек* formula bo'yicha aniqlanadi. Jism harakat boshlangandan 4 sekund vaqt o'tganda qancha yo'l bosib o'tadi?

15.3. Jismning harakat tezligi $v = \sqrt{5t+4}$ *м/сек* formula bo'yicha aniqlanadi. Jism harakat boshlangandan 9 sek vaqt o'tganda qancha yo'l bosib o'tadi?

15.4. Jismning harakat tezligi $v = (4t - \frac{6}{t^2})$ *см/сек*.Uning uchinchi sekundda bosib o'tgan yo'lini aniqlang.

15.5. 6 kG miqdordagi kuch prujinani 8 sm ga cho'zganda, u qanday ish bajarishini aniqlang.

15.6. Prujinani 4 sm ga cho'zganda 10 kGm miqdordagi ish bajarilishi ma'lum. Prujinani 10 sm ga cho'zish uchun qanday ish bajarilishini toping.

15.7. Radiusi R ga teng bo'lgan yarim shar shaklidagi qog'ozdan suvni chiqarishda sarf bo'ladigan ish miqdorini aniqlang.

15.8. Silindrda diametri 20 sm va uzunligi 80 sm bo'lgan porshen harakat qiladi. Bu silindr $P_0 = 10$ *кг/см*^{2} bosim ostida bug' bilan to'ldirilgan bo'lsa, temperaturani o'zgartirmasdan qanday ish bajarilganda, bug'ning hajmi ikki baravar kamayadi?

15.9. Agar 5 kГ kuch prujinani 25 sm ga cho'zsa, u holda prujinani 6 sm ga cho'zish uchun qanday ish bajarish kerak?

15.10. Agar 1 kГ kuch elastik prujinani 1 sm ga cho'zsa, u holda, prujinani 10 sm ga cho'zish uchun qanday ish bajarish kerak (Guk qoidasidan foydalanish kerak)?

15.11. Agar korxonadagi ishchining mehnat mahsuldorligi $f(t) = \frac{4}{4t+1} + 1$ funksiya orqali ifodalansa, ishchi ish kunining to'rtinchi soati davomida qancha hajmdagi mahsulot ishlab chiqarishini aniqlang.

15.12. Agar magazinga yangi tovarlar olib kelinishi $f(t) = 2t + 5$ funksiya orqali ifodalansa, magazinda 6 kun mobaynida qancha hajmda tovar to'planishini aniqlang.

15.13. Tarmoq tomonidan $x = 0$ dan $x = 1$ gacha o'tgan, bir yilda energiya sarflanishi $w = 3 + 2\cos(2\pi(t + 0{,}025))$ funksiya orqali ifodalansin. 2

yilda, ya'ni $t=0$ dan $t=2$ gacha o'tgan davrda sarflangan energiya miqdorini aniqlang.

15.14. Har bir lampa va fonarning $x=0$ dan $x=1$ gacha o'tgan vaqtda energiya sarflashi $w=3+2\cos(2\pi(t+0{,}025))$ funksiya orqali ifodalangan bo'lsin. bunda b va $c-$ qandaydir sonlar. Tumanning yoritish tarmog'i $u=2+5t$ chiziqli qonun bo'yicha o'ssin (ortsin), bu erda $t-$ yillar bilan o'lchanadi. U holda tarmoqning $t=0$ dan $t=1$ gacha o'tgan 1 yilda sarflagan energiyasini hisoblang.

15.15. Har bir lampa va fonarning $x=0$ dan $x=1$ gacha energiya sarflashi $w=3+2\cos(2\pi(t+0{,}025))$ funksiya orqali ifodalangan bo'lsin,bunda b va $c-$ qandaydir sonlar to'plamining yorilishi tarmog'i $u=1+2t^2$ kvadratik qonun bo'yicha o'ssin (ortsin), bu erda $t-$ yillar bilan o'lchanadi. U holda, tarmoqning $t=0$ dan $t=1$ gacha o'tgan 1 yilda sarflagan energiyasini hisoblang.

Mustaqil yechish uchun misollarning javobiari

15.1. $490\,м$. **15.2.** $48\,м$. **15.3.** $\frac{134}{3}\,м$. **15.4.** $9\,см$ **15.5.** $0{,}24\,кГм$.

15.6. $62{,}5\,кГм$. **15.7.** $\frac{\pi R^4}{4}\,кГм$.. **13.8.** $1740\,кГ\,см$. **13.9.** $A=3{,}6\,кГ\,м$.

13.10. $A=0{,}5\,kJ$. **15.11.** $\ln\frac{17}{13}+1$. **15.12.** 66. **15.13.** 4 birlik quvvat.

15.14. $9{,}25$ birlik quvvat.**15.15.** $5{,}26$ birlik quvvat.

IV bob. KO‘P O‘ZGARUVCHILI FUNKSIYALAR

16-§. Evklid tekisligi va Evklid fazosi. Evklid fazosidagi muhim to‘plamlar

16.1. Evklid tekisligi va Evklid fazosi. x va y haqiqiy sonlarning mumkin bo‘lgan har qanday tartiblangan (x,y) juftlari - to‘plamiga *koordinatalar tekisligi* deyiladi. Bunda har bir (x,y)- juftlik koordinatlar tekisligining *nuqtasi* deyiladi va qisqacha, M harfi orqali belgilanadi. x va y sonlar- M nuqtaning *koordinatalari* deyiladi. $M(x,y)$ yozuvda, x va y - M nuqtaning koordinatalarini anglatadi.

16.1-ta’rif. Agar koordinatalar tekisligining ixtiyoriy $M_1(x_1;y_1)$ va $M_2(x_2;y_2)$ nuqtalari orasidagi $\rho(M_1,M_2)$ masofa tushunchasi kiritilgan bo‘lib, u

$$\rho(M_1,M_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

formula bo‘yicha aniqlansa, u holda koordinatalar tekisligi *Evklid tekisligi* deyiladi va R^2 orqali belgilanadi.

m o‘lchovli koordinatalar fazosi va Evklid fazosi tushunchalari ham shunga o‘xshash kiritiladi. $x_1,x_2,...,x_m$ haqiqiy sonlarning mumkin bo‘lgan har qanday tartiblangan $x_1,x_2,...,x_m$ qiymatlar to‘plamiga *koordinatlar fazosi* deyiladi. Bunda har bir koordinatalar fazosining $(x_1,x_2,...,x_m)$ nuqtasi M yoki $M(x_1,x_2,...,x_m)$ orqali belgilanadi, $x_1,x_2,...,x_m$ sonlar esa, M nuqtaning *koordinatalari* deyiladi.

Agar koordinatalar fazosining $M'(x_1',x_2',...,x_m')$ va $M''(x_1'',x_2'',...,x_m'')$ nuqtalari orasidagi $\rho(M',M'')$masofa tushunchasi kiritilgan bo‘lib, u

$$\rho(M',M'')=\sqrt{(x_1''-x_1')^2+(x_2''-x_2')^2+...+(x_m''-x_m')^2}$$

formula bo‘yicha aniqlansa, u holda koordinatalar fazosi, *Evklid fazosi* deyiladi va R^m kabi belgilanadi.

R^m fazoda x va y nuqtalar orasidagi $\rho(x,y)$ masofa quyidagi xossalarga ega:

16. $\rho(x,y)\geq 0$, $\rho(x,y)=0$ faqat va faqat $x=y$ bo‘lganda;

2^0. $\rho(x,y)=\rho(y,x)$;

3^0. $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$.

16.2. Evklid fazosidagi muhim to‘plamlar. Evklid fazosidagi $\{M\}$ to‘plamga doir misollar keltiramiz.

1) $\{M\}=\{(x,y): x\in R,\ y\in R, (x-a)^2+(y-b^2)\le r^2\}$ to‘plamga, R^2 fazoda, markazi $M_0(a;b)$ nuqtada, radiusi r ga teng bo‘lgan *yopiq doira* ($(x-a)^2+(y-b)^2<r^2$ ga - *ochiq doira*) deyiladi. Mos ravishda, (x_1,x_2), (x_1,x_2,x_3) va $(x_1,x_2,....,x_m)$ tartiblangan juftlik, uchlik va hokazolarni ikki o‘lchovli, uch o‘lchovli va m o‘lchovli vektorlarning koordinatalari deb ham qarash mumkin:

$$x=(x_1,x_2),\ x=(x_1,x_2,x_3),\ x=(x_1,x_2,.....,x_m).$$

Ko‘p hollarda, $x=(x_1,x_2,.....,x_m)$ simvolga, koodinatalari $x_1,x_2,....x_m$ bo‘lgan M nuqta yoki $\overline{OM}$ vektor, deb qaraladi.

16.1-eslatma. Agar R^m koordinatalar fazosini koordinatalari $x_1,x_2,....x_m$ bo‘lgan x *vektorlar fazosi* deb qaralsa, u holda, $x=(x_1,x_2,...,x_m)$, $y=(y_1,y_2,...,y_m)$ vektorlarning *yig‘indisi* deb, koordinatalari $x_1+y_1,\ x_2+y_2,....,x_m+y_m$ bo‘lgan vektorga aytiladi; $x=(x_1,x_2,...,x_m)$ vektorning biror $\lambda-$ haqiqiy songa λx *ko‘paytmasi* deb, koordinatalari $\lambda x_1,\ \lambda x_2,...,\lambda x_m$ bo‘lgan vektorga aytiladi.

Agar R^m- *vektorlar fazosi* deb qaralib, uning elementlarini qo‘shish va biror haqiqiy songa ko‘paytirish amallari aniqlangan bo‘lib, bu amallar, quyidagi: 1) $x+y=y+x$; 2) $(x+y)+z=x+(y+z)$; 3) shunday 0 nul element mavjud bo‘lib, $\forall x$ element uchun $x+0=x$; 4) $\forall x$ uchun unga qarama – qarshi x' vektor mavjud bo‘lib, $x+x'=0$; 5) $\lambda(x+y)=\lambda x+\lambda y$; 6) $(\lambda+\mu)x=\lambda x+\mu x$; 7) $\lambda(\mu x)=(\lambda\mu)x$; 8) $1\cdot x=x$ aksiomalarni qanoatlantirsa, u holda, R^m- *chiziqli vektor fazo* deyiladi.

R^m chiziqli vektor fazoda ikkita $x=(x_1,x_2,...,x_m)$, $y=(y_1,y_2,...,y_m)$ vektorga $(x,y)=\sum_{i=1}^{m}x_iy_i$ sonni mos qo‘yish orqali, ikki vektorning (x,y) *skalyar ko‘paytmasi* aniqlanadi. $\sqrt{(x,x)}$ son x vektorning *uzunligi* deyiladi va y $|x|$ kabi belgilanadi.

Agarda $(x,y)=0$ bo‘lsa, u holda, x va y vektorlar *o‘zaro ortogonal* deyiladi.

Agar x va y vektorlar nuldan farqli vektorlar bo‘lsa, u holda, ular orasidagi φ $(0\le\varphi\le\pi)$ burchak

$$\cos\varphi=\frac{(x,y)}{|x||y|}$$

formula orqali topiladi.

2) $\{M\}=\{(x,y,z): x\in R, y\in R, z\in R, (x-a)^2+(y-b)^2+(z-c)^2\le r^2\}$ to'plam R^3 fazoda markazi $M_0(a,b,c)$ nuqtada, radiusi r ga teng bo'lgan *yopiq shar* ($(x-a)^2+(y-b)^2+(z-c)^2<r^2$ – *ochiq shar*) deyiladi va u qisqacha, $\rho(M,M_0)<r$, $(\rho(M,M_0)<y)$ kabi yoziladi.

3) $\{M\}=\{(x,y): x\in R, y\in R, |x-a|\le d_1, |y-b|\le d_2\}$ to'plam markazi $M_0(a;b)$ nuqtada bo'lgan koordinatalar *to'g'ri to'rtburchagi,*

$$\{M\}=\{(x,y,y): x\in R, y\in R, z\in R,\ |x-a|\le d_1, |y-b|\le d_2, |z-c|\le d_3\}$$

to'plam esa, markazi $M_0(a,b,c)$ nuqtada bo'lgan koordinatalar *parallelepipedi* deyiladi.

4) $\{M\}=\{(x_1,x_2,...,x_m): x_i\in R, i=\overline{1,m}, (x_1-x_1^0)^2+(x_2-x_2^0)^2+....+(x_m-x_m^0)^2\le r^2\}$ to'plam, R^m fazoda, markazi $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada, radiusi r ga teng bo'lgan *yopiq shar* deyiladi va qisqacha, $\rho(M,M_0)\le r$ shaklida yoziladi ($\rho(M,M_0)<r$ *-ochiq shar*).

5) $\{M\}=\{(x_1,x_2,...,x_m): x_i\in R, i=\overline{1,m}, (x_1-x_1^0)^2+(x_2-x_2^0)^2+....+(x_m-x_m^0)^2=r^2\}$ to'plam, R^m fazoda, markazi $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada, radiusi r teng bo'lgan *sfera* deyiladi.

16.2-eslatma. Markazi $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada, radiusi r ga teng bo'lgan ochiq sharga, markazi $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada radiusi r ga teng bo'lgan sferani qo'shsak, natijada markazi M_0 nuqtada radiusi r ga teng bo'lgan yopiq shar hosil bo'ladi.

16.2-ta'rif. Markazi $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada radiusi $\varepsilon>0$ bo'lgan ochiq $\rho(M,M_0)<\varepsilon$ sharga M_0 nuqtaning *atrofi* deyiladi.

$$\{M\}=\{(x_1,x_2,...,x_m): x_i\in R, i=\overline{1,m},\ |x_1-x_1^0|^2<d_1, |x_2-x_2^0|^2<d_2,...,|x_m-x_m^0|<d_m\}$$

to'plam ($d_1,d_2,..,d_m$ – lar biror o'zgarmas sonlar), m- o'lchovli ochiq koordinatalar parallelepipedi yoki M_0 nuqtaning *to'g'ri burchakli atrofi* deyiladi.

Quyidagi elementar tasdiqlar o'rinli: M_0 nuqtaning ixtiyoriy ε - atrofi, M_0 nuqtaning biror to'g'ri burchakli atrofida joylashadi; M_0 nuqtaning ixtiyoriy to'g'ri bo'rchakli atrofi, uning biror ε - atrofida joylashadi.

Haqiqatan ham, belgilangan $\varepsilon>0$ da $d_1=d_2=...=d_m=\frac{\varepsilon}{\sqrt{m}}$ deb olinsa, u holda ko'rsatilgan $d_1,d_2,..,d_m$ – larda M_0 nuqtaning to'g'ri burchakli atrofi uning ε- atrofida joylashadi.

Agar $d_1 > 0, d_2 > 0, ..., d_m > 0$ lar belgilanib, $\varepsilon = \min\{d_1, d_2, ..., d_m\}$ deb olinsa, u holda, M_0 nuqtaning (ko'rsatilgan $d_1, d_2, .., d_m$ – larda) to'g'ri burchakli atrofida, uning ε - atrofi yotadi.

16.3-ta'rif. Agar $M \in \{M\} \subset R^m$ nuqtaning shunday ε - atrofi mavjud bo'lib, bu atrofning barcha nuqtalari $\{M\}$ to'plamga qarashli bo'lsa, u holda, M nuqta. $\{M\}$ to'plamning *ichki nuqtasi* deyiladi.

Agar $M \in R^m$ nuqtaning shunday ε - atrofi mavjud bo'lib, bu atrofning barcha nuqtalari $\{M\}$ to'plamga qarashli bo'lmasa, M nuqta, $\{M\}$ to'plamning *tashqi nuqtasi* deyiladi.

16.4-ta'rif. Agar $M \in \{M\} \subset R^m$ nuqta, $\{M\}$ to'plamning ichki nuqtasi ham, tashqi nuqtasi ham bo'lmasa, u holda, M nuqta, $\{M\}$ to'plamning *chegara nuqtasi* deyiladi.

16.5-ta'rif. Agar $\{M\} \subset R^m$ to'plamning hamma elementlari uning ichki nuqtalari, ya'ni uning ixtiyoriy M nuqtasi, o'zining ixtiyoriy ε – atrofi bilan $\{M\}$ ga qarashli, bo'lsa, u holda, $\{M\}$ to'plam, *ochiq to'plam* deyiladi.

16.6-ta'rif. M_0 nuqtani o'zida saqlovchi ixtiyoriy ochiq to'plamga M_0 nuqtaning *atrofi* deyiladi.

16.7-ta'rif. Agar ixtiyoriy $\{M\} \subset R^m$ to'plam o'zining hamma chegara nuqtalarini o'zida saqlasa, u holda, $\{M\}$ to'plam, *yopiq to'plam* deyiladi.

16.8-ta'rif. Agar $A \in R^m$ nuqtaning ixtiyoriy ε – atrofida $\{M\} \subset R^m$ to'plamning, hech bo'lmaganda, A dan farqli bitta nuqtasi mavjud bo'lsa, u holda, A nuqta, $\{M\}$ to'plamning *limit nuqtasi* deyladi.

$\{M\}$ to'plam yopiq bo'lishi uchun, uning hamma limit nuqtalari unga qarashli bo'lishi zarur va etarlidir.

16.9-ta'rif. Agar shunday m- o'lchovli shar mavjud bo'lib, $\{M\}$ to'plamning barcha elementlari shu sharga qarashli bo'lsa, u holda, $\{M\}$ to'plam, *chegaralangan* deyiladi.

16.10-ta'rif. Ushbu

$$L = \{M(x_1 x_2, ..., x_m) \in R^m : x_1 = \varphi(t), x_2 = \varphi_2(t), ..., x_m = \varphi_m(t),\ \alpha \le t \le \beta\},$$

to'plam, bunda $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ funksiyalar $[\alpha; \beta]$ segmentda uzluksiz funksiyalar, *uzluksiz chiziq* deyiladi. $A(\varphi_1(\alpha), ..., \varphi_m(\alpha))$ va $B(\varphi_1(\beta), ..., \varphi_m(\beta))$ nuqtalar L chiziqning *uchlari* deyiladi.

Ushbu

$$\{M(x_1, x_2, ..., x_m) \in R^m : x_1 = x_1^0 + \alpha_1 t,\ x_2 = x_2^0 + \alpha_2 t, ..., x_m = x_m^0 + \alpha_m t;\ -\infty < t < \infty\}$$

(bunda, $x_1^0,...,x_m^0;\alpha_1,\alpha_2,...,\alpha_m$ – qandaydir sonlar) to‘plam, R^m fazoda *to‘g‘ri chiziq* deyiladi. Ma’lumki, bu to‘g‘ri chiziq $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtadan o‘tadi (M_0 nuqta $t=0$ ga mos keladi).

16.11-ta’rif. Agar $\{M\}$ to‘plamning ixtiyoriy ikkita nuqtasini, shu to‘plamda to‘liq yotuvchi uzluksiz chiziq bilan tutashtirish mumkin bo‘lsa, u holda, $\{M\}$ to‘plamga *bog‘lamli to‘plam* deyiladi.

16.12-ta’rif. Agar $E\subset R^m$ to‘plam ochiq bog‘lamli to‘plam bo‘lsa, u holda, E to‘plam, R^m fazoda *soha* deyiladi.

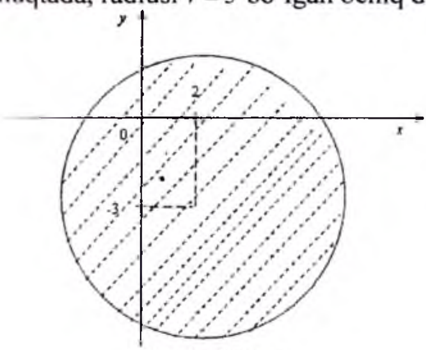
Soha va uning chegarasining birlashmasi – *yopiq soha* deyiladi.

16.1-misol. Tekislikda koordinatalari, ushbu

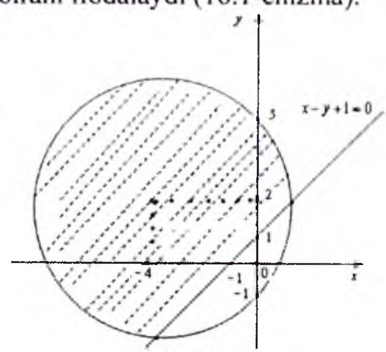
$$(x-2)^2+(y+3)^2<25 \qquad (16.1)$$

tengsizlikni qanoatlantiradigan nuqtalar to‘plamining geometrik o‘rnini aniqlang.

Yechilishi. Ma’lumki, $\{M\}=\{(x,y):x\in R,y\in R:(x-a)^2+(y-b)^2<r^2\}$ to‘plam, R^2 fazoda, markazi $M(a,b)$ nuqtada, radiusi r ga teng bo‘lgan ochiq doirani ifodalaydi. Shuning uchun, koordinatalari (16.1) tengsizlikni qanoatlantiruvchi nuqtalar to‘plami - markazi $M_0(2;-3)$ nuqtada, radiusi $r=5$ bo‘lgan ochiq doirani ifodalaydi (16.1-chizma).



16.1-chizma.



16.2-chizma.

16.2-misol. Tekislikda koordinatalari

$$\begin{cases}(x+4)^2+(y-2)^2\le 25,\\ x-y+1\le 0\end{cases} \qquad (16.2)$$

tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to‘plamining geometrik o‘rnini aniqlang.

Yechilishi. Markazi $M_0(-4;2)$ nuqtada, radiusi 5 ga teng bo'lgan yopiq doirada yotgan barcha nuqtalarning koordinatalari $(x+4)^2+(y-2)^2 \le 25$ tengsizlikni qanoatlantiradi.

$x-y+1 \le 0$ tengsizlikni $y \ge x+1$ ko'rinishda yozib olamiz. $y=x+1$ to'g'ri chiziqda va undan yuqorida joylashgan nuqtalarning koordinatalari $x-y+1 \le 0$ tengsizlikni qanoatlantiradi (16.2-chizma).

$(x+4)^2+(y-2)^2=25$ aylana bilan $y=x+1$ to'g'ri chiziqning kesishish nuqtasini topish uchun $\begin{cases}(x+4)^2+(y-2)^2=25\\ x-y+1=0\end{cases}$ sistemani birgalikda Yechib, $A(1;2)$, $B(-4;-3)$ ekanligini topamiz. (16.2) tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to'plami, $(x+4)^2+(y-2)=5^2$ aylana va uning ichida joylashgan, hamda $y=x+1$ to'g'ri chiziq va undan yuqorida joylashgan nuqtalarning kesishmasidir.

16.3-misol. Fazoda koordinatalari,

$$\begin{cases}x^2+y^2+z^2 \ge 36,\\ 2x-3y+z-2 \ge 0\end{cases}$$

tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to'plamining geometrik o'rnini aniqlang.

Yechilishi. $x^2+y^2+z^2 \ge 36$ tengsizlikni, $x^2+y^2+z^2=36$ sferada va undan tashqarida yotgan, $2x-3y+z-2 \ge 0$ tengsizlikni esa, $z=2-2x+3y$ tekislik va undan yuqorida joylashgan nuqtalar to'plamining koordinatalari qanoatlantiradi. Demak, sistemani, $z \ge 2-2x+3y$ fazo bilan $x^2+y^2+z^2 \ge 36$ sharning kesishgan qismini olib tashlash natijasida hosil bo'lgan qismidagi nuqtalarning koordinatalari qanoatlantiradi.

16.4-misol. Markazi $M_0(1;2)$ nuqtada, radiusi 4 ga teng bo'lgan doiradagi nuqtalar to'plamini, $\begin{cases}a \le x \le b,\\ \varphi(x) \le y \le \psi(x)\end{cases}$ tengsizliklar sistemasi ko'rinishida tasvirlang.

Yechilishi. Ravshanki, berilgan doiradagi nuqtalarning absissasi – 3 dan 5 gacha o'zgaradi. $(x-1)^2+(y-2)^2=16$ aylana tenglamasini, $y=2\pm\sqrt{15-x^2+2x}$ ko'rinishda yozamiz.

Bunda, $y=2+\sqrt{15-x^2+2x}$ aylananing yuqori qismini, $y=2-\sqrt{15+2x-x^2}$ esa, aylananing pastki qismini ifodalaydi. Demak, x $[-3;5]$ segmentda o'zgarganda, y ning qiymatlari $2-\sqrt{15+2x-x^2}$ dan $2+\sqrt{15+2x-x^2}$ gacha o'zgaradi. Shunday qilib, berilgan doiraning nuqtalari to'plami,

$$\begin{cases} -3 \le x \le 5, \\ 2-\sqrt{12+2x-x^2} \le y \le 2+\sqrt{15+2x-x^2} \end{cases}$$

tengsizliklar sistemasi yordamida berilgan ekan.

16.5-misol. Ushbu $\frac{x^2}{25}+\frac{y^2}{16}+z^2=1$ ellipsoid bilan chegaralangan to'plamni, quyidagi, $\begin{cases} a \le x \le b, \\ \varphi(x) \le y \le \psi(x) \\ \Phi(x,y) \le z \le \Psi(x,y) \end{cases}$ tengsizliklar sistemasi ko'rinishida tasvirlang.

Yechilishi. Berilgan ellipsoid tenglamasidan $z=\pm\sqrt{1-\frac{x^2}{25}-\frac{y^2}{16}}$ ekanligini topamiz. Demak, z ning o'zgarish sohasi, $1-\frac{x^2}{25}-\frac{y^2}{16} \ge 0$ dan iborat. Bu soha, $\frac{x^2}{25}+\frac{y^2}{16}=1$ ellips bilan chegaralangan. Ellips tenglamasidan, $y=\pm 4\sqrt{1-\frac{x^2}{25}}$ bo'lishini topamiz. Bundan x ning - 5 dan 5 gacha o'zgarishi kelib chiqadi. x $[-5;5]$ segmentda o'zgarganda y o'zgaruvchi, $-4\sqrt{1-\frac{x^2}{25}}$ dan $4\sqrt{1-\frac{x^2}{25}}$ gacha o'zgaradi.

Agar M nuqta ellipsning ichida yotsa, uchlari $-\sqrt{1-\frac{x^2}{25}-\frac{y^2}{16}}$ va $\sqrt{1-\frac{x^2}{25}-\frac{y^2}{16}}$ bo'lgan qismi ellipsoidning ichida yotadi. Demak, ellipsoid,

$$\begin{cases} -5 \le x \le 5, \\ -4\sqrt{1-\frac{x^2}{25}} \le y \le \sqrt{1-\frac{x^2}{25}}, \\ -\sqrt{1-\frac{x^2}{25}-\frac{y^2}{16}} \le z \le \sqrt{1-\frac{x^2}{25}-\frac{y^2}{16}} \end{cases}$$

tengsizliklar sistemasi yordamida beriladi.

16.6-misol. $\{M\}$ - tekislikda koordinatalari $x^2+y^2<25$ tengsizlikni qanoatlantiruvchi nuqtalar to'plami bo'lsin. U holda, $A(3;4)$ nuqta, $\{M\}$ to'plamning limit nuqtasi ekanligini isbotlang.

Yechilishi. $A(3;4)$ nuqtaning, $|x-3|<\delta, |y-4|<\delta$ $(0<\delta<1)$ tengsizliklar orqali berilgan ixtiyoriy atrofini olamiz. $B\left(3-\frac{\delta}{2}, 4-\frac{\delta}{2}\right)$ nuqta bu atrofda yotadi va $\left(3-\frac{\delta}{2}\right)^2+\left(4-\frac{\delta}{2}\right)^2<3^2+4^2=25$ tengsizlik o'rinli bo'ladi. Demak,

B nuqta $\{M\}$ to'plamga qarashli ekan. Shunday qilib, 16.8-ta'rifga ko'ra, $A(3;4)$ nuqta - $\{M\}$ to'plamning limit nuqtasi bo'ladi.

16.7-misol. Tekislikda $y \geq x^2$ tengsizlikni qanoatlantiruvchi barcha nuqtalar to'plami $\{M\}$ ning yopiq to'plam ekanligini isbotlang.

Yechilishi. Ma'lumki, $\{M\}$ to'plam yopiq bo'lishi uchun, uning barcha limit nuqtalari o'ziga qarashli bo'lishi kerak. Shuning uchun, ixtiyoriy $A(a,b)\overline{\in}\{M\}$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lmasligini ko'rsatish etarli. $A(a;b)\overline{\in}\{M\}$ bo'lsin. U holda, $b<a^2$ tengsizlikka ega bo'lamiz. $\varepsilon = a^2 - b > 0$ deb olamiz. Shunday $\delta < \frac{\varepsilon}{2}$ topiladiki, $|x-a|<\delta$ tengsizlikdan $|x^2-a^2|<\frac{\varepsilon}{2}$ bo'lishi kelib chiqadi. U holda, $|x-a|<\delta, |y-b|<\delta$ tengsizliklardan

$$x^2 - y = (a^2-b)+(b-y)+(x^2-a^2) \geq (a^2-b)-|b-y|-|x^2-a^2| > \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 0.$$

Demak, A nuqtaning δ atrofi $\{M\}$ to'plamga qarashli emas. Shuning uchun, A nuqta - to'plamning limit nuqtasi bo'la olmaydi. Shunday qilib, $\{M\}$ to'plam yopiq ekan.

16.8-misol. Ushbu $x^2 + y^2 < 100$ tengsizlik orqali berilgan $\{M\}$ to'plamning chegara nuqtalarini toping.

Yechilishi. Berilgan tengsizlikni qanoatlantiruvchi nuqtalar to'plami – markazi koordinatalar boshida, radiusi 10 ga teng bo'lgan aylananing ichidagi nuqtalar to'plamidan iborat (aylanadagi nuqtalar $\{M\}$ to'plamga qarashli emas).

Geometrik nuqtai nazaridan, ravshanki, $\{M\}$ to'plamning chegarasi $x^2 + y^2 = 100$ aylanadan iborat. $x^2 + y^2 < 100$ va $x^2 + y^2 > 100$ tengsizliklar bilan berilgan to'plamlar ochiq to'plamlardan iborat. Shuning uchun, $x^2 + y^2 < 100$ to'plamning nuqtalari ham $\{M\}$ to'plamning chegara nuqtalari bo'la olmaydi.

Endi $x^2 + y^2 = 100$ aylananing nuqtalarini qaraymiz. $\forall A(a;b)$ nuqta shu aylanada yotsin. A nuqtaning ixtiyoriy atrofini qaraymiz. Bu atrofda koordinatalari $|x|<|a|, |y|<|b|$ tengsizliklarni va $|x|>|a|, |y|>|b|$ tengsizliklarni qanoatlantiruvchi $B(x,y)$ nuqtalar mavjud. Birinchi tengsizliklarni qanoatlantiruvchi nuqtalar uchun, $x^2 + y^2 < a^2 + b^2 = 100$, ikkinchi tengsizliklarni qanoatlantiruvchi nuqtalar uchun $x^2 + y^2 > 100$.

Demak $A(a.b)$ nuqta $\{M\}$ to'plam uchun, 16.4-ta'rifga ko'ra, chegara nuqta bo'ladi. Ma'lumki, berilgan to'plamning hamma chegara nuqtalari, uning chegarasi bo'ladi.

16.9-misol. $\{M\}$ - tekislikda ikkala koordinatalari ham rasional bo‘lgan nuqtalar to‘plami bo‘lsin. U holda tekislikdagi ixtiyoriy $A(a;b)$ nuqta $\{M\}$ to‘plamning limit nuqtasi bo‘lishini isbotlang.

Yechilishi. $A(a,b)$ nuqtaning ixtiyoriy $|x-a|<\delta$, $|y-b|<\delta$ atrofini qaraymiz. a dan farqli shunday r_1 rasional son topiladiki, $|r_1-a|<\delta$ xuddi shunday, b dan farqli r_2 shunday rasional son topiladiki, $|r_2-b|<\delta$ bo‘ladi.

U holda, $B(r_1;r_2)$ nuqta $\{M\}$ to‘plamga qarashli bo‘lib, u $A(a;b)$ nuqtaning belgilangan atrofida yotadi $(B(r_1;r_2)\neq A(a,b))$. Demak, A ning ixtiyoriy atrofida $\{M\}$ to‘plamning A dan farqli nuqtasi mavjud ekan. Shuning uchun, 16.8- ta’rifga ko‘ra, $A(a;b)$ nuqta, $\{M\}$ to‘plamning limit nuqtasi bo‘ladi.

Mustaqil yechish uchun misollar

Quyidagi tengsizliklar bilan berilgan nuqtalar to‘plamining geometrik o‘rnini aniqlang:

16.1. $y\le 2x+4$.

16.2. $y^2\ge 6x$.

16.3. $(x-4)^2+(y+6)^2\le 25$.

16.4. $x^2+6x+y^2 2y-26>0$

16.5. $\begin{cases} x^2+y^2>9, \\ x^2+y^2<16. \end{cases}$

16.6. $\begin{cases} 0<x^2+y^2<25, \\ y>2x^2. \end{cases}$

16.7. A to‘plam, $x^2+y^2\ge 1$ tengsizlik, $B-x^2+y^2\le 4$ tengsizlik, $D-y\le 8-x^2$ tengsizlik, $C-y\ge x^2$ tengsizlik bilan aniqlanganda,

$a)\ (A\cup B)\cap(C\cup D)$; $\qquad b)\ (A\cap B)\cup(C\cap D)$;

to‘plamlarning geometrik o‘rnini aniqlang.

16.8. $x^2-4x+y^2+6y=0$ aylana va $x+2y+1=0$ to‘g‘ri chiziq bilan chegaralangan aylanma segmentning nuqtalari to‘plamini tengsizliklar sistemasi yordamida ifodalang.

16.9. Uchlari, $A(-1;2)$, $B(3;7)$, $C(6;4)$, $D(0;-2)$ nuqtalarda bo‘lgan to‘rtburchakning nuqtalari to‘plamini tengsizliklar sistemasi yordamida ifodalang.

16.10. Uchlari $A(-3;1)$, $B(2;4)$, $C(6;2)$, $D(1;-1)$ nuqtalarda bo‘lgan $ABCD$ parallelogrammning nuqtalari to‘plamini tengsizliklar sistemasi yordamida ifodalang.

16.11. $y=x^2-6$ paraboladan $y=2x+1$ to‘g‘ri chiziq yordamida kesib olingan segmentning nuqtalari to‘plamini tengsizliklar sistemasi yordamida ifodalang.

16.12. Tekislikda $x-3y+4=0$ to‘g‘ri chiziqdan yuqorida, markazi $M_0(2;-1)$ nuqtada, radiusi 10 ga teng bo‘lgan doiradan tashqarida joylashgan nuqtalar to‘plamini tengsizliklar sistemasi yordamida ifodalang.

16.13. Ushbu $\frac{x^2}{25}+\frac{y^2}{16}=1$ ellipsda va uning ichida yotgan nuqtalar, uchlari $A(0;5)$, $B(-3;6)$, $C(3;0)$ nuqtalarda bo‘lgan uchburchakning nuqtalari hamda $y=x^2$ paraboladan yuqorida yotgan nuqtalar to‘plamining umumiy qismidan iborat bo‘lgan, uchta to‘plamning nuqtalari to‘plamini tengsizliklar sistemasi yordamida ifodalang.

Quyidagi berilgan to‘plamlarni bitta yoki bir nechta, $\begin{cases} a\le x\le b \\ \varphi(x)\le y\le \psi(x) \end{cases}$ ko‘rinishdagi tengsizliklar sistemasi yordamida ifodalang:

16.14. Tomonlari: $x=3$, $x=5$, $3x-2y+4=0$, $6x-4y+2=0$ bo‘lgan parallelogram.

16.15. $x\ge 0$, $y\ge 0$, $x^2+y^2\le 4$ tengsizliklar bilan berilgan soha.

16.16. $\frac{x^2}{25}+\frac{y^2}{16}\le 1-$ ellipsning ichki qismi.

16.17. $y=x^2$, $y=\sqrt{x}$ parabolalar bilan chegaralangan soha.

16.18. $y^2=6x$ parabola va $x=2$ to‘g‘ri chiziq bilan chegaralangan soha.

16.19. $x=2$, $y=x$ to‘g‘ri chiziqlar va $xy=1$ giperbola bilan chegaralangan soha.

Fazoda tengsizlik yoki tengsizliklar sistemasi yordamida berilgan nuqtalar to‘plamining geometrik o‘rnini aniqlang:

16.20. $\begin{cases} x>0, \\ y>0, \\ z>0. \end{cases}$

16.21. $xyz>0.$

16.22. $x^2+y^2+z^2\le 9.$

16.23. $4\le x^2+y^2+z^2\le 16.$

16.24. $\begin{cases} x^2+y^2+z^2<25, \\ x^2+y^2<3. \end{cases}$

16.25. Fazodagi $A(4;1;5)$ nuqtadan $2x+6y+3z-12=0$ tekislikka parallel tekislik o‘tkazilgan. Aylanma paraboloidni shu tekislik bilan kesganda hosil bo‘lgan sohani tengsizliklar sistemasi orqali ifodalang.

Quyidagi berilgan to‘plamlarni:

$$\begin{cases} a \le x \le b, \\ \varphi(x) \le y \le \psi(x,) \\ \Phi(x, y) \le z \le \Psi(x, y) \end{cases}$$

tengsizliklar sistemasi ko'rinishida ifodalang.

16.26. $\frac{x^4}{25}+\frac{y^4}{16}+\frac{z^4}{9}=1$ sirt bilan chegaralangan soha.

16.27. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2+\frac{z^2}{c^2}=1$ sirt bilan chegaralangan soha.

16.28. $x=0,\ y=0,\ z=0,\ x=2,\ y=4,\ x+y+z=8$ tekisliklar bilan chegaralangan soha.

16.29. $2az=x^2+y^2$ paraboloid va $x^2+y^2+z^2=3a^2$ shar bilan chegaralangan soha.

16.30. $x^2+y^2+z^2=R^2$ shar va $x^2=y^2+z^2(y>0)$ konus bilan chegaralangan soha.

16.31. $\{M\}$ - tekislikdagi ikkala koordinatasi ham irrasional bo'lgan nuqtalar to'plami bo'lsin. U holda, tekislikning ixtiyoriy $A(a;b)$ nuqtasi, $\{M\}$ to'plamning limit nuqtasi ekanligini isbotlang.

16.32. $\{M\}$ - tekislikda absissasi rasional, ordinatasi esa, irrasional bo'lgan barcha nuqtalar to'plami bo'lsin. U holda, tekislikning ixtiyoriy $A(a;b)$ nuqtasi $\{M\}$ to'plamning limit nuqtasi ekanligini isbotlang.

16.33. Tekislikning $A\left(\frac{1}{m};\frac{1}{n}\right)$ $(m,n=1,2,......)$ ko'rinishdagi barcha nuqtalaridan iborat bo'lgan $\{M\}$ to'plamning limit nuqtalarini toping. Koordinatalar boshi $\{M\}$ to'plamning limit nuqtasi bo'ladimi?

16.34. $A(3;4)$ va $B(-1;2)$ nuqtalar va faqat shu nuqtalar limit nuqtalari bo'lgan, hech bo'lmaganda, bitta to'plam tuzing.

16.35. Hamma limit nuqtalari faqat $A(0;n)$ ($n-$ butun son) ko'rinishda bo'lgan to'plamni tuzing.

16.36. Ikkita ochiq to'plamlarning yig'indisi ochiq to'plam ekanligini isbotlang.

16.37. Ikkita yopiq to'plamlarning yig'indisi yopiq to'plam ekanligini isbotlang.

16.38. Koordinatalari quyidagi $x+y>5,\ x^2+y^2<100$ tengsizliklarni qanoatlantiradigan nuqtalar to'plamining ochiq to'plam ekanligini isbotlang.

16.39. Koordinatalari ushbu $x+y\ge 5,\ x^2+y^2\le 100$ tengsizliklarni qanoatlantiradigan nuqtalar to'plamining yopiq to'plam ekanligini isbotlang.

Quyidagi tengsizliklar orqali aniqlangan to‘plamlarning qaysi biri soha bo‘ladi?

16.40. $4 < x^2 + y^2 < 25.$ **16.41.** $\begin{cases} 4 < x^2 + y^2 < 10, \\ -1 < y < 1. \end{cases}$

Mustaqil yechish uchun misollarning javoblari

16.1. $y = 2x + 4$ to‘g‘ri chiziq va undan pastdagi nuqtalar.

16.2. $y^2 = 6x$ parabola ustida yotuvchi va undan tashqaridagi nuqtalar to‘plami.

16.3. Markazi $M_0(4;-6)$ nuqtada, radiusi 5 ga teng bo‘lgan aylanada va uning ichida yotgan nuqtalar to‘plami. **16.4.** Markazi $M_0(-3;1)$ nuqtada, radiusi 6 ga teng bo‘lgan aylanadan tashqarida joylashgan nuqtalar to‘plami. **16.5.** Markazi $M_0(0;0)$ nuqtada, radiuslari 3 va 4 bo‘lgan konsentrik aylanalar orasidagi nuqtalar to‘plami. **16.6.** Markazi $M_0(0;0)$ nuqtada, radiusi 5 ga teng bo‘lgan aylananing ichidagi hamda $y = 2x^2$ paraboladan yuqorida joylashgan nuqtalar to‘plami.

16.8. $\begin{cases} (x-2)^2 + (y+3)^2 \le 13, \\ x \le -2y - 1. \end{cases}$ **16.9.** $\begin{cases} y \le \frac{5}{4}x + \frac{13}{4}, \\ y \le -x + 10, \\ y \ge x - 2, \\ y \ge -4x - 2. \end{cases}$

16.16. $\begin{cases} \frac{1}{9}x + 1\frac{1}{3} \le y \le \frac{1}{9}x + 3\frac{7}{9}, \\ \frac{3}{5}x - 1\frac{3}{5} \le y \le \frac{3}{5}x + 2\frac{4}{5}. \end{cases}$ **16.11.** $\begin{cases} y \ge x^2 - 6x - 4, \\ y \le 2x + 1. \end{cases}$

16.12. $\begin{cases} y > \frac{1}{3}(x+4), \\ (x-2)^2 + (y+1)^2 > 100. \end{cases}$ **16.13.** $\begin{cases} \frac{x^2}{25} + \frac{y^2}{16} \le 1, \\ y > x^2, \\ y \le -\frac{1}{3}x + 5, \\ y \le -\frac{5}{3}x + 5, \\ y \ge -x + 3. \end{cases}$

16.14. $\begin{cases} 3 \le x \le 5, \\ \frac{3}{2}x + \frac{1}{2} \le y \le \frac{3}{2}x + 2. \end{cases}$ **16.15.** $\begin{cases} 0 \le x \le 2, \\ 0 \le y \le \sqrt{4 - x^2}. \end{cases}$

16.16. $\begin{cases} -5 \le x \le 5, \\ -\frac{4}{5}\sqrt{25-x^2} < y < \frac{4}{5}\sqrt{25-x^2}. \end{cases}$ **16.17.** $\begin{cases} 0 \le x \le 1, \\ x^2 \le y \le \sqrt{x}. \end{cases}$

16.18. $\begin{cases} 0 \le x \le 2, \\ -\sqrt{6x} \le y \le \sqrt{6x}. \end{cases}$ **16.19.** $\begin{cases} 0 \le x \le 1, \\ 0 \le y \le x, \end{cases}$ $\begin{cases} 1 \le x \le 2, \\ 0 \le y \le \frac{1}{x}. \end{cases}$

16.20. Birinchi oktantdagi nuqtalar (koordinatalar tekisliklaridan tashqari).**16.21.** Birinchi, uchinchi, oltinchi va sakkizinchi oktantdagi nuqtalar (koordinatalar tekisliklaridan tashqari) . **16.22.** Markazi koordinatalar boshida, radiusi 3 ga teng bo'lgan sharning nuqtalari.

16.23. Radiuslari 2 va 4 ga teng bo'lgan konsentrik sferalar bilan chegaralangan soha (bu sferalar ham kiradi). **16.24.** Markazi koordinatalar boshida, radiusi 5 ga teng bo'lgan sfera hamda o'qi Oz va radiusi 3 ga teng bo'lgan doiraviy silindr bilan chegaralangan soha (soha chegaralaridan tashqari).

16.25. $x^2 + y^2 \le z \le 2x + 6y + 3z - 29$.**16.26.** $\begin{cases} -5 \le x \le 5, \\ -\sqrt[4]{16\left(1-\frac{x^4}{25}\right)} \le y \le \sqrt[4]{16\left(1-\frac{x^4}{25}\right)}, \\ -\sqrt[4]{9\left(1-\frac{x^4}{25}-\frac{y^4}{16}\right)} \le z \le \sqrt[4]{9\left(1-\frac{x^4}{25}-\frac{y^4}{16}\right)}. \end{cases}$

16.27. $\begin{cases} -a \le x \le a, \\ -\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}, \\ -c\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2} \le z \le c\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2} \end{cases}$ **16.28.** $\begin{cases} 0 \le x \le 2, \\ 0 \le y \le 4, \\ 0 \le z \le 8-x-y. \end{cases}$

16.29. $\begin{cases} -a \le x \le a, \\ -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}, \\ \frac{1}{2a}(x^2+y^2) \le z \le \sqrt{3a^2-x^2-y^2}. \end{cases}$ **16.30.** $\begin{cases} 0 \le x \le R\frac{\sqrt{2}}{2}, \\ -x \le y \le x, \\ -\sqrt{x^2-y^2} \le z \le \sqrt{x^2-y^2}. \end{cases}$

16.34. $A_n\left(3+\frac{1}{n};4\right)$ va $B_n\left(-1+\frac{1}{n};2\right)$ $n=1,2,\ldots$. **16.35.** Misol uchun, $A_{mn}\left(\frac{1}{m};n\right)$.

16.40. Soha. **16.41.** Soha emas.

17-§. Ko‘p o‘zgaruvchili funksiya tushunchasi va uning aniqlanish sohasi

$\{M\}$ ($\{M\}\subset R^2$) to‘plam berilgan bo‘lsin.

17.1-ta'rif. $\{M\}$ to‘plamning har bir $M(x,y)$ nuqtasiga biror qonun yoki qoida yordamida u son ($u\in R$) mos qo‘yilgan bo‘lsa, $\{M\}$to‘plamda ikki o‘zgaruvchili $u=u(M)$ yoki $u=f(M)=f(x,y)$ *funksiya aniqlangan* deyiladi. Bunda $\{M\}$ to‘plam- funksiyaning aniqlanish sohasi, $\{u\}$ to‘plam esa, funksiyaning *qiymatlar to‘plami* yoki *o‘zgarish sohasi* deyiladi.

$\{M\}\subset R^m$ to‘plam berilgan bo‘lsin.

17.2-ta'rif. $\{M\}$ to‘plamning har bir $M(x_1,x_2,...,x_m)$ nuqtasiga biror qonun yoki qoida yordamida biror u son $(u\in R)$ mos qo‘yilgan bo‘lsa, $\{M\}$to‘plamda m o‘zgaruvchili *funksiya aniqlangan* deyiladi va u $u=u(M)$ yoki $u=f(M)=f(x_1,x_2,...,x_m)$ kabi belgilanadi. Bunda $\{M\}$ to‘plam funksiyaning *aniqlanish sohasi* deyiladi. $\{u\}$ to‘plam esa, uning *qiymatlar to‘plami yoki o‘zgarish sohasi* deyiladi. Bundan buyon, funksiyaning aniqlanish sohasini $D(f)$, o‘zgarish sohasini esa, $E(f)$ orqali belgilaymiz.

17.3-ta'rif. $f(M)=f(x_1,x_2,...,x_m)=C\,(C\in R)$ shartni qanoatlantiruvchi $M\in E\,(E\subset R^m)$ nuqtalar to‘plamiga, $f(M)$ funksiyaning $C-$ *sathi* deyiladi. Ko‘p hollarda, ikki o‘zgaruvchili $f(x,y)$ funksiyaning C – sathi, *sath chizig‘i*, uch o‘zgaruvchili $f(x,y,z)$ funksiyaning $C-$ sathi esa, *sath sirti* deb yuritiladi.

$u=f(M)$ funksiya E sohada $(M\in E\subset R^m)$ berilgan bo‘lsin.

17.4–ta'rif. Agar $\forall M\in E,\ \lambda M\in E,\ \lambda\in R$ uchun

$$f(\lambda M)=f(\lambda x_1,\lambda x_2,...,\lambda x_m)=\lambda^\alpha f(x_1,x_2,...,x_m)$$

tenglik o‘rinli bo‘lsa, u holda $f(M)-$ α darajali *bir jinsli funksiya* deyiladi.

Agar yuqoridagi shartlarda $f(\lambda M)=|\lambda|^m f(M)$ tenglik o‘rinli bo‘lsa, $f(M)$ $m-$ darajali *musbat birjinsli funksiya* deyiladi.

Masalan, $f(x)=x,\ x\in R,$ 1- darajali birjinsli funksiya, $f(x)=|x|,\ x\in R,$ 1- darajali *musbat birjinsli* fusnksiyadir.

17.1-misol . Quyida berilgan funksiyalarning aniqlanish sohasi va o‘zgarish sohasini toping:

1) $u=\sqrt{16-x^2-y^2}$; 2) $u=\ln xy$; 3) $u=x_1^2+x_2^2+...+x_m^2$.

Yechilishi. 1) Berilgan funksiya tekislikning koordinatalari $16-x^2-y^2\geq 0$ tengsizlikni qanoatlantiradigan $M(x,y)$ nuqtalar to‘plamida aniqlangan. Bu tengsizlik, $x^2+y^2\le 16$ tengsizlikka teng kuchli. Oxirgi tengsizlik, markazi $M_0(0,0)$ nuqtada, radiusi 4 ga teng bo'lgan doirani ifoda qiladi. Demak, funksiyaning aniqlanish sohasi $D(u)$: tekislikdagi markazi koordinatalar boshida, radiusi esa, 4 ga teng bo'lgan yopiq doiradan, qiymatlar to'plami yoki o'zgarish sohasi $E(u)$: $[0;4]$ – segmentdan iborat ekan.

2) $u=\ln xy$ funksiya, tekislikning koordinatalari $xy>0$ tengsizlikni qanoatlantiradigan $M(x,y)$ nuqtalar to'plamidan iborat. Oxirgi tengsizlik, $a)\, x>0, y>0;\ b)\, x<0,\ y<0$ tengsizliklar sistemasiga teng kuchli.

Demak, berilgan funksiyaning aniqlanish sohasi, koordinatalar tekisligining birinchi va uchinchi choraklaridan iborat (koordinatalar o'qlari kirmaydi), o'zgarish sohasi $E(u)$ esa, $-\infty<u<\infty$ son o'qidan iborat.

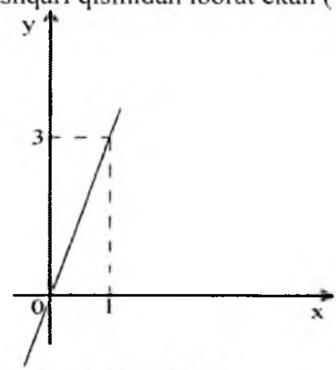
3) $u=x_1^2+x_2^2+\ldots+x_m^2$ funksiya R^m fazoda aniqlangan bo'lib, uning o'zgarish sohasi, $[0,\infty)$ dan iborat.

17.2-misol. Quyida berilgan funksiyalarning aniqlanish sohasi toping va uni chizmada ko'rsating.

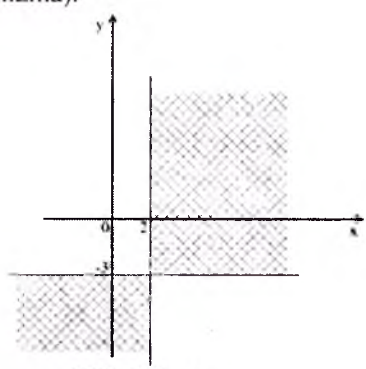
1) $u=\dfrac{x+y}{3x-y}$; 2) $u=\sqrt{(x-2)(y+3)}$; 3) $u=\ln x-\ln y$;

4) $u=\sqrt{\dfrac{x^2+y^2-x}{2x-x^2-y^2}}$; 5) $u=\dfrac{x^2+y-z}{\sqrt{9-x^2-y^2-z^2}}$.

Yechilishi. 1) $u=\dfrac{x+y}{3x-y}$ funksiya, x va y o'zgaruvchilarning, kasrning maxrajini nolga aylantirmaydigan qiymatlari to'plamida aniqlangan, ya'ni $D(u)$ butun tekislikning $y=3x$ to'g'ri chiziqdan tashqari qismidan iborat ekan (17.1-chizma).



17.1-chizma.

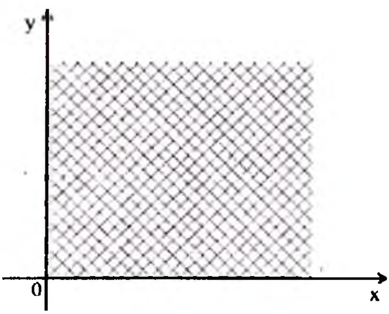


17.2-chizma.

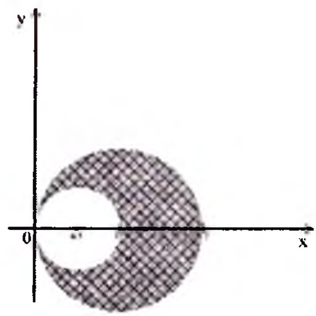
2) $u=\sqrt{(x-2)(y+3)}$ funksiyaning ma'noga ega bo'lishi uchun, $(x-2)(y+3)\geq 0$ tengsizlik o'rinli bo'lishi kerak. Bu tengsizlik, $\begin{cases} x-2\geq 0, \\ y+3\geq 0 \end{cases}$ yoki $\begin{cases} x-2\leq 0 \\ y+3\leq 0 \end{cases}$ tengsizliklar sistemasiga teng kuchli.

Birinchi sistemaning Yechimi, $x\geq 2, y\geq -3$ dan iborat; ikkinchi sistemaning Yechimi esa, $x\leq 2,\ y\leq -3$ dan iborat. Izlanayotgan sohani, ya'ni berilgan funksiyaning aniqlanish sohasini, topish uchun koordinatalar tekisligida $x=2$ va $y=-3$ to'g'ri chiziqlarni chizish etarli (17.2- chizma).

3) $u=\ln x-\ln y$ funksiyaning ma'noga ega bo'lishi uchun, $x>0, y>0$ tengsizliklar o'rinli bo'lishi kerak. Demak, berilgan funksiyaning aniqlanish sohasi, koordinatalar tekisligining birinchi choragidan iborat (koordinatalar o'qlari kirmaydi) (17.3-chizma)



17.3-chizma.



17.4-chizma.

4) $u=\sqrt{\dfrac{x^2+y^2-x}{2x-x^2-y^2}}$ funksiyaning aniqlanish sohasi, tekislikda koordinatalari, $\dfrac{x^2+y^2-x}{2x-x^2-y^2}\geq 0$ tengsizlikni qanoatlantiradigan (x,y) nuqtalar to'plamidan iborat. Bu tengsizlik quyidagi

$$\begin{cases} x^2+y^2-x\geq 0, \\ 2x-x^2-y^2>0; \end{cases} \quad \begin{cases} x^2+y^2-x\leq 0, \\ 2x-x^2-y^2<0 \end{cases}$$

yoki

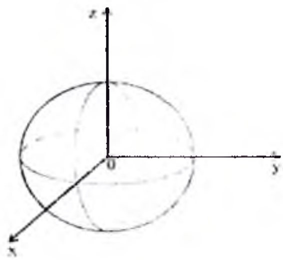
$$\begin{cases} \left(x-\dfrac{1}{2}\right)^2+y^2\geq \dfrac{1}{4}, \\ (x-1)^2+y^2<1; \end{cases} \quad \begin{cases} \left(x-\dfrac{1}{2}\right)^2+y^2\leq \dfrac{1}{4}, \\ 1<(x-1)^2+y^2; \end{cases}$$

ikkita tengsizliklar sistemasiga teng kuchli.

Birinchi tengsizliklar sistemasini markazi $M_0\left(\frac{1}{2};0\right)$ nuqtada, radiusi $\frac{1}{2}$ ga teng bo'lgan yopiq doiraning tashqarisidagi nuqtalarning, va markazi $M_1(1;0)$ nuqtada, radiusi 1 ga teng bo'lgan ochiq doiraning ichkarisidagi nuqtalarning, koordinatalari qanoatlantiradi; ikkinchi tengsizliklar sistemasini esa, markazi $M_0\left(\frac{1}{2};0\right)$ nuqtada, radiusi $\frac{1}{2}$ ga teng bo'lgan yopiq doiraning ichidagi nuqtalarning va markazi $M_1(1;0)$ nuqtada, radiusi 1 ga teng bo'lgan ochiq doiraning tashqarisidagi nuqtalarning koordinatalari qanoatlantiradi (17.4-chizma).

Shunday qilib, berilgan funksiyaning aniqlanish sohasi: koordinatalari $x \le x^2 + y^2 < 2x$ tengsizlikni qanoatlantiruvchi nuqtalar to'plamidan iborat ekan.

5) $u = \dfrac{x^2 + y - z}{\sqrt{9 - x^2 - y^2 - z^2}}$ funksiya ma'noga ega bo'lishi uchun, x, y, z o'zgaruvchilar $9 - x^2 - y^2 - z^2 > 0$ yoki $x^2 + y^2 + z^2 < 9$ tengsizlikni qanoatlantirishi kerak. Bu tengsizlik, fazoda markazi koordinatalar boshida, radiusi esa, 3 ga teng bo'lgan ochiq sharni ifodalaydi (17.5-chizma).



17.5-chizma.

17.3-misol $u = \sqrt{-1 - x^2 - y^2}\left(\sin^2 \pi x + \sin^2 \pi y\right)$ funksiyaning aniqlanish sohasini toping.

Yechilishi. $\sqrt{-1 - x^2 - y^2}$ – ifoda x va y ning ixtiyoriy haqiqiy qiymatlarida mavhum sonni ifodalaydi. Shuning uchun berilgan funksiyaning aniqlanish sohasi x va y ning

$$\sin^2 \pi x + \sin^2 \pi y = 0 \qquad (*)$$

tenglamani qanoatlantiradigan qiymatlar to'plamidan iborat bo'ladi. (*) dan $\sin \pi x = 0$, $\sin \pi y = 0$. Bu tenglamalarni, x va y ning butun qiymatlari qanoatlantiradi, ya'ni $A(n;m)$ nuqtalar to'plami (n va m butun sonlar), berilgan funksiyaning aniqlanish sohasini ifodalaydi.

17.4 -misol. Quyidagi berilgan funksiyalarning C – sath chiziqlari topilsin:

1) $u = xy$, 2) $u = \sqrt{y - x}$,

3) $u = \sqrt{36 - 4x^2 - 9y^2}$; 4) $u = \dfrac{2z}{x^2 + y^2 + z^2}$ $\left(x^2 + y^2 + z^2 \ne 0\right)$

Yechilishi. 1) $u=xy$ funksiyaning sath chiziqlari oilasi - x va y larning, $xy=C$ tenglamani qanoatlantiradigan (x,y) qiymatlari to'plamidan iborat bo'ladi, bunda $C\in R$. C ga har xil qiymatlar berish natijasida, har xil sath chiziqlarini hosil qilamiz. Masalan, $C=1,2,...,n$ qiymatlarda $xy=1, xy=2,...,xy=n$, birinchi va uchinchi choraklarda joylashgan giperboloidlar oilasini; $C=-1,-2,...,-n$ qiymatlarda esa, $xy=-1,\ xy=-2,....,\ xy=-n$ ikkinchi va to'rtinchi choraklarda joylashgan giperboloidlar oilasini hosil qilamiz. $c=0$ da $xy=0$, ya'ni sath chiziqlari $x=0$ va $y=0$ to'g'ri chiziqlardan iborat bo'ladi.

2) $u=\sqrt{y-x}, \sqrt{y-x}=C$, bundan $y=x+C^2$. Demak, berilgan $u=\sqrt{y-x}$ funksiyaning sath chizig'i, $C\geq 0$ bo'lganda, $(0;C^2)$ va $(1;\ 1+C^2)$ nuqtalardan o'tuvchi to'g'ri chiziqdan, $C<0$ bo'lganda esa, ϕ - bo'sh to'plamdan iborat.

3) $\sqrt{36-4x^2-9y^2}=C,\ 36-4x^2-9y^2=C^2,\ 4x^2+9y^2=36-C^2$. Agar $C\in[0;6]$ bo'lsa, berilgan funksiyaning sath chizig'i- markazi $(0;0)$ nuqtada, fokusi Ox o'qda, yarim o'qlari esa, mos ravishda, $\frac{\sqrt{36-c^2}}{2}, \frac{\sqrt{36-c^2}}{3}$ bo'lgan ellipsdan; $C=6$ bo'lganda, $(0;0)$ nuqtadan; $C\notin[0;6]$ bo'lganda, ϕ - bo'sh to'plamdan iborat bo'ladi.

4) $\frac{2z}{x^2+y^2+z^2}=C,\ x^2+y^2+\left(z-\frac{1}{C}\right)^2=\frac{1}{C^2}$

Agar $C\neq 0$ bo'lsa, u holda, berilgan funksiyaning sath sirti – radiusi $\frac{1}{|C|}$, markazi $\left(0;0;\frac{1}{C}\right)$ nuqtada bo'lgan sferadan ((0;0;0) nuqtadan tashqari) iborat; agar $C=0$ bo'lsa, $(0;0;0)$ nuqtadan tashqari $z=0$ tekislikdan iborat bo'ladi.

17.5-misol. Ushbu $u=x^{\sqrt{2}}\sin\frac{y}{x}+y^{\sqrt{2}}\cos\frac{x}{y}$ funksiyaning birjinsli ekanligini isbotlang va uning birjinslilik darajasini toping.

Yechilishi. x ni λx ga, y ni λy ga almashtirib, topamiz:

$$u=(\lambda x)^{\sqrt{2}}\sin\frac{(\lambda y)}{(\lambda x)}+(\lambda y)^{\sqrt{2}}\cos\frac{(\lambda x)}{(\lambda y)}=\lambda^{\sqrt{2}}\cdot x^{\sqrt{2}}\cdot\sin\frac{y}{x}+\lambda^{\sqrt{2}}\cdot y^{\sqrt{2}}\cos\frac{x}{y}=$$

$$=\lambda^{\sqrt{2}}\left(x^{\sqrt{2}}\cdot\sin\frac{y}{x}+y^{\sqrt{2}}\cdot\cos\frac{y}{x}\right)=\lambda^{\sqrt{2}}\cdot u.$$

Demak, 17.4-ta'rifga ko'ra, berilgan funksiya birjinsli va uning birjinslilik darajasi $\lambda=\sqrt{2}$.

17.5-misol. Ushbu $u=\frac{xy+zt}{xyz+yzt}$ funksiyaning birjinsli ekanligi isbotlang va uning birjinslilik darajasini toping.

Yechilishi. x ni λx ga, y ni λy ga, z ni λz ga, t ni λt ga almashtirib, topamiz:

$$u=\frac{\lambda x\cdot\lambda y+\lambda z\cdot\lambda t}{\lambda x\cdot\lambda y\cdot\lambda z+\lambda y\cdot\lambda z\cdot\lambda t}=\frac{\lambda^2 xy+\lambda^2 zt}{\lambda^3 xyz+\lambda^3 yzt}=\frac{1}{\lambda}\cdot\frac{xy+zt}{xyz+yzt}=\lambda^{-1}u.$$

Demak, berilgan funksiya birjinsli va uning bir jinslilik darajasi $\lambda=-1$.

Mustaqil yechish uchun misollar

Quyidagi funksiyaning aniqlanish sohasini toping va chizmasini chizing:

17.1. $u=\sqrt{1-x^2-y^2}$.

17.2. $u=\sqrt{1-x^2}+\sqrt{y^2-1}$.

17.3. $u=\frac{1}{x+y}$.

17.4. $u=\frac{x}{\sqrt{x-y}}+\frac{y}{\sqrt{x+y}}$.

17.5. $u=arc\cos\frac{x^2+y^2}{9}$.

17.6. $u=\ln(y^2-4x+8)$.

17.7. $u=\sqrt{R^2-x^2-y^2}+\frac{1}{\sqrt{x^2+y^2-r^2}}$.

17.8. $u=\frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$.

17.9. $u=\sqrt{(x^2+y^2-1)(4-x^2-y^2)}$.

17.10. $u=\sqrt{\frac{x^2+y^2-x}{2x-x^2-y^2}}$.

Quyidagi funksiyaning aniqlanish sohasini toping:

17.17. $u=\ln(1-x-y-z)$.

17.12. $u=\frac{1}{\sqrt{1-x^2-y^2-z^2}}$.

17.13. $u=arc\cos\frac{z}{\sqrt{x^2+y^2}}$.

17.14. $u=\sqrt{8-x^2-2y^2-4z^2}$.

17.15. $u=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}$.

17.16. $u=\arcsin\frac{x}{a}+\arcsin\frac{y}{b}+\arcsin\frac{z}{c}$.

17.17. $u=xy+\sqrt{\ln\frac{9}{x^2+y^2}}+\sqrt{x^2+y^2-9}$.

17.18. $u=\frac{1}{\sqrt{z-x^2-y^2}}$.

17.19. $u=\sqrt{2(x^2+y^2+z^2)-(x^2+y^2+z^2)^2-1}$.

17.20. $u=\frac{\ln x+\ln z}{\sqrt{y-1}}+\ln(5-x-y-z)$.

Quyidagi $u=f(x,y)$ funksiyaning $(x,y)\in E$ to‘plamdagi qiymatlari to‘plamini (o‘zgarish sohasi) toping $((x,y)\in E\subset R^2)$:

17.21. $u=x-2y-3;\ E=\{(x,y),x+y=1,\ x\geq 0,\ y>0\}$.

17.22. $u = x^2 - xy + y^2, E = \{(x, y): |x| + |y| = 1\}$.

17.23. $u = x^2 + y^2 - 12x + 16y + 25, E = \{(x, y): x^2 + y^2 = 25\}$

17.24. $u = \ln(2x^2 + 3y^2), E = \{(x, y): x + y = 2, x \geq 0, y \geq 0\}$.

Quyidagi funksiyaning sath chiziqlarini toping:

17.25. $u = y - x$.

17.26. $u = \sqrt{y - x}$.

17.27. $u = x^2 + y^2$.

17.28. $u = x^2 - y^2$.

17.29. $u = \dfrac{1}{\sqrt{x^2 - y^2}}$.

17.30. $z = \dfrac{1}{x^2 + 2y^2}$.

17.31. $u = \sqrt{y - \sin x}$.

Quyidagi misollarda berilgan funksiyaning aniqlanish va o'zgarish sohalarini toping hamda uning sath chiziqlarini aniqlang. Sath chiziqlaridan birini chizing:

17.32. $f(x, y) = 9x^2 + y^2$.

17.33. $f(x, y) = \dfrac{1}{xy}$..

Quyidagi funksiyaning sath sirtlarini toping.

17.34. $u = x + y + z$.

17.35. $u = x^2 + y^2 + z^2$.

17.36. $u = \dfrac{1}{x^2 + y^2 + z^2 + 2x}$.

17.37. $u = \ln(x^2 + y^2 + z^2)$

17.38. $u = \dfrac{2z}{x^2 + y^2 + z^2}$. **17.39.** $u = \sqrt{(x+1)^2 + y^2 + z^2} + \sqrt{(x-1)^2 + y^2 + z^2}$.

Quyidagi misollarda berilgan funksiyaning aniqlanish va o'zgarish sohalarini toping hamda uning sath sirtlarini aniqlang. Sath sirtlaridan birini chizing:

17. 40. $f(x, y, z) = x^2 + y^2 - z$.

17.41. $g(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

Quyidagi funksiyani bir jinslilikka tekshiring va uning birjinslilik darajasini toping:

17.42. $u = \dfrac{\sqrt{x^2 + y^2}}{x}, x > 0$.

17.43. $u = \dfrac{x^2 - 3xy + y^2}{\sqrt{x^2 + 2xy - y^2}}$.

17.44. $u = \sqrt[4]{x^6 z^2 + 2x^3 y^4 z + xy^2 \cdot z^5}$.

17.45. 4) $u = \sqrt{x^2 + y^2 + z^2 - t^2}$.

17.46. $u = \dfrac{xy + zt}{xyz + yzt}$.

17.47. $u = \sum_{i=1}^{n-1} x_i^k (\ln x_{i+1} - \ln x_i)$.

17.48. $u = x^2 y + z^2 + yz$.

17.49. x, y o'zgaruvchilarning har qanday λ juft darajali birjinsli funksiyasini, $f(x; y) = |x|^\lambda \varphi\left(\dfrac{y}{x}\right)$ ko'rinishda tasvirlash mumkinligini ko'rsating.

17.50. x,y o'zgaruvchilarning har qanday λ toq darajali bir jinsli funksiyasini $f(x;y)=|x|^{\lambda-1}x\varphi\left(\frac{y}{x}\right)$ ko'rinishda tasvirlash mumkinligini ko'rsating.

17.51. Ushbu $f(x,y,z)=|x|^{\lambda}\varphi\left(\frac{y}{x},\frac{z}{x}\right)$ funksiyaning λ juft darajali birjinsli ekanligini isbotlang.

17.52. x,y,z o'zgaruvchilarning har qanday toq darajali birjinsli funksiyasini $f(x;y,z)=|x|^{\lambda-1}x\cdot f\left(\frac{y}{x};\frac{z}{x}\right)$ ko'rinishda tasvirlash mumkinligini isbotlang.

Mustaqil yechish uchun misollarning javoblari

17.1. Markazi $(0,0)$ nuqtada, radiusi 1 ga teng bo'lgan yopiq doira. **17.2.** $|x|\le 1,\ |y|\ge 1$. **17.3.** $y=-x$ to'g'ri chiziq nuqtalaridan tashqari tekislikning hamma nuqtalari. **17.4.** Burchak bissektrisalari bilan chegaralangan o'ng vertikal burchakning ichki qismi. **17.5.** Markazi koordinatalar boshida, radiusi 3 ga teng bo'lgan doira. **17.6.** Uchi (2;0) nuqtada va fokusi (3;0) nuqtada bo'lgan parabolaning tashqi qismi. **17.7.** Tekislikning $x^2+y^2=R^2$ va $x^2+y^2=r^2$ aylanalar orasidagi qismi.**17.8.**Tekislikning $y^2=4x$ parabolaning ichi bilan $x^2+y^2=1$ aylana orasidagi qismi (parabola yoyi kiradi, aylana yoyi kirmaydi). **17.9.** $1\le x^2+y^2\le 4$ - halqa.**17.10.** $x\le x^2+y^2<2x$ - oycha. **17.11.** $(1;0;0)$, $(0;1;0)$, $(0;0;1)$ nuqtalardan o'tuvchi tekisliklar bilan chegaralangan va $(0;0;0)$ nuqtani o'zida saqlovchi ochiq yarim fazo. **17.12.** Markazi $(0;0;0)$ nuqtada, radiusi 1 ga teng bo'lgan ochiq shar. **17.13.** $x^2+y^2-z^2=0$ konusning tashqi tomoni (chegarasi kiradi, uchi kirmaydi). **17.14.** Fazoning $\frac{x^2}{8}+\frac{y^2}{4}+\frac{z^2}{2}=1$-ellipsoid bilan chegaralangan qismi (ellipsoidning nuqtalari kiradi). **17.15.** Birinchi oktantda.**17.16.** $\begin{cases}|x|\le a,\\ |y|\le b,\\ |z|\le c\end{cases}$

17.17. Faqat $x^2+y^2=9$ aylananing nuqtalari. **17.18.** $z=x^2,\ y=0$ parabolaning Oz o'q atrofida aylanishi natijasida hosil bo'lgan aylanma paraboloidning ichki qismi.**17.19.** Markazi $(0;0;0)$ nuqtada, radiusi 1 ga teng bo'lgan sfera. **17.20.** Uchlari (4;1;0) , (0;1;0), (0;5;0), (0;1;4)

nuqtalarda bo‘lgan ochiq piramida.**17.21.** $[-5;-2]$. **17.22.** $\left[\frac{1}{4};1\right]$.
17.23. $[-50;150]$. **17.24.** $\left[\ln\frac{24}{5};\ln 12\right]$. **17.25.** $(0;C)$ va $(1;1+C)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq . **17.26.** Agar $C\geq 0$ bo‘lsa, $(0;C^2)$ va $(1;1+C^2)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq; agar $C<0$ bo‘lsa ϕ- bo‘sh to‘plam.

17.27. Konsentrik aylanalar. **17.28.** $y=\pm x$ umumiy asimptotaga ega bo‘lgan teng tomonli giperbolalar oilasi. **17.29.** $C>0$ bo‘lganda, markazi $(0,0)$ nuqtada, fokusi Ox o‘qda, yarim o‘qi $\frac{1}{C}$ bo‘lgan giperbola; $C\leq 0$ bo‘lganda, ϕ- bo‘sh to‘plam. **17.30.** Ellipsga o‘xshash figuralar oilasi. **17.31.** Agar $C\geq 0$ bo‘lsa, $y=C^2+\sin x$-sinusoida, agar $C<0$ bo‘lsa, ϕ- bo‘sh to‘plam.**17.32.** $D(f)=R^2$, $E(f)=[0,\infty)$, ellipslar. **17.33.** $D(f)=\{(x,y): x\neq 0 \ va \ y\neq 0\}$, $E(f)=(-\infty;0)\cup(0,\infty)$, giperbolalar. **17.34.** Parallel tekisliklar oilasi. **17.35.** Markazi koordinatalar boshida bo‘lgan sferalar oilasi. **17.36.** Agar $C<-1$ yoki $C>0$ bo‘lsa markazi $(-1,0,0)$, radiusi $\sqrt{\frac{C+1}{C}}$ bo‘lan sfera; $C=-1$ bo‘lsa, $(-1,0,0)$ nuqta, $-1<C\leq 0$ bo‘lsa, ϕ- bo‘sh to‘plam. **17.37.** Markazi $(0;0;0)$ nuqtada, radiusi $e^{C/2}$ ga teng bo‘lgan sfera. **17.38.** $C\neq 0$ bo‘lganda, markazi $\left(0;0;\frac{1}{C}\right)$ nuqtada, radiusi $\frac{1}{|c|}$ ga teng bo‘lgan sfera ($(0;0;0)$ nuqta kirmaydi), $C=0$ bo‘lganda, $(0;0;0)$ nuqtadan tashqari $z=0$ tekislik. **17.39.** Agar $C>2$ bo‘lsa, $\frac{x^2}{(C/2)^2}+\frac{y^2+z^2}{(C/2)^2-1}=1$ ellipsoid; $C=2$ bo‘lsa, $[-1;1]$ kesma; $C<2$ bo‘lsa, ϕ- bo‘sh to‘plam. **17. 40.** $D(f)$: $Oxyz$ fazoning barcha nuqtalari, $E(f)$: barcha haqiqiy sonlar, elliptik paraboloid $z=x^2+y^2+1$.
17.41. $D(f)=R^3\setminus\{(0,0,0)\}$, $E(f)$: musbat haqiqiy sonlar, sfera $x^2+y^2+z^2=1$.
17.42. $\lambda=0$. **17.43.** $\lambda=1$. **17.44.** $\lambda=4/3$. **17.45.** $\lambda=1$. **17.46.** $\lambda=-1$. **17.47.** π. **17.48.** Birjinsli emas.

18-§. R^m fazoda sonlar ketma-ketligi va uning limiti

R^m fazo va N- natural sonlar to‘plami berilgan bo‘lsin. Har bir n $(n\in N)$ natural songa, biror qonun yoki qoida yordamida, R^m fazoning biror muayyan $M_n = M_n\left(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\right)\left(M_n \subset R^m\right)$ nuqtasi mos qo'yilgan, ya'ni

$$1 \to \quad M_1\left(x_1^{(1)}, x_2^{(1)}, ..., x_m^{(1)}\right),$$
$$2 \to \quad M_2\left(x_1^{(2)}, x_2^{(2)}, ..., x_m^{(2)}\right),$$
$$\dots\dots\dots\dots$$
$$n \to \quad M_n\left(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\right)$$
$$\dots\dots\dots\dots$$

bo'lsa, R^m fazoda $M_1, M_2, ..., M_n, ...$ *sonlar ketma – ketligi aniqlangan* deyiladi va qisqacha $\{M_n\}\left(M_n \subset R^m\right)$ kabi belgilanadi.

Misollar: 1) $M_n = M_n\left(\frac{1}{n}, \frac{1}{n}\right)$: $M_1(1,1)$, $M_2\left(\frac{1}{2}, \frac{1}{2}\right)$, ..., $M_n\left(\frac{1}{n}, \frac{1}{n}\right)$, ...

2) $M_n = M_n\left(0, \frac{1}{n}\right)$: $M_1(0,1)$, $M_2\left(0, \frac{1}{2}\right)$, ..., $M_n\left(0, \frac{1}{n}\right)$, ...,

3) $M_n = M_n\left(\frac{1}{n}, 0, \frac{1}{n}\right)$: $M_1(1,0,1)$, $M_2\left(\frac{1}{2}, 0, \frac{1}{2}\right)$, ..., $M_n\left(\frac{1}{n}, 0, \frac{1}{n}\right)$, ...

Bu ketma-ketliklarning birinchi va ikkinchisi, R^2 fazoning nuqtalaridan, uchinchisi esa, R^3 fazoning nuqtalaridan tashkil topgan sonlar ketma-ketliklaridir.

R^m fazoda

$$M_1, M_2, ..., M_n, ... \tag{18.1}$$

sohalar ketma-ketligi va $A = A(a_1, a_2, ..., a_m) \in R^m$ nuqta berilgan bo'lsin.

18.1-ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $n_0 \in N$ topilsaki, barcha $n > n_0$ lar uchun

$$\rho(M_n, A) < \varepsilon \tag{18.2}$$

tengsizlik bajarilsa, A nuqta $\{M_n\}$ *ketma-ketlikning limiti* deyiladi va $\lim_{n\to\infty} M_n = A$ yoki $n \to \infty$ da $M_n \to A$ kabi belgilanadi.

Agar (18.1) ketma-ketlik limitga ega bo'lsa, u, *yaqinlashuvchi ketma-ketlik* deyiladi. Limit ta'rifidagi shartni qanoatlantiruvchi A nuqta mavjud bo'lmasa, $\{M_n\}$ ketma-ketlik *limitga ega emas* deyiladi, ketma-ketlikning o'zi esa, *uzoqlashuvchi* deb ataladi. Ketma-ketlikning limiti ta'rifidagi ε ixtiyoriy musbat son bo'lib, izlanayotgan n_0 $(n_0 \in N)$ esa, shu ε ga bog'liq ravishda topiladi, shuning uchun, ba'zi hollarda, $n_0 = n_0(\varepsilon)$ kabi yoziladi.

18.1- misol. R^2 fazoda berilgan $\{M_n\} = \left\{M_n\left(\frac{3}{n}, \frac{1}{n^2}\right)\right\}$ ketma-ketlikning limiti $A = A(0; 0)$ ekanligini ko'rsating.

Yechilishi. $\forall\varepsilon>0$ sonni olaylik. Berilgan ε ga ko'ra, $n_0=\left[\frac{\sqrt{10}}{\varepsilon}\right]+1$ desak, unda $\forall n>n_0$ lar uchun,

$$\rho(M_n;\ A)=\rho\left(M_n\left(\frac{3}{n},\frac{1}{n^2}\right);\ A(0,0)\right)=\sqrt{\left(\frac{3}{n}-0\right)^2+\left(\frac{1}{n^2}-0\right)^2}=\sqrt{\frac{9}{n^2}+\frac{1}{n^4}}<\sqrt{\frac{10}{n^2}}=$$

$$=\frac{\sqrt{10}}{n}<\frac{\sqrt{10}}{n_0}=\frac{\sqrt{10}}{\left[\frac{\sqrt{10}}{\varepsilon}\right]+1}<\varepsilon$$

tengsizlik bajariladi. Demak, $\rho(M_n;\ A)<\varepsilon$. 18.1- ta'rifga ko'ra,

$$\lim_{n\to\infty}M_n=\lim_{n\to\infty}\left(\frac{3}{n},\frac{1}{n^2}\right)=A(0,\ 0)$$

bo'ladi.

18.2-misol. R^2 fazoda, berilgan $\{M_n\}=\{M_n((-1)^{n+1},(-1)^{n+1})\}$ ketma-ketlikning limiti mavjud emasligini ko'rsating.

Yechilishi. Teskarisidan faraz qilaylik, ya'ni berilgan ketma-ketlik limitga ega va uning limiti $A=A(a_1,a_2)$ bo'lsin. Limitning ta'rifiga ko'ra, $\forall\varepsilon>0$, jumladan, $\varepsilon=1$ uchun shunday $n_0\in N$ nomer topiladiki, $\forall n>n_0$ dan boshlab,

$$\rho((1;\ 1),\ (a_1;\ a_2))<\varepsilon,\quad \rho((-1;-1),\ (a_1;\ a_2))<\varepsilon$$

tengsizliklar bajariladi. Bu tengsizliklar yordamida

$$2\sqrt{2}=\rho((1;\ 1),(-1;\ -1),)\le\rho((1;\ 1),(a_1;a_2),)+\rho((-1;-1),(a_1;a_2),)<\varepsilon+\varepsilon=2\varepsilon=2\ \ (\varepsilon=1)$$

ziddiyatni hosil qilamiz.

Bu ziddiyatga sabab, qaralayotgan ketma-ketlik limitga ega, degan farazimizdir. Demak, berilgan ketma-ketlik limitga ega emas.

18.1-teorema. R^m fazoda $\{M_n(x_1^{(n)},x_2^{(n)},...,x_m^{(n)})\}$ ketma-ketlikning $A=A(a_1,a_2,...,a_m)\in R^m$ limitga ega bo'lishi, ya'ni $\lim\limits_{n\to\infty}M_n=A$ uchun, bir vaqtda

$$\lim_{n\to\infty}x_1^{(n)}=a_1$$
$$\dots\dots\dots\dots\dots\dots,$$
$$\lim_{n\to\infty}x_m^{(n)}=a_m$$

bo'lishi zarur va yetarli. Bundan,

$$\lim_{n\to\infty}M_n=A\quad\Rightarrow\begin{cases}\lim\limits_{n\to\infty}x_1^{(n)}=a_1\\ \dots\dots\dots\dots\dots\dots,\\ \lim\limits_{n\to\infty}x_m^{(n)}=a_m\end{cases}$$

ekanligi kelib chiqadi.

18.2-ta'rif. $M_n \subset R^m$ bo'lsin. Agar $\forall\varepsilon>0$ olinganda ham, shunday $n_0 \in N$ topilib, barcha $n>n_0$, $p>n_0$ $(p \in N)$ lar uchun $\rho(M_n, M_{p+n})<\varepsilon$ tengsizlik bajarilsa, $\{M_n\}$ ketma-ketlik R^m fazoda *fundamental ketma-ketlik* deb ataladi.

Ravshanki, agar $\{M_n\}(M_n \subset R^m)$ ketma-ketlik R^m fazoda fundamental ketma-ketlik bo'lsa, M_n ning koordinatalari hosil qilgan $\{x_1^{(n)}\}, \{x_2^{(n)}\}, ..., \{x_m^{(n)}\}$ ketma-ketliklarning har biri fundamental ketma-ketlik bo'ladi va aksincha.

$\{x_1^{(n)}\}, \{x_2^{(n)}\}, ..., \{x_m^{(n)}\}$ ketma-ketliklarning har biri fundamental ketma-ketlik bo'lsa, $\{M_n\}(M_n \subset R^m)$ ketma-ketlik R^m fazoda fundamental ketma-ketlik bo'ladi, ya'ni quyidagi teorema o'rinli.

18.2-teorema. R^m fazoda $M_n = M_n(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})$ ketma-ketlikning fundamental bo'lishi uchun, uning koordinatalaridan hosil bo'lgan $\{x_1^{(n)}\}, \{x_2^{(n)}\}, ..., \{x_m^{(n)}\}$ ketma-ketliklarning har biri fundamental ketma-ketlik bo'lishi zarur va yetarli.

18.3-teorema(Koshi prinsipi). $\{M_n\}$ ketma-ketlikning yaqinlashuvchi bo'lishi uchun, uning fundamental ketma-ketlik bo'lishi zarur va yetarli.

18.4-teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, uning limiti yagonadir.

18.3-ta'rif. Agar $\{M_n\}$ ketma-ketlikning barcha hadlaridan tuzilgan to'plam chegaralangan bo'lsa, $\{M_n\}$ ketma-ketlik *chegaralangan ketma-ketlik* deb ataladi.

18.4-ta'rif. Agar $\exists R>0$ mavjud bo'lib, $\forall n$ lar uchun $\rho(M_n, O) \le R$ tengsizlik bajarilsa, $\{M_n\}$ ketma-ketlik *chegaralangan ketma-ketlik* deyiladi, bunda $O(0, 0, ..., 0)$.

18.5-teorema. R^n fazoda $\{M_n\}$ ketma-ketlikning chegaralangan bo'lish uchun, bu ketma-ketlikning koordinatalaridan iborat $\{x_1^{(n)}\}, \{x_2^{(n)}\}, ..., \{x_m^{(n)}\}, ...$ sonlar ketma-ketliklari har birining chegaralangan bo'lishi zarur va yetarlidir.

18.6-teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

18.7-teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti A bo'lsa, u holda $\{kM_n\}$ $(k \in R)$ ketma-ketlik ham yaqinlashuvchi bo'lib, uning limiti kA ga teng bo'ladi, ya'ni $\lim_{n\to\infty} kM_n = k\lim_{n\to\infty} M_n = kA$.

18.8-teorema. Agar $\{M_n\}$ va $\{N_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lib, ularning limitlari, mos ravishda, A va B bo‘lsa, u holda $\{M_n \pm N_n\}$ ketma-ketlik ham yaqinlashuvchi bo‘ladi va uning limiti $A \pm B$ ga teng bo‘ladi, ya’ni

$$\lim_{n\to\infty}(M_n \pm N_n) = \lim_{n\to\infty} M_n \pm \lim_{n\to\infty} N_n = A \pm B$$

R^m fazoda $M_n = M_n(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})$ ketma-ketlik berilgan bo‘lsin. Bu ketma-ketlikning $n_1, n_2, ..., n_k, ...$ $(n_1 < n_2 < ... < n_k < ...\ n_k \in N,\ k = 1,2,..)$ nomerli hadlaridan tashkil topgan, ushbu $M_{n_1}, M_{n_2}, ..., M_{n_k}, ...$ $(M_{n_k} \in R^m,\ k = 1,2,...)$ ketma-ketlik, berilgan ketma-ketlikning *qismiy ketma-ketligi* deyiladi va u $\{M_{n_k}\}$ kabi belgilanadi.

Masalan, R^2 fazoda

$$(1;1), \left(\frac{1}{2^2}, \frac{1}{2^2}\right), \left(\frac{1}{3^2}, \frac{1}{3^2}\right), ..., \left(\frac{1}{n^2}; \frac{1}{n^2}\right), ...$$

$$\left(\frac{1}{2}; \frac{1}{2}\right), \left(\frac{1}{4}; \frac{1}{4}\right), \left(\frac{1}{6}; \frac{1}{6}\right), ..., \left(\frac{1}{2n}; \frac{1}{2n}\right), ...$$

ketma-ketliklar, $(1;1), \left(\frac{1}{2}; \frac{1}{2}\right), \left(\frac{1}{3}; \frac{1}{3}\right), ..., \left(\frac{1}{n}; \frac{1}{n}\right), ...$ ketma-ketlikning qismiy ketma-ketliklari bo‘ladi.

18.9-teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo‘lib, uning limiti A $(A \in R^m)$ bo‘lsa, u holda, bu ketma-ketlikning har bir $\{M_{n_k}\}$ qismiy ketma-ketligi ham yaqinlashuvchi bo‘ladi va uning limiti ham A ga teng bo‘ladi.

18.1-eslatma. Ketma-ketlik qismiy ketma-ketliklarining limiti mavjud bo‘lishidan, berilgan ketma-ketlikning limiti mavjud bo‘lishi har doim ham kelib chiqavermaydi. Masalan, ushbu $(1;1), (-1;-1), (1;1), (-1;-1), ..., ((-1)^{n+1}; (-1)^{n+1})$ ketma-ketlik limitga ega emas, lekin uning qismiy ketma-ketliklari, mos ravishda, $(1;1)$ va $(-1;-1)$ limitlarga ega. Shunday qilib, $\{M_n\}$ ketma-ketlik limitga ega bo‘lmasa ham, uning qismiy ketma-ketliklari limitga ega bo‘lishi mumkin ekan.

18.10-teorema (Bolsano-Veyershtrass). Har qanday chegaralangan ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin.

18.3-misol. R^2 fazoda $\{M_n\} = \left\{M_n\left(\frac{2^{n+2} + 3^{n+3}}{2^n + 3^n}, \frac{5 \cdot 2^n - 3 \cdot 5^{n+1}}{100 \cdot 2^n + 2 \cdot 5^n}\right)\right\}$ ketma-ketlikning limitini toping.

Yechilishi. Berilgan ketma-ketlikning koordinatalaridan tashkil topgan ketma-ketliklar, sonlar ketma-ketliklari bo‘lib, ular quyidagi ko‘rinishda bo‘ladi:

$$x_1^{(n)}=\frac{2^{n+2}+3^{n+3}}{2^n+3^n},\quad x_2^{(n)}=\frac{5\cdot 2^n-3\cdot 5^{n+1}}{100\cdot 2^n+2\cdot 5^n}.$$

Ravshanki,

$$\lim_{n\to\infty}x_1^{(n)}=\lim_{n\to\infty}\frac{2^{n+2}+3^{n+3}}{2^n+3^n}=\lim_{n\to\infty}\frac{4\cdot(2/3)^n+27}{(2/3)^n+1}=27,$$

$$\lim_{n\to\infty}x_2^{(n)}=\lim_{n\to\infty}\frac{5\cdot(2/5)^n-15}{100\cdot(2/5)^n+2}=-\frac{15}{2}.$$

Demak, 18.1-teoremaga ko'ra, berilgan ketma-ketlikning limiti

$$\lim_{n\to\infty}M_n=\lim_{n\to\infty}\left(\frac{2^{n+2}+3^{n+3}}{2^n+3^n},\frac{5\cdot 2^n-3\cdot 5^{n+1}}{100\cdot 2^n+2\cdot 5^n}\right)=A\left(27;-\frac{15}{2}\right)$$

18.4-misol. R^4 fazoda $\{M_n\}=\left\{M_n\left(\sqrt{n+1}-\sqrt{n};\frac{n-1}{n};\ \frac{2n^2-1}{n^2};\left(1+\frac{2}{n}\right)^n\right)\right\}$ ketma-ketlikning limitini toping.

Yechilishi. Berilgan ketma-ketlikning koordinatalaridan tashkil topgan ketma-ketliklar sonlar ketma-ketligi bo'lib, ular

$$x_1^{(n)}=\sqrt{n+1}-\sqrt{n};x_2^{(n)}=\frac{n-1}{n};\ x_3^{(n)}=\frac{2n^2-1}{n^2};x_4^{(n)}=\left(1+\frac{2}{n}\right)^n$$

ko'rinishda bo'ladi. Bular uchun

$$\lim_{n\to\infty}x_1^{(n)}=\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0,\qquad \lim_{n\to\infty}x_2^{(n)}=\lim_{n\to\infty}\frac{n-1}{n}=1;$$

$$\lim_{n\to\infty}x_3^{(n)}=\lim_{n\to\infty}\frac{2n^2-1}{n^2}=2;\qquad \lim_{n\to\infty}x_4^{(n)}=\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2.$$

Demak, 18.1-teoremaga ko'ra, berilgan ketma-ketlikning limiti- $A(0;1;2;e^2)$ bo'ladi.

18.5-misol. R^2 fazoda $\{M_n\}=\left\{M_n\left(\frac{1}{n};\ \frac{1}{n}\right)\right\}$ ketma-ketlikning fundamental bo'lishini ko'rsating.

Yechilishi. Berilgan $\forall\varepsilon>0$ ga ko'ra, $n_0=\left[\frac{2\sqrt{2}}{\varepsilon}\right]+1$ deb olinsa, u holda $\forall n>n_0,\forall p\ (p\in N)$ lar uchun.

$$\rho(M_n;M_p)=\rho\left(M_n\left(\frac{1}{n},\frac{1}{n}\right);M_p\left(\frac{1}{p},\frac{1}{p}\right)\right)=\sqrt{\left(\frac{1}{p}-\frac{1}{n}\right)^2+\left(\frac{1}{p}-\frac{1}{n}\right)^2}=\sqrt{2}\left|\frac{1}{p}-\frac{1}{n}\right|<$$

$$<\sqrt{2}\left(\frac{1}{p}+\frac{1}{n}\right)\le\sqrt{2}\left(\frac{1}{n_0}+\frac{1}{n_0}\right)=2\sqrt{2}\frac{1}{n_0}<2\sqrt{2}\frac{1}{2\sqrt{2}}\varepsilon=\varepsilon$$

tengsizlik o'rinli bo'ladi. Demak, 18.2- ta'rifga ko'ra, berilgan ketma-ketlik fundamental bo'lar ekan.

18.6-misol. R^2 fazoda $\{M_n\}=\left\{M_n\left(1+\frac{1}{2}+\frac{1}{3}....+\frac{1}{n};0\right)\right\}$ ketma-ketlikning fundamental emasligini ko'rsating.

Yechilishi. $\forall n_0$ lar uchun, berilgan $\exists\varepsilon_0=\frac{1}{4}$ ga ko'ra, $n>p\ \ (p\in N)$, deb olinsa, u holda,

$$\rho(M_{n+p}(x_{n+p},0);\,M_n(x_n,0))=\sqrt{(x_{n+p}-x_n)^2}=|x_{n+p}-x_n|=$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{n+p}>\frac{p}{n+p},$$

Bundan, $n=2p$ bo'lganda,

$$\rho(M_{n+p}(x_{n+p},0);\,M_n(x_n,0))>\frac{2}{3}>\frac{1}{4}=\varepsilon_0$$

ekanligi kelib chiqadi. Bu esa, berilgan ketma-ketlikning fundamental emasligini ko'rsatadi.

18.7- misol. R^2 fazoda

$$\{M_n\}=\left\{M_n\left(\frac{\sin 1}{2}+\frac{\sin 2}{4}+....+\frac{\sin n}{2^n};\frac{\cos 1!}{1\cdot 2}+\frac{\cos 2!}{2\cdot 3}+....+\frac{\cos n!}{n(n+1)}\right)\right\}$$

ketma-ketlikning yaqinlashuvchi ekanligini isbotlang.

Yechilishi. Berilgan ketma-ketlikning yaqinlashuvchi ekanligini ko'rsatish uchun, uning fundamental ketma-ketlik ekanligini ko'rsatish etarli. Buning uchun $\{M_n(x_n,y_n)\}$ ketma-ketlikning koordinatalari hosil qilgan $\{x_n\},\{y_n\}$ ketma-ketliklarning har biri fundamental ekanligini ko'rsatamiz. Berilgan $\forall\varepsilon>0$ ga ko'ra, $n_0=\left[\log_2\frac{1}{\varepsilon}\right]+1$ deb olinsa, u holda,

$\forall n>n_0$ va $\forall p\ \ (p\in N)$ lar uchun,

$$|x_{n+p}-x_n|=\left|\frac{\sin(n+1)}{2^{n+1}}+\frac{\sin(n+2)}{2^{n+2}}+....+\frac{\sin(n+p)}{2^{n+p}}\right|\le\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+....+\frac{1}{2^{n+p}}<$$

$$<\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+....+\frac{1}{2^{n+p}}+.....=\frac{1}{2^{n+1}}\frac{1}{1-\frac{1}{2}}=\frac{1}{2^n}<\frac{1}{2^{n_0}}<\varepsilon$$

tengsizlik o'rinli bo'ladi.

Demak, $\{x_n\}$ ketma-ketlik fundamental bo'lar ekan.

Endi $\{y_n\}=\left\{\frac{\cos 1!}{1\cdot 2}+\frac{\cos 2!}{2\cdot 3}+....+\frac{\cos n!}{n(n+1)}\right\}$ ketma-ketlikning fundamental ketma-ketlik ekanligini ko'rsatamiz. Berilgan $\forall\varepsilon>0$ ga ko'ra, $n_0=\left[\frac{1}{\varepsilon}\right]+1$ deb olinsa, u holda $\forall n>n_0$ va $\forall p\ \ (p\in N)$ lar uchun,

$$|y_{n+p}-y_n|=\left|\frac{\cos(n+1)!}{(n+1)\cdot(n+2)}+\frac{\cos(n+2)!}{(n+2)\cdot(n+3)}+....+\frac{\cos(n+p)!}{(n+p)\cdot(n+p+1)}\right|\le$$

$$\le \frac{1}{(n+1)\cdot(n+2)}+\frac{1}{(n+2)\cdot(n+3)}+....+\frac{1}{(n+p)\cdot(n+p+1)}=$$

$$=\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+2}-\frac{1}{n+3}...+\frac{1}{n+p}-\frac{1}{n+p+1}=\frac{1}{n+1}-\frac{1}{n+p+1}<\frac{1}{n+1}<\frac{1}{n_0+1}<\varepsilon$$

tengsizlik o'rinli bo'ladi.

Shunday qilib, 18.2- teoremaga ko'ra, berilgan ketma-ketlik fundamental ketma-ketlik bo'ladi, bundan esa, 18.4- teoremaga asosan, berilgan ketma-ketlikning yaqinlashuvchi ekanligi kelib chiqadi.

Mustaqil yechish uchun misollar

R^2 fazoda quyidagi ketma-ketlikning limiti $A\left(A\in R^2\right)$ ekanligini isbotlang:

18.1. $\{M_n\}=\left\{M_n\left(\frac{1}{n};\frac{10}{n}\right)\right\}$, $A=(0;0)$.

18.2. $\{M_n\}=\left\{M_n\left(\frac{4n^2}{n^2-1};\frac{2-n}{3+n}\right)\right\}$, $A=(4;-1)$.

18.3. $\{M_n\}=\left\{M_n\left(\frac{5}{n};\frac{2}{n^2}\right)\right\}$, $A=(0;0)$.

18.4. $\{M_n\}=\left\{M_n\left(\frac{1}{n^2};\frac{\sin n}{n}\right)\right\}$, $A=(0;0)$.

18.5. $\{M_n\}=\left\{M_n\left(\frac{n}{5^n};\frac{3}{n}\right)\right\}$, $A(0;0)$.

18.6. $\{M_n\}=\left\{M_n\left(\sqrt[n]{3};\frac{\log_3 n}{n}\right)\right\}$, $A,(1;0)$.

18.7. $\{M_n\}=\left\{M_n\left(\sqrt[n]{n};\frac{n^3}{3^n}\right)\right\}$, $A(1;0)$.

R^2 fazoda quyidagi ketma – ketlikning limiti A $\left(A\in R^2\right)$ ekanligini isbotlang:

18.9. $\{M_n\}=\left\{M_n\left(\sqrt[3n]{9};\ \sqrt[2n]{0{,}25}\right)\right\}$, $A(1;1)$.

18.10. $\{M_n\}=\left\{M_n\left(\frac{\sqrt[n]{8}-1}{\sqrt[n]{2}-1};3^{\frac{n^2+2}{n^2+1}}\right)\right\}$, $A(3;3)$.

18.11. $\{M_n\}=\left\{M_n(\sqrt[n]{n};\frac{a^n}{n!})\right\}$, $(a>0)$, $A(1;0)$.

18.18. $\{M_n\}=\left\{M_n\left(\frac{n+1}{n^2-1};\frac{n^2-3n+4}{n^3+4n^2-5n+6}\right)\right\}$, $A(0;0)$.

18.15. $\{M_n\}=\left\{M_n\left(\sqrt[n]{n^2};\sqrt[n^2]{n}\right)\right\}$, $A(1;1)$.

18.16. $\{M_n\}=\left\{M_n\left(\sqrt[n]{3n-2};\sqrt[n]{n^3+3n}\right)\right\}$, $A(1;1)$.

18.17. $\{M_n\}=\left\{M_n\left(\frac{\log_5(n^2+1)}{n};\frac{n-\lg n}{\log_2(4^n+1)}\right)\right\}$, $A\left(0;\frac{1}{2}\right)$.

18.18. $\{M_n\}=\left\{M_n\left(\sqrt{n+2}-\sqrt{n};\frac{\log_a n}{n}\right)\right\}(a>1)$, $A(0;0)$.

18.19. $\{M_n\}=\left\{M_n\left(\frac{(-2)^n}{(n+2)!};\frac{1}{(0,3)^n\cdot n!}\right)\right\}$, $A(0;0)$.

18.20. $\{M_n\}=\left\{M_n\left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+....+\frac{2n-1}{2^n};\frac{\sqrt[3]{n^2}\cdot\sin n!}{n+1}\right)\right\}$, $A(3;0)$.

18.21. $\{M_n\}=\left\{M_n\left(\frac{1^2}{n^3}+\frac{3^2}{n^3}+...+\frac{(2n-1)^2}{n^3};\frac{n}{2^n}\right)\right\}$, $A\left(\frac{4}{3};0\right)$.

19-§. Ko'p o'zgaruvchili funksiyaning limiti

19.1. Ko'p o'zgaruvchili funksiya limitining ta'riflari. $u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

19.1-ta'rif (Geyne ta'rifi). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan va A ga intiluvchi har qanday $\{M_n\}$ $(M_n\neq A, n=1,2...)$ ketma-ketlik olinganda ham, funksiyaning unga mos kelgan $\{f(M_n)\}$ qiymatlari ketma-ketligi hamma vaqt, yagona B (chekli yoki cheksiz) limitga intilsa, shu B ga $f(M)$ *funksiyaning* A *nuqtadagi (yoki* $M\to A$ *dagi) limiti* deyiladi va u

$$\lim_{M\to A} f(M)=B \text{ yoki } \lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \cdots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=B \text{ yoki } M\to A \text{ da } f(M)\to B$$

kabi belgilanadi.

19.2.-ta'rif (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun, $\exists\delta>0$ bo'lib, $0<\rho(M;A)<\delta$ tengsizliklarni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda

$$|f(M)-B|<\varepsilon$$

tengsizlik bajarilsa, shu B songa $f(x)$ *funksiyaning* A *nuqtadagi (* $M\to A$ *dagi) limiti* deyiladi.

$u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda aniklangan bo'lib, ∞ esa, $\{M\}$ to'plamning limit nuqtasi bo'lsin.

19.3-ta'rif (Geyne ta'rifi). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan har qanday $\{M_n\}$ ketma-ketlik uchun $M\to\infty$ da funksiyaning unga mos kelgan $\{f(M_n)\}$ qiymatlari ketma-ketligi hamma vaqt yagona B

songa intilsa, shu B songa $f(M)$ funksiyaning *$M\to\infty$ dagi limiti* deyiladi va $\lim\limits_{M\to\infty} f(M)=B$ kabi belgilanadi.

19.4-ta'rif. Agar $\forall\varepsilon>0$ son uchun, shunday $\exists E>0$ bo'lib, $\rho(M,O)>E$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda $|f(M)-B|<\varepsilon$ tengsizlik bajarilsa, B son $f(M)$ *funksiyaning $M\to\infty$ dagi limiti* deyiladi va $\lim\limits_{M\to\infty} f(M)=B$ yoki $\lim\limits_{\substack{x_1\to\infty\\ x_2\to\infty\\ \dots\\ x_m\to\infty}} f(x_1,x_2,\dots,x_m)=B$ kabi belgilanadi.

19.5-ta'rif (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun $\exists\delta>0$ bo'lib, $0<\rho(M,A)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda $|f(M)|>\varepsilon$ $(f(M)>\varepsilon;\ f(M)<-\varepsilon)$ bo'lsa, $f(M)$ *funksiyaning A nuqtadagi ($M\to A$ dagi) limiti* $+\infty$ $(-\infty)$ deyiladi.

19.1- misol. Ushbu

$$f(x,y)=\begin{cases}\dfrac{x\cdot y}{\sqrt{x^2+y^2}}, & x^2+y^2>0 \ \text{bo'lganda},\\ 0, & x^2+y^2=0 \ \text{bo'lganda},\end{cases}$$

funksiyaning $M(x,y)\to A(0;0)\ (x\to 0, y\to 0)$ dagi limiti nolga teng ekanligini ko'rsating.

Yechilishi. $f(x,y)$ funksiya R^2 da berilgan bo'lib, $A(0;0)$ nuqta shu to'plamning limit nuqtasidan iborat.

1) Geyne ta'rifi buyicha: R^2 to'plamdan $A(0;0)$ nuqtaga intiluvchi ixtiyoriy $\{M_n\}=\{M_n(x^{(n)};y^{(n)})\}$ $(M_n\neq A(0;0),\ n=1,2,\dots)$ ketma-ketlikni olamiz. Funksiyaning unga mos kelgan $\{f(M_n)\}$ qiymatlari ketma-ketligi uchun,

$$f(M_n)=f(x^{(n)},y^{(n)})=\frac{x^{(n)}\cdot y^{(n)}}{\sqrt{(x^{(n)})^2+(y^{(n)})^2}}=\sqrt{\frac{x^{(n)}y^{(n)}}{(x^{(n)})^2+(y^{(n)})^2}}\cdot\sqrt{x^{(n)}y^{(n)}}\le\frac{1}{\sqrt2}\sqrt{x^{(n)}\cdot y^{(n)}}$$

bo'ladi. Bundan $x^{(n)}\to 0, y^{(n)}\to 0$ da $\sqrt{x^{(n)}\cdot y^{(n)}}\to 0$.

Demak, $\lim\limits_{M\to A} f(M)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{\sqrt{x^2+y^2}}=0$.

2) *Koshi ta'rifi bo'yicha:* $\forall\varepsilon>0$ songa ko'ra, $\delta=2\varepsilon$ deyilsa, u holda, $0<\rho(M;A(0,0))<\delta$ tengsizliklarni qanoatlantiruvchi barcha $M(x;y)$ nuqtalarda

$$|f(x,y)-0|=\left|\frac{xy}{\sqrt{x^2+y^2}}\right|\le\frac{1}{2}\sqrt{x^2+y^2}=\frac{1}{2}\rho(M;A)<\frac{1}{2}\delta=\varepsilon$$

tengsizlik o'rinli bo'ladi.

19.2-misol. Ushbu $f(x,y)=\frac{2xy}{x^2+y^2}$ funksiyaning $M(x,y)\to A(0;0)$ $(x\to 0, y\to 0)$ dagi limiti mavjud emasligini ko'rsating.

Yechilishi. Bu funksiya $R^2\setminus\{(0;0)\}$ to'plamda aniqlangan bo'lib, $A(0;0)$ nuqta shu to'plamning limit nuqtasidir. Ravshanki,

$$M_n\left(x^{(n)};y^{(n)}\right)=M_n\left(\frac{1}{n};\frac{1}{n}\right),\ \overline{M_n}\left(\overline{x}^{(n)},\overline{y}^{(n)}\right)=\overline{M_n}\left(\frac{1}{n};\frac{1}{n^2}\right)$$

ketma-ketliklar $n\to\infty$ da $M_n\to A(0;0), \overline{M}_n\to A(0;0)$. Funksiyaning bu ketma-ketliklarga mos kelgan $\{f(x^{(n)},y^{(n)})\}, \{f(\overline{x}^{(n)},\overline{y}^{(n)})\}$ qiymatlari ketma-ketligi uchun, mos ravishda,

$$f\left(x^{(n)},y^{(n)}\right)=\frac{2\cdot\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}}\underset{n\to\infty}{\to}1,\qquad f\left(\overline{x}^{(n)},\overline{y}^{(n)}\right)=\frac{\frac{2}{n^3}}{\frac{1}{n^2}+\frac{1}{n^4}}\to 0$$

bo'ladi. Bu esa, $M(x,y)\to A(0;0)$ da berilgan funksiyaning limiti mavjud emasligini anglatadi.

19.3-misol. Ushbu $\lim\limits_{\substack{y\to\infty\\ x\to\infty}}\frac{x+y}{x^2-xy+y^2}$ limitni hisoblang.

Yechilishi. Ravshanki, $x^2+xy+y^2\geq xy$ tengsizlik doimo o'rinli. Bundan, $x\neq 0,\ y\neq 0$ bo'lganda,

$$0\leq\left|\frac{x+y}{x^2-xy+y^2}\right|\leq\left|\frac{x+y}{xy}\right|\leq\frac{1}{|y|}+\frac{1}{|x|}$$

tengsizlikni olamiz. Bu yerdan,

$$0\leq\lim_{\substack{x\to\infty\\ y\to\infty}}\left|\frac{x+y}{x^2-xy+y^2}\right|\leq\lim_{\substack{x\to\infty\\ y\to\infty}}\left(\frac{1}{|x|}+\frac{1}{|y|}\right)=0.$$

Shunday qilib, $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{x+y}{x^2-xy+y^2}=0$.

19.4-misol. Ushbu $f(x,y)=\frac{x^4+y^2}{x^2+y^4}$ funksiya $x\to\infty, y\to\infty$ da limitga ega bo'lmasligini isbotlang.

Yechilishi. $x=t,\ y=t^4$ deb olinsa, $t\to\infty$ da $x\to\infty,\ y\to\infty$ va

$$\lim_{t\to\infty}f\left(t,t^4\right)=\lim_{t\to\infty}\frac{t^4+t^8}{t^2+t^{16}}=0.$$

Ikkinchi tomondan $x=t^2,\ y=t$ deyilsa, $\lim\limits_{t\to\infty}f\left(t^2;t\right)=\lim\limits_{t\to\infty}\frac{t^8+t^2}{t^4+t^4}=\infty$.

Demak, $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{x^4+y^2}{x^2+y^4}$ - mavjud emas.

19.5-misol. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 3}} f(x,y)$, bunda

$$f(x,y)=\begin{cases}\dfrac{\sin(x^2y)}{x^2}, & x\neq 0 \ \ bo'lganda,\\ 3, & x=0 \ \ bo'lganda,\end{cases}$$

limitni hisoblang.

Yechilishi. $x=0,\ y=3$ bo'lsa, berilgan kasrning surati ham, maxraji ham nolga aylanadi. Shuning uchun, $x^2y=\alpha$ deb olinsa, u holda, $x\to 0$ va $y=3$ bo'lganda $\alpha\to 0$ va

$$\lim_{\substack{x\to 0\\ y\to 3}}\frac{\sin(x^2y)}{x^2}=\lim_{\substack{x\to 0\\ y\to 3}}\frac{\sin(x^2y)}{x^2\cdot y}\cdot y=\lim_{\substack{\alpha\to 0\\ y\to 3}}\frac{\sin\alpha}{\alpha}\cdot y=1\cdot 3=3.$$

Agar $x=0$ bo'lsa, $f(x,y)=3$ va $\lim\limits_{y\to 3} f(x,y)=3$.

19.5, a-misol. Ushbu

$$a)\ \lim_{\substack{x\to+\infty\\ y\to+\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2};\qquad b)\lim_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{x^2y^2}$$

limitlarni hisoblang

Yechilishi. $a)$ Ravshanki, $x^2+y^2\geq 2xy$ tengsizlik o'rinli. Bundan $\dfrac{xy}{x^2+y^2}\leq\dfrac{1}{2}$ o'rinlidir. Bularni e'tiborga olgan holda,

$$0<\left(\frac{xy}{x^2+y^2}\right)^{x^2}\leq\left(\frac{1}{2}\right)^{x^2}\underset{x\to\infty}{\to}0$$

ekanligini olamiz.

Demak, bu tengsizlikdan, $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}\left(\dfrac{xy}{x^2+y^2}\right)^{x^2}=0$ ekanligi kelib chikadi.

$b)$ $0<x^2+y^2<1$ bo'lganda,

$$x^2y^2\leq\frac{1}{4}\left(x^2+y^2\right)^2,\ 1\geq\left(x^2+y^2\right)^{x^2y^2}\geq\left(x^2+y^2\right)^{\frac{1}{4}\left(x^2+y^2\right)^2}$$

tengsizliklar o'rinli. Bu tengsizliklarni e'tiborga olib,

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{\frac{1}{4}\left(x^2+y^2\right)^2}=\lim_{t\to 0+0}t^{\frac{1}{4}t^2}=\lim_{t\to 0+0}e^{\frac{1}{4}t^2\ln t}=1$$

ekanligini topamiz.

Demak, (*) tengsizlikka asosan, $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{x^2y^2}=1$ ekanligi kelib chiqadi.

$u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

19.2. Cheksiz kichik va cheksiz katta funksiyalar. $\alpha(M)$ funksiya $\{M\}\subset R^m$ to'plamda aniqlangan bo'lib, $A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

19.6-ta'rif. Agar $M\to A$ da $\alpha(M)$ funksiyaning limiti nol, ya'ni $\lim\limits_{M\to A}\alpha(M)=0$ bo'lsa, u holda, $\alpha(M)$ funksiya, $M\to A$ da *cheksiz kichik funksiya* deyiladi.

19.1-eslatma. Berilgan $f(M)$ funksiya $M\to A$ da B limitga ega bo'lishi uchun, $\alpha(M)=f(M)-B$ ning cheksiz kichik funksiya bo'lishi zarur va yetarli.

Demak, $M\to A$ da $f(M)$ funksiya B limitga ega bo'lsa, bu funksiyani, har doim,

$$f(M)=B+\alpha(M) \qquad (19.1)$$

ko'rinishda ifodalash mumkin, bunda $\alpha(M)$ cheksiz kichik funksiya.

Cheksiz kichik funksiyalar quyidagi xossalarga ega:

1- xossa. Agar $M\to A$ da $\alpha(M)$ va $\beta(M)$ cheksiz kichik funksiyalar bo'lsa, u holda, $\alpha(M)\pm\beta(M)$ ham cheksiz kichik funksiya bo'ladi.

2- xossa. Agar $M\to A$ da $\alpha(M)$- cheksiz kichik funksiya, $\beta(M)$ funksiya esa, chegaralangan bo'lsa, u holda, $\alpha(M)\cdot\beta(M)$ (ularning ko'paytmasi) ham cheksiz kichik bo'ladi.

19.7-ta'rif. Agar $\{M\}\subset R^m$ to'plamda aniqlangan $F(M)$ funksiya uchun $\lim\limits_{M\to A}F(M)=\infty$ bo'lsa, $F(M)$ funksiya $M\to A$ da *cheksiz katta funksiya* deyiladi.

3- xossa. Agar $M\to A$ da $\alpha(M)$ cheksiz kichik $(\alpha(M)\neq 0)$ funksiya bo'lsa, u holda $\frac{1}{\alpha(M)}$ - $M\to A$ da cheksiz katta funksiya bo'ladi.

4- xossa. Agar $M\to A$ da $F(M)$ cheksiz katta funksiya bo'lsa, $\frac{1}{F(M)}$ funksiya $M\to A$ da cheksiz kichik funksiya bo'ladi.

19.3. Limitga ega bo'lgan funksiyalarning xossalari. $\{M\}\subset R^m$ to'plamda $f(M)$ funksiya aniqlangan bo'lib, $A(A\in R^m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin

1- xossa. Agar $\lim\limits_{M\to A}f(M)=B$ mavjud bo'lib, $B>p$ $(B<q)$ bo'lsa, u holda A nuqtaning etarli kichik atrofidagi $M\in\{M\}$ $(M\neq A)$ nuqtalarda $f(M)>p$ $(f(M)<q)$ bo'ladi.

Xususan, $B\neq 0$ bo'lsa, u holda, A nuqtaning yetarli kichik atrofida $f(M)\neq 0$ bo'ladi.

2- xossa. Agar $\lim_{M\to A} f(M)=B$ mavjud bo'lsa, A nuqtaning yetarli kichik atrofidagi $M\in\{M\}(M\neq A)$ nuqtalarda $f(M)$ funksiya chegaralangan bo'ladi.

$\{M\}\subset R^m$ to'plamda $f(M)$ va $g(M)$ funksiyalar berilgan bo'lsin.

3- xossa. Agar $f(M)$ va $g(M)$ funksiyalar, $M\to A$ da, mos ravishda, B va C limitlarga ega bo'lsa, u holda $f(M)\pm g(M)$, $f(M)\cdot g(M)$ va $\frac{f(M)}{g(M)}$ $(C\neq 0)$ funksiyalar ham, mos ravishda, $B\pm C$, $B\cdot C$, $\frac{B}{C}$ limitlarga ega bo'ladi.

19.2- eslatma. $f(M)$ va $g(M)$ funksiyalarning yig'indisi, ko'paytmasi va nisbati limitga ega bo'lishidan, bu funksiyalardan har birining limitga ega bo'lishi har doim ham kelib chiqavermaydi.

19.3- eslatma. Agar:

1) $\lim_{M\to A} f(M)=0$, $\lim_{M\to A} g(M)=0$ bo'lsa, $\frac{f(M)}{g(M)}$ ifoda;

2) $\lim_{M\to A} f(M)=\infty$, $\lim_{M\to A} g(M)=\infty$ bo'lsa, $\frac{f(M)}{g(M)}$ ifoda;

3) $\lim_{M\to A} f(M)=0$, $\lim_{M\to A} g(M)=\infty$ bo'lsa, $f(M)\cdot g(M)$ ifoda;

4) $f(M)$ va $g(M)$ funksiyalar $M\to A$ da turli xil ishorali cheksiz limitga ega bo'lsa, $f(M)+g(M)$ ifoda; mos ravishda, $\frac{0}{0},\frac{\infty}{\infty},0\cdot\infty,\infty-\infty$ ko'rinishidagi aniqmasliklarni ifodalaydi.

19.4- eslatma. Agar:

1) $\lim_{M\to A} f(M)=0$, $\lim_{M\to A} g(M)=0$ bo'lsa;

2) $\lim_{M\to A} f(M)=1$, $\lim_{M\to \infty} g(M)=\infty$ bo'lsa;

3) $\lim_{M\to A} f(M)=\infty$, $\lim_{M\to A} g(M)=0$ bo'lsa,

u holda, $[f(M)]^{g(M)}$ – ifoda, mos ravishda, $0^0, 1^\infty, \infty^0$ ko'rinishdagi aniqmasliklarni ifodalaydi.

19.1-teorema (Koshi kriteriysi). $u=f(M)$ funksiyaning chekli limitga ega bo'lishi uchun, $\forall\varepsilon>0$ son olinganda ham, shunday $\delta>0$ bo'lib, $0<\rho(M',A)<\delta$, $0<\rho(M'',A)<\delta$ tengsizliklarni qanoatlantiruvchi barcha $M',M''\in\{M\}$ nuqtalarda $|f(M')-f(M'')|<\varepsilon$ tengsizlikning bajarilishi zarur va yetarli.

19.4. Takroriy limitlar. Biz yuqorida $u=f(M)=f(x_1,x_2,...,x_m)$ funksiyaning $A=A(a_1,a_2,...,a_m)$ nuqtadagi limiti $\lim\limits_{M\to A} f(M)=B$ yoki $\lim\limits_{\substack{x_1\to a_1\\ x_2\to a_2\\ \dots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=B$ bilan tanishdik.

Demak, funksiyaning limiti, uning argumentlarining $x_1,x_2,...,x_m$ bir yo'la, mos ravishda, $a_1,a_2,...,a_m$ sonlarga intilgandagi limitidan iborat ekan. Biz bundan buyon, bu limitni, *karrali limit* deb ataymiz.

Ko'p o'zgaruvchili funksiyalargagina xos bo'lgan, boshqa ko'rinishdagi, limit tushunchasini kiritamiz. $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A=A(a_1,a_2,...,a_m)$ nuqta- $\{M\}$ to'plamning limit nuqtasi bo'lsin. Berilgan funksiyaning $x_1\to a_1$ (qolgan barcha argumentlarini tayinlab) dagi limiti $\lim\limits_{x_1\to a_1} f(x_1,x_2,...,x_m)$ ni qaraylik, bu limit $x_2,x_3,...,x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)=\varphi_1(x_2,x_3,...,x_m).$$

Endi, $\varphi_1(x_2,...,x_m)$ funksiyaning $x_2\to a_2$ (qolgan barcha argumentlarni tayinlab) dagi limitini qaraymiz, bu $\lim\limits_{x_2\to a_2}\varphi_1(x_2,x_3,...,x_m)$ limit $x_3,x_4,...,x_m$ o'zgaruvchilarga bog'lik bo'ladi: $\lim\limits_{x_2\to a_2}\varphi_1(x_2,x_3,...,x_m)=\varphi_2(x_3,x_4,...,x_m)$.

Xuddi shunday, birin ketin, $x_3\to a_3,\ x_4\to a_4,...,x_m\to a_m$ da limitga o'tib, $\lim\limits_{x_m\to a_m}\lim\limits_{x_{m-1}\to a_{m-1}}.....\lim\limits_{x_1\to a_1} f(x_1,x_2,...,x_m)$ ni hosil qilamiz. Bu limitga $f(x_1,x_2,...,x_m)$ funksiyaning *takroriy limiti* deyiladi.

Xuddi shunday, $f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,a_{i_k}$ argumentlari, mos ravishda, $a_{i_1},a_{i_2},...,a_{i_k}$ larga intilgandagi $\lim\limits_{x_{i_k}\to a_{i_k}}...\lim\limits_{x_{i_1}\to a_{i_1}} f(x_1,x_2,...,x_m)$ takroriy limitni ham qarash mumkin.

Ravshanki, $f(x_1,x_2,...,x_m)$ funksiyaning $x_1,x_2,...,x_m$ argumentlari, mos ravishda, $a_{i_1},a_{i_2},...,a_{i_k}$ sonlarga, turli tartibda intilganda, funksiyaning turli takroriy limitlari hosil bo'ladi.

19.6-misol. $f(x,y)=(x+y)\sin\frac{1}{x}\sin\frac{1}{y}$ funksiyaning $M(x,y)\to O(0;0)$ da cheksiz kichik funksiya ekanligini isbotlang.

Yechilishi. Cheksiz kichik funksiyaning ta'rifiga ko'ra, $\lim\limits_{\substack{x\to0\\ y\to0}} f(x,y)=0$ tenglikning o'rinli ekanligini ko'rsatish yetarli. Ravshanki, berilgan $f(x,y)$ funksiya koordinatalar o'qlarida aniqlanmagan, lekin $O(0,0)$ nuqta funksiya aniqlanish sohasining limit nuqtasidan iborat. Shuning uchun, berilgan funksiyaning $O(0;0)$ nuqtadagi limitini qarash mumkin. Bu limitni, limitning Koshi ta'rifi bo'yicha qaraymiz. $\forall\varepsilon>0$ songa ko'ra, $\delta=\varepsilon/2$ deb olinsa, u holda, $\rho(M(x,y),O(0;0))=\sqrt{x^2+y^2}<\delta$ dan $|x|<\delta$ va $|y|<\delta$ bo'lishi kelib chiqadi. Demak, $|f(x,y)-0|=\left|(x+y)\sin\frac{1}{x}\sin\frac{1}{y}\right|\le|x|+|y|<2\delta=\varepsilon$. Bundan, ta'rifga ko'ra, $\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}(x+y)\sin\frac{1}{x}\sin\frac{1}{y}=0$ ekanligi kelib chiqadi.

Shunday qilib, berilgan funksiyaning $M\to0$ da cheksiz kichik funksiya ekanligi isbotlanadi.

19.7- misol. Ushbu $f(x,y)=(x+y)\sin\frac{1}{x}\cdot\sin\frac{1}{y}$ funksiyaning $O(0;0)$ nuqtada takroriy limitlari mavjudmi?

Yechilishi. $\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)-$ takroriy limitning $\lim\limits_{\substack{x\to0\\y\neq0,\ \text{белгиланган}}}f(x,y)-$ichki limitini qaraymiz. Avvalo, berilgan $f(x,y)$ funksiyani quyidagicha tasvirlaymiz: $f(x,y)=x\sin\frac{1}{x}\cdot\sin\frac{1}{y}+y\sin\frac{1}{x}\sin\frac{1}{y}$. Bunda, belgilangan $y\neq0$ uchun, birinchi qo'shiluvchining $x\to0$ dagi limiti, ya'ni $\lim\limits_{x\to0}x\cdot\sin\frac{1}{x}\cdot\sin\frac{1}{y}=0$; ikkinchi qo'shiluvchidagi $y\cdot\sin\frac{1}{y}$ ko'paytuvchi, $y\neq\frac{1}{\pi n}(n\in Z)$ bo'lganda, noldan farqli o'zgarmas son, $\sin\frac{1}{x}$ ko'paytuvchi esa, $x\to0$ da limitga ega emas, demak, ikkinchi qo'shiluvchi limitga ega bo'lmaydi. U holda, $f(x,y)=x\sin\frac{1}{x}\cdot\sin\frac{1}{y}+y\sin\frac{1}{x}\sin\frac{1}{y}$ funksiya, belgilangan y ($y\neq0,\ y\neq n\pi$) uchun, $x\to0$ da limitga ega bo'lmaydi. Shunday qilib, ichki limit mavjud emas, shunga ko'ra, $\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)$ takroriy limitning mavjud emasligi kelib chiqadi. Xuddi shunday, $\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)$ takroriy limitning ham mavjud emasligini ko'rsatish mumkin.

19.5- eslatma. 19.6- misolda $f(x,y)=(x+y)\sin\frac{1}{x}\cdot\sin\frac{1}{y}$ funksiyaning $O(0;0)$ nuqtadagi karrali limiti mavjud va uning nolga teng ekanligi isbotlangan edi. 19.7-misolda esa, uning takroriy limitlarining mavjud emasligi ko'rsatildi. Demak, funksiyaning nuqtada karrali limiti har doim mavjudligidan, uning shu nuqtada takroriy limitlari mavjudligi kelib chiqmasligi to'g'risida xulosa chiqarish mumkin ekan.

19.8- misol. $f(x,y)=\frac{2xy}{x^2+y^2}$ funksiyaning $O(0;0)$ nuqtadagi takroriy limitlarini hisoblang.

Yechilishi. Takroriy limitlarni topamiz:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{x\to 0}\left(\lim_{\substack{y\to 0\\ x-\text{fiksir}\\ x\neq 0}}\frac{2xy}{x^2+y^2}\right)=\lim_{x\to 0}\left(\frac{0}{x^2}\right)=0.$$

Xuddi shunday, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)=0$ ekanligini ko'rsatish mumkin.

19.6-eslatma. 19.2- misolda $f(x,y)=\frac{2xy}{x^2+y^2}$ funksiyaning $O(0;0)$ nuqtada karrali limiti mavjud emasligi ko'rsatilgan edi. Demak, 19.2 va 19.8 misollarga asosan, berilgan nuqtada takroriy limitlarning mavjudligi va ularning tengligidan, bu nuqtada funksiyaning karrali limiti mavjudligi har doim kelib chiqavermas ekan, degan xulosa chiqarish mumkin.

19.6- misol. Ushbu

$$f(x,y)=\begin{cases}\dfrac{x-y+x^2+y^2}{x+y}, & y\neq -x \quad bo'\text{lg}anda,\\[2ex] 0, & y=-x \quad bo'\text{lg}anda,\end{cases}$$

funksiyaning $x\to 0,\ y\to 0$ dagi takroriy limitlarini hisoblang.

Yechilishi. y belgilangan va $y\neq 0$ bo'lsin. U holda,

$$\lim_{x\to 0} f(x,y)=\lim_{x\to 0}\frac{x-y+x^2+y^2}{x+y}=\frac{-y+y^2}{y}=-1+y,$$

bundan, $\lim_{y\to 0}[\lim_{x\to 0} f(x,y)]=\lim_{y\to 0}(-1+y)=-1$ bo'lishi kelib chiqadi. Endi x belgilangan va $x\neq 0$ bo'lsin. U holda,

$$\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\frac{x-y+x^2+y^2}{x+y}=\frac{x+x^2}{x}=1+x,$$

bundan $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)]=\lim_{x\to 0}(1+x)=1$ bo'lishi kelib chiqadi.

Shunday qilib, berilgan funksiyaning takroriy limitlari mavjud, lekin ular o'zaro teng emas, ya'ni $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)]\neq \lim_{y\to 0}[\lim_{x\to 0} f(x,y)]$.

19.7-eslatma. 19.6- misoldan ko'rinadiki, funksiyaning karrali va takroriy limitlarining teng bo'lishi uchun, ma'lum shartlarning bajarilishi kerak bo'lar ekan.

19.2-teorema. $u=f(M)$ funksiya $\{M\}=\{(x,y)\in R^2 : |x-x^0|<d_1, |y-y^0|<d_2\}$ to'plamda berilgan bo'lib, y quyidagi shartlarni qanoatlantirsin:

1) $(x,y)\to(x^0,y^0)$ da $f(x,y)$ funksiyaning $\lim\limits_{\substack{x\to x^0\\ y\to y^0}} f(x,y)=B$ karrali limiti mavjud;

2) har bir tayinlangan x da $\lim\limits_{y\to y^0} f(x,y)=\varphi(x)$ mavjud, har bir tayinlangan y da $\lim\limits_{x\to x^0} f(x,y)=\varphi(y)$ mavjud bo'lsin. U holda $\lim\limits_{x\to x^0}\lim\limits_{y\to y} f(x,y)$ va $\lim\limits_{x\to x^0}\lim\limits_{y\to y^0} f(x,y)$ takroriy limitlar ham mavjud va ular B ga teng bo'ladi.

Natija. 19.2- teoremaning shartlari bajarilganda,

$$\lim_{\substack{x\to x^0\\ y\to y^0}} f(x,y)=\lim_{x\to x^0}\lim_{y\to y^0} f(x,y)=\lim_{y\to y^0}\lim_{x\to x^0} f(x,y)$$

munosabat o'rinli.

Mustaqil yechish uchun misollar

Quyidagi limitlarni hisoblang:

19.1. $\lim\limits_{\substack{x\to \pi\\ y\to \ln 2}} e^{y}\cos x.$

19.2. $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{x-y}{x^2-y^2}.$

19.3. $\lim\limits_{\substack{x\to 1\\ y\to -1\\ z\to e}} \ln|x+y+z|.$

19.4. $\lim\limits_{\substack{x\to 0\\ y\to a}} \dfrac{\sin xy}{x}.$

19.5. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{tg\,2xy}{x^2 y}.$

19.6. $\lim\limits_{\substack{x\to 0\\ y\to 3}} \left(1+xy^2\right)^{\frac{y}{x^2y+xy^2}}.$

19.7. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{ax+by}{x^2+xy+y^2}.$

19.8. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{x^4+y^4}.$

19.9. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{|x^3|+|y^3|}.$

19.10. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} (x+y)^{-(x^2+y^2)}.$

19.11. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(x^2+y^2\right)^{x^2y^2}.$

19.12. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 1}} f(x,y)$ ni hisoblang, bunda

$$f(x,y)=\begin{cases} \dfrac{x^2 y}{\sqrt{1+x^2y}-1}, & x^2y\neq 0 \ bo'lganda,\\ 2, & x^2y=0 \ bo'lganda.\end{cases}$$

Quyidagi karrali limitlarni hisoblang:

19.13. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}.$

19.14. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sin\left(x^4y^2\right)}{\left(x^2+y^2\right)^2}.$

19.15. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\left(x^2+y^2\right)}{1-\cos\left(x^2+y^2\right)}.$

19.16. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{e^{\frac{1}{x^4+y^4}}}{x^4+y^4}.$

19.17. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(1+x^2y^2)^{\frac{1}{x^2+y^2}}$.

19.18. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{(x+y)^2}{(x^2+y^4)^2}$.

19.19. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}(x^2+y^2)\sin\frac{1}{x^2+y^2}$.

19.20. Ushbu $\lim\limits_{\substack{x\to 1\\ y\to 1}} f(x,y)$ ni hisoblang, bunda

$$f(x,y)=\begin{cases}\dfrac{x^2+2xy-3y^2}{x^3-y^3}, & x\neq y \text{ bo'lganda},\\ 4/3\ , & x=y \text{ bo'lganda}.\end{cases}$$

Quyidagi karrali limitlarning mavjud emasligini isbotlang:

19.21. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x}{x+y}$.

19.22. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x-y}{x+y}$.

19.23. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{x^2+y^2}$.

19.24. $\lim\limits_{\substack{x\to 1\\ y\to 0}}\frac{\ln(x+y)}{y}$.

19.25. Ushbu $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^4+y^2}, & (x,y)\neq(0,0) \text{ bo'lganda},\\ 0\ , & (x,y)=(0,0) \text{ bo'lganda}.\end{cases}$

funksiyaning $(0,0)$ nuqtada karrali limiti mavjud emasligini ko‘rsating.

19.26. Ushbu

$$f(x,y)=\begin{cases}\left(1+\dfrac{1}{x+y}\right)^{x+y}, & x+y\neq 0 \text{ bo'lganda},\\ 1\ , & x+y=0 \text{ bo'lganda}\end{cases}$$

funksiyaning $x\to\infty,\ y\to\infty$ dagi karrali limiti mavjud emasligini isbotlang.

Quyidagi $\lim\limits_{x\to x^0}\lim\limits_{y\to y^0} f(x,y)$ va $\lim\limits_{y\to y^0}\lim\limits_{x\to x^0} f(x,y)$ takroriy limitlarni hisoblang:

19.27. $f(x,y)=\frac{x^2+xy+y^2}{x^2-xy+y^2}$, $x^0=0,\ y^0=0$.

19.28. $f(x,y)=\frac{\sin(x+y)}{2x+3y}$, $x^0=0,\ y^0=0$.

19.29. $f(x,y)=\frac{\cos x-\cos y}{x^2+y^2}$, $x^0=0,\ y^0=0$.

19.30. $f(x,y)=\frac{x^2+y^2}{x^2+y^4}$, $x^0=\infty,\ y^0=\infty$.

19.31. $f(x,y)=\frac{x^y}{1+x^y}$, $x^0=\infty,\ y^0=0$.

19.32. $f(x,y)=\sin\frac{\pi x}{2x+y}$, $x^0=\infty,\ y^0=\infty$.

19.33. $f(x,y)=\frac{1}{xy}tg\frac{xy}{1+xy}$, $x^0=0,\ y^0=\infty$.

19.34. $f(x,y)=\log_x(x+y),\quad x^0=1,\ y^0=0.$

19.35. $f(x,y)=\dfrac{\sin 3x+tg2y}{6x+3y}tg\dfrac{xy}{1+xy},\quad x^0=0,\ y^0=0.$

Quyidagi berilgan funksiyalarning (x^0,y^0) nuqtada karrali va takroriy limitlari mavjudmi?

19.36. $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2},\quad x^0=0,\ y^0=0.$

19.37. $f(x,y)=\log_x(x+y),\quad x^0=1,\ y^0=0.$

19.38. $f(x,y)=\dfrac{\sin x+\sin y}{x+y},\quad x^0=0,\ y^0=0.$

Mustaqil yechish uchun misollarning javoblari

19.1. -2. **19.2.** $0{,}5$. **19.3.** 1. **19.4.** a. **19.5.** 2. **19.6.** e^3. **19.7.** 0. **19.8.** 0. **19.9.** 0. **19.10.** 0. **19.11.** 0. **19.12.** 2. **19.19.** 0. **19.14.** 0. **19.15.** 0. **19.16.** 0. **19.17.** 1. **19.18.** 0. **19.19.** 1. **19.20.** $\frac{4}{3}$. **19.27.** $1,1$. **19.28.** $\frac{1}{2},\frac{1}{3}$. **19.29.** $\frac{1}{2}$ va $\frac{1}{2}$. **19.30.** 0 va 1. **19.31.** $\frac{1}{2}$ va 1. **19.32.** 0 va 1. **19.33.** 0 va 1. **19.34.** 1 va ∞. **19.35.** $\frac{1}{2}$ va $-\frac{2}{3}$. **19.36.** Karrali limit mavjud emas, $\lim\limits_{x\to 0}\lim\limits_{y\to 0}f(x,y)=-1$ va $\lim\limits_{y\to 0}\lim\limits_{x\to 0}f(x,y)=1$. **19.37.** Karrali limit mavjud emas, $\lim\limits_{x\to 1}\lim\limits_{y\to 0}f(x,y)=1$ va $\lim\limits_{y\to 0}\lim\limits_{x\to 1}f(x,y)=\infty$.

19.38. $\lim\limits_{\substack{x\to 0\\ y\to 0}}f(x,y)=\lim\limits_{x\to 0}\lim\limits_{y\to 0}f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 0}f(x,y)=1$.

20-§. Ko'p o'zgaruvchili funksiyaning uzluksizligi

20.1. Uzluksiz funksiyaning ta'riflari. $u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $A=A(a_1,a_2,...,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi va $A\in\{M\}$ bo'lsin.

20.1-ta'rif. Agar $M\to A$ da $u=f(M)$ funksiyaning limiti mavjud bo'lib,

$$\lim_{M\to A}f(M)=f(A)\ \text{yoki}\ \lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \cdots\\ x_m\to a_m}}f(x_1,x_2,...,x_m)=f(a_1,a_2,...,a_m)$$

bo'lsa, u holda $f(M)$ funksiya A *nuqtada uzluksiz* deb ataladi, $A=\lim\limits_{M\to A}M$ bo'lgani uchun, funksiyaning uzluksizlik shartini,

$$\lim_{M\to A} f(M) = f\left(\lim_{M\to A} M\right) \qquad (20.1)$$

ko'rinishda ham yozish mumkin.

$\{M\}$ to'plamning funksiya uzluksizligi shartini qanoatlantirmaydigan nuqtalari funksiyaning *uzilish nuqtalari* deyiladi.

20.2-ta'rif (Geyne ta'rifi). Agar $\{M\}\subset R^m$ to'plamning nuqtalaridan tuzilgan, $A\in\{M\}$ ga intiluvchi har qanday $\{M_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(M_n)\}$ ketma-ketlik, hamma vaqt $f(A)$ ga teng bo'lsa, $f(M)$ funksiya A *nuqtada uzluksiz* deb ataladi.

20.3-ta'rif. (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun, shunday $\delta>0$ topilsaki, $\rho(M,A)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda,

$$|f(M)-f(A)|<\varepsilon$$

tengsizlik bajarilsa, $f(M)$ funksiya A *nuqtada uzluksiz* deb ataladi.

Agar $f(M)$ funksiya $\{M\}$ to'plamning har bir nuqtasida uzluksiz bo'lsa, u holda $f(M)$ funksiya $\{M\}$ *to'plamda uzluksiz* deyiladi.

Ushbu $\Delta u = f(M)-f(A)$ ayirmaga $f(M)$ funksiyaning A nuqtadagi *orttirmasi yoki to'liq orttirmasi* deyiladi.

A va M nuqtalar, mos ravishda, $a_1,a_2,...,a_m$ va $x_1,x_2,...,x_m$ koordinatalarga ega bo'lsin. Ushbu

$$x_1-a_1=\Delta x_1,\quad x_2-a_2=\Delta x_2,...,x_m-a_m=\Delta x_m$$

belgilashlardan foydalanib, funksiya argumentlarining $\Delta x_1,\Delta x_2,...,\Delta x_m$ orttirmalariga moc keluvchi orttirmasi uchun,

$$\Delta u = f(a_1+\Delta x_1,\quad a_2+\Delta x_2,...,a_m+\Delta x_m)-f(a_1,a_2,...,a_m)$$

ifodani hosil qilamiz. Ravshanki, $u=f(M)$ funksiyaning A nuqtada uzluksiz bo'lishi uchun, uning orttirmasi A nuqtada cheksiz kichik bo'lishi zarur va yetarli, ya'ni:

$$\lim_{M\to A}\Delta u = \lim_{M\to A}\Delta f = \lim_{M\to A}(f(M)-f(A))=0 \text{ yoki } \lim_{\substack{\Delta x_1\to 0\\ \Delta x_2\to 0\\ \dots\\ \Delta x_m\to 0}}\Delta u = 0 \qquad (20.2)$$

bo'lishi zarur va yetarli. (20.2) shartga $u=f(M)$ funksiyaning A nuqtada uzluksizligining *ayirma shakli deyiladi.*

Ko'p o'zgaruvchili funksiyaning bitta argumenti bo'yicha (qolganlari belgilangan deb) uzluksizlik tushunchasini ham kiritish mumkin. Bu tushunchani kiritish uchun, $u=f(M)=f(x_1,x_2,...,x_m)$ funksiyaning, uning aniqlanish sohasiga tegishli $M(x_1,x_2,...,x_m)$ nuqtadagi, *xususiy orttirmalari* deb ataluvchi tushunchalarini kiritamiz. Boshqa argumentlari belgilangan deb qarab, funksiyaning birinchi argumentiga Δx_1 orttirma beramiz. Ushbu

$$x_1+\Delta x_1,\quad x_2,\dots,x_m$$

koordinatalarga ega bo'lgan nuqta funksiyaning aniqlanish sohasida yotsin.

Bu orttirma, funksiyaning $M(x_1,x_2,\dots,x_m)$ nuqtadagi, x argumentning Δx_1 orttirmasiga mos keluvchi, *xususiy orttirmasi* deyiladi va $\Delta_{x_1}u$ kabi belgilanadi:

$$\Delta_{x_1}u=f(x_1+\Delta x_1,x_2,\dots,x_m)-f(x_1,x_2,\dots,x_m).$$

Funksiyaning qolgan argumentlari orttirmalariga mos keluvchi xususiy orttirmalari ham, shunga o'xshash aniqlanadi:

$$\left.\begin{array}{l}\Delta_{x_2}u=f(x_1,x_2+\Delta x_2,\dots,x_m)-f(x_1,x_2,\dots,x_m)\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \Delta_{x_m}u=f(x_1,x_2,\dots x_m+\Delta x_m)-f(x_1,x_2,\dots,x_m).\end{array}\right|$$

Agar $\Delta x_k\to 0$ da funksiyaning $\Delta x_k f$ xususiy orttirmasi ham nolga intilsa, ya'ni $\lim\limits_{\Delta x_k\to 0}\Delta x_k f=0$ bo'lsa, $f(x_1,x_2,\dots,x_m)$ funksiya $M(x_1,x_2,\dots,x_m)$ nuqtada x_k *o'zgaruvchi bo'yicha uzluksiz* deyiladi.

20.1-eslatma. Agar $f(x_1,x_2,\dots,x_m)$ funksiya $M_0(x_1^0,x_2^0,\dots,x_m^0)\in\{M\}$ nuqtada (bir yo'la) uzluksiz bo'lsa, funksiya shu nuqtada har bir o'zgaruvchi bo'yicha ham uzluksiz bo'ladi, lekin funksiyaning biror nuqtada har bir o'zgaruvchi bo'yicha xususiy uzluksiz bo'lishidan, uning shu nuqtada (bir yo'la) uzluksiz bo'lishi har doim ham kelib chiqavermaydi.

20.1- misol. Ushbu

$$f(x,y)=\begin{cases}\dfrac{x^4+x^2y^2+y^4}{x^2+y^2}, & (x,y)\neq(0,0)\ bo'lganda,\\ 0, & (x,y)=(0,0)\ bo'lganda\end{cases}$$

funksiyaning R^2 da uzluksiz ekanligini ko'rsating.

Yechilishi. $\forall(a,b)\in R^2$ $((a,b)\neq(0,0))$ nuqtani olamiz:

$$\lim_{\substack{x\to a\\ y\to b}}f(x,y)=\lim_{\substack{x\to a\\ y\to b}}\frac{x^4+x^2y^2+y^4}{x^2+y^2}=\frac{a^4+a^2b^2+b^4}{a^2+b^2}=f(a,b)$$

tenglik o'rinli bo'ladi. Bu esa, berilgan funksiyaning 20.1-ta'rifga ko'ra, (a,b) nuqtada uzluksizligini bildiradi.

Endi, berilgan funksiyaning (0;0) nuqtada uzluksiz ekanligini ko'rsatamiz. Buning uchun quyidagi almashtirishni olamiz: $x = r\cos\varphi,\ y = r\sin\varphi$.

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{r\to 0} f(r\cos\varphi, r\sin\varphi) = \lim_{r\to 0} r^2\left(\cos^2\varphi + \sin^4\varphi\right) = 0$$

Demak, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0;0) = 0$. Bundan $f(x,y)$ funksiyaning (0;0) nuqtada uzluksiz ekanligi kelib chiqadi.

20.2- misol. Ushbu $f(x,y) = 2x - 3y + z$ funksiyaning R^2 da uzluksiz ekanligini ko'rsating.

Yechilishi $\forall\varepsilon > 0$ songa ko'ra, $\delta = \dfrac{\varepsilon}{6}$ deb olinsa, u holda,

$$\rho(M(x,y,z), M_0(x_0,y_0,z_0)) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$$

tengsizlikni qanoatlantiruvchi $\forall M(x,y,z) \in R^m$ nuqtalarda,

$$|f(x,y,z) - f(x_0,y_0,z_0)| = |(2x - 3y + z) - (2x_0 - 3y_0 + z_0)| \le |2(x-x_0)| +$$

$$+ 3|y - y_0| + |z - z_0| \le 6\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < 6\delta = \varepsilon$$

tengsizlik o'rinli bo'ladi. Bundan, 20.3- ta'rifga ko'ra, $f(x,y) = 2x - 3y + z$ funksiyaning $\forall M_0(x_0,y_0,z_0)$ nuqtada uzluksizligi kelib chiqadi.

20.3-misol. Ushbu $f(x,y) = \dfrac{x - 2y + 4}{x^2 + y^2 + 3}$ funksiyaning $\forall M_0(x_0,y_0) \in R^2$ nuqtada uzluksiz ekanligini ko'rsating.

Yechilishi. $M_0(x_0;y_0)$ nuqtaga Δx, Δy ortirmalar berib, berilgan funksiyaning to'liq orttirmasini topamiz:

$$\Delta f(x_0,y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0) =$$

$$\frac{(x_0 + \Delta x) - 2(y_0 + \Delta y) + 4}{(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 3} - \frac{x_0 - 2y_0 + 4}{x_0^2 + y_0^2 + 3} =$$

$$= \frac{\left(x_0^2 + y_0^2 + 3\right)[(x_0 + \Delta x) - 2(y_0 + \Delta y) + 4] - (x_0 - 2y_0 + 4)[(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 3]}{\left(x_0^2 + y_0^2 + 3\right)[(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 3]}$$

Bu tenglikdan $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta f(x_0,y_0) = 0$ ekanligi kelib chiqadi. (20.2) shartga ko'ra, berilgan funksiyaning $\forall M_0(x_0,y_0)$ nuqtada uzluksiz ekanligi kelib chiqadi.

20.4- misol. Ushbu

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4 + y^4}, & x^4 + y^4 \ne 0 \ \ bo'lganda, \\ 0, & x^4 + y^4 = 0 \ \ bo'lganda \end{cases}$$

funksiyaning $O(0,0)$ nuqtada har bir o'zgaruvchisi bo'yicha uzluksiz, ikkala o'zgaruvchisi bo'yicha bir yo'la uzluksiz emasligini ko'rsating.

Yechilishi. $f(x,y)$ funksiyaning $O(0,0)$ nuqtadagi xususiy orttirmalarini qaraymiz: x bo'yicha xususiy orttirmasi

$$\Delta_x u = f(\Delta x, 0) - f(0,0) = 0 - 0 = 0.$$

Ravshanki, $\lim\limits_{\Delta x \to 0} \Delta_x u = 0$. Bundan $f(x,y)$ funksiyaning $O(0,0)$ nuqtada x argument bo'yicha uzluksizligi kelib chiqadi.

Xudi shunday, $f(x,y)$ funksiyaning $O(0,0)$ nuqtada y argument bo'yicha ham uzluksizligi ko'rsatiladi.

Endi, $f(x,y)$ funksiyaning $O(0,0)$ nuqtada ikkala o'zgaruvchi bo'yicha bir yo'la uzluksiz emasligini ko'rsatamiz. $M(x,y)$ nuqta $O(0,0)$ nuqtaga, $O(0,0)$ nuqtadan o'tuvchi $y = kx$ to'g'ri chiziqlar bo'ylab intilsin. U holda,

$$\lim_{\substack{x \to 0 \\ y \to 0 \\ y = kx}} \frac{x^2 y^2}{x^4 + y^4} = \lim_{x \to 0} \frac{k^2 x^4}{x^4 + k^4 x^4} = \frac{k^2}{1 + k^4}.$$

Demak, $M(x,y)$ nuqta, turli to'g'ri chiziqlar (k ning har xil qiymatida) bo'yicha, $O(0,0)$ nuqtaga intilganda limitning qiymati turlicha bo'ladi. Bu hol, qaralayotgan limitning mavjud emasligini bildiradi. Shunday qilib, berilgan funksiyaning $O(0,0)$ nuqtadagi karrali limiti mavjud emas ekan.

20.5- misol Ushbu

$$u = f(x,y) = \begin{cases} \dfrac{\cos x - \cos y}{x - y}, & x - y \neq 0 \text{ bo'lganda}, \\ 0, & x - y = 0 \text{ bo'lganda} \end{cases}$$

funksiyani $O(0,0)$, $A(\frac{\pi}{4}; \frac{\pi}{4})$ nuqtalarda har bir argumenti va ikkala argumenti bo'yicha bir yo'la uzluksizlikka tekshiring.

Yechilishi. Kosinuslar ayirmasi formulasidan foydalanib, berilgan funksiyani quyidagi ko'rinishda yozib olamiz:

$$f(x,y) = \begin{cases} \dfrac{-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{x-y}, & x - y \neq 0 \text{ bo'lganda}, \\ 0, & x - y = 0 \text{ bo'lganda}, \end{cases}$$

$$1)\ \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = -\lim_{\substack{x \to 0 \\ y \to 0}} \sin\frac{x+y}{2} \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin\frac{x-y}{2}}{\frac{x-y}{2}} = -\lim_{\substack{x \to 0 \\ y \to 0}} \sin\frac{x+y}{2} \lim_{t \to 0} \frac{\sin\frac{t}{2}}{\frac{t}{2}} = 0 \cdot 1 = 0 = f(0,0).$$

Demak, $f(x,y)$ funksiya $O(0;0)$ nuqtada ikkala argumenti bo'yicha bir yo'la uzluksiz. U holda, 20.1-eslatmaga asosan, har bir argumenti bo'yicha ham uzluksiz bo'ladi.

2) $$\lim_{\substack{x\to\frac{\pi}{4}\\ y\to\frac{\pi}{4}}} f(x,y) = -\lim_{\substack{x\to\frac{\pi}{4}\\ y\to\frac{\pi}{4}}}\sin(x+y)\lim_{\substack{x\to\frac{\pi}{4}\\ y\to\frac{\pi}{4}}}\frac{\sin\frac{x-y}{2}}{\frac{x-y}{2}} = -1,$$

lekin $f\left(\frac{\pi}{4};\frac{\pi}{4}\right)=0$ bo'lgani uchun, $f(x,y)$ funksiya $A\left(\frac{\pi}{4};\frac{\pi}{4}\right)$ nuqtada ikkala argumenti bo'yicha bir yo'la uzluksiz emas.

Ushbu $$f\left(x,\frac{\pi}{4}\right)=\begin{cases}\dfrac{-2\sin\dfrac{x+\frac{\pi}{4}}{2}\sin\dfrac{x-\frac{\pi}{4}}{2}}{x-\frac{\pi}{4}}, & x\neq\frac{\pi}{4}\ \textit{bo'lganda},\\ 0, & x=\frac{\pi}{4}\ \textit{bo'lganda}\end{cases}$$

funksiyani qaraymiz. Funksiyaning berilishiga ko'ra, $\forall x$ uchun $f\left(x,\frac{\pi}{4}\right)=0$, jumladan, u $x=\frac{\pi}{4}$ da ham nolga teng bo'ladi.

$$\lim_{x\to\frac{\pi}{4}} f\left(x,\frac{\pi}{4}\right)=\lim_{x\to\frac{\pi}{4}}\frac{-2\sin\dfrac{x+\frac{\pi}{4}}{2}\sin\dfrac{x-\frac{\pi}{4}}{2}}{x-\frac{\pi}{4}}=-\lim_{x\to\frac{\pi}{4}}\sin\frac{x+\frac{\pi}{4}}{2}\lim_{x\to\frac{\pi}{4}}\frac{\sin\dfrac{x-\frac{\pi}{4}}{2}}{(x-\frac{\pi}{4})/2}=$$
$$=-\frac{\sqrt{2}}{2}\cdot 1=-\frac{\sqrt{2}}{2}\neq f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=0.$$

Demak, $f\left(x,\frac{\pi}{4}\right)$ funksiya $x=\frac{\pi}{4}$ da uzluksiz emas.

Endi

$$f\left(\frac{\pi}{4},y\right)=\begin{cases}\dfrac{-2\sin\dfrac{\frac{\pi}{4}+y}{2}\sin\dfrac{\frac{\pi}{4}-y}{2}}{\frac{\pi}{4}-y}, & y\neq\frac{\pi}{4}\ \textit{bo'lganda},\\ 0, & y=\frac{\pi}{4}\ \textit{bo'lganda}\end{cases}$$

funksiyani qaraymiz. Funksiyaning berilishiga ko'ra, $\forall y$ uchun,

$$\lim_{y\to\frac{\pi}{4}} f\left(\frac{\pi}{4},y\right)=\lim_{y\to\frac{\pi}{4}}\frac{-2\sin\dfrac{\frac{\pi}{4}+y}{2}\sin\dfrac{\frac{\pi}{4}-y}{2}}{\frac{\pi}{4}-y}=-\lim_{y\to\frac{\pi}{4}}\sin\frac{\frac{\pi}{4}+y}{2}\lim_{y\to\frac{\pi}{4}}\frac{\sin\dfrac{\frac{\pi}{4}-y}{2}}{(\frac{\pi}{4}-y)/2}=$$

$$= -\frac{\sqrt{2}}{2}\cdot 1 = -\frac{\sqrt{2}}{2} \neq f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = 0.$$

Demak, $f\left(\frac{\pi}{4}, y\right)$ funksiya $y = \frac{\pi}{4}$ nuqtada uzluksiz emas ekan.

20.6- misol Ushbu

$$f(x,y) = \frac{x+y}{x^3+y^3}$$

funksiyaning uzilish nuqtalarini toping.

Yechilishi. Berilgan funksiya, kasr rasional funksiya bo‘lib, uning surati ham, maxraji ham uzluksiz funksiyalardan iborat. Shuning uchun, funksiyaning uzilish nuqtalari, $x^3 + y^3 = 0$ shartni qanoatlantiruvchi nuqtalardan iborat bo‘ladi. Bu tenglamani y ga nisbatan echamiz: $y = -x$.

Shunday qilib, berilgan funksiyaning uzilish nuqtalari, $y = -x$ to‘g‘ri chiziqning nuqtalaridan iborat ekan.

$x_0 \neq 0$, $y_0 \neq 0$ va $x_0 + y_0 = 0$ bo‘lsin. U holda,

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{x+y}{x^3+y^3} = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{1}{x^2 - xy + y^2} = \frac{1}{x_0^2 - x_0 y_0 + y_0^2}.$$

Demak, $y = -x\ (x \neq 0)$ to‘g‘ri chiziqning nuqtalari, $f(x,y) = \frac{x+y}{x^3+y^3}$ funksiyaning yo‘qotilishi mumkin bo‘lgan (birinchi tur) uzilish nuqtalari bo‘lar ekan.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+y}{x^3+y^3} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1}{x^2 - xy + y^2} = +\infty$$

munosabatdan esa, $O(0,0)$ nuqta berilgan funksiyaning cheksiz (ikkinchi tur) uzilish nuqtasi ekanligi kelib chiqadi.

20.2. Uzluksiz funksiyalar ustida arifmetik amallar. Murakkab funksiyaning uzluksizligi. $u = f(M)$ **va** $v = g(M)$ **funksiyalar** $\{M\} \subset R^m$ **to‘plamda berilgan,** $A \in R^m$ **nuqta esa,** $\{M\}$ **to‘plamning limit nuqtasi va** $A \in \{M\}$ **bo‘lsin.**

20.1-teorema. Agar $f(M)$ va $g(M)$ funksiyalar A nuqtada uzluksiz bo‘lsa, $f(M) \pm g(M)$, $f(M)\cdot g(M)$ hamda $\frac{f(M)}{g(M)}(g(A) \neq 0)$ funksiyalar ham shu nuqtada uzluksiz bo‘ladi.

20.2-eslatma. Ikkita funksiyaning yig'indisi, ayirmasi, ko'paytmasi va nisbati uzluksizligidan, ulardan har birining uzluksizligi har doim ham kelib chiqa vermaydi.

20.7-misol. Ushbu $\{M\}=\{(x,y)\in R^2 : |x|\le 1, y\le 1\}\subset R^2$ to'plamning rasional nuqtalari (har ikkala koordinatasi ham rasional son) to'plamini $\{M_p\}$ deb belgilaymiz. $\{M\}$ to'plamda

$$f_1(x,y)=\begin{cases}1, & (x,y)\in\{M_p\},\\ 0, & (x,y)\in\{M\}/\{M_p\}\end{cases}$$

va

$$f_2(x,y)=\begin{cases}-1, & (x,y)\in\{M_p\}\\ 0, & (x,y)\in\{M\}/\{M_p\}\end{cases}$$

funksiyalarni qaraymiz. Bu funksiyalarning yig'indisi

$$f_1(x;y)+f_2(x;y)=0 \quad ((x;y)\in\{M\})$$

bo'lib, u shu to'plamda uzluksiz, lekin $f_1(x;y)$, $f_2(x;y)$ funksiyalarning har biri $\{M\}$ to'plamda uzluksiz emas.

20.3-eslatma. Yuqoridagi 20.1-teorema qo'shiluvchilar soni ixtiyoriy chekli bo'lgan holda ham o'rinli.

$u=f(M)=f(x_1,x_2,\dots,x_m)$ funksiya $\{M\}$ $(\{M\}\subset R^m)$ to'plamda berilgan bo'lib, $x_1,x_2,\dots,x_m$ o'zgaruvchilarning har biri, o'z navbatida, $\{N\}$ $(\{N\}\subset R^k\,(k=1,2\dots))$ to'plamda,

$$\left.\begin{array}{l} x_1=\varphi_1(t_1,t_2,\dots,t_k),\\ x_2=\varphi_2(t_1,t_2,\dots,t_k),\\ \dots\dots\dots\dots\dots\dots\\ x_m=\varphi_m(t_1,t_2,\dots,t_k)\end{array}\right| \qquad (20.3)$$

ko'rinishda berilgan bo'lsin. $t=(t_1,t_2,\dots,t_k)\in\{N\}\subset R^k$ bo'lganda $M=M(x_1,x_2,\dots,x_m)\in\{M\}\subset R^m$ deb qaraymiz. Natijada, har bir $M(x_1,x_2,\dots,x_m)\in\{M\}$ nuqtaga, (20.3) formula yordamida,

$$u=f(\varphi_1(t_1,t_2,\dots,t_k),\varphi_2(t_1,t_2,\dots,t_k),\dots,\varphi_m(t_1,t_2,\dots,t_k))=F(t_1,t_2,\dots,t_k)=F(t)$$

nuqta mos qo'yiladi, ya'ni murakkab funksiya hosil bo'ladi.

20.2-teorema. Agar $\varphi_i(t_1,t_2,\dots,t_k)$ $(i=1,2,\dots,m)$ funksiyalarning har biri $t_0=(t_1^0,t_2^0,\dots,t_k^0)$ nuqtada uzluksiz bo'lib, $f(M)=f(x_1,x_2,\dots,x_m)$ funksiya esa, $t_0=(t_1^0,t_2^0,\dots,t_k^0)$ nuqtaga mos,

$$M_0(x_0)=(x_1^0,x_2^0,\dots,x_m^0)(x_1^0=\varphi_1(t_1^0,t_2^0,\dots,t_k^0),\ x_2^0=\varphi_2(t_1^0,\dots,t_k^0),\dots,x_m^0=\varphi_n(t_1^0,t_2^0,\dots,t_k^0))$$

nuqtada uzluksiz bo'lsa, $u=F(t_1,t_2,\dots,t_k)$ murakkab funksiya $t_0=(t_1^0,t_2^0,\dots,t_k^0)$ nuqtada uzluksiz bo'ladi.

20.3. Nuqtada uzluksiz bo'lgan funksiyalarning lokal xossalari
$u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lsin. Bu to'plamdan $M_0(x_1^0,...,x_m^0)\in\{M\}$ nuqtani olib, uning etarli kichik atrofini qaraymiz.

1-*xossa*. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, u holda, funksiya M_0 nuqtaning etarli kichik atrofida chegaralangan bo'ladi.

2-*xossa*. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lib, $f(M_0)>0$ $(f(M)<0)$ bo'lsa, M_0 nuqtaning etarli kichik atrofidagi M nuqtalarda ham $f(M)>0$ $(f(M)<0)$ bo'ladi.

3-*xossa*. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, M_0 nuqtaning etarli kichik atrofidagi ixtiyoriy $M_1\in\{M\}, M_2\in\{M\}$ nuqtalar uchun, $|f(M_1)-f(M_2)|<\varepsilon$ tengsizlik o'rinli bo'ladi.

20.4.Uzluksiz bo'lgan funksiyalarning global xossalari.

20.4-ta'rif. Agar shunday c va C sonlar mavjud bo'lib, $\forall M\in\{M\}$ uchun $c\le f(M)\le C$ tengsizlik o'rinli bo'lasa, $u=f(M)$ funksiya shu $\{M\}$ to'plamda *chegaralangan* deyiladi.

20.3-teorema(Bolsano-Koshining birinchi teoremasi). $u=f(M)$ funksiya $\{M\}\subset R^m$ bog'lamli to'plamda berilgan va uzluksiz bo'lsin. Agar bu funksiya to'plamning ikkita $A(a_1,a_2,...,a_m)\in\{M\}$, $B(b_1,b_2,...,b_m)\in\{M\}$ nuqtasida har xil ishorali qiymatlarga ega bo'lsa, u holda, shunday $C(c_1,c_2,...,c_m)\in\{M\}$ nuqta topiladiki, shu nuqtada funksiya nolga aylanadi, ya'ni $f(C)=f(c_1,c_2,...,c_m)=0$.

20.4-teorema(Bolsano-Koshining ikkinchi teoremasi). $u=f(M)$ funksiya $\{M\}\subset R^m$ bog'lamli to'plamda berilgan va uzluksiz bo'lib, $\{M\}$ to'plamning ikkita $A(a_1,a_2,...,a_m)$, $B(b_1,b_2,...,b_m)$ nuqtasida $f(A)\ne f(B)$ bo'lsa, u holda, $f(A)$ va $f(B)$ qiymatlar orasida har qanday C son olinsa ham, $\{M\}$ to'plamda shunday $C(c_1,c_2,...,c_m)$ nuqta topiladiki, $f(C)=f(c_1,c_2,...,c_m)=C$ bo'ladi.

20.5-teorema (Veyershtrassning birinchi teoremasi). Agar $u=f(M)$ funksiya, $\{M\}\subset R^m$–chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, funksiya shu $\{M\}$ to'plamda chegaralangan bo'ladi.

20.6- teorema(Veyershtrassning ikkinchi teoremasi). Agar $f(M)$ funksiya chegaralangan yopiq $\{M\}\subset R^m$ to'plamda uzluksiz bo'lsa, u shu to'plamda o'zining aniq yuqori hamda aniq quyi chegarasiga erishadi.

20.5. Ko'p o'zgaruvchili funksiyaning tekis uzluksizligi. $f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lsin.

20.4- ta'rif. Agar $\forall\varepsilon>0$ son uchun, shunday $\delta>0$ topilsaki, $\{M\}$ to'plamning $\rho(M',M'')<\delta$ tengsizlikni qanoatlantiruvchi $\forall M',M''\in\{M\}$ nuqtalarida

$$f(M')-f(M'')<\varepsilon$$

tengsizlik bajarilsa, $f(M)$ funksiya $\{M\}$ to'plamda *tekis uzluksiz* deyiladi.

20.4-eslatma. Funksiyaning tekis uzluksizligi ta'rifidagi $\delta>0$ son faqat $\varepsilon>0$ ga bog'liq bo'ladi.

20.7-teorema (Kantor teoremasi). Agar $f(M)$ funksiya chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, funksiya shu to'plamda tekis uzluksiz bo'ladi.

20.5-eslatma. Chegaralangan $\{M\}$ to'plamning $\forall M',M''\in\{M\}$ nuqtalari orasidagi $\rho(M',M'')$ masofaning aniq yuqori chegarasiga $\{M\}$ to'plamning *diametri* deyiladi.

Natija. Agar $f(M)$ funksiya chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, u holda $\forall\varepsilon>0$ son uchun shunday $\delta>0$ topilib, $\{M\}$ to'plamni, *diametri* δ dan kichik bo'lgan, $\{M_i\}\subset\{M\}(i=1,2,..,k)$ bo'laklarga bo'lganimizda, bu bo'laklarning har birida ham, $f(M)$ funksiyaning tebranishi ε dan kichik bo'ladi.

20.8-misol. Ushbu $u=\dfrac{x^3+y^3}{x^2+y^2}$ funksiyani $\{M\}=\{(x,y):0<x^2+y^2\le 1\}$ to'plamda tekis uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiyaning surati x^3+y^3 va maxraji x^2+y^2 $\{M\}$ to'plamda uzluksiz. Demak, funksiya $\{M\}$ da uzluksiz $(O(0,0)\notin\{M\})$. $\{M\}$ to'plam yopiq to'plam emas. Shuning uchun, bu holda Kantor teoremasini qo'llab bo'lmaydi, lekin $u(x,y)$ funksiyani $O(0,0)$ nuqtada qayta aniqlasak, ya'ni $u(0;0)=0$ desak, u holda $u(x,y)$ funksiya $O(0;0)$ nuqtada uzluksiz bo'ladi. Haqiqatan ham, $x=\rho\cos\varphi,\ y=\rho\sin\varphi$ almashtirishni olsak, $x\to 0, y\to 0$ da $\rho\to 0$, ya'ni $\lim\limits_{\substack{x\to 0\\ y\to 0}}u(x,y)=\lim\limits_{\rho\to 0}u(\rho\cos\varphi,\rho\sin\varphi)=\lim\limits_{\rho\to 0}\rho=0$ Shunday qilib, $u(x,y)$ funksiyani $O(0;0)$ nuqtada qayta aniqlasak, ya'ni $u(0;0)=0$ desak, u holda berilgan funksiya $\{\overline{M}\}=\{(x,y):x^2+y^2\le 1\}$ yopiq to'plamda uzluksiz bo'ladi. Demak, Kantor teoremasiga asosan, berilgan funksiya, markazi $O(0;0)$ olib tashlangan $\{\overline{M}\}$- doirada, ya'ni $\{M\}$ to'plamda tekis uzluksiz bo'ladi.

20.9- misol Ushbu $u=f(x,y)=arc\sin\dfrac{x}{y}$ funksiyani aniqlanish sohasida tekis uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiyaning aniqlanish sohasi:

$$\{M\}=\{(x,y):|x|\le|y|,\ y\ne 0\}.$$

Bu sohada $f(x,y)=arc\sin\frac{x}{y}$ funksiya uzluksiz funksiyalarning superpozisiyasi sifatida, uzluksiz. Endi bu funksiyani $\{M\}$ sohada tekis uzluksizlikka tekshiramiz: $M_n\left(\frac{1}{n};\frac{1}{n}\right)$, $M'_n\left(\frac{1}{n};-\frac{1}{n}\right)$ $(n=1,2...)$ nuqtalar uchun

$$\rho(M_n;M'_n)=\sqrt{\left(\frac{1}{n}-\frac{1}{n}\right)^2+\left(\frac{1}{n}+\frac{1}{n}\right)^2}=\frac{2}{n}\underset{n\to\infty}{\to}0.$$

Funksiyaning bu nuqtalardagi qiymatlari,

$$|u(M_n)-u(M'_n)|=|arc\sin 1-arc\sin(-1)|=2arc\sin 1=\pi>1=\varepsilon$$

munosabatni qanoatlantiradi, ya'ni 20.4- ta'rifning sharti bajarilmaydi.

Demak, berilgan funksiya $\{M\}$ to'plamda tekis uzluksiz emas.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning uzilish nuqtalarini toping:

20.3. $u=\ln(9-x^2-y^2)$ **20.4.** $u=\frac{xy}{x+y}$.

20.5. $u=\sin\frac{x}{y}$. **20.6.** $u=\sin\frac{1}{xy}$.

Quyidagi funksiyalarning hamma uzilish nuqtalarini toping va ular ichida ikkala argument bo'yicha yo'qotilishi mumkin bo'lgan uzilish nuqtalarini ko'rsating:

20.7. $u=\frac{x^3}{x^2+y^2}$. **20.8.** $u\frac{x^2}{x^2+y^2}$.

20.9. $u=\begin{cases}\frac{x^3+y^3}{x+y}, & x+y\ne 0\ bo'lganda,\\ 3, & x+y=0\ bo'lganda.\end{cases}$

20.10. $u\frac{1}{\sin^2 x+\sin^2 y}$. **20.11.** $u=x\sin\frac{x^2}{x^2+y^2}$.

20.12. $u=\frac{\sin^2 x\cdot\sin y}{\sin^4 x+\sin^2 y}$.

20.13. $u=\begin{cases}(x^2+y^2-1)\sin\frac{1}{1-x^2-y^2}, & x^2+y^2\ne 1 bo'lganda,\\ 0, & x^2+y^2=1\ bo'lganda.\end{cases}$

Quyidagi funksiyalarni ko'rsatilgan nuqtalarda har bir argumenti va hamma argumentlari bo'yicha bir yo'l uzluksizlikka tekshiring:

20.14. $u=\begin{cases}\dfrac{x^2y^2}{x^4+y^4}, & x^4+y^4\neq 0 \ bo'lganda,\\ 0, & x^4+y^4=0 \ bo'lganda,\end{cases}$ $O(0;0)$ va $A(1;2)$.

20.15. $u=\begin{cases}\dfrac{x^3y^2}{x^4+y^4}, & x^4+y^4\neq 0 \ bo'lganda,\\ 0, & x^4+y^4=0 \ bo'lganda.\end{cases}$ $O(0;0)$ va $A(10^{-4};10^{-5})$.

20.16. $u=\begin{cases}\dfrac{x^2+y^2}{x+y}, & x^4+y^4\neq 0 \ bo'lganda,\\ 0, & x^4+y^4=0 \ bo'lganda,\end{cases}$ $O(0;0)$ va $A(1;-1)$.

20.17. $u=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \ bo'lganda,\\ 1, & x^2+y^2=0 \ bo'lganda,\end{cases}$ $O(0;0)$ ва $A(0;1)$.

20.18. $u=\begin{cases}\dfrac{\sin x+\sin y}{x+y}, & x+y\neq 0 \ bo'lganda,\\ 1, & x+y=0 \ bo'lganda,\end{cases}$ $0(0;0)$ va $A\left(\dfrac{\pi}{3};-\dfrac{\pi}{3}\right)$

Quyidagi funksiyalarning ko'rsatilgan sohada chegaralangan yoki chegaralanmagan ekanligini aniqlang:

20.19. $u=x^2-y^2,\ \{M\}=\{(x,y): x^2+y^2\leq 25\}$.

20.20. $u=x^2-y^2,\ \{M\}=\{(x,y): x^2+y^2>25\}$.

20.21. $u=\dfrac{ax^2+by^2}{x^2+y^2}, x^2+y^2\neq 0$, ($a$ va b haqiqiy sonlar).

20.22. $u=\dfrac{\sin(x+y)-\sin(x-y)}{xy},\quad xy\neq 0$.

20.23. Ushbu $u=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0 \ bo'lganda,\\ 0, & x^2+y^2=0 \ bo'lganda\end{cases}$

funksiya $O(0,0)$ nuqtada: 1) x bo'yicha uzluksiz; 2) y bo'yicha uzluksiz; 3) ikkala argumenti bo'yicha bir yo'la uzluksiz bo'ladimi?

20.24. a ning qanday qiymatida ushbu

$$u=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \ bo'lganda,\\ a, & x^2+y^2=0 \ bo'lganda\end{cases}$$

funksiya $O(0,0)$ nuqtada: 1) x bo'yicha uzluksiz; 2) y bo'yicha uzluksiz; 3) $y=k\sqrt{x}$, $(k\neq 0)$ chiziq bo'yicha uzluksiz; 4) ikkala argumenti bo'yicha bir yo'la uzluksiz bo'ladimi?

20.25. a ning qanday qiymatida ushbu

$$u=\begin{cases}\dfrac{x^3-xy^2}{x^2+y^2}, & x^2+y^2\neq 0 \text{ bo'lganda},\\ a, & x^2+y^2=0 \text{ bo'lganda}\end{cases}$$

funksiya $O(0,0)$ nuqtada: 1) $x=\alpha t,\ y=\beta t\ (\alpha^2+\beta^2\neq 0)$ chiziq bo'yicha uzluksiz; 2) ikkala argumenti bo'yicha bir yo'la uzluksiz bo'ladi?

20.26. Ushbu $$u=\begin{cases}\dfrac{x^2y}{x^4+y^4}, & x^4+y^4\neq 0 \text{ bo'lganda},\\ 0, & x^4+y^4=0 \text{ bo'lganda}\end{cases}$$

funksiyaning $O(0,0)$ nuqtada: $x=t\cos\alpha,\ y=t\sin\alpha\ (0\le t<\infty)$ chiziq bo'yicha uzluksizligini; ikkala argumenti bo'yicha bir yo'la uzilishga ega ekanligini ko'rsating.

20.27. a ning qanday qiymatida ushbu

$$u=\begin{cases}\dfrac{1}{x+y}e^{-\frac{1}{|x+y|}}, & x+y\neq 0 \text{ bo'lganda},\\ a, & x+y=0 \text{ bo'lganda}\end{cases}$$

funksiya R^2 da uzluksiz bo'ladi.

20.28. a va b ning qanday qiymatlarida ushbu

$$u=\begin{cases}a, & x^2+y^2\le 4 \text{ bo'lganda},\\ \sqrt{9-x^2-y^2}-\sqrt{x^2+y^2-4}, & 4<x^2+y^2\le 9 \text{ bo'lganda},\\ b, & x^2+y^2>9 \text{ bo'lganda}\end{cases}$$

funksiya R^2 da uzluksiz bo'ladi?

Quyidagi funksiyalarning ko'rsatilgan sohada chegaralanganligini isbotlang, uning aniq chegaralarini toping hamda funksiyaning aniq chegaralariga erishishi yoki erishmasligini aniqlang:

20.29. $u=\dfrac{x^2-y^2}{x^2+y^2},\ x^2+y^2\neq 0$.

20.30. $u=\dfrac{x^6-y^6}{x^2+y^2},\ \{M\}=\{(x,y):0<x^2+y^2\le 9\}$.

20.31. $u=\dfrac{x^2y^2}{x^4+y^4},\ x^4+y^4\neq 0$.

20.32. $u=xye^{-xy},\ \{M\}=\{(x,y):x\ge 0,\ y\ge 0\}$.

Quyidagi funksiyalarning ko'rsatilgan to'plamda tekis uzluksiz ekanligini ta'rifga asosan isbotlang:

20.33. $u=ax+by+c,\ \ (a\neq 0,\ b\neq 0),\ \ R^2$.

20.34. $u=x^2+y^2,\ \{M\}=\{(x,y):x^2+y^2\le 1\}$.

20.35. $u=\sqrt{x^2+y^2},\ \{M\}=\{(x,y):|x|<\infty,\ |y|<\infty\}$.

20.36. $u=\sqrt{x^2+y^2+z^2},\ \ R^3$.

20.37. $u=x^3-y^3,\ \{M\}=\{(x,y):1\le x\le 2,\ 0\le y\le 2\}$.

20.38. $u=\frac{x^4+y^4}{x^2+y^2}$, $\{M\}=\{(x,y):0<x^2+y^2\le 25\}$.

20.39. $u=xy\sin\frac{1}{y}$, $\{M\}=\{(x,y):0<x<1,\ 0<y<1\}$.

20.40. $u=\ln(x^2+y^2)$, $\{M\}=\{(x,y):x^2+y^2\ge 1\}$ t.

Quyidagi funksiyalarning ko'rsatilgan to'plamda tekis uzluksiz emasligini ko'rsating:

20.41. $u=\frac{\sqrt{x^4+y^4}}{x^2+y^2}$, $\{M\}=\{(x,y):0<x^2+y^2\le 1\}$.

20.42. $u=x\sin\frac{1}{y}$, $\{M\}=\{(x,y):0<x<1,\ 0<y<1\}$.

20.43. $u=\frac{x^2+y^2}{x^4+y^4}$, $\{M\}=\{(x,y):0<x^2+y^2<1\}$.

20.44. $u=\sin\frac{\pi}{1-x^2-y^2}$, $\{M\}=\{(x,y):x^2+y^2<1\}$.

20.45. $u=\sin\frac{\pi}{x^2+y^2-1}$, $\{M\}=\{(x,y):x^2+y^2<1\}$.

Quyidagi funksiyalarni ko'rsatilgan to'plamda tekis uzluksizlikka tekshiring:

20.46. $u=2x-3y+5$, $\{M\}=\{(x,y):|x|<\infty,\ |y|<\infty\}$.

20.47. $u=\sqrt{4-x^2-y^2}$, $\{M\}=\{(x,y):x^2+y^2\le 4\}$.

20.48. $u=\arcsin\frac{y}{x}$, $\{M\}=\{(x,y):|y|<x\}$.

20.49. $u=f(x,y)$ funksiya R^2 da uzluksiz bo'lsa, $u=f(x,y)$ R^2 da tekis uzluksiz bo'ladimi?

Mustaqil yechish uchun misollarning javoblari

20.1. $O(0;0)$.

20.2. $0(0;0)$.

20.3. $x^2+y^2=9-$ aylananing hamma nuqtalari.

20.4. $x+y=0$ chiziqning hamma nuqtalari.

20.5. $y=0$ to'g'ri chiziqning hamma nuqtalari.

20.6. Koordinata o'qlarining hamma nuqtalari.

20.7. $0(0;0)$- yo'qotilishi mumkin bo'lgan uzilish nuqtasi.

20.8. $0\ (0;0)$.

20.9. $x+y=0$ chiziqning (1;-1) va (-1;1) nuqtalardan boshqa hamma nuqtalari uzilish nuqtalari bo'ladi, (1;-1) va (-1;1) nuqtalar esa, yo'qotilishi mumkin bo'lgan uzilish nuqtalari bo'ladi.

20.10. $(\pi k; \pi n)$, $k, n \in Z$.

20.11. $0(0;0)$ yo'qotilishi mumkin bo'lgan uzilish nuqtasi.

20.12. $(\pi k; \pi n)$, $k, n \in Z$.

20.13. Uzilish nuqtalari yo'q.

20.14. O va A nuqtalarida funksiya har bir argumenti bo'yicha uzluksiz.

20.15. O va A nuqtalarda funksiya har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzluksiz.

20.16. O nuqtada funksiya har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzluksiz, A nuqtada esa, har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzilishga ega.

20.17. O nuqtada funksiya x argumenti bo'yicha uzluksiz, y argumenti va hamma argumentlari bo'yicha bir yo'la uzilishga ega. A nuqtada har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzluksiz.

20.18. O nuqtada funksiya har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzluksiz; A nuqtada har bir argumenti va hamma argumentlari bo'yicha bir yo'la uzilishga ega.

20.19. Chegaralangan.

20.20. Chegaralanmagan.

20.21. Chegaralangan.

20.22. Chegaralanmagan.

20.23. 1) Ha. 2) Ha. 3) Yo'q.

20.24. 1) 1. 2) -1. 3) -1. 4) Mavjud emas.

20.25. 1) 0. 2) 0.

20.27. 0.

20.28. $a = \sqrt{5}$, $b = -\sqrt{5}$.

20.29. $\sup u = 1$, bunga erishadi, masalan, $(1, 0)$ nuqtada, $\inf u = -1$, bunga erishadi, masalan, $(0, 1)$ nuqtada.

20.30. $\sup u = 81$, bunga erishadi, masalan, $(0; 3)$ nuqtada, $\inf u = 0$, bunga erishmaydi.

20.31. $\sup u = 0{,}5$, bunga erishadi, masalan, $(1, 1)$ nuqtada, $\inf u = 0$, bunga erishadi, masalan, $(0,1)$ nuqtada.

20.32. $\sup u = \frac{1}{e}$, bunga erishadi, masalan, $(1; 1)$ nuqtada, $\inf u = 0$, bunga erishadi, masalan, $(0,0)$ nuqtada.

20.33. Tekis uzluksiz.

20.34. Tekis uzluksiz.

20.35. Tekis uzluksiz emas.

20.36. Yo'q.

21 -§. Ko'p o'zgaruvchili funksiyaning xususiy hosilalari va differensiallari

21.1. Ko'p o'zgaruvchili funksiyaning xususiy hosilalari.

$u=f(M)=f(x_1,x_2,...,x_m)$ funksiya ochiq $\{M\}$ ($\{M\}\subset R^m$) to'plamda aniqlangan bo'lsin. Bu to'plamdan $M(x_1,x_2,...,x_m)$ nuqtani olamiz va funksiyaning x_k argumentiga Δx_k orttirma beramiz (qolgan argumentlarini o'zgarmas, deb hisoblaymiz). Natijada, funksiya ham $\Delta_{x_k} u$ orttirma oladi. Ushbu

$$\frac{\Delta_{x_k} u}{\Delta x_k}=\frac{f(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)-f(x_1,x_2,...,x_m)}{\Delta x_k} \qquad (21.1)$$

nisbatni qaraymiz, bunda $M(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)\in\{M\}$.

21.1-ta'rif. Agar $\Delta x_k\to 0$ da (21.1) nisbatning limiti mavjud va chekli bo'lsa, bu limit $f(x_1,x_2,...,x_m)$ funksiyaning $M(x_1,x_2,...,x_m)$ nuqtadagi x_k argumenti bo'yicha *xususiy hosilasi* deyiladi va

$$\frac{\partial f(x_1,...,x_m)}{\partial x_k},\ \frac{\partial f}{\partial x_k},\ U'_{x_k},\ f'_{x_k}(x_1,x_2,...,x_m),\ f'_{x_k}$$

kabi belgilarning biri orqali yoziladi. Ta'rifga ko'ra,

$$\frac{\partial u}{\partial x_k}=\lim_{\Delta x_k\to 0}\frac{\Delta_{x_k} u}{\Delta x_k}$$

ko'rinishda yozish mumkin.

21.1-misol. 21.1-ta'rifdan foydalanib, ushbu

$$f(x,y)=3^{2x+y}$$

funksiyaning $O(0,0)$ nuqtadagi f'_x, f'_y xususiy hosilalarini hisoblang.

Yechilishi. 21.1-ta'rifga ko'ra, f'_x, f'_y xususiy hosilalarini topamiz:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{3^{2\Delta x}-1}{\Delta x}=2\ln 3=\ln 9,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{3^{\Delta y}-1}{\Delta y}=\ln 3.$$

Demak, $f'_x(0,0)=\ln 9$, $f'_y(0,0)=\ln 3$.

21.2-misol. Ushbu

$$f(x,y)=\sqrt{(x-1)^2+(y-1)^2}$$

funksiyaning $A(1,1)$ nuqtadagi f'_x, f'_y xususiy hosilalari mavjud emasligini ko'rsating.

Yechilishi. Faraz qilaylik, $(x,y)\neq(1;1)$ bo'lsin, u holda, f'_x, f'_y xususiy hosilalarni topamiz:

$$f_x(x,y)=\frac{x-1}{\sqrt{(x-1)^2+(y-1)^2}},\quad f_y'(x,y)=\frac{y-1}{\sqrt{(x-1)^2+(y-1)^2}}.$$

Demak, $(x,y)\neq(1;1)$ da berilgan funksiyaning xususiy hosilalari mavjud.

21.1-ta'rifga ko'ra, berilgan funksiyaning $f_x(1;1)$, $f_y'(1;1)$ xususiy hosilalarini topamiz:

$$\frac{\partial f(1,1)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(1+\Delta x,1)-f(1,1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x},$$

$$\frac{\partial f(1,1)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(1,1+\Delta y)-f(1,1)}{\Delta y}=\lim_{\Delta y\to 0}\frac{|\Delta y|}{\Delta y}.$$

Bunda $\lim\limits_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$, $\lim\limits_{\Delta y\to 0}\frac{|\Delta y|}{\Delta y}$ limitlar har xil qiymatlarga ega bo'lganligi uchun, limitlar mavjud emas.

Demak, berilgan funksiyaning $A(1,1)$ nuqtada xususiy hosilalari mavjud emas.

21.3-misol. Quyidagi:

1) $f(x,y)=\arcsin\frac{x}{y}$; 2) $f(x,y,z)=e^{xyz}$

funksiyalarning xususiy hosilalarini toping.

Yechilishi. 1) $(\arcsin u)'=\frac{u'}{\sqrt{1-u^2}}$ formulaga asosan, $f_x'(x,y)$ va $f_y'(x,y)$ xususiy hosilalarni topamiz:

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}=\frac{1}{\sqrt{y^2-x^2}},\quad \frac{\partial f}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{yx}{\sqrt{y^2-x^2}}.$$

2) $(e^u)'=e^u u'$ formulaga asosan, berilgan funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x}=e^{xyz}\cdot yz,\quad \frac{\partial u}{\partial y}=e^{xyz}\cdot xz,\quad \frac{\partial u}{\partial z}=e^{xyz}\cdot xy.$$

21.4-misol. Ushbu $u=\frac{1}{2}(x^2+y^2)+\varphi(x-y)$ funksiya $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=x+y$ tenglamani qanoatlantiradimi?

Yechilishi. Dastlab, berilgan funksiyaning $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x}=x+\varphi'(x-y),\quad \frac{\partial u}{\partial y}=y-\varphi'(x-y).$$

Endi bu xususiy hosilalarni tenglamaga olib borib qo'yamiz:

$$x+\varphi'(x-y)+y-\varphi'(x-y)=x+y\Rightarrow x+y\equiv x+y.$$

Demak, $u=\frac{1}{2}(x^2+y^2)+\varphi(x-y)$ funksiya birinchi tartibli xususiy hosilali differensial tenglamaning yechimi ekan.

21.1-eslatma. Berilgan nuqtada funksiyaning hamma xususiy hosilalarining mavjudligidan, uning shu nuqtada uzluksizligi kelib chiqavermaydi. Masalan,

$$u=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0 \;\; bo'lganda,\\ 0\;, & x^2+y^2=0\;\; bo'lganda\end{cases}.$$

funksiya $M_0(0,0)$ nuqtada uzluksiz emas, lekin bu funksiya koʻrsatilgan nuqtada x va y argumentlari boʻyicha xususiy hosilalarga ega:

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)}=0;\quad \left.\frac{\partial u}{\partial y}\right|_{(0,0)}=0,$$

chunki $f(x,0)=0,\; f(0,y)=0$.

21.2. Koʻp oʻzgaruvchili funksiyaning differensiallanuvchanligi sharti. $u=f(M)$ funksiya $\{M\}$ ($\{M\}\subset R^m$) toʻplamda berilgan boʻlsin. Ma'lumki, $u=f(M)$ funksiyaning $M(x_1,x_2,...,x_m)$ nuqtadagi toʻliq orttirmasi:

$$\Delta u=f(x_1+\Delta x_1,x_2+\Delta x_2,...,x_m+\Delta x_m)-f(x_1,x_2,...,x_m)$$

ifodadan iborat (20-§ ga qarang).

21.2-ta'rif. Agar $u=f(M)$ funksiyaning M nuqtadagi toʻliq orttirmasini

$$\Delta u=A_1\Delta x_1+A_2\Delta x_2+...+A_m\Delta x_m+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m, \qquad (21.2)$$

bunda, $A_1,A_2,...,A_m$ lar, $\Delta x_1,\Delta x_2,...,\Delta x_m$ larga bogʻliq boʻlmagan oʻzgarmaslar, $\alpha_1,\alpha_2,...,\alpha_m$ lar esa, $\Delta x_1,\Delta x_2,...,\Delta x_m$ larga bogʻliq va $\Delta x_1\to 0,\Delta x_2\to 0,...,\Delta x_m\to 0$ da $\alpha_1\to 0,\alpha_2\to 0,...\alpha_m\to 0$ ($\Delta x_1=\Delta x_2=...\Delta x_m=0$ boʻlganda esa, $\alpha_1=\alpha_2=...=\alpha_m=0$ deb olinadi), koʻrinishda ifodalash mumkin boʻlsa, $f(M)$ funksiya $M(x_1,x_2,...,x_m)$ *nuqtada differensiallanuvchi* deyiladi.

21.1-teorema. Agar $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $M(x_1,x_2,...,x_m)$ nuqtada differensiallanuvchi boʻlsa, bu funksiyaning $M(x_1,x_2,...,x_m)$ nuqtada barcha argumentlari boʻyicha xususiy hosilalari mavjud va $\frac{\partial u_i}{\partial x_i}=A_i\;(i=1,2,...,m)$ boʻladi, bunda $A_i\;(i=1,2,...,m)$ lar (21.2) shartdan topiladi.

21.1-natija. (21.2) differensiallanuvchanlik shartini,

$$\Delta u=\frac{\partial u}{\partial x_1}\Delta x_1+\frac{\partial u}{\partial x_2}\Delta x_2+...+\frac{\partial u}{\partial x_m}\Delta x_m+o(\rho) \qquad (21.3)$$

ko'rinishda yozish mumkin, bunda ρ- $M(x_1,x_2,...,x_m)$ va $M_1(x_1+\Delta x_1, x_2+\Delta x_2,...,x_m++\Delta x_m)\in\{M\}$ nuqtalar orasidagi masofa, ya'ni $\rho=\sqrt{\Delta x_1{}^2+\Delta x_2{}^2+...+\Delta x_m{}^2}$.

21.2-natija. Agar $u=f(M)$ funksiya $M(x_1,x_2,...,x_m)$ nuqtada differensiallanuvchi bo'lsa, uning to'liq orttirmasi (21.2) shaklida tasvirlanishi yagonadir. Agar $u=f(M)$ funksiya $M(x_1,x_2,...,x_m)$ nuqtada differensiallanuvchi bo'lsa, u shu nuqtada uzluksiz bo'ladi.

21.2-teorema. Agar $u=f(M)$ funksiya $M_0(x_1^0,...,x_m^0)$ nuqtaning biror atrofida barcha argumentlari bo'yicha xususiy hosilalarga ega bo'lib, bu hosilalar M_0 nuqtada uzluksiz bo'lsa, u holda, berilgan funksiya M_0 nuqtada differensiallanuvchi bo'ladi.

21.6-misol. Ushbu $f(x,y)=\sqrt[3]{xy}$ funksiyaning $O(0,0)$ nuqtada xususiy hosilalarga egaligini va u shu nuqtada differensiallanuvchi emasligini ko'rsating.

Yechilishi. Ta'rifdan foydalanib, berilgan funksiyaning $O(0,0)$ nuqtadagi xususiy hosilalarini topamiz:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt[3]{\Delta x\cdot 0}}{\Delta x}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,\ 0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\sqrt[3]{\Delta y\cdot 0}}{\Delta y}=0$$

$O(0,0)$ nuqtada berilgan funksiyaning to'la orttirmasini topamiz:

$$\Delta f(0,0)=f(\Delta x,\Delta y)-f(0,0)=\sqrt[3]{\Delta x\Delta y}\ . \qquad (*)$$

Faraz qilaylik, berilgan funksiya $O(0,0)$ nuqtada differensiallanuvchi bo'lsin. U holda, (*) orttirma ushbu

$$\Delta f(0,0)=f'_x(0,0)\Delta x+f'_y(0,0)\Delta y+o(\rho)$$

ko'rinishda ifodalanadi, bunda $\rho=\sqrt{\Delta x^2+\Delta y^2}$. Quyidagi limitlarni qaraymiz:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta f(0,0)-f'_x(0,0)\Delta x+f'_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\sqrt[3]{\Delta x\Delta y}}{\sqrt{\Delta x^2+\Delta y^2}}\neq 0.$$

Agar $\Delta x=\Delta y\to 0$ da $\dfrac{\sqrt[3]{\Delta x\Delta y}}{\sqrt{\Delta x^2+\Delta y^2}}=\dfrac{\sqrt[3]{\Delta x^2}}{|\Delta x|\sqrt{2}}\to+\infty.$

Demak, berilgan funksiya $O(0,0)$ nuqtada differensiallanuvchi emas.

21.3. Ko'p o'zgaruvchili funksiyaning differensiali. $u=f(M)$ funksiya $\{M\}$ ($\{M\}\subset R^m$) to'plamda berilgan bo'lib, bu funksiya

$M(x_1, x_2, ..., x_m) \in \{M\}$ nuqtada differensiallanuvchi bo'lsin. U holda $u = f(M)$ funksiyaning Δu to'liq orttirmasi uchun (21.2) formula o'rinli:

$$\Delta u = A_1 \Delta x_1 + ... + A_m \Delta x_m + \alpha_1 \Delta x_1 + ... + \alpha_m \Delta x_m .$$

21.3-ta'rif. $u = f(M)$ funksiya Δu orttirmasining $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli bosh qismi, $u = f(M)$ funksiyaning M nuqtadagi *differensiali (to'liq differensiali)* deb ataladi va y, du, df yoki $df(x_1, x_2, ..., x_m)$ kabi belgilanadi.

Demak,

$$du = df = df(x_1, x_2, ..., x_m) = A_1 \Delta x_1 + A_2 \Delta x_2 + ... + A_m \Delta x_m \qquad (21.4)$$

21.1-teoremani e'tiborga olsak, u holda, (21.4) funksiya differensialini quyidagi,

$$du = \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \Delta x_2 + ... + \frac{\partial u}{\partial x_m} \Delta x_m \qquad (21.5)$$

ko'rinishda ham yozish mumkin. $x_i \ (i = 1,2,...,m)$ o'zgaruvchining differensiali $dx_i \ (i = 1,2,...,m)$ deb, ixtiyoriy $(x_1, x_2, ..., x_m)$ larga bog'liq bo'lmagan son tushuniladi. Bu sonni, bundan keyin, $\Delta x_i \ (i = 1,2,...,m)$ ga teng deb olishga kelishib olamiz, ya'ni $dx_i = \Delta x_i \ (i = 1,2,...,m)$. Bu kelishuvni e'tiborga olsak, (21.5) ni quyidagi,

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + ... + \frac{\partial u}{\partial x_m} dx_m \qquad (21.6)$$

ko'rinishda yozish mumkin.

5.6-misol. Ushbu $z = e^{x^2 y^2}$ fuksiyaning dz birinchi tartibli to'liq differensialini toping.

Yechilishi. Berilgan funksiyadan x va y o'zgaruvchilar bo'yicha xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x} = \left(e^{x^2 y^2}\right)'_x = e^{x^2 y^2} 2xy^2, \quad \frac{\partial z}{\partial y} = \left(e^{x^2 y^2}\right)'_y = e^{x^2 y^2} 2x^2 y.$$

Topilgan birinchi tartibli xususiy hosilalarni $dz = z_x dx + z_y dy$ formulaga keltirib qo'yamiz

$$dz = e^{x^2 y^2} 2xy^2 dx + e^{x^2 y^2} 2x^2 y dy = e^{x^2 y^2} 2xy(ydx + xdy).$$

21.7- misol. Funksiyaning orttirmasini uning differensialiga almashtirib, ushbu $(0{,}98)^{2{,}01}$ sonni taqribiy hisoblang.

Yechilishi. Quyidagi $u = x^y$ funksiyani qaraymiz. Bu funksiyaning $(1;2)$ nuqtadagi qiymati $u(1;2) = 1$ bo'ladi. $u = x^y$ funksiyaning $(0{,}98; 2{,}01)$ nuqtadagi qiymatini hisoblaymiz.

Dastlab $u = x^y$ funksiyaning $(1;2)$ nuqtadagi u'_x, u'_y xususiy hosilalarini topamiz:

$$u_x' = yx^{y-1},\ \ u_x'(1;2) = 2;\ \ u_y' = x^y \ln x,\ \ u_y'(1;2) = 0.$$

Berilgan sonni $u(x+\Delta x, y+\Delta y) \approx u(x,y) + u_x'(x,y)\Delta x + u_y'(x,y)\Delta y$ formula bo'yicha hisoblaymiz, bunda $\Delta x = -0{,}02,\ \Delta y = 0{,}01$,

$$(0{,}98)^{2{,}01} = u(1-0{,}02, 2+0{,}01) \approx u(1;2) + u_x'(1;2)\Delta x + u_y'(1;2)\Delta y =$$

$$= 1 + 2\cdot(-0{,}02) + 0\cdot 0{,}01 = 0{,}96.$$

21.4. Ko'p o'zgaruvchili murakkab funksiyaning hosilasi. $u = f(M) = f(x_1, x_2, ..., x_m)$ funksiya $\{M\}$ ($\{M\} \subset R^m$) to'plamda berilgan bo'lib, $x_1, x_2, ..., x_m$ o'zgaruvchilarning har biri, o'z navbatida, $t_1, t_2, ..., t_k$ o'zgaruvchilarning funksiyasi sifatida, $\{N\}$ ($\{N\} \subset R^k$) to'plamda berilgan funksiyalar bo'lsin:

$$\begin{aligned} x_1 &= \varphi_1(t_1, t_2, ... t_k), \\ x_2 &= \varphi_2(t_1, t_2, ... t_k), \\ &\ldots\ldots\ldots\ldots\ldots \\ x_m &= \varphi_m(t_1, t_2, ... t_k) \end{aligned} \qquad (21.7)$$

Bunda, $(t_1, t_2, ..., t_m) \in \{N\}$ bo'lganda, unga mos kelgan $(x_1, x_2, ..., x_m) \in \{M\}$ bo'lsin, deb faraz qilinadi. Natijada, ushbu

$$u = f(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k))$$

ko'p o'zgaruvchili murakkab funksiyaga ega bo'lamiz.

21.3-teorema. Agar (21.7) fnksiyalarning har biri $N_0(t_1^0, t_2^0, ..., t_k^0) \in \{N\}$ nuqtada differensiallanuvchi bo'lib, $u = f(x_1, x_2, ..., x_m)$ funksiya esa, unga mos $M_0(x_1^0, x_2^0, ..., x_k^0) \in \{M\}$ $\left(x_1^0 = \varphi_1(t_1^0, t_2^0, ..., t_k^0), x_2^0 = \varphi_2(t_1^0, t_2^0, ..., t_k^0), ..., x_m^0 = \varphi_m(t_1^0, t_2^0, ..., t_k^0)\right)$ nuqtada differensiallanuvchi bo'lsa, u holda, $f(\varphi_1(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k))$ murakkab funksiya ham $N_0(t_1^0, t_2^0, ..., t_k^0)$ nuqtada differensiallanuvchi bo'ladi va uning xususiy hosilalari

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + ... + \frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_1}, \\ \frac{\partial u}{\partial t_2} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + ... + \frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{\partial u}{\partial t_k} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_k} + ... + \frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_k} \end{aligned} \qquad (21.8)$$

formulalar orqali topiladi.

21.4-eslatma. Xususiy holda, (21.8) dagi funksiyalarning har biri faqat bitta t ga bog'liq bo'lsa, u holda, biz faqat t ga bog'liq bo'lgan $u = f(x_1, x_2, ..., x_m),\ x_i = \varphi_i(t)$ ($i = 1,2,...,k$) murakkab funksiyaga ega bo'lamiz. Bu murakkab funksiyaning hosilasi

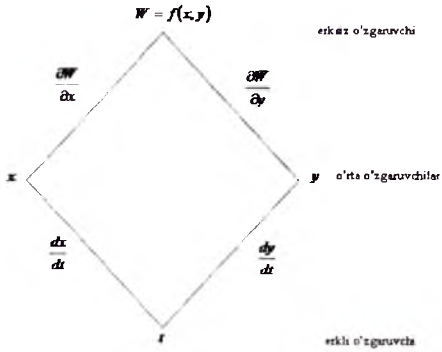
$$\frac{du}{dt} = \frac{du}{dx_1}\cdot\frac{dx_1}{dt} + \frac{du}{dx_2}\cdot\frac{dx_2}{dt} + ... + \frac{du}{dx_m}\cdot\frac{dx_m}{dt}$$

formula bo‘yicha topiladi.

Ikki o‘zgaruvchili funksiya uchun zanjir qoidasi: Agar $W = f(x, y)$ diferensiallanuvchi funksiya, x va y lar esa, t erkli o‘zgaruvchining diferensiallanuvchi funksiyalari bo‘lsa, u holda, W *funksiya ham* t erkli o‘zgaruvchining diferensiallanuvchi funksiyasi bo‘ladi va

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt}$$

formula o‘rinli.



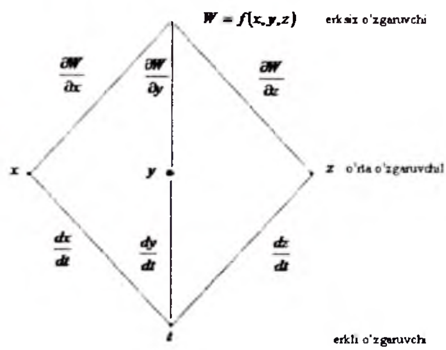
Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab, har bir yo‘l bo‘yicha pastga qarab harakat qilinib, yo‘lda uchragan hosilalar ko‘paytirilib, so‘ngra ular qo‘shiladi:

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt}.$$

Uch o‘zgaruvchili murakkab funksiya uchun zanjir qoidasi: Agar $W = f(x, y, z)$ diferensiallanuvchi funksiya, x, y va z lar esa, t erkli o‘zgaruvchining diferensiallanuvchi funksiyalari bo‘lsa, u holda, W funksiya ham t erkli o‘zgaruvchining diferensiallanuvchi funksiyasi bo‘ladi va

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt} + \frac{\partial W}{\partial z}\frac{dz}{dt}$$

formula o‘rinli.



Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab har bir yo‘l bo‘yicha pastga qarab harakat qilinib, yo‘lda uchragan hosilalar ko‘paytirilib, so‘ngra ular qo‘shiladi:

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt} + \frac{\partial W}{\partial z}\frac{dz}{dt}.$$

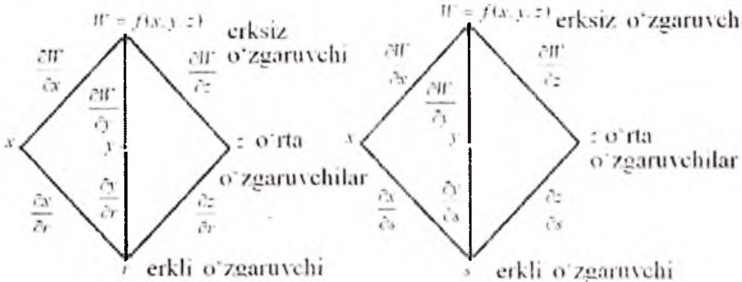
Ikkita erkli o‘zgaruvchi va uchta o‘rta o‘zgaruvchilar uchun zanjir qoidasi: *Agar* $W = f(x, y, z)$ $x = g(r, s)$, $y = h(r, s)$ va $z = k(r, s)$

diferensiallanuvchi funksiyalar bo'lsa, u holda, W funksiya ham r va s erkli o'zgaruvchilarga nisbatan xususiy hosilalarga ega bo'ladi va ular uchun,

$$\frac{\partial W}{\partial r}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial r}, \quad \frac{\partial W}{\partial s}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial s}$$

formulalar o'rinli.

Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{\partial W}{\partial r}$, $\frac{\partial W}{\partial s}$ ni topish uchun, W dan boshlab har bir yo'l bo'yicha pastga qarab harakat qilinib, yo'lda uchragan hosilalar ko'paytirilib, so'ngra ular qo'shiladi:



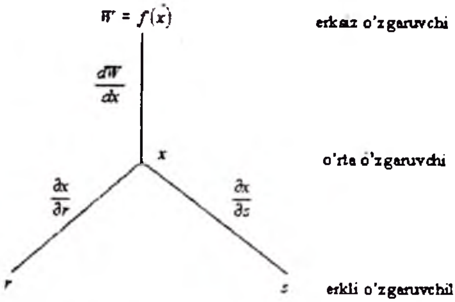
$$\frac{\partial W}{\partial r}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial r}, \qquad \frac{\partial W}{\partial s}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial s}.$$

Agar $W=f(x)$, $x=g(r,s)$ diferensiallanuvchi funksiyalar bo'lsa, u holda, W funksiya h*am* r va s erkli o'zgaruvchilarga nisbatan xususiy hosilalarga ega bo'ladi va ular uchun,

$$\frac{\partial W}{\partial r}=\frac{dW}{dx}\frac{\partial x}{\partial r}, \quad \frac{\partial W}{\partial s}=\frac{dW}{dx}\frac{\partial x}{\partial s}$$

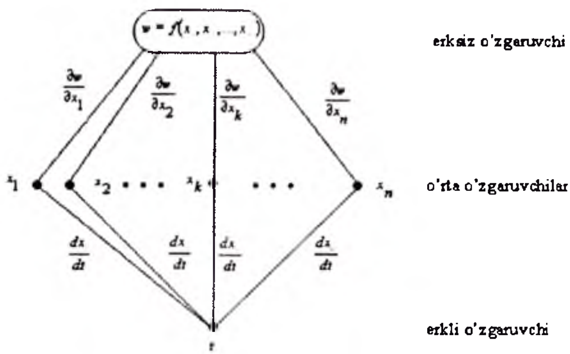
formulalar o'rinli.

Bu tasdiqning "daraxt diagrammasi" quyidagicha: $\frac{\partial W}{\partial r}$, $\frac{\partial W}{\partial s}$ larni topish uchun, W dan boshlab harakat qilinib, $\frac{dW}{dx}$ hosila x o'rta o'zgaruvchining r va s erkli o'zgaruvchilar bo'yicha xususiy hosilalariga ko'paytiriladi:



$$\frac{\partial W}{\partial r}=\frac{dW}{dx}\frac{\partial x}{\partial r}, \qquad \frac{\partial W}{\partial s}=\frac{dW}{dx}\frac{\partial x}{\partial s}$$

$w=f(x_1,x_2,...,x_n)$, $x_1=x_1(t), x_2=x_2(t),.....,x_n=x_n(t)$ (n ta o‘rta o‘zgaruvchilar, bitta erkli o‘zgaruvchi holi) bo‘lsa, daraxt diagrammasi



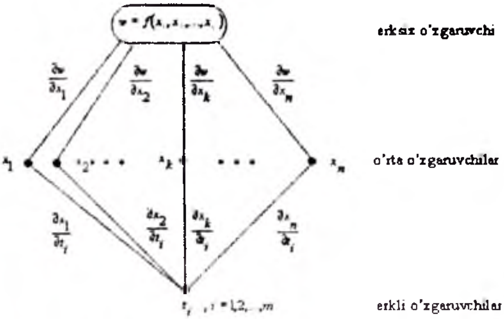
ko‘rinishni oladi va w funksiyaning t erkli o‘zgaruvchi bo‘yicha to‘liq hosilasi,

$$\frac{dw}{dt}=\frac{\partial f}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial f}{\partial x_2}\frac{dx_2}{dt}+...+\frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

formula bo‘yicha hisoblanadi.

$$w=f(x_1,x_2,...,x_n),\ x_1=\varphi_1(t_1,t_2,...,t_m),\ x_2=\varphi_2(t_1,t_2,...,t_m),.....,x_n=\varphi_n(t_1,t_2,...,t_m)$$

funksiya uchun “daraxt diagrammasi”



ko'rinishni oladi va w funksiyaning t_i, $i=1,2,...,m$, erkli o'zgaruvchilar bo'yicha xususiy hosilalari,

$$\frac{\partial w}{\partial t_1}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1}+...+\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_1},$$

$$\frac{\partial w}{\partial t_2}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2}+...+\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_2},$$

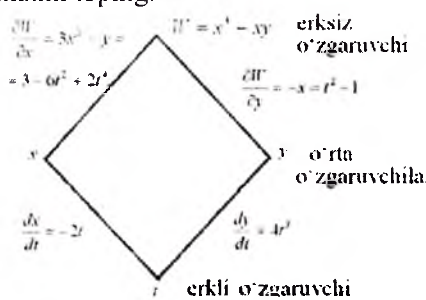
....

$$\frac{\partial w}{\partial t_m}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_m}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_m}+...+\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

formulalar yordamida hisoblanadi.

21.8-misol. Ushbu $W=x^3-xy,\ x=1-t^2,\ y=t^4$ murakkab funksiyaning $a)$ zanjir qoidasidan foydalanib; $b)$ bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t=1$ nuqtadagi qiymatini toping.

Yechilishi. $a)\ \frac{dW}{dt}$ ni topish uchun, «daraxt diagrammasi»ga asosan, W dan boshlab, har bir yo'l bo'yicha pastga qarab harakat qilib, yo'lda uchragan hosilalarni ko'paytiramiz, so'ngra ularni qo'shamiz:



$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}=\left(3-6t^2+2t^4\right)\left(-2t\right)+\left(t^2-1\right)4t^3=2t\left(4t^2-3\right)$$

Murakkab funksiyaning $\frac{dW}{dt}$ hosilasini $\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}$ formula bo'yicha topamiz:

$$\frac{\partial W}{\partial x}=3x^2-y=3(1-t^2)^2-t^4=3-6t^2+2t^4,\quad \frac{\partial W}{\partial y}=-x=t^2-1,\quad \frac{dx}{dt}=-2t,\quad \frac{dy}{dt}=4t^3,$$

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}=(3-6t^2+2t^4)(-2t)+(t^2-1)4t^3=2t(4t^2-3)$$

Endi, $\frac{dW}{dt}$ ning berilgan $t=1$ nuqtadagi qiymatini topamiz:

$$\left.\frac{dW}{dt}\right|_{t=1}=2t(4t^2-3)\Big|_{t=1}=2.$$

21.9-misol. Ushbu $z=e^x y^2,\ x=u^2-v^2,\ y=u\cdot v$ murakkab funksiyaning xususiy hosilasini toping.

Yechilishi. $z(x(u,v),y(u,v))$ murakkab funksiyaning xususiy hosilasini topish formulasidan foydalanamiz:

$$\frac{\partial z}{\partial x}=e^x y^2,\qquad \frac{\partial x}{\partial u}=2u,\qquad \frac{\partial x}{\partial v}=-2v,\qquad \frac{\partial x}{\partial y}=2ye^x,\qquad \frac{\partial y}{\partial u}=v,\qquad \frac{\partial y}{\partial v}=u,$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=e^x y^2\cdot 2u+2ye^x\cdot v=2e^{u^2-v^2}\cdot u^3v^2+2uv^2e^{u^2-v^2}=$$

$$=2uv^2(u^2+1)e^{u^2-v^2},$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=e^x y^2\cdot(-2v)+2ye^x\cdot u=2e^x y(u-vy)=2e^{u^2-v^2}\cdot uv(u-v^2u)=$$

$$=2e^{u^2-v^2}u^2v(1-v^2).$$

$u=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lsin.

21.4- ta'rif. Agar $\{M\}$ ($\{M\}\subset R^m$) to'plamning har bir nuqtasida va har bir t lar uchun

$$f(tx_1,tx_2,...,tx_m)=t^p f(x_1,x_2,...,x_m)\tag{21.9}$$

tenglik bajarilsa, $u=f(x_1,x_2,...,x_m)$ funksiya $\{M\}$ ($\{M\}\subset R^m$) to'plamda p-*darajali birjinsli funksiya* deyiladi.

21.4-teorema (birjinsli funksiyalar haqida Eyler teoremasi). Agar $u=f(x_1,x_2,...,x_m)$ funksiya $\{M\}$ to'plamda differensiallanuvchi p-darajali birjinsli funksiya bo'lsa, u holda $\{M\}$ to'plamning har bir $M(x_1,x_2,...,x_m)$ nuqtasida $\frac{\partial u}{\partial x_1}x_1+\frac{\partial u}{\partial x_2}+...+\frac{\partial u}{\partial x_m}x_m=pu$ tenglik o'rinli.

21.10-misol. Agar $f(x,y,z)$ funksiya $\{M\}$ to'plamda differensiallanuvchi p-darajali birjinsli funksiya bo'lsa, $f'_x(x,y,z),\ f'_y(x,y,z),\ f'_z(x,y,z)$ xususiy hosilalari $p-1$-darajali birjinsli funksiyalar ekanligini isbotlang.

Yechilishi. Shartga ko'ra,

$$f(tx,ty,tz)=t^p f(x,y,z)$$

bo'lib, tenglikning chap tomoni differensiallanuvchi. Tenglikni x bo'yicha differensiallaymiz:

$$f_x(tx,ty,tz)t = t^p f_x'(x,y,z) \text{ yoki } f_x'(tx,ty,tz) = t^{p-1} f_x'(x,y,z).$$

Bundan $f_x(x,y,z)$ funksiya $p-1$- darajali birjinsli funksiya ekanligi kelib chiqadi. $f_y'(x,y,z)$ va $f_z'(x,y,z)$ funksiyalarning ham $p-1$- darajali birjinsli funksiya ekanligi xudi yuqoridagidek ko'rsatiladi.

21.5. Differensial shakli invariantligining saqlanishi.

$u = f(M) = f(x_1, x_2, ..., x_m)$ funksiya $\{M\}$ ($\{M\} \subset R^m$) to'plamda berilgan bo'lsin. Biz yuqorida ko'rgan edikki, agar $x_1, ..., x_m$ argumentlar erkli o'zgaruvchilar bo'lsa, funksiyaning differensiali (to'liq differensiali)

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + ... + \frac{\partial u}{\partial x_m}dx_m \qquad (21.10)$$

ko'rinishda tasvirlanadi. Endi $x_1, x_2, ..., x_m$ argumentlar erksiz o'zgaruvchili, ya'ni biror $N_0(t_1^0, t_2^0, ..., t_k^0) \in \{N\}$ nuqtada, differensiallanuvchi $x_i = \varphi_i(t_1, t_2, ..., t_k)$ $(i = 1,2,...,m)$ funksiyalar, $u = f(x_1, x_2, ..., x_m)$ funksiya esa, $M_0(x_1^0, x_2^0, ..., x_m^0)$ nuqtada differensiallanuvchi bo'lsin, deb faraz qilaylik. Bu holda, $u = f(x_1, x_2, ..., x_m)$ funksiyani, $t_1, t_2, ..., t_k$ o'zgaruvchilarning murakkab funksiyasi deb qaraymiz. 21.3-teoremaga asosan, bu murakkab funksiya N_0 nuqtada differensiallanuvchi bo'ladi va uning differensiali,

$$du = \frac{\partial u}{\partial t_1}dt_1 + \frac{\partial u}{dt_2}dt_2 + ... + \frac{\partial u}{\partial t_m}dt_m \qquad (21.11)$$

ko'rinishda tasvirlanadi, bunda $\frac{\partial u}{\partial t_i}$ lar (21.8) formulalar orqali topiladi. $\frac{\partial u}{\partial t_i}$ larning ifodalarini (21.8) dan (21.11) ga keltirib qo'yib va $\frac{\partial u}{\partial x_i}$ larning koeffisientlarini jamlab, natijada

$$du = \frac{\partial u}{\partial x_1}\left(\frac{\partial x_1}{\partial t_1}dt_1 + \frac{\partial x_1}{\partial t_2}dt_2 + ... + \frac{\partial x_1}{\partial t_k}dt_k\right) + ... + \frac{\partial u}{\partial x_m}\left(\frac{\partial x_m}{\partial t_1}dt_1 + \frac{\partial x_m}{\partial t_2}dt_2 + ... + \frac{\partial x_m}{\partial t_k}dt_k\right)$$

(21.12)

munosabatni hosil qilamiz. Ma'lumki, $\frac{\partial u}{\partial x_i}$ $(i = 1,2,...,m)$ ning koeffitsiyenti, $x_i = \varphi_i(t_1, t_2, ..., t_k)$ $(i = 1,2,...,m)$ funksiyaning dx_i $(i = 1,2,...,m)$ differensialini ifodalaydi. Shuning uchun, (21.12) ning ko'rinishini, (21.11) ko'rinishda yozish mumkin. Demak, $x_1, x_2, ..., x_m$ lar erksiz o'zgaruvchilar bo'lganda ham, $u = f(x_1, x_2, ..., x_m)$ funksiyaning differensiali (21.11) ko'rinishda bo'lar ekan, ya'ni ko'p o'zgaruvchili murakkab funksiyaning birinchi tartibli differensiali shakli invariantligi (ko'rinishi) saqlanar ekan.

$u=f(M)$ va $v=g(M)$ funksiyalar ochiq $\{M\}$ ($\{M\}\subset R^m$) to'plamda berilgan bo'lib, $\{M\}$ to'plamda differensiallanuvchi bo'lsa, u holda, $u\pm v, uv, \frac{u}{v}$ $(v\neq 0)$ funksiyalar ham shu M nuqtada differensiallanuvchi va ularning differensiali uchun, quyidagi,

$$d(cu)=cdu \qquad (c=const)$$

$$d(u\pm v)=du\pm dv,$$

$$d(uv)=udv+vdu,$$

$$d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

formulalar o'rinli bo'ladi.

21.6. Yo'nalish bo'yicha hosila. Gradient. Ma'lumki, bir o'zgaruvchili $y=f(x)$ $(x\in R,\ y\in R)$ funksiyaning $\frac{df}{dx}$ hosilasi, berilgan funksiyaning o'zgarish tezligini bildiradi. Ko'p o'zgaruvchili $u=f(x_1,x_2,...,x_m)=f(M)$ $((x_1,x_2,..,x_m)\in R^m, u\in R)$ funksiyaning xususiy hosilalari ham bir o'zgaruvchili funksiyaning hosilasi kabi ekanligini e'tiborga olib, $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2},...,\frac{\partial f}{\partial x_m}$ funksiyalar ham, $u=f(x_1,x_2,..,x_m)$ funksiyaning, mos ravishda, $Ox_1,Ox_2,...,Ox_m$ o'qlar bo'yicha R^m da o'zgarish tezligini ifodalaydi, deb qarash mumkin.

Endi ko'p o'zgaruvchili funksiyaning ixtiyoriy yo'nalish bo'yicha o'zgarish tezligini ifodalovchi tushuncha bilan tanishamiz. Soddalik uchun, uch o'zgaruvchili funksiyani qaraymiz. $u=f(x,y,z)=f(M)$ funksiya ochiq $\{M\}\subset R^3$ to'plamda berilgan bo'lsin. $\vec{r}=\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ birlik vektor bilan aniqlanadigan biror $\vec{l}$ yo'nalishni qaraylik. $\{M\}$ to'plamda ixtiyoriy $M_0(x_0,y_0,z_0)$ nuqtani olib, bu nuqta orqali o'tuvchi, yo'nalishi, $\vec{r}=\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ vektor yo'nalishiga mos kelgan $\vec{l}$ yo'nalishni qaraylik. $\vec{l}$ yo'nalishda $M_0(x_0,y_0,z_0)$ nuqtaga yaqin yotgan ixtiyoriy o'zgaruvchi $M(x,y,z)$ nuqtani olamiz $(M\in\{M\})$. $\overrightarrow{M_0M}$ yo'nalgan kesma $\{M\}\subset R^3$ to'plamga tegishli, u holda,

$$\frac{x-x_0}{\rho(M_0,M)}=\cos\alpha,\quad \frac{y-y_0}{\rho(M_0,M)}=\cos\beta,\quad \frac{z-z_0}{\rho(M_0,M)}=\cos\gamma \qquad (21.13)$$

bo'ladi.

21.5-ta'rif. Agar $M(x,y,z)$ nuqta $\vec{l}$ yo'nalgan to'g'ri chiziq bo'ylab $M_0(x_0,y_0,z_0)$ nuqtaga intilganda $(M\to M_0)$, ushbu

$$\frac{f(M)-f(M_0)}{\rho(M_0,M)}=\frac{f(x,y,z)-f(x_0,y_0,z_0)}{\rho((x_0,y_0,z_0),(x,y,z))}$$

nisbatning limiti mavjud bo‘lsa, bu limitga $f(x,y,z)=f(M)$ funksiyaning $M_0(x_0,y_0,z_0)$ nuqtada $\vec{l}$ *yo‘nalish bo‘yicha hosilasi* deb ataladi va u $\frac{\partial f(M_0)}{\partial \vec{l}}$ yoki $\frac{\partial f(x_0,y_0,z_0)}{\partial \vec{l}}$ kabi belgilanadi.

21.5-ta’rifga ko‘ra, uni

$$\frac{\partial f}{\partial \vec{l}}=\lim_{M\to M_0}\frac{f(M)-f(M_0)}{\rho(M_0,M)}.$$

ko‘rinishda yozish mumkin.

21.6-teorema. Agar $u=f(x,y,z)=f(M)$ funksiya ochiq $\{M\}\subset R^3$ to‘plamda berilgan bo‘lib, u $M_0(x_0,y_0,z_0)\in\{M\}$ nuqtada differensiallanuvchi bo‘lsa, u holda, $u=f(x,y,z)=f(M)$ funksiya shu nuqtada har qanday $\vec{l}$ yo‘nalish bo‘yicha hosilaga ega bo‘ladi va bu hosila,

$$\frac{\partial u(M_0)}{\partial \vec{l}}=\frac{\partial f(x_0,y_0,z_0)}{\partial \vec{l}}=\frac{\partial f(x_0,y_0,z_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0,z_0)}{\partial y}\cos\beta+$$
$$+\frac{\partial f(x_0,y_0,z_0)}{\partial z}\cos\gamma \qquad (21.14)$$

formula orqali topiladi.

21.11-misol. Ushbu $f(x,y)=arctg\frac{x}{y}$ funksiyaning $O(0,0)$ nuqtadan $M_0(1,1)$ nuqtaga qarab yo‘nalgan $\vec{l}$ yo‘nalish bo‘yicha hosilasini toping.

Yechilishi. $\vec{l}$ birinchi kvadratning $M_0(1,1)$ nuqtasidan o‘tuvchi va $O(0,0)$ nuqtadan $M_0(1,1)$ nuqtaga qarab yo‘nalgan bissektrisadan iborat bo‘ladi. (21.13) formulaga asosan, $\varphi=\frac{\pi}{4}$. Berilgan funksiya $M_0(1,1)$ nuqtada differensiallanuvchi bo‘lgani uchun, uning yo‘nalish bo‘yicha hosilasini (21.14) formula bo‘yicha topamiz:

$$\frac{\partial f(x,y)}{\partial \vec{l}}=\frac{\partial f(x,y)}{\partial x}\cos\frac{\pi}{4}+\frac{\partial f(x,y)}{\partial y}\cos\frac{\pi}{4}=\left(\frac{y}{x^2+y_2}-\frac{x}{x^2+y^2}\right)\frac{\sqrt{2}}{2}$$

Demak, $\frac{\partial f(1,1)}{\partial \vec{l}}=0$.

21.12- misol. Ushbu $f(x,y)=\sqrt{x^2+y^2}$ funksiyaning $M_0(0,0)$ nuqtada ixtiyoriy $\vec{l}$ yo‘nalish bo‘yicha hosilasi $\frac{\partial f(0,0)}{\partial \vec{l}}=1$ ekanligini ko‘rsating.

Yechilishi. Quyidagi nisbatni tuzamiz:

$$\frac{f(M)-f(M_0)}{\rho(M_0,M)}=\frac{\sqrt{x^2+y^2}}{\rho}=\frac{\rho}{\rho}=1,$$

bundan,

$$\frac{\partial f(0,0)}{\partial \vec{l}}=\lim_{\rho\to 0}\frac{f(M)-f(M_0)}{\rho(M_0,M)}=1.$$

21.13- misol. Ushbu $f(x,y)=x+|y|$ funksiyaning $M_0(0,0)$ nuqtada Ox va Oy koordinatalar o‘qlari bo‘yicha hosilasi mavjudmi?

Yechilishi. Berilgan $f(x,y)=x+|y|$ funksiyaning $M_0(0,0)$ nuqtada Ox koordinatalar o‘qi bo‘yicha hosilasi 1 ga teng bo‘lib, Oy koordinatalar o‘qi bo‘yicha hosilasi mavjud emas.

21.4-eslatma. Funksiya biror nuqtada differensiallanuvchi bo‘lmasa ham, u shu nuqtada biror yo‘nalish bo‘yicha va hatto har qanday yo‘nalish bo‘yicha hosilaga ega bo‘lishi ham mumkin. Masalan, ushbu $f(x,y)=\sqrt{x^2+y^2}$ funksiya $M_0(0,0)$ nuqtada differensiallanuvchi emas, lekin biz yuqorida ko‘rdikki, bu funksiya $M_0(0,0)$ nuqtada ixtiyoriy yo‘nalish bo‘yicha hosilaga ega.

21.7- ta’rif. Komponentalari (koordinatalari) $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ bo‘lgan vektor, $u=f(x,y,z)$ funksiyaning $M_0(x_0,y_0,z_0)$ *nuqtadagi gradienti,* deb ataladi va u

$$grad\,u(M_0)=\frac{\partial u(M_0)}{\partial x}\vec{i}+\frac{\partial u(M_0)}{\partial y}\vec{j}+\frac{\partial u(M_0)}{\partial z}\vec{k} \qquad (21.15)$$

kabi belgilanadi, bunda $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ xususiy hosilalar $M_0(x_0,y_0,z_0)$ nuqtada hisoblangan. $\vec{r}=\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ ekanligini e’tiborga olsak, u holda (21.14) ni, ushbu $\frac{\partial u}{\partial \vec{l}}=\left(\vec{r},\, grad\,u\right)$ (21.14) ko‘rinishda yozish mumkin.

$$|grad\,u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2} \qquad (21.16)$$

ekanligini hisobga olib, (21.15) formulani,

$$\frac{\partial u}{\partial \vec{l}}=|\vec{r}|\cdot|grad\,u|\cdot\cos\varphi$$

ko‘rinishda ham yozish mumkin, bunda $\varphi-\vec{r}$ bilan gradientning orasidagi burchak. $|\vec{r}|=1$ bo‘lgani uchun,

$$\frac{\partial u}{\partial \vec{l}}=|grad\,u|\cdot\cos\varphi.$$

Bu formuladan ko'rinadiki, yo'nalish bo'yicha hosila o'zining maksimum $\left(\frac{\partial u}{\partial \vec{l}}\right)_{max}$ qiymatiga $\cos\varphi = 1$ bo'lganda, ya'ni $\vec{v}$ vektorning yo'nalishi gradientning yo'nalishiga mos tushganda erishadi, $\left(\frac{\partial u}{\partial \vec{l}}\right)_{max} = |grad u|$. Agar $\vec{r}$ ning yo'nalishi gradientga perpendikulyar yo'nalgan bo'lsa, $\frac{\partial u}{\partial \vec{l}} = 0$ bo'ladi, chunki $\varphi = \frac{\pi}{2}$, $\cos\frac{\pi}{2} = 0$.

21.14- misol. Ushbu $u = x^2 + y^2 + z^2$ skalar funksiyaning $P(1,1,1)$ va $Q(1,-1,1)$ nuqtalardagi gradientlari orasidagi burchakni toping.

Yechilishi. Dastlab, berilgan funksiyaning $P(1,1,1)$ va $Q(1,-1,1)$ nuqtalardagi xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = 2x, \quad \left.\frac{\partial u}{\partial x}\right|_P = 2, \quad \left.\frac{\partial u}{\partial x}\right|_Q = 2,$$

$$\frac{\partial u}{\partial y} = 2y, \quad \left.\frac{\partial u}{\partial y}\right|_P = 2, \quad \left.\frac{\partial u}{\partial y}\right|_Q = -2,$$

$$\frac{\partial u}{\partial z} = 2z, \quad \left.\frac{\partial u}{\partial z}\right|_P = 2, \quad \left.\frac{\partial u}{\partial z}\right|_Q = 2.$$

(21.15), (21.16) formulalarga asosan, $u = x^2 + y^2 + z^2$ skalar funksiya uchun $grad u(P)$, $grad u(Q)$ hamda $|grad u(P)|$, $|grad u(Q)|$ larni hisoblaymiz:

$$grad u(P) = 2\vec{i} + 2\vec{j} + 2\vec{k}, \quad grad u(Q) = 2\vec{i} - 2\vec{j} + 2\vec{k},$$

$$|grad u(P)| = 2\sqrt{3}, \quad |grad u(Q)| = 2\sqrt{3}.$$

Endi gradientlar orasidagi burchakni topamiz:

$$\cos\varphi = \frac{grad u(P) grad(Q)}{|grad(P)||grad(Q)|} = \frac{2\cdot 2 + 2\cdot(-2) + 2\cdot 2}{2\sqrt{3}\ 2\sqrt{3}} = \frac{1}{3}.$$

Demak, berilgan u skalyar funksiyaning $P(1,1,1)$ va $Q(1,-1,1)$ nuqtalardagi gradientlari orasidagi burchak $\varphi = \arccos\frac{1}{3}$.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning xususiy hosilalarini toping:

21.1. $u = x^2 + y^2 + 3x^2y^3$. **21.2.** $u = \frac{x(x-y)}{y^2}$.

21.3. $u = xyz + \frac{x}{yz}$. **21.4.** $u = \sin(xy + yz)$.

21.5. $u = tg(x+y)\cdot e^{x/y}$. **21.6.** $u = \sin\frac{x}{y}\cdot\cos\frac{y}{x}$.

21.7. $u = e^x(\cos y + x\sin y)$.

21.8. $u = x^y$.

21.9. $u = \left(\frac{y}{x}\right)^z$.

21.10. $u = \ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

21.11. $u = \arcsin\sqrt{\frac{x^2-y^2}{x^2+y^2}}$.

21.12. $u = (1+\sin^2 x)^{\ln y}$.

21.13. $u = x^y y^z z^x$.

21.14. $g(r,\theta) = r\cos\theta + r\sin\theta$.

21.21. $f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

21.16. $P(n,R,T,V) = \frac{nRT}{V}$.

Quyidagi funksiyalarning berilgan nuqtadagi xususiy hosilalarini toping:

21.17. $u = \frac{x}{y^2}$, $(1;1)$.

21.18. $u = \ln\left(1+\frac{x}{y}\right)$, $(1;2)$.

21.19. $u = xye^{\sin xy}$, $(1;1)$.

21.20. $u = (2x+y)^{2x+y}$, $(1;-1)$.

21.21. Ushbu $u = \sqrt[3]{xy}$ funksiyaning $O(0;0)$ nuqtadagi xususiy hosilalarini toping. Bu funksiya $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

Quyidagi berilgan $u(x,y)$ funksiyalar $O(0;0)$ nuqtada xususiy hosilalarga egami; $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

21.22. $u = \sqrt{x^2+y^4}$.

21.23. $u = \sqrt{x^4+y^4}$.

21.24. $u = \sqrt[5]{xy}$

21.25. $u = \sqrt[3]{x^2y^2}$.

21.26. $u = \begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2 \neq 0 \text{ bo'lganda}, \\ 0, & x^2+y^2 = 0 \text{ bo'lganda}. \end{cases}$

21.27. $u = \begin{cases} \frac{x^4+y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \text{ bo'lganda}, \\ 0, & x^2+y^2 = 0 \text{ bo'lganda}. \end{cases}$

21.28. $u(x,y)$ funksiya:

a) $u = \frac{x}{\sqrt{x^2+y^2}}$; b) $u = \ln(x^2+xy+y^2)$

ko'rinishlarda bo'lganda, $\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ ifodani hisoblang.

21.29. $u(x,y,z)$ funksiya:

a) $u = (x-y)(y-z)(z-x)$; b) $u = x + \frac{x-y}{y-z}$

ko'rinishlarda bo'lganda, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ ifodani hisoblang.

21.30-21.35- misollarda $u = f(x_1, x_2, \ldots, x_m)$ funksiya uchun quyidagi tasdiqlarning qaysi biri to'g'ri, qaysi biri noto'g'ri?

21.30. $f(x_1, x_2, ..., x_m)$ funksiya biror nuqtada hamma argumentlari bo'yicha xususiy hosilalarga ega bo'lsa, u shu nuqtada uzluksiz bo'ladi.

21.31. Agar funksiya R^m fazoning har bir nuqtasida hamma argumentlari bo'yicha xususiy hosilalarga ega bo'lsa, u R^m da uzluksiz bo'ladi.

21.32. Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u shu nuqtada hamma argumentlari bo'yicha xususiy hosilalarga ega bo'ladi.

21.33. Agar funksiyaning biror nuqtada hamma argumentlari bo'yicha xususiy hosilalari mavjud bo'lsa, u shu nuqtada differensiallanuvchi bo'ladi.

21.34. Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u holda, shu nuqtada funksiyaning hamma argumentlari bo'yicha uzluksiz xususiy hosilalari mavjud bo'ladi.

21.35. Agar funksiyaning biror nuqtada uzluksiz xususiy hosilalari mavjud bo'lsa, u holda, funksiya shu nuqtada differensiallanuvchi bo'ladi.

21.36. Agar $f(x,y)-xOy$ tekislikdagi G ochiq sohada aniqlangan, uning f_x va f_y xususiy hosilalari G da chegaralangan bo'lsa, u holda, $f(x,y)$ funksiyasining G da uzluksizligini isbotlang.

Quyidagi berilgan funksiyalarning differensialini toping:

21.37. $u = 2x^4 - 3x^2y^2 + x^3y$. **21.38.** $u = (y^3 + 2x^2 + 3)^4$.

21.39. $u = \frac{y}{x} + \frac{x}{y}$. **21.40.** $u = -\frac{x}{\sqrt{x^2 + y^2}}$.

21.41. $u = a^{-\frac{y}{x}}$. **21.42.** $u = \ln\left(x + \sqrt{x^2 + y^2}\right)$.

21.43. $u = \ln\sin\frac{x+1}{\sqrt{y}}$. **21.44.** $u = arctg\frac{x+y}{x-y}$.

21.45. $u = (1 + xy)^y$.

Quyidagi funksiyalarning berilgan nuqtalardagi differensialini toping:

21.46. $u = \frac{x^2 - y^2}{x^2 + y^2}$. *a)* $(1;1)$; *b)* $(0;1)$. **21.47.** $u = \sqrt{xy + \frac{x}{y}}$, $(2;1)$.

21.48 $u = \cos(xy + xz)$, $M(x,y,z)$ va $N\left(1; \frac{\pi}{6}; \frac{\pi}{6}\right)$ nuqtalarda.

21.49. $u = e^{xy}$ $M(x,y)$ va $O(0,0)$ nuqtada.

21.50. $u = x^y$, $M(x,y)$ va $M_0(2;3)$nuqtalarda.

21.51. $u = x\ln(xy)$, $M(x,y)$ va $M_0(-1;-1)$ nuqtalarda.

21.52. $u=\frac{x}{x^2+y^2+z^2}$, $M(1,0,1)$. **21.53.** $u=arctg\frac{xy}{z^2}$, $M(3,2,1)$.

21.54. $u=\left(xy+\frac{x}{y}\right)^z$, $M(1,1,1)$.

Quyidagi berilgan $f(u)$ funksiyani differensiallanuvchi va uning f_u hosilalani aniq deb faraz qilib, $f(u)$ funksiya funksiya uchun f_x, f_y xususiy hosilalarni toping:

21.55. $u=x^2+e^y$. **21.56.** $u=\sqrt[3]{x^3+xy^2}$. **21.57.** $u=arctg(x+\ln y)$.

Quyidagi berilgan $f(u)$, $f(u,v)$, $f(u,v,w)$ funksiyalarni differensialla-nuvchi va ularning f_u, f_v, f_w xususiy hosilalari aniq deb faraz qilib, quyidagi φ funksiyaning differensialini toping:

21.58. $\varphi=f(u)$, $u=xy+\frac{y^2}{x}$.

21.59. 2) $\varphi=f(u,v)$, $u=\frac{y}{x+y}$, $v=x^2-y^3$.

21.60. $\varphi=f(u,v,w)$, $u=x^2+y^2+z^2$, $v=x+y+z$, $w=xyz$.

21.61. Agar $W=\sin(xy+\pi), x=e^t$ va $y=\ln(t+1)$ bo'lsa, $t=0$ da $\frac{dw}{dt}$ hosilani hisoblang.

21.66. Agar $W=\sin(2x-y), x=r+\sin s$, $y=rs$ bo'lsa, $r=\pi$ va $s=0$ bo'lganda, mos ravishda, $\frac{\partial w}{\partial t}$ va $\frac{\partial w}{\partial s}$ xususiy hosilalarni toping.

21.67. Ushbu $W(x,y,z)=xy+yz+xz$ funksiyaning $x=\cos t$, $y=\sin t$, $z=\cos 2t$ egri chiziqdagi t bo'yicha hosilasining $t=1$ dagi qiymatini toping.

21.68. $w=f(z,\sigma)$, $r=\sqrt{x^2+y^2}$, $\sigma=arctg\frac{y}{x}$ bo'lsin. U holda, $\frac{\partial w}{\partial x}$ va $\frac{\partial w}{\partial y}$ larni toping va javobingizni r va σ orqali ifodalang.

Quyidagi misollarda: a) zanjir qoidasidan foydalanib; b) bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t=t_0$ nuqtadagi qiymatini toping:

21.69. $W=x^2+y^2$, $x=\cos t$, $y=\sin t$; $t_0=\pi$.

21.70. $W=\frac{x}{z}+\frac{y}{z}$, $x=\cos^2 t$, $y=\sin^2 t$, $z=\frac{1}{t}$; $t_0=3$.

21.71. $W=2ye^x-\ln z$, $x=\ln(t^2+1)$, $y=arctgt$, $z=e^t$; $t_0=1$.

21.72. Agar $W=(x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$, $z=\sin(r+s)$ bo‘lsa, $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1,-1)}$ toping.

21.73. Agar $W=x^2+\frac{y}{x}$, $x=u-2v+1$, $y=2u+v-2$ bo‘lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0,0)}$ ni toping.

21.74. Agar $W=arctgx$ va $x=e^u+\ln v$ bo‘lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(\ln 2,1)}$, $\left.\frac{\partial W}{\partial v}\right|_{(u,v)=(\ln 2,1)}$ larni toping.

21.75. Agar a va b - o‘zgarmas sonlar, $w=u^3+thu+\cos u$ va $u=ax+by$ bo‘lsa, $a\frac{\partial w}{\partial y}=b\frac{\partial w}{\partial x}$ munosabat o‘rinli ekanligini ko‘sating.

21.76. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo‘lsa, u holda, $\varphi(x,y)=yf(x^2-y^2)$ funksiya, $y^2\frac{\partial\varphi}{\partial x}+xy\frac{\partial\varphi}{\partial y}=x\varphi$ tenglamani qanoatlantirishini isbotlang.

21.77. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo‘lsa, u holda, $\varphi(x,y)=xy+xf\left(\frac{y}{x}\right)$ funksiya, $x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}=xy+\varphi$ tenglamani qanoatlantirishini isbotlang.

21.78. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo‘lsa, u holda, $\varphi(x,y)=\sin x+f(\sin y-\sin x)$ funksiya, $\cos y\frac{\partial\varphi}{\partial x}+\cos x\frac{\partial\varphi}{\partial y}=xy+\varphi$ tenglamani qanoatlantirishini isbotlang.

21.79. Agar $f(u,v)$ ixtiyoriy differensiallanuvchi funksiya bo‘lsa, u holda, $\varphi(x,y,z)=f\left(\frac{x}{y},x^2+y-z^2\right)$ funksiya, $2xz\frac{\partial\varphi}{\partial x}+2yz\frac{\partial\varphi}{\partial y}+(2x^2+y)\frac{\partial\varphi}{\partial z}=0$ tenglamani qanoatlantirishini isbotlang.

21.80. Agar $w=f(s)-s$ ning differensiallanuvchi funksiyasi, $s=y+5x$ bo‘lsa, u holda $\frac{\partial w}{\partial x}-5\frac{\partial w}{\partial y}=0$ musbat bajarilishini ko‘rsating.

21.81. Agar a va b - o‘zgarmas sonlar, $w=u^3+thu+\cos u$ va $u=ax+by$ bo‘lsa, $a\frac{\partial w}{\partial y}=b\frac{\partial w}{\partial x}$ musbat o‘rinli ekanligini ko‘sating.

21.82. Agar $f(u,v,w)$ diferensiallanuvchi funksiya va $u=x-y,\ v=y-z$ hamda $w=z-x$ bo'lsa, $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0$ ekanligini ko'rsating.

21.83. Faraz qilaylik, $W=f(x,y)$ diferensiallanuvchi funksiyada $x=r\cos\theta$ va $y=r\sin\theta$ qutb koordinatalariga o'tish amalga oshirilgan (qutb almashtirishlari bajarilgan) bo'lsin. U holda

$$a)\ \frac{\partial W}{\partial r}=f'_x\cos\theta+f'_y\sin\theta,\quad \frac{1}{r}\frac{\partial W}{\partial\theta}=-f'_x\sin\theta+f'_y\cos\theta,$$

ekanligini ko'rsating;

$b)$ $a)$ banddagi tenglamalarni f'_x va f'_y larga nisbatan Yechib, ularni $\frac{\partial W}{\partial r}$ va $\frac{\partial W}{\partial\theta}$ lar orqali ifodalang;

$$c)\ (f'_x)^2+(f'_y)^2=\left(\frac{\partial W}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial W}{\partial\theta}\right)^2$$

ekanligini ko'rsating.

21.84. f va $g-x$ va y ning shunday funksiyalardan iboratki, $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$ va $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial y}$ munosabat o'rinli bo'lsin. Faraz qilaylik, $\frac{\partial f}{\partial x}=0,\ f(1,2)=g(1,2)=5$ va $f(0,0)=4$ bo'lsin. U holda, $f(x,y)$ va $g(x,y)$ larni toping.

21.85. Birinchi tartibli xususiy hosilalardan $\frac{\partial w}{\partial x}=1+e^x\cos y$ va $\frac{\partial w}{\partial y}=2y-e^x\sin y$ hamda $(\ln 2,0)$ nuqtadagi qiymatini $2+\ln 2$ ga teng bo'lgan $(f(\ln 2,0)=2+\ln 2)$, $w=f(x,y)$ funksiyani toping.

21.86. Agar $u=f(x,y,z)$ funksiya biror E sohada differensiallanuvchi bo'lib, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=pu$ tenglamani qanoatlantirsa, u holda, uning p- darajali birjinsli funksiya bo'lishini isbotlang.

21.87. Agar $u=f(x,y,z)$ funksiya biror E sohada ikki marta differensiallanuvchi bo'lsa, u holda,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2u=p(p-1)u$$

tenglikning o'rinli ekanligini isbotlang.

21.88. Ushbu $u=x^yy^x$ funksiya $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(x+y+\ln u)u$ tenglamani qanoatlantirishini isbotlang.

21.89. Ushbu $u=\frac{x-y}{z-t}+\frac{t-x}{y-z}$ funksiya $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=0$ tenglamani qanoatlantirishni isbotlang.

Quyidagi funksiyaning M_0 nuqtada $\overrightarrow{M_0M}$ yo‘nalish bo‘yicha hosilasini toping:

21.90. $f(x,y)=5x+10x^2y+y^5$, $M_0(1,2)$, $M(5,-1)$.

21.91. $f(x,y)=xy^2z^3$, $M_0(3,2,1)$, $M(7,5,1)$.

21.92. $f(x,y,z)=\arcsin\dfrac{z}{\sqrt{x^2+y^2}}$, $M_0(1,1,1)$, $M(1,\ 5,\ 4)$.

21.93. Ushbu $f(x,y)=3x^4+y^3+xy$ funksiyaning $M_0(1,2)$ nuqtada, Ox o‘q bilan 135° burchak tashkil qilgan nurning yo‘nalishi bo‘yicha hosilasini toping.

21.94. Ushbu $f(x,y)=arctg\dfrac{y}{x}$ funksiyaning $x^2+y^2=2x$ aylananing $M_0\left(\dfrac{1}{2},\dfrac{\sqrt{3}}{2}\right)$ nuqtasiga o‘tkazilgan tashqi normalning yo‘nalishi bo‘yicha hosilasini toping.

21.95. Quyidagi $f(x,y,z)=\ln\left(e^x+e^y+e^z\right)$ funksiyaning $M_0(0,0,0)$ nuqtada, Ox, Oy, Oz koordinatalar o‘qlari bilan, mos ravishda, $\dfrac{\pi}{3}$, $\dfrac{\pi}{4}$ va $\dfrac{\pi}{3}$ burchaklarni tashkil qilgan nurning yo‘nalishi bo‘yicha hosilasini toping.

21.96-21.97- misollarda berilgan $f(x,y)$ funksiyaning P_0 nuqtada kamayish va o‘sish yo‘nalishlarini toping va har bir yo‘nalish bo‘yicha hosilasini toping. Shuningdek, $f(x,y)$ funksiyaning P_0 nuqtada $\vec{v}$ vektor yo‘nalishidagi hosilasini toping.

21.96. $f(x,y)=\cos x\cos y$, $P_0\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$, $\vec{v}=3\vec{i}+4\vec{j}$.

21.97. $f(x,y,z)=\ln(2x+3y+6z)$, $P_0(-1,-1,1)$, $\vec{v}=2\vec{i}+3\vec{j}+6\vec{k}$.

Quyidagi skalar maydonning berilgan nuqtadagi gradientini toping:

21.98. $u(x,y)=x^2-2xy+3y-1$, $\quad grad\,u\,\big|_{M(x,y)}=?$

21.99. $u(x,y)=5x^2y-3xy^3+y^4$, $grad\,u\,\big|_{M(x,y)}=?$

21.100. $u=x^2+y^2$, $grad\,u\big|_{(3,2)}=?$

21.101. $u=\sqrt{4+x^2+y^2}$. $grad\,u\big|_{(2,1)}=?$

21.102. $u=\operatorname{arctg}\dfrac{y}{x}$. $grad\,u\big|_{(1,1)}=?$

21.103. $u=arctg\dfrac{x}{y}$ skalyar maydonning $(1;1)$ va $(-1;-1)$ nuqtalardagi gradientlari orasidagi burchakni toping.

21.104. $z_1=\sqrt{x^2+y^2}$, $z_2=x-3y+\sqrt{3xy}$ funksiyalarning $(3,4)$ nuqtadagi gradientlari orasidagi burchakni toping.

21.105. $grad(\varphi\psi)=\varphi\, grad\psi+\psi\, grad\varphi$ tenglikni isbotlang.

21.106. $z=\varphi(u,v),\ u=\psi(x,y),\ v=\zeta(x,y)$ funksiyalar berilganda, $grad\, z=\frac{\partial\varphi}{\partial u}gradu+\frac{\partial\varphi}{\partial v}gradv$ tenglikning to‘g‘riligini ko‘rsating.

21.107-21.111 misollarda funksiyaning orttirmasini uning differensialiga almashtirib, quyida berilgan ifodalarni taqribiy hisoblang:

21.107. $(1{,}02)^{4{,}05}$. **21.108.** $\sqrt{8{,}04^2+6{,}03^2}$. **21.109.** $(1{,}02)^3\cdot(0{,}97)^2$.

21.110. $\sin 32^0\cos 59^0$. **21.111.** $\ln(0{,}9^3+0{,}99^3)$. **21.112.** $\sqrt{2{,}03^2+5e^{0{,}02}}$.

Mustaqil yechish uchun misollarning javoblari

21.1. $u'_x=2x+6xy^3,\ u'_y=3y^2+9x^2+y^2$. **21.2.** $u'_x=\frac{2x-y}{y^2},\ u'_y=\frac{xy-2x^2}{y^3}$.

21.3. $u'_x=yz+\frac{1}{yz},\ u'_y=xz-\frac{x}{y^2z},\ u'_z=xy-\frac{x}{yz^2}$. **21.4.** $u'_x=y\cdot\cos(xy+yz)$, $u'_y=(x+z)\cos(xy+yz),\ u'_z=y\cdot\cos(xy+yz)$.

21.5. $u'_x=\frac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)\cdot e^{x/y}\cdot\frac{1}{y}$,

$u'_y=\frac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)e^{x/y}\left(-\frac{x}{y^2}\right)$. **21.6.** $u'_x=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\cdot\sin\frac{y}{x}$,

$u'_y=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}$. **21.7.** $u'_x=e^x(x\sin y+\sin y+\cos y)$,

$u'_y=e^x(x\cos y-\sin y)$. **21.8.** $u'_x=yx^{y-1},\ u'_y=x^y\cdot\ln x$.

21.9. $u'_x=z\left(\frac{y}{x}\right)^{z-1}\cdot\left(-\frac{y}{x^2}\right)=-\frac{z}{x}\left(\frac{y}{x}\right)^z,\quad u'_y=\frac{z}{y}\left(\frac{y}{x}\right)^z,\ u'_z=\left(\frac{y}{x}\right)^z\cdot\ln\frac{y}{x}$.

21.10. $u'_x=-\frac{2}{\sqrt{x^2+y^2}},\ u'_y=\frac{2x}{y\sqrt{x^2+y^2}}$.

21.11. $u'_x=\frac{xy^2\sqrt{2x^2-2y^2}}{|y|(x^4-y^4)},\ u'_y=\frac{yx^2\sqrt{2x^2+2y^2}}{|y|(y^4-x^4)}$.

21.12. $u'_x=\sin 2x\ln g(1+\sin^2 x)^{\ln y-1},\quad u'_y=\frac{1}{y}(1+\sin^2 x)^{\ln y}\cdot\ln(1+\sin^2 x)$

21.13. $u'_x=x^{y-1}y^{z+1}z^x+x^yy^zz^x\cdot\ln z$, $u'_y=x^yy^zz^x\ln x+x^yy^{z-1}z^{x+1}$,

$u'_z=x^yy^zz^x\ln y+x^{y+1}y^zz^{x-1}$. **21.14.** $\frac{\partial g}{\partial r}=\cos\theta+\sin\theta,\ \frac{\partial g}{\partial\theta}=-r\sin\theta+r\cos\theta$.

21.21. $\frac{\partial f}{\partial R_1}=-\frac{1}{R_1^2};\ \frac{\partial f}{\partial R_2}=-\frac{1}{R_2^2};\ \frac{\partial f}{\partial R_3}=-\frac{1}{R_3^2}$. **21.16.** $\frac{\partial P}{\partial n}=\frac{RT}{V};\ \frac{\partial p}{\partial R}=\frac{nT}{V}$;

$\frac{\partial P}{\partial T}=\frac{nR}{V};\ \frac{\partial P}{\partial V}=-\frac{nRT}{V^2}$.

21.17. $u'_x(1;1)=1,\quad u'_y(1;1)=-2$. **21.18.** $u'_x(1;2)=\frac{1}{3},\quad u'_y(1;2)=-\frac{1}{6}$.

21.19. $u'_x(1;1)=1-\pi,\ u'_y(1;1)=1-\pi$. **21.20.** $u'_x(1;-1)=2,\ u'_y(1;-1)=1$.

21.21. $u'_x(0;0)=0,\ u'_y=(0;0)=0$. Funksiya $O(0;0)$ nuqtada differensiallanuvchi emas.

21.22. $u'_x(0;0),\ u'_y(0;0)$lar mavjud emas, $u(x,y)$ funksiya 0(0;0) nuqtada differensiallanuvchi.

21.23. $u'_x(0;0)=u'_y(0;0)=0;$ $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **21.24.** $u'_x(0,0)=u'_y(0,0)=0;$ $u(x,y)$ funksiya 0(0;0) nuqtada diferensiallanuvchi emas.

21.25. $u'_x(0,0)=u'_y(0,0)=0;\ u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi.

21.26. $u'_x(0,0)=u'_y(0,0);\ u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **21.27.** $u'_x(0,0)=u'_y(0,0);$ $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **21.28.** *a*) 0; *b*) 2.

21.29. *a*) 0, *b*) 1. **21.30.** Noto'g'ri. **21.31.** Noto'g'ri ($n>1$ bo'lganda). **21.32.** To'g'ri. **21.33.** Noto'g'ri ($n>1$ bo'lganda). **21.34.** Noto'g'ri. **21.35.** To'g'ri. **21.37.** $(8x^3-6xy^2+3x^2y)dx+(x^3-6x^2y)dy$. **21.38.** $4(y^3+2x^2y+3)\cdot(4xydx+(3y^2+2x^2)dy)$.

21.39. $\dfrac{x^2-y^2}{xy}\left(\dfrac{dx}{x}-\dfrac{dy}{y}\right)$ **21.40.** $y(x^2+y^2)^{-3/2}(ydx-xdy)$.

21.41. $2^{-y/x}\dfrac{\ln a}{x^2}(ydx-xdy)$. **21.42.** $\dfrac{1}{\sqrt{y}}\left(dx+\dfrac{ydy}{x+\sqrt{x^2+y^2}}\right)$.

21.43. $\dfrac{1}{\sqrt{y}}ctg\dfrac{x+1}{\sqrt{y}}\left(dx-\dfrac{x+1}{2y}\right)dy$. **21.44.** $\dfrac{xdy-ydx}{x^2+y^2}$.

21.45. $(1+xy)^{y-1}(y^2dx+(xy+(1+xy)\ln(1+xy))dy)$. **21.46.** *a*) $dx-dy$, *b*) 0.

21.47. $\frac{1}{2}dx$

21.48. $du|_N=-\sin x(y+z)\cdot[(y+z)dx+xdy+xdz],\ du|_N=-\dfrac{\sqrt{3}}{2}\left(\dfrac{\pi}{3}dx+dy+dz\right)$.

21.49. $du|_M=e^{xy}(ydx+xdy),\ du|_O=0$.]**21.50.** $du|_M=x^y\left(\dfrac{y}{x}dx+\ln x\,dy\right)$,

$du|_{M_0}=12dx+8\ln 2\,dy$. **21.51.** $du|_M=(1+\ln xy)dx+\dfrac{x}{y}dy,\ du|_{M_0}=dx+dy$.

21.52. $du|_M=-\dfrac{1}{2}dz$. **21.53.** $du|_M=\dfrac{1}{37}(2dx+3dy-12dz)$.

21.54. $du|_M=(2dx+\ln 4dz)$.

21.55. $f'_x=2xf'_u,\ f'_y=e^yf'_u$. **21.56.** $f'_x=\dfrac{3x^2+y^2}{3\sqrt[3]{(x^3+xy^2)^2}}f'_u,\ f'_y=\dfrac{2xy}{3\sqrt[3]{(x^3+xy^2)^2}}f'_u$.

21.57. $f'_x = \frac{1}{1+(x+\ln y)^2} f'_u$, $f'_y = \frac{1}{y\left(1+(x+\ln y)^2\right)} f'_u$.

21.58. $d\varphi = \left(y - \frac{y^2}{x^2}\right) f'_u dx + \left(x + \frac{2y}{x}\right) f'_u dy$.

21.59. $d\varphi = \left(2x f'_v - \frac{y}{(x+y)^2} f'_u\right) dx + \left(\frac{x}{(x+y)^2} f'_u - 3y^2 f'_v\right) dy$.

21.60. $d\varphi = (2x f'_u + f'_v + yz f'_w)dx + (2y f'_u + f'_v + xz f'_w)dy + (2z f'_u + f'_v + yx f'_w)dz$.

21.61. $\left.\frac{dw}{dt}\right|_{t=0} = -1$. **21.66.** $\frac{\partial w}{\partial r}\Big|_{(r,s)=(\pi,0)} = 2; \frac{\partial w}{\partial s}\Big|_{(r,s)=(\pi,0)} = 2 - \pi$.

21.67. $\left.\frac{dw}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)\sin 1 + (\cos 1 + \cos 2)\cos 1 + 2(\sin 1 + \cos 1)\sin 2$.

21.68. $\frac{\partial w}{\partial x} = \cos\sigma \frac{\partial w}{\partial r} - \frac{\sin\sigma}{r} \cdot \frac{\partial w}{\partial \sigma}$; $\frac{\partial w}{\partial y} = \sin\sigma \frac{\partial w}{\partial r} + \frac{\cos\sigma}{r} \frac{\partial w}{\partial \sigma}$.

21.69. $\frac{dW}{dt} = 0$, $\left.\frac{dW}{dt}\right|_{t_0=\pi} = 0$. **21.70.** $\frac{dW}{dt} = 1$, $\left.\frac{dW}{dt}\right|_{t_0=3} = 1$.

21.71. $\frac{dW}{dt} = arctgt + 1$, $\left.\frac{dW}{dt}\right|_{t_0=1} = \pi + 1$. **21.72.** $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1,-1)} = 12$.

21.73. $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0,0)} = -7$.

21.74. $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(\ln 2,1)} = 2$, $\left.\frac{\partial W}{\partial v}\right|_{(u,v)=(\ln 2,1)} = 1$. **21.84.** $f(x,y) = \frac{y}{2} + 4$; $g(x,y) = \frac{x}{2} + \frac{9}{2}$.

21.85. $w = f(x,y) = x + y^2 + e^x \cos y$. **21.90.** -18. **21.91.** $\frac{52}{5}$. **21.92.** $\frac{1}{5}$.

21.93. $-\frac{\sqrt{2}}{2}$. **21.94.** $\frac{\sqrt{3}}{2}$. **21.95.** $\frac{2+\sqrt{2}}{6}$. **21.96.** $\vec{u} = -\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$ yo‘nalishda o‘sadi, $-\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$ yo‘nalishda kamayadi. $f'\left(P_0;\vec{u}\right) = \frac{\sqrt{2}}{2}$; $f'\left(P_0;-\vec{u}\right) = -\frac{\sqrt{2}}{2}$; $f'\left(P_0;\vec{u}_1\right) = -\frac{7}{10}, \vec{u}_1 = \frac{\vec{v}}{|\vec{v}|}$.

21.97. $\vec{u} = \frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$ yo‘nalishda o‘sadi; $-\vec{u} = -\frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} - \frac{6}{7}\vec{k}$ yo‘nalishda kamayadi; $f'\left(P_0;\vec{u}\right) = 7$; $f'\left(P_0;-\vec{u}\right) = -7$; $f'\left(P_0;\vec{u}_1\right) = 7$; $\vec{u}_1 = \frac{\vec{v}}{|\vec{v}|}$.

21.98. $grad u\,|_{M(x,y)} = 2(x-y)\vec{i} + (3-2x)\vec{j}$

21.99. $grad u\,|_{M(x,y)} = (10xy - 3y^3)\vec{i} + (4y^3 - 9xy^2)\vec{j}$ **21.100.** $grad u|_{(3,2)} = 6\vec{i} + 4\vec{j}$.

21.101. $grad u|_{(2;1)} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j}$. **21.102.** $grad u|_{(1,1)} = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$.

21.102. $\varphi=\pi$. **21.107.** 1,08. **21.108.** 10,05. **21.109.** 1,00. **21.110.** $-0{,}03$. **21.111.** 0,273. **21.112.** 3,037.

22 -§. Ko'p o'zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differensiallari

22.1. Yuqori tartibli xususiy hosilalar. $u=f(x_1,x_2,...,x_m)=f(M)$ funksiya $\{M\}\subset R^m$ ochiq to'plamda berilgan bo'lib, uning har bir $M(x_1,x_2,...,x_m)$ nuqtasida $f'_{x_1}, f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsin. Bu xususiy hosilalar, o'z navbatida, $x_1,x_2,...,x_m$ o'zgaruvchilarning funksiyasi sifatida, $\{M\}$ to'plamda aniqlangan bo'lsin.

$\frac{\partial u}{\partial x_i}$ $(i=1,2,...,m)$ funksiya ham, biror $M\in\{M\}$ nuqtada x_k argument bo'yicha xususiy hosilaga ega bo'lishi mumkin. Bu x_k argument bo'yicha xususiy hosila- berilgan $u=f(x_1,x_2,...,x_m)$ funksiyaning *ikkinchi tartibli xususiy hosilasi* deyiladi va u $\frac{\partial^2 u}{\partial x_k\partial x_i}$, $f^{(2)}_{x_i x_k}$, $u^{(2)}_{x_i x_k}$ $(i=1,2,...,m;\ k=1,2,...,m)$ kabi belgilanadi, bunda, $i\neq k$ bo'lsa, u holda, $\frac{\partial^2 u}{\partial x_k\partial x_i}$ xususiy hosilaga, *aralash xususiy hosila* deyiladi, $k=i$ bo'lganda $\frac{\partial^2 u}{\partial x_k\partial x_k}=f''$ deb yozish o'rniga, $\frac{\partial^2 u}{\partial x_k^2}=f''_{x_k^2}$ kabi yoziladi. Xuddi shunday, $f(x_1,x_2,...,x_m)$ funksiyaning uchinchi, to'rtinchi, va xokazo, tartibli xususiy hosilalarining ta'rifi beriladi. $f(x_1,x_2,...,x_m)$ funksiya $x_{i_1},x_{i_2},...,x_{i_n}$ argumentlari bo'yicha $(n-1)$-tartibli xususiy hosilalarga ega bo'lsin. Bu $(n-1)$ tartibli xususiy hosilalar ham, $M(x_1,x_2,...,x_m)\in\{M\}$ nuqtada x_{i_n} argumenti bo'yicha xususiy hosilaga ega bo'lsin. Bu hosila, $u=f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,x_{i_n}$ argumentlar bo'yicha M nuqtadagi n-*tartibli xususiy hosilasi* deyiladi. Shunday qilib, $x_{i_1},x_{i_2},...,x_{m-1},x_m$ argumentlar bo'yicha n-tartibli xususiy hosilani,

$$\frac{\partial^n u}{\partial x_m\partial x_{m-1},...,\partial x_{i_2}\partial x_{i_1}}=\frac{\partial}{\partial x_m}\left(\frac{\partial^{n-1}u}{\partial x_{m-1}...\partial x_{i_2}\partial x_{i_1}}\right)$$

kabi yozish mumkin. Agar $i_1,i_2,...,i_n$ indekslarning hammasi birdaniga bir-biriga teng bo'lmasa, u holda $\frac{\partial^n u}{\partial x_m...\partial x_{i_2}\partial x_{i_1}}$ xususiy hosila n-*tartibli aralash xususiy hosila* deyiladi.

22.1-misol. Ushbu $f(x,y)=\ln(x^2+y^2)$ funksiya $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ tenglamani qanoatlantirishini ko'rsating.

Yechilishi. Berilgan funksiyaning x va y bo'yicha ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial f(x,y)}{\partial x}=\frac{2x}{x^2+y^2},\quad \frac{\partial f(x,y)}{\partial y}=\frac{2y}{x^2+y^2},$$

$$\frac{\partial^2 f(x,y)}{\partial x^2}=\frac{2(y^2-x^2)}{(x^2+y^2)^2},\quad \frac{\partial^2 f(x,y)}{\partial y^2}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}.$$

$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ tenglamaga ikkinchi tartibli xususiy hosilalarni keltirib qo'yamiz:

$$\frac{\partial^2 f(x,y)}{\partial x^2}+\frac{\partial^2 f(x,y)}{\partial y^2}=0\Rightarrow\frac{2(y^2-x^2)}{(x^2+y^2)^2}+\frac{2(x^2-y^2)}{(x^2+y^2)^2}=0$$

Demak, berilgan funksiya tenglamani qanoatlantirar ekan.

22.1-teorema. $u=f(x,y)$ funksiya $\{M\}\subset R^2$ ochiq to'plamda aniqlangan bo'lib, shu to'plamda $f'_x, f'_y, f''_{xy}, f''_{yx}$ xususiy hosilalarga ega bo'lsin. Agar aralash hosilalar $M_0(x_0,y_0)\in\{M\}$ nuqtada uzluksiz bo'lsa, u holda, shu nuqtada,

$$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$$

bo'ladi.

22.2-misol. Ushbu $f(x,y)=x^2-2xy^2$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping hamda $\frac{\partial^2 f}{\partial x\partial y}$ va $\frac{\partial^2 f}{\partial y\partial x}$ aralash xususiy hosilalarning o'zaro tengligini ko'rsating.

Yechilishi. Berilgan funksiyaning birinchi va ikkinchi tartibli xususiy hosilalarini topamz:

$f'_x(x,y)=2x-2y^2,\ f'_y=-4xy,$

$f''_{x^2}(x,y)=2,\quad f''_{y^2}=-4x,\ f''_{xy}(x,y)=-4y,\ f''_{yx}=-4y.$

Endi, aralash xususiy hosilalarni $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ tenglikka keltirib qo'yamiz:

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-4y.$$

22.3-misol. Ushbu $u=arctg\frac{x}{y}$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

Yechilishi. Birinchi va ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x}=\frac{y}{x^2+y^2},\quad \frac{\partial u}{\partial y}=-\frac{x}{x^2+y^2},$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2+y^2}{\left(x^2+y^2\right)^2},\quad \frac{\partial^2 u}{\partial x^2}=-\frac{2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2},\quad \frac{\partial^2 u}{\partial y^2}=-\frac{2x}{\left(x^2+y^2\right)^2}.$$

Bu misolda $\frac{\partial^2 u}{\partial y\partial x}$ va $\frac{\partial^2 u}{\partial x\partial y}$ aralash xususiy hosilalar bir-biriga teng. Umumiy holda, bu aralash xususiy hosilalar bir-biriga teng bo‘lmasligi ham mumkin.

Misolni Maple tizimidan foydalanib yechish:

> z:=arctan(x/y):diff(z,x,y);diff(z,y,x);

$$-\frac{1}{y^2\left(1+\frac{x^2}{y^2}\right)}+\frac{2x^2}{y^4\left(1+\frac{x^2}{y^2}\right)^2}$$

$$-\frac{1}{y^2\left(1+\frac{x^2}{y^2}\right)}+\frac{2x^2}{y^4\left(1+\frac{x^2}{y^2}\right)^2}$$

22.4-misol. Ushbu

$$u=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

funksiyaning $\frac{\partial^2 u}{\partial y\partial x}$ va $\frac{\partial^2 u}{\partial x\partial y}$ aralash xususiy hosilalarining $M_0(0,0)$ nuqtada mavjudligi va ularning bir-biriga teng emasligini ko‘rsating.

Yechilishi. Birinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x}=\begin{cases}\dfrac{y\left(x^4-y^4+4x^2y^2\right)}{\left(x^2+y^2\right)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

bo‘lgani uchun,

$$\frac{\partial^2 u}{\partial y\partial x}\Big|_{\substack{x=0\\ y=0}}=\lim_{y\to 0}\frac{\frac{\partial u}{\partial x}\big|_{x=0,\ y\neq 0}-\frac{\partial u}{\partial x}\big|_{x=0,\ y=0}}{y}=-1.$$

Xuddi shunday, $\frac{\partial^2 u}{\partial x\partial y}\big|_{x=0,\ y=0}=1$ ekanligini topamiz. Shunday qilib, $M_0(0,0)$ nuqtada $\frac{\partial^2 u}{\partial x\partial y}\neq\frac{\partial^2 u}{\partial x\partial y}$ ekan.

22.2-teorema. $u=f(x_1,x_2,\dots,x_m)$ funksiya $\{M\}\subset R^m$ ochiq to‘plamda aniqlangan, bu to‘plamda mumkin bo‘lgan hamma $(n-1)$- tartibgacha xususiy hosilalarga va n-tartibli aralash hosilalarga ega bo‘lib, bu hosilalar $\{M\}$ to‘plamda uzluksiz bo‘lsa, u holda ixtiyoriy n-tartibli aralash hosilaning ifodasi, hosilani topish tartibiga bog‘liq bo‘lmaydi.

22.2. Yuqori tartibli differensiallar. Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli differensiali tushunchasini kiritishdan avval, funksiyaning n marta differensiallanuvchanligi tushunchasini kiritamiz.

$u=f(x_1,x_2,\dots,x_m)=f(M)$ funksiya $\{M\}\subset R^m$ ochiq to‘plamda berilgan bo‘lib, $M_0(x_1^0,x_2^0,\dots,x_m^0)\in\{M\}$ bo‘lsin. $u=f(x_1,x_2,\dots,x_m)$ funksiya $\{M\}$ to‘plamda $f'_{x_1},f'_{x_2},\dots,f'_{x_m}$ xususiy hosilalarga ega bo‘lsin. Agar $f'_{x_1},f'_{x_2},\dots,f'_{x_m}$ funksiyalar M_0 nuqtada differensiallanuvchi bo‘lsa, $f(M)$ funksiya M_0 nuqtada ikki *marta differensiallanuvchi* deyiladi.

Agar $f(M)$ funksiya $\{M\}$ to‘plamda $(n-1)$–tartibli xususiy hosilalarga ega bo‘lib, bu xususiy hosilalar M_0 nuqtada differensialla-nuvchi bo‘lsa, $f(M)$ funksiya n *marta differensiallanuvchi* deb ataladi.

22.3-teorema. Agar $\{M\}\subset R^m$ ochiq to‘plamda $f(M)$ funksiyaning n-tartibligacha barcha xususiy hosilalari mavjud va $M_0\in\{M\}$ nuqtada uzluksiz bo‘lsa, $f(M)$ funksiya M_0 nuqtada n-marta differensiallanuvchi bo‘ladi. $u=f(x_1,x_2,\dots,x_m)=f(M)$ funksiya $\{M\}\subset R^m$ ochiq to‘plamda berilgan bo‘lib, $M\in\{M\}$ nuqtada differensiallanuvchi bo‘lsa, u holda, uning M nuqtadagi differensiali

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+\dots+\frac{\partial u}{\partial x_m}dx_m \tag{22.1}$$

ko‘rinishda bo‘ladi, bunda $dx_1,dx_2,\dots,dx_m$ lar $x_1,x_2,\dots,x_m$ o‘zgaruvchilarning ixtiyoriy orttirmalaridir.

Faraz qilaylik, $f(M)$ funksiya $M\in\{M\}$ nuqtada ikki marta differensiallanuvchi bo‘lsin. $f(M)$ funksiyaning M nuqtadagi differen-siali $df(M)$ ning differensiali, berilgan $f(M)$ funksiyaning *ikkinchi tartibli differensiali* deb ataladi va u $d^2f=d(df)$ kabi belgilanadi. (22.1) formulani e’tiborga olib, differensiallash formulalaridan foydalansak, quyidagini topamiz:

$$\begin{aligned}d^2u&=d(du)=d\left(\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+\dots+\frac{\partial u}{\partial x_m}dx_m\right)=d\left(\frac{\partial u}{\partial x_1}\right)dx_1+d\left(\frac{\partial u}{\partial x_2}\right)dx_2+\dots+d\left(\frac{\partial u}{\partial x_m}\right)dx_m=\\&=\left(\frac{\partial^2 u}{\partial x_1^2}dx_1+\frac{\partial^2 u}{\partial x_1\partial x_2}dx_2+\dots+\frac{\partial^2 u}{\partial x_m^2}dx_m\right)dx_1+\dots+\left(\frac{\partial^2 u}{\partial x_m\partial x_1}dx_1+\dots+\frac{\partial^2 u}{\partial x_m^2}dx_m\right)dx_m=\\&=\frac{\partial^2 u}{\partial x_1^2}dx_1+\frac{\partial^2 u}{\partial x_2^1}dx_2^2+\dots+2\frac{\partial^2 u}{\partial x_{m-1}^2\partial x_m}dx_{m-1}dx_m\end{aligned} \tag{22.2}$$

$u = f(x_1, x_2, \ldots, x_3) = f(M)$ funksiyaning uchinchi, to‘rtinchi va hokazo tartibli differensiallari ham xuddi yuqoridagidek ta’riflanadi. Shunday qilib, $f(M)$ funksiyaning M nuqtadagi $(n-1)$- tartibli differensiali $d^{n-1}u$ ning differensialiga berilgan $f(M)$ funksiyaning *n-tartibli differensiali* deyiladi va $d^n u = d(d^{n-1}u)$ kabi belgilanadi. (22.2) formuladan ko‘rinadiki, yuqori tartibli differensialning tartibi oshgan sari uning xususiy hosilalar orqali ifodasi murakkablashib boradi. Shu sababli, yuqori tartibli differensiallarni soddaroq shaklda ifodalash uchun, $f(M)$ funksiyaning

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots + \frac{\partial u}{\partial x_m}dx_m$$

differensialini, simvolik ravishda (u ni formal ravishda qavsdan tashkariga chiqarib), quyidagicha

$$du = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)u$$

yozamiz. Unda funksiyaning ikkinchi tartibli differensial,

$$d^2 u = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)^2 u$$

kabi yozilishi mumkin. Bunda, simvolik ravishda. qavs ichidagi yig‘indi kvadratga ko‘tarilib, so‘ngra u ga «*ko‘paytiriladi*», bunda daraja ko‘rsatkichlari xususiy hosilalarning tartibi, deb qaraladi. Xuddi shunday, simvolik ravishda, funksiyaning n-tartibli differensiali

$$d^n u = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)^n u$$

kabi yoziladi.

Xususiy holda, x va y erkli o‘zgaruvchilarga bog‘liq bo‘lgan $u = f(x, y)$ funksiyaning ikkinchi va uchinchi tartibli to‘liq differensiallarini, quyidagi

$$d^2 u = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2 =$$

$$= u''_{x^2}dx^2 + 2u''_{xy}dxdy + u''_{y^2}dy^2, \qquad (22.3)$$

$$d^3 u = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^3 u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 u}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 u}{\partial y^3}dy^3 =$$

$$= u'''_{x^3}dx^3 + 3u'''_{x^2y}dx^2 dy + 3u'''_{xy^2}dxdy^2 + u'''_{y^3}dy^3.$$

ko‘rinishlarda yozish mumkin.

22.9-misol. Ushbu $u = x^2 - y^2 - \ln\frac{x}{y}$ funksiyaning ikkinchi tartibli to‘liq differensialini toping.

Yechilishi. Dastlab berilgan funksiyaning birinchi tartibli xususiy hosilalarini topamiz: $\frac{\partial u}{\partial x}=2x-\frac{1}{x}$, $\frac{\partial u}{\partial y}=-2y-\frac{1}{y}$. Ikkinchi tartibli xususiy hosilalarini topib, (22.3) formulaga keltirib qo'yamiz:

$$\frac{\partial^2 u}{\partial x^2}=2+\frac{1}{x^2}, \quad \frac{\partial^2 u}{\partial y^2}=-2+\frac{1}{y^2}, \quad \frac{\partial^2 u}{\partial x^2}=2+\frac{1}{x^2}, \quad \frac{\partial^2 u}{\partial x\partial y}=0$$

$$d^2u=u''_{x^2}dx^2+2u''_{xy}dxdy+u''_{y^2}dy^2=(2+\frac{1}{x^2})\cdot dx^2+(\frac{1}{y^2}-2)\cdot dy^2.$$

22.3. Murakkab funksiyaning yuqori tartibli differensiallari. Biz yuqorida qaragan, $u=f(x_1,x_2,...,x_m)\ (x_i=\varphi(t_1,t_2,...,t_k),i=1,2,...,m)$ murakkab funksiyaning yuqori tartibli differensialini topamiz.

Ma'lumki, $x_i=\varphi_i(t_1,t_2,...,t_k),(i=1,2,...,m)$ funksiyalarning har biri $N_0(t_1^0,t_2^0,...,t_k^0)\in\{N\}\subset R^k$ nuqtada differensiallanuvchi bo'lib, $u=f(x_1,x_2,...,x_m)$ funksiya esa, mos ravishda, $M_0(x_1^0,x_2^0,...,x_m^0)\in\{N\}\subset R^m$ nuqtada differensiallanuvchi bo'lsa, u holda, 21.3-teoremaga asosan, murakkab funksiya $N_0\in\{N\}\subset R^k$ nuqtada differensiallanuvchi bo'ladi va differensial shaklining invariantlik xossasiga asosan, murakkab funksiyaning differensiali,

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+...+\frac{\partial u}{\partial x_m}dx_m$$

ko'rinishda bo'ladi. Faraz qilaylik, $x_i=\varphi_i(t_1,t_2,...,t_k),(i=1,2,...,m)$ funksiyalarning har biri $N_0(t_1^0,t_2^0,...,t_k^0)\in\{N\}\subset R^k$ nuqtada ikki marta differensiallanuvchi, $f(x_1,x_2,...,x_m)$ funksiya esa, unga mos, $M_0(x_1^0,x_2^0,...,x_m^0)\in\{N\}\subset R^m$ nuqtada, ikki marta differensiallanuvchi bo'lsin. U holda murakkab funksiya ham, $N_0(t_1^0,t_2^0,...,t_k^0)$ nuqtada ikki marta differensiallanuvchi bo'ladi. Differensiallash qoidalaridan foydalanib, funksiyaning ikkinchi tartibli differensialini topamiz:

$$d^2u=d(du)=d\left(\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+...+\frac{\partial u}{\partial x_m}dx_m\right)=d\left(\frac{\partial u}{\partial x_1}\right)dx_1+\frac{\partial u}{\partial x_1}d(dx_1)+d\left(\frac{\partial u}{\partial x_2}\right)dx_2+$$
$$+\frac{\partial u}{\partial x_2}d(dx_2)+...+d\left(\frac{\partial u}{\partial x_m}\right)dx_m+\frac{\partial u}{\partial x_m}d(dx_m)=\left(\frac{\partial}{\partial x}dx_1+\frac{\partial}{\partial x}dx_2+...+\frac{\partial}{\partial x_m}dx_m\right)^2u+ \quad (22.4)$$
$$+\frac{\partial u}{\partial x_1}d^2x_1+\frac{\partial u}{\partial x_2}d^2x_2+...+\frac{\partial u}{\partial x_m}d^2x_m$$

Xuddi shunday usulda murakkab funksiyaning, keyingi, yuqori tartibli differensiallari ham topiladi.

(22.1) va (22.4) formulalarni solishtirish natijasida, ikkinchi tartibli differensiallarda differensial shakli invariantligining saqlanmasligini ko'ramiz.

22.5-eslatma. Agar $x_i = \varphi_i(t_1, t_2, ..., t_k), (i = 1, 2, ..., m)$ funksiyalarning har biri, $(t_1, t_2, ..., t_k)$ o'zgaruvchilarning

$$x_1 = a_{11}t_1 + a_{12}t_2 + ... + a_{1k}t_k + \beta_1,$$

$$..$$

$$x_m = a_{m1}t_1 + a_{m2}t_2 + ... + a_{mk}t_k + \beta_m$$

chiziqli funksiyalari bo'lsa, u holda, $f(x_1, x_2, ..., x_m)$ murakkab funksiyaning yuqori tartibli differensiallari shakli invariantligi saqlanishini ko'rish qiyin emas.

22.10-misol. Ushbu $W = f(u, v),\ u(x, y) = x \sin y,\ v(x, y) = y \cos x$ murakkab funksiyaning ikkinchi tartibli differensialini toping.

Yechilishi. Ma'lumki, $W = f(u, v)$ funksiyaning birinchi tartibli differensiali $dW = f_u du + f_v dv$. murakkab funksiyaning ikkinchi tartibli differensiali esa,

$$d^2W = d(dW) = d(f'_u du + f'_v dv) = (f''_{uu} du + f''_{uv} dv)du + f'_u \cdot d^2u + (f''_{uv} du + f''_{vv} dv)dv + f'_v \cdot d^2v,$$

ko'rinishda bo'ladi, bunda $du = \sin y\ dx + x \cos y\ dy,\ dv = -y \sin x\ dx + \cos x\, dy,$

$$d^2u = 2\cos y \cdot dxdy - x \sin y\, d^2y,\ d^2v = -\cos x\, d^2x - 2\sin x\, dxdy.$$

Endi, bu ifoda-larni murakkab funksiyaning ikkinchi tartibli differensialini topish formulasiga keltirib qo'yamiz:

$$d^2W = [f''_{uu}(\sin y\ dx + x\cos y\ dy) + f''_{uv}(-y\sin x\ dx + \cos x\, dy)](\sin y\ dx + x\cos y\ dy) +$$
$$+ f'_u \cdot (2\cos y \cdot dxdy - x\sin y\, d^2y)^2 + [f''_{uv}(\sin y\ dx + x\cos y\ dy) +$$
$$+ f''_{vv}(-y\sin x\ dx + \cos x\, dy))](-y\sin x\ dx + \cos x\, dy) + f'_v \cdot (-\cos x\, d^2x - 2\sin x\, dxdy)^2 =$$
$$= [\sin^2 y \cdot f''_{uu} - 2y\sin x \sin y \cdot f''_{uv} + y^2 \sin^2 x \cdot f''_{vv} - y\cos x \cdot f'_v]\, dx^2 +$$
$$+ [x\sin 2y \cdot f''_{uu} + 2(\sin y \cos x - xy \sin x \cos y) \cdot f''_{uv} - y\sin 2x \cdot f''_{vv} + 2(\cos y \cdot f'_u - \sin x \cdot f'_v)]dxdy +$$
$$+ [x^2\cos^2 y \cdot f''_{uu} + 2x\cos x \cos y \cdot f''_{uv} + \cos^2 x \cdot f''_{vv} - x\sin y \cdot f'_u]\, dy^2.$$

22.4. O'rta qiymat haqidagi teorema. $u = f(x_1, x_2, ..., x_m) = f(M)$ funksiya $\{M\}$ $(\{M\} \subset R^m)$ to'plamda berilgan bo'lsin. Bu to'plamda shunday $A(a_1, a_2, ..., a_m)$ va $B(b_1, b_2, ..., b_m)$ nuqtalarni olaylikki, bu nuqtalarni birlashtiruvchi,

$$E = \{(x_1, x_2, ..., x_m) \in R^m : a_1 + t(b_1 - a_1), x_2 = a_2 + t(b_2 - a_2), ..., x_m = a_m + t(b_m - a_m); 0 \le t \le 1\}$$

to'g'ri chiziq kesmasi, shu $\{M\}$ to'plamga qarashli, ya'ni $E \subset M$ bo'lsin.

22.4-teorema. Agar $f(M)$ funksiya E kesmaning A va B nuqtalarida uzluksiz bo'lib, kesmaning qolgan nuqtalarida differensiallanuvchi bo'lsa, u holda, E kesmada shunday C nuqta topiladiki $(C = c(c_1, c_2, ..., c_m))$, $f(B) - f(A) = f_{x_1}(C)(b_1 - a_1) + f_{x_2}(C)(b_2 - a_2) + ... + f_{x_m}(C)(b_m - a_m)$ bo'ladi.

22.5. Ko'p o'zgaruvchili funksiyaning Teylor formulasi. $u = f(M)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lsin. $f(M)$ funksiyaning $M \in \{M\}$ nuqtadagi k-tartibli differensialini $d^k u\big|_M$ deb belgilaymiz.

22.5-teorema. $u = f(x_1, x_2, ..., x_m) = f(M)$ funksiya $M_0(x_1^0, x_2^0, ..., x_m^0)$ nuqtaning ($M_0 \in \{M\}$) biror $U_\delta(M_0)$ atrofida $n+1$ marta differensialanuvchi bo'lsin. U holda, berilgan funksiyaning $M_0(x_1^0, x_2^0, ..., x_m^0)$ nuqtadagi $\Delta u = f(M) - f(M_0)$ to'liq orttirmasi quyidagi

$$\Delta u = du\big|_{M_0} + \frac{1}{2!} d^2 u\big|_{M_0} + ... + \frac{1}{n!} d^n u\big|_{M_0} + \frac{1}{(n+1)!} d^{n+1} u\big|_N \tag{22.5}$$

ko'rinishda tasvirlanadi, bunda $N - U_\delta(M_0)$ atrofdagi $M(x_1, x_2, ..., x_m)$ nuqtaga bog'liq bo'lgan biror nuqta, $d^k u\big|_{M_0}$ va $d^{n+1} u\big|_N$ ifodalarda qatnashuvchi dx_i lar $\Delta x_i = x_i - x_i^0$ ga teng. (22.5) formulaga $u = f(x_1, x_2, ..., x_m)$ funksiyaning *Teylor formulasi* deb ataladi.

Agar $\Delta x_i = x_i - x_i^0$ $(i = 1, 2, ..., m)$ deb belgilab, $d^k u\big|_{M_0}$ $(k = 1, 2, ..., n+1)$ to'liq differensial ifodani ochib yozsak, u holda, (22.5) formulani quyidagicha yozish mumkin:

$$f(x_1, x_2, ..., x_m) = f(M_0) + \frac{\partial f(M_0)}{\partial x_1}(x_1 - x_1^0) + ... + \frac{\partial f(M_0)}{\partial x_m}(x_m - x_m^0) +$$
$$\frac{1}{2!}\frac{\partial^2 f(M_0)}{\partial x_1^2}(x_1 - x_1^0)^2 + ... + \frac{1}{n!}\frac{\partial^n f(M_0)}{\partial x_m^n}(x_m - x_m^0)^n + R_{n+1} \equiv P_n(x_1, x_2, ..., x_m) + R_{n+1}, \tag{22.6}$$

bunda $P_n(x_1, x_2, ..., x_m)$ - $x_1, x_2, ..., x_m$ o'zgaruvchilarga bog'liq bo'lgan n darajali ko'phad, $R_{n+1} = \frac{1}{(n+1)!} d^{n+1} u\big|_N$ qoldiq ko'phad. $P_n(x_1, x_2, ..., x_m)$ - ko'phadga *Teylor ko'phadi* deyiladi.

$n = 0$ bo'lganda, (22.6) formula, ko'p o'zgaruvchili funksiya uchun chekli orttirmalar haqidagi *Lagranj formulasi* deyiladi, u quyidagi

$$f(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2, ..., x_m^0 + \Delta x_m) - f(M_0) = \frac{\partial f(N)}{\partial x_1}\Delta x_1 + ... + \frac{\partial f(N)}{\partial x_m}\Delta x_m$$

ko'rinishga ega.

Ushbu $\rho = \rho(M_0, M) = \sqrt{\Delta x_1^2 + \Delta x_2^2 + ... + \Delta x_m^2}$ belgilashni kiritsak, u holda (22.6) formuladagi qoldiq had $R_{n+1} = o(\rho^n)$ Peano ko'rinishdagi qoldiq hadga ega bo'ladi.

22.6-misol. Ushbu $f(x, y) = x^3 - 2xy^2 + y^3 + 4x$ funksiyani $A(1; -1)$ nuqta atrofida Teylor formulasi bo'yicha yoying.

Yechilishi. Dastlab berilgan funksiyaning $A(1;-1)$ nuqtadagi qiymatini hisoblaymiz: $f(1;1)=1-2-1+4=2$. Endi berilgan funksiyaning xususiy hosilalarini topib, ularning $A(1;-1)$ nuqtadagi qiymatlarini hisoblaymiz:

$$f'_x(x,y)=3x^2-2y^2+4,\ f'_x(1;-1)=5;$$
$$f'_y(x,y)=-4xy+3y^2,\ f'_y(1;-1)=7;$$
$$f''_{x^2}(x,y)=6x,\ f''_{x^2}(1,-1)=6;$$
$$f''_{y^2}(x,y)=-4x+6y,\ f''_{y^2}(1,-1)=-10;$$
$$f''_{xy}(x,y)=-4y,\ f''_{xy}(1,-1)=4;$$
$$f'''_{x^3}(x,y)=6,\quad f'''_{x^3}(1,-1)=6;$$
$$f'''_{y^3}(x,y)=6,\quad f'''_{y^3}(1,-1)=6;$$
$$f'''_{xy^2}(x,y)=-4,\quad f'''_{xy^2}(1,-1)=-4$$

Berilgan funksiyaning qolgan xususiy hosilalari nolga teng. Teylor formulasi bo'yicha quyidagi izlanayotgan yoyilmaga ega bo'lamiz:

$$f(x,y)=2+5(x-1)+7(y+1)+3(x-1)^2+4(x-1)(y+1)-5(y+1)^2+$$
$$+(x-1)^3-2(x-1)(y+1)^2+(y+1)^3.$$

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning ko'rsatilgan tartibdagi xususiy hosilalarini toping:

22.1. $u=\sin xy \cdot \dfrac{\partial^3 u}{\partial x^2 \partial y}=?,\ \dfrac{\partial^3 u}{\partial x \partial y^2}=?$ **22.2.** $u=x^4\cos y+y^4\cos x,\ \dfrac{\partial^8 u}{\partial x^4 \partial y^4}=?$

22.3. $u=\sin x\cos 2y,\ \dfrac{\partial^{10} u}{\partial x^4 \partial y^6}=?.$ **22.4.** $u=x^m y^n,\ \dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}?.$

22.5. $u=(x^2+y)^{10} tgx,\ \dfrac{\partial^{10} u}{\partial x \partial y^9}=?.$ **22.6.** $u=\dfrac{x+y}{x-y},\ \dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}=?,\ \dfrac{\partial^n u}{\partial y^n}=?.$

22.7. $u=\ln\dfrac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}},\ \dfrac{\partial^4 u}{\partial x \partial y \partial \xi \partial \eta}=?$

22.8. $u=(x^2+y^2)e^{x+y},\ \dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}=?$ **22.9.** $u=f(x,y)=e^x\sin y,\ f^{(m+n)}_{x^m y^n}(0,0)=?.$

Quyidagi funksiyalarning ko'rsatilgan nuqtalardagi ikkinchi tartibli xususiy hosilalarini toping:

22.10. $u=\dfrac{x}{x+y},\ (1;0).$ **22.11.** $u=y^2(1-e^x),\ (0;1).$

22.12. $u \ln(x^2 + y)$, $(0;1)$. **22.13.** $u = y\sin\frac{y}{x}$, $(2;\pi)$.

22.14. $u = arcsin\frac{x}{\sqrt{x^2+y^2}}$, $(1;-1)$. **22.15.** $u = y + \frac{x}{y}$, $(1;1)$.

22.16. $u = x + xy - 5x^3 + \ln(x^2+1)$, $(1;1)$.

22.17. $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \ bo'lganda, \\ 0, & (x,y) = (0,0) \ bo'lganda \end{cases}$

funksiya $(0,0)$ nuqta uzluksiz ekanligi ma'lum (ko'rsating!). U holda, uning $f_{xy}(0,0)$ va $f_{yx}(0,0)$ xususiy hosilalarini toping:

Quyida berilgan $f(x,y)$ funksiyaning ko'rsatilgan nuqtada ikkinchi tartibli differensialini toping:

22.18. $u = f(x,y) = e^{xy}$, $(1;-1)$. **22.19.** $u = f(x,y) = \frac{x}{y}e^{x^2}$, $(0;1)$.

22.20. $u = f(x,y) = x\cos xy$, $\left(\frac{\pi}{2};-1\right)$. **22.21.** $u = f(x,y) = arctg(x^2 - 2y)$, $(1;0)$.

22.22. $u = (\sin x)^{\cos y}$, $\left(\frac{\pi}{6};\frac{\pi}{2}\right)$. **22.23.** $u = e^{xyz}$, $(1, 1, 1)$.

Quyidagi funksiyalarning ko'rsatilgan tartibdagi differensial-larini toping.

22.24. $u = x^3 + y^3 - 3xy(x-y)$, $d^3u = ?$ **22.25.** $u = \sin(x^2+y^2)$, $d^3u = ?$

22.26. $u = \ln(x^x y^y z^z)$, $d^4u = ?$ **22.27.** $u = e^{ax+by}$, $d^n u = ?$

22.28. $u = X(y)\cdot Y(y)$, $d^n u = ?$. **22.29.** $u = \sin x\, chy$, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - ?$

Quyidagi murakkab funksiyalarning birinchi va ikkinchi tartibli differensiallarini toping (x, y va z lar – erkli o'zgaruvchilar):

22.30. $u = f(t)$, $t = x + y$. **22.31.** $u = f(t)$, $t = \frac{y}{x}$.

22.32. $u = f\left(\sqrt{x^2+y^2}\right)$. **22.33.** $u = f(t)$, $t = x^2 + y^2 + z^2$.

22.34. $u = f(\xi;\eta)$, $\xi = ax, \eta = by$. **22.35.** $W = f(u,v)$, $u = \frac{1}{2}(x^2 - y^2)$, $v = xy$.

22.36. Ushbu $u = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-b)^2}{4a^2t}}$ funksiyaning ushbu $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ issiqlik o'tkazuvchanlik tenglamasini qanoatlantirishini ko'rsating.

22.37. Ushbu $u = \frac{1}{r}$, $r = \sqrt{(x-a)^2 + (y-b)^3 + (z-c)^2}$ funksiyaning, $r \neq 0$ bo'lganda, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ Laplas tenglamasini qanoatlantirishini isbotlang.

22.38. Ushbu $u=\frac{c_1e^{-ar}+C_2e^{ar}}{r}$ funksiya, bunda $r=\sqrt{x^2+y^2+z^2}$, c_1, c_2 o'zgarmas sonlar, quyidagi $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=Cu$ Gelьmgolьs tenglamasini qanoatlantirishini isbotlang.

22.39. Ushbu $u(t,x)=\frac{1}{\sqrt{t}}e^{ix^2/(4t)}$ funksiyaning $i\frac{\partial u}{\partial t}+\frac{\partial^2u}{\partial x^2}=0$ Shryodinger tenglamasini qanoatlantirishini isbotlang.

22.40. $f-r$ argumentning ikki marta differensiallanuvchi funksiyasi, bunda $r=\sqrt{x^2+y^2+z^2}$, bo'lsin va $f_{xx}+f_{yy}+f_{zz}=0$ munosabat o'rinli bo'lsin. U holda, qandaydir a va b o'zgarmaslar uchun, $f(r)=\frac{a}{r}+b$ ekanligini ko'rsating.

Quyidagi ixtiyoriy φ,ψ va hokazo funksiyalarni istalgan marta differensiallanuvchi, deb faraz qilib, quyida berilgan tengliklarni tekshiring:

22.41. $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0,\quad z=\varphi(x^2+y^2).$

22.42. $x^2\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}+y^2=0,\quad z=\frac{y^2}{3x}+\varphi(xy).$

22.43. $\frac{x\partial u}{\partial x}+\alpha y\frac{\partial u}{\partial y}+\beta z\frac{\partial u}{\partial z}=nu,\quad u=x^n\varphi\left(\frac{y}{x^\alpha},\frac{z}{x^\beta}\right).$

22.44. $\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2},\quad u=\varphi(x-at+\varphi(x+at)).$

22.45. $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=n(n-1)u,\quad u=x^n\varphi\left(\frac{y}{x}\right)+x^{1-n}\psi\left(\frac{y}{x}\right).$

Berilgan $f(x,y)$ funksiyani berilgan nuqta atrofida Teylor formulasi bo'yicha yoying:

22.46. $f(x,y)=x^2-3xy+y^2-4x+5y,\ A(1;1).$

22.47. $f(x,y)=-x^2+2xy+3y^2-6x-2y-4,\ A(-2;1).$

22.48. $f(x,y)=x^3+3xy-2y^3,\ A(1;2).$

22.49. $f(x,y)=x^3-5x^2-xy+y^2+10x+5y,\ A(2;-1).$

Berilgan $f(x,y)$ funksiyani berilgan nuqta atrofida Teylor formulasining uchinchi $(n=3)$ hadigacha yoying:

22.50. $f(x,y)=\sqrt{1-x^2-y^2},\ O(0,0).$ **22.51.** $f(x,y)=x^y,\ A(1;1).$

Quyida berilgan funksiyalarni Teylor formulasi bo'yicha ikkinchi hadigacha ($n=2$) yoying:

22.52. $f(x,y)=\frac{1}{x-y}.$ **22.53.** $f(x,y)=\sqrt{x+y}.$ **22.54.** $f(x,y)=e^{x+y}.$

22.55. $f(x,y)=x^3+2y^3-xy$ funksiya berilgan. $f(x+h,y+k)$ funksiyani h va k ning darajalari bo'yicha ikkinchi hadgacha Teylor formulasiga yoying.

22.56. 2) $f(x,y)=e^x\sin y$ funksiya berilgan. $f(x+h,y+k)$ funksiyani h va k ning darajalari bo'yicha uchinchi hadgacha Teylor formulasiga yoying. Bu natijadan foydalanib $e^{0,1}\sin 0{,}49\pi$ ning qiymatini hisoblang.

Berilgan $f(x,y,z)$ funksiyani berilgan nuqta atrofida Teylor formulasiga yoying:

22.57. $f(x,y,z)=(x+y+z)^2$, $(1;1;-2)$.

22.58. $f(x,y,z)=x^2+3z^2-2yz-3z$, $(0;1;2)$.

22.59. $f(x,y,z)=xyz$, $(1;2;3)$.

22.60. $f(x,y,z)=x^3+y^3+z^3-3xyz$, $(1;0;1)$.

22.61. $f(x,y,z)=x^2+y^2+z^2-2(xy+xz+yz)$, $(1;-1;2)$.

Quyida berilgan funksiyalarni Makloren formulasi bo'yicha uchinchi hadgacha ($n=3$) yoying:

22.62. $f(x,y)=e^y\cos x$. **22.63.** $f(x,y)=\sin x\ shy$.

22.64. $f(x,y)-n$ darajali birjinsli funksiya, ya'ni barcha t,x va y lar hamda n- nomanfiy butun son uchun:

$$f(tx,ty)=t^n f(x,y)$$

munosabat o'rinli, bo'lsin. Bunday funksiya uchun,

$$a)\ x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y);\quad b)\ x^2\left(\frac{\partial^2 f}{\partial x^2}\right)+2xy\left(\frac{\partial^2 f}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 f}{\partial y^2}\right)=n(n-1)f(x,y)$$

munosabatlar bajarilishini isbotlang.

Mustaqil yechish uchun misollarning javoblari

22.1. $u'''_{yxx}=-2y\sin xy-xy^2\cos xy$, $u'''_{yxx}=-2x\sin xy-x^2y\cos xy$.

22.2. $\dfrac{\partial^8 u}{\partial x^4\partial y^4}=24(\cos y+\cos x)$. **22.3.** $\dfrac{\partial^{10} u}{\partial x^4\partial y^6}=-2^6\sin x\cdot\cos 2y$.

22.4. $\dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}=m!n!$. **22.5.** $\dfrac{\partial^{10} u}{\partial x\partial y^9}=10!\left(2xtgx+\dfrac{c^2+y}{\cos^2 x}\right)$.

22.6. $\dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}=\dfrac{2(-1)^m(n+m-1)!(nx+my)}{(x-y)^{m+n+1}}$, $\dfrac{\partial^n u}{\partial y^n}=\dfrac{2xn!}{(x-y)^{n+1}}$.

22.7. $\dfrac{\partial^4 u}{\partial x\partial y\partial\xi\partial\eta}=-\dfrac{6}{r^4}+\dfrac{48(x-\xi)^2(y-\eta)^2}{r^8}$, $r=\sqrt{(x-\xi)^2+(y-\eta)^2}$

22.8. $\dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}=e^{x+y}[x^2+y^2+2(mx+ny)+m(m-1)+n(n-1)]$.

22.9. $\sin\frac{n\pi}{2}$.

22.10. $\frac{\partial^2 u}{\partial x^2}=0,\ \frac{\partial^2 u}{\partial x\partial y}=1,\ \frac{\partial^2 u}{\partial y^2}=2$. **22.11.** $\frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial^2 u}{\partial x\partial y}=-2,\ \frac{\partial^2 u}{\partial y^2}=0$.

22.12. $\frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial^2 u}{\partial x\partial y}=0,\ \frac{\partial^2 u}{\partial y^2}=-1$. **22.13.** $\frac{\partial^2 u}{\partial x^2}=-\frac{\pi^3}{16},\ \frac{\partial^2 u}{\partial x\partial y}=\frac{\pi^2}{8},\ \frac{\partial^2 u}{\partial y^2}=-\frac{\pi}{4}$.

22.14. $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=\frac{1}{2},\frac{\partial^2 u}{\partial x\partial y}=0$. **22.15.** $\frac{\partial^2 u}{\partial x^2}=0,\ \frac{\partial^2 u}{\partial y^2}=2,\ \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=-1$.

22.22. $\frac{\partial^2 u}{\partial x^2}=-30,\ \frac{\partial^2 u}{\partial y^2}=0,\ \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=1$. **22.17.** $f_{xy}(0,0)=-1;\ \ f_{yx}(0,0)=1$.

22.18. $d^2u=e^{-1}(dx^2+dy^2)$. **22.19.** $2dx\ \ d^2u=-2dxdy$.

22.20. $u=-2(dx^2-\pi dxdy)$. **22.21.** $d^2u=-dx^2+4dxdy-2dy^2$.

22.22. $d^2u=-2\sqrt{3}\,dxdy+\ln^2 2\ \ dy^2$.

22.23.

$d^2u=e[dx+dy+dz]^2+2[dxdy+dydz+dzdx]$.

22.24. $d^3u=6(dx^3-3dx^2dy+3dxdy^2+dy^3)$.

22.25. $d^3u=-8(xdx+ydy)^3\cos(x^2+y^2)-$.

$-12(xdx+ydy)(dx^2+dy^2)\sin(x^2+y^2)$ **22.26.** $d^4u=2\left(\frac{dx^4}{x^3}+\frac{dy^4}{y^3}+\frac{dz^4}{z^3}\right)$.

22.27. $d^nu=e^{ax+by}(adx+bdy)^n$. **22.28.** $d^nu=\sum_{k=0}^{n}C_n^k X^{(n-k)}(x)\,Y^{(k)}(y)dx^{n-k}dy^k$.

22.29. $\Delta u=0$. **22.30.** $du=f'(t)(dx+dy),\ d^2u=f''(t)(dx+dy)^2$.

22.31. $du=f'(t)\frac{xdy-ydx}{x^2},\quad d^2u=f''(t)\frac{(xdy-ydx)^2}{x^4}-2f'(t)\frac{dx(xdy-ydx)}{x^3}$.

22.32. $du=f'\cdot\frac{xdx+ydy}{\sqrt{x^2+y^2}};\quad d^2u=f''\frac{(xdx+ydy)^2}{x^2+y^2}+f'\frac{(ydx-xdy)^2}{(x^2+y^2)^{3/2}}$.

22.33. $du=2f'(t)[xdx+ydy+zdz]$,

$d^2u=4f''(t)[xdx+ydy+zdz]+2f'(t)[dx^2+dy^2+dz^2]$.

22.34. $du=df'_\xi dx+bf'_\eta dy,\qquad d^2u=a^2f''_{\xi\xi}dx^2+2abf''_{\xi\eta}dx^2+b^2f''_{\eta\eta}dy^2$.

22.35. $dW=(xf'_u+yf'_v)dx+(xf'_v-yf'_u)dy,\ d^2W=$

$[x^2\cdot f''_{uu}+2xy\cdot f''_{uv}+y^2\cdot f''_{vv}+f'_v]\,dx^2++2[xy\cdot f''_{vv}+(x^2-y^2)\cdot f''_{uv}-xy\cdot f''_{uu}+f'_v]dxdy+$

$[y^2\cdot f''_{uu}-2xy\cdot f''_{uv}+x^2\cdot f''_{vv}-f'_u]\,dy^2$.

22.46. $f(x,y)=-5(x-1)+4(y-1)+(x-1)^2-3(x-1)(y-1)+(y-1)^2$.

22.47. $f(x,y)=1-(x+2)^2-2(x+2)(y-1)+3(y-1)^2$.

22.48. $f(x,y)=-9+9(x-1)-21(y-2)+3(x-1)^2+3(x-1)(y-2)-12(y-1)^2+$
$+(x-1)^3-2(y-2)^3$.

22.49. $f(x,y)=6+3(x-2)+(y+1)+(x-2)^2-(x-2)(y+1)+(y+1)^2+$
$+(x-2)^3$. **22.50.** $f(x,y)\approx 1-\frac{1}{2}(\Delta x^2+\Delta y^2)+R_3$.

22.51. $f(x,y)=1+\Delta x+\Delta x\Delta y+\frac{1}{2}\Delta x^2\Delta y+R_3.$

22.52. $\Delta z=\frac{\Delta y-\Delta x}{(x-y)^2}+\frac{\Delta x^2+2\Delta x\Delta y+\Delta y^2}{(x-y)^2}+R_2$,

22.53., $\Delta z=\frac{\Delta x+\Delta y}{2\sqrt{x+y}}+\frac{\Delta x^2+2\Delta x\Delta y+\Delta y^2}{8(x+y)^{3/2}}+R_2$,

22.54. $\Delta z=e^{x+y}(\Delta x+\Delta y)+ +e^{x+1}\frac{(\Delta x+\Delta y)^2}{2}+R_2$.

22.55. $x^3+2y^3-xy+h(3x^2-y)+k(6y^2-x)+3xh^2-hk+ +6yk^2+h^3+2k^3$.

22.56. $f(x+h,y+k)\approx f(x,y)+e^x[\sin y+h\sin y+k\cos y+$

$+\frac{1}{2}(h^2\sin y+2hk\cos y-k^2\sin y)+\frac{1}{6}(h^3\sin y+3h^2k\cos y-3hk^2\sin y-k^3\cos y)]+R_3$,

$f(0{,}1;\,0{,}49\pi)\approx 1{,}1051$.

22.57. $f(x,y,z)=(x-1)^2+(y-1)^2+(z+2)^2+2(x-1)(y-1)+$
$+2(x-1)(z+2)+2(y-1)(z+2)$

22.58. $f(x,y,z)=2-4(y-1)+7(z-2)+x^2+3(z-2)^2- -2(y-1)(z-2)$

22.59. $f(x,y,z)=6+6(x-1)+3(y-2)+2(z-3)+3(x-1)(y-2)+$
$+2(x-1)(z-3)+(y-2)(z-3)+(x-1)(y-2)(z-3)$

22.60. $f(x,y,z)=2+3(x-1)-3y+$
$3(z-1)+3(x-1)^2+3(z-1)^2-3(x-1)y-3y(z-1)+(x-1)^3+y^3+(z-1)^3-3(x-1)y(z-1)$.

22.61. $f(x,y,z)=8-8(y+1)+4(z-2)+(x-1)^2+(y+1)^2+(z-2)^2-2(x-1)(y+1)-$
$-2(x-1)(z-2)-2(y+1)(z-2)$ **22.62.** $f(x,y)=1+y+\frac{1}{2}(y^2-x^2)+\frac{1}{3}(y^3-3x^2y)+o(\rho^3)$,
$\rho=\sqrt{x^2+y^2}$ **22.63.** $f(x,y)=xy+\frac{1}{3!}(xy^3-x^3y)+o(\rho^4)$, $\rho=\sqrt{x^2+y^2}$.

23-§. Ko'p o'zgaruvchili funksiyaning ekstremumlari

23.1. Funksiyaning maksimum va minimum qiymatlari. $u=f(x_1,x_2,...,x_m)=f(M)$ funksiya ochiq $\{M\}$ ($\{M\}\subset R^m$) to'plamda berilgan bo'lib, $M_0(x_1^0,x_2^0,...,x_m^0)\in\{M\}$ bo'lsin.

23.1-ta'rif. Agar M_0 nuqtaning shunday

$$U_\delta(M_0)=\left\{M\in\{M\}:\rho(M,M_0)=\sqrt{(x_1-x_1^0)^2+...+(x_m-x_m^0)^2}<\delta\right\}\subset\{M\}$$

atrofi mavjud bo'lib, $\forall M\in U_\delta(M_0)$ uchun $f(M)\le f(M_0)$ ($f(M)\ge f(M_0)$) tengsizlik bajarilsa, $f(M)$ funksiya M_0 nuqtada *maksimumga (minimumga)* ega deyiladi. $f(M_0)$ qiymat esa, $f(M)$ funksiyaning *maksimum (minimum) qiymati yoki maksimumi (minimumi)* deyiladi va u, $f(M_0)=\max\limits_{M\in U_\delta(M_0)}\{f(M)\}$ $\left(f(M_0)=\{f(M)\}_{M\in U_\delta(M_0)}\right)$ kabi belgilanadi.

23.2-ta'rif. Agar M_0 nuqtaning shunday $U_\delta(M_0)$ atrofi mavjud bo'lib, $\forall M\in U_\delta(M_0)/\{M_0\}$ uchun $f(M)<f(M_0)(f(M)>f(M_0))$ tenglik bajarilsa, $f(M)$ funksiya M_0 nuqtada *qa'tiy maksimumga (qa'tiy minimumga)* ega deyiladi, $f(M_0)$ qiymatga esa, $f(M)$ funksiyaning *qa'tiy maksimum* (*qa'tiy minimum) qiymati* yoki *qa'tiy maksimumi* (*qa'tiy minimumi*) deyiladi.

Funksiyaning *maksimumi va minimumi, umumiy nom bilan, uning ekstremumi deb yuritiladi.*

23.1. va 23.2 - ta'riflardagi M_0 nuqta $f(M)$ funksiyaga *maksimum* (*minimum*) , *qa'tiy maksimum (qa'tiy minimum) qiymat beradigan nuqta* deyiladi. 23.1. va 23.2-ta'riflardan ko'rinadiki, $f(M)$ funksiyaning M_0 nuqtadagi $f(M_0)$ qiymati, uning shu nuqta atrofidagi nuqtalardagi qiymatlari bilan solishtirilar ekan. Shuning uchun, funksiyaning M_0 nuqtadagi ekstremumi, *lokal ekstremum* deb yuritiladi.

Misol. Quyidagi

$$1)\ u=x^2+y^2;\quad 2)\ u=|x-y|;\ 3)\ u=\sqrt{1-x^2-y^2}$$

funksiyalarni qaraymiz. Ravshanki, $M_0(0;0)$ nuqta: 1) funksiya uchun qa'tiy minimum; 2) funksiya uchun, minimum; 3) funksiya uchun esa, qa'tiy maksimum nuqtasi bo'ladi. Haqiqatan ham, $M_0(0;0)$ nuqtaning shunday $U_r(0;0)=\{(x,y)\in R^2 : x^2+y^2<r^2\}$ $(0<r<1)$ atrofini qaraylik. Bu atrofdan olingan $\forall M(x,y)\in U_r(0;0)/\{0;0)\}$ uchun, $u=\sqrt{1-x^2-y^2}<u(0;0)=1$ bo'ladi.

23.2. Funksiya ekstremumining zaruriy sharti.

$u=f(x_1,x_2,...,x_m)=f(M)$ funksiya ochiq $\{M\}\left(\{M\}\subset R^m\right)$ to'plamda aniqlangan bo'lsin.

23.1 - teorema (ekstremumning zaruriy sharti). Agar $f(M)$ funksiya M_0 nuqtada ekstremumga ega bo'lib, shu nuqtada barcha $f'_{x_1}, f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsa, u holda, $f'_{x_1}(M_0)=0,\ f'_{x_2}(M_0)=0,...,f'_{x_m}(M_0)=0$ bo'ladi.

23.1-eslatma. $f(M)$ funksiyaning biror M' nuqtada barcha $f'_{x_1}, f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega va $f'_{x_1}(M')=0,\ f'_{x_2}(M')=0,...,f'_{x_m}(M')=0$

bo'lishidan, uning shu M' nuqtada ekstremumga ega bo'lishi har doim ham kelib chiqavermaydi.

Masalan, $u = f(x,y) = xy$ funksiya R^m to'plamda aniqlangan bo'lib, $\frac{\partial u}{\partial x} = y, \frac{\partial u}{\partial y} = x$ xususiy hosilalarga ega va ular $M_0(0;0)$ nuqtada nolga aylanadi. Ammo, bu funksiya $M_0(0;0)$ nuqtada nolga aylanadi, $\forall U_\delta(0,0)$ atrofda esa, musbat va manfiy qiymatlar qabul qiladi.

Demak, 23.1-teorema funksiya ekstremumga ega bo'lishining zaruriy shartini ifodalar ekan. $u = f(M)$ funksiyaning birinchi tartibli xususiy hosilalarini nulga aylantiruvchi nuqtalarga *stasionar nuqtalar* deyiladi. Stasionar nuqtalarda funksiya ekstremumga ega bo'lishi ham, ega bo'lmasligi ham mumkin.

23.2-eslatma. Agar $f(M)$ funksiya M_0 nuqtada differensiallanuvchi bo'lsa, u holda funksiya ekstremumga ega bo'lishining zaruriy shartini, $df(M_0) = 0$ ko'rinishda ham yozish mumkin.

23.1-misol. Ushbu

$$u = f(x,y) = \sqrt{x^2 + y^2}$$

funksiya $M_0(0;0)$ nuqtada ekstremumga ega bo'ladimi?

Yechilishi. Berilgan funksiya $M_0(0;0)$ nuqtada nolga aylanadi. $M_0(0;0)$ nuqtaning ixtiyoriy $U_\delta(M_0) = \{(x,y) \in R^2 : x^2 + y^2 < \delta\}$ $(\delta > 0)$ atrofini qaraymiz. Unda $\forall M(x,y) \in U_\delta(M_0)$ uchun $f(x,y) = \sqrt{x^2 + y^2} \geq f(0;0) = 0$ bo'ladi.

Demak, berilgan funksiya $M_0(0;0)$ nuqtada minimumga ega va $\min\{f(x,y)\} = 0$ bo'ladi. Lekin, $f(x,y) = \sqrt{x^2 + y^2}$ funksiya $M_0(0;0)$ nuqtada xususiy hosilalarga ega emas. Shunday qilib, $u = f(x_1, x_2, \ldots, x_m)$ funksiya, ochiq $\{M\} \subset R^m$ to'plamning: 1) barcha xususiy hosilalar nolga aylanadigan, ya'ni $\frac{\partial u}{\partial x_1} = 0, \frac{\partial u}{\partial x_2} = 0, \ldots, \frac{\partial u}{\partial x_m} = 0$ tenglamalarni qanoatlantiradigan nuqtalarida;

2) xususiy hosilalar mavjud bo'lmagan nuqtalarida ekstremumga erishishi mumkin.

23.3. Funksiya ekstremumining etarli sharti. 23.3.1. Kvadratik forma to'g'risida qisqacha ma'lumot. Kvadratik formalar algebra kursida batafsil o'rganilsa-da, biz, kelgusida qo'llaniladigan, kvadratik formaga oid ba'zi bir tushunchalarni keltirib o'tamiz. Ushbu,

$$Q(x_1, x_2, \ldots, x_m) = a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{1m}x_1x_m + a_{21}x_2x_1 + a_{22}x_2^2 + \ldots + a_{mm}x_m^2$$

(yoki $Q=\sum_{i,j=1}^{m} a_{ij}x_i x_j$) ifodaga, $x_1, x_2, ..., x_m$ o'zgaruvchilarning *kvadratik formasi* deyiladi, bunda a_{ik} $(a_{ij}=a_{ji})$ lar - kvadratik formaning *koeffitsiyentlari* deyiladi. Bu koeffisientlardan tuzilgan,

$$A=\begin{pmatrix} a_{11} & a_{12} \dots a_{1m} \\ a_{21} & a_{22} \dots a_{2m} \\ a_{m1} & a_{m2} \dots a_{mm} \end{pmatrix}$$

matrisaga – kvadratik formaning *matrisasi* deyiladi. Ushbu

$$\delta_1=a_{11},\ \delta_2=\begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix}, ..., \delta_k=\begin{vmatrix} a_{11} \dots a_{1k} \\ \dots\dots\dots \\ a_{k1} \dots a_{kk} \end{vmatrix}, ..., \delta_m=\begin{vmatrix} a_{11} \dots a_{1m} \\ a_{21} \dots a_{2m} \\ \dots\dots\dots \\ a_{m1} \dots a_{mm} \end{vmatrix}$$

determinantlarga esa, A matrisaning *minorlari* deyiladi. Ravshanki, $x_1=x_2=...=x_m=0$ bo'lganda har qanday kvadratik forma uchun, $Q(0,0,...,0)=0$ bo'ladi. Endi boshqa nuqtalarni qaraylik. Bunda quyidagi hollar bo'lishi mumkin:

1.Barcha $x_1^2+x_2^2+...+x_m^2>0$ nuqtalar uchun, $Q(x_1,x_2,...x_m)>0$ bo'lsa, bu holda kvadratik forma *musbat aniqlangan* deyiladi.

2. Barcha $x_1^2+x_2^2+...+x_m^2>0$ nuqtalar uchun, $Q(x_1,x_2,...x_m)<0$ bo'lsa, bu holda kvadratik forma *manfiy aniqlangan* deyiladi.

3. Ba'zi $(x_1,x_2,...,x_m)$ nuqtalar uchun, $Q(x_1,x_2,...x_m)>0$, ba'zi $(x_1,x_2,...,x_m)$ nuqtalar uchun esa, $Q(x_1,x_2,...x_m)<0$ bo'lsa, bu holda kvadratik forma *noaniq* (aniqlanmagan) deyiladi.

4. Barcha $x_1^2+x_2^2+...+x_m^2>0$ nuqtalar uchun, $Q(x_1,x_2,...x_m)\geq 0$ va ular orasida, $Q(x_1,x_2,...x_m)=0$ bo'ladigan $(x_1,x_2,...,x_m)$ nuqtalar ham bor bo'lsa, kvadratik forma - *yarim musbat aniqlangan* deyiladi.

5. Barcha $x_1^2+x_2^2+...+x_m^2>0$ nuqtalar uchun, $Q(x_1,x_2,...x_m)\leq 0$ va ular orasida, $Q(x_1,x_2,...x_m)=0$ bo'ladigan $(x_1,x_2,...,x_m)$ nuqtalar ham bor bo'lsa, kvadratik forma - *yarim manfiy aniqlangan* deyiladi.

Silvestr alomati. $Q(x_1,x_2,...,x_m)=\sum_{i,k=1}^{m} a_{ik}x_i x_k$ kvadratik formaning musbat aniqlangan bo'lishi uchun, $\delta_1>0, \delta_2>0, ..., \delta_m>0$ tengsizliklarning; manfiy aniqlangan bo'lishi uchun, $\delta_1<0, \delta_2>0, \delta_3<0, \delta_4>0, ..., (-1)^m\delta_m>0$ tengsizliklarning bajarilishi zarur va yetarli.

Funksiya ikki o'zgaruvchiga bog'liq bo'lgan xususiy holni qaraymiz.

$u = f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtaning biror $U_\delta(M_0) = \{(x, y) \in R^2 : \rho(M, M_0) < \delta\}$ $(\delta > 0)$ atrofida aniqlangan, u barcha birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo'lib, M_0 nuqta $u = f(M)$ funksiyaning stasionar nuqtasi, ya'ni

$$f'_x(M_0) = 0,\ f'_y(M_0) = 0$$

bo'lsin. $a_{11} = f''_{x^2}(M_0),\ a_{12} = f''_{xy}(M_0),\ a_{22} = f''_{y^2}(M_0)$ deb belgilaymiz.

1^0. Agar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 > 0$ va $a_{11} > 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada minimumga erishadi.

2^0. Agar $a_{11}a_{22} - a_{12}^2 > 0$ va $a_{11} < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada maksimumga erishadi.

3^0. Agar $a_{11}a_{22} - a_{12}^2 < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4^0. Agar $a_{11}a_{22} - a_{12}^2 = 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin.

23.3-teorema (shartsiz ekstremumning yetarli sharti). $f(M)$ funksiya $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ nuqtaning biror atrofida ikkinchi tartibli uzluksiz xususiy hosilalarga ega va $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ nuqta - $f(M)$ funksiyaning stasionar nuqtasi bo'lsin. U holda:

1) agar

$$B(dx_1, dx_2, \ldots, dx_n) = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(M_0)}{\partial x_k \partial x_j} dx_k dx_j$$

kvadratik forma, ya'ni $f(M)$ funksiyaning $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ nuqtadagi ikkinchi tartibli differensiali $B(dx_1, dx_2, \ldots, dx_n) = d^2 f(M_0)$ musbat (manfiy) aniqlangan bo'lsa, $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ nuqta - $f(M)$ funksiyaning minimum (maksimum) nuqtasi bo'ladi.

2) agar $B(dx_1, dx_2, \ldots, dx_n)$ kvadratik forma aniqlanmagan bo'lsa (ham musbat, ham manfiy qiymatlar qabul qilsa), $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ nuqta - $f(M)$ funksiyaning ekstremum nuqtasi bo'lmaydi.

23.2-misol. Quyidagi

$$a)\ \ u = x^2 - xy + y^2;\ b)\ \ u = x^2 - xy - y^2$$

funksiyalarni ekstremumga tekshiring.

Yechilishi. $a)$ Berilgan funksiyaning xususiy hosilalarini topamiz va nulga tenglashtiramiz. $u'_x = 2x - y = 0,\ u'_y = -x + 2y = 0$. Bu sistemani yechib, ekstremumga shubhali, $M_0(0;0)$ nuqtani topamiz. Endi ikkinchi tartibli xususiy hosilalarni topamiz: $a_{11}=u''_{x^2}=2;\ a_{12}=u''_{xy}=-1;\ a_{22}=u''_{y^2}=2$. Bu xususiy hosilalarning $M_0(0;0)$ nuqtadagi qiymatlarni hisoblaymiz:

$$u''_{x^2}(M_0)=2,\ u''_{xy}(M_0)=-1,\ u''_{y^2}(M_0)=2$$

Shunday qilib, $a_{11}a_{22}-a_{12}^2=2\cdot 2-(-1)^2=4-1=3>0,\ \ a_{11}=2>0$.

Demak, Silvestr alomatining 1^0 – shartiga asosan, funksiya $M_0(0;0)$ nuqtada minimumga erishadi, ya'ni $U_{min}(0;0)=U(0;0)=0$.

Misolni Maple tizimidan foydalanib yechish:

> readlib(extrema):

> extrema(x^2-x*y+y^2,{},{x,y},'z');z;

$$\{0\}$$

$$\{\{y=0, x=0\}\}$$

b) Xuddi *a*) banddagi singari, $u'_x=2x-y,\ u'_y=-x-2y$ xususiy hosilalarni topib, ularni nulga tenglashtiramiz: $u'_x=2x-y=0,\ u'_y=-x+2y=0$. Bu sistemani Yechib, $M_0(0;0)$ stasionar nuqtani, ya'ni ekstremumga shubhali nuqtani topamiz.

Ikkinchi tartibli xususiy hosilalarni topib, ularning $M_0(0;0)$ stasionar nuqtadagi qiymatlarini hisoblaymiz: $a_{11}=u''_{x^2}=2;\ a_{12}=u''_{xy}=-1;$

$a_{22}=u''_{y^2}=-2$. Unda $a_{11}a_{22}-a_{12}^2=2\cdot(-2)-(-1)^2=-4-1=-5<0$.

Demak, Silvestr alomatining 3^0 - shartiga asosan, funksiya $M_0(0;0)$ nuqtada ekstremumga erishmaydi.

23.3-misol. 'Ushbu $u=x^2+2y^2+z^2-2x+4y-6z+1$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiyaning xususiy hosilalarini topib, ularni nulga tenglashtiramiz: $u'_x=2x-2=0,\ u'_y=4y+4=0,\ u'_z=2z-6=0$. Bu sistemani Yechib, $M_0(1;-1;3)$ stasionar nuqtani topamiz.

Endi ikkinchi tartibli xususiy hosilalarni topib, ularning $M_0(1;-1;3)$ stasionar nuqtadagi qiymatlarini hisoblaymiz:

$$u''_{x^2}(M_0)=2,\ u''_{xy}(M_0)=0,\ u''_{yx}(M_0)=0,\ u''_{xz}(M_0)=0,\ u''_{zx}(M_0)=0,$$

$$u''_{y^2}(M_0)=4,\ u''_{yz}(M_0)=0 u''_{zy}(M_0)=0,\ u''_{z^2}(M_0)=0.$$

Unda

$$\delta_1=a_{11}=2>0,\ \delta_2=\begin{vmatrix}a_{11}a_{12}\\ a_{21}a_{22}\end{vmatrix}=\begin{vmatrix}2&0\\0&4\end{vmatrix}=8>0,\ \ \delta_3=\begin{vmatrix}a_{11}&a_{12}&a_{12}\\a_{21}&a_{22}&a_{23}\\a_{21}&a_{32}&a_{33}\end{vmatrix}=\begin{vmatrix}2&0&0\\0&4&0\\0&0&2\end{vmatrix}=16>0$$

bo'ladi.

Demak, $\delta_1 > 0,\ \delta_2 > 0,\ \delta_3 > 0$. Shunday qilib, Silьvestr alomatiga asosan, $M_0(1;-1,3)$ nuqtada funksiya minimumga erishadi: $U_{\min} = u(1;-1;3) = -11$.

23.4. Ko‘p o‘zgaruvchili funksiyaning shartli ekstremumi.

Bizga biror ochiq G $(G \subset R^n)$ to‘plamda aniqlangan, n o‘zgaruvchili

$$z = f(x_1 x_2,\dots,x_n) = f(M) \qquad (23.1)$$

funksiya berilgan bo‘lib, uning $x_1, x_2,\dots,x_n$ argumentlari o‘zaro ushbu

$$\left.\begin{array}{l} \varphi_1(x_1, x_2,\dots x_n) = 0 \\ \varphi_2(x_1, x_2,\dots x_n) = 0 \\ \dots\dots\dots\dots\dots\dots \\ \varphi_{m1}(x_1, x_2,\dots x_n) = 0 \end{array}\right| \qquad (23.2)$$

qo‘shimcha munosabatlar bilan bog‘langan bo‘lsin $(m < n)$.

(23.2) tenglamalar sistemasi – *bog‘lanishlar tenglamalari yoki bog‘lanishlar* deyiladi. Koordinatalari (23.2) sistemani qanoatlantiradigan nuqtalar to‘plamini E $(E \subset G)$ orqali belgilaymiz.

23.3-ta’rif. Agar M_0 $(M_0 \in E)$ nuqtaning shunday $U(M_0)$ atrofi mavjud bo‘lib, $E \cap U(M_0)$ to‘plamdan olingan $\forall M$ nuqtalar uchun,

$$f(M) \le f(M_0) \quad (f(M) \ge f(M_0))$$

tengsizlik bajarilsa, $M_0 \in E$ nuqta, $f(M)$ funksiyaning, (23.2) bog‘lanishlarga nisbatan, *shartli maksimum (minimum) nuqtasi* deyiladi.

23.4-ta’rif. Agar M_0 $(M_0 \in E)$ nuqtaning shunday $U(M_0)$ atrofi mavjud bo‘lib, $E \cap U(M_0)$ to‘plamdan olingan $\forall M$ nuqtalar uchun

$$f(M) < f(M_0) \quad (f(M) > f(M_0))$$

tengsizlik bajarilsa, $M_0 \in E$ nuqta $f(M)$ funksiyaning, (23.2) bog‘lanishlarga nisbatan, *shartli qat’iy maksimum (minimum) nuqtasi* deyiladi.

Bu yerda ham, shartli maksimum va shartli minimum, umumiy nom bilan, *shartli ekstremum* deb yuritiladi.

Boshqacha aytganda, *shartli maksimum (minimum)* - funksiyaning, M_0 $(M_0 \in E)$ nuqtaning $U(M_0)$ atrofdagi barcha nuqtalarga nisbatan emas, balki, $U(M_0)$ atrofdagi (23.2) sistemani qanoatlantiradigan nuqtalarga nisbatan M_0 nuqtadagi eng katta (eng kichik) qiymatidan iboratdir.

Masalan, $u = f(x,y) = xy$ funksiya $\varphi(x,y) = y - x = 0$ bog‘lanishga nisbatan $O(0,0)$ nuqtada shartli qat’iy minimumga ega, chunki $f(0,0) = 0$, ammo $y-x=0$ tenglamani qanoatlantiradigan $(\varepsilon,\varepsilon)$, $\varepsilon\neq 0$, nuqtalarda funksiyaning qiymatlari musbat: $f(\varepsilon,\varepsilon)=\varepsilon^2>0$.

Shartli ekstremumni topish haqidagi masalani Yechishning ikkita usulini qaraymiz.

23.4.1. O'zgaruvchilarning bir qismini yo'qotish usuli. Faraz qilaylik, M_0 ($M_0\in E$) nuqtaning biror $\omega=U(M_0)$ atrofida: 1) $f(M)$ funksiya va (2) sistemadagi $\varphi_i(x), i=\overline{1,m}$ funksiyalar differensiallanuvchi; 2) $\frac{\partial\varphi_i}{\partial x_j}$, $(i=\overline{1,m};\ j=\overline{1,n})$ xususiy hosilalari M_0 nuqtada uzluksiz; 3) ushbu

$$\begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_2} & \cdots & \frac{\partial\varphi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\varphi_m}{\partial x_1} & \frac{\partial\varphi_m}{\partial x_2} & & \frac{\partial\varphi_m}{\partial x_n} \end{pmatrix} \qquad (23.3)$$

matrisaning birorta m- tartibli minori M_0 nuqtada noldan farqli bo'lsin. Masalan, $\frac{D(\varphi_1,\varphi_2,...,\varphi_m)}{D(x_1,x_2,...,x_n)}\neq 0$, ya'ni (23.3) matrisaning M_0 nuqtada rangi m ga teng bo'lsin. U holda, M_0 nuqtaning kichik atrofi, $Q\subset\omega=U(M_0)$ parallelepipedda, (23.2) sistema, qandaydir m ta o'zgaruvchilarga, masalan, $x_1,x_2,...,x_m$ o'zgaruvchilarga nisbatan yagona yechimga ega bo'ladi, ya'ni

$$x_i=\psi_i(x_{m+1},x_{m+1},x_{m+2},...,x_n),\ i=\overline{1,m}, \qquad (23.4)$$

bunda, $\psi_1,\psi_2,...,\psi_m$ lar (23.2) sistemadan aniqlanadigan oshkormas funksiyalar. U holda, Q parallelepipedda (23.2) sistema, $x_{m+1},x_{m+2},...,x_n$ o'zgaruvchilar *erkli* deb qaraladigan, (23.4) bog'lanishlarga teng kuchli bo'ladi.

Agar $x_i=\psi_i(x_{m+1},x_{m+1},x_{m+2},...,x_n)$, $i=\overline{1,m}$, funksiyalarni oshkor ravishda topish mumkin bo'lsa, ularni (1) ga keltirib qo'yib, $n-m$ o'zgaruvchili,

$$z=f(\psi_1(x_{m+1},...,x_n),...,\Psi_m(x_{m+1},...,x_n),x_{m+1},...x_n)\equiv g(x_{m+1},...x_n)=g(M')$$

funksiyani hosil qilamiz. Q parallelepiped bilan chegaralangan sohada $g(M')$ funksiyaning ixtiyoriy $M'(x_{m+1},...,x_n)$ nuqtadagi qiymati, $f(M)$ funksiyaning (23.4) tenglamalarni qanoatlantiradigan, yoki (23.2) tenglamalarni qanoatlantiradigan, unga mos $M(x_1,x_2...x_n)$ nuqtadagi qiymati bilan ustma-ust tushadi. Shu sababli, (23.1) funksiyaning Q parallelepipeddagi (23.2) bog'lanishlar bajarilgandagi shartli ekstremum masalasi, $g(M')$ funksiyaning shartsiz ekstremumi masalasiga keltiriladi.

23.4-misol. O‘zgaruvchilarning bir qismini yo‘qotish usuli bo‘yicha,

$$f(M) = x_1^2 + x_2^2 + x_3^2, \quad (M(x_1, x_2, x_3))$$

funksiyaning ekstremumini

$$x_1 + x_2 + x_3 = 1 \qquad (*)$$

bog‘lanishlar bajarilganda toping.

Yechilishi. Masaladagi $\varphi(M) = x_1 + x_2 + x_3 - 1 = 0$ tenglamadan $x_3 = 1 - x_2 - x_1$ ni topib, $f(x)$ funksiyaga keltirib qo‘yamiz va ikki o‘zgaruvchili

$$g(M') = g(x_1, x_2) = 2x_1^2 + 2x_2^2 + 2x_1x_2 - 2x_1 - 2x_2 + 1 \quad (M' \in R^2)$$

funksiyani hosil qilamiz. Shunday qilib, berilgan $f(M) = x_1^2 + x_2^2 + x_3^2$ funksiyaning (*) tenglamaga nisbatan $M_0(x_1^0, x_2^0, x_3^0) \in E$ nuqtadagi shartli ekstremumini topish masalasini, $g(M')$ funksiyaning $P(x_1^0, x_2^0)$ nuqtadagi shartsiz ekstremumini topish masalasiga keltirdik. Endi oxirgi masalaning stasionar nuqtalarini topamiz:

$$\begin{cases} \dfrac{\partial g}{\partial x_1} = 4x_1 2x_2 - 2 = 0 \\ \dfrac{\partial g}{\partial x_2} = 4x_2 + 2x_1 - 2 = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 - 1 = 0 \\ 2x_2 + x_1 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{1}{3}, \\ x_2 = \dfrac{1}{3}. \end{cases}$$

Demak, $M_0'(x_1^0, x_2^0) = M_0'\left(\frac{1}{3}, \frac{1}{3}\right)$ nuqta, $g(M')$ $(M' \in R^2)$ funksiyaning stasionar nuqtasidir. Bu stasionar nuqta, $g(M')$ $(M' \in R^2)$ funksiyaning (absolyut) minimum nuqtasi bo‘ladi, chunki $g(x_1, x_2)$ qat’iy qavariq funksiyadir. Haqiqatan ham, bu funksiyaning ikkinchi tartibli xususiy hosilalari matrisasi $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ bo‘lib, uning bosh minorlari $\delta_1 = 4 > 0,\ \delta_2 = 12 > 0 \Rightarrow$ matrisa musbat aniqlangan.

Endi $x_1^0 = \frac{1}{3}, x_2^0 = \frac{1}{3}$ qiymatlarni $x_3 = 1 - x_1 - x_2$ ifodaga keltirib qo‘yib, $x_3^0 = \frac{1}{3}$ ekanligini topamiz.

Demak, $M_0(x_1^0, x_2^0, x_3^0) = M_0\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ nuqta - berilgan shartli ekstimum masalasida minimum nuqtasi bo‘ladi. Maksimum nuqtasi mavjud emas, chunki,

$$\sup\left(x_1^2 + x_2^2 + x_3\right) = +\infty, \quad x_1 + x_2 + x_3 = 1, \quad x \in R^3.$$

Agar (23.4) funksiyalarni oshkor ko‘rinishda topish qiyinchilik tug‘dirsa yoki uning iloji bo‘lmasa, quyidagicha ish ko‘rish mumkin. Faraz qilaylik, (23.4) funksiyalar (23.1) ga keltirib qo‘yilgan va uning natijasida $g(M')$ funksiya hosil qilingan, hamda ular (23.2) tenglamalarga keltirib qo‘yilgan, natijada tenglamalar ayniyatlarga aylangan bo‘lsin. U holda $g(M')$ funksiyaning differensialini, birinchi differensial shaklining invariantligiga asosan,

$$dg = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \tag{23.5}$$

ko‘rinishda yozish mumkin, bunda $dx_{m+1}, dx_{m+2},...,dx_n$ – erkli o‘zgaruvchilarning differensiallari, $dx_1, dx_2,...,dx_m$ lar esa, - (23.4) oshkormas funksiyalarning differensiallaridir. Yuqorida hosil qilingan ayniyatlarni differensiallab,

$$\begin{cases} \frac{\partial \varphi_1}{\partial x_1} dx_1 + \frac{\partial \varphi_1}{\partial x_2} dx_2 + ... + \frac{\partial \varphi_1}{\partial x_n} dx_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial \varphi_m}{\partial x_1} dx_1 + \frac{\partial \varphi_m}{\partial x_2} dx_2 + ... + \frac{\partial \varphi_m}{\partial x_n} dx_n = 0 \end{cases} \tag{23.6}$$

chiziqli tenglamalar sistemasini hosil qilamiz. Bu sistemadan (23.4) oshkormas funksiyalarning $dx_1, dx_2,...,dx_m$ differensiallarini, erkli o‘zgaruvchilarning $dx_{m+1},...dx_n$ differensiallari orqali ifodalaymiz. $dx_1, dx_2,...,dx_m$ lar uchun olingan ifodalarni (23.5) munosabatga keltirib qo‘yib,

$$dg = \sum_{i=m+1}^{n} A_i(x_1 x_2,...x_n) dx_i \tag{23.6^1}$$

tenglikni hosil qilamiz, bunda

$$x_j = \psi_j(x_{m+1}, x_{m+2},...,x_n), \quad j = 1,2...m.$$

Agar $g(M')$ funksiya $M_0' = (x_{m+1}^0, x_{m+2}^0,...,x_n^0)$ nuqtada ekstremumga erishsa, $dg|_{M_0'} = 0$, ya’ni

$$\sum_{i=m+1}^{n} A_i(x_1^0, x_2^0,...,x_m^0, x_{m+1}^0,...,x_n^0) dx_i = 0 \tag{23.7}$$

bo‘ladi, bunda

$$x_j^0 = \psi_j(x_{m+1}^0, x_{m+2}^0,...,x_n^0) = \psi_j(M_0'), \quad j = \overline{1,m}.$$

$dx_{m+1}, dx_{m+2},...,dx_n$ lar erkli o‘zgaruvchilarning differensiallari bo‘lganligidan, (23.7) munosabatdan,

$$A_i(x_1^0, x_2^0,...,x_m^0, x_{m+1}^0,...,x_n) = 0, \quad i = \overline{m+1,n} \tag{23.8}$$

ekanligini olamiz.

Shunday qilib, (23.8) tengliklar sistemasi, $g(M')$ funksiyaning M_0' nuqtada ekstremumga ega bo‘lishining zaruriy shartlarini yoki $f(M)$ funksiyaning, (23.2) bog‘lanishlar qatnashganda, $M_0(x_1^0, x_2^0,...,x_n^0)$ nuqtada shartli ekstremumga ega bo'lishning zaruriy shartlarini ifoda qiladi.

Bu erdan, qaralayotgan masalada ekstremumga shubhali bo'lgan $M_0(x_1^0, x_2^0, ..., x_n^0)$ nuqtalarni topish uchun, n ta $x_1, x_2, ..., x_n$ noma'lumlarga nisbatan, n ta,

$$\begin{cases} \varphi_i(x_1, x_2, ..., x_n) = 0, \ i = \overline{1,m} \\ A_i(x_1, x_2, ..., x_n) = 0, \ i = \overline{1, m+1, n} \end{cases} \tag{23.9}$$

tenglamalar sistemasini Yechish lozim bo'ladi.

Agar ekstremumga shubhali bo'lgan $M_0(x_1^0, x_2^0, ..., x_n^0)$ nuqta topilgan bo'lsa, uni ekstremumga tekshirish, $d^2 f\big|_{M_0}$ ikkinchi tartibli to'liq differensialni hisoblashga bog'liq bo'ladi. Buni hisoblashda, f va φ_i ($i = \overline{1,m}$) funksiyalar ikki marta differensiallanuvchi deb faraz qilib, dg uchun olingan (23.) tenglikni differensiallash va (23.4) oshkormas funksiyalarning $dx_1, dx_2, ..., dx_m$ differensiallari uchun (23.6) sistemadan topilgan ifodalardan foydalanish zarur. Natijada, biz, $dx_{m+1}, dx_{m+2}, ..., dx_n$ o'zgaruvchilarga nisbatan $d^2 f\big|_{M_0}$ - kvadratik formani hosil qilamiz.

Agar bu kvadratik forma musbat (manfiy) aniqlangan bo'lsa, (23.1) funksiya M_0 nuqtada shartli minimum (maksimum) ga erishadi.

23.5-misol. O'zgaruvchilarning bir qismini yo'qotish usuli yordamida,

$$u = f(x_1, x_2, x_3) = x_1 + x_2 - x_3 \tag{23.10}$$

funksiyaning ekstremumini,

$$\begin{cases} 4x_1^3 + 4x_2^3 + 4x_3^3 + 12x_1 + 12x_2 + 12x_3 = 13 \\ x_1 + x_2 = 1 \end{cases} \tag{23.11}$$

bog'lanishlar tenglamalari bajarilganda toping.

Yechilishi. Bu masalani yechishda yuqorida keltirilgan ikkinchi usuldan, ya'ni (23.11) tenglamalarda o'zgaruvchilardan qandaydir ikkitasini uchinchisi orqali oshkor ravishda ifodalash mumkin bo'lmagan usuldan foydalanamiz.

(23.11) sistema ikki marta differensiallanuvchi $x_2(x_1)$ va $x_3(x_1)$ oshkormas funksiyalarni aniqlaydi, deb faraz qilamiz va ular bog'lanishlar tenglamalariga keltirib qo'yilgan deb hisoblab, (23.11) tenglamalarni ayniyatlar sistemasi kabi qaraymiz. (23.11) ayniyatlarni har ikkala tomonidagi differensiallarni hisoblab, natijada

$$\begin{cases} (x_1^2 + 1)dx_1 + (x_2^2 + 1)dx_2 + (x_3^2 + 1)dx_3 = 0 \\ dx_1 + dx_2 = 0 \end{cases}$$

munosablarni hosil qilamiz. Bu erdan,

$$\begin{cases} dx_2 = -dx_1, \\ dx_3 = \dfrac{-2x_1+1}{x_3^2+1}dx_1 \end{cases} \qquad (23.12)$$

ekanligini topamiz. Bu ifodalarni, (23.10) funksiyaning $du = dx_1 + dx_2 - dx_3$ ko'rinishdagi differensiali ifodasiga keltirib qo'yib,

$$du = \frac{2x_1 - 1}{x_3^2+1}dx_1 \equiv Adx_1 \qquad (23.13)$$

munosabatni hosil qilamiz.

Endi, (23.11) tenglamalar va $A = 0$ tenglamalardan tashkil topgan,

$$\begin{cases} 4x_1^3 + 4x_2^3 + 4x_3^3 + 12x_1 + 12x_2 + 12x_3 - 13 = 0, \\ x_1 + x_2 - 1 = 0, \\ \dfrac{2x_1 - 1}{x_3^2+1} = 0 \end{cases}$$

tenglamalar sistemasini (bu sistema, qaralayotgan hol uchun (23.9) sistemadan iborat) qaraymiz. Bu sistema, yagona $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 0$ yechimga ega, ya'ni $M_0 = \left(1/2, \frac{1}{2}, 0\right)$ - (23.10) funksiyaning (23.11) bog'lanishlarga nisbatan ekstremumga shubhali bo'lgan yagona nuqtasidir.

Endi du ning (23.13) dagi ifodasini differensiallaymiz:

$$d^2u = \frac{2(x_3^2+1)dx_1 - (2x_1 - 1)2x_3dx_3}{(x_3+1)^2}dx_1 .$$

Oxirgi ifodadan, (23.12) tenglamalarni hisobga olib,

$$d^2u\big|_{M_0} = 2(dx_1)^2$$

munosabatga kelamiz. Modomiki, $2(dx_1)^2 - dx_1$ bir o'zgaruvchining musbat aniqlangan kvadratik formasi ekan, (23.10) funksiya, (23.11) bog'lanishlarda M_0 nuqtada minimumga erishadi.

23.3-eslatma. Biz masalani, (23.11) sistema ikki marta differensiallanuvchi $x_2(x_1)$ va $x_3(x_1)$ funksiyalarni aniqlaydi, deb faraz qilib, echdik. Endi, bu shartning $M_0 = (1/2, 1/2, 0)$ nuqtaning biror atrofida bajarilishini ko'rsatamiz. Buning uchun, 1- banddagi 1)- 3) shartlarni tekshiramiz.

$\varphi_1(x_1, x_2, x_3) = 4x_1^3 + 4x_2^3 + 4x_3^3 + 12x_1 + 12x_2 + 12x_3 - 13$ va $\varphi_2(x_1, x_2, x_3) = x_1 + x_2 - 1$ funksiyalar M_0 nuqtaning ixtiyoriy atrofida differensiallanuvchi; $\frac{\partial\varphi_1}{\partial x_1} = 12(x_2^2+1)$, $\frac{\partial\varphi_1}{\partial x_3} = 12(x_3^2+1)$, $\frac{\partial\varphi_2}{\partial x_2} = 1$, $\frac{\partial\varphi_2}{\partial x_3} = 0$ xususiy hosilalar M_0 nuqtada uzluksiz; $\varphi_1(x_1,x_2,x_3)$ va $\varphi_2(x_1,x_2,x_3)$funksiyalar M_0 nuqtada nolga aylanadi va

$$\left.\frac{D(\varphi_1,\varphi_2)}{D(x_2,x_3)}\right|_{M_0} = -12 \neq 0.$$

Demak, 1)- 3) shartlarga asosan, (23.11) sistema M_0 nuqtaning biror atrofida yagona $x_2(x_1)$ va $x_3(x_1)$ funksiyalarni aniqlaydi. Bundan tashqari, $\varphi_1(x_1,x_2,x_3)$ va $\varphi_2(x_1,x_2,x_3)$ funksiyalar M_0 nuqtaning ixtiyoriy atrofida ikki marta differensiallanuvchi, u holda $x_2(x_1)$ va $x_3(x_1)$ funksiyalar ham ikki marta differensiallanuvchi bo'ladi.

23.4.2. Ko'p o'zgaruvchili funksiyaning shartli ekstremumini topishning Lagranj ko'paytuvchilari qoidasi. Shartli ekstremum masalasini yechishda yuqorida ko'rib o'tilgan usullarni ko'p hollarda qo'llab bo'lmaydi, ya'ni bog'lanishlardan o'zgaruvchilarning bir qismini qolganlari orqali oshkor yoki oshkormas shaklda ifodalash mumkin bo'lmaydi.

Bunday masalalarni yechishda quyidagicha ish ko'riladi.

Bizga, (23.1) funksiyaning ekstremumini, (23.2) bog'lanishlar bajarilganda, topish masalasi qo'yilgan bo'lsin, hamda $f(M)$, $\varphi_i(M), (i=\overline{1,m})$ funksiyalar $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqtada va uning biror atrofida uzluksiz differinsiallanuvchi, va nihoyat,

$$\begin{pmatrix} \frac{\partial\varphi_1(M_0)}{\partial x_1} & \frac{\partial\varphi_1(M_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_1(M_0)}{\partial x_n} \\ \frac{\partial\varphi_2(M_0)}{\partial x_2} & \frac{\partial\varphi_2(M_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_2(M_0)}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\varphi_m(M_0)}{\partial x_1} & \frac{\partial\varphi_m(M_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_m(M_0)}{\partial x_n} \end{pmatrix}$$

matrisaning rangi m ga teng bo'lsin. Quyidagi $n+m$ o'zgaruvchili

$$L(M,\lambda) = f(M) + \lambda_1\varphi_1(M) + \lambda_2\varphi_2(M) + ... + \lambda_m\varphi_m(M) \qquad (23.14)$$

funksiyani kiritamiz, bunda $M \in G\ (G \subset R^n), \lambda \in R^m,\ \lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$.

$\lambda_1, \lambda_2, ..., \lambda_m$- *Lagranj ko'paytuvchilari,* λ- *Lagranj vektori.* $L(M,\lambda)$ funksiya esa, *Lagranj funksiyasi* deb ataladi.

23.4-teorema (Lagranj ko'paytuvchilari qoidasi). Faraz qilaylik, $M_0(x_1^0, x_2^0, ..., x_n^0)$ - $f(M)$ funksiyaning (23.2) bog'lanishlar bajarilgandagi shartli ekstremum nuqtasi bo'lsin. U holda, shunday yagona $\lambda^0 = (\lambda_1^0, \lambda_2^0, ..., \lambda_m^0)$ Lagranj vektori topiladiki, (M_0, λ^0) juftlik uchun,

$$\begin{cases} \dfrac{\partial L(M_0,\lambda^0)}{\partial M} = 0 \quad \Leftrightarrow \quad \dfrac{\partial L(M_0,\lambda^0)}{\partial x_j} = 0,\ j = \overline{1,m}, & (23.15) \\ \dfrac{\partial L(M_0,\lambda^0)}{\partial \lambda_1} = \varphi_1(M_0) = 0,\dots,\dfrac{\partial L(M_0,\lambda^0)}{\partial \lambda_m} = \varphi_m(M_0) = 0 & (23.16) \end{cases}$$

munosabatlar bajariladi.

(23.15), (23.16) munosabatlarni qanoatlantiruvchi (M_0,λ^0) juftlik-Lagranj funksiyasining stasionar nuqtasi, shartli ekstremum masalasining *shartli stasionar nuqtasi* deyiladi.

23.4-teoremadan kelib chiqadiki, (23.1) funksiyaga, (23.2) bog'lanishlar qatnashganda, shartli ekstremum berishi mumkin bo'lgan nuqtalarni topish uchun, $n+m$ ta, $x_1,x_2,\dots,x_n,\ \lambda_1,\ \lambda_2,\dots,\lambda_m$, noma'lumlarga nisbatan,

$$\begin{cases} \varphi_i(x_1,x_2,\dots,x_n) = 0,\ i = \overline{1,m} \\ \dfrac{\partial L(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m)}{\partial x_j} = 0,\ j = \overline{1,n} \end{cases} \tag{23.17}$$

$n+m$ ta tenglamalar sistemasini yechish lozim bo'ladi. (23.17) tenglamalar sistemasi- *stasionarlik* tenglamalari sistemasi deyiladi.

Shunday qilib, agar (M_0,λ^0) juftlik (23.17) tenglamalar sistemasining yechimi bo'lsa (bunday yechimlar bir nechta bo'lishi ham mumkin), $M_0(x_1^0,x_2^0,\dots,x_n^0)$ nuqta - (23.1) funksiyaga, (23.2) bog'lanishlar qatnashganda, shartli ekstremum berishi mumkin bo'lgan nuqta bo'ladi.

Bu nuqtaning shartli ekstremum nuqtasi bo'lishini aniqlashda, (23.1) funksiya va (23.2) bog'lanishlarda qatnashayotgan funksiyalar ikkinchi tartibli uzluksiz xususiy hosilalarga ega, deb faraz qilinib, Lagranj funksiyasining, belgilangan $\lambda_1^0,\ \lambda_2^0,\dots,\lambda_m^0$ lar uchun, $M_0(x_1^0,x_2^0,\dots,x_n^0)$ nuqtada $x_1,x_2,\dots,x_n$ o'zgaruvchilar bo'yicha hisoblangan,

$$d^2L(M_0,\lambda^0) = \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 L(M_0,\lambda^0)}{\partial x_k \partial x_j}dx_k dx_j \tag{23.18}$$

ikkinchi tartibli differensiali qaraladi.

23.5-teorema (shartli ekstremumning ikkinchi tartibli zaruriy sharti). Faraz qilaylik, $M_0(x_1^0,x_2^0,\dots,x_n^0)$ - $f(M)$ funksiyaning (23.2) bog'lanishlar bajarilgandagi shartli minimum (maksimum) nuqtasi, $f(M)$, $\varphi_i(M),(i=\overline{1,m})$ funksiyalar $M_0(x_1^0,x_2^0,\dots,x_n^0)$ nuqtaning biror atrofida ikkinchi tartibli uzluksiz xususiy hosilalarga ega, hamda $M_0(x_1^0,x_2^0,\dots,x_n^0)$ nuqtada (23.3) matrisaning rangi m ga teng bo'lsin. U holda, shunday

$\lambda^0=(\lambda_1^0,\lambda_2^0,...,\lambda_m^0)$ Lagranj vektori topiladiki, (M_0,λ^0) juftlik - Lagranj funksiyasining stasionar nuqtasi bo'ladi va

$$\begin{cases} \frac{\partial\varphi_1(M_0)}{\partial x_1}dx_1+\frac{\partial\varphi_1(M_0)}{\partial x_2}dx_2+....+\frac{\partial\varphi_1(M_0)}{\partial x_n}dx_n=0, \\ \\ \frac{\partial\varphi_m(M_0)}{\partial x_1}dx_1+\frac{\partial\varphi_m(M_0)}{\partial x_2}dx_2+....+\frac{\partial\varphi_m(M_0)}{\partial x_n}dx_n=0 \end{cases} \quad (23.19)$$

tenglamalar sistemasini qanoatlantiruvchi $dx_1,dx_2,....,dx_n$ lar uchun

$$d^2L(M_0,\lambda^0)\geq 0 \qquad (d^2L(M_0,\lambda^0)\leq 0)$$

munosabat bajariladi.

23.6-teorema (shartli ekstremumning yetarli sharti). $f(M)$, $\varphi_i(M),(i=\overline{1,m})$, funksiyalar $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqtaning biror atrofida ikkinchi tartibli uzluksiz xususiy hosilalarga ega, $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqtada (23.3) matrisaning rangi m ga teng, hamda (M_0,λ^0) juftlik- Lagranj funksiyasining stasionar nuqtasi bo'lsin. U holda:

1) agar $d^2L(M_0,\lambda^0)$, $dx_1,dx_2,....,dx_n$ lar (23.19) tenglamalar sistemasini qanoatlantirganda, musbat aniqlangan kvadratik forma bo'lsa, $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqta - $f(M)$ funksiyaning, (23.2) bog'lanishlar bajarilganda, shartli minimum nuqtasi bo'ladi.

2) agar $d^2L(M_0,\lambda^0)$, $dx_1,dx_2,....,dx_n$ lar (23.19) tenglamalar sistemasini qanoatlantirganda, manfiy aniqlangan kvadratik forma bo'lsa, $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqta - $f(M)$ funksiyaning, (23.2) bog'lanishlar bajarilganda, shartli maksimum nuqtasi bo'ladi.

3) agar $d^2L(M_0,\lambda^0)$, $dx_1,dx_2,....,dx_n$ lar (23.19) tenglamalar sistemasini qanoatlantirganda, aniqlanmagan kvadratik forma (ham musbat, ham manfiy qiymatlar qabul qiluvchi) bo'lsa, $M_0(x_1^0,x_2^0,...,x_n^0)$ nuqta - $f(M)$ funksiyaning shartli ekstremum nuqtasi bo'lmaydi.

Xususiy holda, $u=f(x,y)$ funksiya $\varphi(x,y)=0$ bog'lanishlar tenglamasi bilan berilgan bo'lsa, Lagranj funksiyasi quyidagi ko'rinishda bo'ladi:

$$L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y).$$

(23.17) tenglamalar sistemasi, uchta tenglamadan iborat bo'ladi:

$$\begin{cases} \frac{\partial L}{\partial x}=0, \\ \frac{\partial L}{\partial y}=0, \\ \varphi(x,y)=0, \end{cases}$$

(M_0,λ^0) juftlik- Lagranj funksiyasining stasionar nuqtasi bo‘lsin va

$$\Delta=-\begin{vmatrix} 0 & \varphi_x'(M_0) & \varphi_y'(M_0) \\ \varphi_x'(M_0) & L_{xx}''(M_0,\lambda_0) & L_{xy}''(M_0,\lambda_0) \\ \varphi_y'(M_0) & L_{xy}''(M_0,\lambda_0) & L_{yy}''(M_0,\lambda_0) \end{vmatrix}.$$

Agar $\Delta<0$ bo‘lsa, $M_0(x^0,y^0)$ nuqta - $u=f(x,y)$ funksiyaning, $\varphi(x,y)=0$ bog‘lanishlar bajarilganda, shartli maksimum nuqtasi bo‘ladi.

Agar $\Delta>0$ bo‘lsa, $M_0(x^0,y^0)$ nuqta - $u=f(x,y)$ funksiyaning, $\varphi(x,y)=0$ bog‘lanishlar bajarilganda, shartli minimum, nuqtasi bo‘ladi.

Izohlar.

1. Funksiyalarning ekstremumini bog‘lanishlar qatnashganda topish masalalari juda keng tarqalgan, ular ekstremal masalalarning bir qismidan iborat bo‘lib, ularni o‘rganuvchi bo‘lim – *ekstremal masalalar nazariyasi*, hozirgi vaqtda, ko‘p amaliy masalalarda qo‘llanilmoqda.

2. Biz yuqorida ko‘rgan masalalarda, ham ekstremumini topish talab qilingan funksiyani, ham bog‘lanishlar funksiyalarini etarli silliq, deb faraz qilgan edik. Lekin bog‘lanishlar faqat tengliklar ko‘rinishida emas, balki tengsizliklar ko‘rinishida ham bo‘lishi, ulardagi funksiyalar odatdagi ma’noda differensiallanuvchi bo‘lmasligi ham mumkin.

3. Bog‘lanishlar funksiyalari yordamida tuzilgan (23.3) matrisaning rangi m ga teng bo‘lsin, degan talab ham ancha kuchli, ko‘p masalalarda bunday talabni qo‘yish mumkin emas. Bu holda, Lagranj ko‘paytuvchilari usulining umumlashmasidan, aniqrog‘i, umumlashgan Lagranj funksiyasidan foydalaniladi.

23.6-misol. Lagranj usulidan foydalanib, ushbu $u=2x+4y-5$ funksiyaning $x^2+y^2=125$ bog‘lanishlarni qanoatlantiruvchi shartli ekstremumlarini toping.

Yechilishi. Dastlab Lagranj funksiyasini quramiz:

$$L(x,y,\lambda)=2x+4y-5+\lambda(x^2+y^2-125).$$

Endi Lagranj funksiyasidan xususiy hosilalarni olib, (23.17) tenglamalar sistemasini tuzamiz:

$$\begin{cases} \dfrac{\partial L}{\partial x}=2+2\lambda x, \\ \dfrac{\partial L}{\partial y}=4+2\lambda y, \\ x^2+y^2=125, \end{cases} \Rightarrow \begin{cases} 2+2\lambda x=0, \\ 4+2\lambda y=0, \\ x^2+y^2=125. \end{cases}$$

Bu sistema ikkita Yechimga ega:

$$x_1=-5,\ y_1=-10,\ \lambda_1=0{,}2;\quad x_1=5,\ y_1=10,\ \lambda_1=-0{,}2.$$

$L(x,y,\lambda)$ Lagranj funksiyasining ikkinchi tartibli xususiy hosilalarini topib, ikkinchi tartibli differensialini tuzamiz:

$$\frac{\partial^2 L}{\partial x^2}=2\lambda,\ \frac{\partial^2 L}{\partial x\partial y}=0\ \frac{\partial^2 L}{\partial y^2}=2\lambda,\ d^2L=2\lambda\left(dx^2+dy^2\right).$$

$\lambda_1=0{,}2$ bo'lganda $d^2L>0$. Shuning uchun, 23.6-teoremaga asosan, $M_1(-5,-10)$ nuqtada u funksiya shartli minimumga ega va $u_{min}=u(-5,-10)=-55.$

$\lambda_2=-0{,}2$ bo'lganda $d^2L<0$. Shuning uchun, 23.6-teoremaga asosan, $M_2(5;10)$ nuqtada u funksiya shartli maksimumga ega va uning maksimum qiymati 45 ga teng, $u_{max}=u(5,10)=45$

u funksiyani shartli ekstremumga tekshirishning boshqa yo'lini qaraymiz.

$\lambda_1=0{,}2$ bo'lganda $\varphi(x,y)=x^2+y^2-125,\ \varphi'_x=2x,\ \varphi'_y=2y,\ \varphi'_x(-5,-10)=-10,$

$$\varphi'_y(-5,-10)=-20,\ \frac{\partial^2 L}{\partial x^2}=0{,}4,\ \frac{\partial^2 L}{\partial x\partial y}=0\ \frac{\partial^2 L}{\partial y^2}=0{,}4.$$

Demak,

$$\Delta=-\begin{vmatrix}0 & -10 & -20\\ -10 & 0{,}4 & 0\\ -20 & 0 & 0{,}4\end{vmatrix}=200>0$$

Demak, $M_1(-5;-10)$ nuqtada u funksiya shartli minimumga ega va uning maksimum viymati -55 ga teng, ya'ni $u_{min}=u(-5,-10)=-55.$

Xuddi shunga o'xshash, $\lambda_2=-0{,}2$ bo'lgan holda va $M_2(5;10)$ nuqtada

$$\Delta=-\begin{vmatrix}0 & 10 & 20\\ 10 & -0{,}4 & 0\\ 20 & 0 & -0{,}4\end{vmatrix}=-200<0.$$

Demak, $M_2(5;10)$ nuqtada u funksiya shartli maksimumga ega va uning qiymati 45 ga teng, ya'ni $u_{max}=u(5,10)=45.$

Misolni Maple tizimidan foydalanib yechish:

```
> readlib(extrema):
> extrema(2*x+4*y-5,{x^2+y^2-125=0},{x,y},'u');u;
                        {-55, 45}
        {{y = 10, x = 5}, {y = -10, x = -5}}
```

23.7-misol. Lagranj usulidan foydalanib, $u=xy+yz$ funksiyaning $x^2+y^2=2,\ y+z=2\ (x>0,\ y>0,\ z>0)$ bog'lanishlarni qanoatlantiruvchi ekstremumlarini toping.

Yechilishi. Dastlab Lagranj funksiyasini tuzamiz:

$$L(x,y,z,\lambda,\mu)=xy+yz+\lambda\left(x^2+y^2-2\right)+\lambda(x+z-2)$$

Bu Lagranj funksiyasidan xususiy hosilalarni olib,

$$\begin{cases} \dfrac{\partial L}{\partial x} = y + 2\lambda x, \\ \dfrac{\partial L}{\partial y} = x + z + 2\lambda y + \mu, \\ \dfrac{\partial L}{\partial z} = y + \mu, \\ x^2 + y^2 = 2, \\ y + z = 2. \end{cases} = \begin{cases} y + 2\lambda x = 0, \\ x + z + 2\lambda y + \mu = 0, \\ y + \mu = 0, \\ x^2 + y^2 = 2, \\ y + z = 2. \end{cases}$$

tenglamalar sistemasini tuzamiz. Bu sistemadan λ va μ sonlarni va stasionar nuqtaning koordinatalarini topamiz: $x = y = z = 1,\ \lambda = -\frac{1}{2},\ \mu = -1$. $L(x, y, z, \lambda, \mu)$ Lagranj funksiyasining ikkiinchi tartibli differensialini topamiz va unga $\lambda = -\frac{1}{2}$ qiymatni keltirib qo'yamiz:

$$d^2L = 2\lambda\left(dx^2 + dy^2\right) + 2dxdy + 2dydz = -dx^2 - dy^2 + 2dxdy + 2dydz.$$

Bog'lanishlar tenglamasidan, $dy = -dx = -dz$ ekanligi kelib chiqadi. Buni e'tiborga olsak, u holda, $d^2L = -dx^2 - 3dy^2 - 2dz^2 < 0$ bo'ladi.

Demak, 23.6-teoremaga asosan, $M_0(1, 1, 1)$ nuqtada u funksiya shartli maksimumga ega va uning qiymati 2 ga teng, ya'ni $u_{max} = u(1, 1, 1) = 2$.

23.5. Ko'p o'zgaruvchili funksiyaning eng kichik va eng katta qiymatlari. Agar $u = f(M)$ funksiya chegaralangan yopiq $\{M\}$ $(\{M\} \subset R^m)$ to'plamda uzluksiz bo'lsa, u holda bu funksiya shu to'plamda o'zining aniq yuqori va aniq quyi chegaralariga erishadi (**Veyershtrass teoremasi**).

$u = f(M)$ funksiya chegaralangan sohada differensiallanuvchi va sohaning chegarasida uzluksiz bo'lsa, bu funksiya, yoki stasionar nuqtalarda, yoki sohaning chegara nuqtalarida o'zining eng katta va eng kichik qiymatlariga erishadi.

Ko'p o'zgaruvchili funksiyaning eng kichik va eng katta qiymatlarini topish qoidasi quyidagicha :

1) berilgan sohada joylashgan stasionar nuqtalarni topish va funksiyaning bu nuqtalardagi qiymatlarini hisoblash;

2) sohaning chegarasini tashkil qiluvchi chiziqda funksiyaning eng kichik va eng katta qiymatlarini topish;

3) funksiyaning topilgan barcha qiymatlarini solishtirish: bularning eng kichigi va eng kattasi, mos ravishda, funksiyaning qaralayotgan sohada eng kichik va eng katta qiymati bo‘ladi.

23.8-misol. Ushbu $u=x^2-4x-y^2$ funksiyaning $x^2+y^2\le 16$ doiradagi eng kichik va eng katta qiymatlarini toping.

Yechilishi. 1. $u=x^2-4x-y^2$ funksiyaning birinchi tartibli xususiy hosilalarini topamiz:

$$u'_x=2x-4, \qquad u'_y=-2y.$$

Bu xususiy hosilalarni nulga tenglashtirib, sistemani Yechib, bitta $P_0(2;0)$ stasionar nuqtani topamiz, unda funksiyaning qiymati -4 ga teng, ya’ni $z(2;0)=4-8-0=-4$

2. Endi funksiyaning chegaradagi, ya’ni $x^2+y^2=16$ aylanadagi eng kichik va eng katta qiymatlarini topamiz. $u=x^2-4x-y^2$ funksiyani bu aylana nuqtalarida bitta x o‘zgaruvchining funksiyasi sifatida ifodalash mumkin. $y^2=16-x^2$ funksiyani u funksiyaga keltirib qo‘yamiz: $u=x^2-4x+y^2=2x^2-4x-16$. $y=\pm\sqrt{16-x^2}$ funksiyaning aniqlanish sohasi $[-4;4]$ bo‘ladi.

Shunday qilib, ikki o‘zgaruvchili funksiyaning $x^2+y^2=16$ aylanadagi eng kichik va eng katta qiymatlarini topish, bir o‘zgaruvchili, $u=2x^2-4x-16$ funksiyaning $[-4;4]$ kesmadagi eng kichik va eng katta qiymatlarini topishga keltirildi. Bu funksiyaning $(-4;4)$ oraliqdagi kritik nuqtalarini topamiz va oraliqning chetlaridagi qiymatlarini topamiz: $u'=4x-4$, $4x=4$, bundan $x=1$ kritik nuqtani olamiz: $u/_{x=1}=-18$, so‘ngra $u/_{x=4}=0$, $u/_{x=-4}=32$ ekanligini topamiz.

3. Berilgan funksiyaning stasionar va chegara nuqtalardagi qiymatlarini hisoblab, ularni solishtiramiz:

$$u(-2;0)=-4,\quad u(1;-2\sqrt{2})=-11,\quad u(4;0)=0,\quad u(1;2\sqrt{2})=11,\quad u(-4;0)=32$$
$$u(0;4)=-16,\quad u(0;-4)=-16.$$

Demak, berilgan funksiyaning eng kichik qiymati -16, eng katta qiymati esa, 32 bo‘ladi,

$$u_{e.kich}=u(0;4)=u(0,-4)=-16,\qquad u_{e.kat}=u(-4;0)=32.$$

23.9-misol. To‘g‘ri burchakli parallelepipeddan iborat usti ochiq bak V litr hajmga ega. Bakning o‘lchovlari qanday bo‘lganda, unga sarflanadigan material eng kam bo‘ladi?

Yechilishi. To‘g‘ri burchakli parallelepipedning o‘lchovlarini $x,y,z\ (x>0,y>0,z>0)$ deb belgilaymiz. U holda, uning hajmi, $V=xyz$

bo‘ladi. Bundan $z=\frac{V}{xy}$. Masalaning shartiga ko‘ra, to‘g‘ri burchakli parallelepipedning to‘la sirti,

$$S=S(x,y)=2(xz+2yz)+xy=2(x+y)\frac{V}{xy}+xy=2V(\frac{1}{x}+\frac{1}{y})+xy$$

bo‘ladi. $S(x,y)$ funksiyaning eng kichik qiymatini topamiz. Ravshanki,

$$S'_x=-\frac{2V}{x^2}+y,$$

$$S'_y=-\frac{2V}{y^2}+x$$

Stasionar nuqtalarni topish uchun, xususiy hosilalarni nulga tenglashtirib, hosil qilingan sistemani echamiz:

$$\left.\begin{array}{l}-\frac{2V}{x^2}+y=0\\ -\frac{2V}{y^2}+x=0\end{array}\right|\Rightarrow \begin{array}{l}x_0=\sqrt[3]{2V},\\ y_0=\sqrt[3]{2V},\end{array}\quad P_0(x_0,y_0)=(\sqrt[3]{2V},\sqrt[3]{2V}).$$

Ikki o‘zgaruvchili funksiya minimumining yetarli shartini tekshiramiz:

$$S''_{xx}=\frac{4V}{x^3},\qquad S''_{xx}(\sqrt[3]{2V},\sqrt[3]{2V})=2$$

$$S''_{y^2}=\frac{4V}{y^3},\qquad S''_{y^2}(\sqrt[3]{2V},\sqrt[3]{2V})=2$$

$$S''_{xy}=1,\qquad S''_{xy}(\sqrt[3]{2V},\sqrt[3]{2V})=1$$

$$\Delta=S''_{x^2}(P_0)S''_{y^2}(P_0)-S''_{xy}(P_0)=4-1=3>0,\quad S''_{xx}(P_0)>0\,.$$

Demak, $S(x,y)$ funksiya, $P_0(\sqrt[3]{2V},\sqrt[3]{2V})$ nuqtada minimumga ega bo‘ladi:

$$z(P_0)=\frac{V}{x_0y_0}=\frac{V}{\sqrt[3]{4V^2}}=\frac{\sqrt[3]{2V}}{2}\ ;\quad S_{\min}=S(P_0)=2V(\frac{1}{\sqrt[3]{2V}}+\frac{1}{\sqrt[3]{2V}})+\sqrt[3]{4V^2}=3\sqrt[3]{4V^2}\,.$$

Mustaqil yechish uchun misollar

Quyidagi ikki o‘zgaruvchili funksiyalarni ekstremumga tekshiring:

23.1. $u=-x^2-xy$.

23.2. $u=2x^2+xy+y^2$.

23.3. $u=x^3-3xy-3y$.

23.4. $u=x^4+y^4-4xy$.

23.5. $u=-2x^2+xy-2y^2+6x+6y$.

23.6. $u=x^3-9xy+y^3$.

23.7. $u=x^3+6xy+8y^3-1$.

23.8. $u=xy(1-x-y)$

23.9. $u=x^2-3xy+y^2-4x+5y+6$.

23.10. $u=x^2+y^2-8x-2$.

23.11. $u=x^2+xy-y^2-3x-6y$.

23.12. $u=3x^2-x^3+3y^2+4y$.

23.13. $u=3x^2-y^2+4y+5$.

23.14. $u=x^2+xy+2y^2-x+y$.

23.15. $u=-x^2-xy-y^2+3x+6y.$ **23.16.** $u=(x+y^2)e^{x/2}.$

23.23. $u=x^3-3axy+y^3.$ **23.18.** $u=x^2+xy+y^2-4\ln x-10\ln y.$

23.19. $u=\sin x+\sin y+\sin(x+y)$, bunda $0\le x\le\frac{\pi}{2},\quad 0\le y\le\frac{\pi}{2}.$

23.20. $u=xe^{y+x\sin y}.$ **23.21.** $u=\frac{8}{x}+\frac{x}{y}+y.$

23.22. $f(x,y)=x^2-xy+y^2+2x+2y-4.$

23.23. $f(x,y)=2x^3+3xy+2y^3.$ **23.24.** $f(x,y)=x^3+y^3+3x^2-3y^2.$

Quyidagi uch o‘zgaruvchili funksiyalarni ekstremumga tekshiring:

23.25. $u=x^2+y^2+z^2-4x+6y-2z.$ **23.26.** $u=x^2+y^2+z^2-xy+x-2z.$

23.27. $u=x^2+y^2+(z+1)^2-xy+x.$ **23.28.** $u=x^3+y^2+z^2+6xy-4z.$

23.29. $u=xyz(16-x-y-2z).$ **23.30.** $u=\frac{226}{x}+\frac{x^2}{y}+\frac{y^2}{z}+z^2.$

23.31. $u=\frac{1}{z}+\frac{z}{y}+\frac{y}{x}+x+1.$ **23.32.** $u=x^{2/3}+y^{2/3}+z^{2/3}.$

Quyidagi ikki o‘zgaruvchili funksiyalarni shartli ekstremumga tekshiring:

23.33. $u=xy,\ x+y-2=0.$ **23.34.** $u=x^2+y^2,\ x+y-1=0.$

23.35. $u=x^2+y^2,\ 3x+4y-12=0.$ **23.36.** $u=xy,\ 2x+3y-5=0.$

23.37. $u=xy^2,\ x+2y-1=0.$ **23.38.** $u=x^2+y^2-xy+x+y-4,\ x+y+3=0.$

23.39. $u=\cos^2 x+\cos^2 y,\quad x-y-\frac{\pi}{4}=0.$ **23.40.** $u=5-3x-4y,\ x^2+y^2=25.$

23.41. $u=1-4x-8y,\ x^2-8y^2=8.$ **23.42.** $u=x^2+xy+y^2,\ x^2+y^2=1.$

Quyidagi uch o‘zgaruvchili funksiyalarni shartli ekstremumga tekshiring.

23.43. $u=2x^2+3y^2 4z^2,\ x+y+z-13=0.$

23.44. $u=xy^2z^3,\ x+y+z-12=0,\ x>0,\ y>0,\ z>0.$

23.45. $u=x-2y+2z,\ x^2+y^2+z^2-9=0.$

23.46. $u=xy+2xz+2yz,\ xyz=108.$

23.47. $u=x^2+y^2+z^2,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a>0,\ b>0,\ c>0.$

23.48. $f(x,y)=x^3+y^2$ funksiyaning $x^2+y^2=1$ aylanadagi ekstremum qiymatlarini toping.

23.49. $f(x,y)=x^2+3y^2+2y$ funksiyaning $x^2+y^2\le 1$ doiradagi ekstremum qiymatlarini toping.

23.50. $f(x,y,z)=x-y+z$ funksiyaning $x^2+y^2+z^2=1$ birlik sferadagi ekstremum qiymatlarini toping.

23.51. $f(x,y,z)=x(y+z)$ funksiyaning $x^2+y^2=1$ to‘g‘ri doiraviy konus va $xz=1$ giperbolik silindrlarning kesishish chizig‘idagi ekstremum qiymatlarini toping.

Quyidagi funksiyalarning ko‘rsatilgan D to‘plamda eng katta va eng kichik qiymatlarini toping.

23.52. $u=x^3-3xy+y^3,\ D=\{(x,y)\in R^2: 0\le x\le 2\ -1\le y\le 2\}$

23.53. $u=x-2y+5,\ D=\{(x,y)\in R^2: x\ge 0,\ y\ge 0,\ x+y\le 1\}$

23.54. $u=x^2-4x-y^2,\ D=\{(x,y)\in R^2: x^2+y^2\le 9\}$

23.55. $u=xy(4-x-y),\ D=\{(x,y)\in R^2: x\ge 0,\ y\ge 0,\ x+y\le 8\}$

23.56. $u=x^3-9xy+y^3+27,\ D=\{(x,y)\in R^2: 0\le x\le 4\ 0\le y\le 4\}$

23.57. $u=\sin x+\sin y+\sin(x+y),\ D=\{(x,y)\in R^2: 0\le x\le \pi/2\ 0\le y\le \pi/2\}$

Quyidagi funksiyalarning ko‘rsatilgan D to‘plamda eng katta va eng kichik qiymatlarini toping:

23.58. $u=xy+x+y,\ D=\{(x,y)\in R^2: -2\le x\le 2\ \ -2\le y\le 2\}$

23.59. $u=x^3-6xy+8y^3+1,\ D=\{(x,y)\in R^2: 0\le x\le 2\ -1\le y\le 1\}$

23.60. $u=3+2xy,\ D=\{(x,y)\in R^2: -4\le x^2+y^2\le 9\}$

23.61. $u=x^4-y^4,\ D=\{(x,y)\in R^2: x^2+y^2\le 9\}$

23.62. $u=x^2+y^2,\ D=\{(x,y)\in R^2: (x-\sqrt{2})^2+(y-\sqrt{2})^2\le 9\}$

23.63. $u=\cos x\cos y\cos(x+y),\ D=\{(x,y)\in R^2: 0\le x\le \pi\ \ 0\le y\le \pi\}$

Berilgan funksiyalarning berilgan R sohada maksimum va minimum qiymatlarini toping.

23.64. $f(x,y)=x^2+xy+y^2-3x+3y,\ R:$ birinchi kvadrantda $x+y=4$ to‘g‘ri chiziq bilan kesilgan uchburchakli soha.

23.65. $f(x,y)=y^2-xy-3y+2x,\ \ R: x=\pm 2$ va $y=\pm 2$ to‘g‘ri chiziqlar bilan chegaralangan kvadratik soha.

23.66. $f(x,y)=x^2-y^2-2x+4y,\ R:$ pastdan Ox o‘q, yuqoridan $y=x+2$ to‘g‘ri chiziq va o‘ngdan $x=2$ to‘g‘ri chiziq bilan chegaralangan uchburchakli soha.

23.67. $f(x,y)=x^3+y^3+3x^2-3y^2,\ R: x=\pm 1$ va $y=\pm 1$ to‘g‘ri chiziqlar bilan chegaralangan kvadratik soha.

Mustaqil yechish uchun misollarning javoblari

23.1. $u_{max}=u(0,0)=0$. **23.2.** $u_{min}=u(0,0)=0$. **23.3.** $(-1;1)$ stasionar nuqtada ekstremum yo‘q. **23.4.** $u_{min}=u(1;\ 1)=u(-1;-1)=-2,\ O(0;\ 0)$ stasionar nuqtada ekstremum yo‘q. **23.5.** $u_{max}=u(2;\ 2)=12$.

23.6. $u_{min}=u(3;3)=-27,\ O(0;\ 0)$ stasionar nuqtada ekstremum yo‘q.

23.7. $u_{max}=u(-1;-0{,}5)=0$. **23.8.** $u_{max}=u\left(\frac{1}{3};\frac{1}{3}\right)=\frac{1}{27}$.

23.9. $\left(\frac{7}{5};-\frac{2}{5}\right)$ stasionar nuqtada ekstremum yo'q. **23.10.** $u_{min}=u(4;0)=-18$.

23.11. $u_{min}=u(0;3)=-9$. **23.12.** $u_{min}=u\left(0;-\frac{2}{3}\right)=-\frac{4}{3}$, $\left(2;-\frac{2}{3}\right)$ stasionar nuqtada ekstremum yo'q. **23.13.** Ekstremum yo'q. **23.14.** $u_{min}=u(1;-1)=0$.

23.15. $u_{max}=u(0;3)=9$. **23.16.** $u_{min}=u(-2;0)=-\frac{2}{e}$. **23.23.**

$a<0 \quad da \quad u_{max}=u(a;a)=-a^3$, .

$a>0 \ da \ u_{min}=u(a;a)=-a^3$. **23.18.** $u_{min}=u(1;2)=7-10\ln 2$. **23.19.** $u_{max}=u(\frac{\pi}{3},\frac{\pi}{3})=\frac{3}{2}\sqrt{3}$. **23.20.** Ekstremum yo'q. **23.21.** $u=u_{min}(4;2)=6$.

23.22. $f_{min}=f(-2,-2)=-8$. **23.23.** $f_{max}=f\left(-\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{4}$.

23.24. $f_{min}=f(0,2)=-4$, $f_{max}=f(-2,0)=4$. **23.25.** $u_{min}=u(2;-3;1)=-14$.

23.26. $u_{min}=u\left(-\frac{2}{3};-\frac{1}{3};1\right)=-\frac{4}{3}$. **23.27.** $u_{min}=u\left(-\frac{2}{3};-\frac{1}{3};-1\right)=-\frac{1}{3}$.

23.28. $u_{min}=u(6;-18;2)=-112$. **23.29.** $u_{max}=u(4;4;2)=128$.

23.30. $u_{min}=u(8;4;2)=60$. **23.31.** $u_{min}=u(1;1;1)=5$, $u_{max}=u(-1;1;-1)=-3$.

23.32. $u_{min}=u(0;0;0)=0$. **23.33.** $u_{max}=u(1;1)=1$.

23.34. $u_{min}=u(0{,}5;0{,}5)=0{,}5$. **23.35.** $u_{min}=u\left(\frac{36}{25};\frac{48}{25}\right)=\frac{144}{25}$.

23.36. $u_{max}=u\left(\frac{5}{4};\frac{5}{6}\right)=\frac{25}{24}$. **23.37.** $u_{max}=u(1;0)=0 \quad u_{min}=u\left(\frac{1}{3};\frac{1}{3}\right)=\frac{1}{27}$.

23.38. $u_{min}=u\left(-\frac{3}{2};-\frac{3}{2}\right)=-\frac{19}{4}$.

23.39. $u_{min}=u\left(\frac{5\pi}{8}+\pi k;\ \frac{3\pi}{8}+\pi k\right)=1-\frac{\sqrt{2}}{2}$, $u_{max}=u\left(\frac{\pi}{8}+\pi k;\ -\frac{\pi}{8}+\pi k\right)=1+\frac{\sqrt{2}}{2}$, $k\in Z$.

23.40. $u=u_{min}(3;4)=-20$, $u=u_{max}(-3;-4)=30$.

23.41. $u=u_{min}(-4;1)=9$, $u=u_{max}(4;-1)=-7$.

23.42. $u=u_{min}\left(\pm\frac{\sqrt{2}}{2};\mp\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$, $u=u_{max}\left(\pm\frac{\sqrt{2}}{2};\pm\frac{\sqrt{2}}{2}\right)=\frac{3}{2}$.

23.43. $u_{min}=u(6;4;3)=156$. **23.44.** $u_{max}=u(2;4;6)=6912$.

23.45. $u_{max}=u(1;-2;2)=9$, $u_{min}=u(-1;2;-2)=-9$. **23.46.** $u_{min}=u(6;6;3)=108$.

23.47. $u_{max}=u(\pm a;0;0)=a^2$, $u_{min}=u(0;0;\pm c)=c^2$.

23.48. $f_{max}=f(0,\pm1)=f(1,0)=1$, $f_{min}=f(-1,0)=-1$.

23.49. $f_{\max}=f(0,1)=5,\ f_{\min}=f(0,-1/3)=-\frac{1}{3}.$

23.50. $f_{\max}=f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\sqrt{3},\quad f_{\min}=f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=-\sqrt{3}.$

23.51. $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\sqrt{2}\right)$ va $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\sqrt{2}\right)$- maksimum nuqtalari, ularda funksiya $\frac{3}{2}$ qiymat qabul qilidi; $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\sqrt{2}\right)$ va $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\sqrt{2}\right)$- minimum nuqtalari, ularda funksiya $\frac{1}{2}$ qiymat qabul qiladi. **23.52.** $u_{e\ kich}=u(1;1)=u(0;-1)=-1,$

$u_{e\ kat}=u(2;-1)=13.$ **23.53.** $u_{e\ kat}=u(1;0)=6.$

23.54. $u_{e\ kich}=u(1;-2\sqrt{2})=u(1;2\sqrt{2})=-11,\quad u_{e\ kat}=u(-3;0)=21.$

23.55. $u_{e\ kich}=u(4;4)=-64,\quad u_{e\ kat}=u\left(\frac{4}{3};\frac{4}{3}\right)=\frac{64}{27}.$

23.56. $u_{e\ kich}=u(3;3)=0,\ u_{e\ kat}=u(4;0)=u(0;4)=91.$

23.57. $u_{e\ kat}=u(\pi/3;\pi/3)=\frac{3}{2}\sqrt{3}.$ **23.58.** $u_{e\ kich}=-6,\ u_{e\ kat}=14.$

23.59. $u_{e\ kich}=-7,\ u_{e\ kat}=9+4\sqrt{2}.$ **23.60.** $u_{e\ kich}=-6,\ u_{e\ kat}=12.$

23.61. $u_{e\ kich}=-81,\ u_{e\ kat}=81.$ **23.62.** $u_{e\ kich}=0,\ u_{e\ kat}=25.$

23.63. $u_{e\ kich}=-\frac{1}{8},\ u_{e\ kat}=1.$ **23.64.** $f_{\max}=f(0,4)=28;\ f_{\min}=f\left(\frac{3}{2},0\right)=-\frac{9}{4}.$

23.65. $f_{\max}=f(2,-2).\ f_{\min}=f\left(-2;\frac{1}{2}\right)=-\frac{17}{4}.$ **23.66.** $f_{\max}=f(-2,0)=8,$ $f_{\min}=f(1,0)=-1.$

23.67. $f_{\max}=f(1,0)=4,\ f_{\min}=f(0,-1)=-4.$

24-§. Oshkormas funksiyalar

24.1. Oshkormas funksiya haqida tushuncha. Faraz qilaylik, x va y o'zgaruvchilarning qiymatlari, o'zaro, ushbu

$$F(x,y)=0 \tag{24.1}$$

tenglama orqali bog'langan bo'lsin. Bunda $F(x,y)$, ikki o'zgaruvchining funksiyasi sifatida, $\{M\}=\{(x,y)\in R^2 : a<x<b,\ c<y<d\}$ to'plamda berilgan bo'lsin. Biror x_0 sonni $(x_0\in(a,b))$ olib, uni (24.1) tenglamadagi x ning o'rniga qo'ysak, natijada, y ni topish uchun, quyidagi

$$F(x_0,y)=0 \tag{24.2}$$

tenglamaga ega bo'lamiz. Bu tenglamani yechishda quyidagi hollar bo'lishi mumkin:

1^0. (24.2) tenglama yagona y_0 yechimga ega ,

2^0. (24.2) tenglama bitta ham yechimga ega emas ,

3^0. (24.2) tenglama bir nechta, hatto, cheksiz ko'p, yechimga ega bo'lishi mumkin.

***Masalan*:** 1) $F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ tenglama $[-a,a]$ kesmada y ni, x ning ikki qiymatli funksiyasi sifatida aniqlaydi: $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. Buni $F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ tenglamadagi y ning o'rniga qo'ysak, natijada ayniyat hosil bo'ladi.

2) $F(x,y)=y\sqrt{x^2-3}-2=0$ tenglama, x ning $R\setminus\{x\in R:-\sqrt{3}\le x\le\sqrt{3}\}$ dan olingan har bir qiymatida, yagona $y=\frac{2}{\sqrt{x^2-3}}$ yechimga ega, bundan $F\left(x,\frac{2}{\sqrt{x^2-3}}\right)\equiv 0$. Yuqoridagi keltirilgan 1) va 2) misollarda, y ni x orqali ifodalash mumkin bo'ldi. Lekin, har doim ham bunday oson bo'lavermaydi. Masalan, ushbu

$$F(x,y)=y-x-\varepsilon\sin y=0\ \ (0<\varepsilon<1) \qquad (24.3)$$

tenglamani qaraylik. Bunda y ni, x orqali elementar funksiyalar yordamida aniqlab bo'lmaydi. Lekin, $F(x,y)=y-x-\varepsilon\sin y=0$ $(0<\varepsilon<1)$ tenglama, umumiy holda, y ni, x ning bir qiymatli funksiyasi sifatida aniqlaydi. Haqiqatan ham, tenglamani $x=y-\varepsilon\sin y=\varphi(y)$ $(y\in(-\infty;+\infty))$ ko'rinishda yozib olib, $\varphi(y)$ funksiyaning $(-\infty;\infty)$ oraliqda uzluksizligiga, hamda $\varphi'(y)=1-\varepsilon\cos y>0$ hosilaga ega ekanligiga ishonch hosil qilamiz. Unda, teskari funksiyaning mavjudligi to'g'risidagi teoremaga asosan, yagona $y=\varphi^{-1}(x)$ funksiya mavjud. (24.3) tenglama - Kepler tenglamasini ifodalaydi.

Demak, (24.3) tenglama, yagona $y=\varphi^{-1}(x)$ yechimga ega, bunda $F(x,\varphi^{-1}(x))\equiv 0$.

3) $F(x,y)=x^2+y^2-\ln y=0$ $(y>0)$ tenglama, x ning $(-\infty;\infty)$ oraliqdan olingan hech qanday qiymatida yechimga ega emas. Chunki, doimo $y^2-\ln y>0$ munosabat o'rinli. Bu holda berilgan tenglama bitta ham yechimga ega emas.

$F(x,y)=0$ tenglama uchun, 1-holning o'rinli bo'lishi muhim ahamiyatga ega. Uning yordamida funksiya aniqlanishi mumkin.

x o'zgaruvchining qiymatlaridan iborat, x ning undan olingan har bir qiymatida $F(x,y)=0$ tenglama yagona yechimga ega bo'lgan, X to'plamni qaraymiz. X to'plamdan ixtiyoriy x sonni olib, unga, $F(x,y)=0$ tenglamaning yagona yechimi bo'lgan, y ni mos qo'yamiz. Natijada, X to'plamdan olingan har bir x ga, yuqorida ko'rsatilgan qoidaga ko'ra, bitta y mos qo'yilib, funksiya hosil bo'ladi. Bunda x va y o'zgaruvchilar orasidagi bog'lanish, $F(x,y)=0$ tenglama yordamida ifodalangan bo'ladi. Odatda, bunday aniqlangan funksiya, *oshkormas funksiya* deyiladi va u $x \to y : F(x,y)=0$ kabi belgilanadi. Masalan, 2), 3) misollardagi oshkormas funksiyalar,

$$x \to y = \frac{2}{\sqrt{x^2-3}},\ F\left(x, \frac{2}{\sqrt{x^2-3}}\right)=0,\ x \to y = \varphi^{-1}(x):\ F\left(x, \varphi^{-1}(x)\right)=0$$

kabi yoziladi.

24.1-eslatma. Agar $F(x,y)=0$ tenglama oshkormas ko'rinishdagi funksiyani aniqlamasa, ba'zi hollarda, y ga ma'lum shart qo'yish natijasida berilgan tenglama, oshkormas funksiyani aniqlashi mumkin. Masalan, $F(x,y)=x^2+y^2-1=0$ tenglama x ning $(-1;1)$ oraliqdan olingan har bir qiymatida, ikkita, $y=-\sqrt{1-x^2}$, $y=\sqrt{1-x^2}$ yechimlarga ega. Agar y ga, uning qiymatlari $[-1,0]$ kesmada bo'lsin, degan shart qo'yilsa, u holda, berilgan tenglama yordamida aniqlangan,

$$x \to y = -\sqrt{1-x^2},\ F\left(x,-\sqrt{1-x^2}\right)=0$$

ko'rinishdagi oshkormas funksiya hosil bo'ladi.

24.2. Oshkormas funksiyaning mavjudligi.

24.1-teorema. $F(x,y)$ funksiya $M_0(x_0,y_0)\in R^2$ nuqtaning biror $U_{h,k}((x_0,y_0))=\{(x,y)\in R^2 : x_0-h<x<x_0+h,\ y_0-k<y<y_0+k\}$ $(h>0,\ k>0)$ atrofida berilgan bo'lib, u quyidagi shartlarni qanoatlantirsin:

1) $U_{h,k}((x_0,y_0))$ da uzluksiz;

2) x o'zgaruvchining (x_0-h, x_0+h) oraliqdan olingan har bir tayin qiymatida, y o'zgaruvchining funksiyasi sifatida, o'suvchi;

3) $F(x_0,y_0)=0$ bo'lsin.

U holda $M_0(x_0,y_0)$ nuqtaning shunday

$$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2 : x_0-\delta<x<x_0+\delta,\ y_0-\varepsilon<y<y_0+\varepsilon\}\ (0<\delta<h,\ 0<\varepsilon<k)$$

atrofi topiladiki:

a) $\forall x \in (x_0-\delta, x_0+\delta)$ uchun $F(x,y)=0$ tenglama, yagona y yechimga $(y \in (y_0-\varepsilon, y_0+\varepsilon))$ ega, ya'ni $F(x,y)=0$ tenglama yordamida $x \to y : F(x,y)=0$ oshkormas ko'rinishdagi funksiya aniqlanadi;

b) $x = x_0$ bo'lganda, $y = y_0$ unga mos keladi;

c) $x \to y : F(x,y)=0$ oshkormas funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz bo'ladi.

24.3. Oshkormas funksiyaning hosilasi.

24.2-teorema. $F(x,y)$ funksiya $M_0(x_0,y_0) \in R^2$ nuqtaning biror $U_{h,k}((x_0,y_0)) = \{(x,y) \in R^2 : x_0-h < x < x_0+h,\ y_0-k < y < y_0+k\}\ (h>0,\ k>0)$ atrofida berilgan bo'lib, u quyidagi shartlarni qanoatlantirsin:

1) $U_{h,k}((x_0,y_0))$ da uzluksiz;

2) $U_{h,k}((x_0,y_0))$ da uzluksiz $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalarga ega va $F'_y(x_0 y_0) \neq 0$;

3) $F(x_0,y_0)=0$.

U holda, $M_0(x_0,y_0)$ nuqtaning shunday $U_{\delta,\varepsilon}((x_0,y_0)) = \{(x,y) \in R^2 : x_0-\delta < x < x_0+\delta,\ y_0-\varepsilon < y < y_0+\varepsilon\}\ (0<\delta<h,\ 0<\varepsilon<k)$ atrofi topiladiki:

a) $\forall x \in (x_0-\delta,\ x_0+\delta)$ uchun $F(x,y)=0$ tenglama yagona, y Yechimga $(y \in (y_0-\varepsilon,\ y_0+\varepsilon))$ ega, ya'ni, $F(x,y)=0$ tenglama yordamida, $x \to y : F(x,y)=0$ oshkormas funksiya aniqlanadi;

b) $x = x^0$ bo'lganda, $y = y_0$ unga mos keladi;

c) $x \to y : F(x,y)=0$ oshkormas ko'rinishda aniqlangan funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz bo'ladi;

d) oshkormas ko'rinishdagi funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz hosilaga ega bo'ladi va uning hosilasi

$$y'_x = -\frac{F_x(x,y)}{F_y(x_0,y_0)} \tag{24.4}$$

formula bo'yicha hisoblanadi.

$F(x,y)$ funksiya, $U_{\delta,\varepsilon}((x_0,y_0))$ atrofda uzluksiz ikkinchi tartibli $F''_{x^2}(x,y)$, $F''_{xy}(x,y)$, $F''_{y^2}(x,y)$ hususiy hosilalarga ega bo'lsin. y ning x ga bog'liqligini e'tiborga olib, (24.4) tenglikni x bo'yicha differensiallab, quyidagini topamiz:

$$y'' = \frac{2F'_x F'_y F''_{xy} - F'^2_y F''_{x^2} - F'^2_x F''_{y^2}}{(F'_y)^3}.$$

Xuddi shunday, oshkormas ko'rinishdagi funksiyaning uchinchi va hokazo tartibdagi hosilalari topiladi.

24.4. Ko'p o'zgaruvchili oshkormas funksiyalar.

$F(x,y)=F(x_1,x_2,...,x_m,y)$ $(x=(x_1,x_2,...,x_m)\in R^m)$ funksiya

$$\{M\}=\{(x,y)\in R^{m+1}: a_1<x_1<b_1,\ a_2<x_2<b_2,...,a_m<x_m<b_m,\ c<y<d\}$$

to'plamda aniqlangan bo'lsin. Ushbu

$$F(x,y)\equiv F(x_1,x_2,...,x_m,y)=0 \qquad (24.5)$$

tenglamani qaraymiz. $x\in R^m$ nuqtalardan iborat, undan olingan har bir x nuqtada (24.5) tenglama yagona haqiqiy yechimga ega bo'lgan, $X\ (X\subset R^m)$ to'plamni qaraymiz. X to'plamdan ixtiyoriy x nuqtani olib, unga (24.5) tenglamaning yagona haqiqiy yechimi y ni mos qo'yamiz. Natijada, X to'plamdan olingan har bir x nuqtaga, yuqorida ko'rsatilgan qoidaga ko'ra, bitta y mos qo'yilib, funksiya hosil qilinadi. Bunday aniqlangan funksiya, ko'p o'zgaruvchili (m o'zgaruvchili) *oshkormas ko'rinishda aniqlangan funksiya* deb ataladi va u,

$$x=(x_1,x_2,...,x_m)\to y: F(x_1,x_2,...,x_m,y)=0$$

kabi belgilanadi.

24.3-teorema. $F(x,y)=F(x_1,x_2,...,x_m,y)$ funksiya $(x^0,y_0)\in R^{m+1}$ nuqtaning biror

$$U_{h_1,h_2,...,h_m,k}(x^0,y_0)=\{(x,y)\in R^{m+1}: x_1^0-h_1<x_1<x_1^0+h_1,\ x_2^0-h_2<x_2<x_2^0+h_2,...,$$
$$x_m^0-h_m<x_m<x_m^0+h_m,\ y_0-k<y<y_0+k\}\ (h_i>0,\ i=1,2,...,m,\ k>0)$$

atrofida aniqlangan va u quyidagi shartlarni qanoatlantirsin:

1) $U_{h_1,h_2,...,h_m,k}((x^0,y_0))$ da uzluksiz;

2) $x=(x_1,x_2,...,x_m)$ o'zgaruvchining

$$\{x\in R^m: x_1^0-h_1<x_1<x_1^0+h_1,\ x_2^0-h_2<x_2<x_2^0+h_2,...,x_m^0-h_m<x_m<x_m^0+h_m\}$$

to'plamdan olingan har bir tayin qiymatlarida, y o'zgaruvchining funksiyasi sifatida, o'suvchi (kamayuvchi);

3) $F(x^0,y_0)=0$. U holda, $(x^0,y_0)\in R^{m+1}$ nuqtaning shunday

$$U_{\delta_1,\delta_2,...,\delta_m,\varepsilon}(x^0,y_0)=\{(x,y)\in R^{m+1}: x_1^0-\delta_1<x_1<x_1^0+\delta_1,\ x_2^0-\delta_2<x_2<x_2^0+\delta_2,...,$$
$$x_m^0-\delta_m<x_m<x_m^0+\delta_m,\ y_0-\varepsilon<y<y_0+\varepsilon\}\ (0<\delta_i<h_i,\ i=1,2,...,m,\ 0<\varepsilon<k)$$

atrofi topiladiki: *a)* $\forall x\in\{(x_1,x_2,...,x_m)\in R^m: x_1^0-\delta_1<x_1<x_1^0+\delta_1,$ $x_2-\delta<x_2<x_2^0+\delta_2,...,x_m^0-\delta_m<x_m<x_m^0+\delta_m\}$ uchun, $F(x,y)=0$ tenglama, $y\ (y\in(y_0-\varepsilon,y_0+\varepsilon)$ yagona haqiqiy yechimga ega, ya'ni (24.5) tenglama, $x=(x_1,x_2,...,x_m)\to y: F(x_1,x_2,...,x_m,y)=0$ oshkormas ko'rinishdagi funksiyani aniqlaydi;

b) $x=x^0$ bo'lganda, $y=y_0$ unga mos keladi;

$c)$ oshkormas ko'rinishda aniqlangan $(x_1,x_2,...,x_m)\to y: F(x_1,x_2,...,x_m,y)=0$ funksiya, $\{(x_1,x_2,...,x_m)\in R^m : x_1^0-\delta_1<x_1<x_1^0+\delta_1,$
$x_2-\delta<x_2<x_2^0+\delta_2,...,x_m^0-\delta_m<x_m<x_m^0+\delta_m\}$
to'plamda uzluksiz bo'ladi.

24.4-teorema. $F(x,y)=F(x_1,x_2,...,x_m,y)$ funksiya $(x^0,y_0)\in R^{m+1}$ nuqtaning biror

$$U_{h_1,h_2,...,h_m,k}(x^0,y_0)=\{(x,y)\in R^{m+1}: x_1^0-h_1<x_1<x_1^0+h_1,\ x_2^0-h_2<x_2<x_2^0+h_2,...,$$
$$x_m^0-h_m<x_m<x_m^0+h_m, y_0-k<y<y_0+k\}\ (h_i>0,\ i=1,2,...,m, k>0)$$

atrofida aniqlangan bo'lib, u quyidagi shartlarni qanoatlantirsin:

1) $U_{h_1,h_2,...,h_m,k}(x^0,y_0)$ da uzluksiz;

2) $U_{h_1,h_2,...,h_m,k}(x^0,y_0)$ da $F_{x_i}(x_1,x_2,...,x_m,y)\ (i=1,2,...,m),\ F_y(x_1,x_2,...,x_m,y)$ uzluksiz xususiy hosilalarga ega va $F_y'(x;y)\neq 0$.

3) $F(x^0,y_0)=0$.

U holda, $(x^0,y_0)\in R^{m+1}$ nuqtaning shunday

$$U_{\delta_1,\delta_2,...,\delta_m,\varepsilon}(x^0,y_0)=\{(x,y)\in R^{m+1}: x_1^0-\delta_1<x_1<x_1^0+\delta_1,\ x_2^0-\delta_2<x_2<x_2^0+\delta_2,...,$$
$$x_m^0-\delta_m<x_m<x_m^0+\delta_m, y_0-\varepsilon<y<y_0+\varepsilon\}\ (0<\delta_i<h_i,\ i=1,2,...,m,\ 0<\varepsilon<k)$$

atrofi topiladiki:

$a)$
$\forall x\in\{(x_1,x_2,...,x_m)\in R^m : x_1^0-\delta_1<x_1<x_1^0+\delta_1, x_2-\delta<x_2<x_2^0+\delta_2,...,x_m^0-\delta_m<x_m<x_m^0+\delta_m\}$
uchun, $F(x,y)=0$ tenglama, $y\ (y\in(y_0-\varepsilon,y_0+\varepsilon)$ yagona haqiqiy yechimga ega, ya'ni (24.5) tenglama, $x=(x_1,x_2,...,x_m)\to y: F(x_1,x_2,...,x_m,y)=0$ oshkormas ko'rinishdagi funksiyani aniqlaydi;

$b)$ $x=x^0$ bo'lganda, $y=y_0$ unga mos keladi;

$c)$ oshkormas ko'rinishda aniqlangan $(x_1,x_2,...,x_m)\to y: F(x_1,x_2,...,x_m,y)=0$ funksiya,
$\{(x_1,x_2,...,x_m)\in R^m : x_1^0-\delta_1<x_1<x_1^0+\delta_1, x_2-\delta<x_2<x_2^0+\delta_2,...,x_m^0-\delta_m<x_m<x_m^0+\delta_m\}$
to'plamda uzluksiz bo'ladi.

$d)$ oshkormas ko'rinishdagi funksiya uzluksiz xususiy hosilalarga ega bo'ladi.

24.1-misol. Ushbu

$$y^4-4x^2y^2+\sin x=0$$

tenglamadan y ni, oshkor shaklda, x orqali ifodalang.

Yechilishi. $y^2=z$ almashtirish olib, $z^2-4x^2z+\sin x=0$ kvadrat tenglamaga ega bo'lamiz. Bundan, $y^2=z_{1,2}=2x^2\pm\sqrt{4x^4-\sin x}$, $y^2>0$ bo'lgani uchun, kvadrat ildiz oldidagi "+" ishorani olamiz:
$y=\pm\sqrt{2x^2+\sqrt{4x^4-\sin x}}$.

24.2-misol. Ushbu

$$y^2-2x^3y+x^6-x^4+x^2=0 \qquad (24.6)$$

tenglama, y ni $M_0(0,0)$ nuqta atrofida $x$ning bir qiymatli uzluksiz funksiyasi sifatida aniqlaydimi?

Yechilishi. $F(x,y)=y^2-2x^3y+x^6-x^4+x^2$, $F'_y=2y-2x^3$. Bu hosila $M_0(0,0)$ nuqtada $F'_y(0;0)=0$ bo'ladi. Demak, oshkormas funksiyaning mavjudligi va uzluksizligi haqidagi 24.1 – teoremaning shartlari bajarilmaydi. Shuning uchun, berilgan tenglama, $M_0(0;0)$ nuqtaning atrofida oshkormas funksiyani aniqlamaydi. Haqiqatan ham, (24.6) tenglamani y ga nisbatan yechib, $y=x^3\pm x\sqrt{x^2-1}$ ni hosil qilamiz. Bu funksiyalar $x=0$ va $|x|>1$ da aniqlangan bo'lib, $M_0(0;0)$ nuqtaning atrofida, (24.6) tenglamani qanoatlantiruvchi $M_0(0;0)$ nuqtadan boshqa birorta ham nuqta mavjud emas. $M_0(0;0)$ nuqta- egri chiziqning yakkalangan nuqtasi bo'lar ekan.

24.3-misol. Ushbu

$$ye^x-x\ln y-1=0$$

tenglama, $x=0$ nuqta atrofida, uzluksiz $y=f(x)$ funksiyani aniqlaydimi? Agar aniqlasa, bu funksiya hosilaga egami? Agar hosilaga ega bo'lsa, uning hosilasini hisoblang.

Yechilishi. $ye^x-x\ln y-1=0$ tenglamaga $x=0$ qiymatni qo'yib, $y=1$ bo'lishini topamiz. Endi $M_0(0;1)$ nuqta atrofida $F(x,y)=ye^x-x\ln y-1$ funksiyaning 24.2 - teoremaning shartlarini qanoatlantirishini tekshirib ko'ramiz:

1) $F(x,y)=ye^x-x\ln y-1$ funksiya, $y>0$ yarim tekislikda aniqlangan, uzluksiz. Uning xususiy hosilalari mavjud va uzluksiz:

$$F'_x=ye^x-\ln y,\ F'_y=e^x-\frac{x}{y};$$

$$2)\quad F'_y(0;1)=e^0-\frac{0}{1}=1\neq 0.$$

Shunday qilib, 24.2-teoremaning shartlari bajariladi, shuning uchun $ye^x-x\ln y-1=0$ tenglama $M(0;1)$ nuqtaning atrofida $x\to y(x): ye^x-x\ln y-1=0$ ko'rinishdagi oshkormas uzluksiz funksiyani aniqlaydi. Bu oshkormas funksiya, quyidagi uzluksiz hosilaga ega:

$$y'=-\frac{F'_x}{F'_y}=-\frac{ye^x-\ln y}{e^x-\dfrac{x}{y}}=-\frac{y(ye^x-\ln y)}{ye^x-x}.$$

24.4-misol. Ushbu $xe^{2y}-y\ln x-8=0$ oshkormas ko'rinishda berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini toping.

Yechilishi. $xe^{2y}-y\ln x-8=0$ tenglikni x bo'yicha (y ni x ning funksiyasi, deb qarab) ikki marta differensiallaymiz:

$$a)\ e^{2y}+xe^{2y}\cdot 2y'-y'\cdot\ln x-\frac{y}{x}=0;$$

$$b)\ e^{2y}\cdot y'+e^{2y}\cdot 2y'+2x\left(e^{2y}\cdot(y')^2+e^{2y}\cdot y''\right)-y''\cdot\ln x-\frac{y'}{x}-\frac{xy'-y}{x^2}=0.$$

$a)$ dan $y'=\dfrac{\frac{y}{x}-e^{2y}}{2xe^{2y}-\ln x}=\dfrac{e^{2y}-\frac{y}{x}}{\ln x-2xe^{2y}}$, (24.7)

$b)$ dan $y''=\dfrac{2xy'\left(1-xe^{2y}-x^2e^{2y}\cdot y'\right)}{2xe^{2y}-\ln x}$. (24.8)

Agar (24.8) da y' ning o'rniga (24.7) dagi ifodani keltirib qo'ysak, natijada y'' ning x va y orqali ifodasi topiladi.

24.5-misol. Ushbu $z^3-3xyz=8$ tenglamaning $M_0(0;-1;2)$ nuqta atrofida, yagona, $z=f(x,y)$ oshkormas funksiyani aniqlashini isbot qiling. Uning $z'_x(0;-1)$ va $z'_y(0;-1)$ xususiy hosilalarni toping.

Yechilishi. $F(x,y,z)=z^3-3xyz-8$ funksiya $M_0(0;-1;2)$ nuqta atrofida differensiallanuvchi. Uning $F'_z=3z^2-3xy=3(z^2-xy)$ hosilasi $M_0(0;-1;2)$ nuqtada uzluksiz, hamda $F(0;-1;2)=0$, $F'_z(0;-1;2)=12\neq 0$. Demak, 24.4 - teoremaning hamma shartlari bajariladi. Shuning uchun, berilgan tenglama, $M_0(0,-1;2)$ nuqtada yagona, differensiallanuvchi, $z=f(x,y)$ ko'rinishdagi oshkormas funksiyani aniqlaydi.

$z=f(x,y)$ funksiyaning $(0;-1)$ nuqtadagi qiymatini hisoblaymiz: $z^3-3\cdot 0\cdot(-1)\cdot z-8=0$, $z^3=8$, $z=2$.

Endi $z^3-3xyz=8$ tenglamani, mos ravishda, x va y bo'yicha differensiallaymiz:

$$3z^2\cdot z'_x-3yz-3xyz'_x=0, \qquad (24.9)$$

$$3z^2\cdot z'_y-3xz-3xz-3xy\cdot z'_y=0 \qquad (24.10)$$

(24.9) ga $x=0$, $y=-1$, $z=2$ qiymatlarni keltirib qo'yib.

$$3\cdot 4\cdot z'_x-3\cdot(-1)\cdot 2-3\cdot 0\cdot(-1)\cdot z'_x=0, \quad z'_x=-0{,}5$$

bo'lishini topamiz.

(24.10) ga $x=0$, $y=-1$, $z=2$ qiymatlarni keltirib qo'yib,

$$3\cdot 4\cdot z'_y-3\cdot(-1)\cdot 2-3\cdot 0\cdot(-1)\cdot z'_y=0, \quad z'_y=0$$

ekanligini topamiz.

24.6-misol. Ushbu $\cos^2 x+\cos^2 y+\cos^2 z=1$ tenglama bilan aniqlangan **$z(x,y)$ oshkormas funksiyaning xususiy hosilalarini va to'liq** differensialini toping.

Yechilishi. $\cos^2 x+\cos^2 y+\cos^2 z-1=0$ tenglamani, mos ravishda, x va y bo'yicha (z ni x va y ning funksiyasi, deb qarab) differensiallaymiz:

$$-2\cos x\ \sin x-2\cos z\cdot\sin z\cdot z'_x=0$$
$$-2\cos y\cdot\sin y-2\cos z\cdot\sin\cdot z'_y=0.$$

Bu tenglamalardan, $z'_x=-\dfrac{\sin 2x}{\sin 2z},\ z'_y=-\dfrac{\sin 2y}{\sin 2z}$ xususiy hosilalarni topamiz. Ikki o'zgaruvchili funksiyaning to'liq differensialini topish formulasiga asosan,

$$dz=\cdot z'_x dx+z'_y dy=-\frac{\sin 2x\,dx+\sin 2y dy}{\sin 2z}.$$

24.7-misol. Ushbu $x^2+y^2+z^2=2z$ tenglama bilan aniqlangan $z(x,y)$ funksiyaning xususiy hosilalari, birinchi va ikkinchi tartibli to'liq differensiallarini toping.

Yechilishi. $x^2+y^2+z^2=2z$ tenglamani, mos ravishda, x va y bo'yicha differensiallaymiz:

$$2x+2z\cdot z'_x-2z_x=0, \tag{24.11}$$
$$2y+2z\cdot z'_y-2z'_y=0. \tag{24.12}$$

Bundan, $z'_x=-\dfrac{x}{z-1},\ z'_y=-\dfrac{y}{z-1},\qquad dz=\dfrac{xdx+ydy}{1-z}$.

(24.11), (24.12) larni, mos ravishda, x va y bo'yicha differensiallaymiz:

$$1+(z'_x)^2+z\cdot z''_{x^2}-z''_{x^2}=0,$$
$$1+(z'_y)^2+z\cdot z''_{y^2}-z''_{y^2}=0.$$

Bu tenglamalardan,

$$z''_{x^2}=\frac{1+(z'_x)^2}{1-z}=\frac{1+\dfrac{x^2}{(1-z)^2}}{(1-z)}=\frac{(1-z)^2+x^2}{(1-z)^3},$$
$$z''_{y^2}=\frac{(1-z)^2+y^2}{(1-z)^3}.$$

(24.11) ni y bo'yicha, (24.12) ni x bo'yicha differensiallab, $z''_{xy}=\dfrac{xy}{(1-z)^3},\ z''_{yx}=\dfrac{xy}{(1-z)^3}$ aralash hosilalarni topamiz. Ikki o'zgaruvchili funksiyaning ikkinchi tartibli differensialini topish qoidasiga asosan,

$$d^2z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2=$$
$$=\frac{(1-z)^2+x^2}{(1-z)^3}dx^2+\frac{2xy}{(1-z)^3}dxdy+\frac{(1-z)^2+y^2}{(1-z)^3}dy^2.$$

24.5. Tenglamalar sistemasi bilan aniqlanadigan oshkormas funksiyalar. Ushbu $F_i(u_1,u_2,...,u_n,x_1,x_2,...,x_m)$ $(i=\overline{1,m})$ funksiyalar biror $D=\{(u_1,u_2,...,u_n,x_1,x_2,...,x_m)\in R^{m+n};\ a_1<x_1<b_1,...,a_m<x_m<b_m,\ c_1<u_1<d_1,...c_n<u_n<d_n\}$ to'plamda berilgan bo'lsin.

$$\left.\begin{aligned} F_1 &= F_1(u_1,u_2,...,u_n,x_1,x_2,...,x_m)=0,\\ F_2 &= F_2(u_1,u_2,...,u_n,x_1,x_2,...,x_m)=0,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ F_n &= F_n(u_1,u_2,...,u_n,x_1,x_2,...,x_m)=0 \end{aligned}\right\} \tag{24.13}$$

tenlamalar sistemasini qaraymiz. $x=(x_1,x_2,...,x_m)$ o'zgaruvchining qiymatlaridan iborat, undan olingan har bir $x'=(x_1',x_2',...,x_m')$ nuqtada (24.13) sistema yagona $u_1,u_2,....,u_n$ yechimlar sistemasiga ega bo'lgan, $D_x=\{x\in R^m: a_1<x_1<b_1,...,a_m<x_m<b_m\}\subset R^m$ to'plamni qaraymiz. D_x to'plamdan olingan ixtiyoriy $x=(x_1,x_2,...,x_m)$ nuqtani olib, unga (24.13) sistemaning yagona, $u_1,u_2,....,u_n$ yechimlar sistemasini mos qo'yamiz. Natijada, D_x to'plamdan olingan har bir x nuqtaga, yuqorida ko'rsatilgan qoidaga ko'ra, $u_1,u_2,....,u_n$ lar mos qo'yilib, n ta funksiya hosil bo'ladi. Bunday aniqlangan funksiyalar, (24.13) tenglamalar sistemasi yordamida aniqlangan *oshkormas funksiyalar sistema* deb ataladi.

24.5-teorema. $F_i(i=\overline{1,n})$ funksiyalarning har biri, $(u^0,x^0)=(u_1^0,u_2^0,....,u_n^0,x_1^0,x_2^0,....,x_m^0)$ nuqtaning biror

$$U_{k,h}(u^0,x^0)=U_{k_1,k_2,...,k_n,h_1,h_2,...,h_m,k}(x^0,y_0)=$$
$$=\{(u,x)\in R^{m+n}: u_1^0-k_1<u_1<u_1^0+k_1,\ u_2^0-k_2<u_2<u_2^0+k_2,...,u_m^0-k_m<u_m<u_m^0+k_m,$$
$$x_1^0-h_1<x_1<x_1^0+h_1,\ x_2^0-h_2<x_2<x_2^0+h_2,...,x_m^0-h_m<x_m<x_m^0+h_m\ \}$$
$$(h_i)>0,\ i=1,2,...,m,\ k_j>0,\ j=1,2,...,n$$

atrofida aniqlangan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $U_{kh}((u^0,x^0))$ da uzluksiz;

2) $U_{kh}((u^0,x^0))$ da barcha xususiy hosilalarga ega va ular uzluksiz;

3) xususiy hosilalarning (u^0,x^0) nuqtadagi qiymatlaridan tuzilgan determinat nuldan farqli:

$$\begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial u_1} & \frac{\partial F_n}{\partial u_2} & \cdots & \frac{\partial F_n}{\partial u_n} \end{vmatrix} \neq 0;$$

4) (u^0, x^0) nuqtada $F_i(u^0, x^0) = 0$ $(i = \overline{1,n})$.

U holda, (u^0, x^0) nuqtaning shunday $U_{\delta,\varepsilon}((u^0, x^0)) = U_{\delta_1\delta_2 \ldots \delta_m \varepsilon_1\varepsilon_2 \ldots \varepsilon_n}((u^0, x^0))$ $(0 < \delta_i < n_i < h_i,\ i = \overline{1,m},\ 0 < \varepsilon_j < k_j,\ j = \overline{1,n})$ atrofi topiladiki, bu atrofda:

a) (24.5) sistema, oshkormas funksiyalar sistemasini aniqlaydi, ularni $u_i = f_i(x_1 x_2, \ldots, x_m)$, $i = \overline{1,n}$ orqali belgilaymiz;

b) $x = x^0$ nuqtada $f_i(x^0) = y_i^0$, $i = \overline{1,n}$, bo'ladi;

c) oshkormas ko'rinishda aniqlangan $f_i(i = \overline{1,n})$ funksiyalar $\{x \in R^m : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1, \ldots, x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\}$ to'plamda uzluksiz va uzluksiz xususiy hosilalarga ega bo'ladi.

24.8-misol. Ushbu

$$\begin{cases} x + y = u + v \\ xy + yv = 1 \end{cases}$$

sistema, $(1; 1; 2, 0)$ nuqtaning atrofida, oshkormas funksiyalarni aniqlaydimi?

Yechilishi. Ravshanki, $F_1(x, y, u, v) = x + y - u - v$, $F_2(x, y, u, v) = xy + yv - 1$ funksiyalar $(1; 1; 2, 0)$ nuqtaning atrofida uzluksiz hamda barcha

$$\frac{\partial F_1}{\partial x} = 1,\ \frac{\partial F_1}{\partial y} = 1,\ \frac{\partial F_1}{\partial u} = -1,\ \frac{\partial F_1}{\partial v} = -1,$$

$$\frac{\partial F_2}{\partial x} = y,\ \frac{\partial F_2}{\partial y} = v,\ \frac{\partial F_2}{\partial u} = 0,\ \frac{\partial F_2}{\partial v} = y$$

xususiy hosilalarga ega va ular ham uzluksiz. $(1; 1; 2, 0)$ nuqtada

$$\begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0,\quad F_1(1, 1; 2, 0) = 0,\quad F_1(1, 1; 2, 0) = 0.$$

Demak, 24.5-teoremaga ko'ra, berilgan sistema, u va v larni, x va y o'zgaruvchilarning funksiyasi sifatida aniqlaydi:

$$u = 2x + y - \frac{1}{y},\quad v = \frac{1}{y} - x.$$

24.8-misol. $u(x, y)$ va $v(x, y)$ oshkormas funksiyalar, ushbu

$$\begin{cases} xu + yv = 4, \\ yu - v = 0 \end{cases}$$

sistema orqali aniqlangan. Agar $x = 1,\ y = -1$ da $u(x, y)$ va $v(x, y)$ funksiyalar $u(1; -1) = 2,\ v(1; -1) = -2$ qiymatlarni qabul qilsa, u holda ularning barcha xususiy hosilalari, birinchi va ikkinchi tartibli to'liq differensiallarini toping.

Yechilishi. Berilgan sistemani ikki marta differensiallaymiz:

$$\begin{cases} udx + xdu + vdy + ydv = 0, \\ udy + ydu - dv = 0. \end{cases} \qquad (24.14)$$

$$\begin{cases} dudx + dxdu + xd^2u + dvdy + dydv + yd^2v = 0, \\ dudy + dydv + yd^2u - d^2v = 0. \end{cases} \begin{cases} 2dxdu + xd^2u + 2dydv + yd^2v = 0, \\ 2dyd^2u + yd^2u - d^2v = 0. \end{cases} \qquad (24.15)$$

(24.14) sistemaga $x = 1,\ y = -1,\ u = 2,\ v = -2$ qiymatlarni keltirib qo‘yib,

$$\begin{cases} 2dx + du - 2dy - dv = 0, \\ 2dy - du - dv = 0. \end{cases} \text{ yoki } \begin{cases} dv = dx \\ du = 2dy - dx \end{cases}$$

ekanligini topamiz. Bundan, $\frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 0,\ \frac{\partial u}{\partial x} = -1,\ \frac{\partial u}{\partial y} = 2.$ Xuddi shunday, $x = 1,\ y = -1,\ u = 2,\ v = -2$ qiymatlarni, hamda dv va du larning, yuqorida topilgan, qiymatlarini (24.15) sistemaga keltirib qo‘yib,

$$\begin{cases} 2dx(2dy - dx) + d^2u + 2dxdy - d^2v = 0, \\ 2dy(2dy - dx) - d^2u - d^2v = 0. \end{cases} \text{ yoki } \begin{cases} 6dxdy - 2(dx)^2 + d^2u - d^2v = 0, \\ 4(dy)^2 - 2dxdy - d^2u - d^2v = 0 \end{cases}$$

bo‘lishini topamiz. Bundan,

$$d^2u - d^2v = 2(dx)^2 - 6dxdy, \qquad d^2u = (dx)^2 + 4dxdy - 2(dy)^2,$$

$$d^2u + d^2v = 4(dy)^2 - 2dxdy, \qquad d^2v = -(dx)^2 + 2dxdy + 2(dy)^2.$$

Demak, ikki o‘zgaruvchili funksiyaning ikkinchi tartibli differensialini topish formulasini e’tiborga olsak, $\frac{\partial^2 u}{\partial x^2} = 1,\ \frac{\partial^2 u}{\partial x\partial y} = 2,\ \frac{\partial^2 v}{\partial x^2} = -1,\ \frac{\partial^2 v}{\partial x\partial y} = 1,\ \frac{\partial^2 v}{\partial y^2} = 2$ larni topamiz.

Mustaqil yechish uchun misollar

Quyidagi tenglamalarni y ga nisbatan yechib, x ning funksiyasi sifatida ifodalang:

24.1. $y^2 - 5x^2y + 4x^4 = 0.$ **24.2.** $y^4 - 4x^2y^2 + \sin x = 0.$

24.3. $e^{x^4+y^4} - x^8 - 6 = 0.$ **24.4.** $x^2y^4 - 3y^3 + 6x^2y^2 - 3y + x^2 = 0.$

24.2. Quyidagi tenglamalar sistemasini, y va z larga nisbatan yechib, x ning funksiyasi sifatida ifodalang:

24.5. $\begin{cases} y + z = x^2, \\ y^2 - yz + z^2 = 6x. \end{cases}$ **24.6.** $\begin{cases} yz = x^4, \\ y + z + x(y^2 + z^2) = 3x - 1. \end{cases}$

Quyidagi tenglamalar, ko‘rsatilgan nuqta atrofida, oshkormas funksiyani aniqlaydimi?

24.7. $F(x,y) \equiv y^4 + xy + y^3 - 3 = 0,\quad A(1;\ 1).$

24.8. $F(x,y) \equiv x^3 - 3axy + y^3 = 0,\quad A(a\sqrt[3]{4};\ a\sqrt[3]{2})$

24.9. $F(x,y) \equiv e^y + y\sin x - x^3 + 7 = 0,\quad A(2;\ 0).$

24.10. $F(x,y) \equiv x(x^2 + y^2) - a(x^2 - y^2),\quad A(0;\ 0).$

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning berilgan A nuqtada f_x', f_y' xususiy hosilalarini hisoblang:

24.11. $u^3-2u^2x+uxy-2=0,\ A(1;1)$. **24.12.** $u^3+3uxy+1=0,\ A(0;1)$

24.13. $e^u-xyu-2=0\ \ A(1,0)$. **24.14.** $u+\ln(x+y-u)=0,\ A(1;-1)$.

Quyidagi oshkormas ko‘rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping:

24.15. $x^3-2y^2+xy=0,\ (1;1)$. **24.16.** $x^2+xy+y^2-7=0,\ (1;2)$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi va ikkinchi tartibli hosilalarini hisoblang:

24.17. $x^2-y^2-4=0$. **24.24.** $1+xy-\ln\left(e^{xy}+e^{-xy}\right)=0$.

24.19. $2\cos(x-2y)-2y+x=0$. **24.20.** $x^3+y^3-3xy=0$.

Quyidagi oshkormas ko‘rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping:

24.21. $z^3-xy+yz+y^3-2=0,\ (1;1;1)$.

24.22. $\sin(x+y)+\sin(y+z)+\sin(x+z)=0,\ (\pi;\pi;\pi)$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi tartibli xususiy hosilalarini va to‘liq differensiallarini hisoblang:

24.23. $x^2+y^2+z^2-6x=0$. **24.24.** $x^2+y^2+z^2-2xz=a^2$.

24.25. $z^2-xy=0$. **24.26.** $z^3+3x^2z-2xy=0$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi va ikkichi tartibli to‘liq differensiallarini hisoblang:

24.27. $x^2+y^2+z^2-2z=0$. **24.28.** $z^3-3xyz=a^3$. **24.29.** 3) $x-z\ln\dfrac{x}{y}=0$.

24.30. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalar, ushbu $\begin{cases}u+v=x,\\ u-yv=0\end{cases}$ sistema orqali aniqlangan. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalarning birinchi va ikkinchi tartibli to‘liq differensiallarini hisoblang.

24.31. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalar, ushbu $\begin{cases}xu+yv=1,\\ y+x+u+v=0\end{cases}$ sistema orqali aniqlangan. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalarning birinchi va ikkinchi tartibli to‘liq differensiallarini hisoblang.

24.32. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalar, ushbu $\begin{cases}u+v=x+y,\\ y\sin u-x\sin v=0\end{cases}$ sistema orqali aniqlangan. $u(x,y)$ va $v(x,y)$ oshkormas ko‘rinishdagi funksiyalarning birinchi va ikkinchi tartibli to‘liq differensiallarini hisoblang.

Mustaqil yechish uchun misollarning javoblari

24.1. $y=4x^2$, $y=x^2$. **24.2.** $y=\pm\sqrt{2x^2\pm\sqrt{4x^2-\sin x}}$. **24.3.** $y=\sqrt[5]{\ln(x^8+6)-x^4}$.

24.4. $y_{1,2}=\dfrac{3+\sqrt{9-16x^4}\pm\sqrt{\left(3+\sqrt{9-16x^4}\right)^2-16}}{4}$, $y_{3,4}=\dfrac{3-\sqrt{9-16x^4}\pm\sqrt{\left(3-\sqrt{9-16x^4}\right)^2-16}}{4}$.

24.5. $y=\dfrac{x^3\pm\sqrt{8x-\frac{1}{3}x^6}}{2}$, $z=\dfrac{x^3\mp\sqrt{8x-\frac{1}{3}x^6}}{2}$. **24.6.** $y_{1,2}=\dfrac{1}{2}\{\dfrac{-1+\sqrt{1+5x^6}}{2x}\pm$
$\pm\sqrt{\left(\dfrac{-1+\sqrt{1+5x^6}}{2x}\right)^2-4x^2}\}$, $y_{3,4}=\dfrac{1}{2}\{\dfrac{-1+\sqrt{1+5x^6}}{2x}\mp\sqrt{\left(\dfrac{-1+\sqrt{1+5x^6}}{2x}\right)^2-4x^2}\}$

24.7. Aniqlaydi. **24.8.** Aniqlamaydi. **24.9.** Aniqlaydi.
24.10. Aniqlamaydi. **24.11.** $f'_x(1;1)=\dfrac{6}{5}$, $f'_y(1;1)=-\dfrac{2}{5}$.

24.12. $f'_x(0;1)=1$, $f'_y(0;1)=0$. **24.13.** $f'_x(1;0)=1$, $f'_y(1;0)=\dfrac{\ln 2}{2}$.

24.14. $f'_x(1;-1)=f'_y(1;-1)=-\dfrac{1}{1+u_0}$, u_0 – $u+\ln u=0$ tenglamaning ildizi.

24.15. $\dfrac{4}{3}$. **24.16.** $-\dfrac{4}{5}$. **24.17.** $y'_x=\dfrac{x}{y}$, $y''_{x^2}=\dfrac{y^2-x^2}{y^3}$.

24.24. $y'_x=-\dfrac{x}{y}$, $y''_{x^2}=\dfrac{2y}{x^2}$. **24.19.** $y'_x=\dfrac{1}{2}$, $y''_{x^2}=0$.

24.20. $y'_x=\dfrac{x^2-y}{x-y^2}$, $y''_{x^2}=\dfrac{(x^2-y)(y^2-x)+2x(y^2-x)^2+2y(x^2-y)^2}{(x-y^2)^3}$

24.21. $\left.\dfrac{\partial z}{\partial x}\right|_{(1,1,1)}=\dfrac{1}{4}$, $\left.\dfrac{\partial z}{\partial y}\right|_{(1,1,1)}=-\dfrac{3}{4}$. **24.22.** $\left.\dfrac{\partial z}{\partial x}\right|_{(\pi,\pi,\pi)}=-1$, $\left.\dfrac{\partial z}{\partial y}\right|_{(\pi,\pi,\pi)}=-1$.

24.23. $\dfrac{\partial z}{\partial x}=\dfrac{3-x}{z}$, $\dfrac{\partial z}{\partial y}=\dfrac{-y}{z}$, $dz=\dfrac{1}{x}[(3-x)dx-ydy]$.

24.24. $\dfrac{\partial z}{\partial x}=1$, $\dfrac{\partial z}{\partial y}=\dfrac{y}{x-z}$, $dz=dx+\dfrac{y}{x-z}dy$.

24.25. $\dfrac{\partial z}{\partial x}=\dfrac{y}{2z}$, $\dfrac{\partial z}{\partial y}=\dfrac{x}{2z}$, $dz=\dfrac{ydx+xdy}{2z}$.

24.26. $\dfrac{\partial z}{\partial x}=\dfrac{2y-6xz}{3(x^2+y^2)}$, $\dfrac{\partial z}{\partial y}=\dfrac{2x}{3(x^2+y^2)}$, $dz=\dfrac{(2y-6xz)dx+2xdy}{3(x^2+y^2)}$.

24.27. $dz=\dfrac{xdx+ydy}{1-z}$, $d^2z=\dfrac{1-z+x^2}{(1-z)^2}dx^2++\dfrac{2xy}{(1-z)^3}dxdy+\dfrac{1-z+y^2}{(1-z)^2}dy^2$.

24.28. $dz=\dfrac{yzdx+xzdy}{z^2-xy}$,

$$d^2z=\frac{-2xy^3z}{(z-xy)^3}dx^2+2\frac{z(z^4-2xyz^2-x^2y^2)}{(z-xy)^3}dxdy+\frac{2x^3yz}{(z-xy)^3}dy^2.$$

24.29. $dz = \frac{yzdx + z^2dy}{y(x+z)}$, $d^2z = -\frac{z^2(ydx - xdy)}{(z - xy)^3}$.

24.30. $du = \frac{ydx + vdy}{1+y}$, $dv = \frac{dx - vdy}{1+y}$, $d^2u = \frac{2(dxdx - vdy^2)}{1+y^2}$, $d^2v = -d^2u$

24.31. $du = \frac{(y-u)dx + (y-v)dy}{x-y}$, $dv = \frac{(x-u)dx + (x-v)dy}{y-x}$.

$d^2v = -d^2u. = \frac{2}{(x-y)^2}\{(y-x)dx^2 + (y - v + u - x)dxdy + (u-x)dy^2\}$.

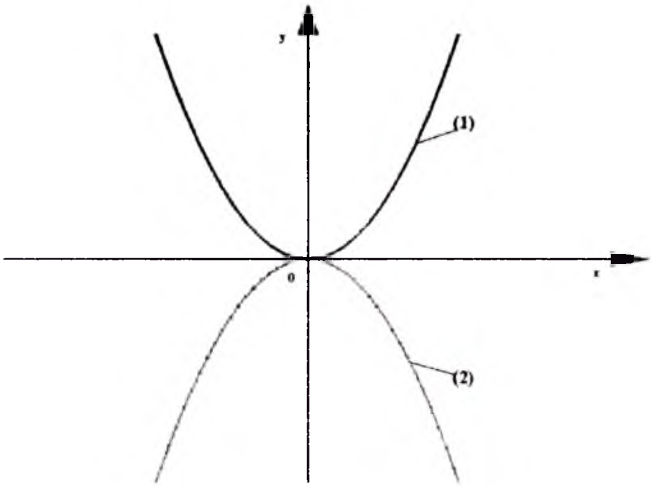
24.32. $du = \frac{(\sin v + x\cos v)dx - (\sin u - x\cos v)dy}{x\cos v + y\cos u}$,

$dv = \frac{(-\sin v + y\cos u)dx + (\sin u + y\cos u)dy}{x\cos v + y\cos u}$,

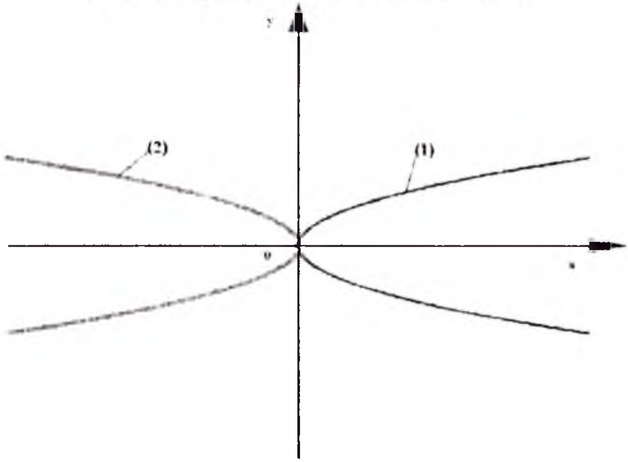
$d^2u = -d^2v = \frac{(2x\cos v + xdv\sin v)dv - (2dy\cos u - ydu\sin u)du}{x\cos v + y\cos u}$.

BA'ZI MUHIM CHIZIQLAR VA SIRTLAR

1. Parabola



1-chizma. $y = ax^2$, (1) $- a > 0$, (2) $- a < 0$.



2-chizma. $ay^2 = x$, (1) $- a > 0$, (2) $- a < 0$.

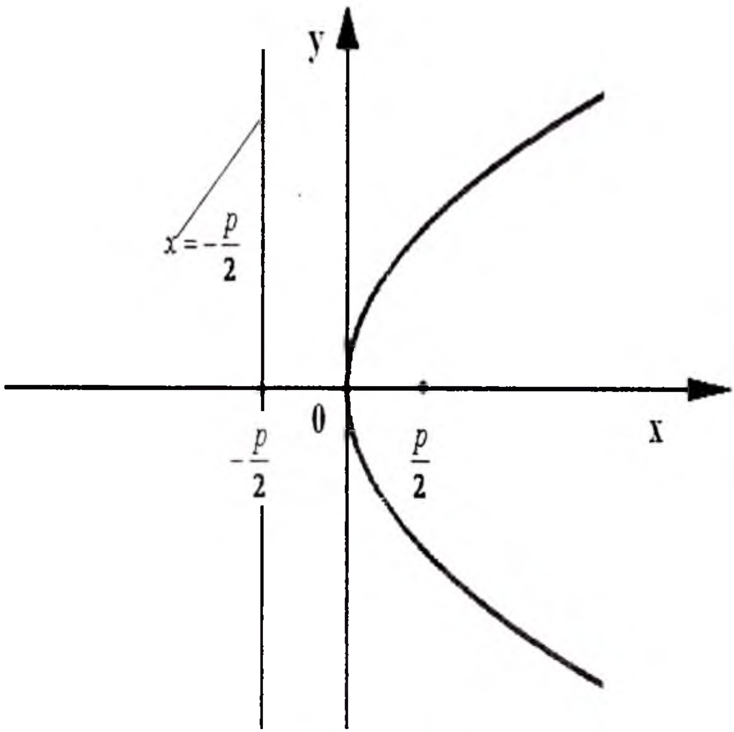
Parabolaning tenglamalari (3-chizma):

1) to‘g‘ri burchakli koordinatalar sistemasida $y^2 = 2px$;

2) qutb koordinatalar sistemasida $\rho = \dfrac{p}{1 - \cos\varphi}$;

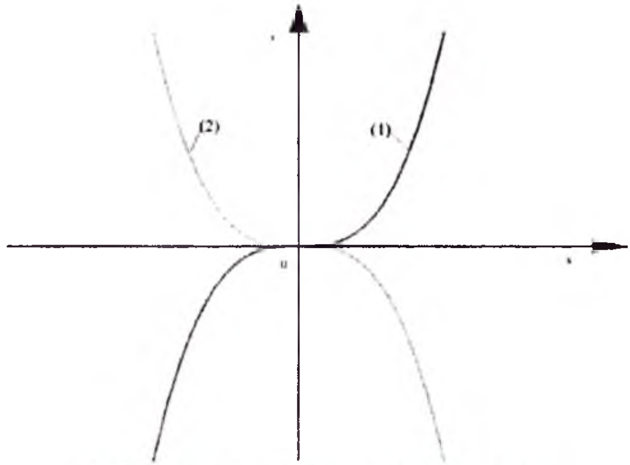
3) parametrik ko‘rinishda $\begin{cases} x = \dfrac{t}{2p} \\ y = t \end{cases}$

kabi ifodalanadi.

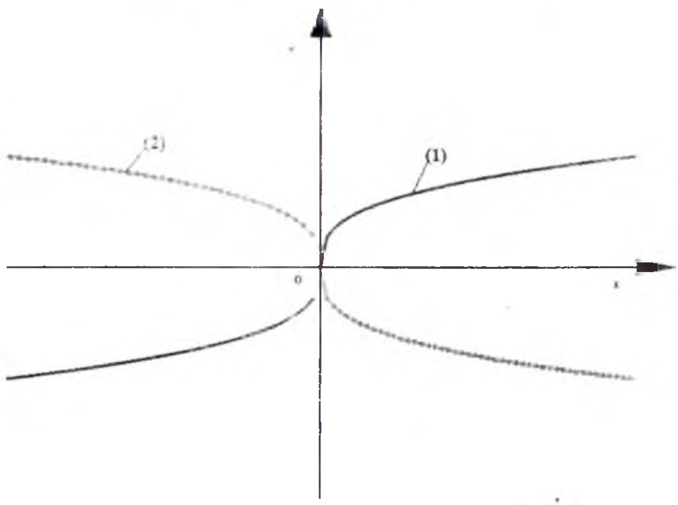


3-chizma.

2. Kubik parabola

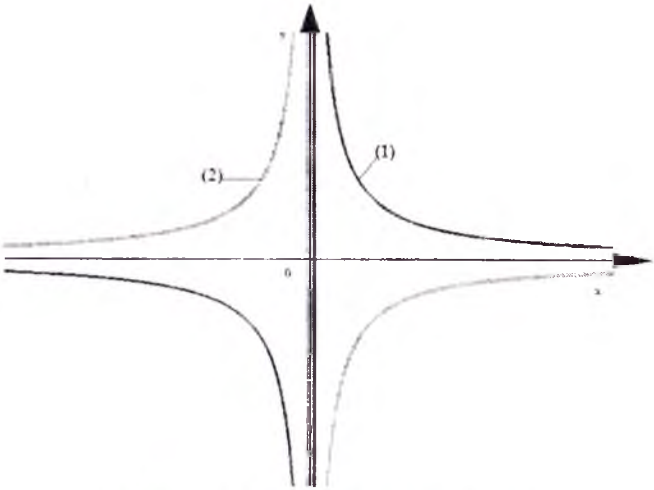


4-chizma. $y = ax^3$, (1) $-a > 0$, (2) $-a < 0$.

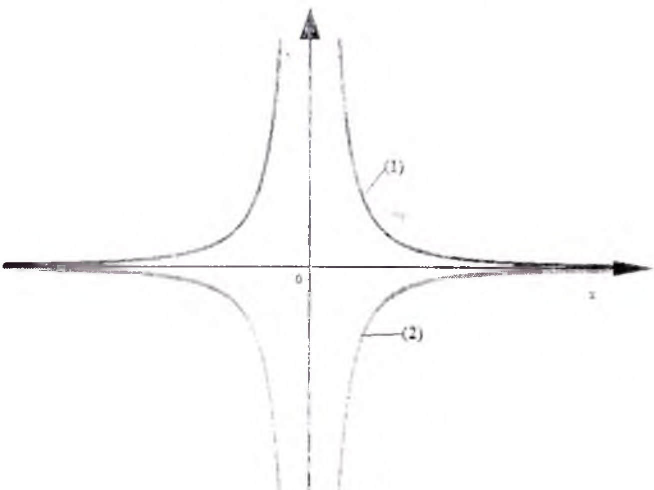


5-chizma. $ay^3 = x$, (1) $-a > 0$, (2) $-a < 0$.

3. Giperbolik funksiya



6-chizma. $y = \frac{a}{x}$, $(1) - a > 0$, $(2) - a < 0$.



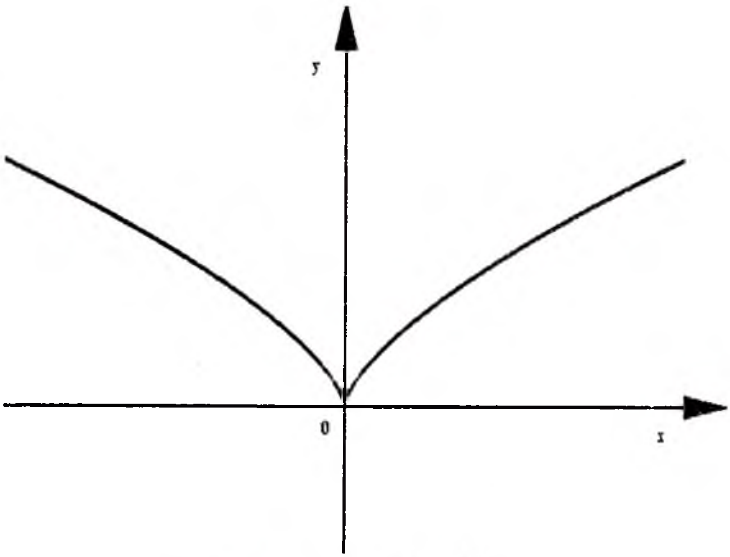
7-chizma. $y = \frac{a}{x^2}$, $(1) - a > 0$, $(2) - a < 0$.

4. Neyl parabolasi (Yarimkubik parabola)

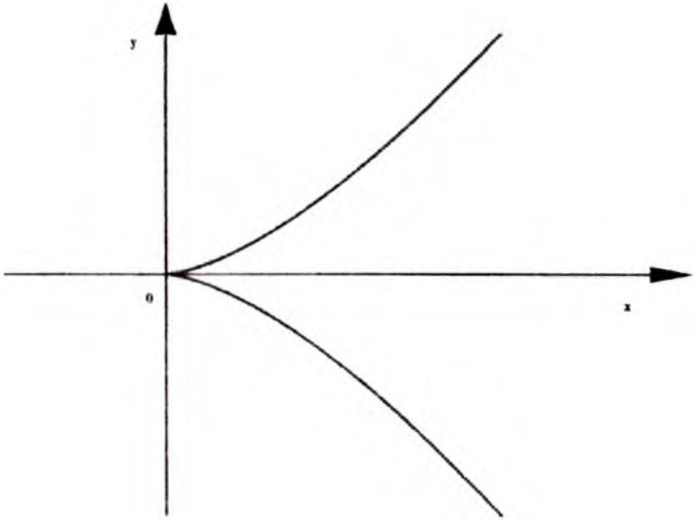
Yarimkubik parabola – qandaydir to‘g‘ri burchakli koordinatalar sistemasida $y^2 = ax^3$ tenglama orqali ifodalangan – tekis algebraik to‘g‘ri chiziqdan iborat.

1657 yilda uning yoy uzunligini topgan olim Neyl nomi bilan ataladi. Uning parametrik tenglamasi $x = t^2$, $y = at^3$ ko‘rinishda yoziladi.

Yarimkubik parabolaning koordinatalar boshidagi egrilik radiusi nolga teng.

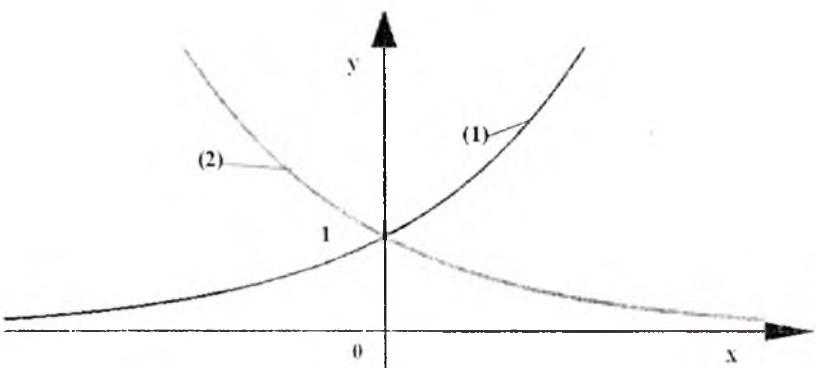


8-chizma. $y^3 = x^2$. $\left.\begin{matrix} x = t^3 \\ y = t^2 \end{matrix}\right\}$



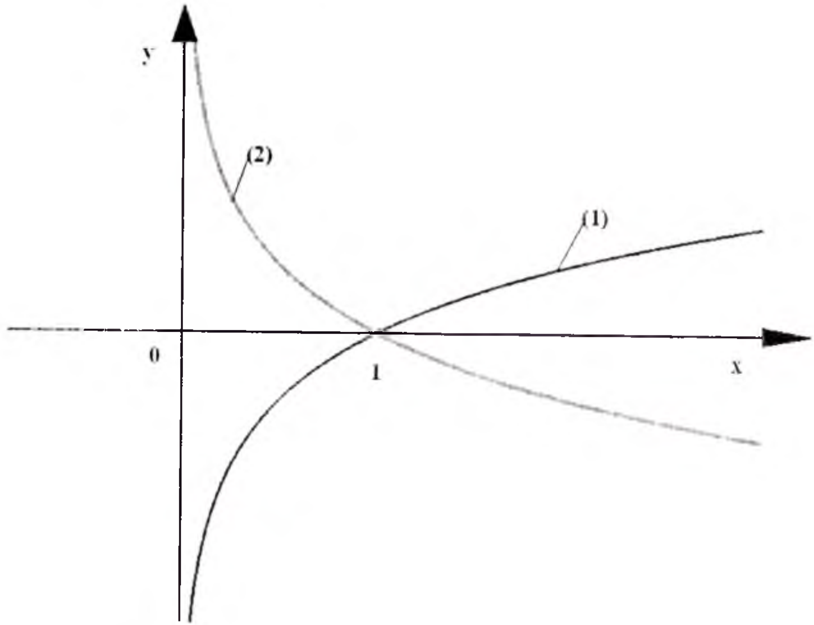
9-chizma. $y^2 = x^3$. $\left.\begin{matrix} x = t^2 \\ y = t^3 \end{matrix}\right\}$

5. Ko'satgichli funksiya



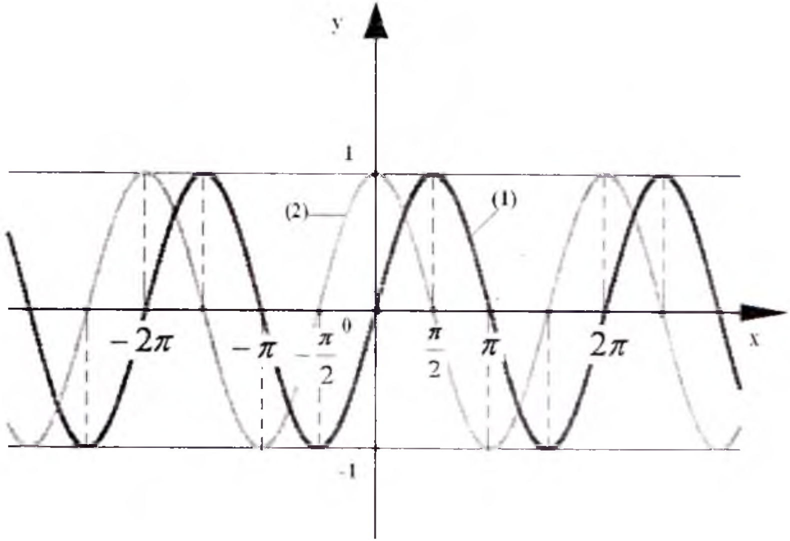
10-chizma. $y = a^x$, $(1) - a > 1$, $(2) - 0 < a < 1$.

6. Logarifmik funksiya



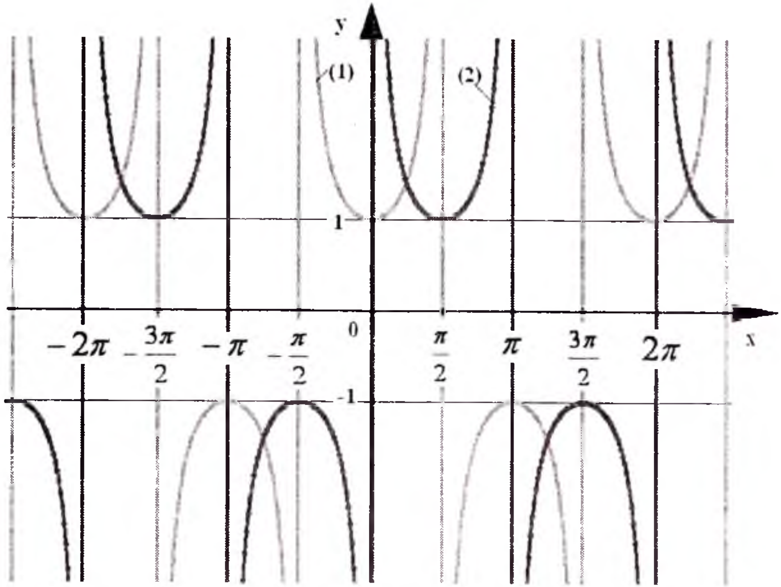
11-chizma. $y = \log_a x$, $(1) - a > 1$, $(2) - 0 < a < 1$.

7. Sinus va kosinus funksiyalar



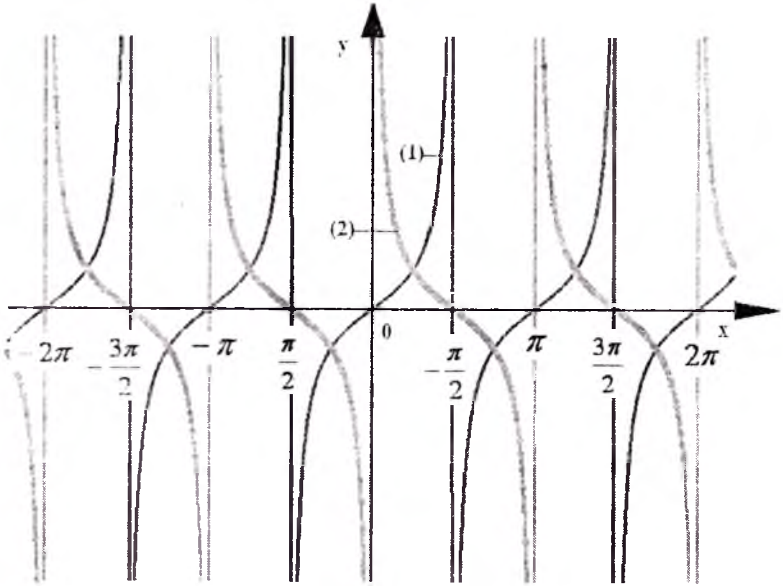
12-chizma. (1) $-\ y = \sin x$, (2) $-\ y = \cos x$.

8. Sekans va kosekans funksiyalar



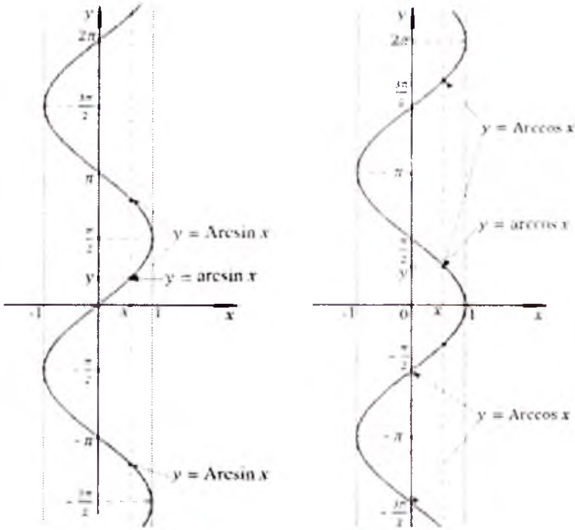
13-chizma. (1) $- y = \sec x = \frac{1}{\cos x}$, (2) $- y = \operatorname{cosec} x = \frac{1}{\sin x}$.

9. Tangens va kotangens funksiya

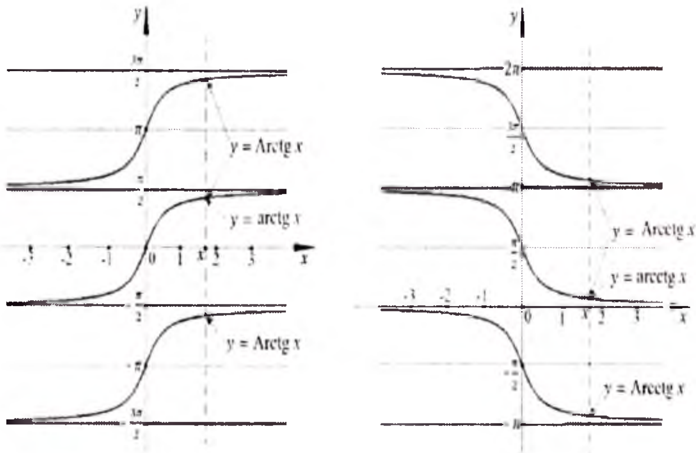


14-chizma. (1) – $y = tgx$, (2) – $y = ctgx$.

10. Teskari trigonometrik funksiyalar



15-chizma. $y = Arc\sin x, \quad y = Arc\cos x$.



16-chizma. $y = Arctgx, \quad y = Arcctgx$.

11. Giperbolik sinus va giperbolik kosinus funksiyalar (Zanjirli chiziq)

Zanjirli chiziq – bir jinsli gravitasion maydonda uchlari mahkamlangan egiluvchan, bir jinsli va cho'zilmaydigan og'ir ip yoki zanjir hosil qiladigan chiziqdan iborat. U tekis transsendent chiziqdir.

Dekart koordinatalar sistemasidagi tenglamasi :

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = ach\frac{x}{a}$$

U quyidagi xossalarga ega :

- uning uchidan ixtiyoriy (x,u) nuqtasigacha bo'lgan yoy uzunligi

$$S = ash\frac{x}{a} = \sqrt{y^2 - a^2}$$

formula orqali topiladi.

- uning egrilik radiusi

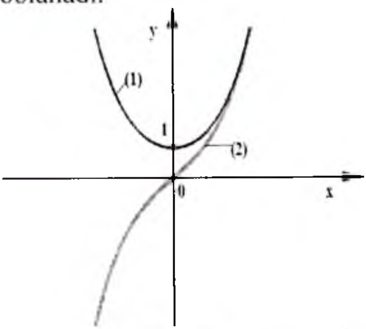
$$R = ach^2\frac{x}{a} = \frac{y^2}{a}$$

formula bo'yicha hisoblanadi.

- zanjirli chiziq, uning ikkita ordinatalari va abssissalar o'qi bilan chegaralangan sohaning

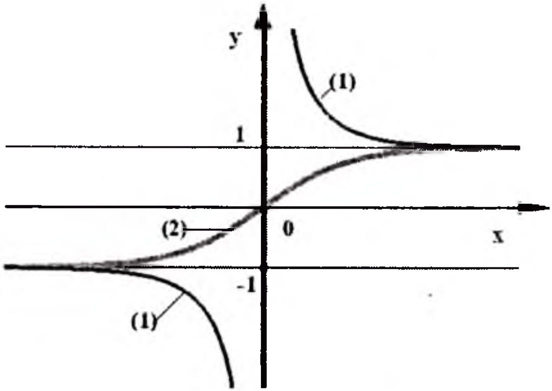
$$S = a^2\left(sh\frac{x_1}{a} - sh\frac{x_2}{a}\right) = a\left(\sqrt{y_2^2 - a^2} - \sqrt{y_1^2 - a^2}\right)$$

formula bo'yicha hisoblanadi.



17-chizma. $(1) - y = chx = \frac{e^x + e^{-x}}{2}$, $(2) - y = shx = \frac{e^x - e^{-x}}{2}$.

12. Giperbolik tangens va giperbolik kotangens funksiyalar



18-chizma. $(1) - y = cthx = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, $(2) - y = thx = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

13. Dekart yaprog'i

Tarixiy ma'lumotlar. Qaralayotgan chiziq matematika tarixida birinchi marta R. Dekartning Fermaga 1638 yilda yuborgan xatida uchraydi. Chiziqning shakli haqida birinchi tadqiqot Robelval tomonidan amalga oshirilgan. Lekin u chiziqni faqat sirtmoqdan iborat, deb tushungan. Sirtmoqni to'rtta kvadratda takrorlab, Roberval to'rt yaproqli gulni eslatuvchi shaklga ega bo'lgan. Shuning uchun chiziqqa «yaprog'i» nomi berilgan. Lekin chiziqning to'la shakli keyinroq X.Gyuygens va I.Bernulli tomonidan aniqlangan. «Dekart yaprog'i» nomi faqat XVIII asrning boshidan boshlab qo'llanila boshlandi.

Dekart yaprog'i – uchinchi tartibli tekis egri chiziqdan iborat va u to'g'ri burchakli koordinatalar sistemasida $x^3 + y^3 = 3axy$ tenglamani qanoatlantiradi. 3a parametr – tomoni sirtmoqning eng katta vatariga teng bo'lgan kvadratning diagonali kabi aniqlanadi.

Uning tenglamalari :

1) to'g'ri burchakli koordinatalar sistemasida: $x^3 + y^3 = 3axy$;

2) qutb koordinatalar sistemasida: $\rho = \dfrac{3a\cos\varphi\sin\varphi}{\cos^3\varphi + \sin^3\varphi}$;

3) parametrik ko'rinishda

$$\begin{cases} x = \dfrac{3at}{1+t^3}, \\ y = \dfrac{3at^2}{1+t^3}, t = tg\ \varphi \end{cases}$$

kabi ifodalanadi.

Ko'pincha 135^0 ga burilgan egra chiziq qaraladi. Uning tenglamalari quyidagicha bo'ladi :

- to'g'ri burchakli sistemada

$$y = \pm x\sqrt{\frac{l+x}{l-3x}}, l = \frac{3a}{\sqrt{2}};$$

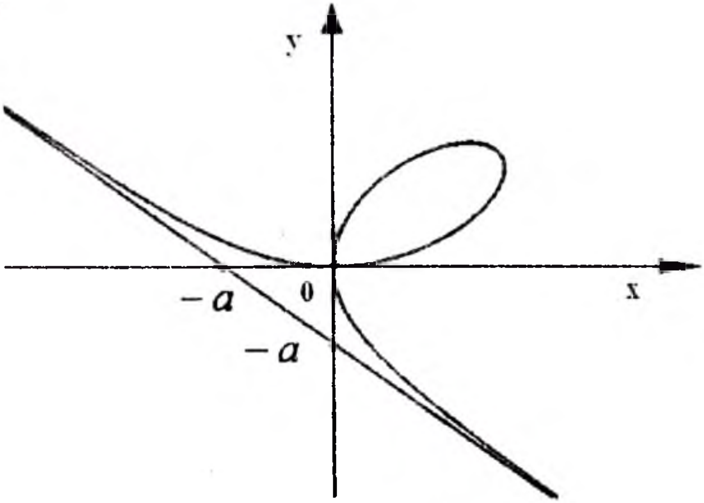
- qutb koordinatalari sistemasida

$$\rho = \frac{l(\sin^2\varphi - \cos^2\varphi)}{\cos\varphi(\cos^2\varphi + 3\sin^2\varphi)};$$

- parametrik shaklda

$$x = l\frac{t^2-1}{3t^2+1}; y = l\frac{t(t^2-1)}{3t^2+1}$$

Dekart yaprog'i – yasmin guli deb ham atalgan (inglizcha jasmine flower).



19-chizma. Dekart yaprog'i.

14. Diokl sissoidasi

Tarixiy ma'lumotlar. Sissoida qadimgi olimlar tomonidan (eramizgacha bo'lgan V-asr) o'sha zamondagi mashhur masalalardan birini yechish jarayonida ochilgan. Bunday masalalar quyidagilardan iborat: doiraning kvadraturasi (izi berilgagn doiraning yuziga teng bo'lgan kvadrat yasash) masalasi, burchakning triseksiyasi (berilgan burchakni teng uchga bo'lish) masalasi, kubni ikkilantirish (hajmi berilgan kub hajmidan ikki marta katta bo'lgan kub yasash) masalasi. Bu qo'yilgan masalalarni sirkul va chizg'ich yordamida yechish talab qilingan.

Kubni ikkilantirish masalasini yechish maqsadida sissondaning nashr qilinishi eramizdan oldingi II asrda ijod qilgan qadimgi dunyo yunon geometri Diokles nomi bilan bog'liq. Shu sababdan chiziq Diokles sissoidasi deb ham yuritiladi.

Diokl sissoidasi – uchinchi tartibli tekis algebraik egri chiziqdan iborat .

Sissoidaning to'g'ri burchakli koordinatalar sistemasidagi tenglamasi quyidagicha yoziladi : $y^2 = \dfrac{x^3}{2a - x}$.

Uning qutb koordinatalar sistemasidagi tenglamasi $\rho = \dfrac{2a\sin^2\varphi}{\cos\varphi}$ ko'rinishda bo'lib, ba'zan

$$\rho = \frac{2a(1-\cos^2\varphi)}{\cos\varphi} = 2a(\frac{1}{\cos\varphi} - \cos\varphi) = 2a(\sin\varphi - \cos\varphi)$$

ko'rinishda ham yoziladi.

Sissoidaning parametrik tenglamasi

$$x = \frac{2a}{1+u^2}, y = \frac{2a}{u(1+u^2)},$$ bunda $u = tg\varphi$

ko'rinishga ega.

Sissoidaning hozirgi ko'rinishi fransuz matematiga J.Robervaya tomonidan 1650 yilda berilgan.

Sissoida – absissalar o'qiga nisbatan simmetrik. U yordamchi aylanani, uning diametrida yotuvchi B va D nuqtalarda kesib o'tadi, hamda bitta UV (chizmaga q.) asimptotaga ega. Asimptotp tengdamasi : $x = 2a$, a – yordamchi aylananig radiusi.

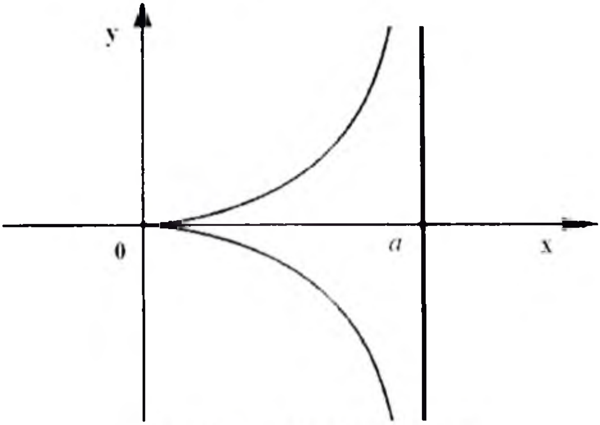
Sissoida va uning asimptotasi orasidagi yuza

$$S_1 = 3\pi a^2$$

Sissoida OL tarmog‘ining (18-chizmaga q.) absissalar o‘qi atrofida aylanishidan hosil bo‘lgan jismning V_1 hajmi quyidagicha hisoblanadi :

$$V_1 = \pi \int_0^{2a} \frac{x^3}{2a-x} dx = \pi \int_0^{2a} (-x^2 - 2ax - 4a^2 + \frac{8a^2}{2a-x}) dx = -\frac{44\pi a^3}{3} - 8\pi a^3 (\ln(2a-x)) \Big|_0^{2a}$$

$x \to 2a$ da $\ln(2a-x) \to -\infty$, y’ani $V_1 \to \infty$.



20-chizma. Diokl sissoidasi.

15. Strofonda

Tarixiy ma’lumotlar. Strofondagi birinchi marta 1645 yilda italiya matematigi va fizigi E. Torrichelli (1608-1647 yy) tadqiq qilgan. Chiziq uzoq vaqt davomida «Torrichelli qanoti» degan nom bilan yuritilgan «Strofonda» atamasi faqat XIX asrning o‘rtalaridan qo‘llanila boshlandi.

E. Torrichelli Faensda tug‘ilib matematik ta’limini Rimda oldi. E. Torrichelli atoqli fizik olim bo‘lishi bilan birga ajoyib matematik kashfiyotlar muallifidir.

U ko‘p jismlarning hajmlarini (jumladan, cheksiz jismlarning ham), egri chiziqlar bilan chegaralangan shakllar yuzlarini hisoblash, chiziqlar yoylari uzunliklarini topish usullari mualifidir.

Strofoida ko‘p sonli geometrik tadbiqlardan tashqari, optika va chizma geometriyaning ba’zi masalalarida ham uchraydi.

Strofoida (yunonchadan - burilish) – uchinchi tartibli algebraik egri chiziq.

To‘g‘ri burchakli koordinatalar sistemasida to‘g‘ri strofoida, yoki strofoida quriladi (yasaladi) (21-chizma).

Qiyshiq burchakli koordinatalar sistemasida qiyshiq strofoida yasaladi (2-chizma).

O – koordinatalar boshi, absissalar o‘qi – OV nur bo‘ylab, ordinatalar o‘qi – OD nur bo‘ylab yo‘nalganda va α = ∠AOD (to‘g‘ri burchakli koordinatalar sistemasi uchun $\alpha = \frac{\pi}{2}$) bo‘lganda, strofoidaning Dekart koordinatalar sistemasidagi tenglamasi

$$y^2(x-a) - 2x^2 y\cos\alpha + x^2(a+x) = 0$$

ko‘rinishida yoziladi.

To‘g‘ri strofoida tenglamasi

$$y = \pm x\sqrt{\frac{a+x}{a-x}}$$

ko‘rinishni oladi.

Strofoidaning qutb koordinatalar sistemasidagi tenglamasi

$$\rho = -\frac{a\cos 2\varphi}{\cos\varphi};$$

Uning parametrik tenglamasi

$$x = a\left(\frac{u^2-1}{u^2+1}\right), \quad y = au\left(\frac{u^2-1}{u^2+1}\right), \quad u = tg\,\varphi$$

ko‘rinishga ega.

Strofoida dastlab fransuz matematigi Jil Roberval tomonidan 1645 yilda qaralgan. Roberval bu egri chiziqni “Pteroida” – qanot deb atagan. Strofoida nomi fanga birinchi marta 1849 yilda kiritilgan.

Strofoidaning to‘g‘ri urinmasini topish quyidagicha amalga oshiriladi. To‘g‘ri strofoidaning Dekart koordinatalar sistemasidagi tenglamasidan

$$y' = \sqrt{\frac{a+x}{a-x}}\left(1 + \frac{ax}{a^2-x^2}\right)$$

ekanligini topamiz va bu hosilaning O(0,0) nuqtalarida qiymatlarini hisoblasak, $y' = \pm 1$ ekanligini, ya’ni O(0,0) ikkita perpendikulyar urinmalar mavjudligini ko‘ramiz, urinmalarning og‘ish burchagi $\pm\frac{\pi}{4}$ ga tengligini olamiz.

Strofoidaning egrilik radiusi, ya'ni R=ON ning O(0,0) nuqtadagi qiymati

$$R = \frac{a}{\cos \angle AON} = \frac{a}{\cos \frac{\pi}{4}} = a\sqrt{2}$$

bo'ladi.

To'g'ri strofoida sirtmog'ining ordinatalar o'qidan chapdagi yuzi

$$S_1 = a^2\left(2 - \frac{\pi}{2}\right)$$

formula orqali, strofoida va ordinatalar o'qidan o'ngda yotuvchi asimtota orasidagi yuza

$$S_1 = a^2\left(2 + \frac{\pi}{2}\right)$$

formula orqali topiladi.

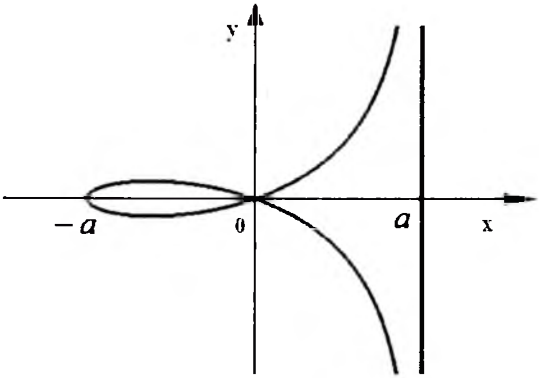
OM_1A yoyning abssissalar o'qi atrofida aylanishidan hosil bo'lgan jismning V_1 hajmi,

$$V_1 = \pi \int_{-a}^{0}\left(x\sqrt{\frac{a+x}{a-x}}\right)^2 dx = \pi \int_{-a}^{0} x^2 \frac{a+x}{a-x} dx =$$

$$-\pi \int_{-a}^{0} x^2 dx - 2\pi a \int_{-a}^{0} x dx - 2\pi a^2 \int_{-a}^{0} dx + 2\pi a^3 \int_{-a}^{0} \frac{dx}{a-x} =$$

$$-\frac{a^3\pi}{3} + a^3\pi - 2a^3\pi + 2a^3\pi \ln 2$$

Demak, $$V_1 = a^3\pi\left(2\ln 2 - \frac{4}{3}\right)$$



21-chizma. Strofoida.

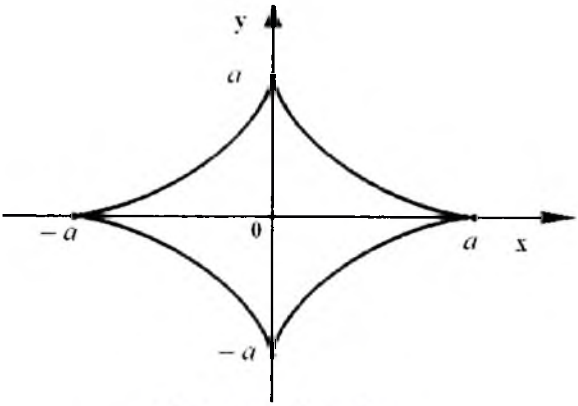
16.Astroida

Astroidaning tenglamalari (22-chizma):

1) to‘g‘ri burchakli koordinatalar sistemasida: $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, $(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0$;

2) parametrik ko‘rinishda $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t \end{cases}$

kabi ifodalanadi.



22-chizma. Astroida.

17. Sikloida

Tarixiy ma'lumotlar. Atoqli italiya olimi Galillo Galiyey (1564-1642yy) birinchi bo'lib, fanga «sikloida» (yoki «doirani eslatuvchi») atamasini kiritgan.

Galiley sikloidaning bitta ravog'i va uning asosi blilan chegaralangan shaklning yuzi, uni yasovchi doira yuzidan uch marta katta ekanligi ko'rsatgan.

XVII asrning 30- yillaridan boshlab sikloida eng ommaviy chiziqlardan biriga aylanadi, ko'p matematiklar bu chiziqda o'zlarining yangi usullari kuchini sinab ko'rishgan. Galiley teoremasining ajoyib isboti Roberval (1634 y) tomonidan va unga bog'liq bo'lmagan holda (va o'zlari bilmagan holda) Ferma va Dekart (1638y) tomonidan berilgan. Ular sikloidaga urinma yasash usullarini ham ko'rsatishgan. B. Paskal tomonidan siklaidaning aylanishidan hosil bo'lgan jismning hajmi va sirtqi hisoblangan hamda ularning og'irlik markazlari topilgan. Siklaodaning to'g'irlanuvchanligi (ya'ni uning yoyi uzunligini topish) Ren (1658y) tomonidan amalga oshirilgan.

Sikloidani tekshirish bilan boshqa olimlar ham shug'ullanishgan. X.Gyuygens va I. Bergulli uning muhim mexanik hissalarini aniqlashgan.

Sikloida (yunon tilida "yumaloq") – tekis **transsendent** egri chiziqdan iborat. U, kinematik tarzda, r radiusli "hosil qiluvchi", sirpanmasdan to'g'ri chiziq bo'ylab dumalaydigan, aylananing belgilangan nuqtasining trayektoriyasi sifatida aniqlanadi (23-chizmaga qarang).

Uning tenglamalarini keltirib chiqarish uchun, koordinatalar sistemasining gorizantal o'qini r radiusli "hosil qiluvchi" aylana aylanadigan to'g'ri chiziq, deb qabul qilamiz.

- Sikloidaning parametrik tenglamasi :

$$\begin{cases} x = rt - r\sin t, \\ y = r - r\cos t \end{cases}$$

- Dekart koordinatalar sistemasidagi tenglamasi :

$$x = r\arccos\frac{r-y}{r} - \sqrt{2ry - y^2}$$

- Sikloida quyidagi

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r-y}{y}$$

ko'rinishidagi oddiy differensial tenglamaning yechimi kabi olinishi ham mumkin.

Sikloida quyidagi xossalarga ega :

1. Sikloida – abussissalar o'qi bo'ylab davriy funksiya va uning davri $2\pi r$ ga teng.

2. Sikloidaga uning ixtiyoriy A nuqtasida urinma o'tkazish uchun, shu nuqtani "hosil qiluvchi" aylananing yuqori nuqtasi bilan tutashtirish yetarli. A nuqtani "hosil qiluvchi" aylananing quyi nuqtasi bilan tutashtirganda biz normalga ega bo'lamiz.

3. Sikloida arkining uzunligi $8r$ ga teng. Bu xossa 1658 yilda Kristofer Rek tomonidan ochilgan.

4. Sikloidaning har bitti arki tagidagi yuza, "hosil qiluvchi" doira yuzidan uch marta katta. Torichellining fikricha, bu xossa Galiley tomonidan ochilgan.

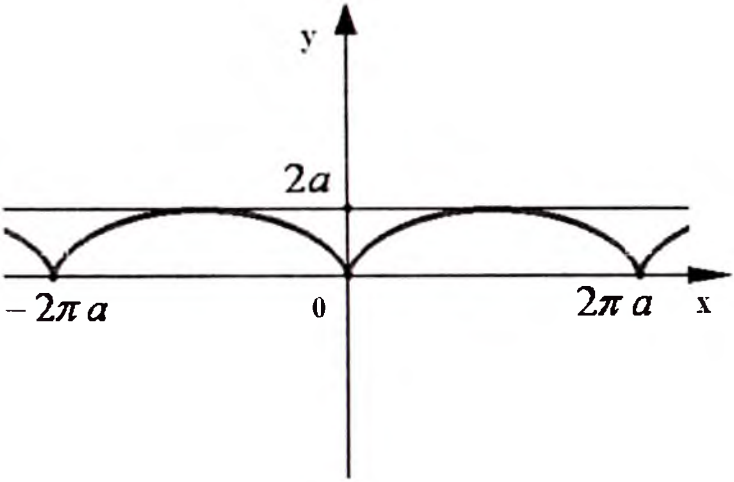
5. Sikloida birinchi arkining egrilik radiusi $4r\sin\frac{t}{2}$ ga teng.

6."To'nkarilgan" sikloida – eng tez **tushish egri** chizig'i (braxistoxrona)dan iborat.

7.Sikloidaning **evolyutasi**, berilgan sikloidaga **kongruent**, sikloidadan iborat.

Olimlar ichida, sikloidaga e'tiborni, birinchi bo'lib, XV asrda Nikolay Kuzanskiy qaratgan, lekin bu egri chiziqni tadqiq qilish, asosan, XVII asrda boshlangan. Sikloida nomini Galiley o'ylab topgan (Fransiyada bu egri chiziqni, dastlab, **ruletta** deb atagan). Sikloidani batafsil tadqiq qilish Galileyning zamondoshi Mersenn tomonidan amalga oshirilgan. U – transsendent egri chiziqlar , ya'ni tenglamasi x,u larga nisbatan ko'phad shaklida ifodalanmaydigan egri chiziqlar, ichida birinchi tadqiq etilganidir.

Sikloidani tadqiq qilishda , XVII – XVIII asrlarda ijod qilgan buyuk fan **darg'alari** Dekart, Ferma, Nyuton, Leybnis hamda aka-uka Bernullilar ham ishtirok etishgan.



23-chizma. Sikloida.

18.Giposikloida

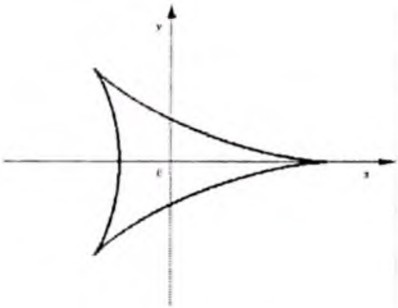
Giposikloida (yunonchadan doiraning, aylananing tagi, pasti) – aylananing nuqtasi, boshqa bir aylananing ichki tomonida sirpanmasdan dumalaganda hosil qilinadigan tekis egri chiziqdan iborat (24 - chizma).

Uning parametrik tenglamasi

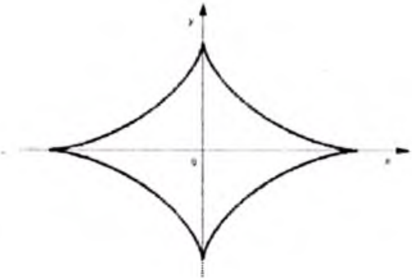
$$\begin{cases} x = r(k-1)\left(\cos t + \dfrac{\cos(k-1)t}{k-1}\right), \\ y = r(k-1)\left(\sin t - \dfrac{\sin(k-1)t}{k-1}\right) \end{cases}$$

ko‘rinishga ega, bunda $k = \frac{R}{r}$, R – qo‘zg‘almas aylananing radiusi, r – aylanayotgan aylananing radiusi.

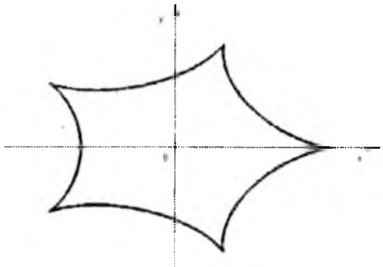
k miqdorning moduli giposikloidaning shaklini aniqlaydi. k=2 bo‘lganda giposikloida qo‘zg‘almas aylananing diametridan iborat, k=4 bo‘lganda esa, u astroidaga aylanadi. k ning har xil qiymatlariga mos kelgan giposikloidalar 2-chizmada keltirilgan.



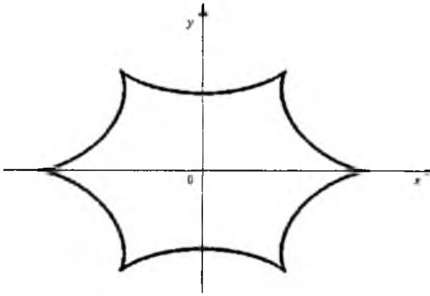
1) $r=1,\ R=3, k=\dfrac{R}{r}=3.$



2) $r=1,\ R=4, k=\dfrac{R}{r}=4.$



3) $r=1,\ R=5, k=\dfrac{R}{r}=5.$



4) $r=1,\ R=6, k=\dfrac{R}{r}=6.$

24-chizma. Giposikloida

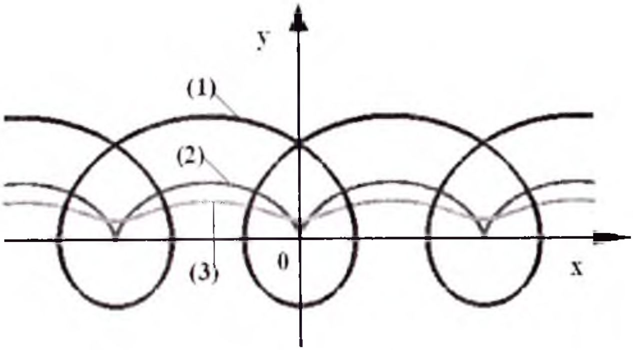
19. Troxoida

Troxoida – quyidagi

$$\begin{cases} x = at - \lambda \sin t \\ y = a - \lambda \cos t \end{cases}$$

parametrik tenglamalar orqali berilgan tekis transsendensi chiziqdan iborat.

$a=\lambda$ bo'lganda troxoida sikloidaga o'tadi. $\lambda > a$ bo'lganda troxoida uzaytirilgan sikloida deb, $\lambda < a$ bo'lganda esa, qisqartirilgan sikloida deb ataladi.



25-chizma. (1)- $a=2,\ \lambda=5$, (2)- $a=2,\ \lambda=2$,(3)- $a=2,\ \lambda=0{,}5$.

20. Ellips

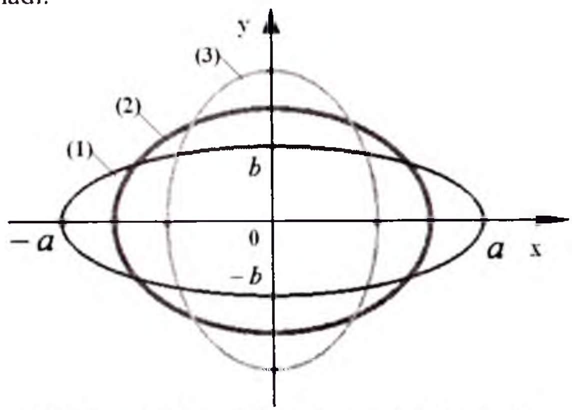
Ellipsning tenglamalari (26-chizma):

1) to'g'ri burchakli koordinatalar sistemasida: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

2) qutb koordinatalar sistemasida: $\rho=\frac{b^2}{a-c\cos\varphi}$;

3) parametrik ko'rinishda: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$

kabi ifodalanadi.



26-chizma. (1)- $a>b$, (2)- $a=b$,(3)- $a<b$.

21.Giperbola

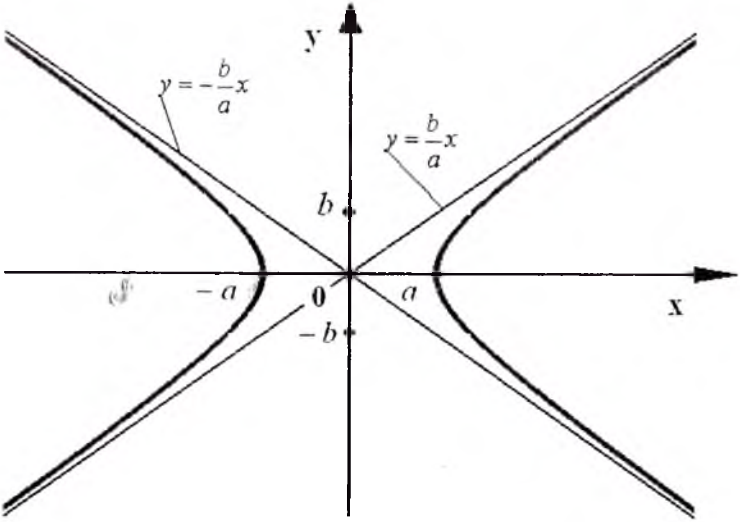
Giperbolaning tenglamalari (27-chizma):

1) to‘g‘ri burchakli koordinatalar sistemasida $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$;

2) qutb koordinatalar sistemasida $\rho=\frac{b^2}{a-c\cos\varphi}$;

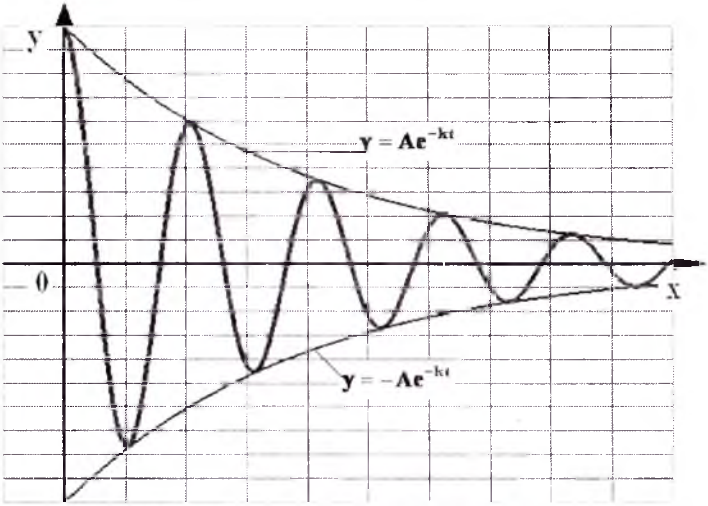
3) parametrik ko‘rinishda $\begin{cases} x = acht \\ y = bsht \end{cases}$

kabi ifodalanadi.



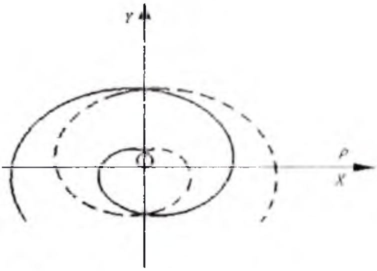
27-chizma.

22. Garmonik tebranish

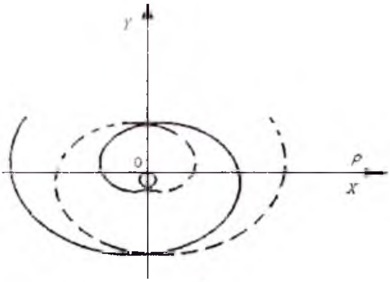


28-chizma. $y = Ae^{-kt}\sin(wt + \alpha)$

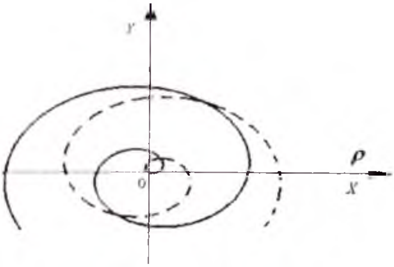
23. Arximed spirallari.



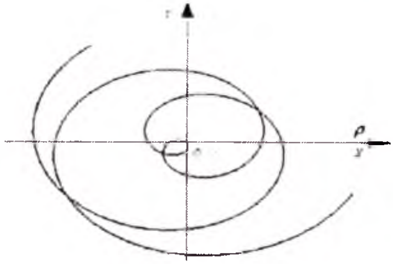
1) $\rho = a\varphi$ $(a > 0)$.



2) $\rho = a\varphi$ $(a < 0)$.



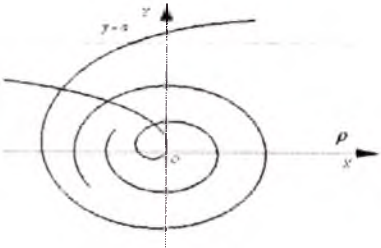
3) $\rho = a\varphi + l \ (a > 0,\ l > 0)$.



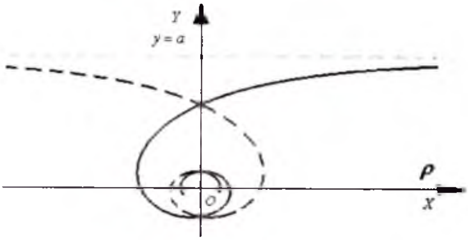
4) $\rho = a\varphi + l \ (a > 0,\ l < 0)$.

29- chizma. Arximed spirallari.

24. Giperbolik spirallar



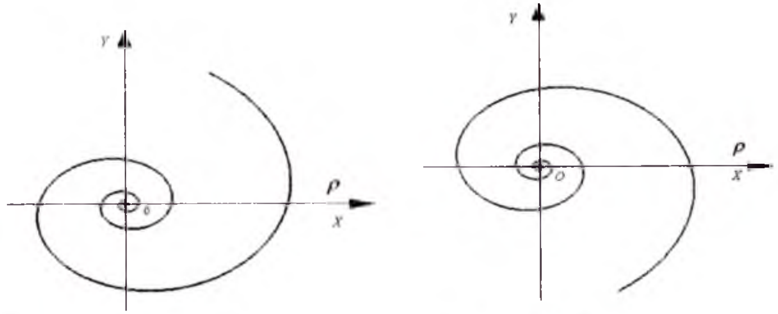
1) $\rho = \dfrac{a}{\varphi} \ (a > 0)$.



1) $\rho = \dfrac{a}{\varphi} + l \ (a > 0, l > 0)$.

30- chizma. Giperbolik spirallar- 1), 2).

25. Logarifmik spirallari

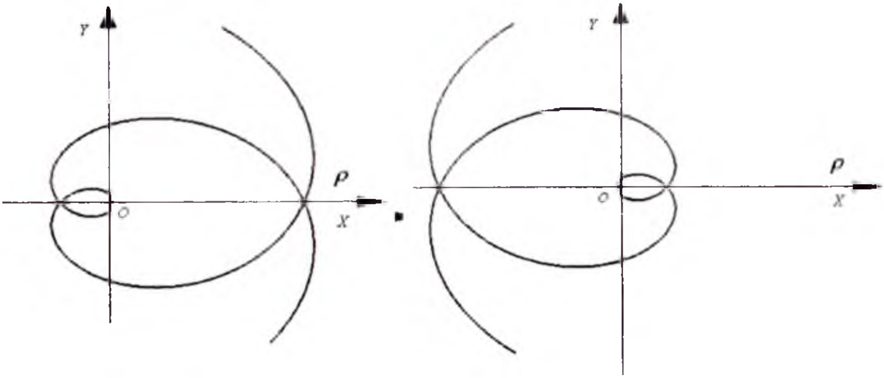


1) $\rho = b^{a\varphi}$ $(b > 0,\ a > 0)$.

2) $\rho = b^{a\varphi}$ $(b > 0,\ a < 0)$.

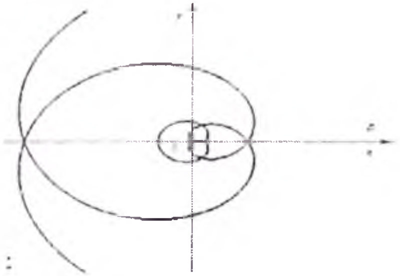
31- chizma. Logarifmik spirallari- 1), 2).

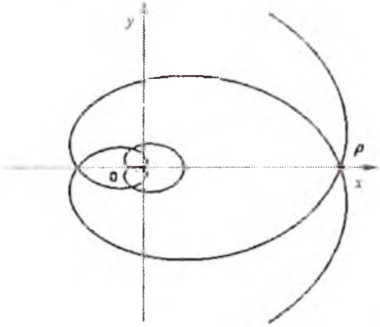
26. Galiley spirallari



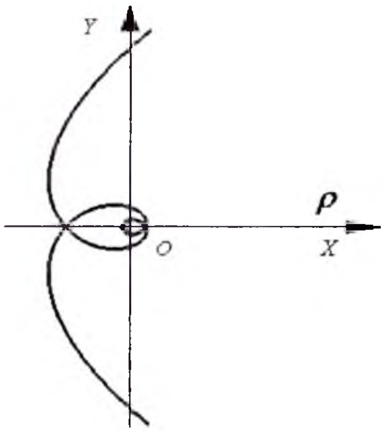
1) $\rho = a\,\varphi^2$ $(a > 0)$.

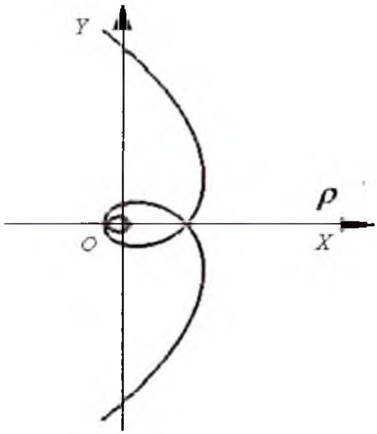
2) $\rho = a\,\varphi^2$ $(a < 0)$.





3) $\rho = a\varphi^2 - l$ $(a = 0{,}2 > 0,\ l = 3 > 0)$. 4) $\rho = a\varphi^2 - l$ $(a = -0{,}2 < 0,\ l = -3 < 0)$.



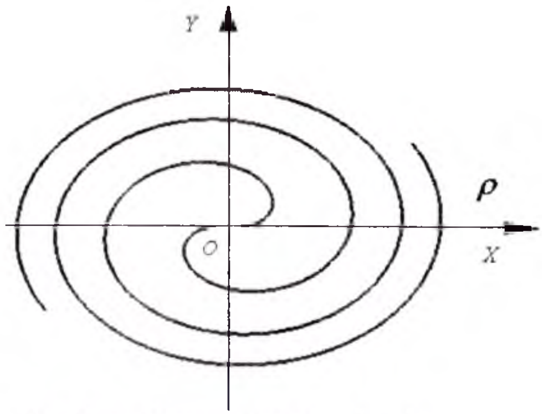


5) $\rho = \dfrac{a}{\varphi^2}$ $(a > 0)$.

6) $\rho = \dfrac{a}{\varphi^2}$ $(a < 0)$.

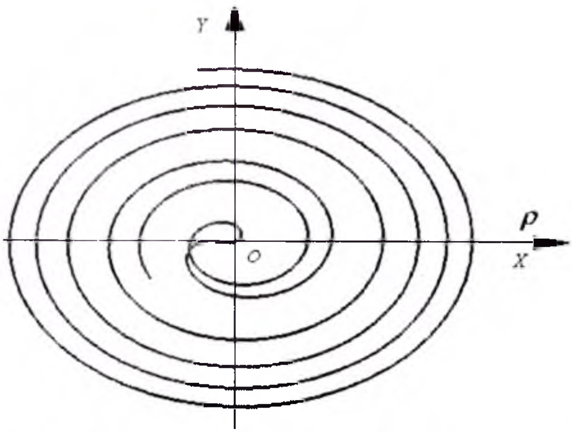
32- chizma. Galiley spirallari- 1), 2), 3), 4) 5) 6).

27. Ferma spirali



33- chizma. Ferma spirali. $\rho^2 = a^2\varphi$.

28. Parabolik spirali

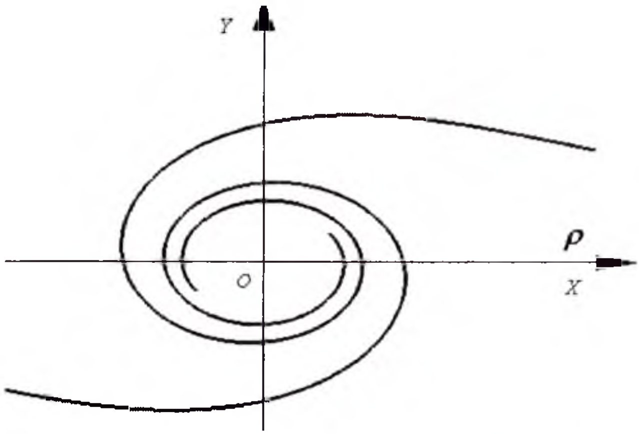


34- chizma. Parabolik spirali. $(\rho - l)^2 = a^2\varphi \ (l > 0)$

29. Jezl spirali

Jezl – qutb koordinatalar sistemasida ushbu $\rho = \frac{a}{\sqrt{\varphi}}$ tenglama bilan ifodalanadigan tekis transsendent egri chiziqdir.

Egri chiziq cheksizlikdan (u yerda u gorizontal o'qqa asimtotik yaqinlashadi) $(0,0)$ nuqtaga, uning atrofida spiral bo'yicha soat miliga teskari yo'nalishda aylangan holda, intiladi. Spiralning kattaligi a koeffisiyent bo'yicha aniqlanadi. Egri chiziq fazasi bitta, $\left(\frac{1}{2}; a\sqrt{2}\right)$ egilish nuqtasiga ega. U algebraik spirallar oilasiga mansub.



35- chizma. Jezl spirali. $\rho^2 = \frac{a^2}{\varphi} (a > 0)$.

30. Sinusoidal spiral

Geometriyada , sinusoidal spiral - qutb koordinatalar sistemasida

$$r^n = a^n \cos(n\Theta)$$

tenglama orqali aniqlanadigan egri chiziqlar oilasidan iborat, bunda a – noldan farqli o'zgarmas son va n – nolga teng bo'lmagan rasional son.

Agar egri chiziqni koordinatalar boshiga nisbatan burish imkoniyati hisobga olinsa, uning tenglamasi

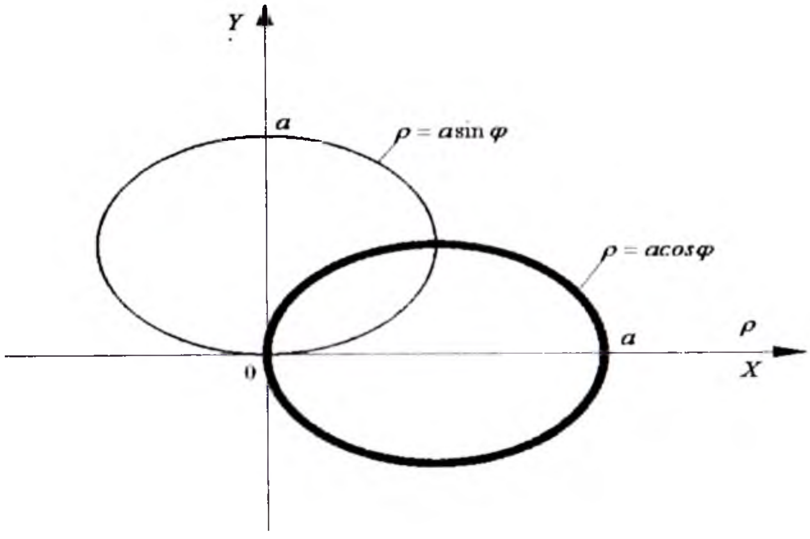
$$r^n = a^n \sin(n\Theta)$$

ko'rinishda yozilishi ham mumkin.

Ko‘p ma’lum egri chiziqlar sinusoidal spiralning xususiy hollaridan iborat :

- to‘g‘ri chiziq ($n=-1$) ;
- aylana ($n=1$) ;
- giperbola ($n=-2$) ;
- parabola $\left(n=-\frac{1}{2}\right)$;
- kardioida $\left(n=\frac{1}{2}\right)$;
- Bernulli lemniskatasi ($n=2$).

Sinusoidal spiral birinchi marta Makloren tomonidan o‘rganilgan.



36- chizma. $\rho=a\sin\varphi,\ \rho=a\cos\varphi,\ a=2.$

31. Atirgul

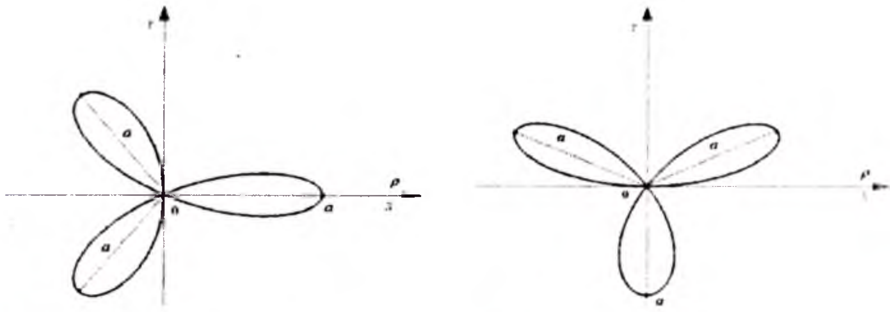
Gulning simvolik tasvirini eslatuvchi tekis egri chiziq – atirguldir. Uning qutb koordinitalar sistemasidagi tenglamasi

$$\rho=a\sin k\varphi$$

Ko‘rinishga ega. Bu yerda a va k lar , berilgan atirgulning o‘lchami (a) va uning yaproqlari soni (k) ni ifodalovchi o‘zgarmas sonlar.

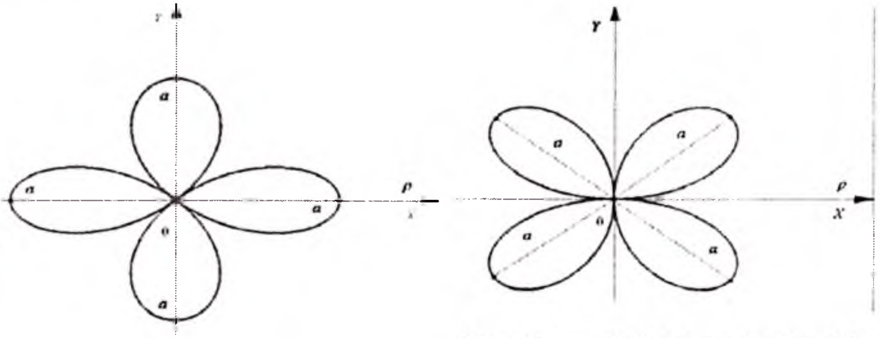
Egri chiziq to‘lasincha a radiusli aylana ichida joylashadi va $k>1$ bo‘lganda bir xil shakl va o‘lchamlardagi yaproqlardan tashkil topadi. Yaproqlarning soni , bu holda , k miqdor orqali aniqlanadi.

k butun son bo‘lganda , yaproqlar soni agar k – toq bo‘lsa, k ga teng bo‘lib, agar k juft bo‘lsa, $2k$ ga teng bo‘ladi. k son o‘zaro tub bo‘lgan m va n sonlarning nisbatidan iborat, ya’ni $k=\frac{m}{n}$ bo‘lsa, m va n lar toq bo‘lganda , atirgulning yaproqlari soni m ga, ulardan hyech bo‘lmaganda bittasi juft bo‘lganda esa, yaproqlar soni $2m$ ga teng bo‘ladi. k - irrasional son bo‘lsa, yaproqlar soni cheksiz ko‘p bo‘ladi.



1) $\rho=a\cos3\varphi$.

2) $\rho=a\sin3\varphi$.



37- chizma. Uch yaproqli atirguli1

1) $\rho=a\cos2\varphi$

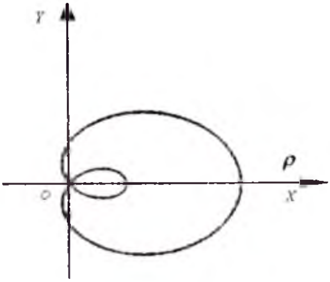
2) $\rho=a\sin2\varphi$

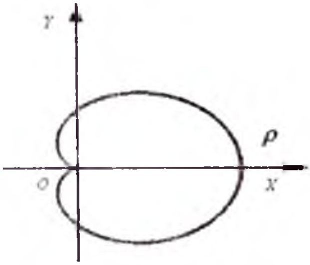
38- chizma. To‘rt yaproqli atirgular.

32. Paskal chig'anog'i

Tarixiy ma'lumotlar. Qaralayotgan chiziq mashhur fransuz olimi B. Paskalning (1623-1663 yy)otasi, ko'p yillar soliq tizimida faoliyat ko'rsatgan, lekin 1666 yilda Parij akademiyasiga aylantirilgan matematiklar va fiziklar kurojogining faol ishtirokchisi, E. Paskal (1588-1651 yy) sharafiga shu nom bilan ataladi. E. Paskal matematik qiziqishlari egri chiziqlar haqidagi ta'minot bilan bevosita bog'liq.

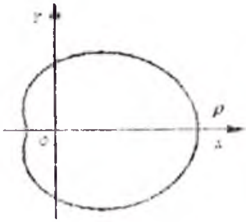
Paskal chig'anog'i texnikada keng qo'llaniladi. Semaforni ko'tarish va tushirish mexaniyasini tashkil etuvchi qismlardan buni Paskal chig'anog'i bo'yicha yasalgan.

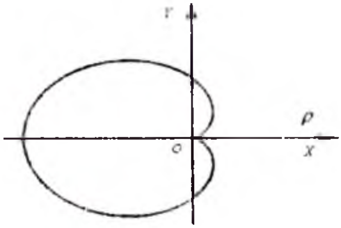




1) $2a > l$

2) $2a = l$





3) $2a < l$ 4) $\rho = 2a(1 - \cos\varphi),\ 2a > 0.$

39- chizma. Paskal chig'anog'i. $\rho = 2a\cos\varphi + l$, 1), 2), 3), 4).

33. Kardioida

Kardioidaning tenglamalari (39-chizma):

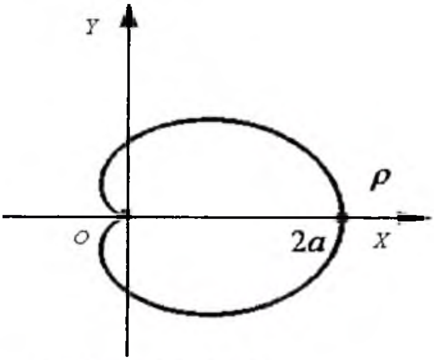
1) to‘g‘ri burchakli koordinatalar sistemasida

$(x^2 + y^2)(x^2 + y^2 - 2ax) - a^2x^2 = 0$;

2) qutb koordinatalar sistemasida $\rho = 2a(1 + \cos\varphi)$;

3) parametrik ko‘rinishda $\begin{cases} x = a\cos t\ (1 + \cos t), \\ y = a\sin t\ (1 + \cos t). \end{cases}$

kabi ifodalanadi.



40- chizma. Kardioida.

34. Beriulli lemniskatasi

Tarixiy ma’lumot. Chiziq uni tahlil qilgan olimning nomi bilan ataladi. Lemniskata tenglamasi birinchi marta Ya.Beriullining (1694y) maqolasida uchradi.

Ya.Beriulli (1654-1705) Shveysiyalik matematik, cheksiz kichik miqdorlar analizi bo‘yicha mashhur olim. Ya. Beriulli lemniskata, lagariflim spiral, zajir chiziq va hokazo chiziqlar xossalarini yangi g‘oyalar qo‘lladi. U matematikaning yangi sohasi variasion hisobiga asos soldi.

Beriulli lemniskatasi qiziq xossalarga ega va keng qo‘llaniladi. Lemniskatadan texnikada, xususan, tog‘li hududlardagi temir yo‘l shahobchalari, tramvay yo‘llardagi kichik radiusli aylanish joylarda o‘tish chizig‘i sifatida foydalaniladi.

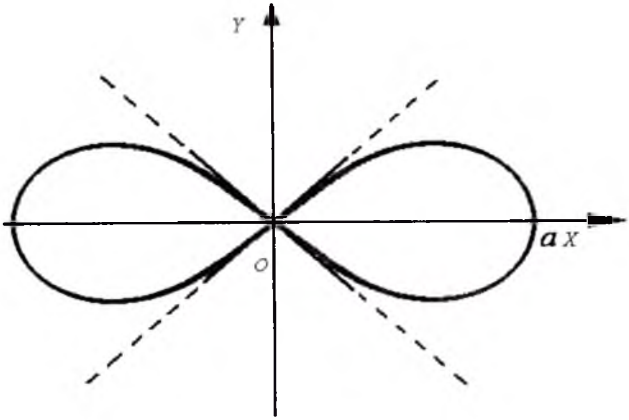
Kardioidaning tenglamalari (40-chizma):

1) to‘g‘ri burchakli koordinatalar sistemasida $(x^2+y^2)^2-a^2(x^2-y^2)=0$;

2) qutb koordinatalar sistemasida $\rho^2=a^2\cos 2\varphi$;

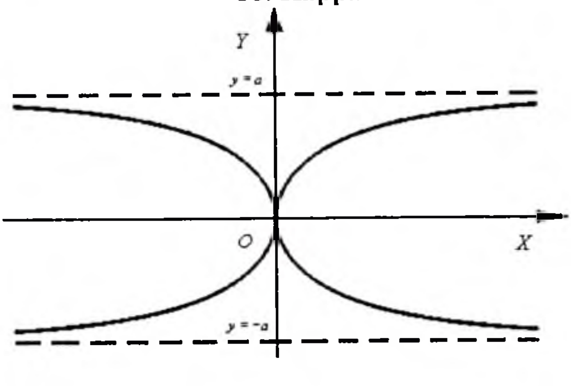
3) parametrik ko‘rinishda $\begin{cases} x=a\sqrt{2}\dfrac{p+p^3}{1+p^4}, \\ y=a\sqrt{2}\dfrac{p-p^3}{1+p^4}, \end{cases}$ bunda $p^2=tg\left(\dfrac{\rho}{4}-\varphi\right)$

kabi ifodalanadi.



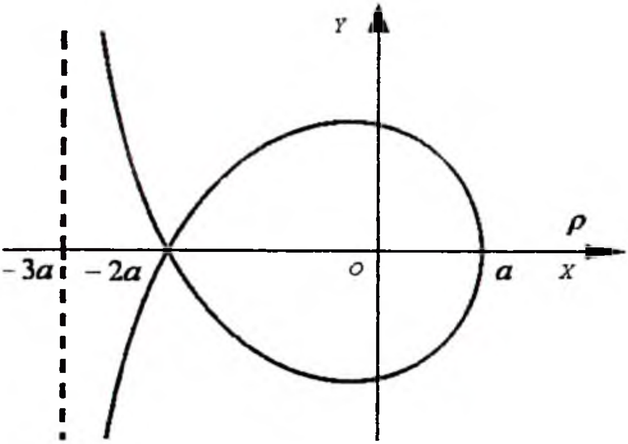
41- chizma. Bernulli lemniskatasi.

35. Kappa



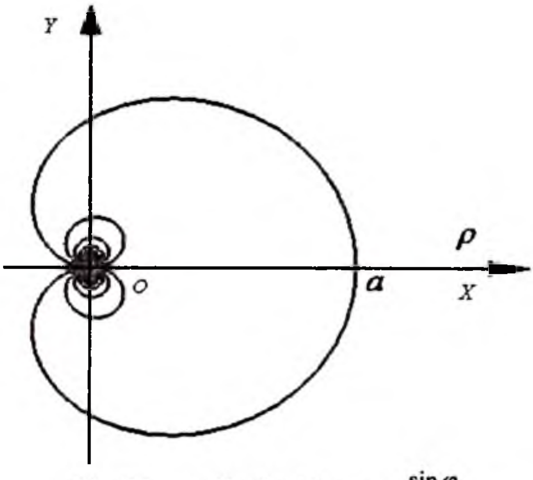
42- chizma. Kappa. $(x^2+y^2)y^2-a^2x^2=0$, $\rho=actg\varphi$.

36. Makloren trisektrisasi



43- chizma. Makloren trisektrisasi. $\rho = \dfrac{a}{\cos\dfrac{\varphi}{3}}$.

37. Koxleoida



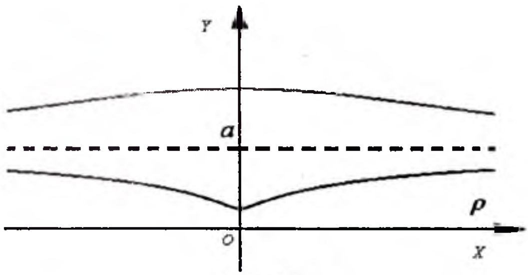
44 -chizma. Koxleoida. $\rho = a\dfrac{\sin\varphi}{\varphi}$.

38. Nikomend konxoidasi

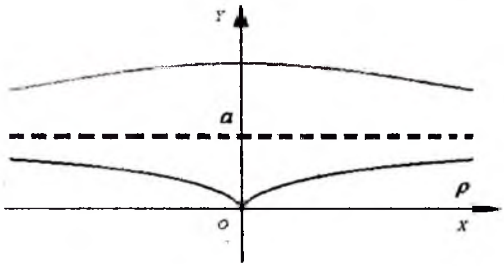
Tarixiy ma'lumotlar. Ushbu chiziqqa «konxoida», ya'ni «chanoqqa o'xshash» degan nom fanga yunon olimi D. Prokl (1410-485y) tomonidan kiritilgan.

Nikomed konxoidasi degan nom uni birinchi marta tahlil qilgan qadimgi yunon geometri, eramizdan avvalgi III-II asrlarda yashagan Nikomend bilan bog'lagandir. Nikomend ushbu chiziqni mexanik ravishda chiza oladigan asbob ham nashr qilgan, shuningdek chiziqning kubni ikkilashtirish hamda burchak triseksiyasi masalalarini yechimiga tadbiq qilgan.

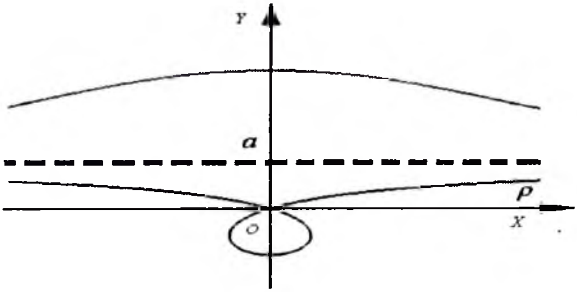
XVII va XVIII asrlarda Nikomed konxoidasini ko'p olimlar o'rganishgan. R. Dekart shu chiziqda o'zi kashf etgan egri chiziqqa normallar va o'rinmalar yasash usulini namoyish etgan. X. Gyuygens tomonidan konxonda va uning ba'zisi bilan chegaralangan shakl cheksiz katta yuzaga ega ekanligi ko'rsatilgan. I.Nyuton konxoidani uchinchi darajali teglamalarni geomayetrik yo'l bilan yechishga tadbiq qilgan.



1) $a > l > 0$



2) $a = l > 0$



3) $l > a > 0$

45-chizma. Nikomeda konxoidasi. $\rho = \dfrac{a}{\sin\varphi} + l$. 1), 2), 3).

39. Anyezi gajagi (versyera)

Tarixiy ma'lumotlar. Bu chiziq yangi davr Yevropasidagi birinchi ayol matematik olim nomli bilan atalgan. Mariya Gaetana Anyezi (1718-1799yy)).
Bolonya universiteti professori oilasida tavallud topdi. Bolonya universiteti bo'lib, u 1088 yilda ochilgan.

M. Anyezi 1748 yilda «Italiya o'smirlari uchun analiz asoslari» nomli ikki tomdan iborat asarni nashr qildirdi. Asarning birinchi tomi Eylergacha bo'lgan analitik geometriyaning batafsil va ravshan bayoniga bag'ishlangan. Shuningdek, kitobda avvalroq Ferma tomonidan o'rganilgan versyera (ya'ni Anyeza gajagi) ham qaralgan.

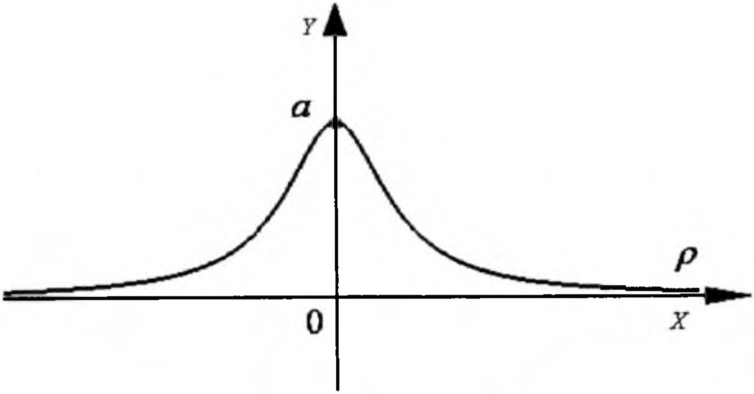
Versyera atamasini birinchi bo'lib, fanga italiya matematigi Gvido Graidi (1671-1742yy) kiritgan. U Piza unversiteti professori bo'lgan, asosiy ilmiy qiziqishlari geometriyaga taluqli, xususan, maxsus silliq chiziqlarni (bargsimon egri chiziqlar yoki «atirgullar»ni) o'rgangan. Bunday chiziqlar nazariyasi G.Graidi tomonidan 1728 yilda nashr qilingan mahallada o'z ifodasini topgan.

Anyezi gajagi (versyera)ning tenglamalari (46-chizma):

1) to'g'ri burchakli koordinatalar sistemasida: $(x^2 + a^2)y - a^3 = 0$;

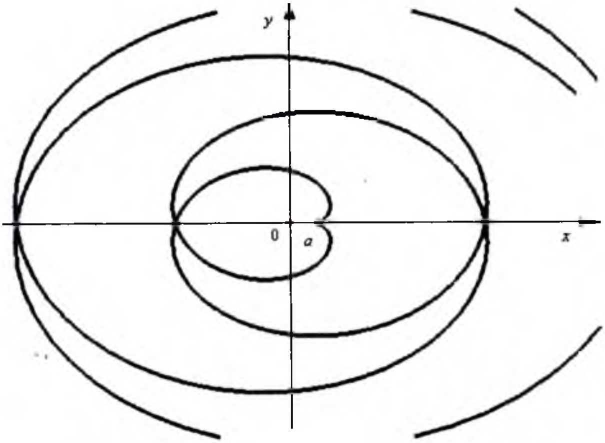
2) qutb koordinatalar sistemasida: $\rho = \dfrac{a\sin^2\varphi}{\cos\varphi}$

kabi ifodalanadi.



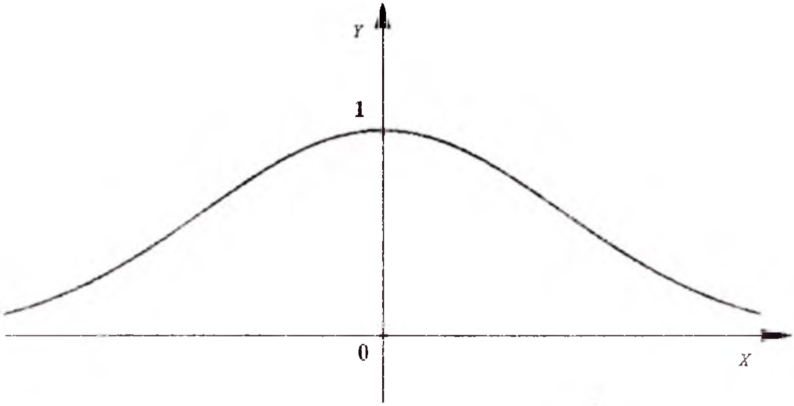
46-chizma. Anyezi gajagi.

40. Aylana evolentasi (yoyunchi)



47-chizma. Aylana evolentasi (yoyunchi).
$x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$.

41. Gauss funksiyasi



48-chizma. Gauss funksiyasi. $y = e^{-x^2}$.

42. Kassini ovali

Kassini ovali (tuxumsimon, cho'ziq, yassi shakl) – shunday nuqtalarning geometrik o'rnidan iboratki, ulardan berilgan ikkita nuqtalar (fokuslar) gacha bo'lgan masofalarning ko'paytmasi o'zgarmas bo'lib, qandaydir a sonning kvadratiga teng.

Kassini ovalining fokuslar orasidagi masofa $2a$ ga teng bo'lgandagi xususiy holi Bernulli lemniskatasidan iborat. Ovalning o'qi ikkita fokusli lemniskatadan iborat.

Bu egri chiziq astronom va muxandis Kassini tomonidan o'ylab topilgan.

Uning tenglamalari :

1) to'g'ri burchakli koordinatalarda $(x^2+y^2)^2 - 2c^2(x^2-y^2) = a^4 - c^4$ ko'rinishga ;

2) to'g'ri burchakli koordinatalardagi oshkor tenglamasi

$$y = \pm\sqrt{\sqrt{a^4 + 4c\,x^2} - x^2 - c^2}$$

ko'rinishga ;

3) qutb koordinatalari sistemasida

$$\rho^4 - 2c^2\rho^2\cos 2\varphi = a^4 - c^4$$

ko'rinishga ega.

Egri chiziq tenglamasida ikkita o‘zaro bog‘liq bo‘lmagan parametrlar: c – fokuslar orasidagi masofaning yarmi va a – fokuslardan egri chiziqning ixtiyoriy nuqtasigacha bo‘lgan masofalarning ko‘paytmasi ishtirok etadi.

U quyidagi xossalarga ega :

- Kassini ovali – to‘rtinchi tartibli algebraik egri chiziqdan iborat ;
- u fokuslar orasidagi kesmaning o‘rtasiga nisbatan simmetrikdir ;
- $0 < a \le c\sqrt{2}$ bo‘lganda ikkita absolyut maksimum va ikkita minimumga ega :

$$x = \pm\frac{\sqrt{4c^4 - a^4}}{2c}, \; y = \pm\frac{a^2}{2c}$$

Uning absolyut maksimum va minimum nuqtalarining geometrik o‘rni – markazi fokuslar orasidagi kesmaning o‘rtasida va radiusi s bo‘lgan aylanadan iborat.

- $-c < a << c\sqrt{2}$ bo‘lganda egri chiziq to‘rtta egilish nuqtalariga ega. Ularning qutb koordinatalari :

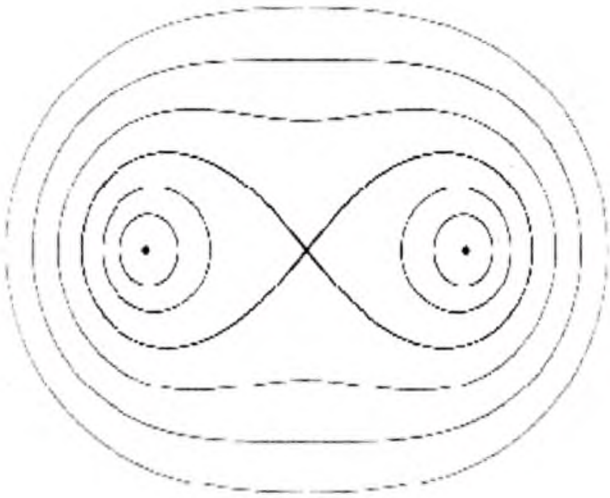
$$\begin{cases} \rho = \sqrt[4]{\dfrac{a^4 - c^4}{3}} \\ \cos 2\varphi = -\sqrt{\dfrac{1}{3}\left(\dfrac{a^4}{c^4} - 1\right)} \end{cases}$$

Uning egilish nuqtalarining geometrik o‘rni – uchlari $(0; \pm c)$ nuqtalarda bo‘lgan lemniskatadan iborat.

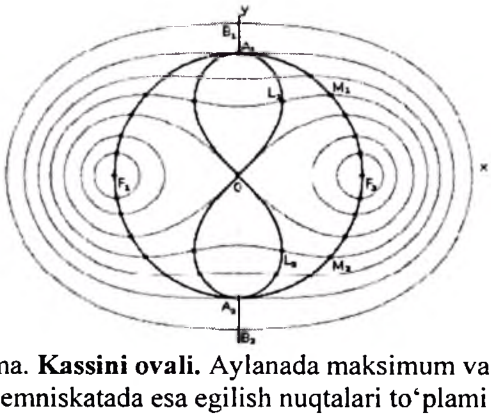
Ovalning egrilik radiusi qutb koordinatalarida

$$R = \frac{a^2\rho}{\rho^2 + c^2\cos 2\varphi} = \frac{2a^2\rho^3}{c^4 - a^4 + 3\rho^4}$$

formula orqali ifodalanadi.



49-chizma. **Kassini ovali.**



50-chizma. **Kassini ovali.** Aylanada maksimum va minimumlar to'plami,(2)- lemniskatada esa egilish nuqtalari to'plami ko'rsatilgan.

43. Kvadratrisa

Kvadratrisa - tekis transsendet egri chiziq bo'lib, Proklning yozishiga ko'ra, Gippiy (e.a. V asr) tomonidan ochilgan va qadim zamonlarda doira kvadraturasi va burchak triseksiyasi masalalarini yechishda undan foydalanilgan.

Egri chiziqning tenglamalari :

1) qutb koordinatalar sistemasida

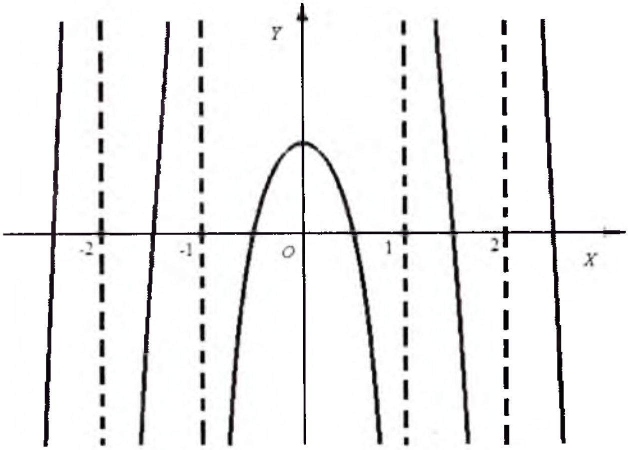
$$\rho = \frac{2R\varphi}{\pi \sin \varphi}$$

ko‘rinishda ;

2) to‘g‘ri burchakli koordinatalar sistemasida esa,

$$x = y ctg \frac{\pi y}{2R}$$

ko‘rinishida yoziladi.



51-chizma. **Kvadratrisa.** $x = y ctg \frac{\pi y}{4}$

44. Jerono lemniskatasi

Jerono lemniskatasi – ushbu

$$x^4 = a^2\left(x^2 - y^2\right)$$

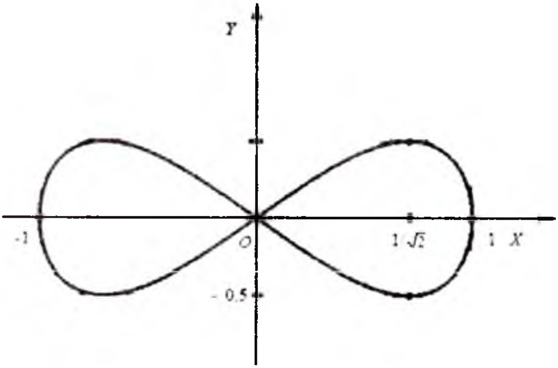
tenglamani qanoatlantiruvchi tekis egri chiziqdan iborat bo‘lib, uning xossalarini XIX asr boshida o‘rgangan fransuz matematig K.- K. Jerono nomi bilan ataladi.

Uning tenglamasini

$$y = \pm\sqrt{\frac{a^2x^2 - x^4}{a^2}}$$

ko'rinishda ham yozish mumkin.

Jerono lemniskatasining parametrik ko'rinishidagi tenglamasi quyidagicha : $x = \cos\varphi,\ y = \sin\varphi\cos\varphi$



52-chizma. **Jerono lemniskatasi,** $x^4 - x^2 + y^2 = 0$.

45. But lemniskatasi

But lemniskatasi – to'rtinchi tartibli algebraik tekis egri chiziq bo'lib, Persey egri chizig'ining xususiy holidir.

Uning to'g'ri burchakli Dekart koordinatalar sistemasidagi tenglamasi

$$\left(x^2 + y^2\right)^2 - \left(2m^2 + c\right)x^2 + \left(2m^2 - c\right)y^2 = 0$$

ko'rinishga ega.

Egri chiziqning shakli m va c parametrlar orasidagi munosabatga bog'liq. Agar $c > 2m^2$ bo'lsa, lemniskata tenglamasi

$$\left(x^2 + y^2\right)^2 = a^2x^2 + b^2y^2$$

ko'rinishni oladi, bunda $a^2 = 2m^2 + c$ va $b^2 = c - 2m^2$.

Agar $c < 2m^2$ bo'lsa, lemniskata tenglamasi

$$\left(x^2 + y^2\right)^2 = a^2x^2 - b^2y^2$$

ko'rinishni oladi, bunda $a^2 = 2m^2 + c$ va $b^2 = c - 2m^2$.

Xususiy hollari:

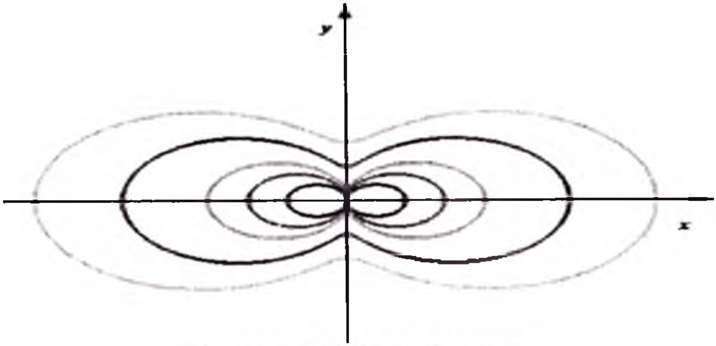
1. Agar $c = 2m^2$ bo'lganda, But lemniskata ikkita

$$x^2 + y^2 \pm 2mx = 0$$

aylanaga aylanadi.

2. Agar $c=0$ bo‘lganda, u Bernulli lemniskatasi aylanadi. Uning qutb koordinatalar sistemasidagi tenglamasi

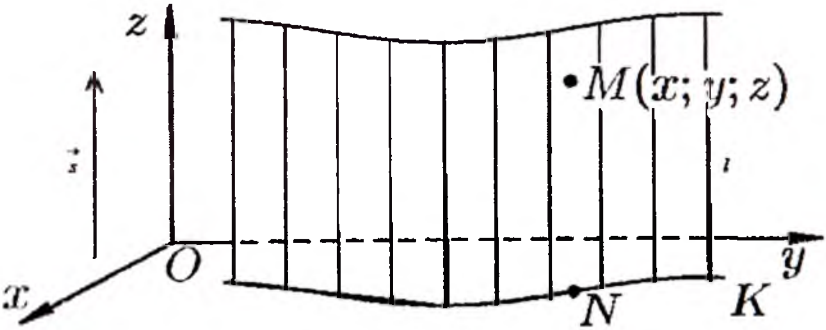
$$\rho^2 = a^2\cos^2\varphi + b^2\sin^2\varphi$$



53-chizma. **But lemniskatasi.**

46. R^3 fazoda ikkinchi tartibli sirtlar

46.1. Silindrik sirtlar. Berilgan $\vec{s}$ vektor yo‘nalishiga parallyelligicha qolib, berilgan K chiziqni kesadigan to‘g‘ri chiziqlar to‘plami **silindirik sirt** deyiladi (54- chizma). Bunda K chiziq silindirlik sirtning yo‘naltiruvchisi , $\vec{s}$ vektorga parallel l chiziqlar silindrik sirtining yasovchilari deyiladi.

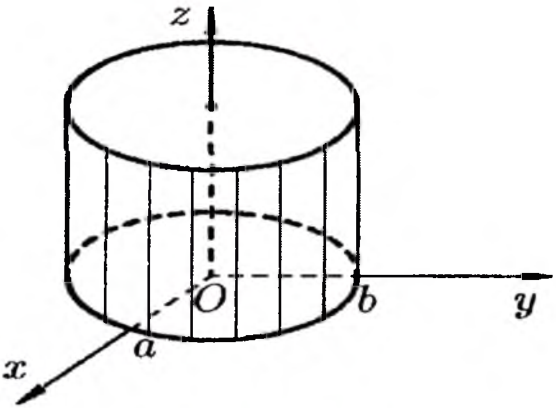


54- chizma.

Silindr – biror egri chiziqning nuqtalaridan o‘tuvchi parallel to‘g‘ri chiziqlar hosil qiladigan sirtdan iborat. Agar Oz silindrning yasovchisi deb qabul qilinsa, silindrik sirt $F(x,y)=0$ tenglana orqali berilishi mumkin.

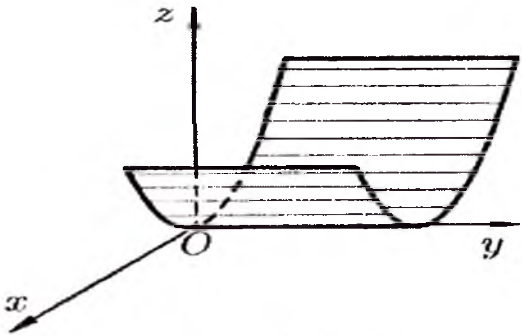
Agar sirtning $F(x,y,z)=0$ tenglamasida qandaydir o‘zgaruvchi qatnashmasa, bu sirt, yashovchi qatnashmayotgan o‘zgaruvchi o‘qiga parallel bo‘lgan silindrdan iborat bo‘ladi.

Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$, $\frac{z^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamalar, elliptik silindrik sirtntng tenglamalari deyiladi. 55-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ sirt tasvirlangan.



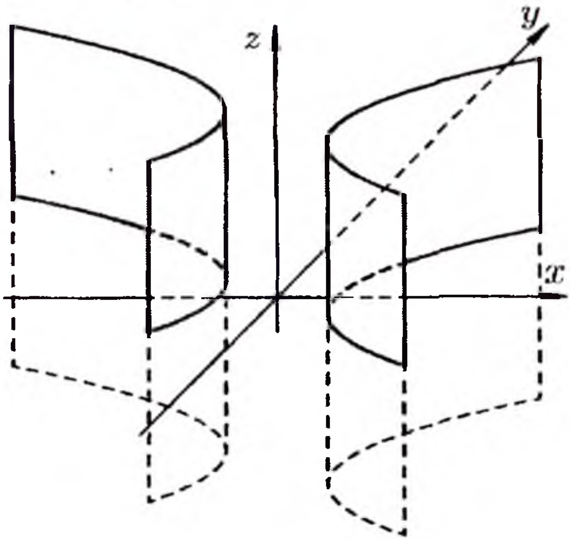
55-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ushbu $y^2=2px$, $x^2=2pz$, $z^2=2px$, $y^2=2pz$, $z^2=2py$, $x^2=2py$ tenglamalarga parabolik silindrik sirtning tenglamalari deyiladi. 56-chizmada $x^2=2pz$ sirt tasvirlangan.



56-chizma. $x^2 = 2pz$.

Ushbu $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$, $-\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$, $\frac{z^2}{a^2}-\frac{y^2}{b^2}=1$ $-\frac{z^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamalarga giperbolik silindrik sirtning tenglamalari deyiladi. 57-chizmada $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ sirt tasvirlangan.

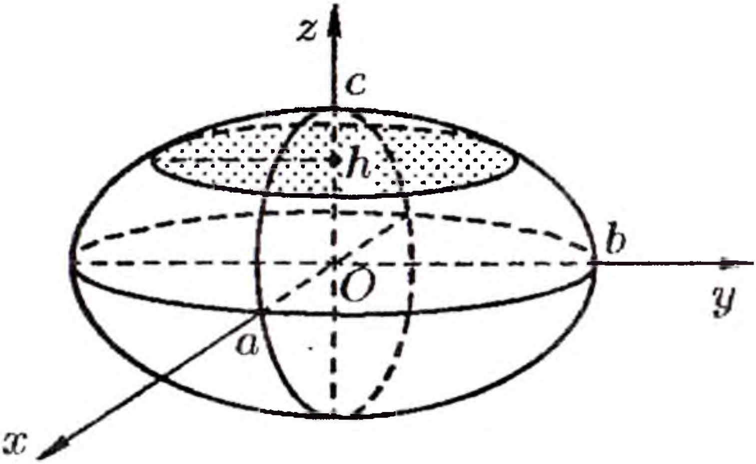


57-chizma. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

46.2. Aylanma sirtlar. Biror tekis L chiziqning l o‘qi atrofida aylanishidan hosil bo‘lgan nuqtalar to‘plami *aylanma sirt* deyiladi.

Ellipsoid. Ellipsning simmetriya o‘qi atrofida aylantirishidan hosil bo‘lgan aylanma sirtga *ellipsoid* deyiladi.

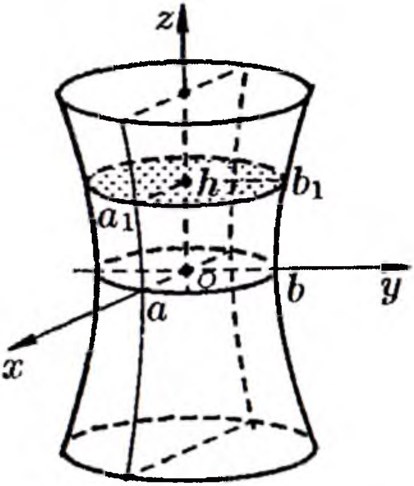
Ellipsoid – tenglamasini o‘zgaruvchilar almashtirish yordamida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ko‘rinishga keltirish mumkin bo‘lgan ikkinchi tartibli sirtdan iborat, bunda a,b,c–ellipsoidning yarim o‘qlari. 58-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ sirt tasvirlangan.



58-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Bir pallali giperboloid. Giperbolaning mavhum o‘q atirofida aylanishidan hosil bo‘lgan sirt *bir pallali aylanma giperboloid* deyiladi.

Bir pallali geperboloid – tenglamasi o‘zgaruvchilarni almashtirish yordamida $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ko‘rinishga keltirish mumkin bo‘lgan ikkinchi tartibli sirtdan iborat. Uning o‘qi – oldida «-» (manfiy) ishora turgan o‘zgaruvchiga mos keladi.



59-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$

Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

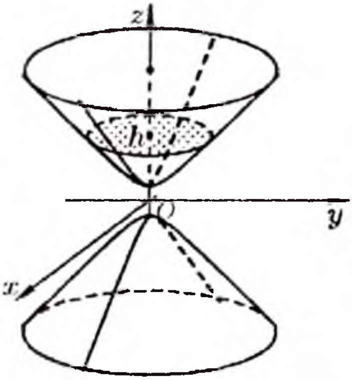
ko'rinishdagi tenglamaga bir pallali giperboloidning kanonink tenglamasi deyiladi. Koordinatalar boshi bir pallali giperboloidning markazi , (1) tenglama uchun a va b sonlar, bir pallali gepirboloidning haqiyqiy yarim o'qlari , c esa, mavhum yarim o'qi deyiladi. 59-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ sirt tasvirlangan.

Ikki pallali giperboloid. Giperbolani o'zining haqiqiy o'qi atrofida aylanishidan hosil bo'lgan sirt, *ikki pallali aylanma giperboloid* deyiladi.

Ikki pallali giperboloid – tenglamasini, o'zgaruvchilarni almashtirish yordamida, $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ ko'rinishga keltirish mumkin bo'lgan ikkinchi tartibli sirtdan iborat. Uning o'qi – oldida «-» (manfiy) ishora turgan o'zgaruvchiga mos keladi. Ushbu

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\ ,\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

ko'rinishdagi tenglama, ikki pallali giperboloidning kanonik tenglamasi deyiladi. 60-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ sirt tasvirlangan.



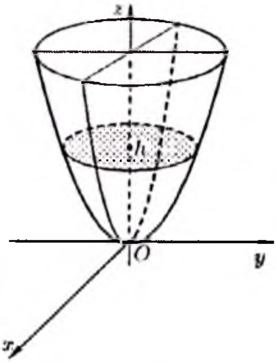
60-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1.$

Elliptik paraboloid. Paraboloidlarning o‘z o‘qlari atirofida aylanishdan hosil bo‘lgan aylanma sirt *yelliptik paraboloid* deyiladi.

Elliptik paraboloid – tenglamasini almashtirish yordamida $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ ko‘rinishga keltirish mumkin bo‘lgan ikkinchi tartibli sirtdan iborat. To‘g‘riburchakli dekart koordinatalari sistemasida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z,\quad \frac{y^2}{a^2}+\frac{z^2}{b^2}=2x,\quad \frac{x^2}{a^2}+\frac{z^2}{b^2}=2y$$

tenglama bilan tasvirlangan sirt *elliptik paraboloid* deyiladi. 61-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ sirt tasvirlangan.



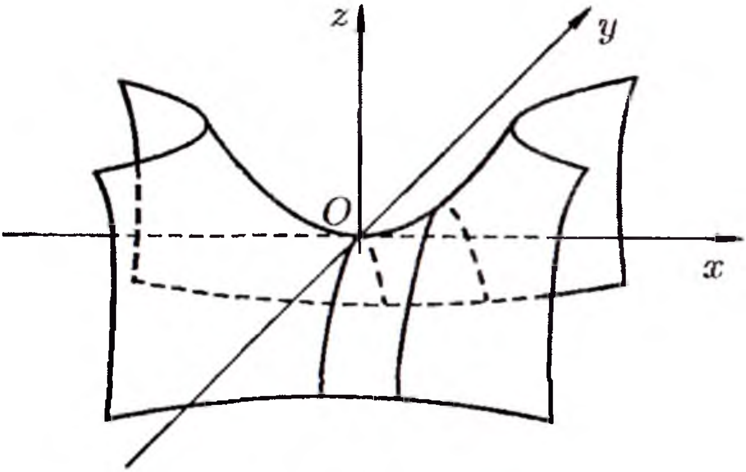
61-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z.$

Giperbolik paraboloid (egar) – tenglamasini o'zgaruvchilarni almashtirish yordamida $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ ko'rinishga keltirish mumkin bo'lgan ikkinchi tartibli sirtdan iborat. U $z=0$ koordinatalar tekisligini ikkita $bx-ay=0$ va $bx+ay=0$ to'g'ri chiziqlar bo'ylab kesib o'tadi. To'g'riburchakli dekart koordinatalari sistemasida

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,\ \frac{y^2}{a^2}-\frac{z^2}{b^2}=2x,\ \frac{x^2}{a^2}-\frac{z^2}{b^2}=2y,$$

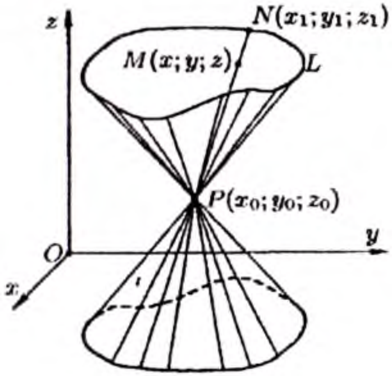
$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z,\ -\frac{y^2}{a^2}+\frac{z^2}{b^2}=2x,\ -\frac{x^2}{a^2}+\frac{z^2}{b^2}=2y,$$

tenglama bilan tasvirlangan sirt eliptik paraboloid deyiladi. 62-chizmada $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ sirt tasvirlangan.



62-chizma. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$.

46.3. Konus sirtlar. Fazoda biror qo'zg'olmas $P(x_0,y_0,z_0)$ nuqtadan o'tib, berilgan L chiziqni kesuvchi l chiziqning harakatidan hosil bo'lgan sirt ***konus sirt*** deyiladi (63-chizma). $P(x_0,y_0,z_0)$ nuqta konus sirtining uchi, L chiziq uning yo'naltiruvchisi, l to'g'ri chiziq esa, konus sirtining yasovchisi deyiladi.

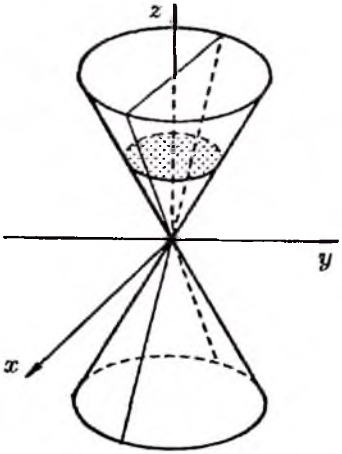


63-chizma.

Konus – bir jinsli $F(x,y,z)=0$ tenglama orqali beriladigan sirtdan iborat. Ikkinchi tartibli konus quyidagi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

ko'rinishga keltirish mumkin bo'lgan ikkinchi darajali bir jinsli tenglama orqali beriladi, bunda konusning o'qi manfiy kvadratga mos kelib, uning uchi – kordinatalar boshida yotadi. 64-chizmada $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ sirt tasvirlangan.



64-chizma. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$.

Tor. Tor – yasovchi aylananing shu aylana tekisligida yotuvchi o‘qi atrofida aylanishidan hosil bo‘lgan aylanish sirtidan iborat.

Tenglamalari

Parametrik tenglamasi:

$$\begin{cases} x(\varphi,\psi) = (R + r\cos\varphi)\cos\psi \\ y(\varphi,\psi) = (R + r\cos\varphi)\sin\psi \\ z(\varphi,\psi) = r\sin\varphi \end{cases}$$

bunda , $\varphi,\psi \in [0,2\pi]$, R – yasovchi aylana markazidan aylanish o‘qigacha bo‘lgan masofa, r – yasovchi aylananing radiusi.

Algebraik tenglamasi.

$$(x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0$$

Tor – to‘rtinchi tartibli sirtdan iborat. 65-chizmada

$$\begin{cases} x(\varphi,\psi) = (3 + \cos\varphi)\cos\psi \\ y(\varphi,\psi) = (3 + \cos\varphi)\sin\psi \\ z(\varphi,\psi) = \sin\varphi \end{cases}$$

sirt tasvirlangan.

Xossalari.

1. Tor sirtning yuzi: $S = 4\pi^2 Rr$ (Guldining 1 – teoremasidan kelib chiqadi).
2. Tor bilan chegaralangan jismning hajmi $V = 2\pi^2 Rr^2$ (Guldining 2 – teoremasidan kelib chiqadi).

Torsimon sirt dastlab qadimgi yunon matematigi Arxit tomonidan kubni ikkilantirish massasini yechishda qaralgan. Boshqa qadimgi yunon matematigi Persey smetrik chiziqlar torni uning o‘qiga paralel tekislik bilan kesganda hosil bo‘ladigan chiziqlar haqida kitob yozadi.

65-chizma.

Gelikoida. Gelikoid – to‘g‘ri chiziqning unga perpendikulyar o‘q atrofida va bir vaqtning o‘zida, shu o‘q yo‘nalishida ilgarilab, harakat qilishi natijasida (bunda harakatlarning tezliklari o‘zaro proporsional) hosil bo‘ladigan sirtdan iborat.

Uning parametrik tenglamasi

$$\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = hv \end{cases}$$

ko‘rinishda bo‘ladi. 66- chizmada $\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = hv \end{cases}$ sirt tasvirlangan.

Xossalari:

1. U minimal sirtdan iborat
2. U chiziqlimon sirtdir.

66- chizma.

Katenoid. Katenoid - $y = a\,ch\frac{x}{a}$ zanjir chiziqning Ox o‘q atrofida aylanishdan hosil bo‘lgan sirtdan iborat.

Uni parametrik koordinatalari yordamida, quyidagicha berish mumkin:

$$\begin{cases} x = ch(u)\cos(v), \\ y = ch(u)\sin(v), \qquad u \in R,\ v \in [0,2\pi] \\ z = u, \end{cases}$$

Katenoid 1744 yilda L.Eyler topib yechilgan (catena – zanjir, eidos - ko‘rinish).

Katenoidning uncha katta bo‘lmagan qismini izometrik ravishda (siqmasdan va cho‘zmasdan) gelikoidning qismiga o‘zgartirish mumkin, va aksincha

FOYDALANILGAN ADABIYOTLAR

1. Azlarov T.A, Mansurov X.T. Matematik analiz. 1- qism. -T.: «O'qituvchi» 1994.

2. Azlarov T.A., Mirzaahmedov M.A., Otaqo'ziyev D.O., Sobirov M.A., To'laganov S.T.- Matematikadan qo'llanma, II qism. T.: «O'qituvchi», 1990.

3. Ataxanov K.U., Yerzin V.A., Xodjayev B. – Matematik analizdan misol va masalalar to'plami, 1-qism, T., 2004.

4. Берман Г.Н. Сборник задач по курсу математического анализа.- М.: Наука,1985.

5. Бруй И.Н., Гаврилюк А.В и др. Лабораторный практикум по математическому анализу. - Минск.: Вышей школа, 1991.

6. Бруй И.Н., Гаврилюк А.В и др. Лабораторный практикум по математическому анализу. - Минск.: Вышей школа, 1991.

7. Gaziyev A., Israilov I., Yaxshiboyev M. Funksiyalar va grafiklar. –T.: "VORIS- NASHRIYOT", 2006.

8. Демидович Б.П. Сборник задач и упражнений по математическому анализу. -М. : Наука,1981.

9. Зорич В.А. Математический анализ. Ч. 1.- М. :Наука,1984.

10. Илин В.А, Садовничий В.А., Сендов Бл. Х. Математический анализ. Т. 1. -М.: Наука, 1979.

11. Коровкин Н.П. - Определенние интеграл и ряди. - М.: Учпедгиз., 1959.

12. Кудрявсев Л.Д. Курс математического анализа.Т.1.- М.: Бысшая школа, 1981.

13. Кудрявсев Л.Д., Кутасов А.Д., Чехов В.И., Шабунин М.И. Сборник задач по математическому анализу: интеграли, ряди.- М.: Наука, 1986.

14. Кудрявсев Л.Д., Кутасов А.Д., Чехов В.И., Шабунин М.И. Сборник задач по математическому анализу: функсии несколких переменных.-Санкт-Петербург, 1994.

15. Виленкин Н.Я., Бохан К.А., Марон И.А., Матвеев И.В., Смолнский М.А., Светков А.Т. Задачник по курсу математического анализа. Ч. 1.- М. : Наука,1971.

16. Виленкин Н.Я., Бохан К.А., Марон И.А., Матвеев И.В., Смолнский М.А., Светков А.Т. Задачник по курсу математического анализа. Ч. И И.- М. : Наука,1971.

17. Ляшко И.И., Боярчук А.К., Гай Я.Г., Голович Г.П. - Справочное пособие по математическому анализу. Киев.: «Висшая школа», 1984.

18. Марон И.А.- Дифференсиалное и интегралное исчисление в примерах и задачах. - М.: «Наука», 1970.

19. Матросов А. Марле-6. Решения задач висшей математики и механики. Питер.Санкт-Петербург, 2000.

20. Николский С.М.Курс математического анализа.Т.1.-М.: «Наука», 1983.

21. Sadullayev A., Mansurov X., Xudoyberganov G., Vorisov A., G'ulamov R. Matematik analiz kursidan misol va masalalar to'plami. 1-qism.- T., 1993.

21. Sadullayev A., Mansurov X., Xudoyberganov G., Vorisov A., G'ulamov R. Matematik analiz kursidan misol va masalalar to'plami. 2-qism.- T.. 1995.

22. Фихтенголс Г.М. Курс дифференсиалного и интегралного исчисления. Т. 1. -М.: Наука,1969.

23.Finney, Weir, Giordano Thomas'CALCULUS. 10-th edition.- Boston, San Francisco, New York, London, Toronto. Sydney, Tokyo, Singapore. Madrid....2001, 1256 pp.

24. Salas Hille Engen. Calculus one variable. 8-th edition.Wiley&Sons. - New York, chichester, Weinheim. Brisbane Singapore Toronto, 1999, 708 pp.

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2 - QISM

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